Service de Physique Théorique – C.E.A.-Saclay

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PhD Thesis

Matrix Quantum Mechanics and Two-dimensional String Theory in Non-trivial Backgrounds

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Abstract

String theory is the most promising candidate for the theory unifying all interactions including gravity. It has an extremely difficult dynamics. Therefore, it is useful to study some its simplifications. One of them is non-critical string theory which can be defined in low dimensions. A particular interesting case is 2D string theory. On the one hand, it has a very rich structure and, on the other hand, it is solvable. A complete solution of 2D string theory in the simplest linear dilaton background was obtained using its representation as Matrix Quantum Mechanics. This matrix model provides a very powerful technique and reveals the integrability hidden in the usual CFT formulation.

This thesis extends the matrix model description of 2D string theory to non-trivial backgrounds. We show how perturbations changing the background are incorporated into Matrix Quantum Mechanics. The perturbations are integrable and governed by Toda Lattice hierarchy. This integrability is used to extract various information about the perturbed system: correlation functions, thermodynamical behaviour, structure of the target space. The results concerning these and some other issues, like non-perturbative effects in non-critical string theory, are presented in the thesis.

Acknowledgements

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Contents

Introduction

This thesis is devoted to application of the matrix model approach to non-critical string theory.

More than fifteen years have passed since matrix models were first applied to string theory. Although they have not helped to solve critical string and superstring theory, they have taught us many things about low-dimensional bosonic string theories. Matrix models have provided so powerful technique that a lot of results which were obtained in this framework are still inaccessible using the usual continuum approach. On the other hand, those results that were reproduced turned out to be in the excellent agreement with the results obtained by field theoretical methods.

One of the main subjects of interest in the early years of the matrix model approach was the $c = 1$ non-critical string theory which is equivalent to the two-dimensional critical string theory in the linear dilaton background. This background is the simplest one for the low-dimensional theories. It is flat and the dilaton field appearing in the low-energy target space description is just proportional to one of the spacetime coordinates.

In the framework of the matrix approach this string theory is described in terms of *Matrix Quantum Mechanics* (MQM). Already ten years ago MQM gave a complete solution of the 2D string theory. For example, the exact S-matrix of scattering processes was found and many correlation functions were explicitly calculated.

However, the linear dilaton background is only one of the possible backgrounds of 2D string theory. There are many other backgrounds including ones with a non-vanishing curvature which contain a dilatonic black hole. It was a puzzle during long time how to describe such backgrounds in terms of matrices. And only recently some progress was made in this direction.

In this thesis we try to develop the matrix model description of 2D string theory in non-trivial backgrounds. Our research covers several possibilities to deform the initial simple target space. In particular, we analyze winding and tachyon perturbations. We show how they are incorporated into Matrix Quantum Mechanics and study the result of their inclusion.

A remarkable feature of these perturbations is that they are exactly solvable. The reason is that the perturbed theory is described by Toda Lattice integrable hierarchy. This is the result obtained entirely within the matrix model framework. So far this integrability has not been observed in the continuum approach. On the other hand, in MQM it appears quite naturally being a generalization of the KP integrable structure of the $c < 1$ models. In this thesis we extensively use the Toda description because it allows to obtain many exact results.

We tried to make the thesis selfconsistent. Therefore, we give a long introduction into the subject. We begin by briefly reviewing the main concepts of string theory. We introduce the Polyakov action for a bosonic string, the notion of the Weyl invariance and the anomaly associated with it. We show how the critical string theory emerges and explain how it is generalized to superstring theory avoiding to write explicit formulae. We mention also the modern view on superstrings which includes D-branes and dualities. After that we discuss the low-energy limit of bosonic string theories and possible string backgrounds. A special attention is paid to the linear dilaton background which appears in the discussion of non-critical strings. Finally, we present in detail 2D string theory both in the linear dilaton and perturbed backgrounds. We elucidate its degrees of freedom and how they can be used to perturb the theory. In particular, we present a conjecture that relates 2D string theory perturbed by windings modes to the same theory in a curved black hole background.

The next chapter is an introduction to matrix models. We explain what the matrix models are and how they are related to various physical problems and to string theory, in particular. The relation is established through the sum over discretized surfaces and such important notions as the $1/N$ expansion and the double scaling limit are introduced. Then we consider the two simplest examples, the one- and the two-matrix model. They are used to present two of the several known methods to solve matrix models. First, the one-matrix model is solved in the large N-limit by the saddle point approach. Second, it is shown how to obtain the solution of the two-matrix model by the technique of orthogonal polynomials which works, in contrast to the first method, to all orders in perturbation theory. We finish this chapter giving an introduction to Toda hierarchy. The emphasis is done on its Lax formalism. Since the Toda integrable structure is the main tool of this thesis, the presentation is detailed and may look too technical. But this will be compensated by the power of this approach.

The third chapter deals with a particular matrix model — Matrix Quantum Mechanics. We show how it incorporates all features of 2D string theory. In particular, we identify the tachyon modes with collective excitations of the singlet sector of MQM and the winding modes of the compactified string theory with degrees of freedom propagating in the non-trivial representations of the SU(N) global symmetry of MQM. We explain the free fermionic representation of the singlet sector and present its explicit solution both in the non-compactified and compactified cases. Its target space interpretation is elucidated with the help of the Das–Jevicki collective field theory.

Starting from the forth chapter, we turn to 2D string theory in non-trivial backgrounds and try to describe it in terms of perturbations of Matrix Quantum Mechanics. First, the winding perturbations of the compactified string theory are incorporated into the matrix framework. We review the work of Kazakov, Kostov and Kutasov where this was first done. In particular, we identify the perturbed partition function with a τ-function of Toda hierarchy showing that the introduced perturbations are integrable. The simplest case of the windings of the minimal charge is interpreted as a matrix model for the 2D string theory in the black hole background. For this case we present explicit results for the free energy. Relying on these description, we explain our first work in this domain devoted to calculation of winding correlators in the theory with the simplest winding perturbation. This work is little bit technical. Therefore, we concentrate mainly on the conceptual issues.

The next chapter is about tachyon perturbations of 2D string theory in the MQM framework. It consists from three parts representing our three works. In the first one, we show how the tachyon perturbations should be introduced. Similarly to the case of windings, we find that the perturbations are integrable. In the quasiclassical limit we interpret them in terms of the time-dependent Fermi sea of fermions of the singlet sector. The second work provides a thermodynamical interpretation to these perturbations. For the simplest case corresponding to the Sine–Liouville perturbation, we are able to find all thermodynamical characteristics of the system. However, many of the results do not have a good explanation and remain to be mysterious for us. In the third work we discuss how to obtain the structure of the string backgrounds corresponding to the perturbations introduced in the matrix model.

The sixth chapter is devoted to our fifth work where we establish an equivalence between the MQM description of tachyon perturbations and the so called Normal Matrix Model. We explain the basic features of the latter and its relation to various problems in physics and mathematics. The equivalence is interpreted as a kind of duality for which a mathematical as well as a physical sense can be given.

In the last chapter we present our sixth work on non-perturbative effects in matrix models and their relation to D-branes. We calculate the leading non-perturbative corrections to the partition function for both $c = 1$ and $c < 1$ string theories. In the beginning we present the calculation based on the matrix model formulation and then we reproduce some of the obtained results from D-branes of Liouville theory.

We would like to say several words about the presentation. We tried to do it in such a way that all the reported material would be connected by a continuous line of reasonings. Each result is supposed to be a more or less natural development of the previous ideas and results. Therefore, we tried to give a motivation for each step leading to something new. Also we explained various subtleties which occur sometimes and not always can be found in the published articles.

Finally, we tried to trace all the coefficients and signs and write all formulae in the once chosen normalization. Their discussion sometimes may seem to be too technical for the reader. But we hope he will forgive us because it is done to give the possibility to use this thesis as a source for correct equations in the presented domains.

Chapter I

String theory

String theory is now considered as the most promising candidate to describe the unification of all interactions and quantum gravity. It is a very wide subject of research possessing a very rich mathematical structure. In this chapter we will give just a brief review of the main ideas underlying string theory to understand its connection with our work. For a detailed introduction to string theory, we refer to the books [1, 2, 3].

1 Strings, fields and quantization

1.1 A little bit of history

String theory has a very interesting history in which one can find both the dark periods and remarkable breakthroughs of new ideas. In the beginning it appeared as an attempt to describe the strong interaction. In that time QCD was not yet known and there was no principle to explain a big tower of particles discovered in processes involving the strong interaction. Such a principle was suggested by Veneziano [4] in the so called *dual* models. He required that the sum of scattering amplitudes in s and t channels should coincide (see fig. I.1).

This requirement together with unitarity, locality and *etc.* was strong enough to fix completely the amplitudes. Thus, it was possible to find them explicitly for the simplest cases as well as to establish their general asymptotic properties. In particular, it was shown that the scattering amplitudes in dual models are much softer then the usual field theory amplitudes, so that the problems of field-theoretic divergences should be absent in these models.

Moreover, the found amplitudes coincided with scattering amplitudes of strings — objects extended in one dimension [5, 6, 7]. Actually, this is natural because for strings the property

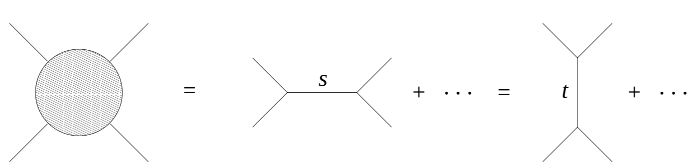


Fig. I.1: Scattering amplitudes in dual models.

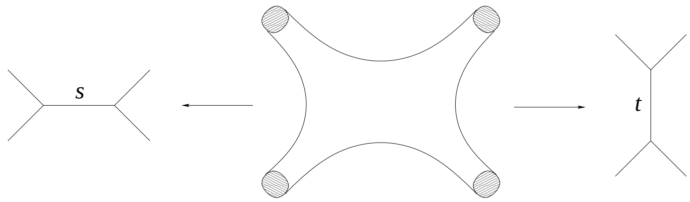


Fig. I.2: Scattering string amplitude can be seen in two ways.

of duality is evident: two channels can be seen as two degenerate limits of the same string configuration (fig. I.2). Also the absence of ultraviolet divergences got a natural explanation in this picture. In field theory the divergences appear due to a local nature of interactions related to the fact that the interacting objects are thought to be pointlike. When particles (pointlike objects) are replaced by strings the singularity is smoothed out over the string world sheet.

However, this nice idea was rejected by the discovery of QCD and description of all strongly interacting particles as composite states of fundamental quarks. Moreover, the exponential fall-off of string amplitudes turned out to be inconsistent with the observed power-like asymptotics. Thus, strings lost the initial reason to be related to fundamental physics.

But suddenly another reason was found. Each string possesses a spectrum of excitations. All of them can be interpreted as particles with different spins and masses. For a closed string, which can be thought just as a circle, the spectrum contains a massless mode of spin 2. But the graviton, quantum of gravitational interaction, has the same quantum numbers. Therefore, strings might be used to describe quantum gravity! If this is so, a theory based on strings should describe the world at the very microscopic level, such as the Planck scale, and should reproduce the standard model only in some low-energy limit.

This idea gave a completely new status to string theory. It became a candidate for the unified theory of all interactions including gravity. Since that time string theory has been developed into a rich theory and gave rise to a great number of new physical concepts. Let us have a look how it works.

1.2 String action

As is well known, the action for the relativistic particle is given by the length of its world line. Similarly, the string action is given by the area of its world sheet so that classical trajectories correspond to world sheets of minimal area. The standard expression for the area of a two-dimensional surface leads to the action [8, 9]

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d\tau d\sigma\,\sqrt{-h}, \qquad h = \det h_{ab}, \tag{I.1}$$

which is called the Nambu–Goto action. Here α' is a constant of dimension of squared length. The matrix h_{ab} is the metric induced on the world sheet and can be represented as

$$h_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{I.2}$$

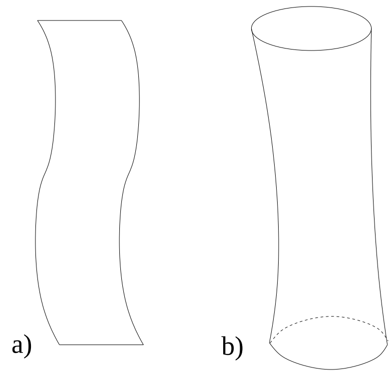


Fig. I.3: Open and closed strings.

where $X^\mu(\tau,\sigma)$ are coordinates of a point (τ,σ) on the world sheet in the spacetime where the string moves. Such a spacetime is called *target space* and $G_{\mu\nu}(X)$ is the metric there.

Due to the square root even in the flat target space the action (I.1) is highly non-linear. Fortunately, there is a much more simple formulation which is classically equivalent to the Nambu–Goto action. This is the Polyakov action [10]:

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\,\sqrt{-h}\,G_{\mu\nu}h^{ab}\partial_a X^\mu\partial_b X^\nu. \tag{I.3}$$

Here the world sheet metric is considered as a dynamical variable and the relation (I.2) appears only as a classical equation of motion. (More exactly, it is valid only up to some constant multiplier.) This means that we deal with a gravitational theory on the world sheet. We can even add the usual Einstein term

$$\chi = \frac{1}{4\pi}\int_\Sigma d\tau d\sigma\,\sqrt{-h}\,\mathcal{R}. \tag{I.4}$$

In two dimensions $\sqrt{-h}\mathcal{R}$ is a total derivative. Therefore, χ depends only on the topology of the surface Σ, which one integrates over, and produces its Euler characteristic. In fact, any compact connected oriented two-dimensional surface can be represented as a sphere with g handles and b boundaries. In this case the Euler characteristic is

$$\chi = 2 - 2g - b. \tag{I.5}$$

Thus, the full string action reads

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\,\sqrt{-h}\left(G_{\mu\nu}h^{ab}\partial_a X^\mu\partial_b X^\nu + \alpha'\nu\mathcal{R}\right), \tag{I.6}$$

where we introduced the coupling constant ν. In principle, one could add also a two-dimensional cosmological constant. However, in this case the action would not be equivalent to the Nambu–Goto action. Therefore, we leave this possibility aside.

