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Cyclic Renormalization and Automorphism Groups of Rooted Trees

Springer

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I am the Lorax, I speak for the trees...[‡]

For the ones we love...

[‡]From *The Lorax*, Dr. Seuss

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Chapter 0

Introduction

The motivation behind this monograph derives from a relation between renormalizability of certain dynamical systems on the unit interval and group actions on rooted trees. Certain classes of maps of the unit interval, when restricted to invariant Cantor sets, have the form of such automorphisms and better understanding the structure of such automorphism groups contributes to a fuller understanding of the types of maps which have these sorts of restrictions. While a priori these two subjects have quite different audiences, it is our hope that the link we draw between the two may persuade others to investigate similar potential bridges between algebra and dynamics.

Some of renormalization group theory can be traced to connections made between the classes of maps $f(x) = Rx(1-x)$ and $g(x) = S\sin(\pi x)$, for varying parameters R and S. For particular parameter values (at the so-called "accumulation of period doubling" or "boundary of chaos") these maps have infinitely many periodic orbits and the periods of these orbits are all of the form 2^n. It is well known that these two maps share many "smooth characteristics". For example, both maps have invariant Cantor sets of the same Hausdorff dimension; the bifurcation structures of the two parametrized families also have similar metric properties near the two maps $f(x)$ and $g(x)$, properties which are shared by all maps and families in the same so-called universality class.

These characteristics were first observed numerically in the physics community [CT, Fe1, TC1]. It was conjectured that an analogous phenomenon played a role in the transition to chaos in certain experiments in fluid dynamics [CT]. Physicists described this phenomenon in terms of a class of techniques which in the areas of quantum field theory and statistical mechanics has come to be called called renormalization group theory.

Much of this monograph can be viewed as part of the ongoing effort directed towards the mathematical development of renormalization group theory. Recent work (cf. [Su, Mc1, Mc2]) has made great strides in this direction, but many open questions remain.

As stated at the opening, the connection with group theory comes from considering the dynamical systems at the accumulation of period doubling when restricted to their invariant Cantor sets. These Cantor sets are naturally viewed as the ends of a rooted tree with corresponding action given by a particular element of the automorphism group of the tree. The structure of the full automorphism group sheds light on the possible dynamical systems obtained in this fashion and conversely, dynamical considerations indicate possible directions for better understanding the group structure. To say things a bit more precisely, recall that dynamical systems are often studied in terms of periodic structure. The basic periodic systems are $\mathbb{Z}_n := (\mathbb{Z}/n\mathbb{Z}, +1)$, where $+1$ sends r to $r+1$ for $r \in \mathbb{Z}/n\mathbb{Z}$. For a dynamical system (K, f) (K a topological space and $f : K \longrightarrow K$ a continuous map), an orbit of period n is an embedding $\mathbb{Z}_n \longrightarrow (K, f)$. Here we study the dual notion of a (necessarily surjective) morphism $R : (K, f) \longrightarrow \mathbb{Z}_n$, which we call an *n-renormalization* of (K, f). Thus K is the disjoint union of the (open and closed) fibers $K_r = R^{-1}(r)$ and f sends K_r to K_{r+1} for $r \in \mathbb{Z}/n\mathbb{Z}$. Let $Per(K, f)$ (resp., $Ren(K, f)$) denote the set of integers n for which (K, f) admits an orbit of period n (resp. an n-renormalization).

Suppose that (I, f) is a dynamical system on a compact real interval I. The linear order of I influences the above notions as follows. An orbit of period n is ordered, $x_1 < x_2 < \ldots < x_n$, and the action of f defines a permutation σ of the indices; σ belongs to the set C_n of n-cycles in the symmetric group S_n. Let $C_n(f)$ denote the set of n-cycles that so occur, and $C(f)$ the union of the $C_n(f)$, contained in the union C of the C_n. Define a "forcing" relation, $\Rightarrow$, on the natural numbers, and on the set C, as follows: Let $n, m \in \mathbb{N}$, and $\sigma, \tau \in C$. Then $n \Rightarrow m$, (resp., $\sigma \Rightarrow \tau$), means that, for all f as above, $n \in Per(f)$ implies that $m \in Per(f)$, (resp., $\sigma \in C(f)$ implies $\tau \in C(f)$). A remarkable theorem of Sharkowskii says that $\Rightarrow$ is an (explicit) total order on the natural numbers. (See II, (11.2) below.) Thus $Per(f)$ is always a terminal segment for the Sharkowskii order; f has entropy zero iff $Per(f)$ consists of a sequence (finite or infinite) $1, 2, 4, 8, \ldots$ of consecutive powers of 2. On the set C of cycles, $\Rightarrow$ can sometimes go opposite to the Sharkowskii order.

For renormalizations, suppose now that K is a minimal closed invariant subset of I, and that $R : (K, f) \longrightarrow \mathbb{Z}_n$ is an n-renormalization, as above. We call R an *interval n-renormalization* if each of the fibers K_r is an interval of K, and write $IRen(K, f)$ for the set of integers n for which (K, f) admits an interval n-renormalization. A simple, but fundamental, result (I, (2.6)) is that $IRen(K, f)$ is totally ordered by divisibility. This permits us to coherently organize the interval renormalizations of (K, f) into an inverse sequence

$$(*) \ldots \longrightarrow (\mathbb{Z}/n_i\mathbb{Z}, +1) \longrightarrow (\mathbb{Z}/n_{i-1}\mathbb{Z}, +1) \longrightarrow \ldots \longrightarrow (\mathbb{Z}/n_0\mathbb{Z}, +1)$$

where $\{n_0 = 1 < n_1 < n_2 < \ldots\} = IRen(K, f)$. In turn this defines a (surjective) morphism ϕ from (K, f) to the inverse limit $(\widehat{\mathbb{Z}}, +1)$ of (*), sometimes called a "q-adic adding machine." When $IRen(K, f)$ is infinite (f is "infinitely interval renormalizable") then we show that both ϕ and f are injective, except perhaps for countably many 2-point fibers (which actually occur in given examples). Moreover each n in $Ren(K, f)$ divides some m in $IRen(K, f)$. (See I, (4.1).) We also show that every set of natural numbers totally ordered by divisibility can be realized as some $IRen(K, f)$. It can happen that $IRen(K, f)$ is finite even when $Ren(K, f)$ is infinite.

Interval renormalization has some relation to periodic structure. For example $IRen(K, f)$ is contained in $Per(I, f)$; in fact the indicated periodic orbits occur in the convex hull of K in I, and K is contained in the closure of their union. Further, the complement of the union of the periodic orbits in its closure is sometimes the place to find a minimal closed invariant set K as above.

Chapter II interprets interval renormalizations for unimodal maps in terms of a $*$-product on "itineraries", in the sense of Milnor-Thurston. This permits us to invoke theorems about the itinerary behavior of quadratic maps to deduce analogous results about the interval renormalization structure of such maps.

In Chapter III we take the point of view that the inverse sequence (*) can be interpreted as a rooted tree X, which is "spherically homogeneous," on which $+1$ acts as a "spherically transitive" automorphism. We show that, in $G = Aut(X)$, the spherically transitive automorphisms form a single conjugacy class. Given an interval n-renormalization $R : (K, f) \longrightarrow \mathbb{Z}_n$, we obtain new dynamical systems from f^n restricted to the fibers K_r of R. The latter may or may not be interval equivalent to the original (K, f). In Chapter III, Section 6, we study a group theoretic analogue of this problem.

Finally, in Chapter IV we investigate the normal subgroup structure of $G = Aut(X)$, using a description of G as an infinite iterated wreath product of symmetric groups. In the course of this we construct certain abelian characters (multi-signatures in the case of the dyadic tree) in terms of which one can characterize the spherically transitive automorphisms. The kernels of restrictions of G to finite radius balls centered at the root of X define natural normal subgroups of G, which are somewhat analogous to principal congruence subgroups in p-adic algebraic groups. These and certain abelian characters defined on them afford a general description of the normal subgroups of G. This analysis applies as well to certain subgroups of G also constructed as iterated wreath products.

It is natural to ask if the ideas in this paper extend to higher dimensions. It appears that one of the fundamental facts which allow group theory to play a role in the combinatorial discussion of interval maps is that periodic orbits of such maps are naturally described by permutations, as determined by the linear order of the orbit in the real line. Furthermore, continuity of the map implies that the permutation representing a periodic orbit in one dimension yields some information about the map as it provides some information about the way in which the intervals between points are mapped. In general, in higher dimensions we appear to lose the natural ordering as well as strong influence of finite orbits on the large scale structure of the map. In two dimensions though, the latter aspect does seem to have a natural counterpart in the form of the mapping class of the map restricted to the punctured manifold obtained by removing the periodic orbit. After selecting a suspension on the map, there is an associated braid, and the braid group in dimension two in some sense replaces the symmetric group used in the one-dimensional dynamics discussed here. These sorts of ideas may be found in [GST] and the references therein where analogous considerations permit the definition and investigation of two-dimensional infinitely renormalizable dynamical systems. Despite this remark it remains unclear as to what sorts of objects would assume that role that trees play in the one-dimensional case.

Remark. Our bibliography is far from exhaustive and we apologize for any instances in which we may not have given proper credit. For a fairly comprehensive bibliography for one-dimensional dynamics see [MS]. Those interested in the more group theoretic aspects of actions on trees might start with [Se]. The book [Rot] is an excellent group theory resource. A classic text for permutation groups is [Wie] while the paper [We] serves as a nice introduction to wreath products and contains many early references to the origins of the subject.

Overview of Chapter I

Let (K, f) be a dynamical system, consisting of a topological space K and a continuous map $f : K \longrightarrow K$. **Renormalization** procedures generally involve a choice of some subspace $H \subset K$ and an appropriate "first return" map of f-orbits, starting in H, back to H. The resulting dynamical system on H is then called a **renormalization of** (K, f). In general, the time of first return varies with the point of departure, and the f-transforms of H need not cover K. (See, for example, [MS] for interval dynamics, [LyMil] for interval dynamics with non-uniform return times, and [BRTT] for renormalization on n-tori.)

The "cyclic renormalizations" that we study here correspond to a fixed time of return, given by some integral power f^n of f. More precisely, an n-**renormalization** of (K, f) is, for us, a morphism of dynamical systems

$$\phi_n : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1). \tag{1}$$

Thus the fibers $K_r = \phi_n^{-1}(r)$ $(r \in \mathbb{Z}/n\mathbb{Z})$ are open-closed sets partitioning K, and $f(K_r) \subset K_{r+1}$ for all r. Then each (K_r, f_r) where $f_r = f^n \,|_{K_r}$ is a renormalization of (K, f) in the sense described above. The fiber K_0 here corresponds to the H above, and its f-transforms cover K.

If (K, f) is **minimal**, i.e. if each f-orbit is dense in K, then the K_r are just the f^n-orbit closures, and each (K_r, f_r) is again minimal. Moreover ϕ_n in (1) above is determined by n, up to a translation of $\mathbb{Z}/n\mathbb{Z}$. We put

$$Ren(K, f) = \{n \geq 1 \mid (K, f) \textit{ admits an } n\textit{-renormalization}\}. \tag{2}$$

Then (cf. (1.5)) the set $Ren(K, f)$ is stable under divisors and LCM's (least common multiples). It is convenient to introduce (cf. (1.6)) the **supernatural number**‡

$$Q = Q(K, f) = LCM(Ren(K, f)). \tag{3}$$

Then

$$Ren(K, f) = Div(Q) = \{\textit{integral divisors of } Q\}. \tag{4}$$

Assume that some $x_0 \in K$ has a dense f-orbit. Choosing our n-renormalization ϕ_n $(n \in Ren(K, f))$ so that $\phi_n(x_0) = 0$, they form an inverse system (with respect to divisibility) and we obtain a morphism

$$\widehat{\phi}_Q : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_Q, +1), \tag{5}$$

where $\widehat{\mathbb{Z}}_Q = \varprojlim_{n|Q} \mathbb{Z}/n\mathbb{Z}$, and $(\widehat{\mathbb{Z}}_Q, +1)$ is called the ***Q*-adic adding machine.**

Now suppose that (K, f) arises from a dynamical system $g : I \longrightarrow I$ on a closed real interval $I = [a, b]$, $a < b$, with K a minimal closed g-invariant

‡A **supernatural number** Q is a formal product, $Q = \prod_p p^{e_p}$, p varying over all primes, and $0 \leq e_p \leq \infty$ for each p. It is clear what it means for one such number to divide another.

subset and $f = g|_K$. Then K inherits a (linear) order structure from I, so we may speak of K-intervals. Moreover the topology on K is the order topology. An n-renormalization ϕ_n of (K, f) as in (1) above is called an **interval n-renormalization** if its fibers K_r are all K-intervals. These form a partition of K by intervals, so they occur in a definite order in K. This defines a unique linear order on $\mathbb{Z}/n\mathbb{Z}$ so that ϕ_n is weak order preserving (i.e. ϕ_n preserves $\leq$). We put

(6) $$\begin{aligned} IRen(K,f) &= \{n \geq 1 \mid (K,f) \textit{ admits an interval } n\textit{-renormalization}\} \\ &\subset Ren(K,f). \end{aligned}$$

The fundamental observation about this (Theorem (2.6)) is:

(7) *$IRen(K, f)$ is totally ordered by divisibility.*

Thus we can write:

(8) $$IRen(K,f) = \{n_0(=1) < n_1 < n_2 < \cdots\}, \textit{ with } n_i \mid n_{i+1}.$$

Put

(9) $$q_i = \frac{n_i}{n_{i-1}},$$

so that

(10) $$n_h = q_1 \cdot q_2 \cdots q_h.$$

Of course $IRen(K, f)$ may be finite or infinite. We put

(11) $$\mathbf{q} = \mathbf{q}(K,f) = (q_1, q_2, q_3, \ldots)$$

and call this the **interval renormalization index** of (K, f). As above, we obtain a natural morphism

(12) $$\widehat{\phi}_{\mathbf{q}} : (K,f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1),$$

where $\widehat{\mathbb{Z}}_{\mathbf{q}} = \varprojlim_h \mathbb{Z}/n_h\mathbb{Z}$, and $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$ is called the $\mathbf{q}$-adic adding machine. In terms of supernatural numbers, we have

(13) $$\widehat{\mathbb{Z}}_{\mathbf{q}} = \widehat{\mathbb{Z}}_{Q(\mathbf{q})},$$

where

(14) $$Q(\mathbf{q}) = LCM(IRen(K,f)) = \prod_h q_h.$$

The K-induced order on each $\mathbb{Z}/n_h\mathbb{Z}$ gives, in the limit, a linear order on $\widehat{\mathbb{Z}}_{\mathbf{q}}$ so that $\widehat{\phi}_{\mathbf{q}}$ is weak order preserving.

If $K = \{1, 2, \ldots, N\}$, with its natural order, then f is just a transitive permutation of K, and $IRen(K, f) = \{n_0 = 1 < n_1 < n_2 < \cdots < n_m = N\}$ with $n_{i-1} \mid n_i \quad (i = 1, \ldots, m)$. In the special case that $N = 2^M$ we have $IRen(K, f) = \{1, 2, 4, 8, \ldots, 2^M\}$ if and only if f is a simple permutation in the sense of [Bl].

We can summarize many of our results in Chapter I as follows. Suppose that we are given

$$Q = \text{ a supernatural number} \tag{15}$$

and

$$\begin{array}{l}\mathbf{q} = (q_1, q_2, q_3, \ldots) \text{ a sequence, finite or infinite, of} \\ \text{integers } q_i \geq 2, \text{ such that } Q(\mathbf{q}) := \prod_i q_i \text{ divides } Q.\end{array} \tag{16}$$

Then the results of Sections 1, 3, and 4 give the following.

Theorem. *(a) There is a compact real ordered minimal dynamical system (K, f) such that $Q(K, f) = Q$ and $\mathbf{q}(K, f) = \mathbf{q}$ iff either $\mathbf{q}$ is finite, or $\mathbf{q}$ is infinite and $Q(\mathbf{q}) = Q$.*

(b) Suppose that (K, f) is a compact real dynamical system with a dense orbit, and that (K, f) is infinitely interval renormalizable, i.e. that $\mathbf{q} = \mathbf{q}(K, f)$ is infinite. Then

$$\widehat{\phi}_{\mathbf{q}} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$$

is surjective, and injective except perhaps for countably many 2-point fibers. Moreover $f : K \longrightarrow K$ is surjective, and injective, except perhaps, for countably many 2-point fibers. Further, K is a Cantor set and (K, f) is minimal.

In (4.6) we construct, using a "Denjoy expansion technique", examples where the 2-point fibers of $\widehat{\phi}_{\mathbf{q}}$ and f do in fact occur.

In Section 4 we anticipate examples (constructed in Ch. II, Section 3) of (K, f) where K is a minimal closed invariant set for a unimodal dynamical system f on a real interval I, and with $IRen(K, f)$ prescribed in advance. In Section 5 we relate $IRen(K, f)$ to periodic points of (I, f). Self-similarity operators are defined in Section 6.

Overview of Chapter II

Let (J, f) be a unimodal map on a real interval $J = [a, b]$, with maximum $M = f(C)$, increasing on $L = [a, C)$, and decreasing on $R = (C, b]$. Then each $x \in J$ has an "address" $A(x) \in \{L, C, R\}$ such that $x \in A(x)$. The f-orbit $f^*(x) = (x, f(x), f^2(x), \ldots)$ then has an address

$$Af^*(x) = (A(x), Af(x), Af^2(x), \ldots),$$

called the "itinerary" of x. The itinerary

$$K(f) = Af^*(M)$$

of the "postcritical orbit" is called the **kneading sequence** of f [MilTh] (see also [MSS, My]). It symbolically encodes much of the dynamics of (J, f), especially on the f-orbit closure $\overline{O_f(M)}$ of M.

Consider the monoid $G = G_0 \cup G_0 C$, where G_0 is freely generated by $\{L, R\}$, and subject to the relations $CX = C$ for all $X \in G$. We interpret itineraries as either finite words in $G_0 C$, or as infinite words, in $\hat{G}_0$, with letters L and R.

The central aim of Chapter III is to show how an interval n renormalization of (K, f), where $K = \overline{O_f(x)}$, is reflected in a "$\star$-product" factorization, $Af^*(x) = \alpha \star \beta$ where $\alpha \in G_0$ has length $n-1$ (cf. Theorem (7.1)). In fact $IRen(K, f)$ can be intrinsically recovered from the itinerary $Af^*(x)$ (cf. (9.4)). By iterated star products, we construct in Section 8, elements $\kappa \in \hat{G}_0$ with prescribed initial renormalization. Then in Section 9 we quote results of [MilTh] (see also [CEc]) affirming that all such κ can be realized in the form $Af^*(x)$, $x \in J$, where (J, f) is a quadratic unimodal map.

Sections 10 and 11 relate the previous discussion to periodic orbits, and cyclic permutations.

Overview of Chapter III

Chapters III and IV are essentially group theoretic. They are partly motivated by Chapters I and II, but are mathematically independent of them.

First the motivation. Let (K, f) be a minimal ordered dynamical system that is infinitely interval renormalizable. Put

$$IRen(K, f) = \{n_0 = 1 < n_1 < n_2 \cdots\} \tag{1}$$

and

$$\mathbf{q} = \mathbf{q}(K, f) = (q_1, q_2, a_3, \ldots) \tag{2}$$

with $q_i = n_i / n_{i-1}$.

Then we have the inverse sequence of sets

$$X_0 \xleftarrow{p} X_1 \xleftarrow{p} X_2 \xleftarrow{p} \cdots \tag{3}$$

where

$$X_m = \mathbb{Z}/n_m\mathbb{Z} \tag{4}$$

and p is the natural projection. The interval renormalizations

$$(K, f) \longrightarrow (X_m, g_m) = (\mathbb{Z}/n_m\mathbb{Z}, +1) \tag{5}$$

induce

$$\widehat{\phi} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1) = \varprojlim_m (X_m, g_m). \tag{6}$$

An inverse sequence of (finite) sets, as in (3), can be interpreted as a (locally finite) rooted tree, X, with vertex set

$$VX = \coprod_{m \geq 0} X_m,$$

its root being the single vertex $x_0 \in X_0$ and with edges joining x to $p(x)$ for all $x \neq x_0$. Then X_m is the sphere of radius m centered at x_0. In our case, each $p : X_m \longrightarrow X_{m-1}$ is a surjective homomorphism with kernel of order q_m. Hence, each fiber of $p : X_m \longrightarrow X_{m-1}$ has q_m elements, so X is what we call a **spherically homogeneous rooted tree of index q**. Moreover, the maps $g_m : X_m \longrightarrow X_m$ assemble to define an automorphism g of the rooted tree X which acts transitively on each of the spheres X_m; i.e. g is what we call a **spherically transitive automorphism of the rooted tree X.**

In Chapter III we study rooted trees X defined by any inverse sequence of finite sets

$$(7) \qquad X_0 = \{x_0\} \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$$

and their automorphism groups

$$(8) \qquad G = Aut(X).$$

Then X admits spherically transitive automorphisms iff X is spherically homogeneous, say of index

$$(9) \qquad \mathbf{q} = \mathbf{q}(X) = (q_1, q_2, q_3, \ldots),$$

where q_m is the cardinal of each fiber of $p : X_m \longrightarrow X_{m-1}$. In this case, $\mathbf{q}$ determines X up to isomorphism so we can write

$$(10) \qquad X = X(\mathbf{q}) \text{ and } G = G(\mathbf{q}) = Aut(X(\mathbf{q})).$$

In Theorem (4.6) we show that the spherically transitive automorphisms of $X(\mathbf{q})$ are all conjugate in $G(\mathbf{q})$. Moreover, if g is one of them, then $Z_{G(\mathbf{q})}(g) = \overline{\langle g \rangle} \cong \widehat{\mathbb{Z}}_{\mathbf{q}}$, where $\overline{\langle g \rangle}$ denotes the closure of the cyclic group $\langle g \rangle$ in the profinite group $G(\mathbf{q})$, and $\widehat{\mathbb{Z}}_{\mathbf{q}}$ denotes, as above, the q-adic integers.

In the course of this discussion we obtain a description of $G(\mathbf{q})$ as an infinite iterated wreath product. This structure is used in Section 6 to analyze some group theoretic "renormalization operators".

Overview of Chapter IV

This chapter gives a fairly detailed analysis of the normal subgroups of

$$(1) \qquad G(\mathbf{q}) = Aut(X(\mathbf{q})),$$

where

$$(2) \qquad \mathbf{q} = (q_1, q_2, a_3, \ldots),$$

and of certain of its other subgroups, defined as follows. Let

$$\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots) \tag{3}$$

be a sequence of sets with

$$|\, Y_m \,| = q_m. \tag{4}$$

Then we can define $X(\mathbf{q})$ by the inverse sequence

$$X_0 \xleftarrow{p} X_1 \xleftarrow{p} X_2 \xleftarrow{p} \cdots \tag{5}$$

where $X_m = Y_1 \times Y_2 \times \cdots \times Y_m$ and $p : X_m \longrightarrow X_{m-1}$ is a projection away from the last factor. Let

$$\mathbf{Q} = (Q_1, Q_2, Q_3, \ldots) \tag{6}$$

be a sequence of groups with Q_m a group of permutations of Y_m. Then we can inductively construct the wreath products

$$Q(m) = Q_m^{X_{m-1}} \rtimes Q(m-1), \tag{7}$$

with a natural action of $Q(m)$ on X_m, starting with $Q(1) = Q_1$ acting as given on $X_1 = Y_1$. Then

$$G((\mathbf{Q}, \mathbf{Y})) = \varprojlim_m Q(m) \tag{8}$$

is naturally a closed subgroup of $G(\mathbf{q}) = G((\mathbf{S}, \mathbf{Y}))$, where S_m is taken to be the full symmetric group on Y_m. We can write $G((\mathbf{Q}, \mathbf{Y}))$ explicitly as an infinite iterated wreath product,

$$G((\mathbf{Q}, \mathbf{Y})) = \cdots \rtimes Q_m^{X_{m-1}} \rtimes \cdots \rtimes Q_2^{X_1} \rtimes Q_1. \tag{9}$$

There is a canonical homomorphism compatible with (9),

$$\bar{\sigma} : G((\mathbf{Q}, \mathbf{Y})) \longrightarrow \cdots \times Q_m^{\mathrm{ab}} \times \cdots \times Q_2^{\mathrm{ab}} \times Q_1^{\mathrm{ab}}, \quad \bar{\sigma}(g) = (\sigma_m(g))_{m \geq 1}, \tag{10}$$

where Q_m^{ab} denotes the abelianization of Q_m, and

(11) *$Ker(\bar{\sigma})$ contains the closure of the commutator subgroup of $G((\mathbf{Q}, \mathbf{Y}))$.*

If each Q_m is transitive on Y_m then each $Q(m)$ is transitive on X_m, and the inclusion (11) is an equality.

Suppose that each Q_m is cyclic (hence abelian), generated by a q_m-cycle on Y_m. Then we have

$$\bar{\sigma} : G((\mathbf{Q}, \mathbf{Y})) \longrightarrow \prod_{m \geq 1} Q_m.$$

In this case (cf. (4.4) and (4.5)), $G((\mathbf{Q},\mathbf{Y}))$ contains spherically transitive elements; $g \in G((\mathbf{Q},\mathbf{Y}))$ is spherically transitive iff $\sigma_m(g)$ generates Q_m for all $m \geq 1$; and two spherically transitive elements g, g' are conjugate in $G((\mathbf{Q},\mathbf{Y}))$ iff $\bar{\sigma}(g) = \bar{\sigma}(g')$.

Note that, in the case of the dyadic tree, $\mathbf{q} = (2,2,2,\ldots)$, the previous paragraph applies to the full group $G(\mathbf{q})$.

Finally, in Theorem (5.4), under the assumption that Q_m acts primitively on Y_m for each m (e.g. when Q_m is the full symmetric group) we give an analysis of all the normal subgroups of $G((\mathbf{Q},\mathbf{Y}))$. The result is too technical to state here.

If H is a rank 1 simple algebraic group (e.g. $H = PSL_2$) over a p-adic field F, then its Bruhat-Tits building X (cf. [Se]) is a tree on which $H(F)$ acts, with quotient $H(F)\backslash X = \circ\!\!-\!\!\circ$. The maximal compact subgroups of $H(F)$ are vertex stabilizers in X. If $x_0 \in X$ then $H(F)_{x_0} \cong H(A)$ where A is the ring of integral elements of F. Thus we have $H(A) \leq Aut(X, x_0)$, the automorphism group of the spherically homogeneous rooted tree (X, x_0). The congruence subgroups,

$$Ker\Big(H(A) \longrightarrow H(A/M^m)\Big),$$

where M is the maximal ideal of A, coincide with the groups

$$Ker\Big(H(A) \xrightarrow{res} Aut(B_m(x_0))\Big),$$

where $B_m(x_0)$ denotes the ball of radius m about x_0 in X.

In this light, we can think of the description of normal subgroup of $G(\mathbf{q})$ as a combinatorial analogue of the local congruence subgroup problem for the groups $H(A)$ (see e.g. [BMS]).

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Chapter I

Cyclic Renormalization

0. Introduction.

This chapter introduces the notions of renormalization and interval renormalization of dynamical systems, which are central to what follows. For a detailed synopsis, see the "Overview of Chapter I" in the main Introduction above.

1. Renormalization.

(1.1) Let (K, f) be a **dynamical system**, *i.e.*, K is a nonempty topological space and $f : K \longrightarrow K$ is a continuous map. We call (K, f) **minimal** if K is the only closed nonempty f-invariant subset of K. Equivalently, for each $x \in K$, the **f-orbit** $\{f^n(x) \mid n \geq 0\}$ of x is dense in K [Bi].

A **morphism** $\phi : (K, f) \longrightarrow (K', f')$ of dynamical systems is a continuous map $\phi : K \longrightarrow K'$ such that $f' \circ \phi = \phi \circ f$. If (K, f) is minimal then two morphisms on (K, f) that agree at a single point must coincide. When ϕ is onto it is often called a **semi-conjugacy** of dynamical systems. Note that if (K', f') is minimal then this is necessarily the case. If furthermore ϕ is injective as well, then ϕ is called a **conjugacy**.

(1.2) **n-renormalization.** For an integer $n \geq 1$, an **n-renormalization** of a dynamical system (K, f) is a morphism $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$, where $\mathbb{Z}/n\mathbb{Z}$ is given the discrete topology and $+1$ denotes the map $x \mapsto x+1 \pmod n$. For $r \in \mathbb{Z}/n\mathbb{Z}$ put $K_r = \phi^{-1}(r)$. Then K is the disjoint union of the open-closed sets K_r and $f(K_r) \subset K_{r+1}$ $(r \in \mathbb{Z}/n\mathbb{Z})$. In particular, each K_r is invariant under f^n. We say that (K, f) is **renormalizable** if it admits an n-renormalization for some $n > 1$.

If $\psi : (K', f') \longrightarrow (K, f)$ is a morphism of dynamical systems then $\phi \circ \psi$ is an n-renormalization of (K', f').

Remark. Various related ideas in mathematical physics and dynamical systems have made use of techniques which have been called "renormalization". Renormalization ideas from statistical mechanics were first explicitly adapted to dynamical systems theory in [CT, TC1, TC2] (see also [Fe1, Fe2] for similar material). Among other applications, renormalization techniques have been used extensively for the study of continuous maps acting on an interval or subinterval. An excellent and fairly comprehensive review of this point of view can be found in [MS]. As will be indicated below, this possibly more familiar approach corresponds to what we will call "interval renormalizability" below. Our framework is designed for applying combinatorial analysis to restrictions of a map on the interval to invariant finite sets or Cantor sets (cf. (B.1) below).

(1.3) **Proposition.** *Let $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ be an n-renormalization of a minimal dynamical system.*

(a) For each $r \in \mathbb{Z}/n\mathbb{Z}$, f^n restricted to $K_r = \phi^{-1}(r)$ is minimal. Thus, the various K_r are just the orbit closures of f^n.

(b) For each $r \in \mathbb{Z}/n\mathbb{Z}$, f defines morphisms

$$(K_r, f^n\,|_{K^r}) \longrightarrow (K_{r+1}, f^n\,|_{K_{r+1}})$$

with dense image ($r \in \mathbb{Z}/n\mathbb{Z}$).

(c) If $\phi' : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ is an n-renormalization then, for some $a \in \mathbb{Z}/n\mathbb{Z}$, $\phi'(x) = \phi(x) + a$ for all $x \in K$.

Proof. Clearly $f : K_r \longrightarrow K_{r+1}$ is f^n-equivariant. If $L \subset K_r$ is f^n-invariant then $L^+ = \bigcup_{0 \leq i < n} f^i(L)$ is f-invariant, and $L^+ \cap K_{r+i} = f^i(L)$. By minimality of (K, f), $f^i(L)$ must be dense in K_{r+i} for each i. Now (a) follows from the case $i = 0$, and (b) from the case $i = 1$.

Let $x_0 \in K_0$. If ϕ' is another n-renormalization and $\phi'(x_0) = a$ then ϕ' and $\phi + a$ agree at x_0 and hence coincide, by the minimality of (K, f).

■

Remark. To simplify matters we sometimes identify the integer $j \in [0, n]$ with its residue class $\pmod n$. Similarly, if $n \in \mathbb{Z}/p\mathbb{Z}$ and $m \in \mathbb{Z}/q\mathbb{Z}$, then we may write $nm \in \mathbb{Z}/pq\mathbb{Z}$ and so forth. In general, the context will always clarify these sorts of distinctions.

(1.4) **Proposition.** *Let (K, f) be a minimal dynamical system and n an integer ≥ 1. The following conditions are equivalent.*

(a) (K, f) admits an n-renormalization $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$.

(b) For some $x \in K$ the f^n-orbit closures of $f^i(x)$ $(0 \leq i < n)$ are pairwise disjoint.

(c) (i) For $x, y \in K$ the f^n-orbit closures of x and y are either equal or disjoint
and

(ii) The resulting partition of K by f^n orbit closures has at least n classes.

Proof. Assume (a). Then (b) follows immediately and (c) follows from (1.3).

Assume (b). Let K_r denote the f^n-orbit closure of $f^r(x)$ $(0 \leq r < n)$. Then clearly $f(K_r) \subset K_{r+1}$ $(r \in \mathbb{Z}/n\mathbb{Z})$, so $K_0 \coprod \cdots \coprod K_{n-1}$ is closed and f-invariant, hence equals K by minimality. Now we have the desired n-renormalization defined by $\phi(K_r) = r$, whence (b) implies (a).

Assume (c). Let $x \in K$. Define K_r as above. Again by minimality the f-invariant set $K_0 \cup \cdots \cup K_{n-1}$ must equal K. Each K_r is one of the equivalence classes defined by (c)(i), and there are at least n such classes by (c)(ii). It follows that $K_0, \ldots, K_{n-1}$ must be distinct, hence pairwise disjoint. Thus (c) implies (b).

■

(1.5) For a dynamical system (K, f) we define the set

$$Ren(K,f) = \{n \geq 1 \mid (K,f) \text{ admits an } n\text{-renormalization}\}.$$

Proposition. *(a) $Ren(K, f)$ contains 1 and is stable under divisors: $n \in Ren(K, f)$ and $d \mid n$ implies that $d \in Ren(K, f)$.*

(b) $Ren(K, f)$ is stable under least common multiples: $n, m \in Ren(K, f)$ implies that $LCM(n, m) \in Ren(K, f)$.

Proof. Clearly $1 \in Ren(K, f)$. If $\phi_n : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ is an n-renormalization, $d \mid n$, and $p : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}$ is the natural projection, then $p \circ \phi_n : (K, f) \longrightarrow (\mathbb{Z}/d\mathbb{Z}, +1)$ is a d-renormalization.

Suppose further that $\phi_m : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ is an m-renormalization. Choose $x_0 \in \phi_n^{-1}(0)$. After modifying ϕ_m by a translation of $\mathbb{Z}/m\mathbb{Z}$ if necessary, we can arrange that $\phi_m(x_0) = 0$ also. Put $M = LCM(n, m)$ and $d = \gcd(n, m)$. Then the natural diagram

$$\begin{tikzcd} \mathbb{Z}/M\mathbb{Z} \arrow[r, "q_n"] \arrow[d, "q_m"'] & \mathbb{Z}/n\mathbb{Z} \arrow[d, "p_n"] \\ \mathbb{Z}/m\mathbb{Z} \arrow[r, "p_m"'] & \mathbb{Z}/d\mathbb{Z} \end{tikzcd}$$

is cartesian (*i.e.*, a fiber product). Moreover $p_m \circ \phi_m$ and $p_n \circ \phi_n$ agree at x_0 and hence are equal, by minimality. Thus the universal property of fiber products gives us a map $\phi_M : K \longrightarrow \mathbb{Z}/M\mathbb{Z}$ such that $q_s \cdot \phi_M = \phi_s$ for $s = n, m$. Then ϕ_M is the desired M-renormalization.

■

(1.6) **Supernatural numbers.** The prime factorization of an integer $n \geq 1$ takes the form

$$n = \prod_{p \text{ prime}} p^{v_p(n)}, \tag{1}$$

where $v_p(n) \geq 0$ and $v_p(n) > 0$ for only finitely many p. By a **supernatural number** we mean an expression of the form

$$Q = \prod_p p^{e_p} \qquad (0 \leq e_p \leq \infty \quad \textit{for all } p). \tag{2}$$

We say that n **divides** Q, written $n \mid Q$, if $v_p(n) \leq e_p$ for all p. The set

$$Div(Q) = \{n \geq 1 \;\big|\; n \mid Q\} \tag{3}$$

of divisors of Q contains 1 and is stable under divisors and LCM's. Conversely any set R of integers ≥ 1 containing 1 and stable under divisors and LCM's is of the form $R = Div(Q)$ for a unique supernatural number Q defined by

$$\begin{array}{rcl} Q & = & LCM(R) := \prod_p p^{e^p} \quad \textit{where} \\ e_p & = & \sup_{n \in R} v_p(n). \end{array} \tag{4}$$

This applies, in particular, to the sets $Ren(K, f)$ in (1.5). We have

$$\begin{array}{l} Ren(K, f) = Div(Q), \;\; \textit{where} \\ Q = Q(K, f) := LCM(Ren(K, f)). \end{array} \tag{5}$$

(1.7) **Example.** Given a supernatural number Q as in (1.6)(2), we can define the **ring of Q-adic integers**

$$\widehat{\mathbb{Z}}_Q = \varprojlim_{q \mid Q} \mathbb{Z}/q\mathbb{Z} = \prod_p \mathbb{Z}/p^{e_p}\mathbb{Z}$$

where the inverse limit is taken over divisors q of Q, ordered by divisibility, and when $e_p = \infty$, $\mathbb{Z}/p^\infty\mathbb{Z}$ denotes the p-adic integers, $\varprojlim_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z}$. The inverse limit of the discrete topologies on each $\mathbb{Z}/q\mathbb{Z}$ gives $\widehat{\mathbb{Z}}_Q$ a topological ring structure for which it is compact and totally disconnected. The dynamical system

$$(\widehat{\mathbb{Z}}_Q, +1),$$

called the **Q-adic adding machine**, is minimal (this follows from (2.8) and (III,(4.5)) and it admits q-renormalizations for each $q \in Div(Q)$. In fact

$$Ren(\widehat{\mathbb{Z}}_Q, +1) = Div(Q).$$

For suppose that $\phi : (\widehat{\mathbb{Z}}_Q, +1) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ is an n-renormalization. We must show that $n \mid Q$. After a translation of $\mathbb{Z}/n\mathbb{Z}$ we can assume that $\phi(a) = 0$. It follows then that $\phi(m) = m$ for all $m \in \mathbb{Z}$, where in the lefthand side $m \in \widehat{\mathbb{Z}}_Q$ and in the righthand side $m \in \mathbb{Z}/n\mathbb{Z}$ and consequently that $\phi \mid_{Im(\mathbb{Z} \longrightarrow \widehat{\mathbb{Z}}_Q)}$ is a ring homomorphism. Since $Im(\mathbb{Z} \longrightarrow \widehat{\mathbb{Z}}_Q)$ is dense in $\widehat{\mathbb{Z}}_Q$, it follows that ϕ is a ring homomorphism. Thus, $Ker(\phi)$ is an ideal of index n in $\widehat{\mathbb{Z}}_Q$. It is easily seen that any such index must divide Q.

Historical Remark. The relevance of adding machines to renormalization was apparently first noticed by Sullivan (unpublished). See also [JR2, Ni, Mis1, Mis3].

(1.8) Let (K, f) be a dynamical system, having a point $x_0 \in K$ with a dense f-orbit. Let

$$Ren(K, f) = Div(Q)$$

where $Q = LCM(Ren(K, f))$, as in (1.6)(5). For each divisor q of Q let $\phi_q : (K, f) \longrightarrow (\mathbb{Z}/q\mathbb{Z}, +1)$ be the q-renormalization with $\phi_q(x_0) = 0$. Then the collection of ϕ_q define a map

$$\widehat{\phi}_Q : K \longrightarrow \widehat{\mathbb{Z}}_Q = \varprojlim_{q \mid Q} \mathbb{Z}/q\mathbb{Z}$$

and $\widehat{\phi}_Q$ is a morphism $(K, f) \longrightarrow (\widehat{\mathbb{Z}}_Q, +1)$. The image of $\widehat{\phi}_Q$ is dense, so $\widehat{\phi}_Q$ is surjective if K is compact.

We call (K, f) **faithfully renormalizable** if $\widehat{\phi}_Q$ injective; equivalently, if whenever $x \neq y$ in K, $\phi_q(x) \neq \phi_q(y)$ for some q.

If $\widehat{\phi}_Q$ is a homeomorphism then K must be compact and totally disconnected and f must be a homeomorphism such that (K, f) is minimal. On the other hand these conditions do not suffice to make $\widehat{\phi}_Q$ a homeomorphism. In (4.6) we give an example with Q infinite yet $\widehat{\phi}_Q$ is not injective.

(1.9) **Proposition.** *Let (K, f) be a dynamical system, $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ an n-renormalization, and for $r \in \mathbb{Z}/n\mathbb{Z}$ put $K_r = \phi^{-1}(r)$ and*

$$f_r = f^n \mid_{K_r} : K_r \longrightarrow K_r.$$

(a) The sets $Ren(K_r, f_r)$ are the same for all $r \in \mathbb{Z}/n\mathbb{Z}$.

(b) For an integer $m \geq 1$, $m \in Ren(K_r, f_r)$ if and only if $nm \in Ren(K, f)$. Thus, in the notation of (1.6)(5),

$$Q(K, f) = n \cdot Q(K_r, f_r).$$

Proof. The morphisms $f : (K_r, f_r) \longrightarrow (K_{r+1}, f_{r+1})$ entail inclusions $Ren(K_{r+1}, f_{r+1}) \subset Ren(K_r, f_r)$, whence (a).

To prove (b) put $N = mn$. For $r, M \in \mathbb{Z}$, let r_M denote the residue class of $r \in \mathbb{Z}/M\mathbb{Z}$. We can write elements of $\mathbb{Z}/N\mathbb{Z}$ uniquely in the form

$$r_N + n \cdot s \qquad (r = 0, 1, \ldots, n-1; s \in \mathbb{Z}/m\mathbb{Z}).$$

If $\phi_N : (K, f) \longrightarrow (\mathbb{Z}/N\mathbb{Z}, +1)$ is an N-renormalization that reduces modulo n to ϕ, then for $0 \leq r < n$ and $x \in K_r$, we can write $\phi_N(x) = r_N + n \cdot \psi(x)$, as above, and it is readily checked that $\psi : (K_r, f_r) \longrightarrow \mathbb{Z}/m\mathbb{Z}$ is an m-renormalization.

Suppose, conversely, that $m \in Ren(K_r, f_r)$. Choose an m-renormalization $\phi_m : (K_{n-1}, f_{n-1}) \longrightarrow (\mathbb{Z}/m\mathbb{Z}, +1)$. Define $\psi : K \longrightarrow \mathbb{Z}/m\mathbb{Z}$ on $x \in K_r$ by $\psi(x) = \phi_m(f^{n-1-r}(x))$ for $0 \leq r < n$. Then define ϕ_N on $x \in K_r$ by $\phi_N(x) = r_N + n \cdot \psi(x)$. A straightforward calculation shows that $\phi_N(f(x)) = \phi_N(x) = 1_N$ as required. The only subtle case is $r = n-1$, when we have $f(x) \in K_0$. Then $\phi_N(f(x)) = 0_N = n\phi_m(f^{n-1}(f(x))) = n \cdot \phi_m(f_{n-1}(x)) = n \cdot (\phi_m(x) = 1_m)$, because ϕ_m is an m-renormalization, whereas

$$\phi_N(x) = 1_N = (n-1)_N + 1_N + n \cdot \phi_m(x) = n \cdot (\phi_m(x) + 1_m).$$

∎

Remark. The analogous restatement of (1.9) for interval renormalization is **not** true.

(1.10) **Example of a nonrenormalizable minimal** (K, f)**.** Consider the circle

$$S = \{z \in \mathbb{C} \;\big|\; |z| = 1\}$$

and let f be an irrational rotation of S. Thus, for some $\alpha \notin \mathbb{Q}$ and writing $e(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$, we have

$$f(e(x)) = e(x + \alpha) = e(x)e(\alpha).$$

Then the cyclic group $\langle f \rangle$ acts freely on S, and for each $N \neq 0$, f^N acts minimally on S. More precisely, given $z, w \in S$, we can find sequences $n_i, m_i \longrightarrow \infty$ such that

$$f^{Nm_i}(z) \uparrow w \textit{ and } f^{Nn_i}(z) \downarrow w.$$

Here the notation $y \uparrow w$ denotes that in the counterclockwise orientation of S, y increases to the limit w. Similarly, $y \downarrow w$ signifies that y decreases to the limit w.

Of course (S, f) is not renormalizable, but this follows already for the trivial reason that S is connected. We now propose to modify (S, f) to a nonrenormalizable system, (K, g) with K totally disconnected. We shall use the Denjoy expansion construction described in Appendix A below.

Let C be an orbit of $\langle f \rangle$ not containing 1: For some $z_0 = e(x_0) \in S$, $C = \{f^n(z_0) = e(x_0 + n\alpha) \mid n \in \mathbb{Z}\}$, and $x_0 \notin \mathbb{Z}\alpha$. Define $\delta_S : S \longrightarrow \mathbb{R}$ by $\delta_S(z_0) = 1/2$, $\delta_S(f^n(z_0)) = 1/2^{|n|+2}$ for $n \neq 0$, and $\delta_S(z) = 0$ for $z \notin C$; note that $\sum_z \delta_S(z) = 1$. As in (A.3), use δ_S to construct a Denjoy expansion of S. This furnishes a continuous surjection $\pi : S \longrightarrow S$ such that, for each $z \in S$,

$$J_z = \pi^{-1}(z) = [\sigma_0(z), \sigma_1(z)]$$

is a counterclockwise oriented interval of length $\delta_S(z) \cdot \pi$. Further f lifts to a locally increasing (*i.e.*, orientation preserving) homeomorphism $g : S \longrightarrow S$ such that $\pi \circ g = f \circ \pi$ (cf. (A.6)).

The set

$$K = \sigma_0(S) \cup \sigma_1(S) \subset S$$

is closed and $\langle g \rangle$-invariant. K looks like a rescaled version of S with each $z \in C$ split into a pair $\{\sigma_0(z), \sigma_1(z)\}$. Since f is locally increasing everywhere (cf. (A.4)) we have $g(\sigma_i(z)) = \sigma_i(f(z))$ $(i = 0, 1)$ for all z ((A.4)(8)), and hence the orbit C has been split into two orbits $\sigma_0(C)$ and $\sigma_1(C)$. Since C is dense in S, we see that K is totally disconnected. To show that $(K, g\,|_K)$ is minimal and has no non-trivial renormalizations we must show that for $N \neq 0$, each g^N-orbit in K is dense in K.

Let $z, w \in S$. We must show that the g^N-orbit closures of $\sigma_0(z)$ and $\sigma_1(z)$ each contain both $\sigma_0(w)$ and $\sigma_1(w)$. We treat only the case of $\sigma_0(z)$ as the case of $\sigma_1(z)$ is similar.

Choose sequences $n_i, m_i \longrightarrow \infty$ so that $f^{Nn_i}(z) \uparrow w$ and $f^{Nm_i}(z) \downarrow w$. Since f is locally increasing, we have

$$g^{Nn_i}(\sigma_0(z)) = \sigma_0(f^{Nn_i}(z)) \uparrow \sigma_0(w)$$

and

$$g^{Nm_i}(\sigma_0(z)) = \sigma_0(f^{Nm_i}(z)) \downarrow \sigma_1(w)$$

(cf. (A.3)(6)).

Summary. K is compact and totally disconnected, $g\,|_K$ is a homeomorphism, and g^N acts minimally on K for all $N \neq 0$. Hence

$$Ren(K, g\,|_K) = \{1\}.$$

Note that by picking a base point $x_0 \in S$ interior to some interval $J_z = [\sigma_0(z), \sigma_1(z)]$, and using x_0 to convert the cyclic order on S to a linear order with initial point x_0, we obtain a linear order on K defining the same topology on K. Then K, with this linear order and topology, can be (order preserving) embedded in $\mathbb{R}$.

Appendix A. Denjoy expansion.

(A.0) **Summary.** We describe here a construction that modifies certain one-dimensional dynamical systems by equivariantly "blowing up" a countable set of points to intervals of positive length. It is the basis for various examples that we construct (cf. (1.10), (4.6)).

Variations of this type of construction can be found in the literature and go back to [Boh, De, Po].

(A.1) **δ-expansion of $\mathbb{R}$.** Let

$$\delta : \mathbb{R} \longrightarrow \mathbb{R}^+$$

be a (not necessarily continuous) function such that

$$(1) \qquad \begin{cases} \delta(x) \geq 0 & \textit{for all } x, \textit{ and} \\ \sum_x \delta(x) & < \infty. \end{cases}$$

It follows that

$$C = Supp(\delta) := \{x \mid \delta(x) > 0\}$$

is countable. Define

$$\sigma_0, \sigma_1 : \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$(2) \qquad \begin{aligned} \sigma_0(x) &= x + \textstyle\sum_{y<x} \delta(y), \\ \sigma_1(x) &= x + \textstyle\sum_{y \leq x} \delta(y) = \sigma_0(x) + \delta(x). \end{aligned}$$

Then each σ_i is a strictly increasing function of x.

We shall write $y \uparrow x$ (resp. $y \downarrow x$) to denote that y strictly increases (resp. decreases) to the limit x. It is easily seen that

$$(3) \qquad \begin{array}{lccl} & y \uparrow x & \Longrightarrow & \sigma_i(y) \uparrow \sigma_0(x) \qquad (i = 0, 1) \\ \textit{and} & & & \\ & y \downarrow x & \Longrightarrow & \sigma_i(y) \downarrow \sigma_1(x) \qquad (i = 0, 1). \end{array}$$

For $x \in \mathbb{R}$, put

$$J_x = [\sigma_0(x), \sigma_1(x)],$$

a closed interval of length $\delta(x) \geq 0$. We have $\mathbb{R} = \coprod_x J_x$, and $x < y$ implies $x' < y'$ for all $x' \in J_x$ and $y' \in J_y$. Define

$$\pi : \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$\pi^{-1}(x) = J_x \qquad \textit{for all } x.$$

Then π is surjective, continuous, nondecreasing, and $\pi \circ \sigma_i = Id \quad (i = 0, 1)$.

(A.2) **The sets $K(\delta)$ and $K'(\delta)$.** For $K \subset \mathbb{R}$ we put

$$K(\delta) = \pi^{-1}(K) = \coprod_{x \in K} J_x.$$

Then $\pi : K(\delta) \longrightarrow K$ has sections σ_0, σ_1, and we put

$$K'(\delta) = \sigma_0(K) \cup \sigma_1(K).$$

Then

$$\pi' : K'(\delta) \longrightarrow K$$

has fibers

$$\pi'^{-1}(x) = \{\sigma_0(x), \sigma_1(x)\}$$

of cardinal ≤ 2.

Suppose that K is an interval. Then $K(\delta)$ is an interval of the same type, and there is an affine isomorphism

$$\alpha : K(\delta) \xrightarrow{\cong} K$$

of the form $\alpha(x) = ax + b$, $a > 0$. Using $\bar{\pi} = \pi \circ \alpha^{-1}$ and $\bar{\sigma}_i = \alpha \circ \sigma_i$ $(i = 0, 1)$ we get maps $K \underset{\bar{\sigma}_i}{\overset{\bar{\pi}}{\rightleftarrows}} K$ with properties like those of π and σ_i.

(A.3) **δ-expansion of the circle S.** Let

$$S = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$$

oriented counterclockwise. For $z, w \in S$, $[z, w]$ denotes the closed interval going counterclockwise from z to w. ($[z, w] = \{z\}$ if $z = w$.)

We have the exponential map

$$e : \mathbb{R} \longrightarrow S, \qquad e(x) = e^{2\pi i x}.$$

Suppose that we are given a function

$$\delta_S : S \longrightarrow \mathbb{R}^+$$

such that

$$\text{(1)} \qquad \begin{cases} \delta_S(z) \geq 0 \textit{ for all } z, \quad \delta_S(1) = 0, \quad \textit{and} \\ \sum_{z \in S} \delta_S(z) = 1. \end{cases}$$

Thus δ_S has countable support

$$C_S = \{z \in S \,\big|\, \delta_S(z) > 0\}.$$

Put

$$L = [0,1) \subset \mathbb{R}$$

and define

$$\delta : \mathbb{R} \longrightarrow \mathbb{R}$$

by

(2)
$$\delta(x) = \begin{cases} \delta_S(e(x)) & x \in L \\ 0 & x \notin L. \end{cases}$$

Note then that, with $\sigma_0, \sigma_1, \pi : \mathbb{R} \longrightarrow \mathbb{R}$ defined as in (A.1), we have

(3)
$$\begin{cases} \sigma_0(x) = \sigma_1(x) = x & \textit{for } x \leq 0, \textit{ and} \\ \sigma_0(x) = \sigma_1(x) = x+1 & \textit{for } x \geq 1. \end{cases}$$

Further, we have for $i = 0, 1$,

(4)
$$\begin{array}{ccccc} & L(\delta) & & = & [0,2) \\ \pi & \downarrow\uparrow & \sigma_i & & \\ & L & & = & [0,1) \end{array}$$

Define $e_2(x) = e(x/2)$. Then $e_2 : L(\delta) \longrightarrow S$ is bijective, so we can define π_S and σ_{i_S} $(i = 0, 1)$ on S by commutativity of the diagrams

(5)
$$\begin{array}{ccccc} & L(\delta) & \xrightarrow{e_2} & S & \\ \pi & \downarrow & & \downarrow & \pi_S \\ & L & \xrightarrow[e]{} & S & \end{array} \quad , \quad \begin{array}{ccccc} & L(\delta) & \xrightarrow{e_2} & S & \\ \sigma_i & \uparrow & & \uparrow & \sigma_{i_S} \\ & L & \xrightarrow[e]{} & S & \end{array}$$

Then π_S is continuous, surjective and weak order preserving; its fibers are intervals.

On S we write $z \uparrow w$ (resp. $z \downarrow w$) to denote that z increases (resp. decreases) to the limit w in some S-interval $[u, w]$ (resp. $[w, v]$). With this notation we have

(6)
$$\begin{array}{l} z \uparrow w \implies \sigma_{i_S}(z) \uparrow \sigma_{0_S}(w) \quad (i = 0,1) \\ \textit{and} \\ z \downarrow w \implies \sigma_{i_S}(z) \downarrow \sigma_{1_S}(w) \quad (i = 0,1). \end{array}$$

For $K \subset S$ we define

$$K'(\delta_S) \subset K(\delta_S) \subset S$$

by $K(\delta_S) = \pi_S^{-1}(K)$ and $K'(\delta_S) = \sigma_{0_S}(K) \cup \sigma_{1_S}(K)$.

(A.4) **Expansion of dynamics.** Let T denote either a real interval, or else the circle S (with counterclockwise orientation). Let

$$\pi : T \longrightarrow T$$

be a continuous weak order preserving surjection such that for all $x \in T$,

$$J_x := \pi^{-1}(x) \text{ is a closed interval } [\sigma_0(x), \sigma_1(x)], \tag{1}$$

of length

$$\delta(x) = \mathit{length}(J_x) \geq 0. \tag{2}$$

Let $f : T \longrightarrow T$ be a continuous map. We make the following **basic assumption**:

(3) *If $x \in T$ and $\delta(f(x)) > 0$, then $\delta(x) > 0$, and f is locally strictly monotone on each side of x; i.e., there are intervals $[y, x]$ and $[x, z]$ of positive length such that on each of $[y, x]$ and $[x, z]$, f is either increasing or decreasing.*

In the setting of (3), there are four possible types of behavior of f near x, denoted

$$\mathit{type}(f, x),$$

characterized and denoted as follows:

$$\begin{cases} \mathit{type}\ / & : \ \text{increasing on } [y,x] \text{ and } [x,z] \\ \mathit{type}\ \backslash & : \ \text{decreasing on } [y,x] \text{ and } [x,z] \\ \mathit{type}\ \cap & : \ \text{increasing on } [y,x], \text{ decreasing on } [x,z] \\ \mathit{type}\ \cup & : \ \text{decreasing on } [y,x], \text{ increasing } [x,z]. \end{cases} \tag{4}$$

Given real intervals $[a, b]$, $[c, d]$, $a < b$, $c < d$, we choose surjective maps $g_\star$ $(\star = /, \backslash, \cap, \cup)$ from $[a, b]$ to $[c, d]$ with the indicated type at the midpoint of $[a, b]$. For example we can take, for $0 \leq t \leq 1$:

$$\begin{cases} g_/(a + t(b-a)) & = & c + t(d-c) \\ g_\backslash(a + t(b-a)) & = & d - t(d-c) \\ g_\cap(a + t(b-a)) & = & c + 4 \cdot t(1-t)(d-c) \\ g_\cup(a + t(b-a)) & = & d - 4 \cdot t(1-t)(d-c) \end{cases} \tag{5}$$

Now we propose to lift f to a continuous map g making the following diagram commute.

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "\pi"'] & T \arrow[d, "\pi"] \\ T \arrow[r, "f"'] & T \end{tikzcd} \tag{6}$$

We construct g such that for each $x \in T$, g must restrict to a continuous map

$$g_x : J_x = [\sigma_0(x), \sigma_1(x)] \longrightarrow J_{f(x)} = [\sigma_0(f(x)), \sigma_1(f(x))].$$

There are two possibilities.
Case 1: $\delta(f(x)) = 0$. Then g_x must be the constant map with value $\sigma_0(f(x)) = \sigma_1(f(x))$.
Case 2: $\delta(f(x)) > 0$. Then by assumption (3), $\delta(x) > 0$ also, and the type of (f, x) is well-defined. We then define

(7), $$g_x = g_{type(f,x)} : J_x \longrightarrow J_{f(x)},$$

where $g_{type(f,x)}$ is defined as in (5).

The following properties are immediate from the construction.

(8) *For each $x \in T$, $g_x : J_x \longrightarrow J_{f(x)}$ is continuous, and according to type(f, x)*

$$(g(\sigma_0(x)), g(\sigma_1(x))) = \begin{cases} (\sigma_0(f(x)), \sigma_1(f(x))) & \textit{type } / \\ (\sigma_1(f(x)), \sigma_0(f(x))) & \textit{type } \setminus \\ (\sigma_0(f(x)), \sigma_0(f(x))) & \textit{type } \cap \\ (\sigma_1(f(x)), \sigma_1(f(x))) & \textit{type } \cup \end{cases}$$

(9) *If $J \subset T$ is any interval then $\pi^{-1}(J)$ is an interval. If f is increasing (resp. decreasing) on J then g is increasing (resp. decreasing) on $\pi^{-1}(J)$.*

(10) *If $K \subset T$ is f-invariant then $\pi^{-1}(K)$ and $K' := \sigma_0(K) \cup \sigma_1(K)$ are g-invariant.*

(A.5) **Lemma.** *g is continuous.*

Proof. Since each $g_x : J_x \longrightarrow J_{f(x)}$ is continuous, it remains only to show that, for $x \in T$,

$$z \uparrow \sigma_0(x) \implies g(z) \longrightarrow g(\sigma_0(x))$$

and

$$w \downarrow \sigma_1(x) \implies g(w) \longrightarrow g(\sigma_1(x)).$$

As z passes through infinitely many intervals J_y, the lengths of J_y and $J_{f(y)} = g(J_y)$ tend to 0. Thus it suffices to treat the case when z is say the initial point $\sigma_0(y)$ of J_y. The condition $z \uparrow \sigma_0(x)$ is then equivalent to the condition $y \uparrow x$. Similarly it suffices to treat the case when $w = \sigma_0(u)$, and $u \downarrow x$. Thus, it suffices to show that

$$y \uparrow x \implies g(\sigma_0(y)) \longrightarrow g(\sigma_0(x)) \quad (i = 0, 1)$$

and

$$u \downarrow x \implies g(\sigma_0(u)) \longrightarrow g(\sigma_1(x)).$$

Case 0: $\delta(f(x)) = 0$, *i.e.*, $\sigma_0(f(x)) = \sigma_1(f(x))$ is the constant value of g on J_x.

Since f is continuous, $y \uparrow x$ implies $f(y) \longrightarrow f(x)$, and so, by (A.1)(3) and (A.3)(6), $\sigma_i(f(y)) \longrightarrow g(\sigma_i(x))$ $(i = 0,1)$ (since $g(\sigma_i(x)) = \sigma_0(f(x)) = \sigma_1(f(x))$, $i = 0,1$). Since $g(\sigma_0(y)) = \sigma_0(f(y))$ or $\sigma_1(f(y))$, it follows that $g(\sigma_0(y)) \longrightarrow g(\sigma_0(x))$, as required. Similarly, one concludes from $u \downarrow x$ that $g(\sigma_0(u)) \longrightarrow g(\sigma_1(x))$.

Case 1: $\delta(f(x)) > 0$. Then, by (3), $\delta(x) > 0$ and *type*$(f,x) \in \{/, \backslash, \cap, \cup\}$. We treat each type separately.

Type /: By (A.4)(8), $g(\sigma_i(y)) = \sigma_i(f(y))$ for y near x. Also near x we have

$$\begin{array}{ccccccc} y \uparrow x & \Longrightarrow & f(y) \uparrow f(x) & \Longrightarrow & \sigma_0(f(y)) & \uparrow & \sigma_0(f(x)) \\ & & & & \| & & \| \\ & & & & g(\sigma_0(y)) & & g(\sigma_0(y)). \end{array}$$

Similarly, $u \downarrow x$ implies $g(\sigma_0(u)) \downarrow g(\sigma_1(x))$, as required.

Type \\: In this case, by (A.4)(8), $g(\sigma_i(y)) = \sigma_{1-i}(f(y))$ for y near x. Also near x we have

$$\begin{array}{ccccccc} y \uparrow x & \Longrightarrow & f(y) \downarrow f(x) & \Longrightarrow & \sigma_1(f(y)) & \downarrow & \sigma_1(f(x)) \\ & & & & \| & & \| \\ & & & & g(\sigma_0(y)) & & g(\sigma_0(y)). \end{array}$$

Similarly,

$$\begin{array}{ccccccc} u \downarrow x & \Longrightarrow & f(u) \uparrow f(x) & \Longrightarrow & \sigma_1(f(u)) & \uparrow & \sigma_0(f(x)) \\ & & & & \| & & \| \\ & & & & g(\sigma_0(u)) & & g(\sigma_1(x)). \end{array}$$

Type ∩: The argument in the case $y \uparrow x$ is like that for type /, and in the case $u \downarrow x$ like that for type \\.

Type ∪: The argument in the case $y \uparrow x$ is like that for type \\, and in the case $u \downarrow x$ like that for type /.

■

(A.6) **Terminology.** Let $\pi : T \longrightarrow T$ and $f : T \longrightarrow T$ be as in (A.4). We call π a **Denjoy expansion of T along** (the countable set)

(1) $$C = \{x \in T \mid \delta(x) > 0\}.$$

We call $g : T \longrightarrow T$ the corresponding **Denjoy expansion of f.** Note that assumption (A.4)(3) implies that

(2) $$f^{-1}(C) \subset C.$$

We summarize some of the basic properties.

(3) *The diagram*
$$\begin{array}{ccc} T & \xrightarrow{g} & T \\ \pi\downarrow & & \downarrow\pi \\ T & \xrightarrow[f]{} & T \end{array}$$
commutes.

(4) *If $J \subset T$ is an interval then $\pi^{-1}(J)$ is an interval. If f is increasing (resp. decreasing) on J then g is increasing (resp. decreasing) on $\pi^{-1}(J)$.*

(5) *$g(T) = \pi^{-1}(f(T))$; thus g is surjective if and only if f is surjective.*

(6) *g is injective if and only if f is injective and [$x \in C$ if and only if $f(x) \in C$].*

For since g is surjective on π-fibers, we see that g is injective if and only if f is injective and each $g_x : J_x \longrightarrow J_{f(x)}$ is injective. In particular, we must have $\delta(x) > 0$ if and only if $\delta(f(x)) > 0$, *i.e.*, $x \in C$ if and only if $f(x) \in C$. Further, when $\delta(x) > 0$, f must have type / or \ at x, and this is automatic when f is injective.

Finally, we recall the basic assumption (A.4)(3) about C and f:

(7) *$f^{-1}(C) \subset C$, and near each $x \in C$, f is of type $/, \backslash, \cap$, or $\cup$.*

(A.7) **Minimality.** Let $\pi : (T, g) \longrightarrow (T, f)$ be a Denjoy expansion as in (A.6). Let $K \subset T$ be an infinite closed f-invariant subset containing C, and put $K' = \sigma_0(K) \cup \sigma_1(K)$. Then K' is closed and g-invariant, and π induces a surjection
$$\pi : (K', g) \longrightarrow (K, f).$$

Claim. *(K', g) is minimal if and only if*

(i) (K, f) is minimal, and

(ii) For all $c \in C$, c is a monotone limit from both directions in $K - \{c\}$.

Proof. First assume that (K', g) is minimal. Then clearly (K, f) is also minimal. If $c \in C$ is say a limit in K from the left, but not from the right, then $\sigma_1(c)$ is an isolated point of K', so it cannot be in the closure of a g-orbit not containing $\sigma_1(c)$, contradicting minimality.

To prove, conversely, that (i) and (ii) imply that (K', g) is minimal, it suffices to show, assuming (ii), that if $H' \subset K'$ and $H = \pi(H')$ is dense in K then H' is dense in K'. (We apply this with H' a g-orbit in K'.)

Let $y \in K$. We must show that $\sigma_0(y)$ and $\sigma_1(y)$ are limits of elements of H'. Let (h_n) be a sequence in H. For each n we have $\sigma_{i_n}(h_n) \in H'$ for some $i_n = 0$

or 1. Consequently, the following two cases are possible:

$$\begin{array}{llll} \text{(L)} & \textit{If } h_n \uparrow y & \textit{then} & \sigma_{i_n}(h_n) \uparrow \sigma_0(y) \\ \text{((R)} & \textit{If } h_n \downarrow y & \textit{then} & \sigma_{i_n}(h_n) \downarrow \sigma_1(y) \end{array}$$

Since by assumption, H is dense in K, we can find a sequence (h_n) in H so that either $h_n \uparrow y$ or $h_n \downarrow y$. If $\sigma_0(y) = \sigma_1(y)$ then $\sigma_0(y) \quad (= \sigma_1(y))$ lies in the closure of H'. If $\sigma_0(y) \neq \sigma_1(y)$, *i.e.*, $y \in C$, then by assumption (ii), we can find sequences (h_n) and (k_n) in H so that $h_n \uparrow y$ and $k_n \downarrow y$. Then from (L) and (R) we see that both $\sigma_0(y)$ and $\sigma_1(y)$ belong to the closure of H'.

■

(A.8) **Remark.** When (T, f) is given with certain smoothness properties, it is natural to see how much of these can be preserved for (T, g). This issue must be addressed in the choice of the local interpolating functions g_* of (A.4)(5), so that they are sufficiently smoothly adapted to the local behaviors of f at the cut points of the Denjoy expansions. The smoothness properties of f would be formulated in a suitable enhancement of hypothesis (A.4)(3). For a celebrated example of the smoothness problem for Denjoy expansion of irrational rotations see [Boh, De].

2. Interval renormalization.

(2.1) **Linear orders and intervals.** Let K be a linearly (*i.e.*, totally) ordered set. For $x, y \in K$ we have the usual notion of intervals: $[x, \infty) = \{z \in K \mid x \leq z\}$, $(x, \infty) = \{z \in K \mid x < z\}$, and similarly $(-\infty, x]$, $(-\infty, x)$, $[x, y] = [x, \infty) \cap (-\infty, y]$ (which is $\emptyset$ unless $x \leq y$), and $(x, y) = (x, \infty,) \cap (-\infty, y)$. When there is need to be precise we put a subscript K, as in $[x, y]_K$, etc.

For $L \subset K$, the K**-support of** L, denoted $[L]_K$ is defined to be

$$[L]_K := \bigcup_{x,y \in L} [x, y]$$

We call L a K**-interval** if $L = [L]_K$ (or simply an **interval** if the choice of K is clear).

If K' is another linearly ordered set then a map $\phi : K \longrightarrow K'$ is said to be **weak order preserving** if $x \leq y$ implies $\phi(x) \leq \phi(y)$. In this case the inverse image of an interval is an interval.

The open intervals $(x, y) \quad (x \in K \cup \{-\infty\}, y \in K \cup \{\infty\})$, form a base for the **order topology** on K. For $K \subset \mathbb{R}$ the order and euclidean topologies need not coincide; for example, $K = [0, 1) \cup \{2\}$ is order isomorphic to $[0, 1]$. Nonetheless:

Claim: *For a closed $K \subset \mathbb{R}$, the order and euclidean topologies coincide.*

Proof. It suffices to show that a euclidean open interval in K is order-open. Say

$$L = (A, B)_{\mathbb{R}} \cap K \neq \emptyset \quad (-\infty \leq A < B \leq \infty).$$

Put $a = \inf L$, $b = \sup L$. Since K is closed we have $a \in K \cup \{-\infty\}$, $b \in K \cup \{\infty\}$. Put $\alpha = \sup(-\infty, A]_{\mathbb{R}} \cap K$ and $\beta = \inf[B, \infty)_{\mathbb{R}} \cap K$. (Note that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.) Then again since K is closed we have $\alpha \in K \cup \{-\infty\}$ and $\beta \in K \cup \{\infty\}$ (see Figure 1).

Moreover $(\alpha, a)_K = \emptyset$ and $(b, \beta)_K = \emptyset$. Put

$$a_0 = \begin{cases} a & \text{if } a \notin L \\ \alpha & \text{if } a \in L \end{cases} \quad \text{and} \quad b_0 = \begin{cases} b & \text{if } b \notin L \\ \beta & \text{if } b \in L. \end{cases}$$

Then we have

$$L = (a_0, b_0)_K,$$

as is easily checked.

∎

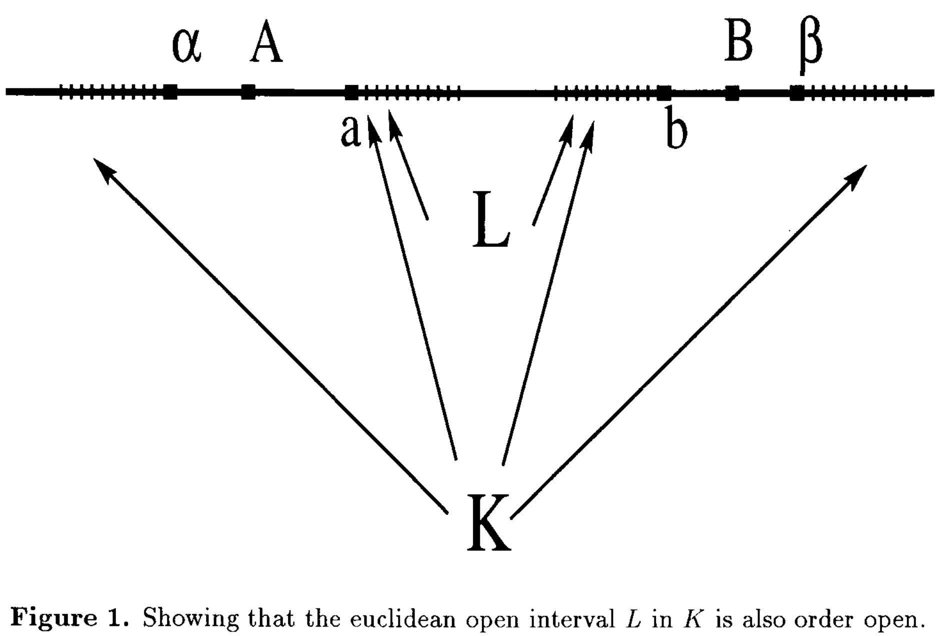

Figure 1. Showing that the euclidean open interval L in K is also order open.

(2.2) A **cyclic ordering** on a set K is defined by a family of subsets called **oriented closed intervals**

$$[x, y] \subset K \qquad (x, y \in K)$$

which are defined as satisfying (1), (2) and (3) below, for all $x, y, z \in K$:

(1) $$[x, y] \cap [y, x] = \{x, y\}$$

$$(2) \qquad y \in [x,z] \Longrightarrow \begin{cases} [x,y] \cup [y,z] & = & [x,z] \\ [x,y] \cap [y,z] & = & \{y\} \end{cases}$$

and

$$(3) \qquad \textit{either } y \in [x,z] \textit{ or } z \in [x,y].$$

We have the following consequences (i)-(xi): First

$$(\mathrm{i}) \qquad [x,x] = \{x\}.$$

$$(ii) \qquad [x,y] = [x,z] \Longrightarrow y = z.$$

Proof (of (i) and (ii)). (i) follows immediately from (1). For (ii), $[x,y] = [x,z] = [x,y] \cup [y,z]$, by (1) and (2), whence $[y,z] \subset [x,y]$ so $z \in [x,y] \cap [y,z] = \{y\}$, by (2).

■

$$(\mathrm{iii}) \qquad x \neq y \Longrightarrow [x,y] \cup [y,x] = K.$$

Proof (of (iii)). Suppose that $z \notin [x,y]$ so that $y \in [x,z] \cup [y,z]$ by (3). We must show that $z \in [y,x]$. If not, then by (3), $x \in [y,z]$, and so $x \in [x,y] \cap [y,z] = \{y\}$, contrary to the assumption that $x \neq y$.

■

$$(\mathrm{iv}) \qquad \textit{Either } y \in [x,z] \textit{ or } x \in [y,z].$$

Proof (of (iv)). By (1) we may assume that $z \neq x \neq y \neq z$. Suppose that $y \notin [x,z]$. By (2), $z \in [x,y] = [x,z] \cup [z,y]$, $\quad [x,z] \cap [z,y] = \{z\}$. Thus $x \notin [z,y]$ so by (iii), $x \in [y,z]$.

■

(v) *The following conditions are equivalent:*

(*a*) $y \in [x,z]$; (*b*) $[x,y] \subset [x,z]$; (*c*) *Either* $x = y$ *or* $x \notin [y,z]$.

Proof (of (v)). First, (a) $\Longrightarrow$ (b) by (2), and (b) $\Longrightarrow$ (a) by (1).

(a) $\Longrightarrow$ (c): Assuming $x \neq y$ we must show that $x \notin [y,z]$. If on the contrary, $x \in [y,z] = [y,x] \cup [x,z]$, from (a) we have $y \in [y,x] \cap [x,z] = \{x\}$, so $y = x$, contrary to assumption.

(c) $\Longrightarrow$ (a): Clearly $y \in [x,z]$ if $x = y$, so assume that $x \neq y$, and so $x \notin [y,z]$, by (c). Then $y \in [x,z]$ by (iv).

■

(vi) *Let $y \leq_x z$ signify the equivalent conditions of (v). Then $\leq_x$ is a linear order on K with least element x.*

Proof (of (vi)). This is immediate from (i), (ii), and (v).

■

(vii) *If $x \notin [y,z]$ then $y \leq_x z$ and $[y,z] = \{w \in K \mid y \leq_x w \leq_x z\}$.*

Proof (of (vii)). By (iv), $y \in [x,z] = [x,y] \cup [y,z]$, so $y \leq_x z$, and $w \in [y,z]$ implies $w \leq_x z$. If, moreover, $w \neq y$, then $w \notin [x,y]$, so $y \in [x,w]$ by (3), hence $y \leq_x w$. Suppose, conversely, that $y \leq_x w \leq_x z$. Then $w \in [x,z] = [x,y] \cup [y,z]$, and either $w = y \in [y,z]$ or else $w \notin [x,y]$, so again $w \in [y,z]$.

■

(viii) *Let $L \subset K$ and $x \in K - L$ be such that L is an interval relative to $\leq_x$. Then for all $x' \in K - L$, $\leq_{x'}$ coincides with $\leq_x$ on L, and L is an interval relative to $\leq_x$.*

Proof (of (viii)). In view of (vii), it suffices to show that if $y, z \in L$ and $y <_x z$,then $y <_{x'} z$. By (vii) we have $[y,z] \subset L$, hence $x' \notin [y,z]$, and so, by (iv), $y \in [x', z]$, as claimed.

■

A subset $L \subset K$ is called an **interval** if either $L = K$ or else L satisfies the condition of (viii) for some (hence every) $x \in K - L$. The open intervals form a base for the **order topology** on K.

Any subset $L \subset K$ inherits an **induced cyclic ordering**, with oriented closed intervals defined by:

$$[x,y]_L := [x,y] \cap L$$

(ix) *Suppose that K is partitioned into a disjoint union of intervals $L_0, L_1, \ldots, L_{n-1}$. Choose $x_i \in L_i$ and give $X = \{x_0, x_1, \ldots x_{n-1}\}$ the induced cyclic ordering. The corresponding cyclic order of $\{L_0, L_1, \ldots, L_{n-1}\}$ is independent of the choice of the x_i's.*

Proof (of (ix)). It suffices to observe that in the linearly ordered set $(K - L_0, \leq_{x_0})$, the linear order is independent of $x_0 \in L_0$, and the linear order induced on $\{x_1, \ldots, x_{n-1}\}$ and on the disjoint intervals $\{L_1, \ldots, L_{n-1}\}$ correspond.

■

(x) *If L and L' are K-intervals then either $L \cap L'$ is a K-interval or $L \cup L' = K$.*

Proof (of (x)). In fact if $x \in K - (L \cup L')$ then L and L' are both intervals relative to $\leq_x$ (cf. (12)) and hence so also is $L \cap L'$.

■

If K' is another cyclically ordered set then a map $\phi : K \longrightarrow K'$ is said to be **weak order preserving** if

$$y \leq_x z \textit{ in } K \textit{ implies } \phi(y) \leq_{\phi(x)} \phi(z) \textit{ in } K'.$$

It is then obvious that

(xi) *If $\phi : K \longrightarrow K'$ is weak order preserving then the inverse image of a K'-interval is a K-interval.*

(2.3) **Examples.** 1. A linear order $\leq$ on a set K defines a cyclic order on K, with

$$[x,y] = \begin{cases} \{z \mid x \leq z \textit{ and } z \leq y\} & \textit{if } x \leq y \\ \{z \mid x \leq z \textit{ or } z \leq y\} & \textit{if } y < x. \end{cases}$$

If K has a least element x then $\leq$ coincides with $\leq_x$, defined relative to the above cyclic ordering.

2. Let S^1 denote the unit circle,

$$S^1 = \{\exp(i\theta) \mid \theta \in \mathbb{R}\}.$$

If $x_0, x_1 \in S^1$, $x_j = \exp(i\theta_j)$, with $0 \leq \theta_1 - \theta_0 < 2\pi$, then we put

$$[x_0, x_1] = \{\exp(i\theta) \mid \theta_0 \leq \theta \leq \theta_1\}.$$

This defines the **counterclockwise cyclic order** on S^1. The intervals in S^1 are precisely the connected subsets.

(2.4) **Interval renormalization.** By an **ordered dynamical system** we understand a dynamical system (K, f) where K is equipped with a linear or cyclic order, and K is given the corresponding order topology. Let

$$\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1) \tag{1}$$

be an n-renormalization, with fibers

$$K_r = \phi^{-1}(r) \qquad (r \in \mathbb{Z}/n\mathbb{Z}). \tag{2}$$

We call ϕ an **interval n-renormalization** if each K_r is a K-interval. In this case there is an induced order (linear or cyclic) on the set of K_r's, and so also, by transport of structure, on $\mathbb{Z}/n\mathbb{Z}$.

In the linearly ordered case we have $x \leq y$ implies $\phi(x) \leq \phi(y)$. In the cyclically ordered case, we have, for any $r \in \mathbb{Z}/n\mathbb{Z}$ and $x_r \in K_r$, $x \leq_{x_r} y$ implies $\phi(x) \leq_r \phi(y)$. Thus in both cases, ϕ is weak order preserving (in the sense of (2.1) and (2.2)(xi)).

Claim. *If K is linearly ordered then ϕ is uniquely determined by n, up to a translation of $\mathbb{Z}/n\mathbb{Z}$.*

Proof. Let ϕ, ϕ' be two interval n-renormalizations, giving rise to intervals $K_r = \phi^{-1}(r)$ and $K'_r = \phi'^{-1}(r)$ $(r \in \mathbb{Z}/n\mathbb{Z})$. After translation of $\mathbb{Z}/n\mathbb{Z}$ we can assume that K_0 is left-most among $K_0, \ldots, K_{n-1}$, and similarly for K'_0 among $K'_0, \ldots, K'_{n-1}$. Then one of the left-most intervals K_0, K'_0 contains the other, say $K_0 \subset K'_0$. Taking successive inverse images under f, we find that $K_r \subset K'_r$ for all $r \in \mathbb{Z}/n\mathbb{Z}$. Hence $K_r = K'_r$, *i.e.*, $\phi = \phi'$, as claimed.

■

Example. The above uniqueness-up-to-translation property can fail when K is cyclically ordered. For example let $K = \mathbb{Z}/6\mathbb{Z}$, with its natural cyclic order, and $f(x) = x + 2$. Define $\phi, \phi' : (K, f) \longrightarrow (\mathbb{Z}/3\mathbb{Z}, +1)$ with fibers $K_r = \phi^{-1}(r)$ and $K'_r = \phi'^{-1}(r)$ defined by $K_0 = \{0, 1\}, K_1 = \{2, 3\}, K_2 = \{4, 5\}, K'_0 = \{5, 0\}, K'_1 = \{1, 2\}, K'_2 = \{3, 4\}$.

The following remains true, even in the cyclically ordered case. Pick a base point $x_0 \in K$. Up to translation of $\mathbb{Z}/n\mathbb{Z}$, any renormalization $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ can be made to satisfy $\phi(x_0) = 0$.

Claim. *All interval n-renormalizations ϕ of (K, f) such that $\phi(x_0) = 0$ induce the same order (linear or cyclic) on $\mathbb{Z}/n\mathbb{Z}$.*

Indeed, the order on $\mathbb{Z}/n\mathbb{Z}$ is such that ϕ is an order preserving bijection on the partial orbit

$$\{x_0, f(x_0), \ldots, f^{n-1}(x_0)\},$$

the latter being given the order induced from K.

We put

$$(3) \qquad IRen(K, f) = \{n \mid (K, f) \textit{ admits an interval } n\textit{-renormalization}\}.$$

Note that when (K, f) is minimal,

$$IRen(K, f) = \{n \mid \textit{ the orbit closures of } f^n \textit{ from } n \textit{ disjoint } K\textit{-intervals}\}.$$

Let (K', f') be another ordered dynamical system, and let

$$(4) \qquad \alpha : (K', f') \longrightarrow (K, f)$$

be a topological morphism. If $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ is a (not necessarily interval) n-renormalization of (K, f), then $\phi \circ \alpha$ is one of $(K'f')$. We call α an **IR-morphism** if whenever ϕ is an interval renormalization, so also is $\phi \circ \alpha$. (This happens, for example, if α is weak order preserving.) Thus:

(5) *α is an IR-morphism if and only if α is a topological morphism, and IRen$(K, f) \subset$ IRen(K', f').*

We call α an **IR-isomorphism** (of minimal ordered dynamical systems) if α is a topological isomorphism and both α and α^{-1} are IR-morphisms. Equivalently:

(6) *α is an IR-isomorphism if and only if α is a topological isomorphism, and IRen$(K, f) =$ IRen(K', f').*

(2.5) **Example.** We claim that, with the natural cyclic order on $\mathbb{Z}/m\mathbb{Z}$,

$$IRen(\mathbb{Z}/m\mathbb{Z}, +1) = \{1, m\}.$$

Indeed, let $\phi : (\mathbb{Z}/m\mathbb{Z}, +1) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ be an interval n-renormalization, say with $\phi(0) = 0$. Then clearly ϕ must be a surjective group homomorphism, so $n \mid m$, say $m = nq$. However the fiber

$$\phi^{-1}(0) = \{0, n, 2n, \ldots, (q-1)n\}$$

clearly cannot be an interval of $\mathbb{Z}/m\mathbb{Z}$ unless $n = 1$ $(q = m)$ or $n = m$ $(q = 1)$.

(2.6) **Theorem.** *Let (K, f) be an ordered dynamical system (cf. (2.4)). Then IRen(K, f) is totally ordered by divisibility; i.e., given $n, m \in$ IRen(K, f), either $n \mid m$ or $m \mid n$.*

Proof. We assume, without loss of generality, that $n, m \geq 2$. For $h = n$ or m, let $\phi_h : (K, f) \longrightarrow (\mathbb{Z}/h\mathbb{Z}, +1)$ be an interval h-renormalization. For $r \in \mathbb{Z}/n\mathbb{Z}$ and $s \in \mathbb{Z}/m\mathbb{Z}$ we put

$$N_r = \phi_n^{-1}(r) \quad \textit{and} \quad M_s = \phi_m^{-1}(s).$$

Define

$$\nu(r) = |\phi_m(N_r)| = \textit{ the number of } \phi_m\textit{-fibers that } N_r \textit{ meets,}$$

and

$$\mu(s) = |\phi_n(M_s)| = \textit{ the number of } \phi_n\textit{-fibers that } M_s \textit{ meets.}$$

If $x \in N_r \cap M_s$ then $f(x) \in N_{r+1} \cap M_{s+1}$. It follows that $\nu(r) \leq \nu(r+1)$ and $\mu(s) \leq \mu(s+1)$. Thus, $\nu(r) = \nu$ is independent of r, and $\mu(s) = \mu$ is independent of s.

Case 1. Some $N_r \cap M_s$ is not an interval.

It follows then from (2.2)(15) that $N_r \cup M_s = K$. Then for $r' \neq r$ and $s' \neq s$ we have $N_{r'} \subset M_s$ and $M_{s'} \subset N_r$. It follows that $\nu = \nu(r') = 1$ and $\mu = \mu(s') = 1$, whence $n = m$, clearly.

Case 2. $\mu = \nu = 2$.

For each r, the two nonempty intervals of the form $N_r \cap M_s$ partition N_r into a left interval $L(N_r)$ and a right interval $R(N_r)$. We similarly define $L(M_s)$ and $R(M_s)$ for each s. Now define

$$\mathbb{Z}/n\mathbb{Z} \underset{R}{\overset{L}{\rightleftarrows}} \mathbb{Z}/m\mathbb{Z}$$

by $L(N_r) = N_r \cap M_{L(r)}$ and $R(M_s) = M_s \cap N_{R(s)}$. Then it is clear that $L \circ R = Id$ and $R \circ L = Id$ so $n = m$.

Case 3. $\mu \geq 3$ or $\nu \geq 3$.

Say $\mu \geq 3$. Then M_0 meets at least 3 ϕ_n-fibers, say $N_{r_1}, N_{r_2}, N_{r_3}$ among them, with $r_1 < r_2 < r_3$ in the induced cyclic ordering. Then for $x \in N_{r_1} \cap M_0$ and $y \in N_{r_3} \cap M_0$ we have $N_{r_2} \subset [x, y] \subset M_0$ and so $\nu = \nu(r_2) = 1$. Similarly $\nu \geq 3$ implies that $\mu = 1$. Thus it remains only to treat:

Case 4. $\mu = 1$ or $\nu = 1$.

It suffices by symmetry to show that:

$$\mu = 1 \Longrightarrow n \mid m.$$

Define $p : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ by $M_s \subset N_{p(s)}$. Modifying ϕ_n by a translation, we can arrange that $p(0) = 0$. Further the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\phi_m"] \arrow[dr, "\phi_n"'] & \mathbb{Z}/m\mathbb{Z} \arrow[d, "p"] \\ & \mathbb{Z}/n\mathbb{Z} \end{tikzcd}$$

shows that p is equivariant for the map $+1$. Hence p is a surjective homomorphism, and so $n \mid m$ as claimed.

■

Remark: Theorem (2.6) should be contrasted with Proposition (1.5) which tells us that $Ren(K, f) = Div(Q)$, the set of divisors of some supernatural number $Q = Q(K, f)$ (cf. (1.6)).

(2.7) **Interval renormalization index.** Let (K, f) be an ordered dynamical system, in the sense of (2.4).

Case 1. *$IRen(K, f)$ is infinite; i.e.,* (K, f) is **infinitely interval renormalizable**. We then list $IRen(K, f)$ as an infinite increasing sequence:

(1) $$IRen(K, f) = (n_0, n_1, n_2, \ldots)$$

with $n_0 = 1 < n_1 < n_2 < \cdots$. By (2.6) n_{i-1} divides n_i and we put

(2) $$q_i = n_i/n_{i-1} \qquad (i \geq 1).$$

The sequence

(3) $$\mathbf{q} \ \ (= \mathbf{q}(K,f)) := (q_1, q_2, q_3, \ldots)$$

is then called the **IR-index** of (K, f). Clearly

(4) $$n_h = \mathbf{q}^{[h]} := q_1 q_2 \cdots q_h \qquad (h \geq 0).$$

Case 2. *IRen*(K,f) *is finite.* Then we list *IRen*(K,f) as a finite sequence

(5) $$\mathit{IRen}(K,f) = (n_0, n_1, \ldots, n_m)$$

with $n_0 = 1 < n_1 < \cdots < n_m$. Again by (2.6) we have the integers

(6) $$q_i = n_i/n_{i-1} > 1 \qquad (1 \leq i \leq m).$$

The IR-index this time is the *infinite* sequence

$$\mathbf{q} = \ \ (\mathbf{q}(K,f)) = (q_1, q_2, q_3 \ldots),$$

where we agree to put

(7) $$q_h = 0 \ \mathit{for} \ h > m.$$

If we define $n_h = \mathbf{q}^{[h]}$ as in (4) we then have $n_h = 0$ for $h > m$.

In case *IRen*$(K,f) = \{1\}$ (*i.e.*, $m = 0$ above) then we say that (K,f) is **non interval-renormalizable**. In all cases, it is clear that

(8) *IRen*(K,f) *is the ascending union of IRen*$(\mathbb{Z}/n\mathbb{Z}, +1)$ *where n increases in IRen*(K,f) *and* $\mathbb{Z}/n\mathbb{Z}$ *is given the order induced by making an interval renormalization* $\phi_n : (K,f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ *weak order preserving.*

We shall describe in (III, Section 5) below a natural rooted tree dynamics associated to the interval renormalizations of (K,f).

(2.8) **The interval renormalizable quotient** $\widehat{\phi} : (K,f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$**.** Let (K,f) be an ordered dynamical system and as in (2.7); write

(1) $$\mathit{IRen}(K,f) = (n_0, n_1, n_2, \ldots) \quad \textit{(finite or infinite)}$$

and

(2) $$\mathbf{q} = \mathbf{q}(K,f) = (q_1, q_2, q_3 \ldots).$$

Here $n_0 = 1 < n_1 < n_2 < \cdots$, $\quad n_{h-1} \,|\, n_h$, $\; q_h = n_h / n_{h-1}$, and

$$n_h = \mathbf{q}^{[h]} := q_1 q_2 \cdots q_h. \tag{3}$$

Assume either that the f-orbit of x_0 is dense, or that K is linearly ordered. Then (cf. (2.4)), there is, for each h, a unique interval n_h-renormalization

$$\phi_h : (K, f) \longrightarrow (X_h, +1), \qquad X_h = \mathbb{Z}/n_h\mathbb{Z} \tag{4}$$

such that $\phi_h(x_0) = 0$. Let

$$p : X_h \longrightarrow X_{h-1} \tag{5}$$

be the canonical projection $\mathbb{Z}/n_h\mathbb{Z} \longrightarrow \mathbb{Z}/n_{h-1}\mathbb{Z}$, which is a surjective ring homomorphism. Then the diagrams

$$\begin{tikzcd} K \arrow[r, "\phi_h"] \arrow[dr, "\phi_{h-1}"'] & X_h \arrow[d, "p"] \\ & X_{h-1} \end{tikzcd} \tag{6}$$

commute, because of uniqueness.

We give each X_h the (linear or cyclic) ordering induced from K, as in (2.4), so that ϕ_h is weak order preserving. Then the commutativity of (6) and the surjectivity of ϕ_h implies that p also is weak order preserving.

Case 1. $IRen(K, f) = (n_0, n_1, \ldots, n_m)$ is finite.

Then we put

$$\begin{cases} \widehat{\mathbb{Z}}_{(K,f)} = \mathbb{Z}/n_m\mathbb{Z} \quad (= X_m), \ \textit{and} \\ \widehat{\phi} \ (= \widehat{\phi}_{(K,f)}) = \phi_m : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{(K,f)}, +1). \end{cases} \tag{7}$$

This $\widehat{\phi}$ is a surjective morphism, and it is weak order preserving for the K-induced ordering on $\widehat{\mathbb{Z}}_{(K,f)}$.

It follows from Proposition (3.1) below that given any integer $N > 1$, for at least one fiber $K_r = \widehat{\phi}^{-1}(r)$ of $\widehat{\phi}$, $(K_r, f^{n_m}|_{K_r})$ is non-interval N-renormalizable.

Case 2. $IRen(K, f)$ is infinite. Then we put

$$\widehat{\mathbb{Z}}_{(K,f)} = \varprojlim_h \mathbb{Z}/n_h\mathbb{Z} = \varprojlim_h X_h. \tag{8}$$

The commutative diagrams (6) furnish the morphisms

$$\widehat{\phi} = \widehat{\phi}_{(K,f)} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{(K,f)}, +1)$$

which has dense image. Moreover $\widehat{\phi}$ is weak order preserving for the given order on K and the inverse limit of the K-induced orderings on each $X_h = \mathbb{Z}/n_h\mathbb{Z}$.

In both cases we call (K, f) **faithfully interval renormalizable** if $\widehat{\phi}$ is injective.

We sometimes (cf. (III, (3.4)) below) call the ring $\widehat{\mathbb{Z}}_{(K,f)}$, the **q-adic integers**, and denote it $\widehat{\mathbb{Z}}_{\mathbf{q}}$ and call $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$ the **q-adic adding machine** (cf. (III, (3.4))). Note that $\widehat{\mathbb{Z}}_{\mathbf{q}} = \widehat{\mathbb{Z}}_Q$ (cf. (1.7)), where Q denotes the supernatural number $Q(\mathbf{q}) = \prod_{h \geq 1} q_h$.

(2.9) **Proposition.** *Let (K, f) be a compact ordered dynamical system which is faithfully interval renormalizable. Then (K, f) is minimal, and is determined, up to IR-isomorphism of ordered dynamical systems (cf. (2.4)) by $IRen(K, f)$.*

Proof. The hypotheses imply that (K, f) is IR-isomorphic to $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$, which is minimal, and is determined by $\mathbf{q} = \mathbf{q}(K, f)$, hence by $IRen(K, f)$.

Minimality of (K, f) follows from that of the adding machine. Let (K', f') be another dynamical system with $IRen(K', f') = IRen(K, f)$. By (2.4)(6) we need only show that (K', f') and (K, f) are topologically isomorphic. But in view of the compactness and faithful interval renormalizable hypotheses, both systems are topologically isomorphic to the $\mathbf{q}$-adic adding machine $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$, where $\mathbf{q} = \mathbf{q}(K, f) = \mathbf{q}(K', f')$.

■

(2.10) **Remark.** Suppose that K is a *finite* totally ordered set, say $K = \{x_1 < x_2 < \cdots < x_n\}$. Then a minimal dynamical system (K, f) is just a transitive permutation f, corresponding to an n-cycle $\sigma \in S_n$ defined by

$$f(x_i) = x_{\sigma(i)} \qquad (1 \leq i \leq n).$$

We define

$$\begin{aligned} IRen(\sigma) &= IRen(K, f) \\ &= (m_0, m_1, \ldots, m_r). \end{aligned}$$

Here $m_0 = 1 < m_1 < \cdots < m_r = n$, and $m_{i-1} \mid m_i$. From (3.5) below it follows that any such sequence of divisors of n can occur this way for suitable σ.

Suppose that $n = 2^m$. Then it is easily seen that σ is a **simple permutation** in the sense of [Bl] if and only if $IRen(\sigma) = (1, 2, 4, 8, \ldots, 2^m)$. Equivalently, for each $r = 0, 1, \ldots, m-1$, the orbits of σ^{2^r} form 2^r disjoint intervals in $\{1, 2, 3, \ldots, 2^m\}$ on each of which σ^{2^r} switches the left and right halves.

If $n = r \cdot 2^m$ with $r > 1$ and odd, and σ is "simple" in the sense of ([Be], Definition 1.12), then $IRen(\sigma) = (1, 2, 4, 8, \ldots 2^m, r \cdot 2^m)$. However, this property does not suffice to make σ simple in general.

3. Systems with prescribed renormalizations.

(3.1) **Proposition.** *Let (K, f) be an ordered dynamical system and*

$$\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$$

an interval n-renormalization with fibers $K_r = \phi^{-1}(r)$.

(a) For each integer $m \geq 1$,

$$nm \in IRen(K, f) \iff m \in IRen(K_r, f^n|_{K^r}) \quad \forall\, r \in \mathbb{Z}/n\mathbb{Z}.$$

(b) Putting $J = \bigcap_{r \in \mathbb{Z}/n\mathbb{Z}} IRen(K_r, f^n|_{K^r})$, we have

$$IRen(K, f) = IRen(\mathbb{Z}/n\mathbb{Z}, +1) \bigcup n \cdot J,$$

where $\mathbb{Z}/n\mathbb{Z}$ is given the order induced from K via ϕ.

Proof. Put $N = nm$. We know from (1.9) that $N \in Ren(K, f)$ if and only if $m \in Ren(K_r, f^n|_{K^r})$ for each r. In this case, both conditions in (a) amount to saying that each of the fibers of an N-renormalization of (K, f) is a K-interval, whence (a). Part (b) follows immediately from (a).

∎

(3.2) **Proposition.** *Suppose in (3.1) that K is finite of cardinal $M > n$ and that f is transitive. Then $M = nq$ for some integer $q > 1$, and each fiber $K_r = \phi^{-1}(r)$ is a (proper) linearly ordered K-interval and an f^n-orbit. Assume that K_r is ordered so that, if x_r is its least element,*

$$x_r < f^n(x_r) < f^{2n}(x_r) < \cdots < f^{(q-1)n}(x_r). \tag{1}$$

Then

$$IRen(K, f) = \{M\} \cup IRen(\mathbb{Z}/n\mathbb{Z}, +1),$$

where $\mathbb{Z}/n\mathbb{Z}$ is given the order that makes ϕ weak order preserving.

Proof. Let $\psi : (K, f) \longrightarrow (\mathbb{Z}/d\mathbb{Z}, +1)$ be an interval d-renormalization. By (2.6) either $d \mid n$ or $n \nmid d$.

If $d \mid n$ then, after modifying ψ by a translation so that $\psi(K_0) = 0$, we obtain a commutative diagram

$$\begin{tikzcd}
(K, f) \arrow[d, "\phi"'] \arrow[dr, "\psi"] & \\
(\mathbb{Z}/n\mathbb{Z}, +1) \arrow[r, "p"'] & (\mathbb{Z}/d\mathbb{Z}, +1)
\end{tikzcd}$$

where p is the natural projection. It is easily seen then that p is weak order preserving, and hence an interval d-renormalization for the order on $\mathbb{Z}/n\mathbb{Z}$ induced from K, whence $d \in IRen(\mathbb{Z}/n\mathbb{Z}, +1)$.

Suppose finally that $n \mid d$, say $d = ne$. It follows then from (3.1) that $e \in IRen(K_r, f^n|_{K^r})$ for each r. By our assumption (cf. (1)), $(K_r, f^n|_{K^r})$ is isomorphic to $(\mathbb{Z}/q\mathbb{Z}, +1)$, with the linear order $0 < 1 < \cdots < (q-1)$, which is compatible with the natural cyclic order on $\mathbb{Z}/q\mathbb{Z}$. It follows then from (2.5) that $IRen(K_r, f^n|_{K^r}) = \{1, q\}$. Thus, $e = 1$ *or* q so $d = n$ *or* M, as was to be shown.

■

(3.3) **Cyclic induced actions.** Let n be an integer ≥ 1. For any dynamical system (H, h) we define the (n-fold) induced system

$$(K, f) = Ind_n(H, h)$$

by defining

(1) $$K = (\mathbb{Z}/n\mathbb{Z}) \times H$$

and

(2) $$f(r, x) = \begin{cases} (r+1, x) & \text{for } 0 \leq r < n-1; \\ (0, h(x)) & \text{for } r = n-1. \end{cases}$$

Here K is given the product topology, with the discrete topology on $\mathbb{Z}/n\mathbb{Z}$. The first coordinate projection

$$\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$$

is an n-renormalization, with fibers

(3) $$K_r = \phi^{-1}(r) = (r, H).$$

Moreover it follows from (2) above that

(4) $$f^n(r, x) = (r, h(x))$$

and so

(5) $$(K_r, f^n|_{K^r}) \cong (H, h) \qquad \text{for all } r \in \mathbb{Z}/n\mathbb{Z}.$$

It follows that if (H, h) is minimal then (K, f) is as well. Moreover, f is a homeomorphism if and only if h is a homeomorphism.

(3.4) **Ordered induction.** Suppose that (H, h) is a linearly ordered dynamical system and that we give $\mathbb{Z}/n\mathbb{Z}$ an order (linear or cyclic). Then we can give

$$K = (\mathbb{Z}/n\mathbb{Z}) \times H$$

the lexicographic order (linear or cyclic, as the case may be), and then

$$\phi : K \longrightarrow \mathbb{Z}/n\mathbb{Z}$$

is an interval n-renormalization of $Ind_n(H, h) = (K, f)$. We write

$$Ind_n^{ord}(H, h)$$

when it is understood that K is given the lexicographic order as above. It follows then from (3.1) that

$$IRen(Ind_n^{ord}(H, h)) = IRen(\mathbb{Z}/n\mathbb{Z}, +1) \cup n \cdot IRen(H, h).$$

Note that Ind_n^{ord} is a functor on linearly ordered dynamical systems and morphisms preserving $\leq$.

(3.5) **The systems** $H(q_1, q_2, \ldots, q_n)$**.** For any integer $q \geq 1$ define the ordered dynamical systems

$$H(q) = (\mathbb{Z}/q\mathbb{Z}, +1), \ \textit{ordered by } 0 < 1 < 2 < \cdots < q-1. \tag{1}$$

Given a sequence $(q_1, q_2, \ldots, q_n)$ of integers ($q_i \geq 2$ for all i) we define $H(q_1, q_2, \ldots, q_n)$ inductively by

$$H(q_1, q_2, \ldots, q_n) = Ind_{q_1}^{ord}(H(q_2, \ldots, q_n)). \tag{2}$$

It follows then by induction from (3.1)(b), using (2.5) for $n = 1$, that

$$IRen(H(q_1, q_2, \ldots, q_n)) = \{1, m_1, \ldots, m_n\}, \ \textit{where } m_k = q_1 \cdot q_2 \cdots q_k. \tag{3}$$

Thus we have constructed finite minimal ordered systems (K, f) with $IRen(K, f)$ any prescribed finite set $\{1 < m_1 < \cdots < m_n\}$, where $m_{i-1} \mid m_i$ for all i.

Note that $H(q_1, q_2, \ldots, q_n) \cong (\mathbb{Z}/m_n\mathbb{Z}, +1)$, relative to a certain ordering on $\mathbb{Z}/m_n\mathbb{Z}$. Writing elements of $\mathbb{Z}/m_n\mathbb{Z}$ in the form

$$r_0 + q_1(r_1 + q_2(r_2 + \cdots + q_{n-2}(r_{n-2} + q_{n-1}r_{n-1}) \cdots))$$

with $0 \leq r_{i-1} < q_i \quad (i = 1, \ldots, n)$ the ordering is lexicographic on n-tuples $(r_0, r_1, \ldots, r_{n-1})$. In particular the natural projection homomorphism

$$p : H(q_1, q_2, \ldots, q_n) \longrightarrow H(q_1, q_2, \ldots, q_{n-1})$$

is weak order preserving.

(3.6) **An infinite, finitely renormalizable system.** Suppose that (H, h) is a minimal linearly ordered dynamical system that is non interval renormalizable, *i.e.*, $IRen(H, h) = \{1\}$. (cf. (1.10) for such an example.) Form

$$(K, f) = Ind_{m_n}^{ord}(H, h),$$

where $\mathbb{Z}/m_n\mathbb{Z}$ is ordered as in $H(q_1, q_2, \ldots, q_n)$ above. Then it follows from (3.1) that

$$IRen(K, f) = IRen(H(q_1, q_2, \ldots, q_n)) = \{1, m_1, \ldots, m_n\}.$$

(3.7) **Examples of (finitely) renormalizable systems which are not interval renormalizable.** Given an integer $Q \geq 1$, we shall produce a linearly ordered minimal dynamical system (K, f) with K compact and totally disconnected such that

$$(1) \qquad Ren(K, f) = Div(Q)$$

and

$$(2) \qquad IRen(K, f) = \{1\}.$$

First take (H, h) a minimal linearly ordered compact totally disconnected system which is infinite yet nonrenormalizable: $Ren(H, h) = \{1\}$. Such an example is constructed (e.g. in (1.10)) using Denjoy expansion of an irrational rotation on the circle, and h is then a homeomorphism.

Next give $\mathbb{Z}/Q\mathbb{Z}$ any linear order. We form the induced system

$$(K, f) = Ind_Q(H, h),$$

but where

$$K = (\mathbb{Z}/Q\mathbb{Z}) \times H$$

is linearly ordered as follows.

Decompose H as

$$H = H_0 \coprod H_1$$

where each H_i is an open and closed (nonempty) H-interval with $x_0 < x_1$ whenever $x_i \in H_i \quad (i = 0, 1)$. Then we have

$$K = K_0 \coprod K_1$$

where each

$$K_i = (\mathbb{Z}/Q\mathbb{Z}) \times H_i$$

is given the lexicographic order, and K_0 precedes K_1 in the ordering of K.

We claim that (K, f) satisfies (1) and (2) above. Clearly

$$\phi_Q : (K, f) \longrightarrow (\mathbb{Z}/Q\mathbb{Z}, +1)$$

is a Q-renormalization (we are not saying "interval"), with fibers topologically isomorphic to (H, h). Since (H, h) is not renormalizable (non-trivially) it follows from (1.9) that

$$Q(K, f) = Q \cdot Q(H, h) = Q,$$

and so (1) follows (cf. (1.8)).

It remains to establish (2). Let $n \in IRen(K, f)$. Then $n \mid Q$, and if

$$\phi_n : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$$

is an interval n-renormalization, the fibers of ϕ_n will be unions of fibers of ϕ_Q. To show, as required, that $n = 1$, it thus suffices to show that a nonempty union U of ϕ_Q-fibers, if not all of K, is not a K-interval.

Now U must be of the form $\phi_Q^{-1}(J)$ for some nonempty $J \subsetneq \mathbb{Z}/Q\mathbb{Z}$, and we have $U = U_0 \coprod U_1$ with $U_i = U \cap K_i = J \times H_i \quad (i = 0, 1)$. Choose $j \in J$, and $r \in (\mathbb{Z}/Q\mathbb{Z}) - J$. Either $r < j$ or $r > j$. Say $r < j$. Choose $x_i \in H_i \quad (i = 0, 1)$. Then

$$(j, x_0) < (r, x_1) < (j < x_1)$$

with $(j, x_i) \in U \quad (i = 0, 1)$, and $(r, x_1) \notin U$. Similarly, when $r > j$,

$$(j, x_0) < (r, x_0) < (j, x_1)$$

shows that U is not an interval.

Combining examples (3.5), (3.6) and (3.7), we obtain the following:

(3.8) **Proposition.** *Let*

$$m_0 = 1 < m_1 < m_2 < \cdots < m_n$$

be a sequence of integers such that $m_{i-1} \mid m_i \quad (i = 1, \ldots, n)$. *Let* q *be any integer* ≥ 1 *and* $Q = m_n \cdot q$. *Then there is a minimal linearly ordered dynamical system* (K, f) *with* K *compact totally disconnected and infinite,* f *a homeomorphism, and such that*

$$Ren(K, f) = Div(Q)$$

and

$$IRen(K, f) = \{1, m_1, \ldots, m_n\}.$$

We next consider *infinite* interval renormalizability.

(3.9) **Theorem.** *Let*

$$\mathbf{q} = (q_1, q_2, q_3 \ldots)$$

be a sequence of integers $q_n \geq 2$. *For* $n \geq 0$ *put*

$$m_n = \mathbf{q}^{[n]} = q_1 \cdot q_2 \cdots q_n.$$

Let $H(q_1, q_2, \dots, q_n) = (\mathbb{Z}/m_n\mathbb{Z}, +1)$, *ordered as in (3.5), and let* $p : H(q_1, q_2, \dots, q_n) \longrightarrow H(q_1, \dots, q_{n-1})$ *be the natural projection (which is weak order preserving by (3.5)). Put*

$$(K, f) = H(\mathbf{q}) := \varprojlim_n H(q_1, q_2, \dots, q_n) = (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1),$$

with the inverse limit ordering. Then (K, f) *is faithfully interval renormalizable,*

$$IRen(K, f) = \{1, m_1, m_2, m_3, \dots\}$$

and

$$Ren(K, f) = \{m \mid m \mid m_n \text{ for some } n\}.$$

Proof. The last assertion follows from (1.7), and it reduces the assertion concerning $IRen(K, f)$ to showing that, for each n, $IRen(H(q_1, q_2, \dots, q_n)) = \{1, m_1, m_2, \dots, m_n\}$; the latter follows from (3.5).

■

(3.10) We next extend Proposition (3.8) to the case where $IRen(K, f)$ is finite, but $Ren(K, f)$ is allowed to be infinite.