To completely define the theory, one should also impose some boundary conditions on the fields $X^\mu(\tau,\sigma)$. There are two possible choices corresponding to two types of strings which one can consider. The first choice is to take Neumann boundary conditions $n^a\partial_a X^\mu = 0$ on $\partial\Sigma$, where n^a is the normal to the boundary. The presence of the boundary means that one considers an *open* string with two ends (fig. I.3a). Another possibility is given by periodic boundary conditions. The corresponding string is called *closed* and it is topologically equivalent to a circle (fig. I.3b).

1.3 String theory as two-dimensional gravity

The starting point to write the Polyakov action was to describe the movement of a string in a target space. However, it possesses also an additional interpretation. As we already mentioned, the two-dimensional metric h_{ab} in the Polyakov formulation is a dynamical variable. Besides, the action (I.6) is invariant under general coordinate transformations on the world sheet. Therefore, the Polyakov action can be equally considered as describing two-dimensional gravity coupled with matter fields X^μ. The matter fields in this case are usual scalars. The number of these scalars coincides with the dimension of the target space.

Thus, there are two *dual* points of view: target space and world sheet pictures. In the second one we can actually completely forget about strings and consider it as the problem of quantization of two-dimensional gravity in the presence of matter fields.

It is convenient to do the analytical continuation to the Euclidean signature on the world sheet $\tau \to -i\tau$. Then the path integral over two-dimensional metrics can be better defined, because the topologically non-trivial surfaces can have non-singular Euclidean metrics, whereas in the Minkowskian signature their metrics are always singular. In this way we arrive at a statistical problem for which the partition function is given by a sum over fluctuating two-dimensional surfaces and quantum fields on them[1]

$$Z = \sum_{\text{surfaces } \Sigma} \int \mathcal{D}X_\mu \, e^{-S_{\rm P}^{(\rm E)}[X,\Sigma]}. \tag{I.7}$$

The sum over surfaces should be understood as a sum over all possible topologies plus a functional integral over metrics. In two dimensions all topologies are classified. For example, for closed oriented surfaces the sum over topologies corresponds to the sum over genera g which is the number of handles attached to a sphere. In this case one gets

$$\sum_{\text{surfaces } \Sigma} = \sum_g \int \mathcal{D}\varrho(h_{ab}). \tag{I.8}$$

On the contrary, the integral over metrics is yet to be defined. One way to do this is to discretize surfaces and to replace the integral by the sum over discretizations. This way leads to matrix models discussed in the following chapters.

In string theory one usually follows another approach. It treats the two-dimensional diffeomorphism invariance as an ordinary gauge symmetry. Then the standard Faddeev–Popov gauge fixing procedure is applied to make the path integral to be well defined. However, the Polyakov action possesses an additional feature which makes its quantization non-trivial.

1.4 Weyl invariance

The Polyakov action (I.6) is invariant under the local Weyl transformations

$$h_{ab} \longrightarrow e^{\phi} h_{ab}, \tag{I.9}$$

where $\phi(\tau, \sigma)$ is any function on the world sheet. This symmetry is very crucial because it allows to exclude one more degree of freedom. Together with the diffeomorphism symmetry,

[1]Note, that the Euclidean action $S_{\rm P}^{(\rm E)}$ differs by sign from the Minkowskian one.

it leads to the possibility to express at the classical level the world sheet metric in terms of derivatives of the spacetime coordinates as in (I.2). Thus, it is responsible for the equivalence of the Polyakov and Nambu–Goto actions.

However, the classical Weyl symmetry can be broken at the quantum level. The reason can be found in the non-invariance of the measure of integration over world sheet metrics. Due to the appearance of divergences the measure should be regularized. But there is no regularization preserving all symmetries including the conformal one.

The anomaly can be most easily seen analyzing the energy-momentum tensor T_{ab}. In any classical theory invariant under the Weyl transformations the trace of T_{ab} should be zero. Indeed, the energy-momentum tensor is defined by

$$T_{ab} = -\frac{2\pi}{\sqrt{-h}}\frac{\delta S}{\delta h^{ab}}. \tag{I.10}$$

If the metric is varied along eq. (I.9) (ϕ should be taken infinitesimal), one gets

$$T_a^{\ a} = \frac{2\pi}{\sqrt{-h}}\frac{\delta S}{\delta \phi} = 0. \tag{I.11}$$

However, in quantum theory T_{ab} should be replaced by a renormalized average of the quantum operator of the energy-momentum tensor. Since the renormalization in general breaks the Weyl invariance, the trace will not vanish anymore.

Let us restrict ourselves to the flat target space $G_{\mu\nu} = \eta_{\mu\nu}$. Then explicit calculations lead to the following anomaly

$$\langle T_a^{\ a}\rangle_{\rm ren} = -\frac{c}{12}\mathcal{R}. \tag{I.12}$$

To understand the origin of the coefficient c, we choose the flat gauge $h_{ab} = \delta_{ab}$. Then the Euclidean Polyakov action takes the following form

$$S_{\rm P}^{\rm (E)} = \nu\chi + \frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\, \delta^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{I.13}$$

This action is still invariant under conformal transformations which preserve the flat metric. They are a special combination of the Weyl and diffeomorphism transformations of the initial action. Thus, the gauged fixed action (I.13) represents a particular case of conformal field theory (CFT). Each CFT is characterized by a number c, the so called *central charge*, which defines a quantum deformation of the algebra of generators of conformal transformations. It is this number that appears in the anomaly (I.12).

The central charge is determined by the field content of CFT. Each bosonic degree of freedom contributes 1 to the central charge, each fermionic degree of freedom gives 1/2, and ghost fields which have incorrect statistics give rise to negative values of c. In particular, the ghosts arising after a gauge fixation of the diffeomorphism symmetry contribute -26. Thus, if strings propagate in the flat spacetime of dimension D, the central charge of CFT (I.13) is

$$c = D - 26. \tag{I.14}$$

This gives the exact result for the Weyl anomaly. Thus, one of the gauge symmetries of the classical theory turns out to be broken. This effect can be seen also in another approaches to string quantization. For example, in the framework of canonical quantization in the flat gauge one finds the breakdown of unitarity. Similarly, in the light-cone quantization one encounters the breakdown of global Lorentz symmetry in the target space. All this indicates that the Weyl symmetry is extremely important for the existence of a viable theory of strings.

2 Critical string theory

2.1 Critical bosonic strings

We concluded the previous section with the statement that to consistently quantize string theory we need to preserve the Weyl symmetry. How can this be done? The expression for the central charge (I.14) shows that it is sufficient to place strings into spacetime of dimension $D_{\rm cr} = 26$ which is called *critical* dimension. Then there is no anomaly and quantum theory is well defined.

Of course, our real world is four-dimensional. But now the idea of Kaluza [11] and Klein [12] comes to save us. Namely, one supposes that extra 22 dimensions are compact and small enough to be invisible at the usual scales. One says that the initial spacetime is *compactified*. However, now one has to choose some compact space to be used in this compactification. It is clear that the effective four-dimensional physics crucially depends on this choice. But *a priori* there is no any preference and it seems to be impossible to find the right compactification.

Actually, the situation is worse. Among modes of the bosonic string, which are interpreted as fields in the target space, there is a mode with a negative squared mass that is a tachyon. Such modes lead to instabilities of the vacuum and can break the unitarity. Thus, the bosonic string theory in 26 dimensions is still a "bad" theory.

2.2 Superstrings

An attempt to cure the problem of the tachyon of bosonic strings has led to a new theory where the role of fundamental objects is played by superstrings. A superstring is a generalization of the ordinary bosonic string including also fermionic degrees of freedom. Its important feature is a supersymmetry. In fact, there are two formulations of superstring theory with the supersymmetry either in the target space or on the world sheet.

Green–Schwarz formulation

In the first formulation, developed by Green and Schwarz [13], to the fields X^μ one adds one or two sets of world sheet scalars θ^A. They transform as Maiorana–Weyl spinors with respect to the global Lorentz symmetry in the target space. The number of spinors determines the number of supersymmetric charges so that there are two possibilities to have $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry. It is interesting that already at the classical level one gets some restrictions on possible dimensions D. It can be 3, 4, 6 or 10. However, the quantization selects only the last possibility which is the critical dimension for superstring theory.

In this formulation one has the explicit supersymmetry in the target space.[2] Due to this, the tachyon mode cannot be present in the spectrum of superstring and the spectrum starts with massless modes.

[2]Superstring can be interpreted as a string moving in a superspace.

RNS formulation

Unfortunately, the Green–Schwarz formalism is too complicated for real calculations. It is much more convenient to use another formulation with a supersymmetry on the world sheet [14, 15]. It represents a natural extension of CFT (I.13) being a two-dimensional super-conformal field theory (SCFT).[3] In this case the additional degrees of freedom are world sheet fermions ψ^μ which form a vector under the global Lorentz transformations in the target space.

Since this theory is a particular case of conformal theories, the formula (I.12) for the conformal anomaly remains valid. Therefore, to find the critical dimension in this formalism, it is sufficient to calculate the central charge. Besides the fields discussed in the bosonic case, there are contributions to the central charge from the world sheet fermions and ghosts which arise after a gauge fixing of the local fermionic symmetry. This symmetry is a superpartner of the usual diffeomorphism symmetry and is a necessary part of supergravity. As was mentioned, each fermion gives the contribution 1/2, whereas for the new superconformal ghosts it is 11. As a result, one obtains

$$c = D - 26 + \frac{1}{2}D + 11 = \frac{3}{2}(D - 10). \tag{I.15}$$

This confirms that the critical dimension for superstring theory is $D_{\rm cr} = 10$.

To analyze the spectrum of this formulation, one should impose boundary conditions on ψ^μ. But now the number of possibilities is doubled with respect to the bosonic case. For example, since ψ^μ are fermions, for the closed string not only periodic, but also antiperiodic conditions can be chosen. This leads to the existence of two independent sectors called Ramond (R) and Neveu–Schwarz (NS) sectors. In each sector superstrings have different spectra of modes. In particular, from the target space point of view, R-sector describes fermions and NS-sector contains bosonic fields. But the latter suffers from the same problem as bosonic string theory — its lowest mode is a tachyon.

Is the fate of RNS formulation the same as that of the bosonic string theory in 26 dimensions? The answer is *not*. In fact, when one calculates string amplitudes of perturbation theory, one should sum over all possible spinor structures on the world sheet. This leads to a special projection of the spectrum, which is called Gliozzi–Scherk–Olive (GSO) projection [16]. It projects out the tachyon and several other modes. As a result, one ends up with a well defined theory.

Moreover, it can be checked that after the projection the theory possesses the global supersymmetry in the target space. This indicates that actually GS and RNS formulations are equivalent. This can be proven indeed and is related to some intriguing symmetries of superstring theory in 10 dimensions.

Consistent superstring theories

Once we have constructed general formalism, one can ask how many consistent theories of superstrings do exist? Is it unique or not?

[3] In fact, it is two-dimensional supergravity coupled with superconformal matter. Thus, in this formulation one has a supersymmetric generalization of the interpretation discussed in section 1.3.

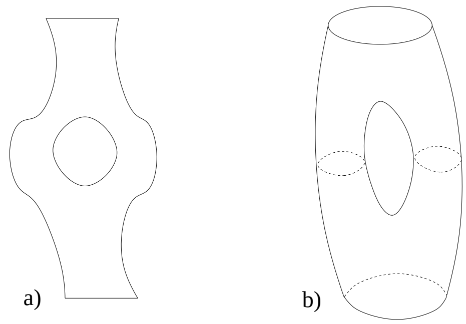


Fig. I.4: Interactions of open and closed strings.

At the classical level it is certainly not unique. One has open and closed, oriented and non-oriented, $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric string theories. Besides, in the open string case one can also introduce Yang–Mills gauge symmetry adding charges to the ends of strings. It is clear that the gauge group is not fixed anyhow. Finally, considering closed strings with $\mathcal{N} = 1$ supersymmetry, one can construct the so called heterotic strings where it is also possible to introduce a gauge group.

However, quantum theory in general suffers from anomalies arising at one and higher loops in string perturbation theory. The requirement of anomaly cancellation forces to restrict ourselves only to the gauge group $SO(32)$ in the open string case and $SO(32)$ or $E_8 \times E_8$ in the heterotic case [17]. Taking into account also restrictions on possible boundary conditions for fermionic degrees of freedom, one ends up with five consistent superstring theories. We give their list below:

- type IIA: $\mathcal{N} = 2$ oriented non-chiral closed strings;
- type IIB: $\mathcal{N} = 2$ oriented chiral closed strings;
- type I: $\mathcal{N} = 1$ non-oriented open strings with the gauge group $SO(32)$ + non-oriented closed strings;
- heterotic $SO(32)$: heterotic strings with the gauge group $SO(32)$;
- heterotic $E_8 \times E_8$: heterotic strings with the gauge group $E_8 \times E_8$.

2.3 Branes, dualities and M-theory

Since there are five consistent superstring theories, the resulting picture is not completely satisfactory. One should either choose a correct one among them or find a further unification. Besides, there is another problem. All string theories are defined only as asymptotic expansions in the string coupling constant. This expansion is nothing else but the sum over genera of string world sheets in the closed case (see (I.8)) and over the number of boundaries in the open case. It is associated with the *string loop expansion* since adding a handle (strip) can be interpreted as two subsequent interactions: a closed (open) string is emitted and then reabsorbed (fig. I.4).