Theorem. *Let Q be an infinite supernatural number. Let $m_0 = 1 < m_1 < \cdots < m_n$ be integers such that $m_{i-1} \mid m_i \quad (1 \le i \le n)$ and $m_n \mid Q$, say $Q = m_n \cdot Q'$. Then there is a minimal compact ordered dynamical system (K, f) which is faithfully renormalizable with $Q(K, f) = Q$ and $IRen(K, f) = \{m_0, m_1, \dots, m_n\}$.*

Proof. We first show that it suffices to establish the case $n = 0$. For assuming this we can find a system (H, h) as in the theorem with $Q(H, h) = Q'$ and $IRen(H, h) = \{1\}$. This done, we take $(K, f) = Ind^{ord}_{m_n}(H, h)$, where $\mathbb{Z}/m_n\mathbb{Z}$ is ordered so as to make $(\mathbb{Z}/m_n\mathbb{Z}, +1) = H(q_1, \dots, q_n)$ with $q_i = m_i/m_{i-1}$, as in (3.5). Then it follows from (1.9) that $Q(K, f) = m_n \cdot Q(H, h) = m_n \cdot Q' = Q$, and it follows from (3.6) that $IRen(K, f) = \{m_0, m_1, \dots, m_n\}$.

It remains to treat the case $n = 0$. We seek a (K, f) topologically isomorphic to $(\widehat{\mathbb{Z}}_Q, +1)$ and ordered so that $IRen(K, f) = \{1\}$.

Let

(1) $$\begin{cases} q = \text{the least prime divisor of } Q; \text{ and} \\ Q = q \cdot Q'. \end{cases}$$

Choose sequences of integers

(2) $$\begin{cases} \mathcal{M} &= \{m_0 = 1 < m_1 < m_2 < m_3 < \cdots\} \\ \mathcal{N} &= \{n_0 = 1 < n_1 < n_2 < n_3 < \cdots\} \end{cases}$$

such that

$$m_{i-1} \mid m_i \text{ and } n_{i-1} \mid n_i \text{ for all } i, \tag{3}$$

$$LCM(\mathcal{M}) = Q' = LCM(\mathcal{N}), \tag{4}$$

and

$$\mathcal{M} \cap \mathcal{N} = \{1\}. \tag{5}$$

Now as in Theorem (3.9), we form systems

$$(H_i, h_i) \quad (i = 0, 1), \ \textit{topologically isomorphic to } (\widehat{\mathbb{Z}}_{Q'}, +1) \tag{6}$$

such that

$$\mathit{IRen}(H_0, h_0) = \mathcal{M} \ \textit{and} \ \mathit{IRen}(H_1, h_1) = \mathcal{N}. \tag{7}$$

Now take

$$K = (\mathbb{Z}/q\mathbb{Z}) \times \widehat{\mathbb{Z}}_{Q'}, \tag{8}$$

with the product topology and assuming the discrete topology on $\mathbb{Z}/q\mathbb{Z}$. We order K as follows. On

$$K_r = (r, \widehat{\mathbb{Z}}_{Q'}) \qquad (r \in \mathbb{Z}/q\mathbb{Z}) \tag{9}$$

the induced order is that corresponding to H_0, for $r = 0, 1, \ldots q-2$, and that corresponding to H_1 for $r = q-1$. For the order on K, we first split K_0 and K_{q-1} each into two nonempty open intervals,

$$\begin{array}{lllll} K_0 & = & K_{0,0} \coprod K_{0,1}, & & K_{0,0} < K_{0,1} \\ K_{q-1} & = & K_{q-1,0} \coprod K_{q-1,1}, & & K_{q-1,0} < K_{q-1,1} \end{array} \tag{10}$$

and so that neither $K_{0,0}$ nor $K_{q-1,1}$ corresponds to a coset of $\widehat{\mathbb{Z}}_{Q'}$ mod $3\widehat{\mathbb{Z}}_{Q'}$.

Finally, we order K as follows:

$$K_{0,0} < K_{q-1,0} < K_1 < \cdots < K_{q-2} < K_{0,1} < K_{q-1,1} \tag{11}$$

where each term in the sequence is a K-interval, with the internal order of each term determined by the order of the K_r which contains it.

Next define

$$f : K \longrightarrow K$$

by

$$f(r, x) = \begin{cases} (r+1, x) & (r \neq q-1) \\ (0, x+1) & (r = q-1). \end{cases} \tag{12}$$

The first coordinate projection then defines a renormalization

$$\begin{aligned} \phi : (K, f) &\longrightarrow (\mathbb{Z}/q\mathbb{Z}, +1), \\ \phi^{-1}(r) &= K_r. \end{aligned} \tag{13}$$

Putting

$$f_r = f^q \mid_{K_r} : K_r \longrightarrow K_r$$

we see that, as ordered dynamical systems,

$$(K_r, f_r) \cong \begin{cases} (H_0, h_0) & (r \neq q-1) \\ (H_1, h_1) & (r = q-1). \end{cases} \tag{14}$$

It follows from (1.9) that

$$Q(K, f) = q \cdot Q(H_i, h_i) = q \cdot Q' = Q, \tag{15}$$

and in fact, (K, f) is topologically isomorphic to $(\widehat{\mathbb{Z}}_Q, +1)$.

It remains to show that

$$IRen(K, f) = \{1\}. \tag{16}$$

We must show that, if N is a divisor of Q and $N > 1$ then some orbit closure of f^N is not an interval.

Case $N = q$. Then (11) shows that, for example, $K_0 = K_{0,0} \coprod K_{0,1}$ is not a K-interval.

Case $N = qM, \quad M > 1$. Then, by (14) each f^N $(= (f^q)^M)$-orbit closure of a point in K_r corresponds to an h_0^M-orbit closure in H_0 (for $r \neq q-1$) or to an h_1^M-orbit closure in H_1 (for $r = q-1$). In view of condition (5), for each $M > 1$, either h_0^M in H_0 or h_1^M in H_1 has an orbit closure that is not an interval.

Case $q \nmid N$. Then, q being prime, N is relatively prime to q. Let L be any f^N-orbit closure. It corresponds to a coset of $\widehat{\mathbb{Z}}_Q$ mod $N\widehat{\mathbb{Z}}_Q$. Since N and q are relatively prime, L meets every K_r $(r \in \mathbb{Z}/q\mathbb{Z})$. Suppose that every f^N-orbit closure is a K interval; we shall derive a contradiction. Since L meets K_1 and K_{q-2} it follows from (11) that the interval L contains K_r for $1 < r < q-2$. But a coset mod N can contain a coset mod q only if $N \mid q$. Since q was chosen to be the least prime divisor of Q this can happen only for $N = 1$. Thus, we are reduced to the case in which no r satisfies $1 < r < q-2$, i.e. $q \leq 4$, and hence $q = 2, 3$.

Case $q = 3$.

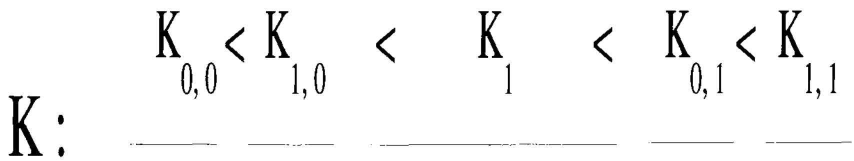

Figure 2.

Let L be the f^N-orbit closure containing the left end point of K, and R the f^N-orbit closure containing the right end point of K. Both L and R meet K_1 and are intervals. Hence L contains $K_{0,0} \cup K_{2,0}$ and R contains $K_{0,1} \cup K_{2,1}$. The complement of $L \cup R$ is contained in K_1, so there can't be any other f^N-orbit closure, since it could not meet K_0 or K_2. Thus $N = 2$. But this contradicts the minimality of q.

Case $q = 2$.

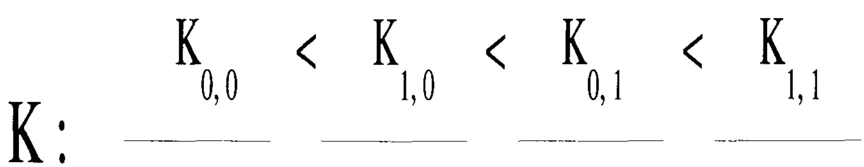

Figure 3.

Let L and R be as above, that is, the f^N-orbit closures containing the left and right end points of K respectively. Since L meets K_1 and R meets K_0 we must have $K_{0,0} \subset L$ and $K_{1,1} \subset R$ Since $N \neq 2$ there must be a third f^N-orbit closure S, which is an interval meeting $K_{1,0}$ and $K_{0,1}$, as indicated in Figure 4.

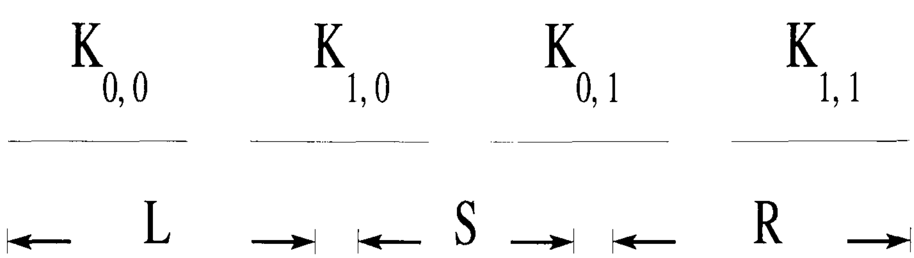

Figure 4.

But then clearly any interval in the complement of $L \cup S \cup R$ is contained in either $K_{1,0} \subset K_1$ or in $K_{0,1} \subset K_0$. Hence we must have $N = 3$. Further $K_{0,0} = L \cap K_0$ is a $\widehat{\mathbb{Z}}_Q$-coset mod $6\widehat{\mathbb{Z}}_Q$, or a $\widehat{\mathbb{Z}}_{Q'}$ coset mod $3\widehat{\mathbb{Z}}_{Q'}$ (identifying K_0 with $\widehat{\mathbb{Z}}_{Q'}$). Similarly $K_{1,1} = R \cap K_1$ corresponds to a $\widehat{\mathbb{Z}}_{Q'}$-coset mod $3\widehat{\mathbb{Z}}_{Q'}$. But this contradicts the choice of the decompositions $K_r = K_{r,0} \coprod K_{r,1}$ for $r = 0$ and $q-1$ in (10) above. This concludes the proof of Theorem (3.10).

■

(3.11) **Possible renormalization types.** Let (K, f) be a minimal ordered dynamical system, with

$$\begin{aligned} Q &= Q(K, f), \\ \mathbf{q} &= \mathbf{q}(K, f) = (q_1, q_2, q_3 \ldots), \ \textit{and} \\ Q(\mathbf{q}) &= \textstyle\prod_{n \geq 1} q_n. \end{aligned}$$

Here, Q and $Q(\mathbf{q})$ are supernatural numbers as in (1.6),

$$Ren(K, f) = Div(Q) \qquad (cf.(1.7))$$

and

$$Q(\mathbf{q}) \quad \textit{divides } Q.$$

We can ask whether conversely, given supernatural numbers Q, Q', with Q' dividing Q can we find a (K, f) as above with $Q(K, f) = Q$ and $Q(\mathbf{q}) = Q'$, where $\mathbf{q} = \mathbf{q}(K, f)$? We have affirmed this, even with K compact and f a homeomorphism, in the following cases.

1. Q' and Q are finite. (Proposition (3.8))
2. Q' is infinite and $Q' = Q$, (Theorem (3.9))
3. Q' is finite and Q is infinite. (Theorem (3.10))

It will be shown in Theorem (4.1) that these exhaust *all* cases; namely, if Q' is infinite then we must have $Q' = Q$, at least for compact K contained in $\mathbb{R}$ or in the circle S^1.

4. Infinite interval renormalizability.

(4.1) **Theorem.** *Let K be a compact subset of $\mathbb{R}$ or S^1 and let (K, f) be a dynamical system with dense orbit and which is infinitely interval renormalizable. Let*

$$\widehat{\phi} = \widehat{\phi}_{(K,f)} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{(K,f)}, +1)$$

be the interval renormalizable quotient, as in (2.8).

(a) $\widehat{\phi}$ *is surjective, and* $\widehat{\phi}$ *is injective except perhaps, for a countable set of 2-point-interval fibers.*

(b) f *is surjective, and* f *is injective except, perhaps, for a countable set of 2-point-interval fibers. Moreover* (K, f) *is minimal, and* K *is a Cantor set.*

(c) If $n \in Ren(K, f)$ *then* $n \mid m$ *for some* $m \in IRen(K, f)$*;*

i.e.,

$$\begin{aligned} Q(K,f) &= Q(\mathbf{q}), \text{ where } \mathbf{q} = \mathbf{q}(K,f). \\ &= LCM(IRen(K,f)). \end{aligned}$$

We give examples in (4.6) below showing that the 2-point fibers can in fact occur, either for both $\widehat{\phi}$ and f, or for $\widehat{\phi}$ alone.

Proof of (a). Since $\widehat{\phi}$ and f have dense images and K is compact, $\widehat{\phi}$ and f are surjective.

For $r \in \widehat{\mathbb{Z}}_{(K,f)}$ put $K_r = \phi^{-1}(r)$. The K_r are closed and pairwise disjoint. Since $\widehat{\phi}$ and f are surjective and $\widehat{\phi}$ is equivariant it follows that

$$f(K_r) = K_{r+1} \text{ and } f^{-1}(K_{r+1}) = K_r. \tag{1}$$

Further, $\widehat{\phi}$ is compatible with the order (linear or cyclic) on K, and the induced order on $\widehat{\mathbb{Z}}_{(K,f)}$. It follows that each K_r is a K-interval. Since $\mathbb{R}$ or S^1 cannot contain uncountably many pairwise disjoint intervals of length > 0, it follows that all but countably many of the K_r consist in a single point. It remains to see that no K_r can contain 3 points.

Say $x \in K$ has a dense f-orbit $f^*(x) = \{f^n(x) \mid n \geq 0\}$, and $x \in K_{r_0}$. If some K_r has at least 3 points, then so also does K_{r-1}, by (1). Thus, we can then choose r with $|K_r| \geq 3$ and $r \neq r_0 + n$ for any integer $n \geq 0$. Choose $u < v < w$ in K_r (where "$<$" denotes "$<_u$" in the circle case). For $n \geq 0$, $f^n(x) \in K_{r_0+n}$, and the K-interval K_{r_0+n} is, by choice of r, disjoint from the interval K_r. Hence either $f^n(x) < u$ or $f^n(x) > w$. Thus, $f^*(x)$ never enters the neighborhood (u, w) of v, contradicting denseness of $f^*(x)$. This proves (a).

Proof of (b). If $f(x) = f(y)$ then $\widehat{\phi}(x) + 1 = \widehat{\phi}(f(x)) = \widehat{\phi}(f(y)) = \widehat{\phi}(y) + 1$, so $\widehat{\phi}(x) = \widehat{\phi}(y)$. Thus the f-fibers are contained in $\widehat{\phi}$-fibers so f is injective, except perhaps for countably many 2-point fibers. As already observed, f is surjective.

We next show that (K, f) is minimal. Let $y \in K$ have f-orbit closure $L \subset K$. The minimality of $(\widehat{\mathbb{Z}}_{(K,f)}, +1)$ and compactness of L imply that $\widehat{\phi}|_L$ is surjective, and hence L contains all 1-point fibers of K, and $K - L$ is a countable open subset of K. For $z \in K - L$, there is a unique $z' \in L$ defined by $K_{\widehat{\phi}(z)} = \{z, z'\}$. Say $z < z'$; then since $\{z', z\}$ is a K-interval, z must be a limit from above of points of the orbit $f^*(x)$ which, when sufficiently near z, lie in the open set $K - L$. Thus, we can find $z < u < v$ with $(z, v) \cap K \subset K - L$. But then the K-interval $\{u, u'\}$ must lie in (z, v), thus forcing the contradiction $u' \notin L$.

The minimality of (K, f), just proved, shows that K has no isolated points. Since $\widehat{\mathbb{Z}}_{(K,f)}$ is totally disconnected and $\widehat{\phi}$ has discrete K-interval fibers, it follows that K is totally disconnected. Hence K is a Cantor set (cf. (B.9)), thus concluding the proof of (b).

Proof of (c). Choose $x_0 \in K_0$. For each $n \in \mathit{Ren}(K, f)$, let

$$\phi_n : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$$

be the renormalization such that $\phi_n(x_0) = 0$. Passing to the inverse limit over $n \in \mathit{Ren}(K, f)$, ordered by divisibility, we obtain, as in (1.8),

$$\widehat{\phi}_Q : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_Q, +1) = \varprojlim_{n \in \mathit{Ren}(K,f)} (\mathbb{Z}/n\mathbb{Z}, +1),$$

where $Q = \mathit{LCM}(\mathit{Ren}(K, f)))$, a supernatural number, as in (1.6), and $\widehat{\mathbb{Z}}_Q$ denotes the Q-adic integers (cf. (1.7)).

Since $\mathit{IRen}(K, f) \subset \mathit{Ren}(K, f)$ we have a natural commutative diagram

$$\begin{tikzcd} (K,f) \arrow[r, "\widehat{\phi}_Q"] \arrow[dr, "\widehat{\phi}"'] & (\widehat{\mathbb{Z}}_Q, +1) \arrow[d, "\alpha"] \\ & (\widehat{\mathbb{Z}}_{(K,f)}, +1). \end{tikzcd}$$

Since α is continuous and equivariant, and $\alpha(0) = 0$, it follows easily that α is a (surjective) homomorphism. Moreover $\widehat{\phi}_Q$ is surjective (because it has dense image and K is compact). It follows therefore from (a) that all but countably many fibers of α have 1 point. But the fibers of α are the (uncountably many, since $\widehat{\mathbb{Z}}_{(K,f)}$ is uncountable) cosets of $Ker(\alpha)$. It follows that $Ker(\alpha) = 0$, so α is an isomorphism, whence (c).

■

(4.2) **Corollary.** *Let (K, f) be an ordered dynamical system with a topological isomorphism*

$$\psi : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_Q, +1)$$

for some supernatural number Q. Assume that (K, f) is infinitely interval renormalizable; let $\mathbf{q} = \mathbf{q}(K, f)$. Then the natural projection $p : \widehat{\mathbb{Z}}_Q \longrightarrow \widehat{\mathbb{Z}}_{\mathbf{q}}$ is an isomorphism, and

$$\widehat{\phi}_{(K,f)} = p \circ \psi : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1).$$

In particular, $\widehat{\phi}_{(K,f)}$ is an isomorphism.

Proof. ψ makes K compact and totally disconnected with a countable base for its topology. Then the order structure permits us to construct an order preserving topological embedding of K as a Cantor set in $\mathbb{R}$. (See (B.6) below.)

Then it follows from (4.1)(c) that p is an isomorphism, whence the corollary.

■

(4.3) **Corollary.** *If in (4.1), (K, f) is faithfully interval renormalizable (i.e., if $\widehat{\phi}$ is injective), then $\widehat{\phi}$ defines a topological isomorphism of (K, f) with the* $\mathbf{q}$*-adic adding machine $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$; where $\mathbf{q} = \mathbf{q}(K, f)$ (cf. (2.9) and (2.10)). In particular f is a homeomorphism, and the group $\langle f \rangle$ generated by f acts freely on K. Moreover (cf. (2.11)) (K, f) is determined, up to IR-isomorphism of ordered dynamical systems, by $\mathbf{q}(K, f)$.*

(4.4) **Remarks.** 1. In (4.3) we see that $\mathbf{q}$ which determines $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$, encodes the topological dynamics of (K, f). It does not, however, record the order structure on K (in terms of which $\mathbf{q}$ was defined, via *interval* renormalizations). We shall see in (III, (5.3)) below how $\mathbf{q}$ further determines a rooted tree $X = X(\mathbf{q})$, with an automorphism α, so that $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$ appears as the action induced by α on the space of ends of X. In this setting, the order structure on K then corresponds essentially to a planar embedding of X (cf. (III, (2.4))).

2. It is well known (see for example [BOT]) that the 2-point fibers of $\widehat{\phi}$ and f in Theorem (4.1) can occur. In fact we give examples in (4.6) below where either $\widehat{\phi}$ and f have 2-point fibers, or else $\widehat{\phi}$ does, while f does not have them.

(4.5) **Corollary.** *Let $\alpha : (K', f') \longrightarrow (K, f)$ be an IR-morphism (cf. (2.4)) of minimal compact real (cf. (B.8)) dynamical systems. Assume that (K, f) is infinitely interval renormalizable.*

(a) $Q(K', f') = Q(K, f)$.

(b) α is surjective, and injective except perhaps for countably many 2-point fibers.

(c) If α is weak order preserving then $IRen(K', f') = IRen(K, f)$.

Proof. By definition (2.4) of IR-morphism, α entails an inclusion, $IRen(K, f) \subset IRen(K', f')$. Since $IRen(K, f)$ is infinite, so also is $IRen(K' f')$. Both sets are totally ordered by divisibility (Theorem (2.6)). Hence $IRen(K, f)$ is a cofinal subsequence of $IRen(K', f')$. Since (cf. Theorem (4.1))

$$Q(K, f) = LCM(IRen(K, f))$$

and

$$Q(K', f') = LCM(IRen(K', f'))$$

it follows that $Q(K', f') = Q(K, f)$; denote this common value by Q. Then we have a commutative diagram

$$\begin{tikzcd} (K', f') \arrow[r, "\alpha"] \arrow[dr, "\widehat{\phi}'"'] & (K, f) \arrow[d, "\widehat{\phi}"] \\ & (\widehat{\mathbb{Z}}_Q, +1) \end{tikzcd}$$

in which (cf. Theorem (4.1)) $\widehat{\phi}$ and $\widehat{\phi}'$ are surjective and injective except perhaps for countably many 2-point fibers. It follows that α has the same properties.

It remains to prove (c), so assume that α is weak order preserving. For $n \in IRen(K, f)$, put $(K_n, f_n) = (\mathbb{Z}/n\mathbb{Z}, +1)$, with the ordering on $K_n = \mathbb{Z}/n\mathbb{Z}$ induced from K by an interval n-renormalization $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$. Then it follows easily from Proposition (3.1) that

$$(*) \qquad IRen(K, f) = \textit{the ascending union of } IRen(K_n, f_n) \; (n \in IRen(K, f)).$$

Similarly we have

$$(*') \qquad IRen(K', f') = \textit{the ascending union of } IRen(K'_{n'}, f'_{n'}) \; (n' \in IRen(K', f')).$$

Now if $n \in IRen(K, f) \subset IRen(K', f')$, then the fact that α is weak order preserving implies that $(K'_n, f'_n) = K_n, f_n)$. Since $IRen(K, f)$ is a cofinal subsequence of $IRen(K', f')$ it follows from $(*)$ and $(*')$ above that $IRen(K', f') = IRen(K, f)$, as claimed.

■

(4.6) **Examples with 2-point fibers.** Let

$$T = [a, b], \qquad a < b,$$

be a real closed interval. Consider a map

(1) $g : T \longrightarrow T$ *that is continuous and piecewise monotone,*

i.e., g is monotone on each of a finite set of closed intervals whose union is T. Let

(2) $K \subset T$ *be a minimal closed g-invariant subset.*

Then with $f = g|_K : K \longrightarrow K$, the minimal ordered dynamical system (K, f) has an interval renormalization index

(3) $$\mathbf{q} = \mathbf{q}(K, f) = (q_1, q_2, q_3, \ldots)$$

and a canonical map

$$\widehat{\phi} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1), \tag{4}$$

unique up to a translation in $\widehat{\mathbb{Z}}_{\mathbf{q}}$. In fact (cf. (II,(9.9)) below) every possible $\mathbf{q}$ can occur this way for suitable choice of g and K, even with g unimodal.

We shall be interested in the cases when $\mathbf{q}$ is infinite, *i.e.*, when (K, f) is infinitely interval renormalizable. It follows then from Theorem (4.1) that $\widehat{\phi}$ and $f : K \longrightarrow K$ are each surjective and also injective except perhaps for countably many 2-point fibers. Plainly, if $\widehat{\phi}$ is injective, then so also is f. We propose now to show, using the Denjoy expansion construction of Appendix A above, that the 2-point fibers can indeed occur either for both $\widehat{\phi}$ and f, or for $\widehat{\phi}$ alone; see (15) and (20) below. Examples with f injective but $\widehat{\phi}$ not injective cannot arise when (T, g) is C^2-unimodal; this follows from [BOT], using deep results of Sullivan [Su].

We start by choosing

(5) $C \subset K$, *a countable subset,*

such that

$$f^{-1}(C) \subset C, \tag{6}$$

and

(7) *each* $c \in C$ *is a monotone limit from both directions in* $K - \{c\}$.

Now, as in (A.4)-(A.6), let

$$\pi : (T, h) \longrightarrow (T, g) \tag{8}$$

be a Denjoy expansion of (T, g) along C. Our assumptions (1), (5), and (6) furnish the conditions (A.4)(3) required for this construction. Moreover it follows easily from (A.6)(4) that:

(9) *h is piecewise monotone with the same number of intervals of monotonicity as g.*

Next we take

$$K' = \sigma_0(K) \cup \sigma_1(K) \subset T \tag{10}$$

(cf. (A.4)(1)). Then K' is h-invariant (A.6)(4); putting $f' = h|_{K'} : K' \longrightarrow K'$, we have

(11) *$\pi : (K', f') \longrightarrow (K, f)$ is a weak order preserving surjection with fiber over $x \in K$ the set $\{\sigma_0(x), \sigma_1(x)\}$, which has two points precisely when $x \in C$.*

It follows further from (7) above and (A.7) that:

(12) (K', f') *is minimal.*

Now it follows from (4.5)(c) that

(13) $$\mathbf{q} = \mathbf{q}(K, f) = \mathbf{q}(K', f'),$$

and we have a commutative diagram

(14) $$\begin{tikzcd} (K', f') \arrow[r, "\pi"] \arrow[dr, "\widehat{\phi}_{(K',f')}"'] & (K, f) \arrow[d, "\widehat{\phi}_{(K,f)}"] \\ & (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1) \end{tikzcd}$$

We know from Theorem (4.1) that $\widehat{\phi}_{(K,f)}$ and $\widehat{\phi}_{(K',f')}$ are both surjective and injective except perhaps for countably many 2-point fibers. It follows therefore that:

(15) *For each* $x \in C, \{\sigma_0(x), \sigma_1(x)\}$ *is a 2-point fiber of* $\widehat{\phi}_{(K',f')}$.

We next address the question of whether $f' : K' \longrightarrow K'$ is a homeomorphism. By minimality and compactness, both f and f' are surjective. *Assume* now that:

(16) f *is locally monotone near each* $x \in C$, *and* $f : K \longrightarrow K$ *is a homeomorphism.*

The local monotonicity assumption plus (A.4)(8) implies that

(17) $f' : \{\sigma_0(x), \sigma_1(x)\} \longrightarrow \{\sigma_0(f(x)), \sigma_1(f(x))\}$ *is surjective for all* $x \in T$, *hence* f' *is surjective on the fibers of* $\pi : K' \longrightarrow K$.

Further, since $f : K \longrightarrow K$ is a homeomorphism (and $C \subset K$), we see from (17) that

(18) $f' : K' \longrightarrow K$ *is a homeomorphism if and only if* $f(C) = C$.

Recall from (6) that we have had to choose C so that $f^{-1}(C) \subset C$. Thus:

(19) *Choosing* C *so that* $f^{-1}(C) = C$ *(with* $f : K \longrightarrow K$ *a homeomorphism), we obtain* (K', f') *with* f' *a homeomorphism, but* $\widehat{\phi}_{(K',f')}$ *not injective.*

For example, C could be an f-orbit in K consisting of points where f is locally monotone. (The latter condition excludes, in view of (1), only finitely many orbits.)

(20) *Choosing* C *so that* $f^{-1}(C) \subsetneq C$, *we obtain* (K', f') *with* f' *not a homeomorphism and* $\widehat{\phi}_{(K',f')}$ *not injective.*

Appendix B. Embedding ordered Cantor-like sets in real intervals.

(B.1) **Profinite spaces and partitions.** A **profinite space** is a compact totally disconnected space K. Examples include finite spaces, and also **Cantor sets**, where K is further required to be without isolated points, and have a countable base for its topology.

By a **partition** of a topological space K we mean a continuous surjective map $\phi : K \longrightarrow X$ where X is finite and discrete. Thus each fiber $K_r = \phi^{-1}(r)$ $(r \in X)$ is open-closed, and $K = \coprod_{r \in X} K_r$. Let $\phi' : K \longrightarrow X'$ be another partition. We write $\phi \leq \phi'$ if there is a (necessarily surjective) map $p : X' \longrightarrow X$ such that $\phi = p \circ \phi'$.

In general we can define

$$\phi \wedge \phi' : K \longrightarrow Y$$

to be induced by $(\phi, \phi') : K \longrightarrow X \times X'$, with $Y = Im((\phi, \phi'))$. Then the two projections of $X \times X'$ show that $\phi \leq \phi \wedge \phi'$ and $\phi' \leq \phi \wedge \phi'$. Thus, the partitions (ϕ, X) of K form an inverse system, and we have a canonical continuous map.

$$\widehat{\phi} : K \longrightarrow \widehat{X} = \varprojlim_{(\phi, X)} X$$

with dense image. If K is totally disconnected, $\widehat{\phi}$ is injective. If K is compact, it is surjective. Thus $\widehat{\phi}$ is a homeomorphism when K is profinite.

Conversely, any inverse limit of finite sets is a profinite space.

(B.2) **Countable base.** Let K be a profinite space. Suppose that K has a countable base B for its topology. Each open-closed set of K is, being open and compact, a finite union of elements of B. Since B is countable there are then only countably many open-closed sets, hence also only countably many partitions. In this case we can choose a cofinal sequence (ϕ_n, X_n), $\phi_n \leq \phi_{n+1}$, of partitions of K, and then we have the homeomorphism

$$\widehat{\phi} : K \longrightarrow \varprojlim_{n} X_n.$$

Thus K is the space of ends of a locally finite tree, as in (III, (2.2)) below.

(B.3) **Ordered profinite spaces.** By an **ordered profinite space** we mean a profinite space K with a linear order whose topology is the order topology, having open intervals as a base. For $x \in K$ put

$$(\leftarrow, x) = \{y \mid y < x\}$$

and

$$(\leftarrow, x] = \{y \mid y \leq x\}$$

and similarly $(x, \rightarrow)$ and $[x, \rightarrow)$. If $C \subset K$ is closed and nonempty then C has a least element $\min(C)$, which is the intersection of the compact sets $C \cap (\leftarrow, x]$ $(x \in C)$. Similarly C has a greatest element $\max(C)$. It follows that an interval $C \neq \emptyset$ of K which is topologically closed must be a closed order-interval, $[a, b]$ $(a \leq b)$, and conversely.

If C is open-closed set in K then C is a finite union of open intervals and hence a union of pairwise disjoint closed intervals (amalgamate all subsets of the covering whose union is again an interval). The latter intervals must be open-closed. It follows that

the open-closed intervals of K form a base for the topology.

(B.4) **Interval partitions.** Let K be an ordered profinite space. A partition $\phi : K \longrightarrow X$ is called an **interval partition** if its fibers $K_r = \phi^{-1}(r)$ $(r \in X)$ are intervals. These intervals occur in a certain order in K, and this defines a unique linear order on X so that ϕ is weak order preserving. For such an interval partition we shall understand X to be given this order structure.

If $\phi' : K \longrightarrow X'$ is another interval partition and $\phi \leq \phi'$ then it is readily seen that $p : X' \longrightarrow X$ is also weak order preserving. It follows that $\widehat{X} = \varprojlim_{(\phi, X)}$ inherits an inverse limit order, and $\widehat{\phi} : K \longrightarrow \widehat{X}$ is an isomorphism of ordered sets.

In case K has a countable base for its topology then as in (B.2), it follows that we have an isomorphism

$$\widehat{\phi} : K \longrightarrow \widehat{X} = \varprojlim_{n} X_n,$$

where

$$X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} \cdots \xleftarrow{p} X_{n-1} \xleftarrow{p} X_n \xleftarrow{p} \cdots,$$

the X_n are finite ordered sets, each p is surjective and weak order preserving, $\widehat{X}$ is given the inverse limit order, and $\widehat{\phi}$ is an order preserving homeomorphism.

(B.5) **Question.** Let K be an ordered profinite space. Must K have a countable base for its topology?

It seems plausible that this is the case, but we have neither been able to verify it, nor obtain a counterexample.

(B.6) **Interval embeddings.** Let K be an ordered profinite space with a countable base for its topology. Then as in (B.4), we can identify

$$K = \varprojlim_n X_n,$$

where X_n is a sequence of finite ordered sets and weak order preserving surjections $p : X_n \longrightarrow X_{n-1}$, and X_0 is a single point.

We inductively construct a commutative diagram

$$(*)\quad \begin{array}{ccccccccccccc} J_0 & \xleftarrow{j} & J_1 & \xleftarrow{j} & J_2 & \xleftarrow{j} & \cdots & \xleftarrow{j} & J_{n-1} & \xleftarrow{j} & J_n & \xleftarrow{j} \\ q_0 \downarrow & & q_1 \downarrow & & q_2 \downarrow & & \cdots & & q_{n-1} \downarrow & & q_n \downarrow & \\ X_0 & \xleftarrow[p]{} & X_1 & \xleftarrow[p]{} & X_2 & \xleftarrow[p]{} & \cdots & \xleftarrow[p]{} & X_{n-1} & \xleftarrow[p]{} & X_n & \xleftarrow[p]{} \end{array}$$

with the following properties:

(a) J_0 is the unit interval $I = [0, 1]$.

(b) Each $j : J_n \longrightarrow J_{n-1}$ is the inclusion of a closed subset.

(c) Each $q_n : J_n \longrightarrow X_n$ is surjective, weak order preserving, and each fiber $J_n(x) = q_n^{-1}(x)$ $(x \in X_n)$ is a closed real interval with

$$0 < \ \mathit{length}\ (J_n(x)) \leq \frac{1}{n}.$$

Suppose that, for $i = 0, 1, \ldots, n-1$, $q_i : J_i \longrightarrow X_i$ has been constructed with the above properties. Say

$$X_{n-1} = \{x_1 < x_2 < \cdots < x_m\}$$

Put $L_r = J_{n-1}(x_r) = q_{n-1}^{-1}(x_r)$ $(r = 1, \ldots, m)$. Then L_r is a closed real interval and $0 <$ $\mathit{length}(L_r) \leq 1/(n-1)$. Say

$$p^{-1}(x_r) = \{y_{r_1} < y_{r_2} < \cdots < y_{r_{m_r}}\},$$

an interval in X_n. In the interval L_r, choose

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{m_r} < b_{m_r}$$

with $b_i - a_i \leq 1/n$. Put

$$J_{n,r} = \bigcup_{h=1}^{m_r} [a_h, b_h] \ \subset \ L_r = J_{n-1}(x_r)$$

and define $q_{n,r} : J_{n,r} \longrightarrow p^{-1}(x_r)$ so that $q_{n,r}^{-1}(y_{r_h}) = [a_h, b_h]$. Now put $J_n = J_{n,1} \cup \cdots \cup J_{n,m}$ and let $q_n : J_n \longrightarrow X_n$ be defined by $q_{n,r}$ on $J_{n,r}$. Then clearly $J_n \subset J_{n-1}$ and $q_n : J_n \longrightarrow X_n$ satisfies the required conditions.

Now the commutative diagram $(*)$ defines, on passage to inverse limits, a weak order preserving continuous map

$$\widehat{q} : \widehat{J} = \bigcap_n J_n \longrightarrow K = \varprojlim_n X_n$$

which is surjective since for each n, J_n is compact and q_n is surjective. Each fiber of $\widehat{q}$ is an intersection of intervals of length $1/n$, going to 0. Thus $\widehat{q}$ is injective. Thus:

$$\widehat{q} : \widehat{J} \longrightarrow K$$

is an order preserving homeomorphism. The inverse of $\widehat{q}$ gives the promised order preserving embedding of K into I.

(B.7) **Extending continuous maps.** Let $K \subset \mathbb{R}$ be a real ordered profinite space, say $K \subset [a, b]$, with $a, b \in K$. Let $f : K \longrightarrow \mathbb{R}$ be a continuous function. Then f extends to a continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}$ (in many ways). Since $[a, b] - K$ is a (countable) union of open intervals $[c, d]$ with $c, d \in K$, $c < d$, we can define F on $[c, d]$ to be any continuous function (e.g. linear), taking the prescribed values $f(c)$ and $f(d)$ at c and d respectively. Similarly, on $(-\infty, a]$, F can be any continuous function sending a to $f(a)$, and analogously for $[b, \infty)$.

Note that F above can be chosen so that $F([a, b])$ is contained in the interval spanned by $F(K)$. In particular, if $f(K) \subset K$ then $F([a, b]) \subset [a, b]$.

(B.8) **Real dynamical systems.** We shall call an ordered dynamical system (K, f) **real** if there is an order preserving topological embedding

$$(K, f) \longrightarrow (\mathbb{R}, F)$$

for some continuous map F. When K is an ordered profinite space, we have seen (see (B.6) and (B.7)) that this condition is equivalent to K having a countable base for its topology.

(B.9) **Theorem** *Let (K, f) be a minimal dynamical system.*

(a) If K has an isolated point then K is finite.

(b) If K is profinite and has a countable base (e.g. if K is a subspace of $\mathbb{R}$ or S^1) then K is either finite or a Cantor set.

Proof. Let x be an isolated point of K. Since every orbit is dense, and x is isolated, x belongs to every orbit. Thus, x belongs to the orbit of $f(x)$, so $f^{\circ n}(x) = x$ for some n, and the orbit of x is finite, hence K is finite. This proves (a).

For (b), if x is not finite, then by (a), it has no isolated points. By assumption, it has a countable base, so K is a Cantor set.

■

5. Interval renormalization and periodic points.

(5.1) **Theorem.** *Let (I, f) be a dynamical system on a compact real interval I. Let $K \subset I$ be a minimal closed f-invariant subset. Let*

$$\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$$

be an interval n-renormalization, with fibers $K_r = \phi^{-1}(r)$. Then there exist points $x_r \in [K_r]_I$ such that $f(x_r) = x_{r+1}$ $(r \in \mathbb{Z}/n\mathbb{Z})$. Thus x_0 is a periodic point of period n with $f^r(x_0) = x_r \in [K_r]_I$.

Proof. Let $J_r = [K_r]_I$. Since K is compact and (K, f) is minimal it follows that $f(K_r) = K_{r+1}$, J_r is closed, and $f(J_r) \supset J_{r+1}$. Moreover since the K_r are disjoint K-intervals, the J_r are disjoint I-intervals. Now the Theorem follows from Lemma (5.2) below.

■

(5.2) **Lemma.** *Let J_r $(r \in \mathbb{Z}/n\mathbb{Z})$ be closed subintervals of I such that $f(J_r) \supset J_{r+1}$ for all r. There exists an x_0 such that $f^n(x_0) = x_0$ and $f^r(x_0) \in J_r$ for all $r \in \mathbb{Z}/n\mathbb{Z}$. If the J_r are pairwise disjoint then x_0 has period n (for f).*

Proof. If $[a, b]$ and $[c, d]$ are closed intervals in I such that $f([a, b]) \supset [c, d]$ then we can find $[a_0, b_0] \subset [a, b]$ such that $f([a_0, b_0]) = [c, d]$. In fact first choose $a' \in f^{-1}(c) \cap [a, b]$ and $b' \in f^{-1}(d) \cap [a, b]$. Suppose that $a' \leq b'$; the proof is similar in the other case. Let $a_0 = \sup(f^{-1}(c) \cap [a', b'])$ and $b_0 = \inf(f^{-1}(d) \cap [a', b'])$. Then it is easily checked that $f([a_0, b_0]) = [c, d]$, as required.

Using this we can choose closed intervals $J'_r \subset J_r$ such that $f(J'_r) = J'_{r+1}$ for $r = 0, 1, \ldots, n-2$, and $f(J'_{n-1}) = J_0$. We start by constructing J'_{n-1} as above, and then proceed inductively backward to successively construct $J'_{n-2}, J'_{n-3}, \ldots, J'_0$. Since $f^n(J'_0) = J_0 \supset J'_0$ it follows that f^n has a fixed point $x_0 \in J'_0$. Then $x_r := f^r(x_0) \in J'_r \subset J_r$ for all r. If the J_r are pairwise disjoint then the x_r are all distinct, so x_0 has period n for f.

■

(5.3) **Remark.** Figure 5 represents the graph of a counterexample to the analog of Theorem 5.1 for circle maps: the second iterate of this map, restricted to I_0 or I_1, has a restriction which is a Denjoy expansion over an irrational rotation (see (1.10)), and no further invariant set. In particular, $I_0 \cup I_1$ does not contain any periodic point of period 2.

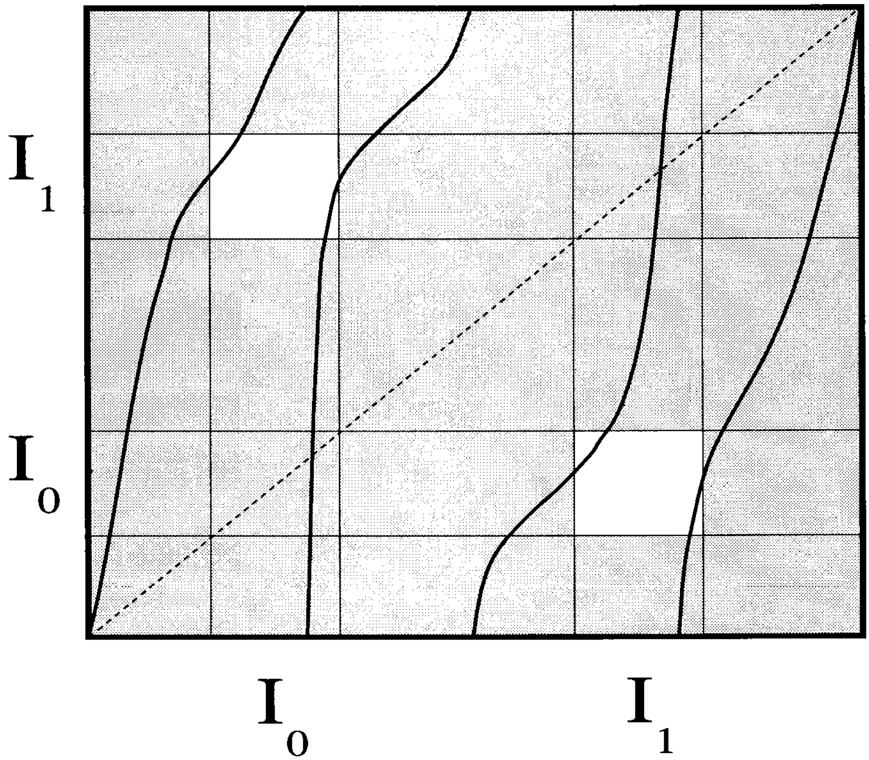

Figure 5. Graph of counterexample the analogue for Theorem 5.1 for the circle.

6. Self-similarity operators.

(6.0) Given a dynamical system (K, f) and a renormalization $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ with fiber $K_r = \phi^{-1}(r)$, we consider here the question of whether there are isomorphisms $\rho_r : (K, f) \longrightarrow (K_r, f^n \mid_{K_r})$. When this is the case, the ρ_r are called **self-similarity operators**. They are analyzed group theoretically in Chapter III, section 6 below, notably Theorem (III, (6.12)).

(6.1) **The context.** Consider a minimal ordered compact dynamical system (K, f) which is faithfully interval renormalizable, *i.e.*,

(1) $$\widehat{\phi} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1),$$

as in (2.8), is a topological isomorphism, where

(2) $$\mathbf{q} = \mathbf{q}(K, f) = (q_1, q_2, q_3 \ldots)$$

We can also write

$$(3)\qquad \begin{aligned} \widehat{\mathbb{Z}}_{\mathbf{q}} &= \widehat{\mathbb{Z}}_Q, \textit{ where} \\ Q &= Q(K,f) = \textstyle\prod_{n\geq 1} q_n, \end{aligned}$$

with Q understood as a supernatural number (cf. (1.6)).

Let $n \in IRen(K,f)$, and let

$$(4)\qquad \begin{aligned} &\phi = p \circ \widehat{\phi} : (K,f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1) \textit{ where} \\ &p : \widehat{\mathbb{Z}}_Q \longrightarrow \mathbb{Z}/n\mathbb{Z} \textit{ is the natural projection.} \end{aligned}$$

For $r \in \mathbb{Z}/n\mathbb{Z}$ we put

$$(5)\qquad K_r = \phi^{-1}(r) \textit{ and } f_r = f^n \,|_{K_r} : K_r \longrightarrow K_r.$$

If $\tilde{r} \in \phi^{-1}(r)$ then $\widehat{\phi}$ induces a topological isomorphism

$$(6)\qquad \widehat{\phi} : (K_r, f_r) \longrightarrow (\tilde{r} + n\widehat{\mathbb{Z}}_Q, +n) \cong (\widehat{\mathbb{Z}}_{Q/n}, +1).$$

It follows then from Corollary (4.2) that

$$(7)\qquad \begin{aligned} &(K_r, f_r) \textit{ is faithfully interval renormalizable,} \\ &\textit{and } Q(K_r, f_r) = Q/n, \textit{ for each } r \in \mathbb{Z}/n\mathbb{Z}. \end{aligned}$$

It follows further from Proposition (3.1) that

$$(8)\qquad \begin{aligned} &IRen(K,f) = IRen(\mathbb{Z}/n\mathbb{Z}, +1) \cup n \cdot J, \textit{ where} \\ &J = \bigcap_{r \in \mathbb{Z}/n\mathbb{Z}} IRen(K_r, f_r), \end{aligned}$$

and $\mathbb{Z}/n\mathbb{Z}$ is given the order induced from K via ϕ.

Assume now that K, and hence $\mathbf{q}$ *and Q, are infinite.* Then it follows from (8) that each (K_r, f_r) is also infinitely interval renormalizable, and so J in (8) is a cofinal subsequence of $IRen(K_r, f_r)$ (ordered by divisibility) for each r.

Write $n = q_1 q_2 \cdots q_t = \mathbf{q}^{[t]}$. Then it follows that each $\mathbf{q}(K_r, f_r)$ is obtained from $(q_{t+1}, q_{t+2}, q_{t+3}, \ldots)$ by replacing each q_{t+m} by a sequence $(q_{m_1}, \ldots, q_{m_{h_m}})$ (depending on r, with each $q_{m_j} \geq 2$ and $q_{t+m} = q_{m_1} \cdots q_{m_{h_m}}$. It is further necessary (by (8)) that the only initial products common to all of the $\mathbf{q}(K_r, f_r)$ are the products $q_{t+1} \cdot q_{t+2} \cdots q_{t+m}$ for $m \geq 0$.

(6.2) **Proposition.** *The following conditions are equivalent.*

(a) (K,f) is topologically isomorphic to (K_r, f_r) for some (hence every) $r \in \mathbb{Z}/n\mathbb{Z}$.

(b) The powers of n, n^e $(e \geq 0)$ belong to and are cofinal (with respect to divisibility) in $Ren(K,f) = Div(Q)$; i.e., $Q =$ "n^∞".

(c) $\widehat{\mathbb{Z}}_Q = \prod_{p|n} \widehat{\mathbb{Z}}_{p^\infty}$, *where p ranges over the prime divisors of n, and* $\widehat{\mathbb{Z}}_{p^\infty}$ *denotes the p-adic integers.*

Proof. We have topological isomorphism $(K, f) \cong (\widehat{\mathbb{Z}}_Q, +1)$ and $(K_r, f_r) \cong (\widehat{\mathbb{Z}}_{Q/n}, +1)$. Here $Q = \prod_p p^{e_p}$, $e_p = \sum_{i \geq 1} v_p(q_i)$, with v_p being the p-adic valuation, and $\widehat{\mathbb{Z}}_Q = \prod_p \mathbb{Z}/p^{e_p}\mathbb{Z}$ where $\mathbb{Z}/p^\infty\mathbb{Z} = \widehat{\mathbb{Z}}_{p^\infty}$ denotes the p-adic integers. We have $Q/n = \prod_p p^{e_p - v_p(n)}$ Now we see that all three conditions (a), (b), (c) are equivalent to the condition: $Q/n = Q$.

■

Next we consider whether (K, f) is IR-isomorphic, not just topologically isomorphic, to the (K_r, f_r).

(6.3) **Proposition.** *The following conditions are equivalent.*

(a) For each $r \in \mathbb{Z}/n\mathbb{Z}$, *there is an IR-isomorphism* $\rho_r : (K, f) \longrightarrow (K_r, f_r)$ *of ordered dynamical systems. (In fact, once* ρ_o *is chosen, we can then take* $\rho_r = f^r \circ \rho_0$ *for* $r = 0, 1, \ldots, n-1$.*)*

(b) IRen$(K, f) =$ *IRen*(K_r, f_r) *for each* $r \in \mathbb{Z}/n\mathbb{Z}$.

(c) (i) IRen(K_r, f_r) *is independent of* r.

(ii) If $n = \mathbf{q}^{[t]} = q_1 q_2 \cdots q_t$, *then* $\mathbf{q} = (q_1, q_2, q_3 \ldots)$ *is periodic of period* t:

$$q_{i+t} = q_i \qquad \textit{for all } i \geq 1.$$

Proof. Clearly (a) implies (b). Conversely, (b) implies that $Q(K, f) = Q(K_r, f_r)$, so that (K, f) and (K_r, f_r) are topologically isomorphic by (6.1)(1), (3), and (6). Hence, by (2.4), (b) implies (a).

In view of (5.1)(8) we see that (b) is equivalent to:

$$(1) \qquad \mathbf{q}(K, f) = (q_1, q_2, q_3 \ldots) = \mathbf{q}(K_r, f_r) = (q_{t+1}, q_{t+2}, q_{t+3}, \ldots)$$

for each r, and clearly (1) is equivalent to (c).

■

Remark. Under the conditions of (6.3), any topological isomorphism $\rho_r : (K, f) \longrightarrow (K_r, f_r)$ is an IR-isomorphism, by (2.4). In view of the $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$ model of these systems, ρ_r is determined by its value $\rho_r(x)$ at a single $x \in K$, and $\rho_r(x)$ may be any element of K_r.

(6.4) **Fixed points of self-similarity.** Fix a system $\rho = (\rho_r)$, $\rho_r : (K, f) \longrightarrow (K_r, f_r)$, of "self-similarity" IR-isomorphisms, as in (6.3). It follows that, for each $e \geq 0$, ρ_r maps the fibers of an interval n^e-renormalization of (K, f) to those of (K_r, f_r) (the latter being fibers of an interval n^{e+1}-renormalization of (K, f)).

Fixing $\rho = (\rho_r)_{r \in \mathbb{Z}/n\mathbb{Z}}$, consider dynamical systems (K, g) such that $g^{-1}(K_r) = K_{r-1}$ (*i.e.*, $\phi : (K, g) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ is an interval n-renormalization) and each ρ_r is also an IR-isomorphism $\rho_r : (K, g) \longrightarrow (K_r, g_r)$, where $g_r = g^n|_{K_r}$. We call such a g a **simultaneous fixed point** of $\rho = (\rho_r)$, and denote the set of them by

$$FP(\rho). \tag{1}$$

For example, consider the closure $\overline{\langle f \rangle}$ of the cyclic group $\langle f \rangle$ generated by f, which is isomorphic to $\widehat{\mathbb{Z}}_{\mathbf{q}}$. Its set of topological generators is

$$\begin{array}{rcl} TopGen\overline{\langle f \rangle} & = & \{f^u \mid u \in \widehat{\mathbb{Z}}_{\mathbf{q}}^{\times}\} \\ & = & \textit{the closure of } \{f^u \mid u \in \mathbb{Z}, \gcd(u, n) = 1\}. \end{array} \tag{2}$$

It is easily seen that

$$TopGen\overline{\langle f \rangle} \subset FP(\rho). \tag{3}$$

In Chapter III, Theorem (6.10)(c), it is shown by group theoretic methods that (3) is an equality iff $n = 2$, in which case $\mathbf{q} = (2, 2, 2, \ldots)$. For related results see also [GLOT, OT].

(6.5) **A classical example.** Consider a unimodal map f on $J = [-1, 1]$ with $f(-1) = -1 = f(1)$ and maximum $f(0) = M > 0$. Then f has a unique fixed point $x_0 \in (0, 1)$ and we define $x_- \in [-1, x_0)$ and $x_+ \in (x_0, 1]$ by $f(x_-) = x_0$ and $f(x_+) = x_-$. This is illustrated in Figure 6.

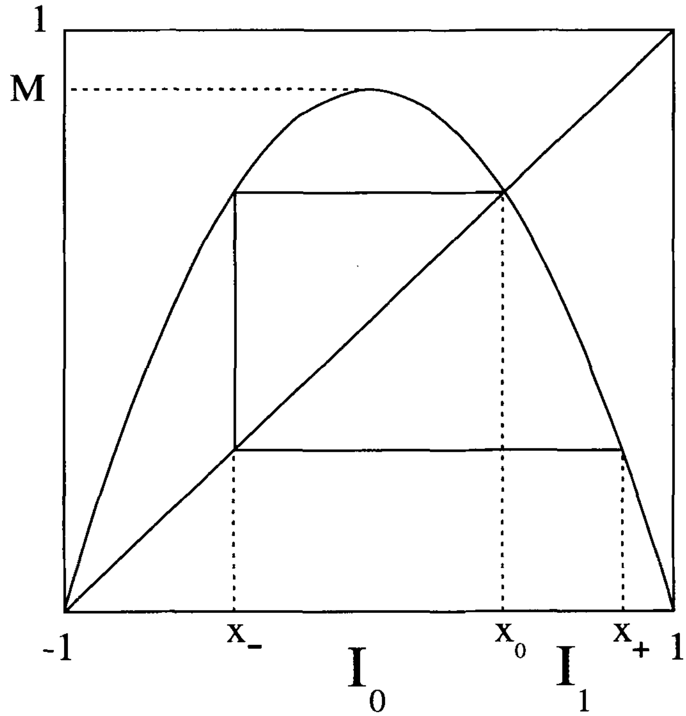

Figure 6. Unimodal map with unique fixed point.

Defining $I_0 = [x_0, x_+]$ and $I_1 = [x_-, x_0]$, it is easy to check that $f(I_0) = I_1$, $f^2|_{I_0}$ is (+)-unimodal, and $f^2|_{I_1}$ is (−)-unimodal. Assume that $M = f(0) \leq x_+$, so that $f(I_1) \subset I_0$.

Let K denote the postcritical orbit closure of f, *i.e.*, the closure of the orbit of $M = f(0)$,

$$O_f(M) = \{M, f(M), f^2(M), \ldots\}.$$

Put $K_r = K \cap I_r$ and $f_r = f^2|_{K_r}$, $r = 0, 1$. Then $\phi(K_r) = r \pmod 2$ defines an interval 2-renormalization of (K, f). In case $\mathbf{q}(K, f) = (2, 2, 2, \ldots)$ and (K, f) is faithfully renormalizable, there will exist IR-isomorphisms $\rho_r : (K, f) \longrightarrow (K_r, f_r)$ $(r = 0, 1)$. Then if (K, g) is a simultaneous fixed point of $\rho = (\rho_0, \rho_1)$, *i.e.*, $g \in FP(\rho)$, we must have $g = f^u$ for some 2-adic unit $u \in \widehat{\mathbb{Z}}_2^\times$.

Chapter II

Itinerary Calculus and Renormalization

0. Introduction.

Consider a unimodal map f on an interval $J = [a, b]$, with maximum at C, increasing on $L = [a, C)$, and decreasing on $R = (C, b]$. Each $x \in J$ then has an "address" $A(x) \in \{L, C, R\}$, and the orbit $f^*(x) = (a, f(x), f^2(x), \ldots)$ has the "itinerary" $Af^*(x) = (A(x), Af(x), Af^2(x), \ldots)$.

If $K \subset J$ is a minimal closed f-invariant subset then K is the closure of an orbit $f^*(x)$, where we can take x to be the maximum element of K, for example. It is known then that the itinerary $Af^*(x)$ encodes much of the combinatorial dynamics of (K, f). The aim of this chapter is to show how one can read the interval renormalizations $Ren(K, f)$ from $Af^*(x)$. Among the applications, we show (Corollary (9.9)) that all possibilities for $IRen(K, f)$ occur already when f is a quadratic map $f(x) = 1 - tx^2$ on $J = [-1, 1]$ $(0 < t \leq 2)$ and K is the critical orbit closure $\overline{f^*(1)}$.

Much of the material in this Chapter is well known (cf. [My, MSS, MilTh, DGP1, DGP2, CEc]). However, several results are presented with full proofs for the first time.

For describing itineraries symbolically, we think of them as infinite words $\alpha = \alpha_1\alpha_2\alpha_3 \ldots$ in the alphabet $\{L, C, R\}$, which we agree to truncate at α_n if, and when, $\alpha_n = C$. In the latter case $\alpha = \alpha' C$ where $\alpha' = \alpha_1 \ldots \alpha_{n-1}$ belongs to the free monoid G_0 with basis $\{L, R\}$ and we put $|\alpha| = n$, the length of α. Otherwise α belongs to the set $\hat{G}_0$ of infinite words in $\{L, R\}$ and we put $|\alpha| = \infty$. Thus, itineraries define a map

$$\mathbf{Af^*} : J \longrightarrow \hat{G}_0 \bigcup G_0 C. \tag{1}$$

In (1.6) and (1.7) we define a linear order on $\hat{G}_0 \cup G_0 C$ so that $\mathbf{Af^*}$ is weak order preserving.

The shift operator σ on $\hat{G}_0 \cup G_0C$ is defined on $\alpha = \alpha_1\alpha_2\alpha_3\dots$ by $\alpha = \alpha_1\sigma(\alpha)$. We put

$$O(\alpha) = \{\sigma^i(\alpha) | 0 \leq i < |\alpha|\} \tag{2}$$

and call α "maximal" if α is the maximal element of $O(\alpha)$. Define

$$\sigma_\alpha : O(\alpha) \longrightarrow O(\alpha) \tag{3}$$

by

$$\sigma_\alpha(\sigma^i(\alpha)) = \sigma^{i+1}(\alpha), \tag{4}$$

unless $i = |\alpha| - 1 < \infty$, in which case $\sigma_\alpha(\sigma^i(\alpha)) = \alpha(= \sigma^0(\alpha))$. Thus, when $|\alpha| < \infty$, σ_α is a transitive (cyclic) permutation of $O(\alpha)$. In all cases, giving $O(\alpha)$ the order topology, $(O(\alpha), \sigma_\alpha)$ is an ordered dynamical system, and $O(\alpha)$ is the σ_α-orbit of α. In particular we have the interval renormalizations

$$IRen(\alpha) := IRen(O(\alpha), \sigma_\alpha). \tag{5}$$

We also put

$$\mathbf{q}(\alpha) := \mathbf{q}(O(\alpha), \sigma_\alpha). \tag{6}$$

To construct α with prescribed $IRen(\alpha)$, we make use of a $\star$-product,

$$\beta \star \gamma \in \hat{G}_0 \cup G_0C \quad (\beta \in G_0, \gamma \in \hat{G}_0 \cup G_0C), \tag{7}$$

with $|\beta \star \gamma| = (|\beta| + 1)|\gamma|$ (cf. (4.1)). The basic result (Theorem (7.1)) asserts that, if βC is maximal and non-quadratic (in the sense of (2.6)) then

$$IRen(\beta \star \gamma) = IRen(\beta C) \cup n \cdot IRen(\gamma) \tag{8}$$

Moreover $\beta\star\gamma$ is maximal if γ is maximal. (The proof of this result is regrettably technical, and requires the analysis of Sections 2 – 5.)

For an integer $q \geq 2$, put $\alpha(q) = \alpha'(q)C$ with $\alpha'(q) = RL^{q-2}$; then $IRen(\alpha(q)) = \{1, q\}$ (cf. (8.1)). Let $\mathbf{q} = (q_1, q_2, q_3, \dots)$ be a sequence of integers $q_i \geq 2$, and define $\alpha(q_1, \dots, q_n) = \alpha'(q_1, \dots, q_n)C$ inductively, by $\alpha(q_1, \dots, q_n) = \alpha'(q_1) \star \alpha(q_2, \dots, q_n)$. It follows from (8) that

$$\mathbf{q}(\alpha(q_1, \dots, q_n)) = (q_1, \dots, q_n). \tag{9}$$

Further, there is a well defined limit

$$\alpha(\mathbf{q}) = \lim_{n \longrightarrow \infty} \alpha(q_1, \dots, q_n)$$

and it follows from (9) that

$$\mathbf{q}(\alpha(\mathbf{q})) = \mathbf{q}. \tag{10}$$

We thus obtain maximal elements α with arbitrarily prescribed $IRen(\alpha)$ (cf. (8.2)).

To relate these results to interval dynamics, consider a unimodal map f on $J = [-1, 1]$ with maximum $f(0) = 1$. Then we have the "kneading sequence"

$$(11) \qquad K(f) = \mathbf{Af}^{\star}(1) \quad \in \quad \hat{G}_0 \cup G_0 C.$$

Define

$$(12) \qquad \kappa(f) = \begin{cases} K(f) & \text{if } K(f) \in \hat{G}_0 \\ \kappa' \star L^{\infty} & \text{if } K(f) = \kappa' C \in G_0 C. \end{cases}$$

Suppose that $\alpha \in \hat{G}_0$ satisfies

$$\mathbf{Af}^{\star}(-1) \leq \alpha \text{ and } \sigma^i(\alpha) < \kappa(f) \quad \text{for all } i \geq 0.$$

Then it follows from [CEc], Theorem II.3.8, that

$$(13) \qquad \alpha = \mathbf{Af}^{\star}(x) \quad \text{for some } x \in J.$$

Let $x \in J$ have f-orbit closure $K = \overline{f^*(x)}$, and put $\alpha = \mathbf{Af}^{\star}(x) \in \hat{G}_0 \cup G_0 C$. Assume that $IRen(\alpha)$ is infinite. Then (cf. (9.4)) K is a minimal f-invariant Cantor set, and

$$(14) \qquad IRen(K, f) = IRen(\alpha).$$

Define $f_t(x) = 1 - tx^2$ $(0 < t \leq 2)$. Given any maximal $\alpha \in \hat{G}_0 \cup G_0 C$, $\alpha \neq L^{\infty}$, it follows from (9.8) that $\alpha = K(f_t)$ for some t, $0 < t \leq 2$. Taking $\alpha = \alpha(\mathbf{q})$ as in (10) above it follows that all possible interval renormalization sets can be realized on the critical orbit closures $\overline{f_t^*(1)}$ for some quadratic map f_t.

1. Preliminaries.

(1.0) **Unimodal maps.** Consider the interval

$$J = [-1, 1] = L \coprod C \coprod R$$

where

$$L = [-1, 0), \qquad C = \{0\}, \qquad R = (0, 1].$$

Define the **address** function

$$A : J \longrightarrow \{L, C, R\}$$

by $x \in A(x)$.

The **unimodal maps** we consider are continuous functions $f : J \longrightarrow J$ such that

(a) f is increasing on L,
(b) f is decreasing on R,

and

(c) $f(0) > 0$.

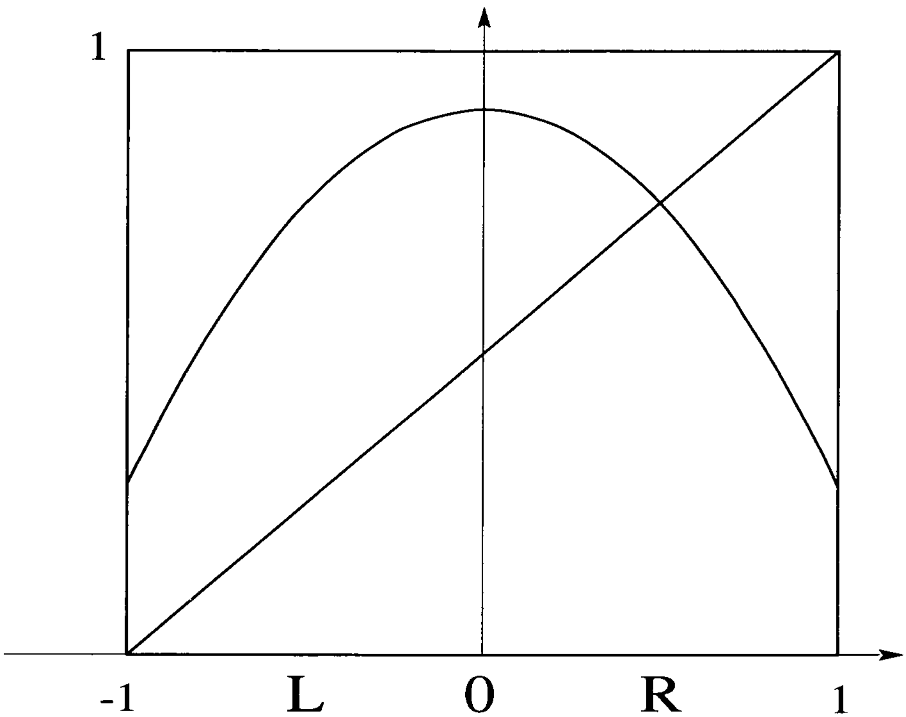

Figure 7. A unimodal map.

The point $C = \{0\}$ is often called the **turning point** of f. For $x \in J$ we denote its **f-orbit** as a sequence

$$f^*(x) = (x, f(x), f^2(x), \ldots).$$

The corresponding address list of the orbit,

$$Af^*(x) = (A(x), Af(x), Af^2(x), \ldots)$$

is called the **itinerary** of x. The itinerary of $M = f(0)$ $(= f(C))$,

$$K(f) = Af^*(M)$$

is called the **kneading sequence** of f. Much of the dynamics of (J, f) is encoded in $K(f)$.

Let $x \in J$ have f-orbit closure K. We would like to determine all interval renormalizations

$$\phi_n : (K, f|_K) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1).$$

Such a ϕ_n, if it exists, is determined by $\phi_n(x)$. Our aim is to determine from $Af^*(x)$, those n for which ϕ_n exists. We shall do this in the case in which x is the maximal element in K.

(1.1) **The monoid** $G = G_0 \coprod G_0C$ is presented by

(1) *generators*: $\quad L, C, R,$

subject to,

(2) *relations*: $\quad CX = C \text{ for } X = L, C, R.$

It follows from (2) that

(2′) $\quad CX = C \text{ for all } X \in G.$

Note that G contains

(3) $\quad G_0 := \text{ the free monoid based on } \{L, R\}.$

By definition, each $\alpha \in G_0$ has a unique expression $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ with each $\alpha_i = L$ or R. We define $|\alpha| = n$. When $n = 0$, $\alpha = 1$, the neutral element of G. By (2), for $\alpha, \beta \in G_0$ we have $\alpha C = \beta C$ if and only if $\alpha = \beta$. Thus

(4) $$G = G_0 \coprod G_0C$$

and each $\alpha \in G$ has a unique expression $\alpha = \alpha_1\alpha_2\cdots\alpha_n$, with

(5) $$\begin{aligned} &\alpha_i \in \{L, C, R\} \text{ and} \\ &\alpha_i \neq C \text{ for } i < n := |\alpha|. \end{aligned}$$

We also write

(6) $$|\alpha|_0 = \begin{cases} |\alpha| & \text{if } \alpha \in G_0 \\ |\alpha| - 1 & \text{if } \alpha \in G_0C. \end{cases}$$

(1.2) R-**parity** refers to the map

$$\rho : G \longrightarrow \{\pm 1\}$$

defined by

$$\rho(L) = \rho(C) = 1, \quad \rho(R) = -1,$$

and

$$\rho(\alpha\beta) = \rho(\alpha)\rho(\beta) \text{ for } \alpha \in G_0, \quad \beta \in G.$$

Thus, $\rho(\alpha)$ measures the parity of the number of R's in α, and $\rho(\alpha C) = \rho(\alpha)$.

(1.3) **The G-set $\hat{G} = G_0 \coprod \hat{G}_0 \coprod G_0 C$.** Let $\hat{G}_0$ denote the set of infinite words (sequences) of the form:

(1) $$\beta = \beta_1\beta_2 \cdots \beta_{n-1}\beta_n \cdots \text{ with each } \beta_n \in \{L, R\}.$$

We put

(2) $$|\beta| = \infty \text{ for } \beta \in \hat{G}_0,$$

and define

(3) $$\hat{G} = G \bigcup \hat{G}_0 = G_0 \coprod \hat{G}_0 \coprod G_0 C.$$

The monoid G acts on the left on $\hat{G}$, using the obvious left multiplication by G_0, and with the rule

(4) $$CX = C \text{ for all } X \in \hat{G}.$$

Each $\beta \in \hat{G}$ has a **normal form**

(5) $$\beta = \beta_1\beta_2\beta_3 \cdots \quad \text{with } \beta_i \in \{L, C, R\} \text{ and } \beta_i \neq C \text{ for } i < |\beta|.$$

We define **truncations** of β as follows. For $i \geq 0$,

(6) $$\beta_{\leq i} = \beta_1 \cdots \beta_i \qquad (= 1 \text{ if } i = 0)$$

and

(7) $$\beta_{i<} = \beta_{i+1}\beta_{i+2} \cdots \qquad (= \beta \text{ if } i = 0 \text{ and } = 1 \text{ if } i \geq |\beta|).$$

Thus, for any nonnegative i, $\beta = \beta_{\leq i}\beta_{i<}$.

We define **divisibility** in $\hat{G}$ as follows. For $\alpha, \beta \in \hat{G}$, define $\alpha \mid \beta$ (read *α divides β*) by

(8) $$\alpha \mid \beta \Longleftrightarrow \alpha \in G \text{ and } \beta \in \alpha\hat{G}.$$

That is, $\alpha \mid \beta$ if and only if $\alpha = \beta_{\leq n}$ for some finite $n \leq |\beta|$.

(1.4) **The involution $\beta \mapsto \bar{\beta}$** on $\hat{G}$ is defined by

(1) $$\bar{L} = R, \quad \bar{C} = C, \quad \bar{R} = L,$$

and for $\beta = \beta_1\beta_2\beta_3 \cdots \in \hat{G}$, $\beta_i \in \{L, C, R\}$,

(2) $$\bar{\beta} = \bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 \cdots .$$

Thus,

(3) $$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta} \textit{ for } \alpha \in G, \beta \in \hat{G}$$

and

(4) $$\bar{\bar{\beta}} = \beta, \qquad |\bar{\beta}| = |\beta|, \quad |\bar{\beta}|_0 = |\beta|_0 .$$

For $\alpha \in G$ it is easily checked that

(5) $$\rho(\bar{\alpha}) = \rho(\alpha) \cdot (-1)^{|\alpha|_0} .$$

(Cf. (1.1) (6).)

(1.5) **The exponential notation** β^α is defined for $\alpha \in G$ and $\beta \in \hat{G}$, and is defined by

(1) $$\beta^\alpha = \begin{cases} \beta & \textit{if } \rho(\alpha) = 1 \\ \bar{\beta} & \textit{if } \rho(\alpha) = -1. \end{cases}$$

Exponentiation has the properties:

(2) $$\begin{cases} (\gamma\beta)^\alpha & = \gamma^\alpha\beta^\alpha, & \textit{for } \gamma \in G \textit{ and} \\ \beta^{\alpha\gamma} & = (\beta^\alpha)^\gamma = \beta^{\gamma\alpha} & \textit{if } \alpha, \gamma \in G_0. \end{cases}$$

From (1) and (1.4)(5) we obtain:

(3) $$\textit{For } \alpha, \gamma \in G, \quad \rho(\gamma^\alpha) = \rho(\gamma) \cdot \rho(\alpha)^{|\gamma|_0} .$$

In particular,

$(3)_0$ $$\rho(\gamma^\alpha) = \rho(\gamma)\rho(\alpha) \textit{ for } \gamma \in \{L, R\}.$$

Note finally that

(4) $$X^X = L \textit{ for } X \in \{L, R\}.$$

(1.6) **The order relations $<$ and $<^\alpha$.** We adopt the following notational conventions. For $\alpha \in G$ and for β, γ in any ordered set, define

(1) $$\beta <^\alpha \gamma \Longleftrightarrow \begin{cases} \beta < \gamma & \textit{if } \rho(\alpha) = 1 \\ \gamma < \beta & \textit{if } \rho(\alpha) = -1. \end{cases}$$

Now we define $<$ on $\hat{G}$ as follows. First,

(2) $$L < C < R.$$

Note then that

(3) $$X <^{\alpha} Y \Longleftrightarrow X^{\alpha} < Y^{\alpha}$$

for $X, Y \in \{L, C, R\}$ and $\alpha \in G$.

Now let $\beta = \beta_1\beta_2\beta_3\cdots$ and $\gamma = \gamma_1\gamma_2\gamma_3\cdots$ be distinct elements of $\hat{G}$ neither of which divides the other (cf. (1.3)(8)). Then, for some index $m \geq 1$, we have $\beta_m \neq \gamma_m$, whereas $\beta_i = \gamma_i$ for $i < m$. Put $\alpha = \beta_1\cdots\beta_{m-1} = \gamma_1\cdots\gamma_{m-1} \in G_0$. Then we define

(4) $$\beta < \gamma \iff \beta_m <^{\alpha} \gamma_m \iff \beta_m^{\alpha} < \gamma_m^{\alpha}.$$

Note that:

(5) $$(\beta < \gamma \textit{ and } \gamma < \delta) \Longrightarrow \beta < \delta.$$

We summarize the most important properties of $<$:

(2) $$L < C < R,$$

(6) $$\beta, \gamma \in \hat{G} \textit{ are related by } < \textit{ unless } \beta \mid \gamma \textit{ or } \gamma \mid \beta \quad (\textit{cf.}(1.3)(8)),$$

and for $\beta, \gamma \in \hat{G}$ and $\alpha \in G_0$,

(7) $$\beta < \gamma \textit{ if and only if } \alpha\beta <^{\alpha} \alpha\gamma.$$

In particular,

(8) $$\hat{G}_0 \coprod G_0 \cdot C \textit{ is linearly ordered by } < .$$

If $\alpha = \alpha' C$ with $\alpha' \in G_0$ then we define α^- and α^+ by

(9) $$\alpha^- := \alpha' L^{\alpha} \; < \; \alpha \; < \; \alpha^+ := \alpha' R^{\alpha}.$$

For purposes of generalization to the multimodal case (cf. Appendix C below) it is convenient to give the following alternative definition of $\alpha^{\pm}$. First define

$$\begin{array}{lll} C^{(1)} & := & R, \\ C^{(-1)} & := & L, \\ C^{(\alpha)} & := & C^{(\rho(\alpha))} \;\; = \;\; R^{\alpha}, \textit{ and} \\ C^{(-\alpha)} & := & C^{(-\rho(\alpha))} \;\; = \;\; L^{\alpha}. \end{array}$$

Then we can write, for $\alpha = \alpha' C$ as above

(10) $$\alpha^+ = \alpha' C^{(\alpha)}, \qquad \alpha^- = \alpha' C^{(-\alpha)}.$$

This formulation adapts better to the multimodal case, in which there are several critical points instead of one (cf. Appendix C.).

(1.7) **The Real meaning of the ordering.** As in (1.0), let

$$f : J \longrightarrow J, \quad J = [-1, 1],$$

be a unimodal map. We have the **itinerary map**

$$\mathbf{Af}^\star : J \longrightarrow \hat{G} \tag{1}$$

defined by

$$\mathbf{Af}^\star(x) = A(x)A(f(x))A(f^2(x)) \cdots . \tag{2}$$

Note that if $f^n(x) = 0$ and $f^i(x) \neq 0$ for $i < n$ then

$$\mathbf{Af}^\star(x) = A(x) \cdots A(f^{n-1}(x))C \in G_0 \cdot C.$$

If $f^n(x) \neq 0$ for all n then $\mathbf{Af}^\star(x) \in \hat{G}_0$. Thus

$$\mathbf{Af}^\star(J) \subset \hat{G}_0 \coprod G_0 \cdot C. \tag{3}$$

The order structure on $\hat{G} \coprod G_0 C$ has been constructed precisely so that

$$\mathbf{Af}^\star : J \longrightarrow \hat{G}_0 \coprod G_0 C \quad \textit{is weak order preserving.} \tag{4}$$

This can be seen as follows. (cf. [MilTh], Lemma 3.1 or [CEc], II.1.2 and II.1.3.) Suppose that $x < y$ in J, and

$$\begin{array}{rcccll} \alpha & := & \mathbf{Af}^\star(x) & = & \alpha_0\alpha_1\alpha_2 \cdots & (\alpha_i = Af^i(x)) \\ \neq \beta & := & \mathbf{Af}^\star(y) & = & \beta_0\beta_1\beta_2 \cdots & (\beta_i = Af^i(y)). \end{array}$$

Say $\alpha_0 \cdots \alpha_{n-1} = \gamma = \beta_0 \cdots \beta_{n-1}$, and $\alpha_n \neq \beta_n$. We must show that $\alpha_n <^\gamma \beta_n$. For $0 \leq i < n$, $f^i(x)$ and $f^i(y)$ lie in an interval $\alpha_i = \beta_i$ on which f is monotone. Let r denote the number of such intervals on which f is decreasing, *i.e.*, order-reversing, *i.e.*, for which $\alpha_i = R$. Then $f^n(x) < f^n(y)$ if r is even and $f^n(y) < f^n(x)$ and r is odd. Since $\rho(\gamma) = (-1)^r$, these two cases are summarized by the condition $\alpha_n <^\gamma \beta_n$.

On the other hand, we can define

$$v : \hat{G} \longrightarrow J \tag{5}$$

as follows: Put

$$\varepsilon(L) = -1, \qquad \varepsilon(C) = 0, \qquad \varepsilon(R) = 1, \tag{6}$$

and for $\alpha = \alpha_1\alpha_2\alpha_3 \cdots \in \hat{G}$,

$$v(\alpha) = \sum_{1 \le n \le |\alpha|} \frac{\epsilon(\alpha_n^{\alpha_{<n}})}{2^n}. \tag{7}$$

The following properties are readily verified.

$$\begin{cases} v(\alpha) = 0 & \iff \alpha = 1 \textit{ or } C \textit{ or } XRL^\infty, \quad X \in \{L, R\}. \\ v(\alpha) = -1 & \iff \alpha = L^\infty. \\ v(\alpha) = 1 & \iff \alpha = RL^\infty. \end{cases} \tag{8}$$

(9) *If $\alpha \in G_0$ and $|\alpha| = n > 0$ then $v(\alpha)$ is rational with denominator 2^n, so $\frac{1}{2^n} \le |v(\alpha)| < 1$.*

For $\delta \in G_0$ and $\beta \in \hat{G}$, with $n = |\delta|$, we have

$$v(\delta\beta) = v(\delta\beta_1\beta_2\beta_3\cdots) = v(\delta) + \frac{1}{2^n}\left(\sum_{1 \le i \le |\beta|} \frac{\epsilon(\beta_i^{\delta\beta_{<i}})}{2^i}\right).$$

For $X \in \{L, C, R\}$, we have $\epsilon(X^\delta) = \rho(\delta)\epsilon(X)$. Thus:

$$\text{For } \delta \in G_0 \quad \text{and} \quad \beta \in \hat{G}, \quad v(\delta\beta) = v(\delta) + \frac{\rho(\delta)v(\beta)}{2^{|\delta|}}. \tag{10}$$

Next observe, using (8), that since $\rho(\delta) = \pm 1$,

$$\begin{aligned} \rho(\delta)v(\beta) = 1 \quad & \iff \quad v(\beta) = \rho(\delta) \\ & \iff \quad \beta = \begin{cases} RL^\infty & \rho(\delta) = 1 \\ L^\infty & \rho(\delta) = -1 \end{cases} \\ & \iff \quad \beta = R^\delta L^\infty. \end{aligned}$$

So that in summary,

$$\rho(\delta)v(\beta) = 1 \iff \beta = R^\delta L^\infty. \tag{11}$$

We now show that v is weak order preserving.

(12) **Claim.** *Let $\alpha, \beta \in \hat{G}$. If $\alpha < \beta$ then $v(\alpha) \le v(\beta)$, with strict inequality except in the following cases: For some $\delta \in G_0$,*

$$\begin{aligned} \alpha &= \delta L^\delta RL^\infty, & \beta &= \delta C \\ \alpha &= \delta L^\delta RL^\infty, & \beta &= \delta R^\delta RL^\infty \\ \alpha &= \delta C, & \beta &= \delta R^\delta RL^\infty. \end{aligned}$$

Remark. These exceptional cases will not arise in the setting that concerns us here.

Proof. Since $\alpha < \beta$ we can write

$$\alpha = \delta X \alpha', \qquad \beta = \delta Y \beta',$$

where $\delta \in G_0$, $X, Y \in \{L, C, R\}$, $X^\delta < Y^\delta$, and $\alpha', \beta' \in \hat{G}$.

Putting $d = |\delta| + 1$, we have, in view of (10),

$$v(\alpha) = v(\delta) + \frac{\varepsilon(X^\delta)}{2^d} + \frac{\rho(\delta X) v(\alpha')}{2^d}$$

and

$$v(\beta) = v(\delta) + \frac{\varepsilon(Y^\delta)}{2^d} + \frac{\rho(\delta Y) v(\beta')}{2^d}.$$

Thus,

$$2^d \Big(v(\beta) - v(\alpha) \Big) = E + D$$

where

$$E = \varepsilon(Y^d) - \varepsilon(X^d), \ \textit{and}$$
$$D = \rho(\delta Y) v(\beta') - \rho(\delta X) v(\alpha').$$

We want to show that $E + D \geq 0$, and to determine when this inequality is strict. Since $X^\delta < Y^\delta$, there are three cases to consider:

$$\begin{array}{ll} (a) & X^\delta = L, \ \ Y^\delta = C, \ \textit{hence} \\ & \alpha = \delta L^\delta \alpha', \ \ \beta = \delta C. \end{array}$$

$$\begin{array}{ll} (b) & X^\delta = L, \ \ Y^\delta = R, \ \textit{hence} \\ & \alpha = \delta L^\delta \alpha', \ \ \beta = \delta R^\delta \beta'. \end{array}$$

$$\begin{array}{ll} (a) & X^\delta = C, \ \ Y^\delta = R, \ \textit{hence} \\ & \alpha = \delta C, \ \ \beta = \delta R^\delta \beta'. \end{array}$$

Case (a). We have $E = \epsilon(C) - \epsilon(L) = 0 - (-1) = 1$, $v(\beta') = v(1) = 0$, $\rho(\delta X) = \rho(\delta L^\delta) = 1$, and so

$$D = \rho(\delta Y) v(\beta') - \rho(\delta X) v(\alpha') = -v(\alpha').$$

Thus $E + D = 1 - v(\alpha') \geq 0$,with equality precisely when $v(\alpha') = 1$, *i.e.*, $\alpha' = RL^\infty$ (cf. (8)). Hence $v(\alpha) \leq v(\beta)$, with equality precisely when

$$\alpha = \delta L^\delta R L^\infty, \qquad \beta = \delta C.$$

Case (b). We have $E = \epsilon(R) - \epsilon(L) = 1 - (-1) = 2$, $\rho(\delta X) = \rho(\delta L^\delta) = 1$, $\rho(\delta Y) = \rho(\delta R^\delta) = -1$, and so

$$D = (-1) v(\beta') - 1 \cdot v(\alpha') = -(v(\beta') + v(\alpha')).$$

Thus $E + D = (1 - v(\alpha')) + (1 - v(\beta')) \geq 0$, with equality iff $v(\alpha') = 1 = v(\beta')$ iff $\alpha' = RL^\infty = \beta'$. Hence $v(\alpha) \leq v(\beta)$, with equality precisely when

$$\alpha = \delta L^\delta R L^\infty, \qquad \beta = \delta R^\delta R L^\infty.$$

Case (c). We have $E = \epsilon(R) - \epsilon(C) = 1 - 0 = 1$, $v(\alpha') = v(1) = 0$, $\rho(\delta Y) = \rho(\delta R^\delta) = -1$, and so

$$D = -v(\beta').$$

Thus $E + D = 1 - v(\beta') \geq 0$, with equality iff $v(\beta') = 1$ iff $\beta' = RL^\infty$. Hence $v(\alpha) \leq v(\beta)$, with equality precisely when

$$\alpha = \delta C, \qquad \beta = \delta R^\delta RL^\infty.$$

■

Appendix C. The multimodal case.

This refers to a continuous map $f : J \longrightarrow J$, where $J = [a, b]$ is a closed interval, f has a finite number of turning points, C_i,

$$(1) \qquad a < C_1 < C_2 < \cdots C_{l-1} < b$$

and f is strictly monotone (with alternating directions) on the intervals J_i defined by

$$(2) \qquad J_1 = [a, C_1), J_2 = (C_1, C_2), \ldots, J_{l-1} = (C_{l-2}, C_{l-1}), J_l = (C_{l-1}, b],$$

We indicate here a notational scheme for this setting which generalizes what we have used above in the unimodal case ($l = 2$). (We shall not deal with the multi-modal case outside of this appendix.)

Let G denote the monoid presented by *generators*

$$(3) \qquad J_1, \ldots J_l, C_1, \ldots C_{l-1}$$

subject to *relations*

$$(4) \qquad C_i X = C_i \textit{ for } i = 1, \ldots l-1 \textit{ and all } X.$$

Note that G contains

$$(5) \qquad G_0 = \textit{ the free monoid on } J_1, \ldots J_l,$$

and thus has the decomposition

$$(6) \qquad G = G_0 \coprod G_0 C_1 \coprod \cdots \coprod G_0 C_{l-1}.$$

For $\alpha \in G_0$ we put

$$(7) \qquad \begin{cases} |\alpha| & = \textit{length of } \alpha, \\ |\alpha C_i| & = |\alpha| + 1, \textit{ and} \\ |\alpha|_0 & = |\alpha C_i|_0 = |\alpha|. \end{cases}$$

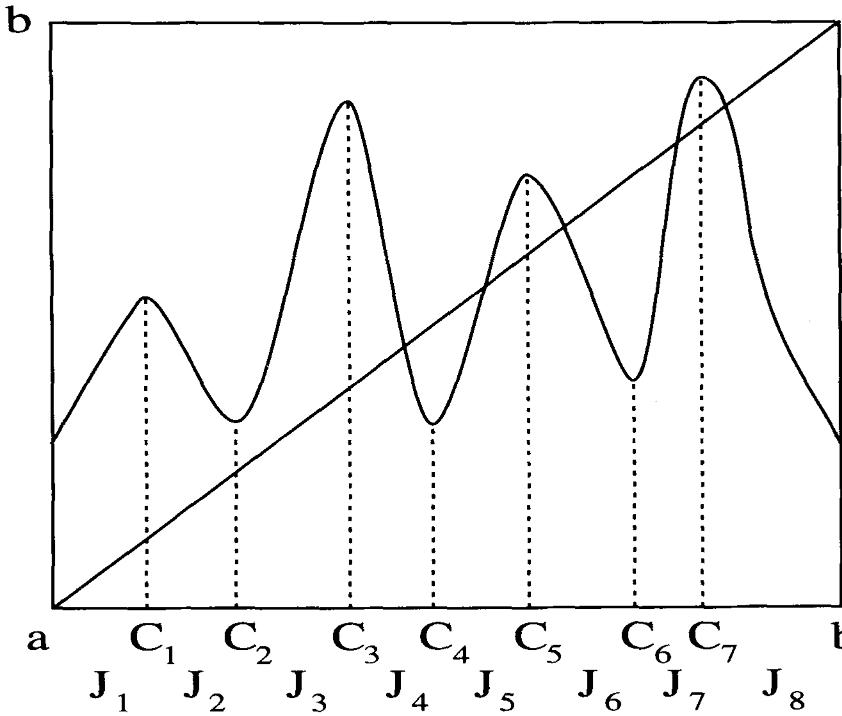

Figure 8. A function with 7 turning points.

Define the *parity* map

$$\rho : G \longrightarrow \{1, -1\}$$

by

$$(8) \qquad \begin{cases} \rho(C_i) &= 1 \qquad (i = 1, \ldots, l-1) \\ \rho(J_i) &= \begin{cases} 1 & \textit{if } f \textit{ is increasing on } J_i, \\ -1 & \textit{if } f \textit{ is decreasing on } J_i. \end{cases} \quad \textit{and} \\ \rho(\alpha\beta) &= \rho(\alpha)\rho(\beta) \qquad \text{for } \alpha \in G_0, \beta \in G. \end{cases}$$

Thus $\rho(\alpha)$ measures the parity of the number of $J_i \in \alpha$ on which f is decreasing. Note that $\rho(J_i)\rho(J_{i+1}) = -1$ for $i = 1, \ldots, l-1$.

Define an *involution* $\alpha \mapsto \bar{\alpha}$ on G by

$$(9) \qquad \begin{cases} \bar{C}_i = C_{(l-1)-i} & (i = 1, \ldots l - i) \\ \bar{J}_i = J_{l-i} & (i = 1, \ldots l) \end{cases}$$

and $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.

For $\alpha \in G$ we have

$$(10) \qquad \begin{cases} \rho(\bar{\alpha}) = \rho(\alpha) & \textit{if } l \textit{ is odd} \\ \rho(\bar{\alpha}) = \rho(\alpha)(-1)^{|\alpha|_0} & \textit{if } l \textit{ is even.} \end{cases}$$

Let $\hat{G}_0$ denote the set of infinite words $\alpha = \alpha_1\alpha_2\alpha_3\cdots$ with each $\alpha_i \in \{J_1,\ldots J_l\}$. The involution extends to

$$\hat{G} = G \cup \hat{G}_0 \tag{11}$$

by $\bar{\alpha} = \bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\cdots$ for $\alpha \in \hat{G}_0$.

For $\beta \in \hat{G}$ and $\alpha \in G$ we define the *exponential*

$$\beta^\alpha = \begin{cases} \beta & \text{if } \rho(\alpha) = 1 \\ \bar{\beta} & \text{if } \rho(\alpha) = -1. \end{cases} \tag{12}$$

We define an *order* on $\hat{G}$ with the following properties.

$$J_1 < C_1 < J_2 < C_2 < \cdots < J_{l-1} < C_{l-1} < J_l. \tag{13}$$

Let $\alpha, \beta, \gamma \in \hat{G}$. Assume that $\alpha < \beta$. Then

$$\begin{cases} \alpha \in G & \Rightarrow & \alpha\gamma < \beta \\ \beta \in G & \Rightarrow & \alpha < \beta\gamma \\ \gamma \in G & \Rightarrow & \gamma\alpha <^\gamma \gamma\beta, \end{cases} \tag{14}$$

where

$$<^\gamma = \begin{cases} < & \text{if } \rho(\gamma) = 1 \\ > & \text{if } \rho(\gamma) = -1. \end{cases} \tag{15}$$

Explicitly, if $\alpha \neq \beta$ then they are comparable unless one is an initial subword of the other. If this is not the case then we can write

$$\begin{cases} \alpha & = & \gamma X \alpha' & \text{and} \\ \beta & = & \gamma Y \beta' \end{cases} \tag{16}$$

where $\gamma \in G_0, \alpha', \beta' \in \hat{G}$ and $X, Y \in \{J_1, \ldots, J_l, C_1, \ldots C_{l-1}\}$ with $X \neq Y$. Then we have

$$\alpha < \beta \iff X <^\gamma Y, \tag{17}$$

and the condition on the right is determined by (13).

Define, for $i = 1, \ldots, l-1$ and $\alpha \in G$

$$\begin{cases} C_i^{(1)} & = & J_{i+1} \\ C_i^{(-1)} & = & J_i \\ C_i^{(\alpha)} & = & C_i^{(\rho(\alpha))} \\ C_i^{(-\alpha)} & = & C_i^{(-\rho(\alpha))}. \end{cases} \tag{18}$$

For $\alpha = \alpha' C_i$, $\alpha' \in G_0$, we put

$$\begin{cases} \alpha^+ & = & \alpha' C_i^{(\alpha)} \\ \alpha^- & = & \alpha' C_i^{(-\alpha)}. \end{cases} \tag{19}$$

Then it is easily checked that

$$(20) \qquad \alpha^- < \alpha < \alpha^+.$$

2. Maximal elements; the quadratic case.

(2.1) A **maximal element** $\alpha \in \hat{G}$ is defined to be one such that

$$(1) \qquad \alpha_{i<} \le \alpha, \quad \text{for } 0 < i < |\alpha|.$$

Put $n = |\alpha| \le \infty$, and write $\alpha = \alpha_1\alpha_2\alpha_3\cdots$ in normal form ((1.3)(8)): $\alpha_i \in \{L, C, R\}$, and $\alpha_i \ne C$ for $i < n$. Then

$$\alpha_{i<} = \alpha_{i+1}\alpha_{i+2}\alpha_{i+3}\cdots.$$

Note that if $n < \infty$, then the inequality (1) must be strict. *Assume that α is maximal.* Since there is no i satisfying $0 < i < 1$ we have, for $n \le 1$:

(2) *Each element of $\{1, L, C, R\}$ is maximal.*

Suppose now that $n \ge 2$; then $\alpha_1 \ne C$, and (1) implies that $\alpha_i \le \alpha_1$ for each $i \ge 1$. It follows that, if $\alpha_1 = L$, then $\alpha = L^n$. For $m < n$, L^m is not less than or equal to L^n (cf. (1.6)(6)). Thus,

$$(3) \qquad \alpha_1 = L \Longrightarrow \alpha = L^\infty.$$

$$(4) \qquad \alpha_1 = R, \Longrightarrow \alpha_n = L \text{ or } C \text{ if } n < \infty.$$

The last assertion follows, since $\alpha_n = \alpha_{n-1<}$, from (1) and (1.6)(6). Also:

$$(5) \qquad \alpha \text{ contains only one } R \Longrightarrow \alpha = \begin{cases} RL^{n-1} & (1 \le n \le \infty) \text{ or} \\ RL^{n-2}C & (2 \le n < \infty). \end{cases}$$

Thus:

$$(6) \qquad \alpha \text{ contains only one } R \text{ and } n = 2 \Longrightarrow \alpha = RL \text{ or } RC.$$

Suppose now that $n \ge 3$ and α contains at least two R's. This means that α has one of the following forms (7) or (8).

$$(7) \qquad \begin{aligned} \alpha &= RL^{a_1}RL^{a_2}\cdots RL^{a_{s-1}}R\beta, \quad \text{where } 0 \le a_i < \infty \quad (1 \le i < s), \text{ and} \\ &\begin{cases} \beta = L^{a_s-1}C & (0 < a_s < \infty), \text{ or} \\ \beta = L^{a_s} & (0 \le a_s \le \infty); \end{cases} \end{aligned}$$

or

$$(8) \qquad \alpha = RL^{a_1}RL^{a_2}\cdots RL^{a_{s-1}}RL^{a_s}\cdots \quad \textit{with } 0 \leq a_i < \infty, \quad (i \geq 1).$$

We have

$$\begin{array}{lcccl} RL^{a_i}\cdots & \leq & RL^{a_1}R\cdots & \Rightarrow & a_i \leq a_1, \textit{ and} \\ RL^{a_{s-1}}C & \leq & RL^{a_1}R\cdots & \Rightarrow & a_s \leq a_1, \end{array}$$

Thus,

$$(9) \qquad a_i \leq a_1 \textit{ for all relevant } i.$$

Since, by assumption, $a_i > 0$ for some i, we have

$$(10) \qquad a_1 > 0.$$

The exact conditions for α as in (7) or (8) to be maximal are, in addition to (9) and (10)

$$(11) \qquad \left\{ \begin{array}{l} \textit{If } j > 1 \textit{ and } a_{j+i} = a_i \quad (1 \leq i < r) \\ \textit{and } a_{j+r} \neq a_r \textit{ then } a_r <^{R^r} a_{j+r}. \end{array} \right.$$

This is easily checked.

In summary:

The maximal elements are those of the following forms:

$$(12) \qquad \left\{ \begin{array}{lcl} \alpha & \in & \{1, L, C, R, L^\infty, R^\infty\}, \textit{ or} \\ \alpha & = & RL^{a_1}RL^{a_2}\cdots RL^{a_{s-1}}RL^{a_s-1}C \textit{ or} \\ \alpha & = & RL^{a_1}RL^{a_2}\cdots RL^{a_{s-1}}RL^{a_s}RL^{a_{s+1}}\cdots, \textit{ with} \\ & & 0 < a_1 < \infty, \quad 0 \leq a_i \leq a_1 \textit{ for all } i, \textit{ and} \\ & & \textit{if } a_{j+i} = a_i \quad (1 \leq i < r) \textit{ and } a_{j+r} \neq a_r, \textit{ then} \\ & & a_r <^{R^r} a_{j+r} \end{array} \right.$$

In the case in which q is an integer greater than 1 and $\alpha = RL^{q-2}C$ we shall make use below of the following observations:

$$(13) \qquad \left\{ \begin{array}{l} \textit{For } 0 \leq i < q \textit{ put} \\ \qquad x_i = \alpha_{i<} = \alpha_{i+1}\cdots\alpha_q. \\ \textit{Thus } x_0 = \alpha, \textit{ and } x_i = L^{(q-2)-(i-1)}C \textit{ for} \\ 0 < i < q-1, \textit{ and } x_{q-1} = C. \textit{ We have} \\ \qquad x_0 > x_1 > x_2 > \cdots > x_{q-1}. \\ \textit{In particular, } \alpha \textit{ is maximal.} \end{array} \right.$$

The rest of this section and Section 3 describe properties of maximal elements, in preparation for the discussion of the $\star$-product in Section 4 and of the main result, the $\star$-product Theorem in Section 5.

(2.2) **The elements $\alpha_i(X)$.** Consider an element

$$(1) \qquad \begin{cases} \alpha &= \alpha' C = \alpha_1\alpha_2\cdots\alpha_n \in G_0C, \quad |\alpha| = n \\ \alpha' &= \alpha_1\cdots\alpha_{n-1} \in G_0, \quad \alpha_n = C. \end{cases}$$

For $0 \le i < n$ and $X \in \{L, C, R\}$ we put

$$(2) \qquad \begin{aligned} \alpha_i(X) &= \alpha'_{i<}\, X\, \alpha_{\le i} \\ &= \alpha_{i+1}\cdots\alpha_{n-1}X\alpha_1\cdots\alpha_i. \end{aligned}$$

Thus, for example,

$$(3) \qquad \begin{aligned} \alpha_0(X) &= \alpha' X, \\ \alpha_{n-1}(X) &= X\alpha', \text{ and} \\ \alpha_i(C) &= \alpha'_{i<}C = \alpha_{<i}. \end{aligned}$$

Note that for $X \neq C$,

$$(4) \qquad \rho(\alpha_i(X)) = \rho(\alpha)\rho(X).$$

Thus, for $X \in \{L, R\}$, $\rho(\alpha_i(X))$ determines X, independently of i. Moreover

$$(5) \qquad \begin{aligned} |\alpha_i(X)| &= n \text{ for } X \neq C, \quad \text{and} \\ |\alpha_i(C)| &= n - i. \end{aligned}$$

Clearly we have

$$(6) \qquad \alpha_i(L^{\alpha'_{\le i}}) < \alpha_i(C) < \alpha_i(R^{\alpha'_{\le i}}).$$

(2.3) **Proposition.** *Let $\alpha = \alpha' C$ as in (2.2) be maximal. Suppose that, for some $X, Y \in \{L, C, R\}$ and $0 \le i < j < n$ we have*

$$\alpha_i(X) = \alpha_j(Y).$$

Put $m = j - i$ $(i > 0)$. Then $n = 2m$, and there is a $\gamma \in G_0$, $|\gamma| = m - 1$, such that

$$\begin{aligned} &\alpha = \gamma R^\gamma \gamma C, \\ &\gamma C \text{ is maximal,} \end{aligned}$$

and

$$X = Y = L^\alpha = R^\gamma \quad (\rho(\gamma) = -\rho(\alpha)).$$

In particular, $\alpha' L^\alpha = (\gamma R^\gamma)^2$.

Proof. Since $i < j$ we cannot have $X = Y = C$ (see (2.2)(5)). Similarly we cannot have one of X or Y equal to C without the other. Thus, $X, Y \in \{L, R\}$. From (2.2)(4) we conclude that $X = Y$, so we now have

$$\alpha_i(X) = \alpha_j(X), \quad X \in \{L, R\}, \quad 0 \le i < j < n.$$

Now, $\alpha_i(X) = \alpha'_{i<}\ X\ \alpha_{\leq i}$ is an i-fold cyclic permutation of the word $\alpha_0(X) = \alpha' X$ in the free monoid G_0, and similarly for $\alpha_j(X)$. It follows that

$$\alpha' X = \alpha_0(X) = \alpha_m(X), \quad 0 < m = j - i < n.$$

Now we use the following elementary fact.

(2.4) **Lemma.** *Let H be a free monoid with basis B, and let $\alpha = \alpha_1\alpha_2 \cdots \alpha_n \in H$, each $\alpha_i \in B$. Suppose, for some m, $0 < m < n$, that*

$$\alpha = \alpha_{m+1} \cdots \alpha_n \alpha_1 \cdots \alpha_m.$$

Put $d = \gcd(n, m) < n$ and $N = n/d > 1$. Then

$$\alpha = \delta^N, \qquad \textit{where } \delta = \alpha_1 \cdots \alpha_d.$$

Proof. Each $\beta \neq 1$ in H has a "primitive root". That is, there exists a $\gamma := rad(\beta) \in H$ such that $\beta = \gamma^r$ for some $r \geq 1$, and the centralizer of β in H is $\{\gamma^s \mid s \geq 0\}$. It follows that β and $\beta' \neq 1$ in H commute if and only if $rad(\beta) = rad(\beta')$.

Now put $\beta = \alpha_1 \cdots \alpha_m \neq 1$ and $\gamma = \alpha_{m+1} \cdots \alpha_n \neq 1$. Our hypothesis says that $\beta\gamma = \gamma\beta$. Thus $rad(\beta) = \delta_0 = rad(\gamma)$, $d_0 = \mid \delta_0 \mid$ divides $\mid \beta \mid = n$ and $\mid \gamma \mid = n - m$ and so d_0 divides d. Then $\delta = \delta_0^{(d/d_0)}$ is as required.

■

Continuing the proof of (2.3), Lemma (2.4) tells us that $\alpha_0(X) = \alpha' X = \delta^N$, where $N = n/\gcd(m, n) > 1$. Clearly $\delta = \gamma X$, so

$$(1) \qquad \begin{cases} \alpha &= (\gamma X)^{N-1}\gamma C, \quad \textit{and} \\ \rho(\alpha) &= \rho(\gamma)^N \rho(X)^{N-1}. \end{cases}$$

If $N \geq 3$ then, since α is maximal,

$$(2) \qquad \gamma X \gamma C < \alpha = \gamma X \gamma X \cdots.$$

Since $\rho(\gamma X \gamma) = \rho(X)$, (2) means that

$$C <^X X,$$

or equivalently (cf. (1.5)(4))

$$C = C^X < X^X = L$$

which is a contradiction. Thus,

$$(3) \qquad \begin{cases} N = 2, \ \textit{hence } m = n/2, \ \textit{and (from (1))} \\ \rho(X) = \rho(\alpha). \end{cases}$$

Now $\alpha = \gamma X \gamma C$ and we have

$$\gamma C < \gamma X \gamma C$$

whence

$$C <^{\gamma} X, \quad i.e., C = C^{\gamma} < X^{\gamma}.$$

It follows that

(4) $$X = R^{\gamma}, \quad \alpha = \gamma R^{\gamma} \gamma C.$$

Hence

$$\begin{aligned} \rho(\alpha) = \rho(X) &= \rho(R^{\gamma}) \\ &= \rho(\gamma)\rho(R) \\ &= -\rho(\gamma), \end{aligned}$$

where the last equality comes from $(1.5)(3)_0$. So

(5) $$\rho(\gamma) = -\rho(\alpha) \text{ and } X = L^{\alpha}.$$

It remains only to observe that $\delta = \gamma C$ is maximal. For $0 < i < m = |\delta|$, we must show that $\delta_{i<} < \delta$. Now $\delta_{i<} = \gamma_{i<} C = \alpha_{(i+m)<} < \alpha = \gamma R^{\gamma} \gamma C$. Since $|\delta_{i<}| < |\delta| = |\gamma R^{\gamma}|$ the condition $\delta_{i<} < \alpha$ implies that $\delta_{i<} < \gamma$, and so $\delta_{i<} < \gamma C = \delta$.

■

We record for reference some consequences of (2.3).

(2.5) **Corollary.** *If $\alpha = \alpha' C$ is maximal, $X \in \{L, R\}$ and $\alpha' X = \delta^N$ for some $N > 0$ then $N = 2$, $\delta = \gamma R^{\gamma} = \gamma L^{\alpha}$, and γC is maximal.*

Proof. This is the case $X = Y$, $i = 0$, and $j = m := n/2$ of (2.3).

■

(2.6) **Terminology.** we call an element $\alpha = \alpha' C$ **quadratic**, if $\alpha = \gamma R^{\gamma} \gamma C$, as in (2.5). Note then that $\rho(\alpha) = -\rho(\gamma)$.

(2.7) **Corollary.** *If $\alpha = \alpha' C$ is maximal and non-quadratic then $\alpha_i(X) \neq \alpha_j(Y)$ whenever $0 \leq i < j < n$ and $X, Y \in \{L, C, R\}$.*

3. Maximal elements; the non-quadratic case.

(3.1) **Proposition.** *Let* $\alpha = \alpha' C, |\alpha| = n$, *be maximal,* $0 \le i,j < n$, *and* $X, Y \in \{L, C, R\}$. *Assume that*

$$(1) \qquad \alpha_{i<} < \alpha_{j<}.$$

We have (cf. (2.2)(6))

$$\alpha_i(X) \le \alpha_i(R^{\alpha_{\le i}}) \text{ and } \alpha_j(L^{\alpha_{\le j}}) \le \alpha_j(Y),$$

and either

$$(2) \qquad \alpha_i(R^{\alpha_{\le i}}) < \alpha_j(L^{\alpha_{\le j}}), \quad \text{and hence} \quad \alpha_i(X) < \alpha_j(Y),$$

or

$$(3) \qquad \begin{cases} \alpha = \gamma R^{\gamma} \gamma C \text{ is quadratic }, \\ \gamma C \text{ is maximal}, \\ |\gamma C| = m = n/2, \quad |j-i| = m, \quad \text{and} \\ \rho(\alpha_{\le i}) = \rho(\gamma) = -\rho(\alpha) = -\rho(\alpha_{\le j}). \end{cases}$$

Proof. If $\alpha_i(X) = \alpha_j(Y)$ then we have (3), in view of (2.3). It suffices therefore to show that (1) implies (2) when

$$(4) \qquad \alpha_i(X) \ne \alpha_j(Y).$$

To economize on notation we shall assume that

$$(5) \qquad 0 \le i < j < n.$$

Then assuming (4) and (5), we must show that (1) implies (2), and also that

$$(1') \qquad \alpha_{j<} < \alpha_{i<}$$

implies

$$(2') \qquad \alpha_j(Y) < \alpha_i(X).$$

We can write

$$(6) \qquad \begin{cases} \alpha_j(Y) = \delta \cdot Y \cdot \gamma \cdot \alpha_{j-i} \cdot \varepsilon & \text{and} \\ \alpha_i(X) = \delta' \cdot \alpha_{n-j+i} \cdot \gamma' \cdot X \cdot \varepsilon', \end{cases}$$

where

$$(7) \qquad \begin{cases} \delta = \alpha'_{j<}, & \delta' = \alpha_{i+1} \cdots \alpha_{n-j+i-1}, & |\delta| = |\delta'|, \\ \gamma = \alpha_{\le (j-i-1)}, & \gamma' = \alpha'_{(n-j+i)<}, & |\gamma| = |\gamma'|, \\ \varepsilon = \alpha_{j-i+1} \cdots \alpha_j, & \varepsilon' = \alpha_{\le i}, & |\varepsilon| = |\varepsilon'|. \end{cases}$$

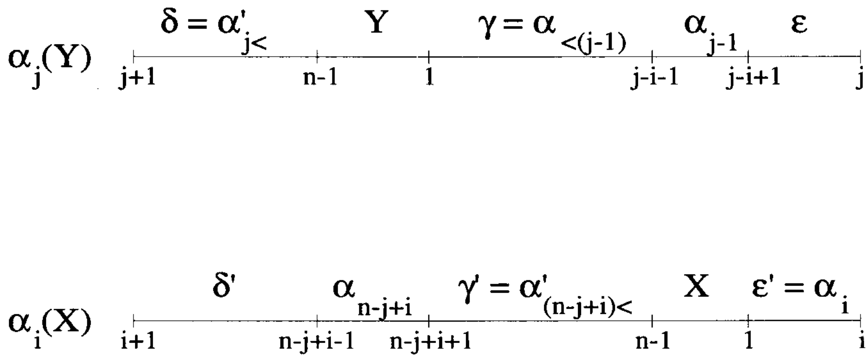

Figure 9. The factorization (6).

We then have

(8) $$\begin{cases} \alpha_{j<} &= \alpha_j(C) = \delta C, \ \textit{and} \\ \alpha_{i<} &= \alpha_i(C) = \delta' \alpha_{n-j+i} \gamma C'. \end{cases}$$

If $\alpha_{i<} < \alpha_{j<}$ then we have either $\delta' < \delta$, or $\delta' = \delta$ and $\alpha^{\delta}_{n-j+i} < C^{\delta} = C$, whence $\alpha_{n-j+i} = L^{\delta}$. Thus, we can restate (1) as:

(1) Either $\delta' < \delta$, *or* $\delta' = \delta$ *and* $\alpha_{n-j+i} = L^{\delta}$.