Note, that from the action (I.13) it follows that each term in the partition function (I.7) is weighted by the factor $e^{-\nu\chi}$ which depends only on the topology of the world sheet. Due

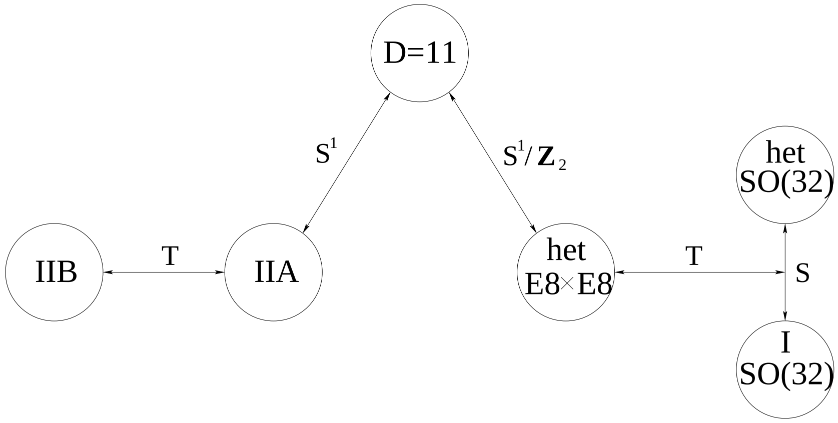


Fig. I.5: Chain of dualities relating all superstring theories.

to this one can associate $e^{2\nu}$ with each handle and e^{ν} with each strip. On the other hand, each interaction process should involve a coupling constant. Therefore, ν determines the closed and open string coupling constants

$$g_{\rm cl} \sim e^{\nu}, \qquad g_{\rm op} \sim e^{\nu/2}. \tag{I.16}$$

Since string theories are defined as asymptotic expansions, any finite value of ν leads to troubles. Besides, it looks like a free parameter and there is no way to fix its value.

A way to resolve both problems came from the discovery of a net of dualities relating different superstring theories. As a result, a picture was found where different theories appear as different vacua of a single (yet unknown) theory which got the name "M-theory". A generic point in its moduli space corresponds to an 11-dimensional vacuum. Therefore, one says that the unifying M-theory is 11 dimensional. In particular, it has a vacuum which is Lorentz invariant and described by 11-dimensional flat spacetime. It is shown in fig. I.5 as a circle labeled D=11.

Other superstring theories can be obtained by different compactifications of this special vacuum. Vacua with $\mathcal{N}=2$ supersymmetry arise after compactification on a torus, whereas $\mathcal{N}=1$ supersymmetry appears as a result of compactification on a cylinder. The known superstring theories are reproduced in some degenerate limits of the torus and cylinder. For example, when one of the radii of the torus is much larger than the other, so that one considers compactification on a circle, one gets the IIA theory. The small radius of the torus determines the string coupling constant. The IIB theory is obtained when the two radii both vanish and the corresponding string coupling is given by their ratio. Similarly, the heterotic and type I theories appear in the same limits for the radius and length of the cylinder.

This picture explains all existing relations between superstring theories, a part of which is shown in fig. I.5. The most known of them are given by T and S-dualities. The former relates compactified theories with inverse compactification radii and exchanges the windings around compactified dimension with the usual momentum modes in this direction. The latter duality says that the strong coupling limit of one theory is the weak coupling limit of another. It is important that T-duality has also a world sheet realization: it changes sign of the right modes on the string world sheet:

$$X_L \to X_L, \qquad X_R \to -X_R. \tag{I.17}$$

The above picture indicates that the string coupling constant is always determined by the background on which string theory is considered. Thus, it is not a free parameter but one of the moduli of the underlying M-theory.

It is worth to note that the realization of the dualities was possible only due to the discovery of new dynamical objects in string theory — D branes [18]. They appear in several ways. On the one hand, they are solitonic solutions of supergravity equations determining possible string backgrounds. On the other hand, they are objects where open strings can end. In this case Dirichlet boundary conditions are imposed on the fields propagating on the open string world sheet. Already at this point it is clear that such objects must present in the theory because the T-duality transformation (I.17) exchanges the Neumann and Dirichlet boundary conditions.

We stop our discussion of critical superstring theories here. We see that they allow for a nice unified picture of all interactions. However, the final theory remains to be hidden from us and we even do not know what principles should define it. Also a correct way to compactify extra dimensions to get the 4-dimensional physics is not yet found.

3 Low-energy limit and string backgrounds

3.1 General σ-model

In the previous section we discussed string theory in the flat spacetime. What changes if the target space is curved? We will concentrate here only on the bosonic theory. Adding fermions does not change much in the conclusions of this section.

In fact, we already defined an action for the string moving in a general spacetime. It is given by the σ-model (I.6) with an arbitrary $G_{\mu\nu}(X)$. On the other hand, one can think about a non-trivial spacetime metric as a coherent state of gravitons which appear in the closed string spectrum. Thus, the insertion of the metric $G_{\mu\nu}$ into the world sheet action is, roughly speaking, equivalent to summing of excitations of this mode.

But the graviton is only one of the massless modes of the string spectrum. For the closed string the spectrum contains also two other massless fields: the antisymmetric tensor $B_{\mu\nu}$ and the scalar dilaton Φ. There is no reason to turn on the first mode and to leave other modes non-excited. Therefore, it is more natural to write a generalization of (I.6) which includes also $B_{\mu\nu}$ and Φ. It is given by the most general world sheet action which is invariant under general coordinate transformations and renormalizable [19]:[4]

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{h}\left[\left(h^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha'\mathcal{R}\Phi(X)\right], \tag{I.18}$$

In contrast to the Polyakov action in flat spacetime, the action (I.18) is non-linear and represents an interacting theory. The couplings of this theory are coefficients of $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ of their expansion in X^μ. These coefficients are dimensionfull and the actual dimensionless couplings are their combinations with the parameter α'. This parameter has dimension of squared length and determines the *string scale*. It is clear that the perturbation expansion of the world sheet quantum field theory is an expansion in α' and, at the same time, it corresponds to the long-range or low-energy expansion in the target space. At large distances compared to the string scale, the internal structure of the string is not important and we should obtain an effective theory. This theory is nothing else but an effective field theory of massless string modes.

3.2 Weyl invariance and effective action

The effective theory, which appears in the low-energy limit, should be a theory of fields in the target space. On the other hand, from the world sheet point of view, these fields represent an infinite set of couplings of a two-dimensional quantum field theory. Therefore, equations of the effective theory should be some constraints on the couplings.

What are these constraints? The only condition, which is not imposed by hand, is that the σ-model (I.18) should define a consistent string theory. In particular, this means that the resulting quantum theory preserves the Weyl invariance. It is this requirement that gives the necessary equations on the target space fields.

With each field one can associate a β-function. The Weyl invariance requires the vanishing of all β-functions [20]. These are the conditions we were looking for. In the first order

[4]In the following, the world sheet metric is always implied to be Euclidean.

in α' one can find the following equations

$$\begin{aligned}
\beta^G_{\mu\nu} &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma} + O(\alpha') = 0, \\
\beta^B_{\mu\nu} &= -\frac{1}{2}\nabla_\lambda H_{\mu\nu}{}^\lambda + H_{\mu\nu}{}^\lambda\nabla_\lambda\Phi + O(\alpha') = 0, \\
\beta^\Phi &= \frac{D-26}{6\alpha'} - \frac{1}{4}R - \nabla^2\Phi + (\nabla\Phi)^2 + \frac{1}{48}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O(\alpha') = 0,
\end{aligned} \tag{I.19}$$

where

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu} \tag{I.20}$$

is the field strength for the antisymmetric tensor $B_{\mu\nu}$.

A very non-trivial fact which, on the other hand, can be considered as a sign of consistency of the approach, is that the equations (I.19) can be derived from the spacetime action [19]

$$S_{\text{eff}} = \frac{1}{2}\int d^DX\,\sqrt{-G}\,e^{-2\Phi}\left[-\frac{2(D-26)}{3\alpha'} + R + 4(\nabla\Phi)^2 - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]. \tag{I.21}$$

All terms in this action are very natural representing the simplest Lagrangians for symmetric spin-2, scalar, and antisymmetric spin-2 fields. The first term plays the role of the cosmological constant. It is huge in the used approximation since it is proportional to α'^{-1}. But just in the critical dimension it vanishes identically.

The only non-standard thing is the presence of the factor $e^{-2\Phi}$ in front of the action. However, it can be removed by rescaling the metric. As a result, one gets the usual Einstein term what means that in the low-energy approximation string theory reproduces Einstein gravity.

3.3 Linear dilaton background

Any solution of the equations (I.19) defines a consistent string theory. In particular, among them one finds the simplest flat, constant dilaton background

$$G_{\mu\nu} = \eta_{\mu\nu}, \qquad B_{\mu\nu} = 0, \qquad \Phi = \nu, \tag{I.22}$$

which is a solution of the equations of motion only in $D_{\text{cr}} = 26$ dimensions reproducing the condition we saw above.

There are also solutions which do not require any restriction on the dimension of spacetime. To find them it is enough to choose a non-constant dilaton to cancel the first term in β^Φ. Strictly speaking, it is not completely satisfactory because the first term has another order in α' and, if we want to cancel it, one has to take into account contributions from the next orders. Nevertheless, there exist *exact* solutions which do not involve the higher orders. The most important solution is the so called *linear dilaton background*

$$G_{\mu\nu} = \eta_{\mu\nu}, \qquad B_{\mu\nu} = 0, \qquad \Phi = l_\mu X^\mu, \tag{I.23}$$

where

$$l_\mu l^\mu = \frac{26-D}{6\alpha'}. \tag{I.24}$$

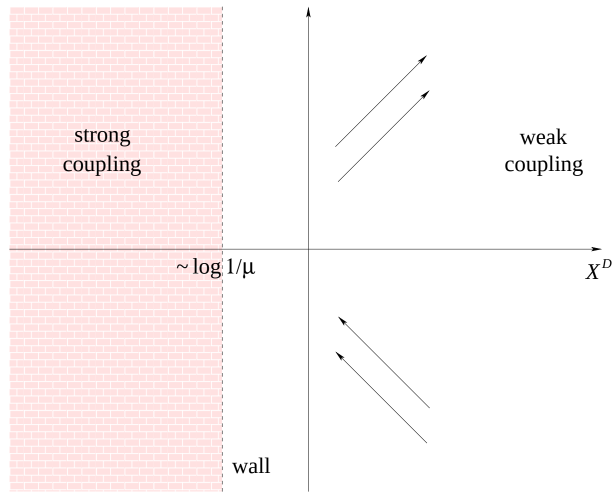


Fig. I.6: String propagation in the linear dilaton background in the presence of the tachyon mode. The non-vanishing tachyon produces a wall prohibiting the penetration into the region of a large coupling constant.

Note that the dilaton is a generalization of the coupling constant ν in (I.6). Therefore, from (I.16) it is clear that this is the dilaton that defines the string coupling constant which can now vary in spacetime

$$g_{\rm cl} \sim e^{\Phi}. \tag{I.25}$$

But then for the solution (I.23) there is a region where the coupling diverges and the string perturbation theory fails. This means that such background does not define a satisfactory string theory. However, there is a way to cure this problem.

3.4 Inclusion of tachyon

When we wrote the renormalizable σ-model (I.18), we actually missed one possible term which is a generalization of the two-dimensional cosmological constant

$$S_\sigma^T = \frac{1}{4\pi\alpha'}\int d^2\sigma \, \sqrt{h}\, T(X). \tag{I.26}$$

From the target space point of view, it introduces a tachyon field which is the lowest mode of bosonic strings. One can repeat the analysis of section 3.2 and calculate the contributions of this term to the β-functions. Similarly to the massless modes, all of them can be deduced from the spacetime action which should be added to (I.21)

$$S_{\rm tach} = -\frac{1}{2}\int d^D X \, \sqrt{-G}\, e^{-2\Phi}\left[(\nabla T)^2 - \frac{4}{\alpha'}T^2\right]. \tag{I.27}$$

Let us consider the tachyon as a field moving in the fixed linear dilaton background. Substituting (I.23) into the action (I.27), one obtains the following equation of motion

$$\partial^2 T - 2l^\mu \partial_\mu T + \frac{4}{\alpha'}T = 0. \tag{I.28}$$

It is easy to find its general solution

$$T = \mu \exp(p_\mu X^\mu), \qquad (p-l)^2 = \frac{2-D}{6\alpha'}. \tag{I.29}$$

Together with (I.23), (I.29) defines a generalization of the linear dilaton background. Strictly speaking, it is not a solution of the equations of motion derived from the common action $S_{\rm eff} + S_{\rm tach}$. However, this action includes only the first order in α', whereas, in general, as we discussed above, one should take into account higher order contributions. The necessity to do this is seen from the fact that the background fields (I.23) and (I.29) involve α' in a non-trivial way. The claim is that they give an *exact* string background. Indeed, in this background the complete σ-model action takes the form

$$S_\sigma^{\rm l.d.} = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{h} \left[h^{ab} \partial_a X^\mu \partial_b X_\mu + \alpha' \mathcal{R} l_\mu X^\mu + \mu e^{p_\mu X^\mu} \right]. \tag{I.30}$$

It can be checked that it represents an exact CFT and, consequently, defines a consistent string theory.

Why does the introduction of the non-vanishing tachyonic mode make the situation better? The reason is that this mode gives rise to an exponential potential, which suppresses the string propagation into the region where the coupling constant $g_{\rm cl}$ is large. It acts as an effective wall placed at $X^\mu \sim \frac{p^\mu}{p^2} \log(1/\mu)$. The resulting qualitative picture is shown in fig. I.6. Thus, we avoid the problem to consider strings at strong coupling.

4 Non-critical string theory

In the previous section we saw that, if to introduce non-vanishing expectation values for the dilaton and the tachyon, it is possible to define consistent string theory not only in the spacetime of critical dimension $D_{\rm cr} = 26$. Still one can ask the question: is there any sense for a theory where the conformal anomaly is not canceled? For example, if we look at the σ-model just as a statistical system of two-dimensional surfaces embedded into d-dimensional space and having some internal degrees of freedom, there is no reason for the system to be Weyl-invariant. Therefore, even in the presence of the Weyl anomaly, the system should possess some interpretation. It is called *non-critical string theory*.