Similarly we can restate (1′) as:

(1′) Either $\delta' > \delta$, *or* $\delta' = \delta$ *and* $\alpha_{n-j+i} = R^{\delta}$.

Now it follows from (6) that:

(9) *If* $\delta \neq \delta'$ *then* (1) $\Longrightarrow$ (2), *and* (1′) $\Longrightarrow$ (2′).

Assume henceforth that:

(9′) $$\delta = \delta'.$$

Then (1) and (1′) become, respectively,

(1_+) $$\alpha_{n-j+i} = L^{\delta},$$

and

($1'_+$) $$\alpha_{n-j+i} = R^{\delta}.$$

Suppose that

(10) $$Y \neq \alpha_{n-j+i}.$$

Then

$$Y^\delta \neq \alpha^\delta_{n-j+i} = \begin{cases} L & \textit{case } (1_+) \\ R & \textit{case } (1'_+) \end{cases}$$

so, in case (1_+), $\alpha^\delta_{n-j+i} < Y^\delta$, whence $\alpha_i(X) < \alpha_j(Y)$, and, in case $(1'_+)$, $Y^\delta < \alpha^\delta_{n-j+i}$, whence $\alpha_j(Y) < \alpha_i(X)$. Thus: assuming $(9')$ and (10), we have (1) implies (2) and $(1')$ implies $(2')$.

So now assume $(9')$ and

$$(10') \qquad Y = \alpha_{n-j+i} = \begin{cases} L^\delta & (\textit{case } (1)) \\ R^\delta & (\textit{case } (1')). \end{cases}$$

Then

$$(11) \qquad \rho(\delta Y) = \rho(\delta)\rho(Y) = \begin{cases} 1 & (\textit{case } (1)) \\ -1 & (\textit{case } (1')). \end{cases}$$

Consider the case:

$$(12) \qquad \gamma \neq \gamma'.$$

Note that (12) implies that $\gamma' < \gamma$ since α is maximal. In view of (11) we then have $\alpha_i(X) < \alpha_j(Y)$ in case (1) and $\alpha_j(Y) < \alpha_i(X)$ in case $(1')$. Thus, assuming $(9')$, $(10')$ and (12), we have (1) implies (2) and $(1')$ implies $(2')$.

So now assume $(9')$, $(10')$ and

$$(12') \qquad \gamma = \gamma'.$$

From $(12')$ and the maximality of α we conclude that $C = C^\gamma < \alpha^\gamma_{j-i}$, whence

$$(13) \qquad \alpha^\gamma_{j-i} = R, \ \textit{and so } \alpha^{\delta Y\gamma}_{j-i} = R^{\delta Y} = \begin{cases} R & (\textit{case } (1)) \\ L & (\textit{case } (1')), \end{cases}$$

using (11).

Thus, in case

$$(14) \qquad X \neq \alpha_{j-i}$$

we have $X^{\delta Y\gamma} < \alpha^{\delta Y\gamma}_{j-i}$, hence $\alpha_i(X) < \alpha_j(Y)$ in case (1). Similarly, $\alpha_j(Y) < \alpha_i(X)$ in case $(1')$.

Now assume $(9')$, $(10')$, $(12')$ and

$$(14') \qquad X = \alpha_{j-i} \qquad (= R^\gamma, \ \textit{by } (13)), \qquad \textit{hence } \rho(\gamma X) = -1.$$

Then in view of (4), we must have

$$(15) \qquad \varepsilon \neq \varepsilon', \qquad \textit{hence } \epsilon < \epsilon',$$

by maximality of α. We have, in view of (11) and $(14')$,

$$(16) \qquad \rho(\delta Y\gamma X) = \begin{cases} -1 & (\textit{case } (1)) \\ 1 & (\textit{case } (1')). \end{cases}$$

In view of (6), (15) and (16) we have $\alpha_i(X) < \alpha_j(Y)$ in case (1), and $\alpha_j(Y) < \alpha_i(X)$ in case (1′). Now, (1) implies (2) and (1′) implies (2′) have been established in all cases, thus concluding the proof of (3.1).

■

4. The $\star$-product.

(4.1) **The product** $\alpha \star \beta$. Let

(1) $$\alpha = \alpha_1 \cdots \alpha_{n-1} \in G_0, \quad |\alpha| = n-1.$$

For

(2) $$\beta = \beta_1\beta_2\beta_3 \cdots \in \hat{G}, \quad \beta_i \in \{L, C, R\}$$

we define (following [DGP1])

(3) $$\alpha \star \beta = (\alpha\beta_1^\alpha)(\alpha\beta_2^\alpha)(\alpha\beta_3^\alpha)\cdots.$$

Note that $\alpha \star 1 = 1$,

(4) $$|\alpha \star \beta| = n\,|\beta| = (|\alpha| + 1)\,|\beta|,$$

and, for $h \geq 0$,

(5) $$(\alpha \star \beta)_{hn<} = \alpha \star \beta_{h<}.$$

For $\gamma \in G$,

(6) $$\alpha \star \gamma\beta = (\alpha \star \gamma)(\alpha \star \beta).$$

For $\gamma \in \{L, C, R\}$,

(7) $$\begin{cases} \alpha \star \gamma & = \quad \alpha\gamma^\alpha, \ \textit{and} \\ \rho(\alpha\gamma^\alpha) & = \quad \rho(\gamma) \ \textit{if}\ \gamma \neq C. \end{cases}$$

Note that:

(8) $$\alpha \star L < \alpha \star C < \alpha \star R.$$

From (6) and (7) we obtain:

(9) $$\rho(\alpha \star \beta) = \rho(\beta) \quad \textit{for}\ \alpha, \beta \in G_0.$$

The $\star$-product is not associative, but we do have the following identity. For $\alpha, \beta \in G_0$ and $\gamma \in \hat{G}$, we have:

(10) $$\alpha \star (\beta \star \gamma) = (\alpha \star \beta)\alpha \star \gamma.$$

In fact, since $\beta\star\gamma = \beta\gamma_1^\beta\beta\gamma_2^\beta\beta\gamma_3^\beta\cdots$,

$$\alpha\star(\beta\star\gamma) = (\alpha\star\beta)\alpha\gamma_1^{\beta\alpha}(\alpha\star\beta)\alpha\gamma_2^{\beta\alpha}(\alpha\star\beta)\alpha\gamma_3^{\beta\alpha}\cdots,$$

and since $\rho((\alpha\star\beta)\alpha) = \rho(\beta)\rho(\alpha)$, by (9), the right side of the displayed equation equals $(\alpha\star\beta)\alpha\star\gamma$.

Suppose that $\alpha C = \gamma R^\gamma\gamma C$ is quadratic (see (2.6)). Then $\alpha = \gamma R^\gamma\gamma = (\gamma\star R)\gamma$, so it follows from (10) that $\alpha\star\beta = \gamma\star(R\star\beta)$: In view of (2.3) we thus have:

$$(11)\qquad \begin{cases} \alpha C = \gamma R^\gamma\gamma C \textit{ is quadratic } \Longrightarrow \alpha\star\beta = \gamma\star(R\star\beta). \\ \alpha C \textit{ is maximal } \Longrightarrow \gamma C \textit{ is maximal.} \end{cases}$$

Let $\alpha', \beta' \in G_0$, $\alpha = \alpha' C$, $\beta = \beta' C$. Then:

$$(12)\qquad \alpha'\star\beta \textit{ is quadratic } \Longleftrightarrow \begin{cases} \beta \textit{ is quadratic, or} \\ |\beta| = 2b+1 \textit{ is odd,} \\ \alpha \textit{ is quadratic and } \beta = L^{2b}C. \end{cases}$$

Proof (of (12)). If $\beta = \delta R^\delta\delta C$ is quadratic then $\alpha'\star\beta = \Delta R^{\delta\alpha'}\Delta C$, where $\Delta = (\alpha'\star\delta)\alpha'$, and $\rho(\Delta) = \rho(\delta)\rho(\alpha')$ (cf. (9) above), so $\alpha'\star\beta$ is quadratic.

If $\alpha = \gamma R^\gamma\gamma C$ is quadratic and $\beta = L^{2b}C$ then, using (11),

$$\begin{aligned} \alpha'\star\beta &= \gamma\star(R\star\beta) = \gamma\star(R^{4b+1}C) \\ &= (\gamma R^\gamma)^{2b}(\gamma R^\gamma)(\gamma R^\gamma)^{2b}(\gamma C) \\ &= \Delta R^\gamma\Delta C \end{aligned}$$

where $\Delta = (\gamma R^\gamma)^{2b}\gamma$, and $\rho(\Delta) = \rho(\gamma)$, so $R^\gamma = R^\Delta$. Thus $\alpha'\star\beta$ is quadratic.

Suppose, conversely, that $\alpha'\star\beta = \Delta R^\Delta\Delta C$ is quadratic. Put $B_i = \beta_i^\alpha$. If $|\beta| = 2b$ we have

$$\begin{aligned} \Delta &= \alpha' B_1\alpha'\cdots\alpha' B_{b-1}\alpha' \\ &= \alpha' B_{b+1}\alpha'\cdots\alpha' B_{2b-1}\alpha', \qquad \text{and} \\ B_b &= R^\Delta,\ \textit{so } \beta_b = R^{\Delta\alpha'}. \end{aligned}$$

It follows that $\beta_i = \beta_{b+i}\quad(1\le i<b)$, so

$$\begin{aligned} \delta &:= \beta_1\cdots\beta_{b-1} = \beta_{b+1}\cdots\beta_{2b-1}, \\ \Delta &= (\alpha'\star\delta)\alpha' \text{ and} \\ \beta' &= \delta R^{\Delta\alpha'}\delta. \end{aligned}$$

Now $\rho(\Delta) = \rho(\alpha'\star\delta)\rho(\alpha') = \rho(\delta)\rho(\alpha')$ ((9) above), so $R^{\Delta\alpha'} = R^\delta$, whence $\beta = \delta R^\delta\delta C$ is quadratic.

Suppose, finally, that $|\beta| = 2b+1$ is odd. Since $|\alpha'\star\beta| = 2(|\Delta|+1) = |\alpha||\beta|$ (cf. (4)), $|\alpha|$ must be even, say $|\alpha| = 2a$. Write $\alpha = \lambda X\mu C$ with $|\lambda| = |\mu| = a-1$ and $X = \alpha_a$. We have

$$\alpha'\star\beta = (\alpha' B_1\cdots\alpha' B_b)\alpha'(B_{b+1}\alpha'\cdots B_{2b}\alpha')C.$$

It follows that

$$\begin{aligned}\Delta &= \alpha' B_1 \cdots \alpha' B_b \lambda = \lambda X \mu B_1 \cdots \lambda X \mu B_b \lambda \\ &= \mu B_{b+1} \alpha' \cdots B_{2b} \alpha' = \mu B_{b+1} \lambda X \cdots \mu B_{2b} \lambda X \mu\end{aligned}$$

and so $\lambda = \mu$; denote this by γ. Then

$$R^\Delta = X = B_i = \beta_i^{\alpha'} \quad (1 \leq i \leq 2b).$$

Hence $\alpha' = \gamma R^\Delta \gamma$, $\beta = (R^{\Delta\alpha'})^{2b} C$, and $\Delta = (\gamma R^\Delta)^{2b}\gamma$, hence $\rho(\Delta) = \rho(\gamma)$. Thus, $\alpha = \gamma R^\gamma \gamma C$ is quadratic, hence $\rho(\gamma) = -\rho(\alpha)$, and $\beta_i = R^{\Delta\alpha'} = L$, hence $\beta = L^{2b} C$.

■

(4.2) **Lemma.** *Let $\alpha \in G_0$, and $\beta, \gamma \in \hat{G}$.*
(a) $\beta < \gamma \iff \alpha \star \beta < \alpha \star \gamma$.
(b) $\gamma \mid \beta \iff (\alpha \star \gamma) \mid (\alpha \star \beta)$.

Proof. Part (b) follows easily from (4.1)(6). For part (a), write $\beta = \delta X \beta'$ and $\gamma = \delta Y \gamma'$ with $\delta \in G_0$ and $X \neq Y$ in $\{L, C, R\}$. Then $\beta < \gamma$ if and only $X^\delta < Y^\delta$. We have

$$\begin{aligned}\alpha \star \beta &= (\alpha \star \delta)(\alpha X^\alpha)(\alpha \star \beta'), \ \textit{and} \\ \alpha \star \gamma &= (\alpha \star \delta)(\alpha Y^\alpha)(\alpha \star \gamma').\end{aligned}$$

Therefore $\alpha \star \beta < \alpha \star \gamma$ if and only if

$$(X^\alpha)^{(\alpha\star\delta)\alpha} < (Y^\alpha)^{(\alpha\star\delta)\alpha}.$$

Thus it suffices to show that $\rho(\alpha(\alpha \star \delta)\alpha) = \rho(\delta)$. We have $\rho(\alpha(\alpha \star \delta)\alpha) = \rho(\alpha)\rho(\alpha)\rho(\alpha \star \delta) = \rho(\alpha \star \delta) = \rho(\delta)$, by (4.1)(9).

■

(4.3) **The $\star$-product as a substitution.** For $\alpha \in G_0$ of length $n-1$ as in (4.1)(1), the map

$$\alpha\star : \hat{G} \longrightarrow \hat{G}, \qquad \beta \mapsto \alpha \star \beta, \tag{1}$$

can be viewed as a "substitution homomorphism," replacing X by αX^α for $X \in \{L, C, R\}$. (Cf. (4.1)(3)). This point of view was originally presented in [PTT].

For an integer $N \geq 0$, denote the N-fold iterate of the operator (1) by

$$\begin{aligned}\alpha^{\star N} &= (\alpha\star)^N; \\ \alpha^{\star N}(\beta) &= \alpha \star (\alpha \star \cdots (\alpha \star \beta)) \cdots)\end{aligned} \tag{2}$$

Claim. *There is an element $\alpha(N) \in G_0$ such that*

$$\alpha^{\star N}(\beta) = \alpha(N) \star \beta \quad \text{for all } \beta \in \hat{G}. \tag{3}$$

If $N = p + q$, $p, q \geq 0$, then

$$\begin{array}{rcll} \alpha(N) & = & (\alpha(p) \star \alpha(q))\alpha(p), & \text{i.e.,} \\ \alpha(N)C & = & \alpha(p) \star \alpha(q)C & \end{array} \tag{4}$$

and

$$|\alpha(N)| = n^N - 1, \quad \text{and} \quad \rho(\alpha(N)) = \rho(\alpha)^N. \tag{5}$$

Proof. We have $\alpha(0) = 1$ and $\alpha(1) = \alpha$. Arguing inductively, we have, for $N = p + q$, $p, q > 0$,

$$\begin{array}{rcll} \alpha^{\star N}(\beta) & = & \alpha^{\star p}(\alpha^{\star q}(\beta)) = \alpha(p) \star (\alpha(q) \star \beta) & \\ & = & (\alpha(p) \star \alpha(q))\alpha(p) \star \beta & \text{(by (10))}, \end{array}$$

whence (3) and (4). For (5), put $E(N) = |\alpha(N)| + 1$. From (4) and (4.1)(4) we have

$$\begin{array}{rcl} E(N) & = & E(p)(E(q) - 1) + (E(p) - 1) + 1 \\ & = & E(p)E(q) \end{array}$$

whence $E(N) = E(1)^N = n^N$. Similarly, from (4) and (4.1)(9) we have $\rho(\alpha(N)) = \rho(\alpha(q))\rho(\alpha(p))$, whence $\rho(\alpha(N)) = \rho(\alpha(1))^N = \rho(\alpha)^N$.

From (4) we see that, for any $\beta \in \hat{G}$,

$$\alpha^{\star N}(\beta)_{\leq p} = \alpha(p) = \alpha(N)_{\leq p} \tag{6}$$

for any $p \leq N$. Fixing p and letting $N \longrightarrow \infty$, we see that the initial segments of $\alpha^{\star N}(\beta)$ stabilize at a value independent of β. Hence we have a well defined limit

$$\begin{array}{rcll} \alpha^{\star\infty} & := & \lim\limits_{N \longrightarrow \infty} \alpha(N) = \lim\limits_{N \longrightarrow \infty} \alpha^{\star N}(\beta) & \text{for all } \beta \in \hat{G} \\ & = & \text{the unique fixed point of } \alpha \star \text{ in } \hat{G}. & \end{array} \tag{7}$$

Examples. Let $X = L$ or R. Then from (4) and (5) we have $X(N + 1) = (X(N) \star X)X(N) = X(N)X^{X^N}X(N)$. Thus $L(N + 1) = L(N)LL(N)$, whence, by induction,

$$L(N) = L^{2^N - 1}, \quad \text{and} \quad L^{\star\infty} = L^{\infty}.$$

On the other hand

$$R(N + 1) = R(N)R^{R^N}R(N) = (R \star R(N))R,$$

so

$$\begin{array}{rcl} R(1) & = & R \\ R(2) & = & RLR \\ R(3) & = & (RLR)R(RLR) \\ R(4) & = & (RLRRRLR)L(RLRRRLR) \end{array}$$

We conclude now with a Lemma needed in Section 5 below.

(4.4) **Lemma.** *Let $\beta, \delta \in \hat{G}$,*

$$\beta = \beta_1\beta_2\beta_3\cdots, \qquad \delta = \delta_1\delta_2\delta_3\cdots$$

and $\delta_0 \in \{L, R\}$. Put

$$B = R \star \beta \text{ and } D = \bar{\delta_0}(R \star \delta).$$

(a) $R^\infty < B$, *unless* $\beta = L^r$, $0 \le r \le \infty$, *when* $B = R^{2r}$.
(b) $D < R^\infty$, *unless* $\delta_0\delta = L^r$, $0 \le r \le \infty$, *when* $D = R^{2r-1}$.
(c) $D < B$, *unless either* $\beta = \delta_0\delta = L^\infty$, *or else the shorter of* β *and* $\delta_0\delta$ *is a power of* L *and divides the other.*

Proof. Put $B_i = \bar{\beta_i}$ and $D_i = \bar{\delta_i}$. Then

$$\begin{array}{rcl} B &=& RB_1RB_2\cdots RB_{r-1}RB_rRB_{r+1}\cdots \\ D &=& D_0RD_1R\cdots D_{r-2}RD_{r-1}RD_rR.\cdots \end{array}$$

If β is not a power of L then we can write $\beta = L^rR\beta'$ with $0 \le r < \infty$ and $\beta' \in \hat{G}$ or $\beta = L^rC$. Then $B = (R \star L^r)(R \star R)(R \star \beta') = R^{2r+1}L(R \star \beta')$ or $B = R^{2r+1}C$, respectively. Since $R <^{R^{2r+1}} X$ for $X = L$ or C, we have $R^\infty < B$, and $R^N\gamma < B$ for any $N > 2r+1$, $\gamma \in \hat{G}$.

If $\delta_0\delta$ is not a power of L then we can write $\delta_0\delta = L^sR\delta'$ with $0 \le s < \infty$, $\delta' \in \hat{G}$ or $\delta_0\delta = L^sC$. Then $D = R^{2s}L(R \star \delta')$ or $R^{2s}C$, respectively. Since $X <^{R^{2s}} R$ for $X = L$ or C, we have $D < R^\infty$ and $D < R^N\gamma$ for any $N > 2s$ and $\gamma \in \hat{G}$.

The assertions of the Lemma follow easily from these observations.

■

5. The $\star$-product theorem

(5.0) Let $\alpha = \alpha'C$ be maximal, where $\alpha' \in G_0$ and $|\alpha| = n$. Recall from (2.6) that α is called quadratic if

$$(1) \qquad \begin{cases} \alpha & = \gamma R^\gamma \gamma C, \\ \gamma C & \textit{is maximal}, \\ |\gamma C| & = m = n/2. \end{cases}$$

we make the following *assumption*:

$$(2) \qquad \alpha_{i<} < \alpha_{j<} \quad \text{for some } i, j \quad 0 \le i, j < n.$$

We would like to conclude that, for $\beta \in \hat{G}$,

$$(\alpha' \star \beta)_{i'<} \;<\; (\alpha' \star \beta)_{j'<}$$

whenever $i' \equiv i \bmod n$, $j' \equiv j \bmod n$, and $0 \leq i', j' < |\alpha' \star \beta|$. The next theorem implies that this is so, but for certain explicit exceptional cases when α is quadratic.

(5.1) **Theorem.** *Keep the notation, and assumption (5.0)(2). Let $\beta \in \hat{G}$, $0 \leq (k-1), (k'-1) < |\beta|$, and put*

$$i' = i + (k-1)n \quad \text{and } j' = j + (k'-1)n.$$

We have in the notation of (2.2),

$$(3) \qquad (\alpha' \star \beta)_{i'<} \;\leq\; \alpha_i(R^{\alpha_{\leq i}}) \;<\; \alpha_j(L^{\alpha_{\leq j}}) \;<\; (\alpha' \star \beta)_{j'<},$$

unless α is quadratic (cf.(1)) and $|j - i| = m$. In the latter case we have

$$(4) \qquad \alpha' \star \beta = \gamma \star (R \star \beta).$$

Moreover, putting $h = \min(i, j)$, $\delta = \gamma_{h<}$, and $\Delta = \delta R^{\gamma}(\gamma \star R^{\infty})$, we have

$$(5i) \qquad (\alpha' \star \beta)_{i'<} \;<\; \Delta \quad \text{unless} \quad \beta_{k\leq} \text{ is a power of } L$$

and

$$(5j) \qquad \Delta < (\alpha' \star \beta)_{j'<} \quad \text{unless} \quad \beta_{k'\leq} \text{ is a power of } L.$$

(Note that if β terminates with L^r, $0 < r \leq \infty$, then $R \star \beta$ in (4) terminates with $R \star L^r = R^{2r}$.) In particular we have

$$(*) \qquad (\alpha' \star \beta)_{i'<} < (\alpha' \star \beta)_{j'<}$$

unless α is quadratic,$|j - i| = m$, and the shorter of $\beta_{k\leq}$ and $\beta_{k'\leq}$ is a power of L and divides the other.

Proof of (5.1). Putting $B_i = \beta_i^{\alpha} = \beta_i^{\alpha'}$ we have

$$\alpha' \star \beta = \alpha' B_1 \alpha' B_2 \alpha' \cdots$$

and so

$$(6) \qquad \begin{cases} I \;:=\; (\alpha' \star \beta)_{i'<} \;=\; \alpha'_{i<} B_k \alpha' B_{k+1} \alpha' \cdots \\ J \;:=\; (\alpha' \star \beta)_{j'<} \;=\; \alpha'_{j<} B_{k'} \alpha' B_{k'+1} \alpha' \cdots. \end{cases}$$

Note that, with the notation of (2.2),

$$I_{\leq n} = \alpha_i(B_k), \text{ and } J_{\leq n} = \alpha_j(B_{k'}).$$

It follows therefore from (3.1) that (2) implies (3), unless α is quadratic, as in (1), and $|j-i| = m = n/2$, which we henceforth assume. Then we have $B_i = \beta_i^\alpha = \bar{\beta_i}^\gamma$ and $\bar{\beta}_i = \beta_i^R$, so

$$\begin{aligned}\alpha' \star \beta &= \gamma R^\gamma \gamma B_1 \gamma R^\gamma \gamma B_2 \gamma \cdots \\ &= \gamma R^\gamma \gamma \bar{\beta_1}^\gamma \gamma R^\gamma \gamma \bar{\beta_2}^\gamma \gamma \cdots \\ &= \gamma \star (R\bar{\beta}_1 R\bar{\beta}_2 R \cdots) \\ &= \gamma \star (R \star \beta),\end{aligned}$$

whence (4).

To economize on notation, let us now assume that

(7) $$0 \leq i < j = i + m.$$

Then we have $\delta = \gamma_{i<}$ and we must show that (2) implies (5i) and (5j) and also that the condition

(2′) $$\alpha_{j<} \ < \ \alpha_{i<}$$

implies, with $\Delta = \delta R^\gamma(\gamma \star R^\infty)$,

(5′i) $$\Delta \ < \ (\alpha' \star \beta)_{i'<} \quad \textit{unless } \beta_{k\leq} \textit{ is a power of } L$$

and

(5′j) $$(\alpha' \star \beta)_{j'<} \ < \ \Delta \quad \textit{unless } \beta_{k'\leq} \textit{ is a power of } L.$$

From (7) and (4) we see that

(8) $$\left\{\begin{aligned} I &= \delta R^\gamma \gamma B_k \gamma R^\gamma \gamma B_{k+1} \gamma \cdots \\ &= \delta R^\gamma(\gamma \star \bar{\beta}_k(R \star \beta_{k<})), \ \textit{and} \\ J &= \delta B_{k'} \gamma R^\gamma \gamma B_{k'+1} \gamma \cdots \\ &= \delta \bar{\beta_{k'}}^\gamma(\gamma \star (R \star \beta_{k'<})).\end{aligned}\right.$$

Moreover we have

$$\alpha_{i<} = \delta R^\gamma \gamma C \ \ \textit{and } \alpha_{j<} = \delta C.$$

Thus, $\alpha_{i<} < \alpha_{j<}$ if and only if $R^{\gamma\delta} < C^\delta = C$, if and only if $\rho(\gamma\delta) = -1$. Similarly, $\alpha_{j<} < \alpha_{i<}$ if and only if $\rho(\gamma\delta) = 1$. Thus, we can rewrite (2) and (2′) as:

(2) $$\rho(\gamma\delta) = -1$$

and

(2′) $$\rho(\gamma\delta) = 1.$$

Now we see that,

$$\begin{aligned}\beta_{\leq k} \textit{ not a power of } L \ &\Rightarrow \ \bar{\beta}_k(R \star \beta_{k<}) < R^\infty \qquad (\textit{by } (4.4)(b)) \\ &\Rightarrow \ \gamma \star \bar{\beta}_k(R \star \beta_{k<}) < \gamma \star R^\infty \quad (\textit{by } (4.2)(a)) \\ &\Rightarrow \ I = \delta R^\gamma(\gamma \star \bar{\beta}_k(R \star \beta_{k<})) <^{\delta R^\gamma} \delta R^\gamma(\gamma \star R^\infty) = \Delta.\end{aligned}$$

Since $\rho(\delta R^\gamma) = -\rho(\gamma\delta)$, the relation "$<^{\delta R^\gamma}$" is "$<$" in case (2), and "$>$" in case (2′). Whence (2) implies (5i) and (2′) implies (5′i).

Next suppose that $\beta_{k'<}$ is not a power of L. If $\beta_{k'} \neq L$ then $\bar{\beta}_{k'} < R$. Thus, the condition (2), $\rho(\gamma\delta) = -1$, implies that $\delta R^\gamma < \delta\bar{\beta}_{k'}{}^\gamma$, which further implies that

$$\Delta = \delta R^\gamma(\gamma \star R^\infty) < \delta\bar{\beta}_{k'}{}^\gamma(\gamma \star (R \star \beta_{k'<})) = J.$$

Similarly, $\rho(\gamma\delta) = 1$ implies that $\delta\bar{\beta}_{k'}{}^\gamma < \delta R^\gamma$, which further implies $J < \delta R^\gamma(\gamma \star R^\infty) = \Delta$. Thus, (2) implies (5j) and (2′) implies (5′j) when $\beta_{k'} \neq L$.

Suppose now that $\beta_{k'} = L$, so that $J = \delta R^\gamma(\gamma \star (R \star \beta_{k'<}))$ (see (8)). Then

$$\begin{aligned}
\beta_{k'<} \textit{ not a power of } L \quad &\Rightarrow \quad R^\infty < R \star \beta_{k'<} \qquad (\textit{ by } (4.4)(a)) \\
&\Rightarrow \quad \gamma \star R^\infty < \gamma \star (R \star \beta_{k'<}) \qquad (\textit{ by } (4.2)(a)), \\
&\Rightarrow \quad \Delta = \delta R^\gamma(\gamma \star R^\infty) <^{\delta R^\gamma} J.
\end{aligned}$$

Thus, since $\rho(\delta R^\gamma) = -\rho(\gamma\delta)$, we have $\Delta < J$ if $\rho(\gamma\delta) = -1$, and $J < \Delta$ if $\rho(\gamma\delta) = 1$. This concludes the proof that (2) implies (5j) and (2′) implies (5′j), and so also the proof of Theorem (5.1).

■

(5.2) **Corollary.** *Let $\alpha = \alpha' C \in G_0 C$ be maximal and $\beta \in \hat{G}$ be as in (5.1). Then $\alpha' \star \beta$ is maximal if and only if β is maximal. In particular $(\alpha^+)^\infty$ and $(\alpha^-)^\infty$ are maximal where $\alpha^+ = \alpha' \star R$ and $\alpha^- = \alpha' \star L$.*

Proof: The last assertion follows from the first since $(\alpha^+)^\infty = \alpha' \star R^\infty$, $(\alpha^-)^\infty = \alpha' \star L^\infty$, and R^∞ and L^∞ are maximal.

For $0 < h < |\beta|$, $(\alpha' \star \beta)_{hn<} = \alpha' \star \beta_{h<}$ ((4.1)(5)), and $\alpha' \star \beta_{h<} \leq \alpha' \star \beta$ iff $\beta_{h<} \leq \beta$ ((4.2)(a)). Thus, β is maximal iff $(\alpha' \star \beta)_{hn<} \leq \alpha' \star \beta$ for $0 < h < |\beta|$, and this is the case if $\alpha' \star \beta$ is maximal.

It remains to show that, if $i' = i + (k-1)n$ with $0 < i < n$ and $0 \leq (k-1) < |\beta|$, and if β is maximal, then $(\alpha' \star \beta)_{i'<} \leq \alpha' \star \beta$. If $n = 1$ then $0 < i < n$ does not occur, so the proof is complete now for $n = 1$, and we can argue by induction on $n = |\alpha|$.

Since α is maximal we have $\alpha_{i<} < \alpha = \alpha_{0<}$. Put $j = 0 = j'$ and $k' = 1$, so $j' = j + (k'-1)n$. Then (5.1) implies that $(\alpha' \star \beta)_{i'<} < (\alpha' \star \beta)_{j'<} = \alpha' \star \beta$, as desired, unless we are in the exceptional case (5.1)(*). In that case $\alpha = \gamma R^\gamma \gamma C$ is quadratic, γC is maximal, and $\alpha' \star \beta = \gamma \star (R \star \beta)$. By the case $n = 1$, $R \star \beta$ is maximal, and so, by induction on n, $\gamma \star (R \star \beta)$ is maximal. This proves (5.2).

■

Remark. Corollary (5.2) is well known to experts, but we were not able to locate a short direct proof. An alternative proof is given in (10.5) below.

6. Shift dynamics on $\hat{G}_0 \cup G_0C$.

(6.1) **The shift maps σ and σ_α.** We define $\sigma : \hat{G} \longrightarrow \hat{G}$ by

$$\sigma(\alpha) = \alpha_2\alpha_3 \cdots = \alpha_{1<} \tag{1}$$

where

$$\alpha = \alpha_1\alpha_2\alpha_3 \cdots$$

in normal form, *i.e.*, $\alpha_i \in \{L, R\}$ for $i <|\alpha|$, and, if $|\alpha|< \infty, \alpha_{|\alpha|} \in \{L, C, R\}$ and $\alpha_i = 1$, and for $i >|\alpha|$. Thus

$$\begin{cases} \sigma(\alpha) = 1 \iff |\alpha|\leq 1, \\ \alpha = \alpha_1\sigma(\alpha) \ \textit{and} \ |\sigma(\alpha)|=|\alpha|-1 \ \textit{if}\ \alpha \neq 1, \ \textit{and} \\ \sigma^i(\alpha) = \alpha_{i<} \ \textit{for}\ i \geq 0. \end{cases} \tag{2}$$

For $\beta, \gamma \in G$ we define the open interval

$$(\beta, \gamma) = \{\alpha \in G \mid \beta < \alpha < \gamma\}. \tag{3}$$

We claim that

$$\sigma^{-1}\Big((\beta,\gamma)\Big) = (L\beta, L\gamma) \cup (R\gamma, R\beta). \tag{4}$$

This follows since

$$\begin{aligned} \beta < \sigma\alpha < \gamma \quad &\iff \quad L\beta < L\sigma\alpha < L\gamma \\ &\iff \quad R\gamma < R\sigma\alpha < R\beta, \end{aligned}$$

and $\alpha = L\sigma\alpha$ or $R\sigma\alpha$ unless $\alpha \in \{1, C\}$, in which case $\sigma\alpha = 1 \notin (\beta, \gamma)$. It follows from (4) that

(5) *$\sigma : \hat{G}_0 \bigcup G_0C \longrightarrow \hat{G}_0 \bigcup G_0C$ is continuous for the topology defined by the linear order on $\hat{G}_0 \bigcup G_0C$.*

Another consequence of (4) is the following. For $\alpha \in U \subset \hat{G}$ we write

$$\alpha = \mathit{Min}\ U \qquad (\mathit{resp.}, \alpha = \mathit{Max}\ U)$$

if

$$\alpha \leq \beta \quad (\mathit{resp.}, \beta \leq \alpha) \quad \mathit{for\ all}\ \beta \in U.$$

If $U \subset \hat{G}$, $\alpha \in \hat{G}$, $|\alpha|\geq 2$, then

$$\begin{cases} \sigma\alpha = \mathit{Min}\ U \implies \alpha = \begin{cases} \mathit{Min}\ \sigma^{-1}(U) & \mathit{if}\ \alpha_1 = L \\ \mathit{Max}\ \sigma^{-1}(U) & \mathit{if}\ \alpha_1 = R, \end{cases} \\ \textit{and} \\ \sigma\alpha = \mathit{Max}\ U \implies \alpha = \begin{cases} \mathit{Max}\ \sigma^{-1}(U) & \mathit{if}\ \alpha_1 = L \\ \mathit{Min}\ \sigma^{-1}(U) & \mathit{if}\ \alpha_1 = R. \end{cases} \end{cases} \tag{6}$$

Let $\alpha \in \hat{G}_0 \bigcup G_0C$ and $|\alpha| = N \leq \infty$. The **σ-orbit** of α is,

$$(7) \qquad O(\alpha) = \{\alpha_{i<} \mid 0 \leq i < N\}.$$

This linearly ordered set is finite of cardinal N *if* $\alpha \in G_0C$. If $\alpha \in \hat{G}_0$ then $O(\alpha)$ is finite iff α is eventually periodic of period p, *i.e.*, there exists $n_0 \geq 0$ such that $\alpha_{n+p} = \alpha_n$ for all $n > n_0$. If n_0 and p are taken to be minimal then $|O(\alpha)| = n_0 + p$.

We give $O(\alpha)$ the topology induced by the order topology on $\hat{G}_0 \bigcup G_0C$. Define

$$\sigma_\alpha : O(\alpha) \longrightarrow O(\alpha),$$

by $\sigma_\alpha = \sigma$ if $\alpha \in \hat{G}_0$; if $\alpha \in G_0C$ then,

$$(8) \qquad \begin{cases} \sigma_\alpha(\alpha_{i<}) = \sigma(\alpha_{i<}) = \alpha_{(i+1)<} & \textit{for } 0 \leq i < N-1, \textit{ and} \\ \sigma_\alpha(\alpha_{(N-1)<} (= C)) = \alpha_{0<} (= \alpha). \end{cases}$$

Thus, in view of (5),

(9) $\qquad (O(\alpha), \sigma_\alpha)$ *is an ordered dynamical system.*

When $N < \infty$ we put an order $<_\alpha$ (or $<_{\alpha,N}$) on $\mathbb{Z}/N\mathbb{Z}$ so that, with

$$(10) \qquad \mathbb{Z}(\alpha) := (\mathbb{Z}/N\mathbb{Z}, \ +1 \ , <_{\alpha,N})$$

we can conclude the following:

(11) *The map $\alpha_{i<} \mapsto i$ (mod N), for $0 \leq i < N = |\alpha|$, defines an order preserving isomorphism $O(\alpha), \sigma_\alpha) \longrightarrow \mathbb{Z}(\alpha)$ of ordered dynamical systems.*

(6.2) Renormalization of $(O(\alpha), \sigma_\alpha)$. As above let $\alpha \in \hat{G}_0 \bigcup G_0C$ and $|\alpha| = N \leq \infty$. Let n be an integer, with $0 < n \leq N$. Define a linear order $<_\alpha$ (or $<_{\alpha,n}$) on $\mathbb{Z}/n\mathbb{Z}$, by

$$(1) \qquad r <_\alpha s \iff \alpha_{r<} < \alpha_{s<} \ \textit{for } 0 \leq r, s < n, r \neq s.$$

Suppose further that

(2) $\qquad n \mid N \qquad$ (*by convention,* $n \mid \infty$ *for all* $n \geq 1$).

Then we have an equivariant map

$$(3) \qquad \begin{cases} \phi_n = \phi_{\alpha,n} : (O(\alpha), \sigma_\alpha) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1) \\ \phi_n(\alpha_{i<}) = (i \mod n). \end{cases}$$

If ϕ_n is continuous then (by definition) ϕ_n is an n-renormalization.

For $r \in \mathbb{Z}/n\mathbb{Z}$ we put

$$(4) \qquad O(\alpha)_r = \phi_n^{-1}(r) = \{\alpha_{i<} \mid 0 \leq i < N, (i \mod n) = r\}.$$

The following conditions are clearly equivalent:

(a) $\phi_n : O(\alpha) \longrightarrow (\mathbb{Z}/n\mathbb{Z},\ \textit{ordered by}\ <_{\alpha,n})$ is weak order preserving. (Hence each fiber $O(\alpha)_r$ is an $O(\alpha)$-interval.)

(b) If $r, s \in \mathbb{Z}/n\mathbb{Z}$ then $r <_{\alpha,n} s$ implies $\alpha_{i<} < \alpha_{j<}$ whenever $(i \bmod n) = r$ and $(j \bmod n) = s$.

If ϕ_n is continuous, then (a) and (b) just say that

(c) ϕ_n is an interval n–renormalization.

In (6.3) below we show that, in the presence of conditions (a) and (b), continuity of ϕ_n can fail only in very special circumstances.

We define

$$
\begin{aligned}
IRen(\alpha) &= IRen(O(\alpha), \sigma_\alpha) \\
&= \{n \geq 1 : \ n \,\big|\, |\alpha| \ \ \textit{and}\ \phi_n : (O(\alpha), \sigma_\alpha) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1) \\
&\qquad \textit{is an interval renormalization}\ \}.
\end{aligned} \tag{5}
$$

It follows from Theorem (I, (2.6)) that:

$$IRen(\alpha)\ \textit{is totally ordered by divisibility.} \tag{6}$$

When $|\,\alpha\,| < \infty$ we have $IRen(\alpha) = IRen(\mathbb{Z}(\alpha))$ (cf. (6.1)(11)).

(6.3) Remarks on the continuity of ϕ_n. First, *suppose that ϕ_n is not continuous.* Then $O(\alpha)$ must be infinite, so $\alpha \in \hat{G}_0$ and $\sigma_\alpha = \sigma\,|_{O(\alpha)}$. The fibers $O(\alpha)_r$ $(r \in \mathbb{Z}/n\mathbb{Z})$ of ϕ_n form a finite partition of $O(\alpha)$, and $\sigma^{-1}(O(\alpha)_r) = O(\alpha)_{r-1}$. It follows that each $O(\alpha)_r$ is neither open nor closed. *Now further, suppose* (a) *and* (b) *of* (6.2), so that each $O(\alpha)_r$ is an $O(\alpha)$-interval. The only way for an interval to be neither open nor closed is for it to have a Min or a Max but not both. Thus each $O(\alpha)_r$ has a unique extreme (Min or Max) element $\alpha_{i_r<}$, $0 \leq i_r$, $(i_r \bmod n) = r$. Suppose that $i_r > 0$, so $\sigma(\alpha_{(i_r-1)<}) = \alpha_{i_r<}$. It follows from (6.1)(6) that $\alpha_{(i_r-1)<}$ is an extreme element of $\sigma_\alpha^{-1}(O(\alpha)_r) = O(\alpha)_{r-1}$. Thus:

$$\textit{For}\ i_r > 0, i_{r-1} = i_r - 1.$$

Since there are exactly n such extreme elements it follows that $i_r = r$, *i.e.*,

$$\alpha_{r<}\ \textit{is the extreme element of}\ O(\alpha)_{(r \bmod n)}\ \textit{for}\ 0 \leq r < n.$$

Write

$$\mu(r) = \begin{cases} 1 & \textit{if}\ \ \alpha_{r<} \ = \ \textit{Max}\ O(\alpha)_{(r \bmod n)} \\ -1 & \textit{if}\ \ \alpha_{r<} \ = \ \textit{Min}\ O(\alpha)_{(r \bmod n)}. \end{cases}$$

Then it is easily seen that

$$\mu(r) = \rho(\alpha_{\leq r}) \cdot \mu(0).$$

Figure 10.

Let us list $\mathbb{Z}/n\mathbb{Z}$ according to $<_\alpha$:

$$r_1 <_\alpha r_2 <_\alpha \cdots <_\alpha r_{n-1} <_\alpha r_n.$$

We cannot have $\mu(r_\nu) = 1$ and $\mu(r_{\nu+1}) = -1$. (See Figure 10.)

For in this case $O(\alpha)_{r_\nu}$ would be open in the order topology on $O(\alpha)$, contrary to assumption. Hence we must have

$$\begin{aligned} \mu(r_1) = \cdots = \mu(r_s) &= -1 \\ \mu(r_{s+1}) = \cdots = \mu(r_n) &= 1 \end{aligned}$$

for some s, $0 \le s \le n$.

7. The $\star$-product renormalization theorem.

(7.1) **Theorem.** *Let $\alpha = \alpha' C$ be maximal, $\alpha' \in G_0$, $|\alpha| = n$, and let $\beta \in \hat{G}$. Put*

$$\Delta = \alpha' \star \beta.$$

Then

$$IRen(\Delta) = IRen(\alpha) \cup n \cdot IRen(\beta), \tag{1}$$

unless $\alpha = \gamma R^\gamma \gamma C$ is quadratic and β terminates with $L^M, 0 < M \le \infty$. In the latter case we have

$$\Delta = \gamma \star (R \star \beta) \tag{2}$$

with γC maximal , $|\gamma C| = m = n/2$, and

$$IRen(\Delta) = IRen(\gamma C) \cup m \cdot IRen(R \star \beta). \tag{3}$$

Proof of (7.1). We have a commutative diagram,

$$\begin{array}{ccc} (O(\Delta), \sigma_\Delta) & & \\ \psi \downarrow & \searrow \phi_{\Delta,n} & \\ (O(\alpha), \sigma_\alpha) & \xrightarrow[\phi_{\alpha,n}]{} & (\mathbb{Z}/n\mathbb{Z}, +1) \end{array} \tag{4}$$

where $\psi(\Delta_{i<}) = \alpha_{r<}$ when $0 \le r < n$ and $i \equiv r \bmod n$. Here $\phi_{\alpha,n}$ is an order preserving isomorphism, using $<_{\alpha,n}$ on $\mathbb{Z}/n\mathbb{Z}$ (cf. (6.1)(11)).

For $0 \le r < n$ put

$$O(\Delta)_r = \phi_{\Delta,n}^{-1}(r) = \{\Delta_{(r+kn)<} \mid 0 \le k <|\beta|\}$$

and

$$\begin{array}{rcl} \sigma_r & = & \sigma_\Delta^n\,|_{O(\Delta)_r} \colon O(\Delta)_r \longrightarrow O(\Delta)_r, \\ \sigma_r(\Delta_{i<}) & = & \Delta_{(i+n)<} \quad \mathit{for}\ 0 \le i \le n(|\beta|-1). \end{array}$$

Define

$$\psi_r : (O(\beta), \sigma_\beta) \longrightarrow (O(\Delta)_r, \sigma_r)$$

by $\psi_r(\beta_{k\le}) = \alpha'_{r<}\beta_k^{\alpha'}(\alpha' \star \beta_{k<}) = \Delta_{(r+(k-1)n)<}$ for $0 < k \le |\beta|$. Putting $\delta = \alpha'_{r<}$ we have

$$\begin{array}{rcl} \Delta_{(r+(k-1)n)<} < \Delta_{(r+(k'-1)n)<} & \Longleftrightarrow & \beta_k^{\alpha'}(\alpha' \star \beta_{k<}) <^{\delta} \beta_{k'}^{\alpha'}(\alpha' \star \beta_{k'<}) \\ & \Longleftrightarrow & (\alpha' \star \beta_{k\le}) <^{\delta\alpha'} (\alpha' \star \beta_{k'\le}) \\ & \Longleftrightarrow & \beta_{k\le} <^{\delta\alpha'} \beta_{k'\le} \end{array}$$

where the last equivalence follows from ((4.2)(a)). Thus ψ_r is an isomorphism of dynamical systems that either preserves or reverses order (according to whether $\rho(\delta\alpha') = 1$ *or* -1, respectively). It follows that

(5) $\qquad \mathit{IRen}(O(\Delta)_r, \sigma_r) = \mathit{IRen}(O(\beta), \sigma_\beta) = \mathit{IRen}(\beta) \quad \mathit{for}\ 0 \le r < n.$

Consider the conditions,

(6) $\qquad \psi : O(\Delta) \longrightarrow O(\alpha)$ *is weak order preserving,*

and

(7) $\qquad$ *Each fiber* $O(\Delta)_r = \psi^{-1}(\alpha_{r<})$ *is an open-closed interval of* $O(\Delta)$, *hence* ψ *is continuous.*

These conditions imply that, in diagram (4),

(8) $\qquad \phi_{\Delta,n}$ *is an interval* n*-renormalization.*

In view of (5), it follows from (I, (3.1)) that (8) implies condition (1). Now it follows from Theorem (5.1) that we have conditions (6) and (7), and hence also (1), except in the case when α is quadratic in the sense of (2.6), *i.e.*,

(9) $$\left\{ \begin{array}{l} \alpha = \gamma R^\gamma \gamma C, \gamma C \text{ is maximal }, \\ |\gamma C| = m = n/2, \text{ and further} \\ \beta \text{ terminates with } L^M, 0 < M \le \infty. \end{array} \right.$$

In this case it further follows from (5.1) that (2) holds true and $R\star\beta$ terminates with $R \star L^M = R^{2M}$. Thus we can apply the discussion for the non-quadratic case to γC and $R\star\beta$ in place of α and β, and conclude that $\phi_{\Delta,m}$ is an interval m-renormalization, and hence (3).

This concludes the proof of (7.1).

∎

The following result is a sort of dynamical converse to Theorem (7.1).

(7.2) **Theorem.** *Let (J, f) be a unimodal map on $J = [-1, 1]$, as in (1.0), with maximum $M = f(0) = f(C)$. Let $K = \overline{O_f(M)}$, the f-orbit closure of M, and let*

$$\Delta = Af^*(M),$$

the kneading sequence of M. Then Δ is a maximal element of $\hat{G}_0 \cup G_0 C$ (cf. (1.7)(4)) and

$$\text{(1)} \qquad IRen(K, f) = IRen(\Delta).$$

If $n \in IRen(K, f)$, $n > 1$, then

$$\Delta = \alpha' \star \beta$$

for some $\alpha' \in G_0$, $|\alpha'| = n - 1$.

Proof. The relation (1) is proved in Proposition (9.4) below.

Let $\phi : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$ be an interval n-renormalization, with fibers $K_r = \phi^{-1}(r)$ $(r \in \mathbb{Z}/n\mathbb{Z})$, and normalized so that $\phi(M) = 1$. The K_r are K-intervals and $f^{-1}(K_r) = K_{r+1}$. Let J_r denote the J-interval spanned by K_r.

Case 1. $C \notin \bigcup_{r \in \mathbb{Z}/n\mathbb{Z}} J_r$. In this case each J_r is contained in either L or R, say

$$J_r \subset \alpha_r \in \{L, R\}.$$

It follows then that $\Delta = Af^*(M)$ is periodic,

$$\begin{aligned} \Delta &= (\alpha_1 \alpha_2 \cdots \alpha_n)^\infty \\ &= \alpha' \star \beta \end{aligned}$$

where $\alpha' = \alpha_1 \alpha_2 \cdots \alpha_{n-1}$ and $\beta = (\alpha_n^{\alpha'})^\infty$.

Case 2. $C \in \bigcup_{r \in \mathbb{Z}/n\mathbb{Z}} J_r$. Then, since $f(C) = M \in K_1 \subset J_1$, it follows that $C \in J_n$. On the other hand, for the same reason as above, we have

$$J_r \subset \alpha_r \in \{L, R\} \quad \textit{for } r = 1, \ldots, n-1.$$

This time, putting $\alpha' = \alpha_1 \alpha_2 \cdots \alpha_{n-1}$ we have $\Delta = \alpha' \gamma_1 \alpha' \gamma_2 \alpha' \gamma_3 \cdots$ for some $\gamma_1, \gamma_2, \gamma_3 \ldots$. Putting $\beta_p = \gamma_p^{\alpha'}$, we then have $\Delta = \alpha' \star \beta$, where $\beta = \beta_1 \beta_2 \beta_3 \cdots$.

∎

8. Iterated $\star$-products.

(8.1) **The elements** $\alpha(q)$. For each integer $q > 1$ we put

$$\alpha'(q) = RL^{q-2}, \ \textit{and} \ \alpha(q) = \alpha'(q)C. \tag{1}$$

The linear order $<_{\alpha(q)}$ on $\mathbb{Z}/q\mathbb{Z}$ is given (cf. (2.1)(12)) by

$$q-1 <_{\alpha(q)} q-2 <_{\alpha(q)} \cdots <_{\alpha(q)} 1 <_{\alpha(q)} 0, \tag{2}$$

which is the reverse of the natural order. It follows therefore from Example (I, (2.5)) that

$$\begin{cases} \alpha(q) \ \textit{is maximal and} \\ \textit{IRen}(\alpha(q)) = \{1, q\}. \end{cases} \tag{3}$$

Moreover it is easy to see that $\alpha(q)$ is not quadratic, *i.e.*, not of the form $\gamma R^{\gamma}\gamma C$ (cf. (2.6)). Hence it follows from (6.1) that

$$\textit{IRen}(\alpha'(q) \star \beta) = \{1\} \bigcup q \cdot \textit{IRen}(\beta) \tag{4}$$

for all $\beta \in \hat{G}$. Moreover, from (6.1), (6.2) and (4.1)(12) we have:

$$\beta = \beta' C \in G_0 C \ \textit{is maximal} \Longrightarrow \begin{cases} \beta' \star \alpha(q) \in G_0 C \ \textit{is maximal}, \\ |\beta' \star \alpha(q)| = |\beta| \, q, \\ \textit{IRen}(\beta' \star \alpha(q)) = \textit{IRen}(\beta) \cup \{|\beta| \, q\}. \\ \beta' \star \alpha(q) \ \textit{is not quadratic}. \end{cases} \tag{5}$$

(8.2) **The elements** $\alpha(q_1, q_2, q_3, \ldots)$. For $\alpha \in G_0C$ we define $\alpha' \in G_0$ by $\alpha = \alpha' C$. For $\beta \in G_0 C$ we have

$$\alpha' \star \beta = (\alpha' \star \beta')(\alpha' \star C) = (\alpha' \star \beta')\alpha' C,$$

and so

$$(\alpha' \star \beta)' = (\alpha' \star \beta')\alpha'. \tag{1}$$

Let $\gamma \in G_0 C$. Then from (4.1)(10) and (1) we have

$$\begin{aligned} \alpha' \star (\beta' \star \gamma) &= (\alpha' \star \beta')\alpha' \star \gamma \\ &= (\alpha' \star \beta)' \star \gamma. \end{aligned} \tag{2}$$

Let

$$\mathbf{q} = (q_1, q_2, q_3, \ldots)$$

be a sequence of integers $q_i \geq 2$. For $n \geq 1$ put

$$\begin{cases} \mathbf{q}_{\leq n} &= (q_1, q_2, \ldots, q_n); \\ m_n &= \mathbf{q}^{[n]} := q_1 q_2 \ldots q_n. \end{cases} \tag{3}$$

We define

$$\alpha(q_1, \ldots, q_n) = \alpha'(q_1, \ldots, q_n)C \in G_0 C$$

inductively, starting, for $n = 1$, with $\alpha(q_1)$ as in (8.1):

$$\alpha(q_i) = RL^{q_i - 2}C. \tag{4}$$

For $n > 1$ we put

$$\alpha(q_1, q_2, \ldots, q_n) = \alpha'(q_1) \star \alpha(q_2, \ldots, q_n). \tag{5}$$

We claim that, for $1 \leq i < n$,

$$\alpha(q_1, q_2, \ldots, q_n) = \alpha'(q_1, \ldots q_i) \star \alpha(q_{i+1}, \ldots, q_n). \tag{6}$$

For $i = 1$ this follows from the definition (5). Suppose that we want to verify (6) for $i > 1$, assuming inductively the analogue for indices $< i$. Then

$$\begin{aligned} \alpha(q_1, \ldots, q_n) &= \alpha'(q_1, \ldots, q_{i-1}) \star \alpha(q_i, \ldots, q_n) \qquad \textit{(by induction)} \\ &= \alpha'(q_1, \ldots, q_{i-1}) \star (\alpha'(q_i) \star \alpha(q_{i+1}, \ldots, q_n)) \quad \textit{(by (5))} \\ &= (\alpha'(q_1, \ldots, q_{i-1}) \star \alpha(q_i))' \star \alpha(q_{i+1}, \ldots, q_n) \quad \textit{(by (2))} \\ &= \alpha'(q_1, \ldots, q_i) \star \alpha(q_{i+1}, \ldots, q_n) \quad \textit{(by induction).} \end{aligned}$$

In particular,

$$\alpha(q_1, q_2, \ldots, q_n) = \alpha'(q_1, \ldots q_{n-1}) \star \alpha(q_n). \tag{7}$$

It follows inductively from (7) and (8.1)(5) that

$$\begin{cases} \alpha(q_1, q_2, \ldots, q_n) \in G_0 C \ \textit{is maximal and nonquadratic}, \\ |\alpha(q_1, q_2, \ldots q_n)| = m_n, \ \textit{and} \\ \mathit{IRen}(\alpha(q_1, q_2, \ldots q_n)) = \{1, m_1, m_2, \ldots m_n\}. \end{cases} \tag{8}$$

To consolidate notation, let us now write

$$\begin{aligned} \Delta(n) &= \alpha(\mathbf{q}_{\leq n}) = \alpha(q_1, \ldots, q_n) \\ &= \Delta'(n)C. \end{aligned} \tag{9}$$

For $n < N$ it follows from (6) that

$$\begin{aligned} \Delta(N) &= \Delta'(n) \star \Delta(n, N], \ \textit{where} \\ \Delta(n, N] &= \alpha(q_{n+1}, \ldots q_N). \end{aligned} \tag{10}$$

In particular,

$$\Delta(N)_{<m_n} = \Delta'(n) = \Delta(n)_{<m_n}. \tag{11}$$

Suppose now that the sequence $\mathbf{q}$ is infinite. Then it follows from (11) that there is a limit

$$\Delta = \varinjlim_n \Delta(n) = \alpha(\mathbf{q}), \tag{12}$$

defined by

$$\Delta_{<m_n} = \Delta'(n) \text{ for all } n \geq 1.$$

It follows further from (10) that for $n \geq 1$,

$$\Delta = \Delta'(n) \star \varinjlim_{N} \Delta(n, N]. \tag{13}$$

We can rewrite (13) more explicitly as:

$$\alpha(\mathbf{q}) = \alpha'(\mathbf{q}_{\leq n}) \star \alpha(\mathbf{q}_{n<}), \tag{14}$$

where $\mathbf{q}_{n<} = (q_{n+1}, q_{n+2}, \ldots)$. It follows from (14) and Theorem (7.1) that

$$IRen(\alpha(\mathbf{q})) = IRen(\alpha(\mathbf{q}_{\leq n})) \bigcup m_n \cdot IRen(\alpha(\mathbf{q}_{<n}))$$

From this it follows that the elements less than or equal to m_n in $IRen(\alpha(\mathbf{q}))$ are just $IRen(\alpha(\mathbf{q}_{\leq n})) = \{1, m_1, m_2, \ldots m_n\}$. Therefore:

$$IRen(\alpha(\mathbf{q})) = \{1, m_1, m_2, \ldots, m_n, \ldots\}. \tag{15}$$

Equivalently, with the notation of (I, (2.7)),

$$\mathbf{q}(O(\alpha(\mathbf{q})), \sigma_{\alpha(\mathbf{q})}) = \mathbf{q}. \tag{16}$$

9. Realization by unimodal maps.

(9.1) **Kneading sequences.** As in section 1, let f be a unimodal map on $J = [-1, 1]$, with maximum $M = f(0)$, and with kneading sequence

$$K(f) = \mathbf{Af}^{\star}(M) \in \widehat{G_0} \cup G_0 C.$$

Define

$$\kappa(f) = \begin{cases} K(f) & \text{if } K(f) \in \widehat{G_0} \\ \kappa' \star L^{\infty} = (\kappa' L^{\kappa'})^{\infty} & \text{if } K(f) = \kappa' C \in G_0 C. \end{cases}$$

For $\alpha, \beta \in \hat{G}$ write

$$\alpha << \beta \iff \alpha_{i<} < \beta \text{ for all } i \geq 0.$$

(9.2) **Theorem.** *([CEc], Theorem II.3.8) Let f be a unimodal map on $J = [-1, 1]$ and let $\alpha \in \widehat{G_0} \cup G_0 C$ satisfy*

$$\mathbf{Af}^{\star}(-1) \leq \alpha << \kappa(f).$$

Then $\alpha = \mathbf{Af}^{\star}(x)$ *for some* $x \in J$.

(9.3) **Interval renormalization, symbolic and real.** Let f as above be a unimodal map on $J = [-1, 1]$, and $x \in J$, with f-orbit

$$O_f(x) = \{x, f(x), f^2(x), \ldots\},$$

and itinerary

$$\alpha = \mathbf{Af}^{\star}(x) \in \widehat{G_0} \cup G_0C.$$

Then (cf. (1.7)(4)) we have a weak order preserving map $\mathbf{Af}^{\star} : O_f(x) \longrightarrow \widehat{G_0} \cup G_0C$.

Case: $\alpha \in \widehat{G_0}$.

Then

$$\mathbf{Af}^{\star} : (O_f(x), f) \longrightarrow (O(\alpha), \sigma)$$

is a weak order preserving surjection of ordered dynamical systems where σ is the shift operator, $\sigma(\alpha_{i<}) = \alpha_{(i+1)<}$. If $O(\alpha)$ is infinite, *i.e.*, if α is not eventually periodic, then $\mathbf{Af}^{\star}$ above is bijective. Consequently (cf. (I, (2.4)(5) and (6)),

(1) $\quad$ *IRen*$(\alpha) \subset$ *IRen*$(O_f(x), f)$, *with equality if* $O(\alpha)$ *is infinite.*

Case: $\alpha = \alpha' C \in G_0C$.

If $|\alpha| = n$, this implies that $f^{n-1}(x) = 0$, hence $f^n(x) = M$. Assume further that,

(2) $\quad$ x *is maximal in* $O_f(x)$, *hence* α *is maximal.*

From (2) and the condition $f^n(x) = M$ we conclude that $x = M$, hence $O_f(x) = O_f(M) = O_f(0)$ is the critical orbit, which is periodic of period n. Further $\alpha = K(f)$, the kneading sequence, and

$$\mathbf{Af}^{\star} : (O_f(M), f) \longrightarrow (O(K(f)), \sigma_{K(f)})$$

is an order preserving isomorphism of finite ordered dynamical systems. In particular *IRen*$(K(f)) =$ *IRen*$(O_f(M), f)$ in this case.

(9.4) **Proposition.** *Let* f *be a unimodal map on* $J = [-1, 1]$, $x \in J$, $O_f(x)$ *the* f*-orbit of* x, $K = \overline{O_f(x)}$ *its closure, and* $\alpha = \mathbf{Af}^{\star}(x) \in \widehat{G_0} \cup G_0C$. *Assume that*

(1) $\quad$ *IRen*(α) *is infinite.*

Then K *is a minimal* f*-invariant Cantor set, and*

(2) $\quad$ *IRen*$(K, f) =$ *IRen*(α).

Proof. Assumption (1) implies that $O(\alpha)$ is infinite, and

$$\mathbf{Af}^* : (O_f(x), f) \longrightarrow (O(\alpha), \sigma)$$

is an order preserving isomorphism of dynamical systems, whence

$$IRen(\alpha) = IRen(O_f(x), f) \supset IRen(K, f)$$

(cf. (I, (2.4)(6)). To show that the latter is an equality we must show that an interval n-renormalization

$$\phi : (O_f(x), f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1)$$

extends to (K, f). Put $L = O_f(x)$ and $L_r = \phi^{-1}(r)$ for $r \in \mathbb{Z}/n\mathbb{Z}$. The various L_r are disjoint closed L-intervals, and $f(L_r) \subset L_{r+1}$ for all $r \in \mathbb{Z}/n\mathbb{Z}$. Put $K_r = \overline{L_r}$. The K_r are closed K-intervals whose union is K and $f(K_r) \subset K_{r+1}$ for all $r \in \mathbb{Z}/n\mathbb{Z}$. If we show that the K_r are pairwise disjoint, then ϕ extends to an interval n-renormalization of K, defined by $\phi^{-1}(r) = K_r$. Say $r \neq s$ and $K_r \cap K_s \neq \emptyset$. Since L_r and L_s are disjoint L-intervals, $K_r \cap K_s$ can contain at most one point, say y. Since $K_r \cap K_s$ is f^n-invariant, $f^n(y) = y$, so y is f-periodic.

Since, by assumption, $IRen(\alpha)$ is infinite, $(O_f(x), f) = (L, f)$ admits an interval nm-renormalization for some $m > 1$. The fibers of the latter partition L_r in to m intervals, which are cyclically permuted by f^n. But then the one of these intervals whose closure contains y ($K_r \cap K_s = \{y\}$) is mapped by f^n to an interval at positive distance from the fixed point y of f^n. This violates the continuity of f on J.

We have now established (2), so (K, f) is infinitely interval renormalizable with the dense orbit $O_f(x)$. It follows therefore from Theorem (I, (4.1)) that (K, f) is minimal, and K is a Cantor set (cf. (B.9)).

(9.5) C^1**-families of unimodal maps**. A unimodal map f on $J = [-1, 1]$ is called C^1**-unimodal** if it is C^1 and $f'(x) \neq 0$ for $x \neq 0$. The C^1-metric on the space of such maps is given by

$$|f - g|_{C^1} = Sup_{x \in J} \Big(|f(x) - g(x)| + |f'(x) - g'(x)| \Big).$$

Let $t \mapsto f_t$ be a curve in the space of C^1-unimodal maps. We quote the following "intermediate value theorem for kneading sequences" from [MilTh] (see also [CEc], Theorem III,1.)

(9.6) **Theorem.** *Say* $t_0 < t_1$ *and* $\alpha \in \widehat{G_0} \cup G_0 C$ *is maximal and* $K(f_{t_0}) \leq \alpha \leq K(f_{t_1})$*. Then* $\alpha = K(f_t)$ *for some* $t \in [t_0, t_1]$.

(9.7) **Corollary.** *Assume further that*

$$(1) \qquad IRen(\alpha) \text{ is infinite.}$$

Then if $K = \overline{O_{f_t}(1)}$ *is the critical orbit closure,* K *is a minimal* f_t*-invariant Cantor set, and* $IRen(K, f_t) = IRen(\alpha)$.

This follows from (9.6) and (9.4).

(9.8) **The quadratic family** is defined by

$$(1) \qquad f_t(x) = 1 - tx^2 \qquad (0 < t \leq 2).$$

We have, for $0 < t < 1$,

$$(2) \qquad K(f_t) = R^\infty < K(f_2) = RL^\infty.$$

If $\alpha \in \widehat{G_0}$ is maximal and $\alpha \neq L^\infty, R^\infty$ or RL^∞ then it follows from (2.1)(12) that $\alpha = RL^a R \cdots$ for some $a > 0$, and hence $R^\infty < \alpha < RL^\infty$. Thus, in view of (2)

$$(3) \qquad \begin{cases} \alpha \in \widehat{G_0} \text{ is maximal and } \alpha \neq L^\infty & \Longrightarrow \\ 0 < t_0 < 1, \quad K(f_{t_0}) \leq \alpha \leq K(f_2), & \Longrightarrow \\ \alpha = K(f_t) \text{ for some } t \in [t_0, 2]. \end{cases}$$

(9.9) **Corollary.** *Let* $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ *be an infinite sequence of integers* $q_i \geq 2$, *and let* $\alpha = \alpha(\mathbf{q}) = \alpha(q_1, q_2, q_3, \ldots)$ *be as in (8.2)(12). For some* $t_{\mathbf{q}} \in [1, 2]$ *and* $f_{t_{\mathbf{q}}}(x) = 1 - t_{\mathbf{q}} x^2$, *the critical orbit closure* $K = \overline{O_{f_{t_{\mathbf{q}}}}(1)}$ *is a minimal* $f_{t_{\mathbf{q}}}$*-invariant Cantor set, and*

$$\mathbf{q}(K, f_{t_{\mathbf{q}}}) = \mathbf{q}.$$

Proof. By (8.2)(8) and (16), α is maximal and $\mathbf{q}(\alpha) = \mathbf{q}$. From (9.8)(3) we get $t_{\mathbf{q}} \in [1, 2]$ with $K(f_{t_{\mathbf{q}}}) = \alpha$. Now (9.9) follows from (9.7).

■

10. A permutation formulation.

(10.0) A permutation $s \in S_n$ can be extended using linear interpolation, to a piecewise linear map f_s from $[0, n+1]$ to itself. Assume that s is an n-cycle, and put $C = s^{-1}(n)$. Then $[n] = \{1, 2, \ldots, n\}$ is an orbit of s (or f_s). Assume further that s (*i.e.*, f_s) is (+)-unimodal. Then we have the kneading sequence

$$K(s) := K(f_s) = \mathbf{Af}^*(n) = \alpha C,$$

where

$$\alpha = \alpha_1 \cdots \alpha_{n-1} \in G_0.$$

It is easily shown (cf. (10.4)) that $s \mapsto K(s)$ defines a bijection from the set of (+)-unimodal n-cycles to the set of maximal elements of length n in G_0C.

This permits us to reformulate some of the preceding theory in terms of permutations. In particular we give new proofs, from this perspective, that if $\alpha C \in G_0C$ is maximal, and $\beta = \alpha M$, with $M = L$ or R, then β^∞ is maximal. Further, if $\beta = \gamma^k$ with $k > 1$, then $k = 2$ and this can happen for at most one choice of $M = L$ or R.

(10.1) **Interpolation.** Let $A = \{a_1 < a_2 < \cdots < a_n\}$ be a finite subset of a real interval $J = [a_0, a_{n+1}]$, and $s : A \longrightarrow J$ a map. The **minimal interpolation** of s is the map $f_s : J \longrightarrow J$ such that $f_s|_A = s$, $f_s|_{[a_{i-1}, a_i]}$ is affine linear $(0 < i \leq n+1)$, and $f_s(a_0)$, $f_s(a_{n+1}) \in \{a_0, a_{n+1}\}$ are chosen so that a_1 and a_n are not turning points of f_s. We call s (and f_s) m-**modal** if f_s has m turning points (which necessarily belong to $\{a_2, \ldots, a_{n-1}\}$, so $m \leq n-2$).

Consider the symmetric group S_n of permutations of

$$[n] = \{1, 2, \ldots, n\}.$$

Each $s \in S_n$ then has a minimal interpolation

$$f_s : J_n = [0, n+1] \longrightarrow J_n,$$

as above, and s is m-modal for some m, $0 \leq m \leq n-2$.

(10.2) **Kneading sequences.** Let $s \in S_n$ be an n-cycle which is unimodal, and whose turning point $C \in [n]$ is the maximum, $f_s(C) = s(C) = n$. We say briefly, that s is **(+)-unimodal**. Then we have the kneading sequence (cf. (1.0))

$$\begin{array}{ll} K(s) := K(f_s) = Af_s^*(n) = \alpha C & \textit{where} \\ \alpha = \alpha_1 \cdots \alpha_{n-1} = K'(s) \in G_0 & \end{array} \tag{1}$$

The following construction shifts the turning point C slightly to the left (denoted $-$) or right (denoted $+$): Put $C^\pm = C \pm \frac{1}{3}$, define $s^\pm(C^\pm) = n \pm \frac{1}{3}$, and then let $f_{s^\pm}$ be the minimal interpolation of $s^\pm : [n] \cup \{C^\pm\} \longrightarrow J_n$; here we make one choice (not both) of $+$ or $-$. (See Figure 11.) Then it is clear that the itineraries of n for $f_{s^\pm}$ take the forms,

$$\left\{ \begin{array}{lcl} K(s^+) & := & Af_{s^+}^*(n) = (\alpha L)^\infty, \\ K(s^-) & := & Af_{s^-}^*(n) = (\alpha R)^\infty. \end{array} \right. \tag{2}$$

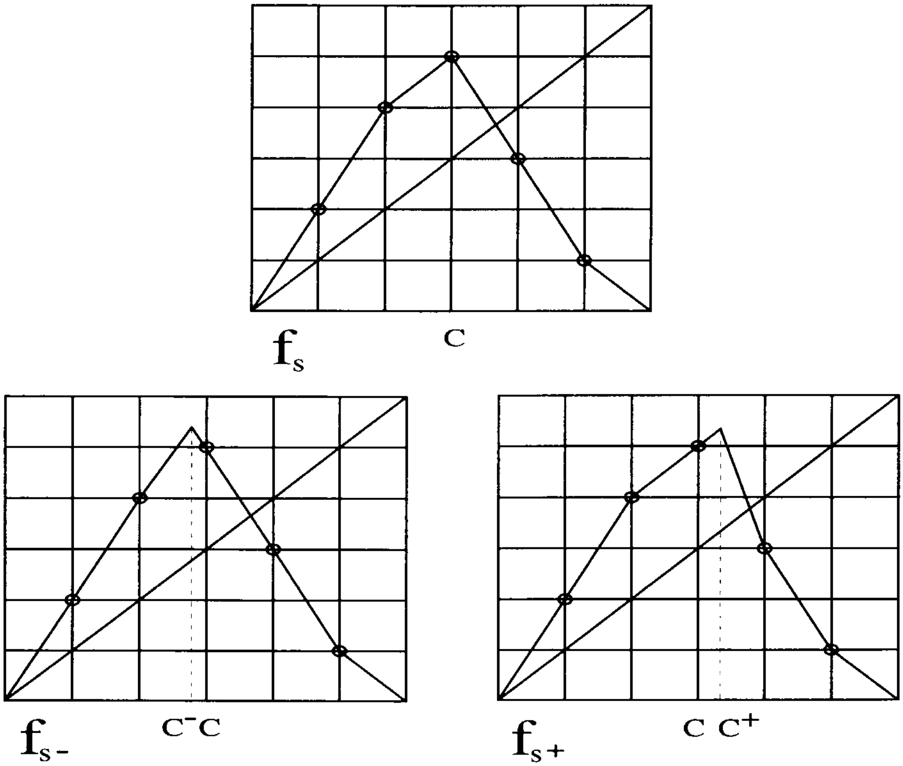

Figure 11. Construction of maps with specified kneading sequence, according to (10.2). Here the permutation is $s = (12435)$.

(10.3) **The (+)-unimodal n-cycle $s \in S_n$ can be recovered from $K(s) = \alpha C$.** In fact, for $1 \leq i \leq n$, put

$$(1) \quad \begin{cases} x_i = (\alpha C)_{\geq i} = \alpha_i \cdots \alpha_{n-1} C \quad (= C \textit{ if } i = n) \\ O(\alpha C) = \{x_1, \ldots, x_n\}, \textit{ and} \\ \sigma = \sigma_{\alpha C} : O(\alpha C) \longrightarrow O(\alpha C), \qquad \textit{defined by} \quad \sigma(x_i) = x_{i+1} \pmod{n}, \end{cases}$$

as in (6.1)(7) and (8). By (1.7)(4), the map $Af_s^* : J_n \longrightarrow \hat{G}_0 \coprod G_0 C$ is weak order preserving, and so it induces an order preserving equivariant bijection

$$(2) \qquad ([n], s) \longrightarrow (O(\alpha C), \sigma).$$

Thus, if we use the total order on $G_0 C$ to make an order preserving identification of $\{x_1, \ldots, x_n\}$ with $[n]$, then the n-cycle $\sigma = (x_1, \ldots, x_n)$ is converted to the n-cycle $s = (n, \ldots, C)$.

Conversely, given any maximal element $\alpha C \in G_0 C$ of length n, we can define $(O(\alpha C), \sigma)$ as above, and use the total order on $G_0 C$ to identify $O(\alpha C)$ with

$[n]$, and $\sigma = (x_1, \ldots, x_n)$ with some $s = (n, \ldots, C) \in S_n$. Using the properties of the order on G_0C, one can verify that this n-cycle s is (+)-unimodal, and then, evidently, that $K(s) = \alpha C$.

In summary, then:

(10.4) **Proposition.** *The map $s \mapsto K(s)$ defines a bijection from the set of (+)-unimodal n-cycles $s \in S_n$ to the set of maximal elements αC of length n in G_0C.*

(10.5) **Theorem.** *Let $\alpha C \in G_0C$ be maximal, and $\beta = \alpha M$, $M \in \{L, R\}$.*

(a) β^∞ is maximal.
(b) If $\beta = \gamma^k$, $k > 1$, then $k = 2$.
(c) αL and αR cannot both be squares.

Remark. Part (a) follows from (5.2), and parts (b) and (c) from (2.3). We shall give direct proofs here.

Proof of (a) Let $s \in S_n$ be the (+)-unimodal n-cycle with $K(s) = \alpha C$ (10.4). Let $f = f_{s\pm}$ (+ if $M = L$, and $-$ if $M = R$), and $I = Af^*$. Then, by (10.2)(2), $I(n) = (\alpha M)^\infty = \beta^\infty$. Since n is maximal in $[n]$ (= the f-orbit of n), and since I is weak order preserving, it follows that β^∞ is maximal.

Proof of (b). Write $n = kr$. Then $I(n) = I(m)$ for the k values of m in the s^r-orbit of n. Since I is weak order preserving, I is constant on the interval spanned by the s^r-orbit of n, hence I is constant on the interval $H = [n - k + 1, n]$. It follows that s^q is monotone on H for all q. But then the turning point $C = s^{-1}(n)$ could never be an interior point of $s^q H$. Hence $k \leq 2$.

Proof of (c). If $k = 2$, then $I(n-1) = I(n)$. Choose m so that $s^m(n-1) = C$. Then C and $s^m(n)$ must lie on the same side of the critical point $C^\pm$ of $f = f_{s\pm}$, and this can happen for only one choice of + or $-$, hence αM can be a square for at most one choice of $M \in \{L, R\}$.

(10.6) **The $*$-product of permutations.** Let $s \in S_n$ be a (+)-unimodal n-cycle with $K(s) = \alpha C$, with $C = s^{-1}(n)$. We shall write C_s in place of C in what follows.

Let $t \in S_m$ and let $r \in S_m$ be the **flip**, $r(b) = m + 1 - b$ for $b \in [m]$. We define the permutation $s \star t$ of $[n] \times [m]$ by

$$(1) \qquad (s \star t)(a, b) = (s(a), u_a(b))$$

where

(2) $$u_a = \begin{cases} Id & 1 \leq a < C_s \\ tr^{n-C_s} & a = C_s \\ r & C_s < a \leq n \end{cases}$$

The projection defines an equivariant map

(3) $$([n] \times [m], s \star t) \longrightarrow ([n], s).$$

On the fibers we have

(4) $$(s \star t)^n(a, b) = (a, r^{\rho(a)} t r^{\rho(a)}(b)),$$

where, if $a = s^q(C_s)$, for $0 \leq q < n$, then

(5) $$\rho(a) = \#\{i \mid 0 < i < q, s^i(C_s) > C_s\}.$$

We linearly order $[n] \times [m]$ by the lexicographic order, and so identify $[n] \times [m]$ with $[nm]$, the correspondence being

(6) $$(a, b) \mapsto (a-1)m + b.$$

This permits us to view $s \star t \in S_{nm}$, with formula (2) then taking the form,

(7) $$(s \star t)((a-1)m + b) = (s(a) - 1)m + u_a(b),$$

where u_a is given by (3).

Suppose that t is (+)-unimodal with $K(t) = \beta C_t$, $C_t \in [m]$. Then $s \star t$ is (+)-unimodal with critical point

(8) $$C_\star = (C_s - 1)m + r^{n-C_s}(C_t) \quad \in \quad [nm],$$

and it can be deduced from (5) and (6) that

(9) $$\begin{aligned} K(s \star t) &= \alpha \star \beta C_t \\ &= \alpha \star K(t) = K'(s) \star K(t). \end{aligned}$$

11. The cycle structure of interval self-maps.

(11.1) **The cycles of (J, f).** For an integer $n \geq 1$, S_n denotes the symmetric group of permutation of $[n] = \{1, 2, \ldots, n\}$, and

(1) $$C_n \subset S_n$$

the set of n-cycles.

Let (J, f) be a dynamical system on a compact real interval J. The **n-cycles of f** form the set

$$(2) \qquad C_n(f) \subset C_n$$

of $\sigma \in C_n$ such that there exist $x_1 < x_2 < \cdots < x_n$ in J with $f(x_i) = x_{\sigma(i)}$ $(1 \leq i \leq n)$. We put

$$(3) \qquad C(f) := \bigcup_{n\geq 1} C_n(f) \quad \subset \quad C := \bigcup_{n \geq 1} C_n.$$

The **periods** of f form the set

$$(4) \qquad Per(f) = \{n \mid C_n(f) \neq \emptyset\}.$$

In the following discussion, we quote from the literature some remarkable results concerning the above objects.

(11.2) **The Sharkowskii (total) order**, which we denote as $=>$, on the positive integers is defined as follows. Each integer $n \geq 1$ can be uniquely written as

$$(1) \qquad n = 2^{e(n)}O(n)$$

with $O(n)$ odd. The non powers-of-2 are lexicographically ordered by $(e(n), O(n))$, and all precede the powers-of-2, the latter ordered by decreasing size. More concretely, say $0 \leq e < E$, and $1 < r < R$ are odd.

$$(2) \qquad \begin{array}{l} 3 => 5 => 7 => \cdots \\ \cdots => 2^e r => 2^e R => \cdots => 2^E r => 2^E R => \cdots \\ \cdots => 2^E => 2^e => \cdots => 4 => 2 => 1. \end{array}$$

(11.3) **Sharkowskii's Theorem** [Sa]. Let (J, f) be a dynamical system on a compact real interval J. Then

$$[n => m] \quad \Longrightarrow \quad [n \in Per(f) \textit{ implies } m \in Per(f).]$$

(11.4) **Forcing** is a relation, denoted $=>$, on $C = \cup_n C_n$, with which to express some refinements of Sharkowskii's Theorem. For $\sigma, \tau \in C$ we define,

$$(1) \qquad \sigma => \tau \Longleftrightarrow \begin{cases} \sigma \in C(f) \textit{ implies } \tau \in C(f) \\ \textit{for all } f \textit{ as in } (11.1) \textit{ above.} \end{cases}$$

We put

$$(2) \qquad \begin{cases} C_m(\sigma) &= \{\tau \in C_m | \sigma => \tau\}, \text{ and} \\ C(\sigma) &= \cup_{m\geq 1} C_m(\sigma). \end{cases}$$

It is known (see for example [Ju] and references therein) that => is a partial order on C.

For $\sigma \in C_n$ Sharkowskii's Theorem says that $C_m(\sigma) \neq \emptyset$ whenever $n => m$. We call σ **primary** if $C_n(\sigma) = \{\sigma\}$, and define

$$(3) \qquad P_n = \{\sigma \in C_n | \sigma \text{ is primary}\}.$$

The following discussion leads to a direct characterization of P_n, in (11.7).

Convention. Let σ be a transitive cycle on an n-element totally ordered set X. Let $\tau : [n] \longrightarrow X$ be the unique order preserving bijection. A property (e.g. "primary") of elements of C_n will be said to hold for σ if it holds for $\tau\sigma\tau^{-1} \in C_n$.

(11.5) **Proposition.** *Suppose that $n = 2^e \cdot r$ and $\sigma \in C_n$. The following conditions are equivalent:*

(a) For $0 \leq d < e$, σ^{2^d} exchanges the left and right halves of each of its orbits.

(b) For $0 \leq d < e$, the orbits of σ^{2^d} are subintervals of $[n]$.

(c) $\{1, 2, 2^2, \ldots, 2^e\} \subset IRen([n], \sigma)$.

Proof. The equivalence of (b) and (c) follows from (I, 2.4), in the remark after (3). That σ exchanges the left and right halves, L and R, of $[n]$, is clearly equivalent to the condition that L and R form the two orbits of σ^2. Now the equivalence (a) and (b) follows from this remark, by induction on e.

■

Under the conditions of (11.5), we shall say that σ **satisfies the Block condition of level 2^e.** (cf. [Bl]).

(11.6) **Stefan cycles.** For *odd* $n = 2m - 1$ $(m \geq 1)$, we define the Stefan cycles to be

$$\sigma, \quad \tau\sigma\tau^{-1}$$

where τ is the flip of $[n]$ given by $\tau(h) = n + 1 - h$, and where σ is the **spiral cycle** illustrated in Figure 12 in the case in which $n = 8$,

It is easily seen that

$$IRen([n], \sigma) = \{1, n\}.$$

(For example, an interval containing $\sigma(\{2m - 2, 2m - 1\}) = \{1, m\}$ must have length $\geq m > n/2$, and hence cannot be a fiber of an interval q-renormalization unless $q = n$.)

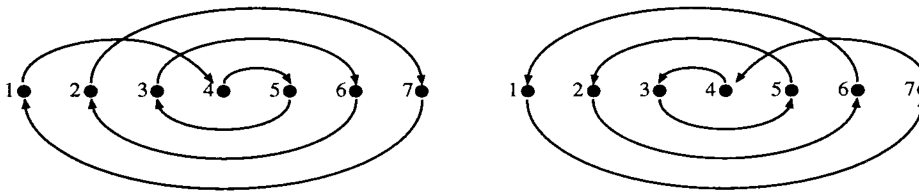

Figure 12. Stefan cycles for $n = 8$.

(11.7) **Theorem** (cf. [ALS], Theorem 1.1). *Let $n = 2^e r$ with r odd, and let $\sigma \in C_n$. The following conditions are equivalent*

(a) $\sigma \in P_n$, i.e., $C_n(\sigma) = \{\sigma\}$.

(b) There is an interval self-map f such that $C(f) = \{\sigma\}$. (Hence $C(\sigma) = \{\sigma\}$.)

(c) σ satisfies the Block condition (11.4) of level 2^e, and, on the (r-element) orbits of σ^{2^e}, σ^{2^e} is a Stefan cycle (11.6) on each of them, and σ is monotone on all but one of them.

Under the above conditions, even without the monotonicity condition in (c), we have

$$\text{(1)} \qquad IRen([n], \sigma) = \{1, 2, 2^2, \ldots, 2^e, n\}.$$

(11.8) **Comments.** For non-primary cycles, forcing can sometimes move contrary to the Sharkowskii ordering. For example Block [Bl] showed that, for $e \geq 2$,

$$\sigma \in (C_{2^e} - P_{2^e}) \Longrightarrow C_{2^{e-2}\cdot 3}(\sigma) \neq \emptyset.$$

Moreover, Jungreis [Ju] has given an algorithm, in terms of itineraries, for deciding whether $\sigma => \tau$ for $\sigma, \tau \in C$ (cf. [BCMM] as well).

Chapter III

Spherically Transitive Automorphisms of Rooted Trees

0. Motivation.

Let (K, f) be a linearly ordered dynamical system, as in (I, (2.4)), with interval renormalization index

$$\mathbf{q} = \mathbf{q}(K, f) = (q_1, q_2, q_3, \ldots) \quad \textit{(finite or infinite)}, \tag{1}$$

as in (I, (2.9)). Put $m_n = \mathbf{q}^{[n]} = q_1 q_2 \cdots q_n$ and $X_n = \mathbb{Z}/m_n\mathbb{Z}$. Then we have the inverse sequence of group homomorphisms

$$X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} \cdots \xleftarrow{p} X_{n-1} \xleftarrow{p} X_n \xleftarrow{p} \cdots \tag{2}$$

Let $k_0 \in K$ have a dense f-orbit. Then there is a unique interval m_n-renormalization

$$\phi_n : (K, f) \longrightarrow (X_n, \alpha_n) := (\mathbb{Z}/m_n\mathbb{Z}, +1), \tag{3}$$

such that $\phi_n(k_0) = 0$. Then we have $p \circ \phi_n = \phi_{n-1}$ and so the interval renormalization quotient

$$\widehat{\phi}_{(K,f)} : (K, f) \longrightarrow (\widehat{X}, \widehat{\alpha}) := (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1), \tag{4}$$

where $\widehat{X} = \widehat{\mathbb{Z}}_{\mathbf{q}}$ is the inverse limit of (2).

In this chapter we take the point of view that (2) defines a "rooted tree" X, with vertex set $\coprod_{n \geq 0} X_n$, root x_0, and for $n > 0$, $x \in X_n$ is a neighbor of $p(x) \in X_{n-1}$. Then we can identify $\widehat{X} = \varprojlim_n X_n$ with Ends(X), the space of ends of X (or leaves of X if X is finite). Further the α_n in (3) assemble to define an automorphism α of the rooted tree X, inducing the adding machine $+1$ on $\widehat{\mathbb{Z}}_{\mathbf{q}}$, identified in (4) with $\widehat{X}$.

So far this identification accounts only for the topological dynamics of (K, f) and not the *order structure*. The linear order on K defines one on each X_n so that $\phi_n : K \longrightarrow X_n$ is weak order preserving, and then each $p : X_n \longrightarrow X_{n-1}$ is likewise. This order structure defines (and is almost equivalent to) a planar embedding of X, up to isotopy (cf. (2.4) below). For example, we may take an order preserving embedding of X_n into the horizontal line $y = -n$, and then join $x \in X_n$ to $p(x) \in X_{n-1}$ by a euclidean line segment (See Figure 13).

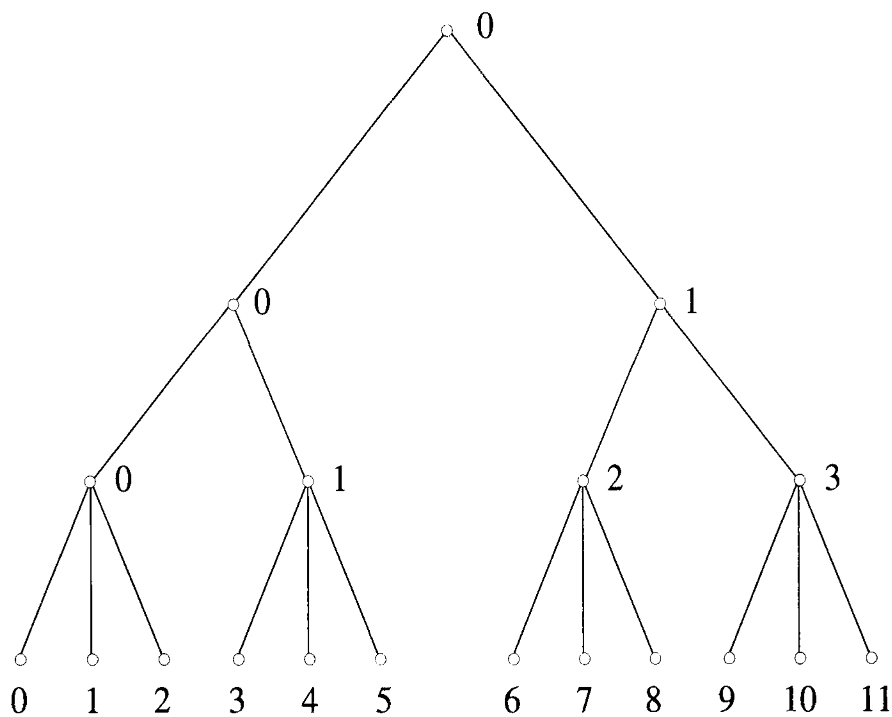

Figure 13. The rooted tree with $\mathbf{q} = (2, 2, 3)$ (see (3.1)).

Let σ be an automorphism of the rooted tree X. Then σ is equivalent to a sequence of permutations σ_n of X_n such that

$$p \circ \sigma_n = \sigma_{n-1} \circ p \quad \textit{for } n > 0. \tag{5}$$

For $N > 0$, if a permutation σ_N of X_N is given, then we can (uniquely) define $\sigma_n \in X_n$ satisfying (5) for $0 \leq n \leq N$, iff

(6) σ_N *permutes the (interval) fibers of* $p^{N-n} : X_N \longrightarrow X_n$ *for* $0 \leq n < N$.

If σ_N is transitive on X_N then the σ_n on X_n $(0 \leq n \leq N)$ will be likewise, and then the condition (6) is clearly equivalent to the condition

$$\{m_1, m_2, \ldots, m_n\} \subset IRen(X_N, \sigma_N). \tag{7}$$

Thus an automorphism σ of X which is "spherically transitive", *i.e.*, transitive on each X_n, defines a homeomorphism $\widehat{\sigma}$ of the linearly ordered space $\widehat{X} = Ends(X)$ such that

$$\{1, m_1, m_2, \ldots, m_n, \ldots\} \subset IRen(\widehat{X}, \widehat{\alpha}). \tag{8}$$

This motivates our group theoretic study, in this and the following chapter, of spherically transitive automorphisms of rooted trees, and of the structure of the automorphism groups of such trees.

1. Relative automorphism groups of trees.

(1.1) **Graphs and trees.** Our graphs are simplicial. Thus a **graph** X consists of a set, also denoted X, of **vertices** and a set EX of two-element subsets of X, called **edges**. Two vertices $x, y \in X$ are said to be **adjacent** if $\{x, y\} \in EX$. A **path** (of **length** n from x_0 to x_n) in X is a sequence of vertices $(x_0, x_1, \ldots, x_n)$ such that x_i and x_{i+1} are adjacent for each i. The path is **reduced** (or non-reversing) if $x_{i-1} \neq x_{i+1}$ for $0 < i < n$. The path is **closed** if $x_0 = x_n$. Infinite paths are defined similarly. For a good general graph theory reference see [Bol].

We call X **connected** if $X \neq \emptyset$ and any two vertices x, y are joined by a path; then the shortest length, $d(x, y)$, of such a path defines a **metric** on X. A graph X is a **tree** if X is connected and contains no reduced closed paths of length > 0. In this case, for any $x, y \in X$, there is a unique reduced path from x to y, of length $d(x, y)$; the underlying (linear) graph of this path will be denoted $[x, y]$:

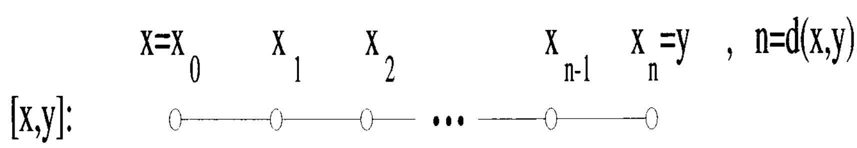

Figure 14. A reduced path $[x, y]$ of length $n = d(x, y)$.

(1.2) **Relative automorphism groups.** Let X be a graph. Then an **automorphism** of X is a bijection from X to itself which preserves edges. The automorphisms of X form a group which will be denoted $G(X)$.

Let $Y \subseteq X$ be a subgraph. Thus, Y is a graph whose vertex and edge sets are subsets of those of X. The **stabilizer of** Y **in** $G(X)$, denoted $G(X, Y)$, is

defined by

$$G(X,Y) = \{g \in G(X) \mid gY = Y\}.$$

There is a naturally defined restriction homomorphism

$$res^X_Y : G(X,Y) \longrightarrow G(Y).$$

Its image and kernel are denoted

$$G(X \mid_Y) = Im(res^X_Y), \qquad and$$

$$\begin{aligned} G^1(X,Y) &= Ker(res^X_Y) \\ &= \{g \in G(X) \mid gy = y \quad for\ all \quad y \in Y\}. \end{aligned}$$

What follows is an analysis of these groups in the case in which X and Y are trees.

(1.3) **Normal trees to a subtree.** Let X be a tree and $Y \subseteq X$ a subtree. For $x \in X$ there is a unique $y = p(x) \in Y$ nearest to x, *i.e.*, , a unique $y \in Y$ such that $[x,y] \cap Y = \{y\}$. This defines a retraction $p : X \longrightarrow Y$ of vertex sets. For $y \in Y$, $p^{-1}(y)$ is the vertex set of a subtree N_y of X such that $N_y \cap Y = \{y\}$. We view N_y as a **rooted tree**, (cf. (2.1)) with root y, and call it the **normal tree to Y in X at y**.

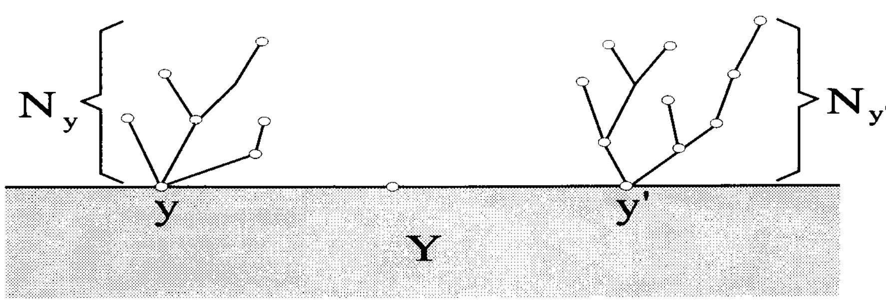

Figure 15. Normal subtrees.

(1.4) **The "wreath product" structure of $G(X,Y)$.** Let X be a tree and Y a subtree. Then there is (cf. (1.2)) an exact sequence

(1) $$1 \longrightarrow G^1(X,Y) \longrightarrow G(X,Y) \xrightarrow{r} Q \longrightarrow 1$$

where $r = res^X_Y$, and $Q = Im(r) = G(X \mid_Y) \leq G(Y)$. We shall show that (1) splits as a kind of multiple wreath product.

First note that, with the notation of (1.3), we evidently have

$$G^1(X,Y) = \prod_{y\in Y} G(N_y, y). \tag{2}$$

Let Y_Q denote a set of representatives of the Q-orbits on (the vertices of) Y; so that, $Y_Q \simeq Q \setminus Y$. Thus

$$Y = \coprod_{y\in Y_Q} Q\cdot y.$$

Using (2) this gives rise to an initial factorization,

$$G^1(X,Y) = \prod_{y\in Y_Q} \prod_{z\in Q\cdot y} G(N_z, z). \tag{3}$$

Now

$$Q\cdot y = G(X,Y)\cdot y \;\cong\; G(X,Y)/G(X,Y)_y$$

where $G(X,Y)_y$ denotes the stabilizer of y in $G(X,Y)$.

Let $S_y \subset G(X,Y)$ denote a set of cosets representatives for $G(X,Y)/G(X,Y)_y$ which contains 1. For $z \in Q\cdot y$ there is a unique $s_z \in S_y$ such that $z = s_z y$. Thus, s_z furnishes an isomorphism of rooted trees

$$s_z : N_y \longrightarrow N_z.$$

Let $g \in G(X,Y)$, $y \in Y_Q$, and $z \in Q\cdot y$. We have the (not necessarily commutative) diagram of isomorphisms

$$(4)_{g,z} \qquad \begin{tikzcd} N_z \arrow[rr, "g"] & & N_{gz} \\ & N_y \arrow[ul, "s_z"] \arrow[ur, "s_{gz}"'] & \end{tikzcd}$$

Let

$$\tilde{Q} = \{g \in G(X,Y) \mid \textit{ all diagrams } (4)_{g,z} \textit{ commute } \}.$$

We have the following properties:

(i) $\tilde{Q}$ is a subgroup of $G(X,Y)$,
(ii) $\tilde{Q} \cap G^1(X,Y) = 1$,
(iii) $r(\tilde{Q}) = Q$.

Properties (i) and (ii) are immediate. For (iii), suppose that we are given $q \in Q \leq G(Y)$. We must extend q to an automorphism $\tilde{q} \in \tilde{Q}$ of X. For $y \in Y_Q$ and $z \in Q\cdot y$ define $\tilde{q}$ on N_z to make

$$\begin{tikzcd} N_z \arrow[rr, "\tilde{q}"] & & N_{qz} \\ & N_y \arrow[ul, "s_z"] \arrow[ur, "s_{qz}"'] & \end{tikzcd}$$

commute. This clearly defines $\tilde{q}$ satisfying our requirements.

The above properties show that $r : \tilde{Q} \longrightarrow Q$ is an isomorphism, and so

$$(5) \qquad G(X,Y) = G^1(X,Y) \rtimes \tilde{Q} \cong G^1(X,Y) \rtimes Q$$

where $\rtimes$ is the usual semidirect product notation.

The isomorphisms $s_z : N_y \longrightarrow N_z$ define conjugation isomorphisms $ad(s_z) : G(N_y, y) \longrightarrow G(N_z, z)$, and these assemble to give an isomorphism

$$(6) \qquad G(N_y, y)^{Q \cdot y} \longrightarrow \prod_{z \in Q \cdot y} G(N_z, z).$$

The isomorphism (6) is Q-equivariant for the conjugation action on the right, and the action by permutations of $Q \cdot y = \tilde{Q} \cdot y$ on the left. Combining (3), (5), and (6), we obtain an isomorphism

$$(7) \qquad G(X,Y) \cong \left(\prod_{y \in Q \backslash Y} G(N_y, y)^{Q \cdot y} \right) \rtimes Q$$

where $Q = G(X\,|_Y) = Im(G(X,Y) \xrightarrow{res} G(Y))$ acts on $G(N_y, y)^{Q \cdot y}$ via its permutation action on $Q \cdot y$. The isomorphism carries $G^1(X,Y)$ to $\prod_{y \in Q \backslash Y} G(N_y, y)^{Q \cdot y}$.

2. Rooted trees and their ends.

(2.1) **Rooted trees, ends, and order structures.** A **rooted tree** (X, x_0) consists of a tree X and a designated vertex x_0. Let d denote the edge path distance on X. Then let X_n denote the n-**sphere**,

$$X_n = \{x \in X \mid d(x_0, x) = n\}$$

and denote by B_n the n-**ball**,

$$B_n = \{x \in X \mid d(x_0, x) \leq n\} = X_0 \coprod X_1 \coprod \cdots \coprod X_n.$$

We view B_n as (the vertices of) a subtree of X.

For $x \in X_n$, $n > 0$, $[x_0, x] \cap X_{n-1}$ is a single vertex, which we denote $p(x)$. Thus we have an inverse sequence

$$(1) \qquad X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} X_2 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{p} X_n \longleftarrow \cdots$$

associated to (X, x_0). Conversely (1) determines $(X, x_0) : X = \coprod_n X_n$, and the edges are all the sets $\{x, p(x)\}, x \in X_n, n > 0$.

If (X', x'_0) is another rooted tree, with inverse sequence

$$(1') \qquad X'_0 = \{x'_0\} \xleftarrow{p'} X'_1 \xleftarrow{p'} X'_2 \longleftarrow \cdots \longleftarrow X'_{n-1} \xleftarrow{p'} X'_n \longleftarrow \cdots$$

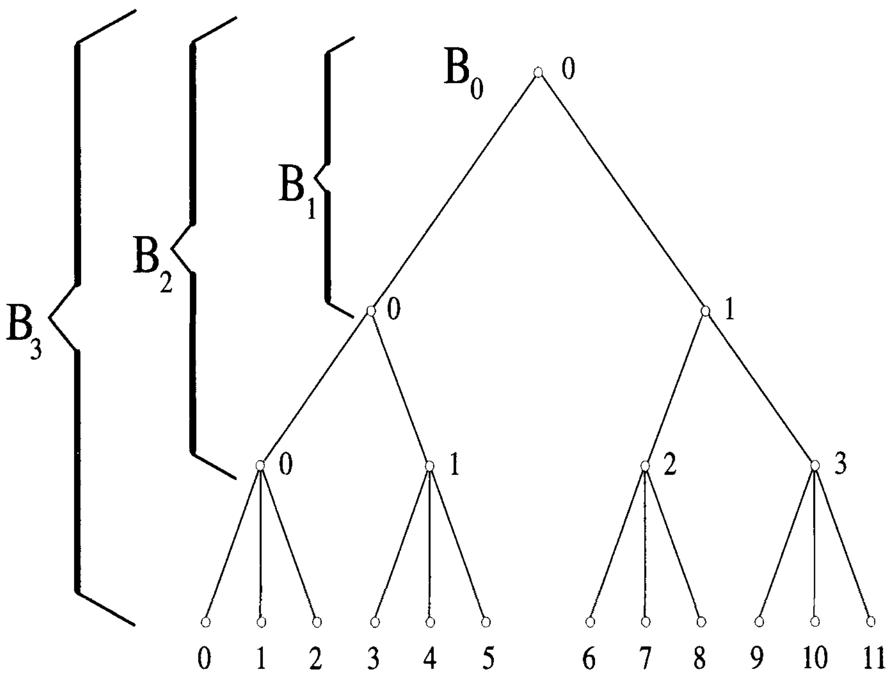

Figure 16. n-balls for the rooted tree $X(2,2,3)$ (see (3.1)).

Then a **morphism** $f : (X, x_0) \longrightarrow (X', x'_0)$ **of rooted trees** is just a sequence of maps

$$f_n : X_n \longrightarrow X'_n (n \geq 0)$$

such that $p' f_n = f_{n-1} p$ for all $n > 0$. In particular an automorphism $g \in G(X, x_0)$ is a sequence of permutations g_n of X_n such that $p g_n = g_{n-1} p$ for all $n > 0$. The restriction monomorphisms $G(X, x_0) \longrightarrow G(B_n, x_0)$ define a map

$$G(X, x_0) \longrightarrow \varprojlim_n G(B_n, x_0), \tag{2}$$

which is evidently an isomorphism. If X is locally finite (all vertices have a finite number of adjacent vertices) then each B_n and $G(B_n, x_0)$ is finite, and so (2) shows that $G(X, x_0)$ is naturally a profinite group.

(2.2) **Ends and leaves of** (X, x_0)**.** By an x_0**-ray in** X we mean a maximal non-reversing path L in X starting from x_0. Thus, $L = (x_0, x_1, x_2, \ldots)$ with $x_n \in X_n$ and $p(x_n) = x_{n-1}$. Either the sequence is infinite, and so represents an **end** (a cofinal set of paths), or else it is finite, $L = (x_0, x_1, \ldots, x_n)$, and, by the maximality of L, $p^{-1}(x_n) = \emptyset$. In the latter case x_n is called an **endpoint** (or **leaf**) of (X, x_0), and x_n determines the x_0-ray: $L = [x_0, x_n]$. We put

$$\begin{aligned} Ends(X) &= \{\textit{infinite } x_0\textit{-rays in } X\} \\ &= \varprojlim_n X_n \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}(X, x_0) &= \{x_0\textit{-rays in } X\} \\ &= Ends(X) \coprod \{\textit{endpoints of } (X, x_0)\}. \end{aligned}$$

While $\mathcal{E}(X, x_0)$ depends on x_0, it is easily shown that $Ends(X)$ does not, up to canonical isomorphism.

(2.3) **The metric on** $\mathcal{E}(X, x_0)$ is defined by

$$d(L, L') = \inf\{\frac{1}{2^n} \mid L \cap B_n = L' \cap B_n, n \geq 0\}.$$

This makes $\mathcal{E}(X, x_0)$ a compact totally disconnected metric space with $Ends(X)$ a closed subspace, and the end points a discrete subspace. The group $G(X, x_0)$ acts faithfully and continuously on $\mathcal{E}(X, x_0)$ as a group of isometries. The above metric on $Ends(X)$ depends on x_0, but the corresponding topology does not.

(2.4) **Planar embeddings and order structures.** Let X be a graph. For $x \in X$, let $E_0(x)$ denote the set of oriented edges $e = (x, y)$ with initial vertex $\partial_o(e) = x$. The map $e \mapsto \partial_1(e) = y$ is a bijection by which we sometimes identify $E_0(x)$ with the set of neighbors of x.

Suppose that X is embedded, as a simplicial 1-complex, in the plane. Then by intersecting each edge $e \in E_0(x)$ with a small circle centered at x we obtain a cyclic (counterclockwise) ordering on $E_0(x)$. This set of cyclic orderings on the various $E_0(x)$ depends only on the isotopy class of the planar embedding.

Suppose that X is a tree. Then if, conversely, we are given a cyclic ordering on each $E_0(x)$, there is a planar embedding, unique up to isotopy, inducing them.

To see this, first fix a base point $x_0 \in X$, so that (X, x_0) is a rooted tree, with inverse sequence

(1) $$X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} \cdots \xleftarrow{p} X_{n-1} \xleftarrow{p} X_n \xleftarrow{p} \cdots$$

Then $X_1 = p^{-1}(x_0) = E_0(x_0)$ has a given cyclic order. On the other hand, if $x \in X_n$, $n > 0$, then $E_0(x) = p^{-1}(x) \coprod \{ p(x) \}$, so relative to the base point $p(x)$, the cyclic order on $E_0(x)$ induces a *linear* order on $p^{-1}(x)$. Indeed, it is thus clear that giving a cyclic order on each $E_0(x)$ is equivalent to giving a cyclic order on $X_1 = p^{-1}(x_0)$ and a linear order on each $p^{-1}(x)$ for $x \neq x_0$.

Now fix a linear order on X_1 consistent with the given cyclic ordering, e.g. use $\leq_x$ for some $x \in X_1$ (cf. (I,2.2)(9)). Then there are unique linear orders on each X_n $(n \geq 0)$ such that $p : X_n \longrightarrow X_{n-1}$ is weak order preserving $(x \leq y \Longrightarrow p(x) \leq p(y))$, and the ordering on X_n induces the given linear order on $p^{-1}(x)$ for each $x \in X_{n-1}$. These are constructed by a straightforward induction.

We shall call such a structure an **order structure** on the rooted tree (X, x_0). It consists in a linear order on each X_n so that $x \leq y$ in X_n implies $p(x) \leq p(y)$ in X_{n-1}. The discussion above shows that such a structure is equivalent to a linear order on $E_0(x_0)$ and a cyclic order on $E_0(x)$ for all $x \neq x_0$.

Given an order structure on (X, x_0) we can embed X in the plane as follows. For $n \geq 0$ let L_n denote the horizontal line $(y = -n)$ at distance n below the x-axis, ordered by its x-coordinate. For each n, choose an order preserving embedding of X_n into L_n. (We assume, say, that X is countable.) For an edge $e = (x, p(x))$, with $x \in X_n$, $n > 0$, embed it using the euclidean segment from $x \in L_n$ to $p(x) \in L_{n-1}$. Since $x \leq y$ implies $p(x) \leq p(y)$, the fibers of p are intervals relative to the given orderings, so the above euclidean segments never meet outside their endpoints, and so furnish the desired planar embeddings of X. The cyclic (counterclockwise) order on $E_0(x)$ defined by this embedding agrees with the one with which we started.

(2.5) ***Ends*(*X*) as a linear profinite space.** Let (X, x_0) be a locally finite rooted tree with inverse sequence

$$X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} \cdots X_{n-1} \xleftarrow{p} X_n \xleftarrow{p} \cdots$$

For $x \in X$ we put

(1) $$q(x) = |\, p^{-1}(x)\, |,$$

a nonnegative integer. Suppose that we are further given an *order structure* on (X, x_0) (cf. (2.4)). This then defines a linear order on the profinite space

$$Ends(X) = \varprojlim_n X_n$$

so that the projections $Ends(X) \longrightarrow X_n$ are weak order preserving.

We propose to relate $Ends(X)$ to a profinite space $K \subset [0,1]$. For this we use **Cantor dissections** defined as follows.

Let q be a positive integer. Given a real closed interval $J = [a, b]$, $a < b$, of length

(2) $$l([a,b]) = b - a,$$

we form q evenly spaced subintervals

(3) $$J_{(q,j)} = [a_j, b_j] \qquad (j = 1, \ldots, q),$$

each of length

$$b_j - a_j = (b-a)/(2q-1) \tag{4}$$

with

$$a = a_1 < b_1 < a_2 < b_2 < \cdots < a_q < b_q = b, \tag{5}$$

and the separating intervals each also of length

$$a_{j+1} - b_j = (b-a)/(2q-1). \tag{6}$$

We put

$$K_q(J) = (J_{(q,1)} \coprod \cdots \coprod J_{(q,q)}) \subset J. \tag{7}$$

Now we inductively define a commutative diagram

$$\begin{array}{ccccccccccc} K_0 & \longleftarrow & K_1 & \longleftarrow & \cdots & \longleftarrow & K_{n-1} & \longleftarrow & K_n & \longleftarrow & \cdots \\ \pi\downarrow & & \pi\downarrow & & & & \pi\downarrow & & \pi\downarrow & & \\ X_0 = \{x_0\} & \underset{p}{\longleftarrow} & X_1 & \underset{p}{\longleftarrow} & \cdots & \underset{p}{\longleftarrow} & X_{n-1} & \underset{p}{\longleftarrow} & X_n & \underset{p}{\longleftarrow} & \cdots \end{array} \tag{8}$$

by the following conditions: $K_0 = [0,1]$; the maps $K_n \longrightarrow K_{n-1}$ are inclusions of the subsets; the maps $\pi : K_n \longrightarrow X_n$ weak order preserving; the fibers $K(x) = \pi^{-1}(x)$ are closed real intervals; and for $x \in X_{n-1}$ with $q(x) > 0$,

$$\pi^{-1}(p^{-1}(x)) = K_{q(x)}(K(x)), \tag{9}$$

with the notation of (7) above. Explicitly, if $p^{-1}(x) = \{x_1 < x_2 < \cdots < x_q\}$, $q = q(x)$, in the given linear order on X_n, then

$$K(x_j) = K(x)_{(q,j)} \qquad (j = 1, \ldots, q), \tag{10}$$

as in (3) above.