When one uses the interpretation we just described, even at the classical level one can introduce terms breaking the Weyl invariance such as the world sheet cosmological constant. Then the conformal mode of the metric becomes a dynamical field and one should gauge fix only the world sheet diffeomorphisms. It can be done, for example, using the conformal gauge

$$h_{ab} = e^{\phi(\sigma)}\hat h_{ab}. \tag{I.31}$$

As a result, one obtains an effective action where, besides the matter fields, there is a contribution depending on ϕ [10]. Let us work in the flat target space. Then, after a suitable rescaling of ϕ to get the right kinetic term, the action is written as

$$S_{\rm CFT} = \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{\hat h}\left[\hat h^{ab}\partial_a X^\mu \partial_b X_\mu + \hat h^{ab}\partial_a\phi\partial_b\phi - \alpha' Q\hat{\mathcal{R}}\phi + \mu e^{\gamma\phi} + \text{ghosts}\right]. \tag{I.32}$$

The second and third terms, which give dynamics to the conformal mode, come from the measure of integration over all fields due to its non-invariance under the Weyl transformations. The coefficient Q can be calculated from the conformal anomaly and is given by

$$Q = \sqrt{\frac{25-d}{6\alpha'}}. \tag{I.33}$$

The coefficient γ is fixed by the condition that the theory should depend only on the full metric h_{ab}. This means that the effective action (I.32) should be invariant under the following Weyl transformations

$$\hat h_{ab}(\sigma) \longrightarrow e^{\rho(\sigma)}\hat h_{ab}(\sigma), \qquad \phi(\sigma) \longrightarrow \phi(\sigma) - \rho(\sigma). \tag{I.34}$$

This implies that the action (I.32) defines CFT. This is indeed the case only if

$$\gamma = -\frac{1}{\sqrt{6\alpha'}}\left(\sqrt{25-d} - \sqrt{1-d}\right). \tag{I.35}$$

The CFT (I.32) is called Liouville theory coupled with $c = d$ matter. The conformal mode ϕ is the Liouville field.

The comparison of the two CFT actions (I.32) and (I.30) shows that they are equivalent if one takes $D = d+1$, $p_\mu \sim l_\mu$ and identifies $X^D = \phi$. Then all coefficients also coincide as follows from (I.24), (I.29), (I.33) and (I.35). Thus, the conformal mode of the world sheet metric can be interpreted as an additional spacetime coordinate. With this interpretation non-critical string theory in the flat d-dimensional spacetime is seen as critical string theory in the $d+1$-dimensional linear dilaton background. The world sheet cosmological constant μ is identified with the amplitude of the tachyonic mode.

5 Two-dimensional string theory

In the following we will concentrate on the particular case of 2D bosonic string theory. It represents the main subject of this thesis. I hope to convince the reader that it has a very rich and interesting structure and, at the same time, it is integrable and allows for many detailed calculations.[5] Thus, the two-dimensional case looks to be special and it is a particular realization of a very universal structure. It appears in the description of different physical and mathematical problems. We will return to this question in the last chapters of the thesis. Here we just mention two interpretations which, as we have already seen, are equivalent to the critical string theory.

From the point of view of non-critical strings, 2D string theory is a model of fluctuating two-dimensional surfaces embedded into 1-dimensional time. The second space coordinate arises from the metric on the surfaces.

Another possible interpretation of this system described in section 1.3 considers it as two-dimensional gravity coupled with the $c = 1$ matter. The total central charge vanishes since the Liouville field ϕ, arising due to the conformal anomaly, contributes $1 + 6\alpha' Q^2$, where Q is given in (I.33), and cancels the contribution of matter and ghosts.

5.1 Tachyon in two-dimensions

To see that the two-dimensional case is indeed very special, let us consider the effective action (I.27) for the tachyon field in the linear dilaton background

$$S_{\rm tach} = -\frac{1}{2}\int d^D X\, e^{2Q\phi}\left[(\partial T)^2 - \frac{4}{\alpha'}T^2\right], \tag{I.36}$$

where ϕ is the target space coordinate coinciding with the gradient of the dilaton. According to the previous section, it can be considered as the conformal mode of the world sheet metric of non-critical strings. After the redefinition $T = e^{-Q\phi}\eta$, the tachyon action becomes an action of a scalar field in the flat spacetime

$$S_{\rm tach} = -\frac{1}{2}\int d^D X\, \left[(\partial \eta)^2 + m_\eta^2\eta^2\right], \tag{I.37}$$

where

$$m_\eta^2 = Q^2 - \frac{4}{\alpha'} = \frac{2-D}{6\alpha'} \tag{I.38}$$

is the mass of this field. For $D > 2$ the field η has an imaginary mass being a real tachyon. However, for $D = 2$ it becomes massless. Although we will still call this mode "tachyon", strictly speaking, it represents a good massless field describing the stable vacuum of the two-dimensional bosonic string theory. As always, the appearance in the spectrum of the additional massless field indicates that the theory acquires some special properties.

In fact, the tachyon is the only field theoretic degree of freedom of strings in two dimensions. This is evident in the light cone gauge where there are physical excitations associated

[5]There is the so called $c = 1$ barrier which coincides with 2D string theory. Whereas string theories with $c \le 1$ are solvable, we cannot say much about $c > 1$ cases.

with $D-2$ transverse oscillations and the motion of the string center of mass. The former are absent in our case and the latter is identified with the tachyon field.

To find the full spectrum of states and the corresponding vertex operators, one should investigate the CFT (I.32) with one matter field X. The theory is well defined when the kinetic term for the X field enters with the + sign so that X plays the role of a space coordinate. Thus, we will consider the following CFT

$$S_{\rm CFT} = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{\hat h}\left[\hat h^{ab}\partial_a X\partial_b X + \hat h^{ab}\partial_a\phi\partial_b\phi - 2\hat{\mathcal{R}}\phi + \mu e^{-2\phi} + \text{ghosts}\right], \tag{I.39}$$

where we chose $\alpha' = 1$ and took into account that in two dimensions $Q = 2$, $\gamma = -2$. This CFT describes the Euclidean target space. The Minkowskian version is defined by the analytical continuation $X \to it$.

The CFT (I.39) is a difficult interacting theory due to the presence of the Liouville term $\mu e^{-2\phi}$. Nevertheless, one can note that in the region $\phi\to\infty$ this interaction is negligible and the theory becomes free. Since the interaction is arbitrarily weak in the asymptotics, it cannot create or destroy states concentrated in this region. However, it removes from the spectrum all states concentrated at the opposite side of the Liouville direction. Therefore, it is sufficient to investigate the spectrum of the free theory with $\mu = 0$ and impose the so called *Seiberg bound* which truncates the spectrum by half [21].

The (asymptotic form of) vertex operators of the tachyon have already been found in (I.29). If $l_\mu = (0, -Q)$ and $p_\mu = (p_X, p_\phi)$, one obtains the equation

$$p_X^2 + (p_\phi + Q)^2 = 0 \tag{I.40}$$

with the general solution ($Q = 2$)

$$p_X = ip, \qquad p_\phi = -2 \pm |p|, \qquad p \in \mathbf{R}. \tag{I.41}$$

Imposing the Seiberg bound, which forbids the operators growing at $\phi\to-\infty$, we have to choose the + sign in (I.41). Thus, the tachyon vertex operators are

$$V_p = \int d^2\sigma\, e^{ipX} e^{(|p|-2)\phi}. \tag{I.42}$$

Here p is the Euclidean momentum of the tachyon. When we go to the Minkowskian signature, the momentum should also be continued as follows

$$X \to it, \qquad p \to -ik. \tag{I.43}$$

As a result, the vertex operators take the form

$$\begin{array}{rcl} V_k^- &=& \int d^2\sigma\, e^{ik(t-\phi)} e^{-2\phi}, \\ V_k^+ &=& \int d^2\sigma\, e^{-ik(t+\phi)} e^{-2\phi}, \end{array} \tag{I.44}$$

where $k > 0$. The two types of operators describe outgoing right movers and incoming left movers, respectively. They are used to calculate the scattering of tachyons off the Liouville wall.

5.2 Discrete states

Although the tachyon is the only target space field in 2D string theory, there are also physical states which are remnants of the transverse excitations of the string in higher dimensions. They appear at special values of momenta and they are called *discrete states* [22, 23, 24, 25].

To define their vertex operators, we introduce the *chiral* fields

$$W_{j,m} = \mathcal{P}_{j,m}(\partial X, \partial^2 X, \ldots) e^{2imX_L} e^{2(j-1)\phi_L}, \tag{I.45}$$
$$\bar{W}_{j,m} = \mathcal{P}_{j,m}(\bar{\partial} X, \bar{\partial}^2 X, \ldots) e^{2imX_R} e^{2(j-1)\phi_R}, \tag{I.46}$$

where $j = 0, \frac{1}{2}, 1, \ldots$, $m = -j, \ldots, j$ and we used the decomposition of the world sheet fields into the chiral (left and right) components

$$X(\tau, \sigma) = X_L(\tau + i\sigma) + X_R(\tau - i\sigma) \tag{I.47}$$

and similarly for ϕ. $\mathcal{P}_{j,m}$ are polynomials in the chiral derivatives of X. Their dimension is $j^2 - m^2$. Due to this, $\mathcal{P}_{j,\pm j} = 1$. For each fixed j, the set of operators $W_{j,m}$ forms an SU(2) multiplet of spin j. Altogether, the operators (I.45) form $W_{1+\infty}$ algebra.

With the above definitions, the operators creating the discrete states are given by

$$V_{j,m} = \int d^2\sigma\, W_{j,m} \bar{W}_{j,m}, \tag{I.48}$$

Thus, the discrete states appear at the following momenta

$$p_X = 2im, \qquad p_\phi = 2(j-1). \tag{I.49}$$

It is clear that the lowest and highest components $V_{j,\pm j}$ of each multiplet are just special cases of the vertex operators (I.42). The simplest non-trivial discrete state is the zero-momentum dilaton

$$V_{1,0} = \int d^2\sigma\, \partial X \bar{\partial} X. \tag{I.50}$$

5.3 Compactification, winding modes and T-duality

So far we considered 2D string theory in the usual flat Euclidean or Minkowskian spacetime. The simplest thing which we can do with this spacetime is to compactify it. Since there is no translational invariance in the Liouville direction, it cannot be compactified. Therefore, we do compactification only for the Euclidean "time" coordinate X. We require

$$X \sim X + \beta, \qquad \beta = 2\pi R, \tag{I.51}$$

where R is the radius of the compactification. Because it is the time direction that is compactified, we expect the resulting Minkowskian theory be equivalent to a thermodynamical system at temperature $T = 1/\beta$.

The compactification restricts the allowed tachyon momenta to discrete values $p_n = n/R$ so that we have only a discrete set of vertex operators. Besides, depending on the radius, the compactification can create or destroy the discrete states. Whereas for rational values of the radius some discrete states are present in the spectrum, for general irrational radius there are no discrete states.

But the compactification also leads to the existence of new physical string states. They correspond to configurations where the string is wrapped around the compactified dimension. Such excitations are called *winding modes*. To describe these configurations in the CFT terms, one should use the decomposition (I.47) of the world sheet field X into the left and right moving components. Then the operators creating the winding modes, the *vortex* operators, are defined in terms of the dual field

$$\tilde{X}(\tau,\sigma) = X_L(\tau + i\sigma) - X_R(\tau - i\sigma). \tag{I.52}$$

They also have a discrete spectrum, but with the inverse frequency: $q_m = mR$. In other respects they are similar to the vertex operators (I.42)

$$\tilde{V}_q = \int d^2\sigma \, e^{iq\tilde{X}} e^{(|q|-2)\phi}. \tag{I.53}$$

The vertex and vortex operators are related by T-duality, which exchanges the radius of compactification $R \leftrightarrow 1/R$ and the world sheet fields corresponding to the compactified direction $X \leftrightarrow \tilde{X}$ (cf. (I.17)). Thus, from the CFT point of view it does not matter whether vertex or vortex operators are used to perturb the free theory. For example, the correlators of tachyons at the radius R should coincide with the correlators of windings at the radius $1/R$.[6]

Note that the self-dual radius $R = 1$ is distinguished by a higher symmetry of the system in this case. As we will see, its mathematical description is especially simple.

[6]In fact, one should also change the cosmological constant $\mu \to R\mu$ [26]. This change is equivalent to a constant shift of the dilaton which is necessary to preserve the invariance to all orders in the genus expansion.

6 2D string theory in non-trivial backgrounds

6.1 Curved backgrounds: Black hole

In the previous section we described the basic properties of string theory in two-dimensions in the linear dilaton background. In this thesis we will be interested in more general backgrounds. In the low-energy limit all of them can be described by an effective theory. Its action can be extracted from (I.21) and (I.27). Since there is no antisymmetric 3-tensor in two dimensions, the B-field does not contribute and we remain with the following action

$$S_{\text{eff}} = \frac{1}{2}\int d^2X\,\sqrt{-G}\,e^{-2\Phi}\left[\frac{16}{\alpha'} + R + 4(\nabla\Phi)^2 - (\nabla T)^2 + \frac{4}{\alpha'}T^2\right]. \tag{I.54}$$

It is a model of dilaton gravity non-minimally coupled with a scalar field, the tachyon T. It is known to have solutions with non-vanishing curvature. Moreover, without the tachyon its general solution is well known and is written as [27] ($X^\mu = (t, r)$, $Q = 2/\sqrt{\alpha'}$)

$$ds^2 = -\left(1 - e^{-2Qr}\right)dt^2 + \frac{1}{1 - e^{-2Qr}}dr^2, \qquad \Phi = \varphi_0 - Qr. \tag{I.55}$$

In this form the solution resembles the radial part of the Schwarzschild metric for a spherically symmetric black hole. This is not a coincidence since the spacetime (I.55) does correspond to a two-dimensional black hole. At $r = -\infty$ the curvature has a singularity and at $r = 0$ the metric has a coordinate singularity corresponding to the black hole horizon. There is only one integration constant φ_0 which can be related to the mass of black hole

$$M_{\text{bh}} = 2Qe^{-2\varphi_0}. \tag{I.56}$$

As the usual Schwarzschild black hole, this black hole emits the Hawking radiation at the temperature $T_H = \frac{Q}{2\pi}$ [28] and has a non-vanishing entropy [29, 30]. Thus, 2D string theory incorporates all problems of the black hole thermodynamics and represents a model to approach their solution. Compared to the quantum field theory analysis on curved spacetime, in string theory the situation is better since it is a well defined theory. Therefore, one can hope to solve the issues related to physics at Planck scale, such as microscopic description of the black hole entropy, which are inaccessible by the usual methods.