From (4) above we see that $l(K(x_j)) = \frac{l(K(x))}{(2q(x)-1)}$ for $x_j \in p^{-1}(x)$. Adjusting notation, we see that for $x \in X_n$,

$$\begin{cases} K(x) \subset K(p(x)) \quad \mathit{and} \\ l(K(x)) = \frac{l(K(p(x)))}{(2q(p(x))-1)} = \prod_{i=1}^n \frac{1}{(2q(p^i(x))-1)}. \end{cases} \tag{11}$$

Now passing to inverse limits in (8), we obtain a surjective continuous map

$$\widehat{\pi} : K(X, x_0) := \bigcap_n K_n \longrightarrow \mathit{Ends}(X) = \varprojlim_n X_n. \tag{12}$$

An element $L \in \mathit{Ends}(X)$ corresponds to an infinite ray

$$L = (x_0, x_1, \ldots, x_n, \ldots), \qquad x_n \in X_n, \qquad p(x_n) = x_{n-1}. \tag{13}$$

We have

$$\widehat{\pi}^{-1}(L) = \bigcap_{n \geq 0} K(x_n). \tag{14}$$

In view of (11) this is an interval of length

$$l(\widehat{\pi}^{-1}(L)) = \prod_{n \geq 0} (2q(x_n) - 1)^{-1}, \tag{15}$$

which equals zero unless, for some n_0 $q(x_n) = 1$ for all $n \geq n_0$. This means that L is an isolated ray of X, *i.e.*, an isolated point of $Ends(X)$.

We now summarize part of the above discussion.

(2.6) **Proposition.** *Let (X, x_0) be a locally finite tree with an order structure (cf. (2.4)). Then there is a closed subset*

$$K = K(X, x_0) \subset [0, 1]$$

and a map

$$\widehat{\pi} : K \longrightarrow Ends(X)$$

which is weak order preserving in the sense of (2.5)(12). If $Ends(X)$ has no isolated points then K is a Cantor set and $\widehat{\pi}$ is a homeomorphism.

(2.7) **Lifting automorphisms to $K = K(X, x_0)$.** Keep the assumptions and notation of (2.5). Let $g \in G(X, x_0)$ be an automorphism of (X, x_0). For each $n \geq 0$, g induces a permutation of X_n, which we lift to a homeomorphism g_n of

$$K_n = \coprod_{x \in X_n} K(x)$$

so that $g_n : K(x) \longrightarrow K(g(x))$ is the unique increasing affine linear homeomorphism.

If $n > 0$ and $y = p(x)$ then $g(y) = g(p(x)) = p(g(x))$ and we have $K(x) \subset K(y)$, and $g_n(K(x)) = K(g(x)) \subset K(g(y)) = g_{n-1}(K(y))$. However g_n and g_{n-1} do not agree on $K(x)$ (unless $g(y) = 1$). If

$$L = (x_0, x_1, \ldots, x_n, \ldots) \in Ends(X)$$

then it follows that, on $\pi^{-1}(L) = \bigcap_{n \geq 0} K(x_n)$, the g_n converge to a well-defined function $g_K : \pi^{-1}(L) \longrightarrow \pi^{-1}(g(L))$ which, when $\pi^{-1}(L)$ is not a point, is an increasing affine linear homeomorphism.

If $h \in G(X, x_0)$ is another automorphism then it is easily seen that $(h \circ g)_n = h_n \circ g_n$ for all n, and so $(h \circ g)_K = h_K \circ g_K$. Thus, $g \mapsto g_K$ defines an action of $G(X, x_0)$ by homeomorphisms on K so that $\pi : K \longrightarrow Ends(X)$ is equivariant.

3. Spherically homogeneous rooted trees, $X(\mathbf{q})$.

(3.1) **Spherical homogeneity.** Let (X, x_0) be a rooted tree with inverse sequence as in (2.1). Assume that each X_n is finite. For $x \in X$ we put $q(x) = | p^{-1}(x) |$ as in (2.5). We call (X, x_0) **spherically homogeneous** if, for each $n > 0$, $q(x)$ takes a constant value, which we denote q_n, for all $x \in X_{n-1}$. We then put

$$\mathbf{q} = (q_1, q_2, q_3, \ldots)$$

and call $\mathbf{q}$ the **spherical index** of (X, x_0).

Note that, for each n, q_n is an integer ≥ 0, and $q_n = 0$ implies that $q_m = 0$ for all $m > n$. We say $\mathbf{q}$ is **finite** if some $q_n = 0$, and **infinite** otherwise.

(3.2) **$\mathbf{q}$ determines (X, x_0).** Indeed let (X, x_0) and (X', x'_0) be spherically homogeneous with the same index $\mathbf{q} = (q_1, q_2, q_3, \ldots)$. Then we can construct an isomorphism

$$f = (f_n)_{n \geq 0} : (X, x_0) \longrightarrow (X'_0, x'_0)$$

by induction on n, starting with $f_0(x_0) = x'_0$. Assume

$$(1) \quad \begin{array}{ccccccccccc} X_0 & \xleftarrow{p} & X_1 & \xleftarrow{p} & \cdots & \xleftarrow{p} & X_{n-1} & \xleftarrow{p} & X_n & \xleftarrow{p} & \cdots \\ f_0 \downarrow & & f_1 \downarrow & & \cdots & & f_{n-1} \downarrow & & f_n \downarrow & & \cdots \\ X_0{}' & \xleftarrow{p'} & X_1{}' & \xleftarrow{p'} & \cdots & \xleftarrow{p'} & X_{n-1}{}' & \xleftarrow{p'} & X_n{}' & \xleftarrow{p'} & \cdots \end{array}$$

where we have bijections $f_0, \ldots, f_{n-1}$ making (1) commute. For $y \in X_{n-1}, p^{-1}(y)$ and $p'^{-1}(f_{n-1}(y))$ both have cardinality q_n, so we can define f_n in such a way as to induce a bijection $p^{-1}(y) \longrightarrow p'(f_{n-1}(y))$ for each $y \in X_{n-1}$.

This observation permits us to denote (X, x_0) by $X(\mathbf{q})$ or $X(q_1, q_2, q_3, \ldots)$. It is a spherically homogeneous rooted tree defined up to (non-unique) isomorphism by $\mathbf{q}$.

(3.3) **The product model of $X(\mathbf{q})$.** For an integer $m \geq 0$, put

$$(1) \qquad C_m = \begin{cases} \mathbb{Z}/m\mathbb{Z} & \textit{if } m > 0 \\ \emptyset & \textit{if } m = 0 \end{cases}$$

so that $| C_m | = m$. Let $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ be a **spherical index**, *i.e.*, , for each n, q_n is a nonnegative integer, such that $q_n = 0$ implies that $q_m = 0$ for all $m > n$. Setting $q_0 = 1$, we put

$$(2) \qquad X_n = C_{q_0} \times C_{q_1} \times \cdots \times C_{q_n} \qquad (n \geq 0)$$

and, for $n > 0$, let

$$p : X_n \longrightarrow X_{n-1}$$

denote projection away from the last factor. Clearly,

$$X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} \cdots \xleftarrow{p} X_{n-1} \xleftarrow{p} X_n \xleftarrow{p} \cdots$$

is a model of $X(\mathbf{q})$.

Note that

$$(3) \qquad |X_n| = \mathbf{q}^{[n]} := q_0 \cdot q_1 \cdots q_n$$

and

$$(4) \qquad Ends(X(\mathbf{q})) = \prod_{n \geq 0} C_{q_n}.$$

If $\mathbf{q}$ is finite, say that $q_n > 0$ and $q_m = 0$ for $m > n$, then $\mathcal{E}(X(\mathbf{q})) = X_n = \{\textit{endpoints of } X(\mathbf{q})\}$. If $\mathbf{q}$ is infinite then $X_n \longrightarrow X_{n-1}$ is surjective for all n, so that $\mathcal{E}(X(\mathbf{q})) = Ends(X(\mathbf{q}))$, and the natural projection $Ends(X(\mathbf{q})) \longrightarrow X_n$ is surjective for all n.

(3.4) **The cyclic model of $X(\mathbf{q})$.** Keep the notation $\mathbf{q} = (q_1, q_2, q_3, \ldots)$, $q_0 = 1, \mathbf{q}^{[n]} = q_0 \cdot q_1 \cdots q_n$, etc. of (3.3). Put

$$Y_n = C_{\mathbf{q}^{[n]}}.$$

Define

$$p : Y_n \longrightarrow Y_{n-1} \ (n > 0)$$

to be the natural projection

$$\mathbb{Z}/\mathbf{q}^{[n]}\mathbb{Z} \longrightarrow \mathbb{Z}/\mathbf{q}^{[n-1]}\mathbb{Z}$$

if $q_n > 0$, and otherwise the unique map from $Y_n = \emptyset$ (cf. (3.3)(1)).

Clearly

$$Y_0 = \{y_0\} \xleftarrow{p} Y_1 \xleftarrow{p} Y_2 \longleftarrow \ldots \longleftarrow Y_{n-1} \xleftarrow{p} Y_n \longleftarrow \ldots$$

is a model of $X(\mathbf{q})$ which we shall here denote as $Y(\mathbf{q})$.

If $\mathbf{q}$ is finite, say $q_n > 0$ and $q_m = 0$ for $m > n$ then

$$\mathcal{E}(Y, y_0) = \{\textit{endpoints of } (Y, y_0)\} = Y_n = \mathbb{Z}/\mathbf{q}^{[n]}\mathbb{Z}.$$

If $\mathbf{q}$ is infinite then

$$(Y, y_0) = Ends(Y) = \varprojlim_n \mathbb{Z}/\mathbf{q}^{[n]}\mathbb{Z}.$$

We shall denote this ring $\widehat{\mathbb{Z}}_{\mathbf{q}}$, and call it the **ring of q-adic integers**.

(3.5) **The q-adic adding machine.** With $Y(\mathbf{q})$ as in (3.4) we define an automorphism α of $Y(\mathbf{q})$ by $\alpha_n(y) = y + 1$ for $y \in Y_n = \mathbb{Z}/\mathbf{q}^{[n]}\mathbb{Z}$ when $q_n > 0$; otherwise $Y_n = \emptyset$. Since, for $Y_n \neq \emptyset$, $p : Y_n \longrightarrow Y_{n-1}$ is a ring homomorphism, we have

$$p\alpha_n(y) = p(y+1) = p(y) + 1 = \alpha_{n-1}(p(y)),$$

so $\alpha = (\alpha_n)$ defines an automorphism of $Y(\mathbf{q})$, which we call the **q-adic adding machine**. Note that α on Y_n is a cyclic permutation of order $\mathbf{q}^{[n]}$.

(3.6) **Proposition.** *Let (X, x_0) be a locally finite rooted tree, with automorphism group $G = G(X, x_0)$. The following conditions are equivalent.*

(a) *(X, x_0) is spherically homogeneous,*
(b) *G acts transitively on X_n for all $n \geq 0$,*
(c) *G acts transitively on $\mathcal{E}(X, x_0)$.*

Proof. If $g \in G$ and $x \in X$ then $q(gx) = q(x)$. Thus, if G acts transitively on X_n then q is constant on X_n, whence (b) implies (a). For the converse, if (X, x_0) has spherical index $\mathbf{q}$, we can use the $\mathbf{q}$-adic adding machine on $Y(\mathbf{q})$ (cf. (3.5)) to show transitivity on each X_n. To show that (b) implies (c), if $\mathbf{q}$ is finite then $\mathcal{E}(X, x_0) = X_n$ for some n, whence (c). If $\mathbf{q}$ is infinite and $E \subseteq Ends(X) = \mathcal{E}(X, x_0)$ is a G-orbit then, by (b), E projects onto X_n for all n. It follows that E is dense in $Ends(X)$. But, since G is compact, E is closed, whence $E = Ends(X)$. Finally, to prove (c) implies (b), we simply observe that each nonempty X_n is a G-equivariant quotient of (X, x_0).

■

4. Spherically transitive automorphisms.

(4.1) **Notation.** We fix a spherically homogeneous rooted tree (X, x_0) of spherical index $\mathbf{q} = (q_1, q_2, q_3 \ldots)$. For $n > 0$ put

$$\begin{aligned} \mathbf{q}[\leq n] &= (q_1, q_2, \ldots, q_n, 0, 0, 0, \ldots) \\ \mathbf{q}[\geq n] &= (q_n, q_{n+1}, \ldots) \end{aligned}$$

Analogously, we write $\mathbf{q}[> n]$ for $\mathbf{q}[\geq n+1]$, and $\mathbf{q}[< n]$ for $\mathbf{q}[\leq n-1]$ (when $n > 1$).

For $x \in X_n$ let N_x denote the normal tree to B_n in X at x (cf. (1.3)). Thus N_x is defined by the inverse sequence:

$$N_x \; : \; \{x\} \longleftarrow p^{-1}(x) \longleftarrow p^{-2}(x) \longleftarrow \cdots \longleftarrow p^{-m}(x) \longleftarrow \cdots$$

Clearly

$$(B_n, x_0) \cong X(\mathbf{q}[\leq n])$$

and

$$(N_x, x) \cong X(\mathbf{q}[> n]).$$

Note that the latter isomorphism depends only on n, not on $x \in X_n$.

Thus, writing $A(\mathbf{q})$ for $G(X, x_0)$ we have $G(B_n, x_0) \cong A(\mathbf{q}[\leq n])$ and $G(N_x, x) \cong A(\mathbf{q}[> n])$.

Put

$$G(n) = Ker(G(X, x_0) \xrightarrow{\text{res}} G(B_n, x_0)).$$

Then

$$\begin{aligned} G(n) &= \prod_{x \in X_n} G(N_x, x) \\ &\cong G(N_x, x)^{X_n} \cong A(\mathbf{q}[> n])^{X_n} \end{aligned}$$

for any $x \in X_n$. Furthermore, we claim that $G(X, x_0) \longrightarrow G(B_n, x_0)$ is surjective. In fact, to extend $g_0 \in G(B_n, x_0)$ to $g \in G(X, x_0)$ we require, for each $x \in X_n$, an isomorphism

$$g_x : (N_x, x) \longrightarrow (N_{g_0 x}, g_0 x).$$

Such g_x exist since all of the rooted trees (N_x, x) with $x \in X_n$ are isomorphic (to $X(\mathbf{q}[> n])$).

We can calculate the finite group order

$$T(n) = | G(B_n, x_0) | = | A(\mathbf{q}[\geq n]) |$$

as follows, starting with $T(0) = 1$. For $n > 0$ we have

$$A(\mathbf{q}[\geq n]) \cong (S_{q_n})^{X_{n-1}} \rtimes A(\mathbf{q}[\geq n-1])$$

and so

$$\begin{aligned} T(n) &= (q_n!)^{|X_{n-1}|} \cdot T(n-1) \\ &= (q_n!)^{\mathbf{q}^{[n-1]}} \cdot T(n-1) \\ &= \prod_{i=1}^{n} (q_i!)^{\mathbf{q}^{[i-1]}}. \end{aligned}$$

Now, from (1.4)(7) we obtain:

(4.2) **Proposition.** *For each $n > 0$, $G = G(X, x_0)$ admits the wreath product decomposition,*

$$\begin{aligned} G &= G(n) \rtimes G(B_n, x_0) \\ &\cong G(N_x, x)^{X_n} \rtimes G(B_n, x_0) \qquad \textit{(for any } x \in X_n\textit{)} \\ &\cong A(\mathbf{q}[> n])^{X_n} \rtimes A(\mathbf{q}[\leq n]). \end{aligned}$$

Moreover, the order of $A(\mathbf{q}[\geq n])$ is

$$|A(\mathbf{q}[\geq n])| = \prod_{i=1}^{n} (q_i!)^{\mathbf{q}^{[i-1]}}.$$

(4.3) **Proposition.** *Let $g \in G(X, x_0)$ generate the cyclic group $\langle g \rangle$. The following conditions are equivalent.*

(a) The cyclic group $\langle g \rangle$ acts transitively on X_n for all $n \geq 0$.

(a′) For all nonempty X_n, $g \mid_{X_n}$ has order

$$|X_n| = \mathbf{q}^{[n]} = q_0 q_1 q_2 \cdots q_n \qquad (q_0 = 1).$$

(b) For all nonempty X_n the cyclic group $\langle g^{\mathbf{q}^{[n-1]}} \rangle$ acts trivially on X_{n-1} and transitively on each fiber of $p : X_n \longrightarrow X_{n-1}$.

(c) Every $\langle g \rangle$-orbit on $\mathcal{E}(X, x_0)$ is dense.

(c′) The cyclic group $\langle g \rangle$ acts on $\mathcal{E}(X, x_0)$ with a dense orbit.

Under these equivalent conditions we call g **spherically transitive.**

Proof. Suppose that $X_n \neq \emptyset$ and put $q = |X_n| = \mathbf{q}^{[n]}$. Then $\langle g \rangle$ is transitive on X_n iff $g \mid_{X_n}$ is a q-cycle iff $g \mid_{X_n}$ has order q, whence $(a) \Longleftrightarrow (a')$. Write $q = q' \cdot q_n$ where $q' = \mathbf{q}^{[n-1]} = |X_{n-1}|$, and observe that $p : X_n \longrightarrow X_{n-1}$ is $\langle g \rangle$-equivariant with each fiber of cardinal q_n. Assuming (a), then $g^{q'}$ is trivial on X_{n-1} and on X_n is a product of q' disjoint q_n-cycles. Since $g^{q'}$ leaves each (cardinal q_n) fiber of $X_n \longrightarrow X_{n-1}$ invariant it must be transitive on each such fiber, whence $(a) \Longrightarrow (b)$.

Assume (b) and put $m_n =$ *order of* $g \mid_{X_n}$. Then $m_0 = 1$ and we prove $(b) \Longrightarrow (a')$ by showing inductively that $m_n = \mathbf{q}^{[n]}$. Assuming that $m_{n-1} = \mathbf{q}^{[n-1]} = q'$, it follows from (b) and the discussion above that $g^{q'} \mid_{X_n}$ has order q_n, whence $m_n = q' q_n = \mathbf{q}^{[n]}$, as claimed.

For $(a) \Longrightarrow (c)$, consider a $\langle g \rangle$-orbit $E \subset \mathcal{E}(X, x_0)$. For every nonempty X_n, $\mathcal{E}(X, x_0) \longrightarrow X_n$ is defined and $\langle g \rangle$-equivariant, so E maps onto X_n by (a). Since $\mathcal{E}(X, x_0)$ is the inverse limit of such X_n, it follows that E is dense.

Trivially $(c) \Longrightarrow (c')$. For $(c') \Longrightarrow (a)$ let $E \subset \mathcal{E}(X, x_0)$ be a dense orbit. If $X_n \neq \emptyset$ then E maps $\langle g \rangle$-equivariantly onto X_n, hence X_n is a $\langle g \rangle$-orbit.

∎

(4.4) **Proposition.** *Let $g \in G = G(X, x_0)$ and for each $n \geq 0$ let (c_n) denote the condition*

(c_n) If $X_n \neq \emptyset$ then $\langle g \rangle$ acts transitively on X_n, and, for all $x \in X_n$, $g^{\mathbf{q}^{[n]}}$ is spherically transitive on (N_x, x).

The following conditions are equivalent:

(a) g is spherically transitive.

(b) The condition (c_n) holds for all $n \geq 0$.

(b') The condition (c_n) holds for some n for which $X_n \neq \emptyset \geq 0$.

Proof. Note that $(a) = (c_0)$. We first show $(c_0) \Longrightarrow (c_1)$: Say $X_1 \neq \emptyset$. Then g is transitive on X_1 and if $X_n \neq \emptyset$ then g is transitive on X_n and $p^{n-1} : X_n \longrightarrow X_1$ is $\langle g \rangle$-equivariant. It follows that g^{q_1} is trivial on X_1 and transitive on the fibers of $X_n \longrightarrow X_1$. (See the proof of $(a) \iff (b)$ in (4.3).) It follows that for each $x \in X_1$, g^{q_1} is transitive on the $(n-1)$-sphere of (N_x, x). This shows, by (4.3) (a), that g^{q_1} is spherically transitive on (N_x, x) for $x \in X_1$, hence (c_1).

$(c_{n-1}) \Longrightarrow (c_n)$: Say $X_n \neq \emptyset$. Put $q = \mathbf{q}^{[n]} = q'q_n$ with $q' = \mathbf{q}^{[n-1]}$. By (c_{n-1}), g is transitive on X_{n-1} and $g^{q'}$ is transitive on each fiber of $X_n \longrightarrow X_{n-1}$, so g is transitive on X_n. Further, for $x \in X_n$, applying $(c_0) \Longrightarrow (c_1)$ to $g^{q'}$ on $(N_{p(x)}, p(x))$, we see that $g^q = (g^{q'})^{q_n}$ on (N_x, x) is spherically transitive, whence (c_n).

$(c_{n-1}) \Longrightarrow (c_n)$ (if $X_n \neq \emptyset$): By ((4.3) (a)) it suffices to show that g acts transitively on each nonempty X_m. For $m = n$ this is part of our hypothesis. For $m < n$, X_m is a g-equivariant quotient of X_n. If $m > n$ then $X_m \longrightarrow X_n$ is g-equivariant, g is transitive on X_n and by (c_n), $g^{\mathbf{q}^{[n]}}$ is transitive on each fiber of $X_m \longrightarrow X_n$, whence g is transitive on X_m.

Clearly the proposition follows from the implications proved above.

■

(4.5) **Proposition.** *(Existence.) The* $\mathbf{q}$*-adic adding machine (3.5) defines a spherically transitive element $g \in G(X, x_0)$ and $\langle g \rangle$ acts freely (with dense orbits) on $\mathcal{E}(X, x_0)$.*

Proof. In the cyclic model $Y = Y(\mathbf{q})$ of (3.4), $Y_n = \mathbb{Z}/\mathbf{q}^{[n]}\mathbb{Z}$ for $q_n > 0$ and $g(y) = y+1$ for $y \in Y_n$. Clearly g is transitive on Y_n, so g is spherically transitive.

If $q_n > 0$ for all n then

$$\mathcal{E}(Y, y_0) = Ends(Y) = \varprojlim_n \mathbb{Z}/\mathbf{q}^{[n]}\mathbb{Z} = \widehat{\mathbb{Z}}_{\mathbf{q}},$$

where $\widehat{\mathbb{Z}}_{\mathbf{q}}$ denotes the ring of $\mathbf{q}$-adic integers (3.4). The action of g on $\widehat{\mathbb{Z}}_{\mathbf{q}}$ is translation by 1, so $\langle g \rangle$ acts freely with dense orbits.

■

(4.6) **Theorem.** *Let $g \in G = G(X, x_0)$ be spherically transitive.*

(a) (Conjugacy) If $g' \in G$ is also spherically transitive then g' is G-conjugate to g.

(b) The centralizer $Z_G(g)$ is the closure $\overline{\langle g\rangle}$ of the cyclic group $\langle g\rangle$ generated by g.

Proof of (b). Let $Z = Z_G(g)$ and $H = \overline{\langle g\rangle}$. Clearly $H \leq Z$ and both are closed subgroups of G. To show their equality it suffices to show that they have the same restriction to B_n, or equivalently, to X_n, for each $n \geq 0$. But on X_n, $g_n = g|_{X_n}$ is a (transitive) $\mathbf{q}^{[n]}$-cycle, so $\langle g_n\rangle$ $(= res_{X_n}(H))$ is already the centralizer of g_n in the full symmetric group on X_n. Thus $res_{X_n}(Z) \leq res_{X_n}(H)$, whence (b).

Proof of (a): Since G is compact its conjugacy classes are closed. Thus it suffices to show that g' can be approximated by conjugates of g. For this it suffices to show that for each $n \geq 0$, g and g' have conjugates that agree on B_n. Since $G \longrightarrow G(B_n, x_0)$ is surjective (4.2) we can reduce to the case $X = B_n$. Then, by induction, $g\ |_{B_{n-1}}$and $g'\ |_{B_{n-1}}$ are conjugate in $G' = G(B_{n-1}, x_0)$. Since $G \longrightarrow G'$ is surjective we can replace g' by a G-conjugate and reduce to the case in which $g'\ |_{B_{n-1}} = g\ |_{B_{n-1}}$. If $X_n = \emptyset$ we are done; thus assume that $X_n \neq \emptyset$. Consider

$$p : T = X_n \longrightarrow S = X_{n-1}.$$

Put $q = \mathbf{q}^{[n-1]} = |S|$. Then g and g' induce the same q-cycle on S and g^q and g'^q each act transitively on all fibers of $p : T \longrightarrow S$. For $s \in S$ put $T_s = p^{-1}(s)$. Then

$$G(n-1) = Ker(G \longrightarrow G') = \prod_{s\in S} Aut(T_s).$$

We conclude the proof by showing that $g' = hgh^{-1}$ for some $h \in G(n-1)$. This results from the next lemma.

■

(4.7) **Lemma** *Let $p : T \longrightarrow S$ be a map of finite sets. Put $T_s = p^{-1}(s)$ for $s \in S$,*

$$G = Aut(p) := \{g = (g_T, g_S) \in Aut(T) \times Aut(S) \mid pg_T = g_S p\},$$

and

$$G^1 = Ker(G \longrightarrow Aut(S)) = \prod_{s\in S} Aut(T_s).$$

Let $q = |S|$. Let $g, g' \in G$ induce the same q-cycle on S and assume that for some $s_0 \in S$, $g^q\ |_{T_{s_0}}$ and $g'^q\ |_{T_{s_0}}$ are conjugate in $Aut(T_{s_0})$. Then $g' = hgh^{-1}$ for some $h \in G^1$.

Proof. Identify S with $\mathbb{Z}/q\mathbb{Z}$ so that $g(s) = g'(s) = s + 1$ for $s \in S$ and $s_0 = 0$. Then $T = \coprod_{s\in S}$, g and g' on T consist of bijections $g_s, g'_s : T_s \longrightarrow T_{s+1}$ and we seek $h = (h_s)_{s\in S}$, $h_s \in Aut(T_s)$, such that the diagrams

$$(1)_s \qquad \begin{array}{ccc} T_s & \xrightarrow{g_s} & T_{s+1} \\ h_s \downarrow & & \downarrow h_{s+1} \\ T_s & \xrightarrow{g'_s} & T_{s+1} \end{array}$$

commute for all $s \in S$.

For integers s, $0 \le s \le q$, define $g_{[s]} : T_0 \longrightarrow T_s$ by

(2)
$$g_{[s]} = \begin{cases} g_{s-1} \cdot g_{s-2} \cdots g_0 & \textit{if } s > 0; \\ \textit{identity} & \textit{if } s = 0 \end{cases}$$

Similarly, define ${g'}_{[s]}$. Then we have

$$g^q \mid_{T_0} = g_{[q]} \textit{ and } {g'}^q \mid_{T_0} = {g'}_{[q]}.$$

By hypothesis there is a $k \in Aut(T_0)$ such that

(3)
$$k g_{[q]} k^{-1} = {g'}_{[q]}.$$

Now define $h_s : T_s \longrightarrow T_s$ (for $0 \le s < q$) by

(4)
$$h_s = {g'}_{[s]} k^{-1} {g'}^{-1}_{[s]}.$$

We complete the proof by showing that $(1)_s$ commutes for $0 \le s < q$.

Case 1: $s \ne q-1$. Then

$$\begin{aligned} h_{s+1} g_s &= {g'}_{[s+1]} k {{g'}_{[s+1]}}^{-1} g_s \\ &= {g'}_s {g'}_{[s]} k {g'}^{-1}_{[s]} = {g'}_s h_s. \end{aligned}$$

Case 2: $s = q-1$. Then $(1)_{q-1}$ takes the form

$$h_{q-1} = {g'}_{[q-1]} k g^{-1}_{[q-1]} \quad \begin{array}{ccc} T_{q-1} & \xrightarrow{g_{q-1}} & T_0 \\ \downarrow & & \downarrow \quad h_0 = k \\ T_{q-1} & \xrightarrow{{g'}_{q-1}} & T_0 \end{array}$$

We require that

$$k g_{q-1} = {g'}_{q-1} {g'}_{[q-1]} k g^{-1}_{[q-1]} \qquad (= {g'}_{[q]} k g^{-1}_{[q-1]})$$

i.e., that

$${g'}_{[q]} = k g_{q-1} g_{[q-1]} k^{-1} \qquad (= k g_{[q]} k^{-1}),$$

which is just (3).

■

(4.8) **Corollary.** *The number of spherically transitive automorphisms of* $(B_n, x_0) = X(\mathbf{q}[\ge n])$ *is*

$$\left(\prod_{i=1}^{n} (q_i!)^{\mathbf{q}^{[i-1]}}\right) / \mathbf{q}^{[n]}.$$

Proof. Let $g \in G(B_n, x_0)$ be spherically transitive. From (4.6) we conclude that the number of spherically transitive elements is $|G(B_n, x_0)|/|\langle g\rangle|$, so the corollary follows from (4.2).

■

(4.9) **Subgroups.** Let H and C be closed subsets of G. Since the subgroups

$$G(n) = Ker(G \longrightarrow G(B_n, x_0))$$

form a base for neighborhoods of 1 in G, it follows that

$$H = \bigcap_n H \cdot G(n),$$

and similarly for C. Hence,

$$H \bigcap C = \bigcap_n (H \cdot G(n) \cap C \cdot G(n)).$$

These sets are all compact. Hence

$$H \cap C \neq \emptyset$$

if and only if

$$H \cdot G(n) \cap C \cdot G(n) \neq \emptyset \qquad \textit{for all } n \geq 1,$$

if and only if

$$r(H) \bigcap r(C) \neq \emptyset \qquad \textit{for all } n \geq 1,$$

where $r : G \longrightarrow G(B_n, x_0)$ is the restriction homomorphism.

Taking H to be a closed subgroup of G and C the (closed) conjugacy class of spherically transitive elements (Theorem (4.6)) we obtain, using (4.3):

(4.10) **Proposition.** *For a closed subgroup $H \subseteq G$, the following conditions are equivalent.*

(a) H contains a spherically transitive element.
(b) For each $n \geq 1$, H contains an element that acts transitively on X_n.

5. Dynamics on the ends of X and interval renormalization.

(5.1) **Dynamics on $\mathcal{E}(X, x_0)$.** Assume that $q_n \geq 2$ for all $n > 0$. We shall use the product model $X(\mathbf{q})$ (cf. (3.3)) for (X, x_0). Thus,

$$X_n = C_{q_0} \times C_{q_1} \times \cdots \times C_{q_n}$$

where $q_0 = 1$ and $C_m = \mathbb{Z}/m\mathbb{Z}$, and $p : X_n \longrightarrow X_{n-1}$ is projection away from the last factor. Order C_m so that $0 < 1 < \cdots < m-1$, and then order each X_n lexicographically, so that $X_n \longrightarrow X_{n-1}$ preserves the relation $\leq$. Relative to this order structure (cf. (2.4)) we have, as in (2.6), a homeomorphism

$$K = K(\mathbf{q}) \longrightarrow \mathcal{E}(X(\mathbf{q})) = Ends(X(\mathbf{q}))$$

from the "$\mathbf{q}$-adic Cantor set" $K \subset [0,1]$. The action of $G(X(\mathbf{q}))$ on $\mathcal{E}(X(\mathbf{q}))$ thus transports to a continuous action on $K(\mathbf{q})$.

(5.2) **Theorem.** *For $g \in G(X(\mathbf{q}))$ the following conditions are equivalent.*

(a) The subgroup $\langle g \rangle$ has a dense orbit in $K(\mathbf{q})$.

(b) The subgroup $\langle g \rangle$ acts freely on $K(\mathbf{q})$ and all orbits are dense.

(c) The action of g on $K(\mathbf{q})$ is topologically conjugate to the $\mathbf{q}$-adic adding machine, α acting on $\widehat{\mathbb{Z}}_{\mathbf{q}} = \varprojlim_n \mathbb{Z}/\mathbf{q}^{[n]}\mathbb{Z}$ by $\alpha(a) = a + 1$.

Proof. That (a) implies (c) follows from (4.6) and (4.5), noting that by (4.3)(c′), (a) is equivalent to g being spherically transitive on $X(\mathbf{q})$. Clearly $(c) \Longrightarrow (b) \Longrightarrow (a)$.

∎

(5.3) Let (K, f) be a *minimal ordered dynamical system*, as in (I, (2.4)), with interval renormalization index

$$\mathbf{q} = \mathbf{q}(K, f) = (q_1, q_2, q_3, \ldots) \tag{1}$$

as in (I, (2.9)). For $n \geq 0$ we put

$$m_n = \mathbf{q}^{[n]} = q_1 q_2 \cdots q_n \tag{2}$$

and

$$X_n = \begin{cases} \mathbb{Z}/m_n\mathbb{Z} & \textit{if } m_n > 0; \\ \emptyset & \textit{if } m_n = 0. \end{cases} \tag{3}$$

Then, as in (3.4), we have the rooted tree $X = X(\mathbf{q})$ with inverse sequence

$$X_0 = \{x_0\} \xleftarrow{p} X_1 \longleftarrow \cdots \longleftarrow X_{n-1} \xleftarrow{p} X_n \longleftarrow \cdots \tag{4}$$

Fix a base point $k_0 \in K$. For each $n \geq 0$ with $m_n > 0$, let

$$\phi_n : (K, f) \longrightarrow (X_n, \alpha) := (\mathbb{Z}/m_n\mathbb{Z}, +1) \tag{5}$$

be the interval m-renormalization such that $\phi_n(k_0) = 0$. Then as in (I, (2.10)), the diagrams

$$
(6) \qquad \begin{array}{ccc} K & \xrightarrow{\phi_n} & X_n \\ & \searrow_{\phi_{n-1}} & \downarrow p \\ & & X_{n-1} \end{array}
$$

commute. Thus, the tree dynamical system (X, α) models the "interval renormalizable quotient" of (K, f).

The order structure (linear or cyclic) on K induces a similar structure on each X_n so that the maps ϕ_n and p are weak order preserving.

Suppose that (K, f) is infinitely interval renormalizable, so that $q_n > 0$ for all n. Passing to the inverse limit in (4) we obtain a morphism

$$\widehat{\phi} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1) = (Ends(X), \alpha)$$

which is weak order preserving for the inverse limit order structure on $Ends(X)$. If K is a compact subset of $\mathbb{R}$ or S^1 then (cf. (I, (2.12))) $\widehat{\phi}$ is surjective, and injective except perhaps for countably many 2-point interval fibers.

6. Some group theoretic renormalization operators.

(6.1) **A wreath product construction.** Let X be a finite set with q elements, and Q a group of permutations of X such that

$$(1) \qquad Q^c := \textit{the set of } q\textit{-cycles in } Q \neq \emptyset.$$

Let H be a group and consider the wreath product

$$G = H^X \rtimes Q.$$

For $h \in H^X$ and $x \in X$, we write $h_{(x)} \in H$ for its x-component. We identify h with its image $(h, 1)$ in G. For $g \in Q$ we have $(1, g)(h, 1)(1, g)^{-1} = (g(h), 1)$, where

$$(2) \qquad g(h)_{(x)} = h_{(g^{-1}x)}$$

Let $\pi : G \longrightarrow Q$ be the natural projection, and put

$$(3) \qquad G^c = \pi^{-1}(Q^c).$$

Let

$$g = (h, g_1) \in G^c \qquad (h \in H^X, g_1 \in Q^c).$$

Then

$$g^q = (h \cdot g_1(h) \cdot \dots \cdot g_1^{(q-1)}(h), 1) \in H^X$$

and

(4) $$g^q_{(x)} = h_{(x)} \cdot h_{(g_1^{-1}x)} \cdots h_{(g_1^{-(q-1)}x)} \in H.$$

Thus, for $i = 0, \dots, q-1$

(5) $$g^q_{(g_1^{-i}x)} = u_i g^q_{(x)} u_i^{-1},$$

where

$$u_i = h_{(x)} \cdot h_{(g_1^{-1}x)} \cdots h_{(g_1^{-(i-1)}x)}.$$

(6.2) **Renormalization operators.** Suppose that we are given a family $\rho = (\rho_x)_{x\in X}$ of group isomorphisms

(1) $$\rho_x : G \xrightarrow{\cong} H \qquad (x \in X).$$

Then for $g = (h, g_1) \in G^c$ as above, we can define **renormalizing operators** $R^\rho_x : G^c \longrightarrow G$ by

(2) $$R^\rho_x(g) = \rho_x^{-1}(g^q_{(x)}) \in G.$$

We call g a **simultaneous fixed point of** ρ if

(3) $$R^\rho_x(g) = g, \qquad i.e., \ \rho_x(g) = g^q_{(x)}, \ for\ all\ x \in X.$$

The set of these fixed points is denoted

(4) $$FP(\rho) = \{g \in G^c \mid \rho_x(g) = g^q_{(x)}, \ for\ all\ x \in X\}.$$

Suppose that $g \in G^c$ and for some base point $x \in X$, we are given an isomorphism $\rho_0 : G \longrightarrow H$ such that $\rho_0(g) = g^q_{(x)}$. Then, with the notation of (6.1)(5), we can define $\rho_{g_1^{-i}x} = ad(u_i) \circ \rho_0$ (for $i = 0, \dots, q-1$), and it follows then from (6.1)(5) that $g \in FP(\rho)$.

(6.3) **Uniqueness of ρ, given g.** Suppose that $g \in G^c$ is a simultaneous fixed point for both $\rho = (\rho_x)_{x\in X}$ and $\rho' = (\rho'_x)_{x\in X}$, *i.e.*, for $x \in X$,

(1) $$\rho_x(g) = g^q_{(x)} = \rho'_x(g).$$

Put

$$\alpha_x = \rho_x^{-1}\rho'_x \in Aut(G).$$

Then

$$\rho'_x = \rho_x \circ \alpha_x \ and \ \alpha_x(g) = g. \tag{2}$$

Conversely, given any family $\alpha_x \in Aut(G)$ such that $\alpha_x(g) = g$ then g is a simultaneous fixed point of ρ' defined by (2). In case α_x is an inner automorphism $ad(z_x)$ then the condition $\alpha_x(g) = g$ becomes

$$z_x \in Z_G(g).$$

(6.4) **Renormalization operators and conjugation.** Let $\rho = (\rho_x)_{x \in X}$ be a family of isomorphisms $\rho_x : G \longrightarrow H$. Let

$$u = (v, u_1) \in G, \qquad v \in H^X, u_1 \in Q. \tag{1}$$

Define $u(\rho) = (u(\rho)_x)_{x \in X}$ by:

$$\begin{array}{rcl} w_{x,u} & = & v_{(x)} \cdot \rho_{u_1^{-1}x}(u)^{-1} \in H \\ and & & \\ u(\rho)_x & = & ad(w_{x,u}) \circ \rho_{u_1^{-1}x} : G \longrightarrow H. \end{array} \tag{2}$$

Proposition. *$ad(u)$ defines a bijection $FP(\rho) \longrightarrow FP(u(\rho))$.*

Proof. Let $g = (h, g_1) \in G^c$. Then

$$\begin{array}{rcl} u(\rho)_x(ugu^{-1}) & = & w_{x,u}\rho_{u_1^{-1}x}(ugu^{-1})w_{x,u}^{-1} \\ & = & (w_{x,u}\rho_{u_1^{-1}x}(u))\rho_{u_1^{-1}x}(g)(w_{x,u}\rho_{u_1^{-1}x}(u))^{-1} \\ & = & v_{(x)}\rho_{u_1^{-1}x}(g)v_{(x)}^{-1} \qquad (cf.\ (2)). \end{array}$$

On the other hand

$$\begin{array}{rcl} (ugu^{-1})^q_{(x)} & = & (vu_1g^qu_1^{-1}v^{-1})_{(x)} \\ & = & v_{(x)}g^q_{(u_1^{-1}x)}v_{(x)}^{-1}. \end{array}$$

Thus we see that

$$u(\rho)_x(ugu^{-1}) = (ugu^{-1})^q_{(x)}$$

if and only if

$$\rho_{u_1^{-1}x}(g) = g^q_{(u_1^{-1}x)},$$

whence the proposition, since u_1 is a permutation of X.

∎

(6.5) **All $\rho = (\rho_x)_{x\in X}$ are equivalent.** The isomorphisms $\rho_x : G \longrightarrow H$ define an isomorphism

$$\rho : G^X \longrightarrow H^X$$

by

$$\rho(k)_{(x)} = \rho_x(k_{(x)}) \qquad (x \in X).$$

This extends to an isomorphism of wreath products

$$\rho : G^X \rtimes Q \longrightarrow G = H^X \rtimes Q,$$

defined by

$$\rho(k, g_1) = (\rho(k), g_1).$$

Let $g \in G^c$ and $\tilde{g} = \rho^{-1}(g)$. Then

$$g^q_{(x)} = \rho(\tilde{g}^q)_{(x)} = \rho_x(\tilde{g}^q_{(x)}),$$

so

$$\tilde{g}^q_{(x)} = \rho_x^{-1}(g^q_{(x)}) = R^\rho_x(g).$$

Thus, ρ induces a bijection

$$\rho : FP \longrightarrow FP(\rho)$$

where

$$FP = \{g \in G^X \rtimes Q \mid \pi(g) \in Q^c \ \textit{and}\ g^q_{(x)} = \rho(g)\ \ \forall x \in X\}.$$

We now propose to analyze this set FP, which as we have just observed, models all of the $FP(\rho)$.

(6.6) **The iterated wreath product.** Inductively define $Q(n)$ acting on X^n starting with

(1) $$Q(1) = Q, \ \textit{with its given action on}\ X,$$

For $n > 1$,

(2) $$Q(n) = Q^{X^{n-1}} \rtimes Q(n-1),$$

where $(h, g) \in Q(n)$ acts on $(y, x) \in X^n = X \times X^{n-1}$ by

(3) $$(h, g)(y, x) = (h_{(gx)}y, gx)$$

(cf. (III, (2.4))). This action is faithful (III, (2.6)). Thus, we have decompositions

(4) $$\begin{aligned} Q(n) &= Q^{X^{n-1}} \rtimes Q^{X^{n-2}} \rtimes \cdots Q^X \rtimes Q \\ &= Q(n-m)^{X^m} \rtimes Q(m) \qquad (m = 1, 2, \ldots, n-1). \end{aligned}$$

The latter defines a projection

$$(5) \qquad {}_n\pi_m : Q(n) \longrightarrow Q(m), \text{ with kernel } Q(n-m)^{X^m}.$$

Writing $\pi = {}_n\pi_{n-1} : Q(n) \longrightarrow Q(n-1)$, and $p : X^n \longrightarrow X^{n-1}$ for projection away from the first factor, we have the inverse system

$$(6) \qquad X \xleftarrow{p} X^2 \xleftarrow{p} \cdots \xleftarrow{p} X^{n-1} \xleftarrow{p} X^n \xleftarrow{p} \cdots$$

which is equivariant for

$$(7) \qquad Q(1) \xleftarrow{\pi} Q(2) \xleftarrow{\pi} \cdots \xleftarrow{\pi} Q(n-1) \xleftarrow{\pi} Q(n) \xleftarrow{\pi} \cdots$$

Put

$$(8) \qquad G = \varprojlim_n Q(n),$$

a profinite group. From the decompositions $Q(n) = Q(n-1)^X \rtimes Q$ we obtain, on letting $n \longrightarrow \infty$, an isomorphism

$$(9) \qquad G \cong G^X \rtimes Q,$$

which we view as an identification. We thus have

$$(10) \qquad \begin{cases} \pi : G \longrightarrow Q, \\ G^c = \pi^{-1}(Q^c), \qquad \text{and} \\ FP(= FP(Q,X)) = \{g \in G^c \mid g^q_{(x)} = g \quad \text{for all } x \in X\}. \end{cases}$$

We further write

$$\pi_n : G \longrightarrow Q(n)$$

for the natural projection. Relative to the decompositions

$$\pi_n : G = G^X \rtimes Q \longrightarrow Q(n) = Q(n-1)^X \rtimes Q$$

we have

$$\pi_n(h, g_1) = (h', g_1)$$

where $h' =$ "π^X_{n-1}"(h) is defined by

$$h'_{(x)} = \pi_{n-1}(h_{(x)}).$$

It follows that

$$(11) \qquad \pi_n(FP) \subset FP(n) := \{g \in Q(n) \mid g^q_{(x)} = {}_n\pi_{n-1}(g) \quad \text{for all } x \in X\}.$$

and similarly it is clear that $\pi(FP(n)) \subset FP(n-1)$. Moreover it is clear that

$$(12) \qquad FP = \varprojlim_n FP(n).$$

(6.7) **Lemma.** *Let $g \in FP$.*

(a) $\pi_n(g)$ is a q^n-cycle in X^n.

(b) The closure $\overline{\langle g\rangle}$ of the cyclic group generated by g is isomorphic to

$$\widehat{\mathbb{Z}}_q := \varprojlim_n \mathbb{Z}/q^n\,\mathbb{Z}$$

(c) The set

$$TopGen(\overline{\langle g\rangle})$$

of topological generators of $\overline{\langle g\rangle}$ consists therefore of elements of the form "g^u", $u \in \widehat{\mathbb{Z}}_q^\times$. We have

$$TopGen(\overline{\langle g\rangle}) \subset FP.$$

Proof. Both (a) and (b) follow once we show by induction on n, that $\pi_n(g)$ has order q^n. For $n = 1$ this follows since $\pi_1(g) \in Q^c$. For $n > 1$ we have from (6.6)(11) that $\pi_n(g)^q_{(x)} = {}_n\pi_{n-1}(\pi_n(g)) = \pi_{n-1}(g)$ for all $x \in X$, so, by induction, $\pi_n(g)^q$ has order q^{n-1}, whence the claim.

If $u \in \mathbb{Z}$ then $(g^u)^q_{(x)} = (g^q)^u_{(x)} = g^u$. By continuity this applies to all $u \in \widehat{\mathbb{Z}}_q$. If $u \in \widehat{\mathbb{Z}}_q^\times$, *i.e.*, if $g^u \in TopGen(\overline{\langle g\rangle})$, then clearly $g^u \in G^c$ and so $g^u \in FP$, whence (c).

■

Note that the proof of (a) shows that each element of $FP(n)$ has order q^n.

(6.8) **Lemma.** *Let $n \geq 2$. Let $g \in Q(n)$, $g' = \pi(g) \in Q(n-1)$, and write*

$$g = (k, g_1) \in Q(n) = Q(n-1)^X \rtimes Q.$$

The following conditions are equivalent.

(a) $g \in FP(n)$.

(b) $g' \in FP(n-1)$, and there exist $e(x) \in \mathbb{Z}/q^{n-1}\,\mathbb{Z}$ such that $\sum_{x\in X} e(x) = 1$ and $k_{(x)} = g'^{e(x)}$ for all $x \in X$.

Proof. Assuming $g' \in FP(n-1)$ and that $k_{(x)} \in \langle g'\rangle$ we can then write $k_{(x)} = g'^{e(x)}$ for a unique $e(x) \in \mathbb{Z}/q^{n-1}\,\mathbb{Z}$, by (6.7). Then we have

$$g^q_{(x)} = k_{(x)} k_{(g_1^{-1}x)} \cdots k_{(g_1^{-(q-1)}x)} = g'^e,$$

where $e = \sum_{0\leq i<q} e(g_1^{-i}x) = \sum_{y\in X} e(y)$. Thus $g \in FP(n)$ if and only if $e = 1$. This shows that (b) implies (a). Assuming (a), we have $g' \in FP(n-1)$. By the above discussion, it suffices to show that $k_{(x)} \in \langle g'\rangle$ for all x. Since g' is a q^{n-1} cycle on X^{n-1}, the centralizer of g' is

$$Z_{Q(n-1)}(g') = \langle g'\rangle. \tag{1}$$

Thus it suffices to show that each $k_{(x)}$ centralizes g'. This follows from (a) since

$$\begin{aligned} g' &= g^q_{(x)} = k_{(x)} k_{(g_1^{-1}x)} \cdots k_{(g_1^{-(q-1)}x)} \\ &= g^q_{(g_1^{-1}x)} = k_{(g_1^{-1}x)} \cdots k_{(g_1^{-(q-1)}x)} k_{(x)}. \end{aligned}$$

∎

Remark. The lemma remains true for $n = 1$ if we define $Q(0) = FP(0) = \{1\}$.

(6.9) **Lemma.** *For $n \geq 2$, each fiber of*

$$\pi : FP(n) \longrightarrow FP(n-1)$$

has cardinal q^{q-1}.

Proof. Let $g' \in FP(n-1)$, $g' = (k', g_1) \in Q(n-2)^X \rtimes Q$. By (6.8) we have $e'(x) \in \mathbb{Z}/q^{n-2}\,\mathbb{Z}$ such that

$$(1') \qquad k'_{(x)} = \pi(g')^{e'(x)} \qquad \textit{for all } x \in X$$

and

$$(2') \qquad \sum_{x \in X} e'(x) = 1 \quad \textit{in } \mathbb{Z}/q^{n-2}\,\mathbb{Z}.$$

An element $g \in Q(n)$ with $\pi(g) = g'$ must have the form $g = (k, g_1) \in Q(n-1)^X \rtimes Q$. with

$$(3) \qquad \pi(k_{(x)}) = k'_{(x)} \qquad (x \in X).$$

For g to belong to $FP(n)$ it is necessary and sufficient, by (6.8), that there exist $e(x) \in \mathbb{Z}/q^{n-1}\,\mathbb{Z}$ such that

$$(1) \qquad k_{(x)} = g'^{e(x)} \qquad \textit{for all } x \in X$$

and

$$(2) \qquad \sum_{x \in X} e(x) = 1 \quad \textit{in } \mathbb{Z}/q^{n-1}\,\mathbb{Z}.$$

Let $p : \mathbb{Z}/q^{n-1}\,\mathbb{Z} \longrightarrow \mathbb{Z}/q^{n-2}\,\mathbb{Z}$ be the natural projection. Then (1) and (3) are satisfied by any choices of

$$(4) \qquad e(x) \in p^{-1}(e'(x)).$$

There are q possible choices for each $e(x)$. It then follows from (2′) that for any such choices of $(e(x))_{x \in X}$, we have $p(\sum_{x \in X} e(x)) = \sum_{x \in X} e'(x) = 1$.

Thus, we can freely choose all but one of the $e(x) \in p^{-1}(e'(x))$, and then the one remaining choice is determined by (2), whence q^{q-1} choices, as claimed.

■

The following theorem now summarizes some of the conclusions that we have drawn.

(6.10) **Theorem.** *Let $G \cong G^X \rtimes Q$ be as in (6.6)(9), and*

$$FP = \{g \in G^c \mid g^q_{(x)} = g \quad \textit{for all } x \in X\}.$$

As in (6.6)(11), put

$$FP(n) = \{g \in Q(n) \mid g^q_{(x)} = {}_n\pi_{n-1}(g) \quad \textit{for all } x \in X\}.$$

(a) $FP(1) = Q^c$, and, for each $n \geq 2$, each fiber of $\pi : FP(n) \longrightarrow FP(n-1)$ has q^{q-1} elements.

(b) $\pi_n(FP) = FP(n)$, which has cardinal $\mid Q^c \mid \cdot (q^{q-1})^{n-1}$.

(c) If $g \in FP$ then $\pi_n(g)$ is a q^n-cycle on X^n, $\overline{\langle g \rangle} \cong \widehat{\mathbb{Z}}_q$, and $TopGen(\overline{\langle g \rangle}) \subset FP$, with equality if and only if $q = 2$.

Proof. The final assertion follows by comparing $\mid \pi_n(FP) \mid = \mid FP(n) \mid$, given by (b), with

$$\mid \pi_n(TopGen(\overline{\langle g \rangle})) \mid = \phi(q^n) = \phi(q) \cdot q^{n-1},$$

where ϕ is the Euler ϕ-function.

■

(6.11) **Renormalization operators on spherically homogeneous trees.** Let

$$\mathbf{X} = X(\mathbf{q})$$

be a spherically homogeneous tree of index

$$\text{(1)} \qquad \mathbf{q} = (q_1, q_2, q_3, \ldots) \qquad (\textit{all } q_n \geq 2)$$

which is **periodic of period** r:

$$\text{(2)} \qquad q_{n+r} = q_n \qquad (\textit{for all } n \geq 1).$$

Put

$$\text{(3)} \qquad q = \mathbf{q}^{[r]} = q_1 q_2 \cdots q_r$$

and

$$X = X_r = \textit{ the } r\textit{-sphere in } \mathbf{X}. \tag{4}$$

For $x \in X$ the normal tree $N(x)$ (denoted N_x in (1.3)) has index $(q_{r+1}, q_{r+2}, \ldots)$, which equals $\mathbf{q}$ in view of (2). Thus

$$N(x) \cong \mathbf{X} \quad \textit{for all } x \in X. \tag{5}$$

Consider the automorphism group

$$G = Aut(\mathbf{X}) = A(\mathbf{q}). \tag{6}$$

As in (4.1) we can write

$$\begin{cases} G &= H^X \rtimes Q, & \textit{where} \\ Q &= A(\mathbf{q}[\leq r]), & \textit{and} \\ H &= A(\mathbf{q}[> r]) \cong A(\mathbf{q}) = G. & \end{cases} \tag{7}$$

Here we identify H with $Aut(N(x))$ for each $x \in X$, via some choice of isomorphisms $N(x) \to \mathbf{X}$ (cf. (5)). Clearly G is isomorphic to the iterated wreath product constructed from (Q, X) as in (6.6).

Recall from (4.8) that

$$Q^c = \textit{the set of } q\textit{-cycles on } X \tag{8}$$

has size

$$\mid Q^c \mid = \Big(\prod_{i=1}^{r} (q_i!)^{\mathbf{q}^{[i-1]}}\Big)/q. \tag{9}$$

For $g \in G$ and $x \in \mathbf{X}$ we write $g_{(x)}$ for the restriction of g to the normal tree $N(x)$:

$$g_{(x)} : N(x) \longrightarrow N(g(x)). \tag{10}$$

If g is spherically transitive on $\mathbf{X}$ then g on X is a q-cycle, and

$$g^q_{(x)} \in H = Aut(N(x)). \tag{11}$$

Moreover (cf. (4.4)) $g^q_{(x)}$ is spherically transitive on $N(x)$. It follows therefore from the Conjugacy Theorem (4.6)(a) that there is an isomorphism

$$t_x : (\mathbf{X}, g) \longrightarrow (N(x), g^q_{(x)}), \tag{12}$$

giving a commutative diagram

$$\begin{array}{ccccc} & \mathbf{X} & \xrightarrow{g} & \mathbf{X} & \\ t_x & \downarrow & & \downarrow & t_x\,. \\ & N(x) & \xrightarrow[g^q_{(x)}]{} & N(x) & \end{array} \tag{13}$$

Putting

$$\rho_x = ad(t_x)\colon G = Aut(\mathbf{X}) \longrightarrow H = Aut(N(x)) \tag{14}$$

we see then that

$$\rho_x(g) = g^q_{(x)}. \tag{15}$$

Thus, with $\rho = (\rho_x)_{x\in X}$, we have

$$g \in FP(\rho). \tag{16}$$

It follows from Theorem (4.6)(6) that

(17) *t_x is unique up to right multiplication by $z_x \in Z_G(g) = \overline{\langle g\rangle}$, the closed cyclic group generated by g.*

Now combining this with (6.5), (6.6), and (6.10) we obtain:

(6.12) **Theorem.** *Keep the notation and assumptions of (6.11).*

(a) Let $g \in G$ be spherically transitive on $\mathbf{X}$. Then there exist rooted tree isomorphisms $t_x : \mathbf{X} \longrightarrow N(x)$, $(x \in X)$, unique up to right composition with an element of $\overline{\langle g\rangle} \cong \widehat{\mathbb{Z}}_q$, such that $g \in FP(\rho)$, where $\rho = (\rho_x)_{x\in X}$ is defined by $\rho_x = ad(t_x)$.

(b) Let $\rho = (\rho_x)_{x\in X}$ be any family of isomorphisms $\rho_x : G \longrightarrow Aut(N(x))$. Let

$$\pi_n : G \longrightarrow G(B_n, x_0)$$

denote restriction to the n-ball B_n in $\mathbf{X}$. We have the following.

(i) $\pi_{nr}(FP(\rho))$ has cardinal

$$\left[\prod_{i=1}^{r}(q_i!)^{\mathbf{q}^{[i-1]}}\right]\cdot q^{(q-1)(n-1)-1}$$

(ii) If $g \in FP(\rho)$ then g is spherically transitive on $\mathbf{X}$ and

$$TopGen(\overline{\langle g\rangle}) \subset FP(\rho),$$

with equality if and only if $q = 2$, i.e., if and only if $\mathbf{q} = (2, 2, 2, \ldots)$ and $r = 1$.

Chapter IV

Closed Normal Subgroups of $Aut(X(\mathbf{q}))$

0. Introduction and notation.

Let $X(\mathbf{q})$ denote the spherically homogeneous rooted tree of index $\mathbf{q} = (q_1, q_2, q_3 \ldots)$, and $G(\mathbf{q}) = Aut(X(\mathbf{q}))$. Our aim is to describe all closed normal subgroups N of $G(\mathbf{q})$. Each such $N \neq \{1\}$ has a "level", the largest $n \geq 0$ such that N acts trivially on the n-ball centered at the root, and the N's at a given level can be described essentially in terms of abelian data; see Theorem (5.4) for a precise statement.

The method used is to present $G(\mathbf{q})$ as an infinite iterated wreath product of symmetric groups. Our analysis applies more generally, to infinite iterated wreath products of appropriate subgroups of symmetric groups. (see Section 4.)

The chapter begins with a general review of normal subgroups of simple wreath products.

1. The symmetric group S_q.

(1.0) We assemble here for reference some familiar facts about the symmetric group S_q of permutations on the set $\{1, \ldots, q\}$. (The reference [Rot] is a good basic group theory resource.)

(1.1) **The alternating group A_q.** The group A_q is the kernel of the signature homomorphism $sgn : S_q \longrightarrow \mathbb{Z}/2\mathbb{Z}$, the latter being identified with the abelianization of S_q, denoted S_q^{ab}, for $q \geq 2$. In fact,

$$A_q = (S_q, S_q) = (S_q, A_q).$$

(1.2) **Small q.** When q is small, there will be "degenerate" behavior. We take care of these special cases in this section.

First, $S_1 = \{1\}$, $A_2 = \{1\}$, and $|A_3| = 3$. For $q = 4$, $A_4' = (A_4, A_4)$, is a Klein 4-group and $A_4 = A_4' \rtimes \langle t \rangle$, where $t = (123)$ is a 3-cycle, cyclically permuting the non-trivial elements of A_4'. Clearly then A_4' is the only proper normal subgroup of A_4 and $(S_4, A_4') = (A_4, A_4') = A_4'$. Moreover,

$$S_4/A_4' \cong S_3,$$

the only non-abelian subgroup of order 6.

For $q \geq 5$, A_q is a simple group (cf. [Rot]).

From the above observations we can deduce,

(1.3) **Proposition.** *Let $H \lhd S_q$.*

(a) Either $H = \{1\}$, A_q, or S_q or else, $q = 4$ and $H = A_4'$.
(b) If $H \neq S_q$, then $H = (S_q, H)$.
(c) If $H \neq \{1\}$ then the action of H on $\{1, 2, \ldots, q\}$ is transitive and even primitive if $H \neq A_4'$.

(1.4) **Proposition.** *Suppose that $\{1\} \neq P \lhd Q \leq S_q$. Let $Y = \{1, 2, \ldots, q\}$.*

(a) If Q acts transitively on Y, then P has no fixed points in Y.
(b) If Q acts primitively on Y, then P acts transitively on Y.

Proof. Since $P \lhd Q$ and Q acts transitively, the orbits of P on Y are permuted transitively by Q, whence both (a) and (b).

■

2. Wreath products.

(2.1) **Semidirect products.** Given groups H, Q and a homomorphism $\alpha : Q \longrightarrow Aut(H)$, we have the semi-direct product $G = H \rtimes_\alpha Q$ (cf. [Rot]). It is the set $H \times Q$ with a multiplication defined by

$$(h, q)(h', q') = (h \cdot \alpha(q)(h'), qq').$$

Identifying $H = (H, 1) \leq G$ and $Q = (1, Q) \leq G$, we have

$$(h, q) = (h, 1)(1, q) = hq$$

and

$$qhq^{-1} = \alpha(q)(h)$$

in G. Thus, $G = H \cdot Q$, $H \cap Q = \{1\}$, $H \lhd G$ and $G/H = Q$.

The commutator subgroup of G is

$$(G, G) = (G, H) \rtimes (Q, Q).$$

This yields

$$\begin{aligned} G^{ab} &= H/(G, H) \times Q^{\text{ab}} \\ &= H^{ab}/(Q, H^{\text{ab}}) \times Q^{\text{ab}}. \end{aligned}$$

Here, H^{ab} is a Q-module, and $H^{\text{ab}}/(Q, H^{\text{ab}})$ is the quotient obtained by trivializing the Q-action. We shall write

$$\sigma_Q : H \longrightarrow H^{\text{ab}}/(Q, H^{\text{ab}})(= H/(Q, H))$$

for the natural projection.

(2.2) **Wreath products $S^X \rtimes Q$.** Let S and Q be groups and X a Q-set. Then we can form the product

$$S^X = \{f : X \longrightarrow S\}.$$

Then Q also acts on S^X by translation,

$$(qf)(x) = f(q^{-1}x).$$

The resulting semidirect product

$$G = S^X \rtimes Q$$

is called the **wreath product** associated to (S, Q, X). (Cf. [We] for historical remarks regarding these constructions and extensive bibliography.)

(2.3) **Commutators in $G = S^X \rtimes Q$.** Assume that X is finite. Then we have

$$(S^X, S^X) = (S, S)^X.$$

Hence,

$$(S^X)^{\text{ab}} = (S^{\text{ab}})^X.$$

(Note that this also holds when X is infinite, provided that for some $n \geq 0$, each element of (S, S) is a product of at most n commutators.) The quotient obtained by trivializing the action of Q on this (permutation) Q-module is clearly just

$$\begin{aligned} S^X/(G, S^X) &= (S^{\mathrm{ab}})^X/(Q, (S^{\mathrm{ab}})^X) \\ &= (S^{\mathrm{ab}})^{Q\backslash X}. \end{aligned}$$

Hence,

$$G^{\mathrm{ab}} = (S^X \rtimes Q)^{\mathrm{ab}} = (S^{\mathrm{ab}})^{Q\backslash X} \times Q^{\mathrm{ab}}.$$

As in (2.1), we shall write

$$\sigma_Q : S^X \longrightarrow (S^{\mathrm{ab}})^{Q\backslash X} = S^X/(G, S^X)$$

for the natural projection. For any $s \in S$, let $\bar{s}$ denote its image in S^{ab}. Then σ_Q is defined by

$$(\sigma_Q f)(Q \cdot x) = \prod_{y \in Q \cdot x} \overline{f(y)}.$$

The kernel

$$K_Q = Ker(\sigma_Q) = (G, S^X)$$

is generated by (S^X, S^X) together with all elements

$$(q, f) = qfq^{-1}f^{-1} = q(f)f^{-1}$$

for all $q \in Q$, $f \in S^X$.

(2.4) **The $(S^X \rtimes Q)$-set $X \times Y$.** Suppose that we are further given an S-set Y. Then the wreath product $G = S^X \rtimes Q$ acts on $X \times Y$ by:

$$(f, q)(x, y) = (qx, f(qx)y).$$

This is easily checked to be a group action. Moreover, the projection $X \times Y \longrightarrow Y$ is equivariant for the projection $G \longrightarrow Q$.

(2.5) **Transitivity.** A quick computation shows that

$$G\backslash(X \times Y) = (Q\backslash X) \times (S\backslash Y).$$

Hence:

Proposition. *The action of $S^X \rtimes Q$ on $X \times Y$ is transitive iff the actions of Q on X and S on Y are both transitive.*

(2.6) **Fidelity.** Let $G_{(x_0,y_0)}$ denote the stabilizer in G of any point $(x_0, y_0) \in X \times Y$. Then

$$G_{(x_0,y_0)} = H(x_0, y_0) \rtimes Q_{x_0}$$

where

$$H(x_0, y_0) = \{f \in S^X \mid f(x_0) \in S_{y_0}\}.$$

Consequently another quick computation shows,

(2.7) **Proposition.** *The action of $S^X \rtimes Q$ on $X \times Y$ is faithful iff the actions of Q on X and S on Y are both faithful.*

(2.8) **Elements of finite order.** Consider a semidirect product $G = H \rtimes Q$, where the action of $q \in Q$ on $h \in H$ is denoted $q(h)$ $(= qhq^{-1}$ *in* $G)$. For $n > 0$ we have, in G,

$$(hq)^n = hq(h) \cdots q^{n-1}(h)q^n.$$

Say q has finite order n. then we put

$$N_q(h) = hq(h) \cdots q^{n-1}(h) \in H,$$

and we see that

$$(1). \qquad order(hq) = order(q) \cdot order(N_q(h))$$

Now, suppose that $H = S^X$, so that $G = S^X \rtimes Q$ is a wreath product, $q \in Q$ has order n, and $h \in S^X$. Then, for $x \in X$,

$$(2) \qquad N_q(h)(x) = h(x)h(q^{-1}x) \cdots h(q^{-(n-1)}x).$$

Moving x in its $\langle q \rangle$-orbit, $\{q^{-i}x \mid 0 \leq i \leq n-1\}$, only affects (2) by a cyclic permutation of factors, hence by a conjugation. Thus,

$$(3) \qquad order(N_q(h)) = LCM\{x \in \langle q \rangle \backslash X \bigm| order(N_q(h)(x))\}.$$

Suppose now that

$$Q = \langle q \rangle \textit{ and } |Q| = n.$$

For $s \in S$, let $\bar{s}$ denote its image in S^{ab}. For $h \in S^X$ we see from (2) that $\overline{N_q(h)(x)}$ is constant on Q-orbits. The resulting $\overline{N_q(h)(x)} \in (S^{\mathrm{ab}})^{Q\backslash X}$ is clearly just $\sigma_Q(h)$, using the notation of (2.3).

Suppose further that S is abelian. Then $N_q(h)$ is constant on Q—orbits, and

$$order(N_q(h)) = order(\sigma_Q(h)).$$

Since $\sigma_Q : S^X \longrightarrow S^{Q\backslash X}$ is surjective, we can choose $h \in S^X$ to make $N_q(h)$ have any order that occurs in $S^{Q\backslash X}$.

(2.9) **Example.** Let $Q = \langle q \rangle$ where q is a transitive n-cycle on X (so $|X| = n$). In addition, let $S = \langle s \rangle$ where s is a transitive m-cycle on the set Y ($|Y| = m$). For $h \in S^X$, write $h(x) = s^{e_x}$ where $e_x \in \mathbb{Z}/m\mathbb{Z}$. Then our map $\sigma_Q : S^X \longrightarrow S^{Q\backslash X} = S$ is given by

$$h \mapsto \prod_{x \in X} h(x) = s^e$$

where

$$e = \sum_{x \in X} e_x.$$

Consequently,

$$order(hq) = n \cdot m_h$$

where

$$m_h = order(\sigma_Q(h)) = order(e)$$

for e considered as an element of $\mathbb{Z}/m\mathbb{Z}$. In particular,

$$order(hq) = nm \quad \Longleftrightarrow \quad order(\sigma_Q(h)) = m \quad \Longleftrightarrow \quad e \in (\mathbb{Z}/m\mathbb{Z})^\times$$

where $(\mathbb{Z}/m\mathbb{Z})^\times$ denotes the group of units in $\mathbb{Z}/m\mathbb{Z}$. Thus, in this case hq defines a transitive cycle on $X \times Y$ (cf. (2.4)). For example, defining h_0 by $h_0(x_0) = s$ (for some $x_0 \in X$) and $h_0(x) = 1$ for $x \neq x_0$, we have $\sigma_Q(h_0) = s$, so $order(h_0 q) = nm$.

(2.10) **Conjugacy.** Let $G = H \rtimes Q$ (semi-direct), $q \in Q$, and $h, h' \in H$. We shall determine when hq and $h'q$ are conjugate in G. Say $f \in H, r \in Q$, and $(fr)(hq)(fr)^{-1} = h'q$. We have

$$\begin{aligned} frhqr^{-1}f^{-1} &= fr(h)rqr^{-1}f^{-1} \\ &= [fr(h)(rqr^{-1})(f)^{-1}](rqr^{-1}) \\ &= h'q. \end{aligned}$$

It follows that

$$r \in Z_Q(q)$$

and

$$h' = fr(h)q(f)^{-1}.$$

Now suppose further that H *is abelian.* Then

$$h' = r(h)fq(f)^{-1}$$

and the elements $fq(f)^{-1}$ form the group $(\langle q \rangle, H) \leq H$. Thus, we conclude:

(2.11) **Proposition.** *Let $G = H \rtimes Q$ with H abelian. Let $q \in Q$, $h, h' \in H$. Then hq and $h'q$ are conjugate in G if and only if the images of h and h' in $H/(\langle q \rangle, H)$ lie in the same $Z_Q(q)$-orbit.*

3. Normal subgroups of wreath products.

(3.1) **Notation.** We fix a wreath product $G = S^X \rtimes Q$ where X is a finite Q-set. For $s \in S$, $x \in X$, define $s_x \in S^X$ by $s_x(x) = s$ and $s_x(x') = 1$ for $x \neq x'$. Thus, $S^X = \prod_x S_x$, where $S_x = \{s_x \mid s \in S\}$ is the copy of S in the x-coordinate of S^X. Note that for $q \in Q$, $q(s_x)$ (or qs_xq^{-1} in G) $= s_{qx}$.

If $N \leq S^X$ then let N_x denote the image of N under the projection into S_x. Clearly $N \leq \prod_x N_x$. Moreover $(S_x, N) = (S_x, N_x) \leq S_x$. Hence:

(3.2) **Lemma.** *If $N \lhd S^X$, then*

$$(S^X, N) = \prod_x (S_x, N_x) \leq N \leq \prod_x N_x.$$

(3.3) **Proposition.** *Suppose that the action of Q on X is transitive. Let $N \lhd G$ and $N \leq S^X$. Then there exists a subgroup $M \lhd S$ such that $N_x = M_x$ for all x, and*

$$(S, M)^X \leq N \leq M^X.$$

Proof. If $q \in Q$, then

$$qN_xq^{-1} = N_{qx}.$$

Since Q acts transitively on X, it follows that the subgroups N_x all define a common subgroup M with $M \lhd S$. Thus, the proposition follows from (3.2).

∎

(3.4) **The groups G_P, K_P, N_P.** For $P \leq Q$, put

$$G_P = S^X \rtimes P \leq G = G_Q.$$

Then as in (2.3) we have

$$\sigma_P : S^X \longrightarrow (S^{\mathrm{ab}})^{P \backslash X},$$

where

$$\sigma_P(f)(P \cdot x) = \prod_{y \in P \cdot x} \overline{f(y)} \in S^{\mathrm{ab}}.$$

If K_p denotes the kernel of σ_P, then

$$K_p = (G_P, S^X)$$

and is generated by the subgroup $(S,S)^X$ together with all elements of the form

$$(p,s_x)=ps_xp^{-1}{s_x}^{-1}=s_{px}{s_x}^{-1}\quad(p\in P,s\in S,x\in X).$$

Note that

$$\begin{aligned}(G_P,G_P)&=K_P\rtimes(P,P)\\(G,G_P)&=K_Q\rtimes(Q,P).\end{aligned}$$

Define

$$N_P=K_P\rtimes(Q,P).$$

Thus,

$$(G_P,G_P)\le N_P\le(G,G_P).$$

Note that σ_P and K_P depend not on P, but only on the quotient $P\backslash X$ of X. For example, suppose that $P\backslash X=Q\backslash X$. Then $K_P=K_Q=(G,S^X)$, and so, $(G,G_P)=N_P$.

(3.5) **Proposition.** *Let $N\lhd G$ have projection $P\lhd Q$ into Q. Thus, $N\le G_P=S^X\rtimes P$. Assume that P has no fixed points in X. Then*

(i) $(N,S^X)=(G_P,S^X)=K_P$,
(ii) $(N,G_P)=(G_P,G_P)=K_P\rtimes(P,P)\le N$, *and*
(iii) $N/(G_P,G_P)\le(G_P)^{ab}=(S^{ab})^{P\backslash X}\times P^{ab}$ *is a G-submodule.*

Suppose that in addition $P\backslash X=Q\backslash X$. Then,

(iv) $(G,N)=(G,G_P)=N_P=K_Q\rtimes(Q,P)\le N$, *and*
(v) $N/N_P\le G_P/(G,G_P)=(S^{ab})^{Q\backslash X}\times(P/(Q,P))$, *where G acts trivially.*

Proof. Put $L=(N,S^X)\le S^X$. Thus, $L\lhd G$. Let $s\in S$ and $x\in X$. As P has no fixed points in X, we can choose $fp\in N,f\in S^X$ and $p\in P$ such that $px=y\neq x$. Note that L contains (fp,s_x). Rewriting, we get

$$(fp,s_x)=fps_xp^{-1}f^{-1}{s_x}^{-1}=fs_yf^{-1}{s_x}^{-1}=(f(y)sf(y)^{-1})_y{s_x}^{-1}.$$

From this it follows that the projection L_x of L into S_x is all of S_x (ie. $L_x=S_x$). It then follows from (3.2) that $(S,S)^X\le L$. Modulo $(S,S)^X$ we have

$$(f(y)sf(y)^{-1})_y{s_x}^{-1}\equiv s_y{s_x}^{-1}=s_{px}{s_x}^{-1}.$$

Such elements, together with $(S,S)^X$, generate $K_P=(G_P,S^X)\ge(N,S^X)=L$, so that $L=K_P$.

Furthermore,

$$N'=N/K_P\lhd G_P/K_P=(S^{\mathrm{ab}})^{P\backslash X}\times P,$$

so that N' projects onto P. From this it follows that $(N', P) = (P, P)$ in G_P/K_P. As $K_P \leq N$, it follows that (N, G_P) contains $K_P \rtimes (P, P) = (G_P, G_P)$, so that

$$(N, G_P) = (G_P, G_P).$$

This proves (i) - (iii).

To prove (iv) and (v) suppose in addition that $P\backslash X = Q\backslash X$. Then (cf. (3.4)) we have

$$K_P = K_Q$$

and

$$(G, G_P) = N_P = K_P \rtimes (Q, P).$$

Moreover, with $N' = N/K_P$ as above, we have

$$N' \lhd G_P/K_P = (S^{\mathrm{ab}})^{Q\backslash X} \times P \lhd G/K_P = (S^{\mathrm{ab}})^{Q\backslash X} \times Q.$$

Thus, $(Q, N') = (Q, P) \leq N'$ in G/K_P, so that it follows from above that

$$N_P = K_P \rtimes (Q, P) = (N, G) \leq N.$$

This completes the proof.

■

(3.6) **Remark.** (cf. (1.4)) In the setting of (3.5) suppose that Q acts faithfully and transitively on X. Then if $1 \neq P \lhd Q$, P has no fixed points in X. If further Q acts primitively on X then P acts transitively on X; in particular $P\backslash X = Q\backslash X$. In this case (3.5) tells us that

$$N_P = K_Q \rtimes (Q, P) \leq N$$

and

$$N/N_P \leq G_P/N_P = S^{\mathrm{ab}} \times (P/(Q, P)) \leq Z(G/N_P).$$

(3.7) **Corollary.** *Assume that Q acts faithfully and primitively on X. Let $N \lhd G = S^X \rtimes Q$, $N \not\leq S^X$. Then there is a unique minimal $P \lhd Q$ such that, with $G_P = S^X \rtimes P$, we have*

$$(G, G_P) = (G, N) \leq N \leq G_P$$

and

$$G_P/(G, G_P) = S^{ab} \times (P/(Q, P)).$$

Proof. Let P denote the projection of N into Q. Then $1 \neq P \lhd Q$. The assumptions imply (cf. (3.6)) that P acts transitively on X, and $|X| \geq 2$ since $P \neq 1$ acts faithfully. Thus, P has no fixed points so that the corollary follows from (iv) and (v) of (3.5)

■

4. Iterated wreath products and rooted trees.

(4.1) **The rooted tree $X = X(\mathbf{Y})$.** Let

$$\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n, \ldots)$$

be a sequence of nonempty sets. We shall assume that each Y_n is finite, say of cardinality $q_n > 0$. Put

$$\mathbf{q} = (q_1, q_2, \ldots, q_n, \ldots).$$

Define sets X_n $(n \geq 0)$ by $X_0 = \{x_0\}$ (a one point set) and for $n > 0$,

$$X_n = Y_1 \times \cdots \times Y_n.$$

Let $p : X_n \longrightarrow X_{n-1}$ denote the natural projection. Then the inverse sequence

$$(1) \qquad X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} \cdots \xleftarrow{p} X_{n-1} \xleftarrow{p} X_n \xleftarrow{p} \cdots$$

defines as in (I, (2.1)), the spherically homogeneous rooted tree

$$X = X(\mathbf{Y}) = X(\mathbf{q})$$

(cf. (II,(3.1) and (3.3))).

(4.2) **The group $G = G((\mathbf{Q}, \mathbf{Y}))$.** Let

$$\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n, \ldots)$$

be a sequence of finite groups such that for each $n \geq 1$, Y_n is a Q_n-set. Given these initial data, we shall define (inductively) a sequence a groups $Q(n)$ $(n \geq 0)$, $Q(n)$ acting on the set X_n and, for $n > 0$, a projection $Q(n) \longrightarrow Q(n-1)$ for which $p : X_n \longrightarrow X_{n-1}$ is equivariant.

Let $Q(0) = \{1\}$, and $Q(1) = Q_1$ acting as given on $X_1 = Y_1$. For $n > 1$,

$$(1) \qquad Q(n) = Q_n^{X_{n-1}} \rtimes Q(n-1)$$

(wreath product as in (2.2)), which acts on $X_n = X_{n-1} \times Y_n$ as in (2.4).

Now we put

$$(2) \qquad G = G((\mathbf{Q}, \mathbf{Y})) = \varprojlim_{Q}(n).$$

Then $G((\mathbf{Q},\mathbf{Y}))$ is a profinite group acting on the tree $X(\mathbf{Y})$ defined by (1).

For a subset H of G let $\overline{H}$ denote its closure. If $H \leq G$ is a closed subgroup then we put

$$H^{\overline{\mathrm{ab}}} = H/\overline{(H,H)},$$

its topological abelianization.

From (2.5) we conclude inductively that

$$G\backslash X_n = Q(n)\backslash X_n = (Q_1\backslash Y_1) \times \cdots \times (Q_n\backslash Y_n). \tag{3}$$

Hence (cf. (2.5)):

(4.3) **Proposition.** *G acts transitively on each X_n iff Q_n acts transitively on Y_n for each n. Furthermore, G acts faithfully on X iff Q_n acts faithfully on Y_n for each n.*

(4.4) From (2.3) we inductively obtain isomorphisms

$$Q(n)^{\mathrm{ab}} \xrightarrow{\cong} (Q_n^{\mathrm{ab}})^{Q(n-1)\backslash X_{n-1}} \times \cdots \times (Q_2^{\mathrm{ab}})^{Q(1)\backslash X_1} \times Q_1^{\mathrm{ab}} \tag{1}$$

which on $Q_n^{X_{n-1}} \leq Q(n)$, induces the homomorphism

$$\sigma_{n-1} = \sigma_{Q(n-1)} : Q_n^{X_{n-1}} \longrightarrow (Q_n^{\mathrm{ab}})^{Q(n-1)\backslash X_{n-1}} \tag{2}$$

of (2.3). Passing to the inverse limit then gives an isomorphism

$$G^{\overline{\mathrm{ab}}} \xrightarrow{\cong} \prod_{n\geq 1} (Q_n^{\mathrm{ab}})^{Q(n-1)\backslash X_{n-1}}. \tag{3}$$

Thus:

Proposition. *If Q_n acts transitively on Y_n for all n then*

$$G^{\overline{ab}} \xrightarrow{\cong} \prod_{n\geq 1} Q_n^{ab} \tag{4}$$

is an isomorphism.

To make the isomorphism (4) explicit, we can express G as the infinite wreath product

$$G = \cdots \rtimes Q_n^{X_{n-1}} \rtimes \cdots Q_2^{X_1} \rtimes Q_1. \tag{5}$$

Write $g \in G$ as its corresponding infinite product,

$$g = \cdots g_n \cdots g_2 \cdot g_1,$$

where $g_n \in Q_n^{X_{n-1}}$. Then we see that

$$\overline{\sigma} : G \longrightarrow \overline{G^{\mathrm{ab}}} = \prod_{n\geq 1} (Q_n^{\mathrm{ab}})^{Q(n-1)\backslash X_{n-1}} \tag{6}$$

sends g to $(\sigma_{n-1}(g_n))_{n\geq 1}$. Here (cf. (2.3)), σ_{n-1} is the homomorphism in (2) above defined by

$$\sigma_{n-1}(g_n)(Q(n-1)\cdot x) = \prod_{y\in Q(n-1)\cdot x} \overline{g_n(y)} \in Q_n^{\mathrm{ab}}, \tag{7}$$

where, for $s \in Q_n$, $\bar{s}$ denotes its image in Q_n^{ab}.

(4.5) **Examples.** 1. Suppose that, for each $n \geq 1$,

$$Q_n = S_{q_n} := \textit{ the permutation group of } Y_n.$$

Then a straightforward inductive argument shows that

$$G = G((\mathbf{Q},\mathbf{Y})) = Aut(X, x_0),$$

the full automorphism group of the rooted tree (X, x_0). In this case G acts transitively on each X_n and we have an isomorphism

$$\overline{G^{\mathrm{ab}}} \xrightarrow{\cong} \prod_{n\geq 1} S_{q_n}^{\mathrm{ab}} = \prod_{n\geq 1} (\mathbb{Z}/2\mathbb{Z}).$$

2. Suppose that each Q_n acts faithfully on Y_n, so that $Q_n \leq S_{q_n}$. Putting

$$\mathbf{q}^{[n]} = |X_n| = q_1 \cdot q_2 \cdots q_n$$

as in (I, (3.3), example 3) we see that

$$|Q(n)| = |Q_1| \cdot |Q_2|^{q_1} \cdots |Q_n|^{\mathbf{q}^{[n-1]}}.$$

Moreover, $G = \varprojlim_n Q(n)$ acts faithfully on X, so $G \leq Aut(X, x_0)$ is a closed subgroup. It is not clear how to characterize which closed subgroups one obtains in this fashion.

3. Consider the case in which all Y_n are the same finite set Y, of cardinality q and all the Q_n are the same subgroup $Q \leq S_q$. Then we have, $\mathbf{q}^{[n]} = q^n$, so that

$$|Q(n)| = |Q|^{1+q+\cdots q^{n-1}} = |Q|^{(q^n-1)/(q-1)}.$$

A particular case of interest is when q is prime and Q is generated by a single q-cycle. Then $Q(n)$ is of order $q^{(q^n-1)/(q-1)}$ and thus is a Sylow-q subgroup of the group $S_{q^{[n]}}$ of permutations of $X_n = Y^n$. Then for each n, $Q(n)$ acts transitively on X_n and Q_n acts primitively on Y_n.

4. Suppose that for each $n \geq 1$, $Q_n \leq S_{q_n}$ is cyclic, generated by some q_n-cycle, c_n. Then it follows by induction, from (2.7), that for each $n \geq 1$, $Q(n)$ contains a transitive cycle on X_n (of order $\mathbf{q}^{[n]}$). Furthermore, this transitivity for each n then implies (from II, (4.10)) that $G((\mathbf{Q}, \mathbf{Y})) = \varprojlim_n Q(n)$ contains spherically transitive elements on $X(\mathbf{Y})$. In fact (2.7) provides an inductive construction of such elements. Indeed, we can give the following method for detecting them.

As in (4.4)(5), write G in as an infinite wreath product,

$$G = \cdots \rtimes Q_n^{X_{n-1}} \rtimes \cdots Q_2^{X_1} \rtimes Q_1.$$

and $g \in G$ as

$$g = \cdots g_n \cdots g_2 \cdot g_1,$$

where $g_n \in Q_n^{X_{n-1}}$. Since each $Q(n)$ acts transitively on X_n and each (cyclic) Q_n is abelian, we have the following simplification of the "abelianization" homomorphism $\bar{\sigma}$,

$$\overline{\sigma} : G \longrightarrow \overline{G^{\mathrm{ab}}} = \prod_{n \geq 1} Q_n$$

given by

$$\overline{\sigma}(g) = (\sigma_{n-1}(g_n))_{n \geq 1}.$$

Here (cf. (4.5)(7)),

$$\sigma_{n-1}(g_n) = \prod_{x \in X_{n-1}} g_n(x)$$

is an element in Q_n.

It follows inductively from the discussion in (2.9) that the following are equivalent:

(1) $g_n \cdots g_2 \cdot g_1 \in Q(n)$ has order $\mathbf{q}^{[n]} = q_1 \cdot q_2 \cdot \cdots \cdot q_n = |X_n|$
(2) $\sigma_{i-1}(g_i) \in Q_i$ has order q_i for each $i \leq n$.
(3) $\sigma_{i-1}(g_i)$ generates Q_i for each $i \leq n$.

Thus, from (I, (4.3)), we obtain:

(4.6) **Proposition.** *For all $n \geq 1$ let $Q_n = \langle s_n \rangle$ be generated by a q_n-cycle s_n. Let $g \in G$ have the expansion*

$$g = \cdots g_n \cdots g_2 \cdot g_1,$$

where $g_n \in Q_n^{X_{n-1}}$ as in (4.3), example 4. For $x \in X_{n-1}$ put $g_n(x) = s_n^{e_x}$, $e_x \in \mathbb{Z}/q^n\,\mathbb{Z}$ and

$$\sigma_{n-1}(g_n) = \prod_{x \in X_{n-1}} g_n(x) = s_n^{e(n)}$$

where $e(n) = \sum_{x \in X_{n-1}} e_x \in \mathbb{Z}/q^n\,\mathbb{Z}$. *Thus,* $\bar{\sigma} : G \longrightarrow \overline{G^{ab}} = \prod_n Q_n$ *sends* g *to* $(\sigma_{n-1}(g_n))_{n \geq 1}$. *Then the following conditions are equivalent.*

(a) g *is a spherically transitive automorphism of* X.
(b) For each n, $\sigma_{n-1}(g_n)$ *generates* Q_n.
(c) For each n, $e(n) \in (\mathbb{Z}/q^n\,\mathbb{Z})^\times$, *the group of units of* $\mathbb{Z}/q^n\,\mathbb{Z}$.

(4.7) **Proposition.** *Let all hypotheses be as in (4.6) Let* $g, g' \in G$ *be spherically transitive elements. Then* g *and* g' *are conjugate in* G *iff they have the same image under* $\bar{\sigma} : G \longrightarrow \overline{G^{ab}} = \prod_{n \geq 1} Q_n$.

Proof. Assume first that $\bar{\sigma}(g) = \bar{\sigma}(g')$. As conjugacy classes in the profinite group are closed, it suffices to show that the images $g(n)$ and $g'(n)$ of g and g' in $Q(n)$ are conjugate, for each $n \geq 1$. For $n = 1$, $Q(1) = Q_1$ is abelian and the hypothesis implies that $g(1) = g'(1)$. Assume now that $n > 1$. Write g as

$$g = g_n g(n-1)$$

and

$$g' = g'_n g'(n-1)$$

with $g_n, g'_n \in Q_n^{X_{n-1}}$. By induction, $g(n-1)$ and $g'(n-1)$ are conjugate. Replacing $g'(n)$ by a conjugate in $Q(n)$, we can then reduce to the case of $g'(n-1) = g(n-1) = q \in Q(n-1)$. By (2.11), $g(n) = g_n q$ and $g'(n) = g'_n q$ are conjugate in $Q(n)$ iff the images of g_n and g'_n in $Q_n^{X_{n-1}}/(\langle q \rangle, Q_n^{X_{n-1}}) = Q_n^{\langle q \rangle \backslash X_{n-1}}$ lie in the same $Z_{Q(n-1)}(q)$-orbit. But, q acts transitively on X_{n-1} so that

$$Q_n^{X_{n-1}}/(\langle q \rangle, Q_n^{X_{n-1}}) = Q_n,$$

with trivial $Q(n-1)$-action. Moreover, the above images of g_n and g'_n in Q_n are just $\sigma_{n-1}(g_n)$ and $\sigma_{n-1}(g'_n)$ respectively. The hypothesis that $\bar{\sigma}(g) = \bar{\sigma}(g')$ implies that these are equal. Hence, $g(n)$ and $g'(n)$ are conjugate.

Conversely, if g and g' are conjugate then clearly $\bar{\sigma}(g) = \bar{\sigma}(g')$.

■

5. Closed normal subgroups of $G = G((\mathbf{Q}, \mathbf{Y}))$.

(5.1) **The "congruence groups".** For all $n \geq 1$ define G_n by

(1) $$G_n = Ker(G \longrightarrow Q(n)).$$

It is clear from the construction of $G = G((\mathbf{Q}, \mathbf{Y})) = \lim_{\overleftarrow{n}} Q(n)$ that

$$G = G_n \rtimes Q(n)$$

and that furthermore

(2) $$G_n = G((\mathbf{Q},\mathbf{Y})_{>n})^{X_{n-1}},$$

where we write

(3) $$(\mathbf{Q},\mathbf{Y})_{>n} = (\mathbf{Q},\mathbf{Y})_{\geq n+1} = ((Q_{n+1},Y_{n+1}),(Q_{n+2},Y_{n+2}),\ldots).$$

In this way we produce the wreath product decomposition

(4) $$G((\mathbf{Q},\mathbf{Y})) = G((\mathbf{Q},\mathbf{Y})_{>n})^{X_n} \rtimes Q(n),$$

which corresponds to the decomposition of $G(X,x_0)$ given in (I, (4.2)). We shall use the abbreviation $G(>n)$ (or $G(\geq n+1)$) for

(5) $$G(>n) = G((\mathbf{Q},\mathbf{Y})_{>n}).$$

Note that $G = G(\geq 1) = G((\mathbf{Q},\mathbf{Y}))$.

Thus, we have for $n > 1$,

$$\begin{aligned} G &= G_{n-1} \rtimes Q(n-1) \\ &= G(\geq n)^{X_{n-1}} \rtimes Q(n-1) \\ &= (G(>n)^{Y_n} \rtimes Q_n)^{X_{n-1}} \rtimes Q(n-1) \\ &= \underbrace{G(>n)^{X_n}}_{G_n} \rtimes \underbrace{Q_n^{X_{n-1}} \rtimes Q(n-1)}_{Q(n)} \end{aligned}$$

(5.2) **The groups $G_P(\geq n)$** . Let $P \leq Q_n$. We put

$$\begin{aligned} G_P(\geq n) &= G(>n)^{Y_n} \rtimes P \\ &\leq G(>n)^{Y_n} \rtimes Q_n = G(\geq n). \end{aligned}$$

Define

$$\overline{\sigma_P} : G(>n)^{Y_n} \longrightarrow (\overline{G(>n)^{\mathrm{ab}}})^{P\backslash Y_n}$$

by

$$(\overline{\sigma_P} f)(P\cdot y) = \prod_{x\in P\cdot y} \overline{f(y)},$$

where for $g \in G(>n)$, $\bar{g}$ denotes its image in $\overline{G(>n)^{\mathrm{ab}}}$. The kernel of $\overline{\sigma_P}$, $\overline{K_P}$ is clearly just the closure of the kernel K_P of the homomorphism σ_P of (3.4). Moreover, we conclude from (3.4) that

$$\overline{(G_P(\geq n), G_P(\geq n))} = \overline{K_P} \rtimes (P,P)$$

and

$$(G(\geq n), G_P(\geq n)) = \overline{K_{Q_n}} \rtimes (Q_n, P).$$

(5.3) **Closed normal subgroups of $G = G((\mathbf{Q},\mathbf{Y}))$.** Assume now that:

(1) *For all $n \geq 1$, Q_n acts both faithfully and transitively on Y_n.*

Note that by (4.3), this implies that the action of $Q(n)$ on X_n is also faithful and transitive.

Let N be a nontrivial closed normal subgroup of G. Thus,

$$1 \neq N = \overline{N} \lhd G. \tag{2}$$

Since $\cap G_n = \{1\}$, it follows that for some $n > 1$, "N has level $n-1$." That is to say,

(3) (a) $N \leq G_{n-1} = G(\geq n)^{X_{n-1}}$, *and*
(b) $N \not\leq G_n$.

We have $G = G(\geq n)^{X_{n-1}} \rtimes Q(n-1)$, so it follows from (1), (2), (3a) and (3.3) that:

(4) *There is an $M = \overline{M} = M(N) \lhd G(\geq n)$, which is the projection of N into each factor of $G_{n-1} = G(\geq n)^{X_{n-1}}$, and*

$$\overline{(G(\geq n), M)}^{X_{n-1}} \leq N \leq M^{X_{n-1}}.$$

Writing $G(\geq n) = G(> n)^{Y_n} \rtimes Q_n$, we put

$$P = P(N) = \textit{the projection of } M \textit{ into } Q_n. \tag{5}$$

From (4) and (3)(b) we have

$$1 \neq P \lhd Q_n. \tag{6}$$

Therefore, it follows from (1) and (1.4)(a) that:

$$P \textit{ has no fixed points on } Y_n. \tag{7}$$

Let $G_P = G(> n)^{Y_n} \rtimes P \leq G(\geq n)$ as in (5.2). Then it follows from (4) and (3.5)(a) that:

(8) (a) $\overline{(G_P(\geq n), G_P(\geq n))} = \overline{(G_P(\geq n), M)} \leq M \leq G_P(\geq n)$,
(b) $M/\overline{(G_P(\geq n), M)} \leq G_P(\geq n)^{\overline{\mathrm{ab}}} = (G(> n)^{\overline{\mathrm{ab}}})^{P\backslash Y_n} \times P^{\mathrm{ab}}$ *is a (Q_n/P)-submodule.*

Moreover, it follows from (1) and (4.4)(5) that:

$$G(> n)^{\overline{\mathrm{ab}}} = \prod_{m>n} Q_m^{\mathrm{ab}}. \tag{9}$$

Now suppose further that:

$$P \textit{ acts transitively on } Y_n. \tag{10}$$

Note that this is automatic if Q_n acts primitively on Y_n (cf. (1.4)(b)). In this case, it follows from (4) and (3.5)(b) that:

(11) (a) $\overline{(G(\geq n), M)} = \overline{(G(\geq n), G_P(\geq n))} = \overline{K_{Q_n}} \rtimes (Q_n, P) \leq M$, *and*
(b) $M/\overline{(G(\geq n), M)} \leq G_P(\geq n)/\overline{(G(\geq n), G_P(\geq n))}$
$= \overline{G(>n)^{\mathrm{ab}}} \times (P/(Q_n, P)) = (\prod_{m>n} Q_m^{\mathrm{ab}}) \times (P/(Q_n, P))$.

Combining (4) and (11) we conclude that:

$$\overline{(G(\geq n), M)}^{X_{n-1}} \leq N \leq M^{X_{n-1}}$$

and,

$$\begin{aligned}
\frac{N}{\overline{(G(\geq n),M)}^{X_{n-1}}} &\leq \left(\frac{M}{\overline{(G(\geq n),M)}}\right)^{X_{n-1}} \\
&\leq \left(\frac{G_P(\geq n)}{\overline{(G(\geq n),G_P(\geq n))}}\right)^{X_{n-1}} \\
&= \left(\overline{G(>n)^{\mathrm{ab}}} \times (P/(Q_n, P))\right)^{X_{n-1}} \\
&= \left((\textstyle\prod_{m>n} Q_m^{\mathrm{ab}}) \times (P/(Q_n, P))\right)^{X_{n-1}}.
\end{aligned} \tag{12}$$

The last term is a $Q(n-1)$-module via the permutation action on X_{n-1} and the first term is an arbitrary closed submodule.

The following theorem summarizes some of these conclusions.

(5.4) **Theorem.** *Assume that for all $n \geq 1$, Q_n acts faithfully and primitively on Y_n. Let N be a non-trivial closed subgroup of G of level $n-1$. Let $P = P(N) \lhd Q_n$ be as defined in (5.3) and $G_P(\geq n) = G(>n)^{Y_n} \rtimes P \leq G(\geq n)$. Put*

$$G'_P(\geq n) = \overline{(G(\geq n), G_P(\geq n))}.$$

Then,

$$\begin{aligned}
V_P &:= G_P(\geq n)/G'_P(\geq n) \\
&= \overline{G(>n)^{ab}} \times (P/(Q_n, P)) \\
&= (\prod_{m>n} Q_m^{ab}) \times (P/(Q_n, P)).
\end{aligned}$$

Moreover,

$$G'_P(\geq n)^{X_{n-1}} \leq N \leq G_P(\geq n)^{X_{n-1}},$$

and

$$N/G'_P(\geq n) \leq V_P^{X_{n-1}}$$

is a closed $Q(n-1)$-submodule where $Q(n-1)$ acts via the permutation action on X_{n-1}.

(5.5) **Remark.** Suppose that $G = Aut(X, x_0)$, *i.e.*, that $Q_n = S_{q_n}$ for all n. Then we distinguish two cases for P,

Case 1: $P = S_{q_n}$. Then $(Q_n, P) = A_{q_n}$. Thus,

$$V_P = \prod_{m \geq n} Q_m^{\mathrm{ab}} = G(\geq n)^{\overline{\mathrm{ab}}} = \prod_{m \geq n} \mathbb{Z}/2\mathbb{Z}.$$

Case 2: $P \neq S_{q_n}$. Then (cf. (1.3)(b)) we have $(Q_n, P) = P$. Thus,

$$V_P = \prod_{m > n} Q_m^{\mathrm{ab}} = G(> n)^{\overline{\mathrm{ab}}} = \prod_{m > n} \mathbb{Z}/2\mathbb{Z}.$$

(5.6) **Remark.** From Theorem (5.4) we see that a closed normal subgroup N of G determines an n and a $P \lhd Q_n$ so that

$$G'_P \leq N \leq G_P$$

and

$$N/G'_P \leq V_P^{X_{n-1}}$$

is any closed submodule of the G-module $V_P^{X_{n-1}}$, where G acts via its permutation action on X_{n-1}. To complete the analysis we would like to determine all of the closed G-modules of $V_P^{X_{n-1}}$. This appears to be too complicated a task. Instead, we shall answer an approximation to this question, when we replace V_P by a field F. In this case we can describe all $F[G]$-submodules of $F^{X_{n-1}}$. This is done in the following section.

6. The G-module V^{X_n}.

(6.1) **Notation.** Let V be an additive group, and $p : X \longrightarrow Y$ a map of sets. Then we have a group homomorphism

$$\gamma_p : V^Y \longrightarrow V^X,$$

defined by

$$(\gamma_p)f(x) = f(px).$$

the image consists of functions constant on the fibers of p, and if p is surjective then γ_p is injective.

Suppose that the fibers of p are finite. Then we have a homomorphism

$$\sigma_p : V^X \longrightarrow V^Y,$$

defined by

$$(\sigma_p)f(y) = \sum_{p(x)=y} f(x).$$

For $h \in V^Y$ we thus have

$$(\sigma_p \gamma_p h)(y) = | p^{-1}(y) | \cdot h(y).$$

(6.2) **Rooted Trees.** Let (X, x_0) be a locally finite rooted tree defined by the inverse system

$$X_0 = \{x_0\} \xleftarrow{p} X_1 \xleftarrow{p} \cdots \xleftarrow{p} X_{n-1} \xleftarrow{p} X_n \xleftarrow{p} \cdots.$$

We put $G = Aut(X, x_0)$.

Let V be an additive group and put

$$V_n = V^{X_n}$$

for $n \geq 0$.

For $0 \leq m \leq n$ the map

$$p^{n-m} : X_n \longrightarrow X_m$$

defines as in (6.1) homomorphisms

$$\gamma_m^n : V_m \longrightarrow V_n$$

and

$$\sigma_n^m : V_n \longrightarrow V_m.$$

Define

$$K_n^m = Ker(\sigma_n^m).$$

These are G-modules and G-homomorphisms.

Define

$$V_n' = K_n^{n-1}$$

for $n \geq 0$ and

$$V_{-1} = V_{-1}' = 0.$$

We also put $\gamma_{-1}^n = 0$ and $\sigma_n^{-1} = 0$.

If all $p : X_n \longrightarrow X_{n-1}$ are surjective (*i.e.*, if X has no terminal vertices) then all γ_m^n are injective, and all σ_n^m are surjective. Suppose further that (X, x_0) is spherically homogeneous of degree

$$\mathbf{q} = (q_1, q_2, \ldots)$$

Then

(1) $\quad \sigma_n^m \cdot \gamma_m^n =$ *multiplication by* $q_{m+1} \cdots q_n = \mathbf{q}^{[n]}/\mathbf{q}^{[m]} =$ "$\mathbf{q}^{[n]-[m]}$".

(6.3) **The modules $V_{(n;n_1,\ldots,n_r)} \leq V_n$.** Given a decreasing sequence of integers

$$n \geq n_1 > n_2 > \ldots > n_r \geq -1, \tag{1}$$

we put

$$V_{(n;n_1,\ldots,n_r)} = \gamma^n_{n_1} W_1 \leq V_n, \tag{2}$$

where $W_1, W_2, \ldots, W_r$ are G-modules,

$$V'_{n_i} = K^{n_i-1}_{n_i} \leq W_i \leq V_{n_i} \tag{3}$$

for $i = 1, \ldots, r$, defined by reverse induction as follows:

First,

$$W_r = V'_{n_r}.$$

Suppose now that $W_{i+1}, \ldots, W_r$ have been defined satisfying (3). Put

$$\gamma = \gamma^{n_i-1}_{n_{i+1}} : V_{n_{i+1}} \longrightarrow V_{n_i-1}$$

and

$$\sigma = \sigma^{n_i-1}_{n_i} : V_{n_i} \longrightarrow V_{n_i-1}.$$

Then we put

$$W_i = \sigma^{-1}(\gamma W_{i+1}). \tag{4}$$

Then (3) follows for W_i from the diagram

$$\begin{array}{ccc}
V_{n_i} & \xrightarrow{\sigma} & V_{n_{i-1}} \\
| & & | \\
| & & \gamma V_{n_{i+1}} \\
| & & | \\
W_i = \sigma^{-1}(\gamma W_{i+1}) & \longrightarrow & \gamma W_{i+1} \\
| & & | \\
V'_{n_i} = Ker(\sigma) & \longrightarrow & \{0\}
\end{array}$$

It is easy to see, inductively, that distinct sequences $(n; n_1, \ldots, n_r)$ produce distinct submodules $V_{(n;n_1,\ldots,n_r)}$.

Note that

$$W_i = V_{(n_i;n_i,\ldots,n_r)},$$

and thus, is characterized by

$$\left\{ \begin{array}{l} V_{(n;n_1,\ldots,n_r)} = \gamma^n_{n_1} W' \\ V'_{n_1} \leq W' \leq V_{n_1} \\ \sigma^{n_1-1}_{n_1}(W') = \gamma^{n_1-1}_{n_2} V_{(n_2;n_2,\ldots,n_r)} = V_{(n_1-1;n_2,\ldots,n_r)} \end{array} \right. \tag{5}$$

We shall show in (6.5) that under special conditions, these are the only G-submodules of V_n.

(6.4) **Irreducible Q-sets.** Let Q be a group and Y a finite Q-set. Let F be a field and $V = F^Y$ the corresponding (permutation) $F[Q]$-module. Then V has two natural submodules: the constant functions $Y \longrightarrow F$; and $V' = Ker(V \xrightarrow{\sigma} F)$ where

$$\sigma(f) = \sum_{y \in Y} f(y).$$

We call the Q-set Y irreducible (over F) if the above are the only $F[G]$-submodules of V other than 0 and V. Clearly irreduciblity implies primitivity and thus transitivity.

The full symmetric group S on Y acts irreducibly on Y. It suffices to show that if $f \in V$ is not constant, then the FS-module W generated by f contains V'. (In characteristic 0 this just says that the permutation representation of S_n on $\{1, \dots, n\}$ splits as two components, trivial representation and its direct summand.) Say $f(x) \neq f(y)$. Let $s \in S$ be the transposition (x, y). Then $h = s(f) - f \in W$ and with $a = f(x) - f(y) \neq 0$, we have $h(x) = -a, h(y) = a$, and $h(z) = 0$ for $z \neq x, y$. Clearly the $F[S]$-module generated by h is V'.

(6.5) **Notation.** Let

$$(\mathbf{Q}, \mathbf{Y}) = ((Q_1, Y_1), (Q_2, Y_2), \dots)$$

be a sequence as in (4.2): Q_n is a finite group and Y_n is a finite Q_n-set of cardinality $q_n > 0$. Put $X = X(\mathbf{Y})$ as in (4.1) and $G = G(\mathbf{Q}, \mathbf{Y})$ as in (4.2).

(6.6) **Theorem.** *Let all notation be as in (6.3). Assume that for each n, Q_n acts irreducibly on Y_n (cf. (6.4)), for example that $G = G((\mathbf{Q}, \mathbf{Y}))$ is the full automorphism group $Aut(X, x_0)$. Let $V = F$, a field, and $V_n = F^{X_n}$ as in (6.2). If $W \leq V_n$ is an $F[G]$-submodule then there is a unique sequence*

$$n \geq n_1 > n_2 > \cdots > n_r \geq -1$$

such that (in the notation of (6.3)) $W = V_{(n; n_1, \dots, n_r)}$.

Proof. Uniqueness was already noted in (6.3). For existence we argue by induction on n. Choose $n_1 \leq n$ minimal so that $W \leq \gamma^n_{n_1} V_{n_1}$. If $n_1 = -1$ then we have $W = 0 = V_{(n;-1)}$. Say $n_1 \geq 0$ and put $m = n_1$. Then $W = \gamma^n_m W'$ with $W' \leq V_m$ and the minimality of n_1 implies that $W' \not\leq \gamma^m_{m-1} V_{m-1}$.

Claim. $V'_m \leq W'$.

We first show that the Claim implies the Theorem. Put $\sigma = \sigma^{m-1}_m : V_m \longrightarrow V_{m-1}$, with kernel V'_m, so that, if $U = \sigma(W')$, then we have $W' = \sigma^{-1}(U)$ by the claim. By induction, there is a sequence

$$m - 1 \geq n_2 > \cdots > n_r \geq -1$$

such that $U = V_{(m-1;n_2,\ldots,n_r)}$ Thus, using (6.3), equation (5),

$$\begin{aligned} W &= \gamma_m^n(\sigma^{-1}(U)) \\ &= \gamma_{n_1}^n(\sigma^{-1}(V_{(n_1-1;n_2,\ldots,n_r)})) \qquad (m = n_1) \\ &= V_{(n;n_1,n_2,\ldots,n_r)}. \end{aligned}$$

Proof. (of Claim) The conditions that $W \le V_m$ and $W \not\le \gamma_{m-1}^m V_{m-1}$ means that some $f \in W \le F^{X_m} = V_m$ is not constant on some fiber $p^{-1}(y)$ of $p : X_m \longrightarrow X_{m-1}$. In $Q_m^{X_{m-1}} \le G$ let H denote the copy of Q_m in the y-coordinate. Then H acts trivially on all fibers of $X_m \longrightarrow X_{m-1}$ other than $p^{-1}(y)$ while by hypothesis, H acts irreducibly on $p^{-1}(y)$. For $h \in H$ and $f' = h(f) - f \in W$, f' vanishes on all fibers of $X_m \longrightarrow X_{m-1}$ except $p^{-1}(y)$ where it takes values $f'(x) = f(h^{-1}x) - f(x)$. Since H on $p^{-1}(y)$ is transitive and f is non-constant on $p^{-1}(y)$ we see that f' is neither constant nor zero (for suitable $h \in H$). Now, by irreducibility of H on $p^{-1}(y)$, the $F[H]$-module generated by f' contains $(V'_m)_y = V'_m \cap (V_m)_y$ where $(V_m)_y$ denotes the functions $X_m \longrightarrow F$ with support in $p^{-1}(y)$. Now

$$V'_m = \prod_{z \in X_{m-1}} (V'_m)_z$$

and G acts transitively on X_{m-1}. Thus, the $F[G]$-module generated by $(V'_m)_y$ is all of V'_m. Thus, $V'_m \le W$ as claimed.

■

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