To accomplish this task, one needs to know the background not only in the low-energy limit but also at all scales. Remarkably, an exact CFT, which reduces in the leading order in α' to the world sheet string action in the black hole background (I.55), was constructed [31]. It is given by the so called $[\mathrm{SL}(2,\mathbf{R})]_k/\mathrm{U}(1)$ coset σ-model where k is the level of the representation of the current algebra. Relying on this CFT, the exact form of the background (I.55), which ensures the Weyl invariance in all orders in α', was found [32]. We write it in the following form

$$ds^2 = -l^2(x)dt^2 + dx^2, \qquad l(x) = \tfrac{(1-p)^{1/2}\tanh Qx}{(1-p\tanh^2 Qx)^{1/2}}, \tag{I.57}$$

$$\Phi = \varphi_0 - \log\cosh Qx - \tfrac{1}{4}\log(1 - p\tanh^2 Qx), \tag{I.58}$$

where p, Q and the level k are related by

$$p = \frac{2\alpha' Q^2}{1 + 2\alpha' Q^2}, \qquad k = \frac{2}{p} = 2 + \frac{1}{\alpha' Q^2} \tag{I.59}$$

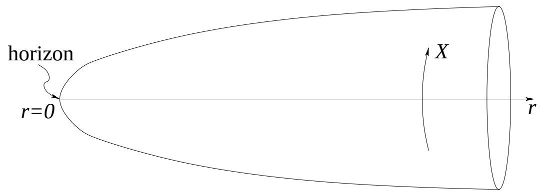


Fig. I.7: The Euclidean black hole.

so that in our case $p = 8/9$, $k = 9/4$. To establish the relation with the background (I.55), one should change the radial coordinate

$$Qr = \ln\left[\frac{\sqrt{1-p}}{1+\sqrt{1-p}}\left(\cosh Qx + \sqrt{\cosh^2 Qx + \frac{p}{1-p}}\right)\right] \tag{I.60}$$

and take $p \to 0$ limit. This exact solution possesses the same properties as the approximate one. However, it is difficult to extract its quantitative thermodynamical characteristics such as mass, entropy, and free energy. The reason is that we do not know any action for which the metric (I.57) and the dilaton (I.58) give a solution.[7] The existing attempts to derive these characteristics rely on some assumptions and lead to ambiguous results [34].

The form (I.57) of the solution is convenient for the continuation to the Euclidean metric. It is achieved by $t = -iX$ what changes sign of the first term. The resulting space can be represented by a smooth manifold if to take the time coordinate X be periodic with the period

$$\beta = \frac{2\pi}{Q\sqrt{1-p}}. \tag{I.61}$$

The manifold looks as a cigar (fig. I.7) and the choice (I.61) ensures the absence of conical singularity at the tip. It is clear that this condition reproduces the Hawking temperature in the limit $p \to 0$ and generalizes it to all orders in α'. The function $l(x)$ multiplied by $R = \sqrt{\alpha' k}$ plays the role of the radius of the compactified dimension. It approaches the constant value R at infinity and vanishes at the tip so that this point represents the horizon of the Minkowskian black hole. Thus, the cigar describes only the exterior of the black hole.

Note that the CFT describing this Euclidean continuation is represented by the coset

$$[H_3^+]_k/\mathrm{U}(1), \quad H_3^+ \equiv \frac{\mathrm{SL}(2,\mathbf{C})}{\mathrm{SU}(2)}, \tag{I.62}$$

where H_3^+ can be thought as Euclidean AdS_3.

Using the coset CFT, two and three-point correlators of tachyons and windings on the black hole background were calculated [32, 35, 36, 37]. By T-duality they coincide with winding and tachyon correlators, respectively, on a dual spacetime, which is called *trumpet* and can be obtained replacing cosh and tanh in (I.57), (I.58) by sinh and coth. This dual spacetime describes a naked (without horizon) black hole of a negative mass [32]. In fact, it appears as a part of the global analytical continuation of the initial black hole spacetime.

[7] It is worth to mention the recent result that (I.57), (I.58) cannot be solution of any dilaton gravity model with only second derivatives [33].

6.2 Tachyon and winding condensation

In the CFT terms, string theory on the curved background considered above is obtained as a σ-model. If one chooses the dilaton as the radial coordinate, then the σ-model looks as CFT (I.39) where the kinetic term is coupled with the black hole metric $G_{\mu\nu}$ and there is no Liouville exponential interaction. The change of the metric can be represented as a perturbation of the linear dilaton background by the gravitational vertex operator. Note that this operator creates one of the discrete states.

It is natural to consider also perturbations by another relevant operators existing in the initial CFT (I.39) defined in the linear dilaton background. First of all, these are the tachyon vertex operators V_p (I.42). Besides, if we consider the Euclidean theory compactified on a circle, there exist the vortex operators $\tilde{V}_q$ (I.53). Thus, both types of operators can be used to perturb the simplest CFT (I.39)

$$S = S_{\rm CFT} + \sum_{n\neq 0}(t_n V_n + \tilde{t}_n \tilde{V}_n), \tag{I.63}$$

where we took into account that tachyons and windings have discrete spectra in the compactified theory.

What backgrounds of 2D string theory do these perturbations correspond to? The couplings t_n introduce a non-vanishing vacuum expectation value of the tachyon. Thus, they simply change the background value of T. Note that, in contrast to the cosmological constant term $\mu e^{-2\phi}$, these tachyon condensates are time-dependent. For the couplings $\tilde{t}_n$ we cannot give such a simple picture. The reason is that the windings do not have a local target space interpretation. Therefore, it is not clear which local characteristics of the background change by the introduction of a condensate of winding modes.

A concrete proposal has been made for the simplest case $\tilde{t}_{\pm 1}\neq 0$, which is called *Sine-Liouville CFT*. It was suggested that this CFT is equivalent to the $H_3^+/{\rm U}(1)$ σ-model describing string theory on the black hole background [38]. This conjecture was justified by the coincidence of spectra of the two CFTs as well as of two- and three-point correlators as we discuss in the next paragraph. Following this idea, it is natural to suppose that any general winding perturbation changes the target space metric.

Note, that the world sheet T-duality relates the CFT (I.63) with one set of couplings $(t_n, \tilde{t}_n)$ and radius of compactification R to the similar CFT, where the couplings are exchanged $(\tilde{t}_n, t_n)$ and the radius is inverse $1/R$. However, these two theories should not describe the same background because the target space interpretations of tachyons and windings are quite different. T-duality allows to relate their correlators, but it says nothing how their condensation changes the target space.

6.3 FZZ conjecture

In this paragraph we give the precise formulation of the conjecture proposed by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov [38]. It states that the coset CFT $H_3^+/{\rm U}(1)$, describing string theory on the Euclidean black hole background, at arbitrary level k is equivalent to the CFT given by the following action

$$S_{\rm SL} = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \left[(\partial X)^2 + (\partial\phi)^2 - \alpha' Q\hat{\mathcal{R}}\phi + \lambda e^{\rho\phi}\cos(R\tilde{X})\right], \tag{I.64}$$

where the field X is compactified at radius R and the parameters are expressed through the level k

$$Q = \frac{1}{\sqrt{\alpha'(k-2)}}, \qquad R = \sqrt{\alpha' k}, \qquad \rho = -\sqrt{\frac{k-2}{\alpha'}}. \tag{I.65}$$

These identifications can be understood as follows. First, the duality requires the coincidence of the central charges of the two theories

$$c_{\mathrm{bh}} = \frac{3k}{k-2} - 1 \quad \text{and} \quad c_{\mathrm{SL}} = 2 + 6\alpha' Q^2. \tag{I.66}$$

This gives the first condition. The second equation in (I.65) allows to identify the two CFTs in the free asymptotic region $\phi \to \infty$ ($r \to \infty$). Indeed, in both cases the target space looks as a cylinder of the radius R so that the world sheet field X coincides with the radial coordinate on the cigar (see fig. I.7). Comparing the expressions for the dilaton, one also concludes that $r \sim \phi$. Finally, the last formula in (I.65) follows from the requirement that the scaling dimension of the interaction term is equal to one and from the first two identifications.

The first evidence for the equivalence is the coincidence of the spectra of the two theories. In both cases the observables $V_{j,n,m}$ are labeled by three indices: j related to representations of SL(2,$\mathbf{R}$), $n \in \mathbf{Z}$ measuring the momentum along the compactified direction, and $m \in \mathbf{Z}$ associated with the winding number. In the free asymptotic region they have the form

$$V_{j,n,m} \sim e^{ip_L X_L + ip_R X_R + 2Qj\phi} \tag{I.67}$$

and their scaling dimensions agree

$$\begin{array}{l} \Delta_{j,n,m} = \frac{\alpha' p_L^2}{4} - \alpha' Q^2 j(j+1) = \frac{n_L^2}{k} - \frac{j(j+1)}{k-2}, \\ \bar{\Delta}_{j,n,m} = \frac{\alpha' p_R^2}{4} - \alpha' Q^2 j(j+1) = \frac{n_R^2}{k} - \frac{j(j+1)}{k-2}, \end{array} \tag{I.68}$$

where

$$\begin{array}{ll} p_L = \frac{n}{R} + \frac{mR}{\alpha'} & p_R = \frac{n}{R} - \frac{mR}{\alpha'}, \\ n_L = \frac{1}{2}(n+km), & n_R = -\frac{1}{2}(n-km). \end{array} \tag{I.69}$$

Note also that in both theories there is a conservation of the momentum n, and the winding number m is not conserved. But the reason for that is different. Whereas in the cigar CFT the winding modes can slip off the tip of the cigar, in the Sine–Liouville CFT (I.64) the winding conservation is broken explicitly by the interaction term.

The next essential piece of evidence in favour of the FZZ conjecture is provided by the analysis of correlators in the two models. The two-point correlators on the cigar in the spherical approximation are written as follows [32]

$$\langle V_{j,n,m} V_{j,-n,-m} \rangle = (k-2)[\nu(k)]^{2j+1} \frac{\Gamma(1-\frac{2j+1}{k-2})\Gamma(-2j-1)\Gamma(j-n_L+1)\Gamma(1+j+n_R)}{\Gamma(\frac{2j+1}{k-2})\Gamma(2j+2)\Gamma(-j-n_L)\Gamma(n_R-j)}, \tag{I.70}$$

where

$$\nu(k) \equiv \frac{1}{\pi}\frac{\Gamma(1+\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}. \tag{I.71}$$

It was shown that they agree with the same correlators calculated in the Sine–Liouville theory [38]. Besides, the same statement was established also for the three point correlators.

Of course, this does not give a proof of the conjecture yet. But this represents a very non-trivial fact which is hardly believed to be accidental. Moreover, there is a supersymmetric generalization of this conjecture proposed in [39]. It relates the $\mathcal{N}=1$ superconformal coset model $\mathrm{SL}(2,\mathbf{R})/\mathrm{U}(1)$ to the $\mathcal{N}=2$ Liouville theory. The former theory has an accidental $\mathcal{N}=2$ supersymmetry which is a special case of the Kazama–Suzuki construction [40]. Therefore, the proposed relation is not quite surprising. As it often happens, supersymmetry simplifies the problem and, in contrast to the original bosonic case, this conjecture was explicitly proven [41].

Finally, one remark is in order. The FZZ conjecture was formulated for arbitrary level k and radius R. However, it is relevant for two-dimensional string theory only when the central charge is equal to 26. Therefore, in our case we have to fix all parameters

$$Q = 2/\sqrt{\alpha'}, \qquad R = 3\sqrt{\alpha'}/2, \qquad k = 9/4. \tag{I.72}$$

This means that there is only one point in the moduli space where we can apply the described duality.

Chapter II

Matrix models

In this chapter we introduce a powerful mathematical technique, which allows to solve many physical problems. Its main feature is the use of matrices of a large size. Therefore, the models formulated using this technology are called *matrix models*. Sometimes a matrix formulation is not only a useful mathematical description of a physical system, but it also sheds light on its fundamental degrees of freedom.

We will be interested mostly in application of matrix models to string theory. However, in the beginning we should explain their relation to physics, their general properties, and basic methods to solve them (for an extensive review, see [42]). This is the goal of this chapter.

1 Matrix models in physics

Working with matrix models, one usually considers the situation when the size of matrices is very large. Moreover, these models imply integration over matrices or averaging over them taking all matrix elements as independent variables. This means that one deals with systems where some random processes are expected. Indeed, this is a typical behaviour for the systems described by matrix models.

Statistical physics

Historically, for the first time matrix models appeared in nuclear physics. It was discovered by Wigner [43] that the energy levels of large atomic nuclei are distributed according to the same law, which describes the spectrum of eigenvalues of one Hermitian matrix in the limit where the size of the matrix goes to infinity. Already this result showed the important feature of universality: it could be applied to any nucleus and did not depend on particular characteristics of this nucleus.

Following this idea, one can generalize the matrix description of statistics of energy levels to any system, which either has many degrees of freedom and is too complicated for an exact description, or possesses a random behaviour. A typical example of systems of the first type is given by mesoscopic physics, where one is interested basically only in macroscopic characteristics. The second possibility is realized, in particular, in chaotic systems.

Quantum chromodynamics

Another subject, where matrix models gave a new method of calculation, is particle physics. The idea goes back to the work of 't Hooft [44] where he suggested to use the $1/N$ expansion for calculations in gauge theory with the gauge group SU(N). Initially, he suggested this expansion for QCD as an alternative to the usual perturbative expansion, which is valid only in the weak coupling region and fails at low energies due to the confinement. However, in the case of QCD it is not well justified since the expansion parameter equals $1/3^2$ and is not very small.

Nevertheless, 't Hooft realized several important facts about the $1/N$ expansion. First, SU(N) gauge theory can be considered as a model of $N \times N$ unitary matrices since the gauge fields are operators in the adjoint representation. Then the $1/N$ expansion corresponds to the limit of large matrices. Second, it coincides with the topological expansion where all Feynman diagrams are classified according to their topology, which one can associate if all lines in Feynman diagrams are considered as double lines. This gives the so called *fat graphs*. In the limit $N \to \infty$ only the *planar* diagrams survive. These are the diagrams which can be drawn on the 2-sphere without intersections. Thus, with each matrix model one can associate a diagrammatic expansion so that the size of matrices enters only as a prefactor for each diagram.

Although this idea has not led to a large progress in QCD, it gave rise to new developments, related with matrix models, in two-dimensional quantum gravity and string theory [45, 46, 47, 48, 49]. In turn, there is still a hope to find a connection between string theory and QCD relying on matrix models [50]. Besides, recently they were applied to describe supersymmetric gauge theories [51].

Quantum gravity and string theory

The common feature of two-dimensional quantum gravity and string theory is a sum over two-dimensional surfaces. It turns out that it also has a profound connection with matrix models ensuring their relevance for these two theories. We describe this connection in detail in the next section because it deserves a special attention. Here we just mention that the reformulation of string theory in terms of matrix models has lead to significant results in the low-dimensional cases such as 2D string theory. Unfortunately, this reformulation has not helped much in higher dimensions.

It is worth to note that there are matrix models of M-theory [52, 53], which is thought to be a unification of all string theories (see section I.2.3). (See also [54, 55, 56] for earlier attempts to describe the quantum mechanics of the supermembranes.) They claim to be fundamental non-perturbative and background independent formulations of Planck scale physics. However, they are based on the ideas different from the "old" matrix models of low-dimensional string theories.

Also matrix models appear in the so called *spin foam* approach to 3 and 4-dimensional quantum gravity [57, 58]. Similarly to the models of M–theory, they give a non-perturbative and background independent formulation of quantum gravity but do not help with calculations.

2 Matrix models and random surfaces

2.1 Definition of one-matrix model

Now it is time to define what a matrix model is. In the most simple case of *one-matrix model* (1MM), one considers the following integral over $N \times N$ matrices

$$Z = \int dM \, \exp\left[-N \,\mathrm{tr}\, V(M)\right], \tag{II.1}$$

where

$$V(M) = \sum_{k>0} \frac{g_k}{k} M^k \tag{II.2}$$

is a potential and the measure dM is understood as a product of the usual differentials of all independent matrix elements

$$dM = \prod_{i,j} dM_{ij}. \tag{II.3}$$

The integral (II.1) can be interpreted as the partition function in the canonical ensemble of a statistical model. Also it appears as a generating function for the correlators of the operators $\mathrm{tr}\, M^k$, which are obtained differentiating Z with respect to the couplings g_k. For general couplings, the integral (II.1) is divergent and should be defined by analytical continuation.

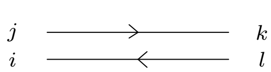
Actually, one can impose some restrictions on the matrix M_{ij} which reflect the symmetry of the problem. Correspondingly, there exist 3 ensembles of random matrices:

- ensemble of hermitian matrices with the symmetry group U(N);
- ensemble of real symmetric matrices with the symmetry group O(N);
- ensemble of quaternionic matrices with the symmetry group Sp(N).

We will consider only the first ensemble. Therefore, our systems will always possess the global U(N) invariance under the transformations

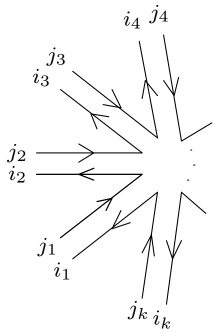
$$M \longrightarrow \Omega^\dagger M \Omega \quad (\Omega^\dagger \Omega = I). \tag{II.4}$$

In general, the integral (II.1) cannot be evaluated exactly and one has to use its perturbative expansion in the coupling constants. Following the usual methodology of quantum field theories, each term in this expansion can be represented as a Feynman diagram. Its ingredients, propagator and vertices, can be extracted from the potential (II.2). The main difference with the case of a scalar field is that the propagator is represented by an oriented double line what reflects the index structure carried by matrices.



$$= \frac{1}{Ng_2}\delta_{il}\delta_{jk}$$

The expression which is associated with the propagator can be obtained as an average of two matrices $\langle M_{ij} M_{kl}\rangle_0$ with respect to the Gaussian part of the potential. The coupling of indices corresponds to the usual matrix multiplication law. The vertices come from the terms of the potential of third and higher powers. They are also composed from the double lines and are given by a product of Kronecker symbols.



$$= N\tfrac{g_k}{k}\delta_{j_1 i_2}\delta_{j_2 i_3}\cdots\delta_{j_k i_1}$$

Note that each loop in the diagrams gives the factor N coming from the sum over contracted indices. As a result, the partition function (II.1) is represented as

$$Z = \sum_{\text{diagrams}} \frac{1}{s}\left(\frac{1}{Ng_2}\right)^E N^L \prod_k (-Ng_k)^{n_k}, \tag{II.5}$$

where the sum goes over all diagrams constructed from the drawn propagators and vertices (fat graphs) and we introduced the following notations:

- n_k is the number of vertices with k legs in the diagram,
- $V = \sum_k n_k$ is the total number of vertices,
- L is the number of loops,
- $E = \frac{1}{2}\sum_k kn_k$ is the number of propagators,
- s is the symmetry factor given by the order of the discrete group of symmetries of the diagram.

Thus, each diagram contributes to the partition function

$$\frac{1}{s} N^{V-E+L} g_2^{-E} \prod_k (-g_k)^{n_k}. \tag{II.6}$$

2.2 Generalizations

Generalizations of the one-matrix model can be obtained by increasing the number of matrices. The simplest generalization is *two-matrix model* (2MM). In a general case it is defined by the following integral

$$Z = \int dA\, dB\, \exp\left[-NW(A,B)\right], \tag{II.7}$$

where $W(A,B)$ is a potential invariant under the global unitary transformations

$$A \longrightarrow \Omega^\dagger A \Omega, \qquad B \longrightarrow \Omega^\dagger B \Omega. \tag{II.8}$$

The structure of this model is already much richer than the structure of the one-matrix model. It will be important for us in the study of 2D string theory.

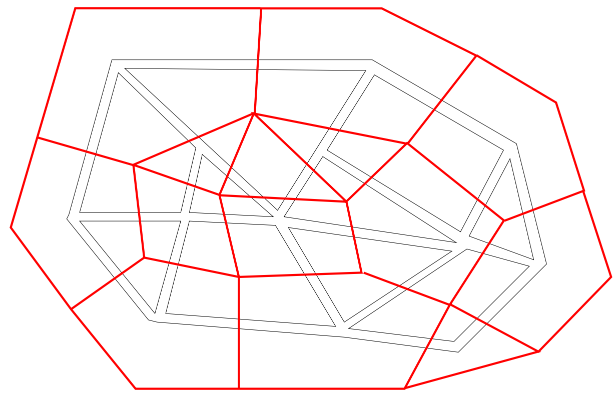


Fig. II.1: Duality between Feynman graphs and discretized surfaces.

Similarly, one can consider 3, 4, *etc.* matrix models. Their definition is the same as (II.7), where one requires the invariance of the potential under the simultaneous unitary transformation of all matrices. A popular choice for the potential is

$$W(A_1, \ldots, A_n) = \sum_{k=1}^{n-1} c_k \, \mathrm{tr}\,(A_k A_{k+1}) - \sum_{k=1}^{n} \mathrm{tr}\, V_k(A_k). \tag{II.9}$$

It represents a linear *matrix chain*. Other choices are also possible. For example, one can close the chain into a circle adding the term $\mathrm{tr}\,(A_k A_1)$. This crucially changes the properties of the model, since it loses integrability which is present in the case of the chain.

When the number of matrices increases to infinity, one can change the discrete index by a continuous argument. Then one considers a one-matrix integral, but the matrix is already a function. Interpreting the argument as a time variable, one obtains a quantum mechanical problem. It is called *matrix quantum mechanics* (MQM). The most part of the thesis is devoted to its investigation. Therefore, we will discuss it in detail in the next chapters.

Further generalizations include cases when one adds new arguments and discrete indices to matrices and combines them in different ways. One can even consider grassmanian, rectangular and other types of matrices. Also multitrace terms can be included into the potential.

2.3 Discretized surfaces

A remarkable fact, which allows to make contact between matrix models and two-dimensional quantum gravity and string theory, is that the matrix integral (II.1) can be interpreted as a sum over discretized surfaces [45, 46, 47]. Each Feynman diagram represented by a fat graph is dual to some triangulation of a two-dimensional surface as shown in fig. II.1. To construct the dual surface, one associates a k-polygon with each k-valent vertex and joins them along edges intersecting propagators of the Feynman diagram.

Note that the partition function is a sum over both connected and disconnected diagrams. Therefore, it gives rise to both connected and disconnected surfaces. If we are interested, as in quantum gravity, only in the connected surfaces, one should consider the free energy, which is the logarithm of the partition function, $F = \log Z$. Thus, taking into account (II.6), the duality of matrix diagrams and discretized surfaces leads to the following representation

$$F=\sum_{\text{surfaces}}\frac{1}{s}N^{V-E+L}g_2^{-E}\prod_k(-g_k)^{n_k}. \tag{II.10}$$

where we interpret:

- n_k is the number of k-polygons used in the discretization,
- $V=\sum_k n_k$ is the total number of polygons (faces),
- L is the number of vertices,
- $E=\frac{1}{2}\sum_k kn_k$ is the number of edges,
- s is the order of the group of automorphisms of the discretized surface.

It is clear that the relative numbers of k-polygons are controlled by the couplings g_k. For example, if one wants to use only triangles, one should choose the cubic potential in the corresponding matrix model.

The sum over discretizations (II.10) (more exactly, its continuum limit) can be considered as a definition of the sum over surfaces appearing in (I.8). Each discretization induces a curvature on the surface, which is concentrated at vertices where several polygons are joint to each other. For example, if at ith vertex there are $n_k^{(i)}$ k-polygons, the discrete counterpart of the curvature is

$$\mathcal{R}_i=2\pi\frac{\left(2-\sum\limits_k\frac{k-2}{k}n_k^{(i)}\right)}{\sum\limits_k\frac{1}{k}n_k^{(i)}}. \tag{II.11}$$

It counts the deficit angle at the given vertex. In the limit of large number of vertices, the discretization approximates some continuous geometry. Varying discretization, one can approximate any continuous distribution of the curvature with any given accuracy.

Note, that the discretization encodes only the information about the curvature which is diffeomorphism invariant. Therefore, the sum over discretizations realizes already a gauge fixed version of the path integral over geometries. Due to this, one does not need to deal with ghosts and other problems related to gauge fixing.

If one considers generalizations of the one-matrix model, the dual surfaces will carry additional structures. For example, the Feynman diagrams of the two-matrix model are drawn using two types of lines corresponding to two matrices. All vertices are constructed from the lines of a definite type since they come from the potential for either the first or the second matrix. Therefore, with each face of the dual discretization one can associate a discrete variable taking two values, say ± 1. Summing over all structures, one obtains Ising model on a random lattice [59, 60]. Similarly, it is possible to get various fields living on two-dimensional dynamical surfaces or, in other words, coupled with two-dimensional gravity.

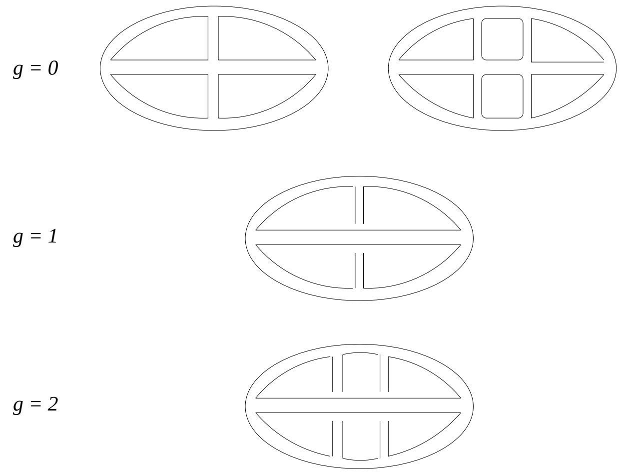


Fig. II.2: Diagrams of different genera.

2.4 Topological expansion

From (II.10) one concludes that each surface enters with the weight N^{V-E+L}. What is the meaning of the parameter N for surfaces? To answer this question, we note that the combination $V-E+L$ gives the Euler number $\chi = 2-2g$ of the surface. Indeed, the Euler number is defined as in (I.4). Under discretization the curvature turns into (II.11), the volume element at ith vertex becomes

$$\sqrt{h_i} = \sum_k n_k^{(i)}/k, \tag{II.12}$$

and the integral over the surface is replaced by the sum over the vertices. Thus, the Euler number for the discretized surface is defined as

$$\chi = \frac{1}{4\pi}\sum_i \sqrt{h_i}\mathcal{R}_i = \frac{1}{2}\sum_i\left(2-\sum_k \frac{k-2}{k}n_k^{(i)}\right) = L - \frac{1}{2}\sum_k (k-2)n_k = L-E+V. \tag{II.13}$$

Due to this, we can split the sum over surfaces in (II.10) into the sum over topologies and the sum over surfaces of a given topology, which imitates the integral over metrics,

$$F = \sum_{g=0}^{\infty} N^{2-2g} F_g(g_k). \tag{II.14}$$

Thus, N allows to distinguish surfaces of different topology. In the large N limit only surfaces of the spherical topology survive. Therefore, this limit is called also the *spherical limit.*

In terms of fat graphs of the matrix model, this classification by topology means the following. The diagrams appearing with the coefficient N^{2-2g}, which correspond to the surfaces of genus g, can be drawn on such surfaces without intersections. In particular, the leading diagrams coupled with N^2 are called *planar* and can be drawn on a 2-sphere or on a plane. For other diagrams, g can be interpreted as the minimal number of intersections which are needed to draw the diagram on a plane (see fig. II.2).

2.5 Continuum and double scaling limits

Our goal is to relate the matrix integral to the sum over Riemann surfaces. We have already completed the first step reducing the matrix integral to the sum over discretized surfaces. And the sum over topologies was automatically included. It remains only to extract a continuum limit.

To complete this second step, let us work for simplicity with the cubic potential

$$V(M) = \frac{1}{2}M^2 - \frac{\lambda}{3}M^3. \tag{II.15}$$

Then (II.10) takes the form

$$F = \sum_{g=0}^{\infty} N^{\chi} \sum_{\substack{\text{genus g} \\ \text{triangulations}}} \frac{1}{s}\lambda^V. \tag{II.16}$$

We compare this result with the partition function of two-dimensional quantum gravity

$$Z_{\rm QG} = \sum_{g=0}^{\infty} \int \mathcal{D}\varrho(h_{ab}) e^{-\nu\chi-\mu A}, \tag{II.17}$$

where g is the genus of the surface, $A = \int d^2\sigma \sqrt{h}$ is its area, and ν and μ are coupling constants. First of all, we see that one can identify $N = e^{-\nu}$. Then, if one assumes that all triangles have unit area, the total area is given by the number of triangles V. Due to this, (II.16) implies $\lambda = e^{-\mu}$.

However, this was a formal identification because one needs the coincidence of the partition function $Z_{\rm QG}$ and the free energy of the matrix model F. It is possible only in a continuum limit where we sum over the same set of continuous surfaces. In this limit the area of triangles used in the discretization should vanish. Since we fixed their area, in our case the continuum limit implies that the number of triangles should diverge $V \to \infty$.

In quantum theory one can speak only about expectation values. Hence we are interested in the behaviour of $\langle V \rangle$. In fact, for the spherical topology this quantity is dominated by non-universal contributions. To see the universal behaviour related to the continuum limit, one should consider more general correlation functions $\langle V^n \rangle$. From (II.16), for these quantities one obtains a simple expression

$$\langle V^n \rangle = \left(\lambda \frac{\partial}{\partial \lambda}\right)^n \log F. \tag{II.18}$$

The typical form of the contribution of genus g to the free energy is

$$F_g \sim F_g^{(\rm reg)} + (\lambda_c - \lambda)^{(2-\gamma_{\rm str})\chi/2}, \tag{II.19}$$

where $\gamma_{\rm str}$ is the so called *string susceptibility*, which defines the critical behaviour, λ_c is some critical value of the coupling constant, and we took into account the non-universal contribution $F_g^{(\rm reg)}$. The latter leads to the fact that the expectation value $\langle V \rangle$ remains finite for $\chi > 0$. But for all $n > 1$, one finds

$$\frac{\langle V^n \rangle}{\langle V^{n-1} \rangle} \sim \frac{1}{\lambda_c - \lambda}. \tag{II.20}$$

This shows that in the limit $\lambda \to \lambda_c$, the sum (II.16) is dominated by triangulations with large number of triangles. Thus, the continuum limit is obtained taking $\lambda \to \lambda_c$. One can renormalize the area and the couplings so that they remain finite in this limit.

Now we encounter the following problem. According to (II.19), the (universal part of the) free energy in the continuum limit either diverges or vanishes depending on the genus. On the other hand, in the natural limit $N \to \infty$ only the spherical contribution survives. How can one obtain contributions for all genera? It turns out that taking both limits not independently, but together in a correlated manner, one arrives at the desired result [61, 62, 63]. Indeed, we introduce the "renormalized" string coupling

$$\kappa^{-1} = N(\lambda_c - \lambda)^{(2-\gamma_{\rm str})/2}, \tag{II.21}$$

and consider the limit $N \to \infty$, $\lambda \to \lambda_c$, where κ is kept fixed. In this limit the free energy is written as an asymptotic expansion for $\kappa \to 0$

$$F = \sum_{g=0}^{\infty} \kappa^{-2+2g} f_g. \tag{II.22}$$

Thus, the described limit allows to keep all genera in the expansion of the free energy. It is called the *double scaling limit*. In this limit the free energy of the one-matrix model reproduces the partition function of two-dimensional quantum gravity. This correspondence holds also for various correlators and is extended to other models. In particular, the double scaling limit of MQM, which we describe in the next chapter, gives 2D string theory.

The fact that nothing depends on the particular form of the potential, which is equivalent to the independence of the type of polygons used to discretize surfaces (triangles, quadrangles, *etc.*), is known as *universality* of matrix models. All of them can be splitted into classes of universality. Each class is associated with some continuum theory and characterized by the limiting behaviour of a matrix model near its critical point [64].

3 One-matrix model: saddle point approach

In the following two sections we review two basic methods to solve matrix models. This will be done relying on explicit examples of the simplest matrix models, 1MM and 2MM. This section deals with the first model, which is defined by the integral over one hermitian matrix

$$Z = \int dM \, \exp\left[-N \,\mathrm{tr}\, V(M)\right]. \tag{II.23}$$

The potential was given in (II.2). Since the matrix is hermitian, its independent matrix elements are M_{ij} with $i \le j$ and the diagonal elements are real. Therefore, the measure dM is given by

$$dM = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}\, M_{ij}\, d\,\mathrm{Im}\, M_{ij}. \tag{II.24}$$

3.1 Reduction to eigenvalues

Each hermitian matrix can be diagonalized by a unitary transformation

$$M = \Omega^\dagger x \Omega, \qquad x = \mathrm{diag}(x_1, \dots, x_N), \;\; \Omega^\dagger \Omega = I. \tag{II.25}$$

Therefore, one can change variables from the matrix elements M_{ij} to the eigenvalues x_k and elements of the unitary matrix Ω diagonalizing M. This change produces a Jacobian. To find it, one considers a hermitian matrix which is obtained by an infinitesimal unitary transformation $\Omega = I + d\omega$ of the diagonal matrix x, where $d\omega$ is antisymmetric. In the first order in ω, one obtains

$$dM_{ij} \approx \delta_{ij} dx_j + [x, d\omega]_{ij} = \delta_{ij} dx_j + (x_i - x_j) d\omega_{ij}. \tag{II.26}$$

This leads to the following result

$$dM = [d\Omega]_{SU(N)} \prod_{k=1}^{N} dx_k \, \Delta^2(x), \tag{II.27}$$

where $[d\Omega]_{SU(N)}$ is the Haar measure on SU(N) and

$$\Delta(x) = \prod_{i<j} (x_i - x_j) = \det_{i,j} x_i^{j-1} \tag{II.28}$$

is the Vandermonde determinant.

Due to the U(N)-invariance of the potential, after the substitution (II.25) into the integral (II.23) the unitary matrix Ω decouples and can be integrated out. As a result, one arrives at the following representation

$$Z = \mathrm{Vol}(SU(N)) \int \prod_{k=1}^{N} dx_k \, \Delta^2(x) \exp\left[-N \sum_{k=1}^{N} V(x_k)\right]. \tag{II.29}$$

The volume of the SU(N) group is a constant depending only on the matrix size N. It is not relevant for the statistics of eigenvalues, although it is important for the dependence of the free energy on N.

The representation (II.29) is very important because it reduces the problem involving N^2 degrees of freedom to the problem of only N eigenvalues. This can be considered as a "generalized integrability": models allowing such reduction are in a sense "integrable".

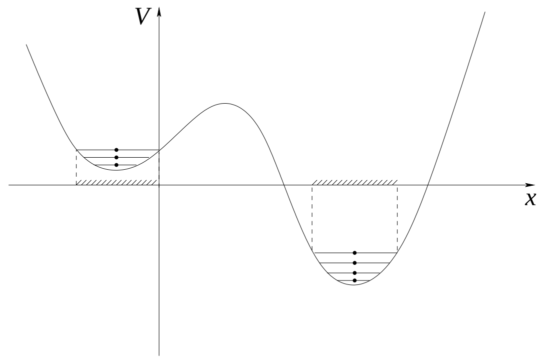


Fig. II.3: In the large N limit the eigenvalues fill finite intervals around the minima of the potential.

3.2 Saddle point equation

The integral (II.29) can also be presented in the form

$$Z=\int\prod_{k=1}^{N}dx_k\, e^{-N\mathcal{E}},\qquad \mathcal{E}=\sum_{k=1}^{N}V(x_k)-\frac{2}{N}\sum_{i<j}\log|x_i-x_j|, \tag{II.30}$$

where we omitted the irrelevant constant factor. It describes a system of N particles interacting by the two-dimensional repulsive Coulomb law in the common potential $V(x)$. In the limit $N\to\infty$, one can apply the usual saddle point method to evaluate the integral (II.30) [65]. It says that the main contribution comes from configurations of the eigenvalues satisfying the classical equations of motion $\partial\mathcal{E}/\partial x_k=0$. Thus, one obtains the following system of N algebraic equations

$$V'(x_k)=\frac{2}{N}\sum_{j\neq k}\frac{1}{x_k-x_j}. \tag{II.31}$$

If we neglected the Coulomb force, all eigenvalues would sit at the minima of the potential $V(x)$. Due to the Coulomb repulsion they are spread around these minima and fill some finite intervals as shown in fig. II.3. In the large N limit their distribution is characterized by the density function defined as follows

$$\rho(x)=\frac{1}{N}\left\langle \operatorname{tr}\delta(x-M)\right\rangle. \tag{II.32}$$

The density contains an important information about the system. For example, the spherical limit of the free energy $F_0=\lim\limits_{N\to\infty}N^{-2}\log Z$ is given by

$$F_0=-\int dx\,\rho(x)V(x)+\iint dxdy\,\rho(x)\rho(y)\log|x-y|. \tag{II.33}$$

The system of equations (II.31) can be also rewritten as one integral equation for $\rho(x)$

$$V'(x)=2\mathcal{P}.v.\int\frac{\rho(y)}{x-y}dy, \tag{II.34}$$

where $\mathcal{P}.v.$ indicates the principal value of the integral. To solve this equation, we introduce the following function

$$\omega(z) = \int \frac{\rho(x)}{z-x} dx. \tag{II.35}$$

Substitution of the definition (II.32) shows that it is the resolvent of the matrix M

$$\omega(z) = \frac{1}{N} \left\langle \mathrm{tr} \frac{1}{z-M} \right\rangle. \tag{II.36}$$

It is an analytical function on the whole complex plane except the intervals filled by the eigenvalues. At these intervals it has a discontinuity given by the density $\rho(x)$

$$\omega(x+i0) - \omega(x-i0) = -2\pi i \rho(x), \qquad x \in \sup[\rho]. \tag{II.37}$$

On the other hand, the real part of the resolvent on the support of $\rho(x)$ coincides with the principal value integral as in (II.34). Thus, one obtains

$$\omega(x+i0) + \omega(x-i0) = V'(x), \qquad x \in \sup[\rho]. \tag{II.38}$$

This equation is already simple enough to be solved explicitly.

3.3 One cut solution

In a general case the potential $V(x)$ has several minima and all of them can be filled by the eigenvalues. It means that the support of the density $\rho(x)$ will have several disconnected components. To understand the structure of the solution, let us consider the case when the support consists of only one interval (a, b). Then the resolvent $\omega(z)$ should be an analytical function on the complex plane with one cut along this interval. On this cut the equations (II.37) and (II.38) must hold. They are sufficient to fix the general form of $\omega(z)$

$$\omega(z) = \frac{1}{2} \left(V'(z) - P(z) \sqrt{(z-a)(z-b)} \right). \tag{II.39}$$

$P(z)$ is an analytical function which is fixed by the asymptotic condition

$$\omega(z) \underset{z\to\infty}{\sim} 1/z \tag{II.40}$$

following from (II.35) and normalization of the density $\int \rho(x)dx = 1$. If $V(z)$ is a polynomial of degree n, $P(z)$ should be a polynomial of degree $n-2$. In particular, for the case of the Gaussian potential, it is a constant. The same asymptotic condition (II.40) fixes the boundaries of the cut, a and b.

The density of eigenvalues is found as the discontinuity of the resolvent (II.39) on the cut and is given by

$$\rho(x) = \frac{P(x)}{2\pi} \sqrt{(x-a)(b-x)}. \tag{II.41}$$

If the potential is quadratic, then $P(x) = const$ and one obtains the famous semi-circle law of Wigner for the distribution of eigenvalues of random matrices in the Gaussian ensemble.

The free energy is obtained by substitution of (II.41) into (II.33). Note that the second term is one half of the first one due to (II.34). Therefore, one has

$$F_0 = -\frac{1}{2} \int_a^b dx\, \rho(x) V(x). \tag{II.42}$$

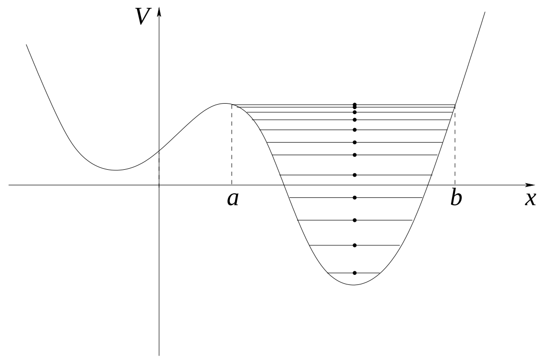


Fig. II.4: Critical point in one-matrix model.

3.4 Critical behaviour

The result (II.41) shows that near the end of the support the density of eigenvalues does not depend on the potential and always behaves as a square root. This is a manifestation of *universality* of matrix models mentioned in the end of section 2.

However, there are degenerate cases when this behaviour is violated. This can happen if the polynomial $P(x)$ vanishes at $x = a$ or $x = b$. For example, if a is a root of $P(x)$ of degree m, the density behaves as $\rho(x) \sim (x-a)^{m+1/2}$ near this point. It is clear that this situation is realized only for special values of the coupling constants g_k. They correspond to the critical points of the free energy discussed in connection with the continuum limit in paragraph 2.5. Indeed, one can show that near these configurations the spherical free energy (II.42) looks as (II.19) with $\chi = 2$ and $\gamma_{\rm str} = -1/(m+1)$ [64, 66].

Thus, each critical point with a given m defines a class of universality. All these classes correspond to continuum theories which are associated with a special discrete series of CFTs living on dynamical surfaces. This discrete series is a part of the so called *minimal conformal theories* which possess only finite number of primary fields [67]. The minimal CFTs are characterized by two relatively prime integers p and q with the following central charge and string susceptibility

$$c = 1 - 6\frac{(p-q)^2}{pq}, \qquad \gamma_{\rm str} = -\frac{2}{p+q-1}. \tag{II.43}$$

Our case is obtained when $p = 2m+1$, $q = 2$. Thus, for $m = 1$ the central charge vanishes and we describe the pure two-dimensional quantum gravity. The critical points with other m describe the two-dimensional quantum gravity coupled to the matter characterized by the rational central charge $c = 1 - 3\frac{(2m-1)^2}{2m+1}$.

From the matrix model point of view, there is a clear interpretation for the appearance of the critical points. Usually, in the large N limit the matrix eigenvalues fill consecutively the lowest energy levels in the well around a minimum of the effective potential. The critical behaviour arises when the highest filled level reaches an extremum of the effective potential. Generically, this happens as shown in fig. II.4. By fine tuning of the coupling constants, one can get more special configurations which will describe the critical behaviour with $m > 1$.

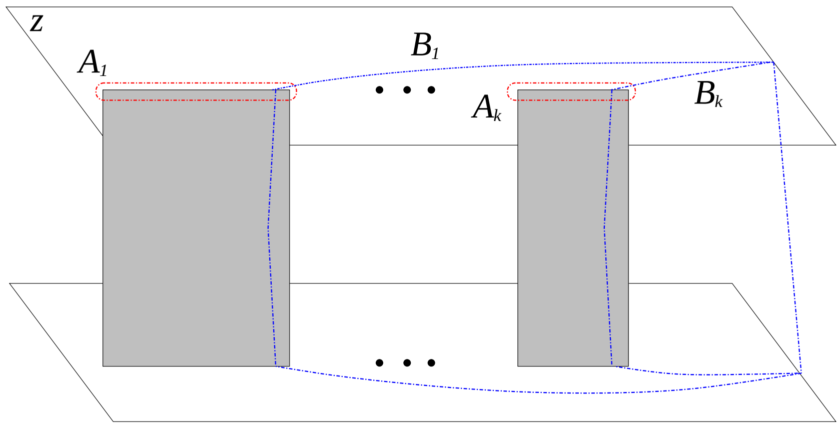


Fig. II.5: Riemann surface associated with the solution of 1MM.

3.5 General solution and complex curve

So far we considered the case where the density is concentrated on one interval. The general form of the solution can be obtained if we note that equation (II.38) can be rewritten as the following equation valid in the whole complex plane

$$y^2(z) = \tilde{Q}(\zeta), \qquad y = \omega(z) - \frac{1}{2}V'(z), \tag{II.44}$$

where $\tilde{Q}(z)$ is some analytical function. Its form is fixed by the condition (II.40) which leads to $\tilde{Q}(z) = V'^2(z) + Q(z)$, where $Q(z)$ is a polynomial of degree $n-2$. Thus, the solution reads

$$y(z) = \sqrt{V'^2(z) + Q(z)}. \tag{II.45}$$

In a general case, the polynomial $\tilde{Q}(z)$ has $2(n-1)$ roots so that the solution is

$$y(z) = \sqrt{\prod_{k=1}^{n-1}(z-a_k)(z-b_k)}, \tag{II.46}$$

where we imply the ordering $a_1 < b_1 < a_2 < \ldots < b_{n-1}$. The intervals (a_k, b_k) represent the support of the density. When a_k coincides with b_k, the corresponding interval collapses and we get a factor $(z-a_k)$ in front of the square root. When $n-2$ intervals collapse, we return to the one cut solution (II.39). Note that at the formal level the eigenvalues can appear around each extremum, not only around minima, of the potential. However, the condensation of the eigenvalues around maxima is not physical and cannot be realized as a stable configuration.

Thus, we see that the solution of 1MM in the large N limit is completely determined by the resolvent $\omega(z)$ whose general structure is given in (II.45). It is an analytical function with at most $n-1$ cuts having the square root structure. Therefore one can consider a Riemann surface associated with this function. It consists from two sheets joint by all cuts (fig. II.5). Similarly, it can be viewed as a genus n_c complex curve where n_c is the number of cuts.

On such curve there are $2n_c$ independent cycles. n_c compact cycles A_k go around cuts and n_c non-compact cycles B_k join cuts with infinity. The integrals of a holomorphic differential along these cycles can be considered as the moduli of the curve. In our case the role of such differential is played by $y(z)dz$. From the definition (II.44) it is clear that the integrals along the cycles A_k give the relative numbers of eigenvalues in each cut

$$\frac{1}{2\pi i}\oint_{A_k} y(z)dz = \int_{a_k}^{b_k} \rho(x)dx \stackrel{\text{def}}{=} n_k. \tag{II.47}$$

By definition $\sum_{k=1}^{n_c} n_k = 1$. The integrals along the cycles B_k can also be calculated and are given by the derivatives of the free energy [68, 69]

$$\int_{B_k} y(z)dz = \frac{\partial F_0}{\partial n_k}. \tag{II.48}$$

Thus, the solution of the one-matrix model in the large N limit is encoded in the complex structure of the Riemann surface associated with the resolvent of the matrix.

4 Two-matrix model: method of orthogonal polynomials

In this section we consider the two-matrix model (II.7) which describes the Ising model on a random lattice. We restrict ourselves to the simplest case of the potential of the type (II.9). Thus, we are interested in the following integral over two hermitian matrices

$$Z = \int dA\, dB\, \exp\left[-N \,\mathrm{tr}\, \left(AB - V(A) - \tilde{V}(B)\right)\right], \tag{II.49}$$

where the potentials $V(A)$ and $\tilde{V}(B)$ are some polynomials and the measures are the same as in (II.24). We will solve this model by the method of orthogonal polynomials. This approach can be also applied to 1MM where it looks even simpler. However, we would like to illustrate the basic features of 2MM and the technique of orthogonal polynomials is quite convenient for this.

4.1 Reduction to eigenvalues

Similarly to the one-matrix case, one can diagonalize the matrices and rewrite the integral in terms of their eigenvalues. However, now one has to use two unitary matrices, which are in general different, for the diagonalization

$$A = \Omega_A^\dagger x \Omega_A, \qquad B = \Omega_B^\dagger y \Omega_B. \tag{II.50}$$

At the same time, the action in (II.49) is invariant only under the common unitary transformation (II.8). Therefore, only one of the two unitary matrices is canceled. As a result, one arrives at the following representation

$$Z = C_N \int \prod_{k=1}^{N} dx_k\, dy_k\, e^{N\left(V(x_k)+\tilde{V}(y_k)\right)} \Delta^2(x)\Delta^2(y) I(x,y), \tag{II.51}$$

where

$$I(x,y) = \int [d\Omega]_{SU(N)} \exp\left[-N \,\mathrm{tr}\, \left(\Omega^\dagger x \Omega y\right)\right] \tag{II.52}$$

and $\Omega = \Omega_A \Omega_B^\dagger$. The integral (II.52) is known as Itzykson–Zuber–Charish-Chandra integral and can be calculated explicitly [70]

$$I(x,y) = \tilde{C}_N \frac{\det e^{-N x_i y_j}}{\Delta(x)\Delta(y)}, \tag{II.53}$$

where $\tilde{C}_N$ is some constant. Substitution of this result into (II.51) leads to the cancellation of a half of the Vandermonde determinants. The remaining determinants make the integration measure antisymmetric under permutations x_k and y_k. Due to this antisymmetry, the determinant $\det e^{N x_i y_j}$ can be replaced by the the product of diagonal terms. The final result reads

$$Z = C'_N \int \prod_{k=1}^{N} d\mu(x_k, y_k)\, \Delta(x)\Delta(y), \tag{II.54}$$

where we introduced the measure

$$d\mu(x,y) = dx\, dy\, e^{-N\left(xy - V(x) - \tilde{V}(y)\right)}. \tag{II.55}$$

4.2 Orthogonal polynomials

Let us introduce the system of polynomials orthogonal with respect to the measure (II.55)

$$\int d\mu(x,y)\Phi_n(x)\tilde{\Phi}_m(y) = \delta_{nm}. \tag{II.56}$$

It is easy to check that they are given by the following expressions

$$\begin{aligned}
\Phi_n(x) &= \frac{1}{n!\sqrt{h_n}} \int \prod_{k=1}^{n} \frac{d\mu(x_k,y_k)}{h_{k-1}} \Delta(x)\Delta(y) \prod_{k=1}^{n}(x-x_k), && \text{(II.57)}\\
\tilde{\Phi}_n(y) &= \frac{1}{n!\sqrt{h_n}} \int \prod_{k=1}^{n} \frac{d\mu(x_k,y_k)}{h_{k-1}} \Delta(x)\Delta(y) \prod_{k=1}^{n}(y-y_k), && \text{(II.58)}
\end{aligned}$$

where the coefficients h_n are fixed by the normalization condition in (II.56). They can be calculated recursively by the relation

$$h_n = \frac{1}{(n+1)!}\left(\prod_{k=1}^{n} h_{k-1}\right)^{-1} \int \prod_{k=1}^{n+1} d\mu(x_k,y_k)\Delta(x)\Delta(y). \tag{II.59}$$

Due to this, the general form of the polynomials is the following

$$\begin{aligned}
\Phi_n(x) &= \frac{1}{\sqrt{h_n}} x^n + \sum_{k=0}^{n-1} c_{n,k}x^k, && \text{(II.60)}\\
\tilde{\Phi}_n(y) &= \frac{1}{\sqrt{h_n}} y^n + \sum_{k=0}^{n-1} d_{n,k}y^k. && \text{(II.61)}
\end{aligned}$$

Using these polynomials, one can rewrite the partition function (II.54). Indeed, due to the antisymmetry the Vandermonde determinants (II.28) can be replaced by determinants of the orthogonal polynomials multiplied by the product of the normalization coefficients. Then one can apply the orthonormality relation (II.56) so that

$$Z = C'_N\left(\prod_{k=0}^{N-1} h_k\right)\int \prod_{k=1}^{N} d\mu(x_k,y_k)\ \det_{ij}\left(\Phi_{j-1}(x_i)\right)\ \det_{ij}\left(\tilde{\Phi}_{j-1}(y_i)\right) = C'_N N! \prod_{k=0}^{N-1} h_k. \tag{II.62}$$

Thus, we reduced the problem of calculation of the partition function to the problem of finding the orthogonal polynomials and their normalization coefficients.

4.3 Recursion relations

To find the coefficients h_n, $c_{n,k}$ and $d_{n,k}$ of the orthogonal polynomials, one uses recursions relations which can be obtained for Φ_n and $\tilde{\Phi}_n$. They are derived using two pairs of conjugated operators which are the operators of multiplication and derivative [71]. We introduce them as the following matrices representing these operators in the basis of the orthogonal polynomials

$$x\Phi_n(x) = \sum_m X_{nm}\Phi_m(x), \qquad \frac{1}{N}\frac{\partial}{\partial x}\Phi_n(x) = \sum_m P_{nm}\Phi_m(x), \tag{II.63}$$

$$y\tilde{\Phi}_n(y) = \sum_m \tilde{\Phi}_m(y)Y_{mn}, \qquad \frac{1}{N}\frac{\partial}{\partial y}\tilde{\Phi}_n(y) = \sum_m \tilde{\Phi}_m(y)Q_{mn}. \tag{II.64}$$

Integrating by parts, one finds the following relations

$$P_{nm} = Y_{nm} - [V'(X)]_{nm}, \qquad Q_{nm} = X_{nm} - [\tilde{V}'(Y)]_{nm}. \tag{II.65}$$

If one takes the potentials $V(x)$ and $\tilde{V}(y)$ of degree p and q, correspondingly, the form of the orthogonal polynomials (II.60), (II.61) and relations (II.65) imply the following properties

$$\begin{gathered} X_{n,n+1} = Y_{n+1,n} = \sqrt{h_{n+1}/h_n}, \\ X_{nm} = 0, \quad m > n+1 \text{ and } m < n-q+1, \\ Y_{mn} = 0, \quad m > n+1 \text{ and } m < n-p+1, \\ P_{n,n-1} = Q_{n-1,n} = \tfrac{n}{N}\sqrt{h_{n-1}/h_n}, \\ P_{nm} = Q_{nm} = 0, \quad m > n-1 \text{ and } m < n-(p-1)(q-1). \end{gathered} \tag{II.66}$$

They indicate that it is more convenient to work with redefined indices

$$X_k(n/N) = X_{n,n-k}, \qquad Y_k(n/N) = Y_{n-k,n} \tag{II.67}$$

and similarly for P and Q. Then each set of functions can be organized into one operator as follows

$$\hat{X}(s) = \sum_{k=-1}^{q-1} X_k(s)\hat{\omega}^{-k}, \qquad \hat{P}(s) = \sum_{k=1}^{(p-1)(q-1)} P_k(s)\hat{\omega}^{-k}, \tag{II.68}$$

$$\hat{Y}(s) = \sum_{k=-1}^{p-1} \hat{\omega}^k Y_k(s), \qquad \hat{Q}(s) = \sum_{k=1}^{(p-1)(q-1)} \hat{\omega}^k Q_k(s), \tag{II.69}$$

where we denoted $s = n/N$ and introduced the shift operator $\hat{\omega} = e^{\frac{1}{N}\frac{\partial}{\partial s}}$. These operators satisfy

$$\hat{X}(n/N)\Phi_n(x) = x\Phi_n(x), \qquad \hat{P}(n/N)\Phi_n(x) = \tfrac{1}{N}\tfrac{\partial}{\partial x}\Phi_n(x), \tag{II.70}$$

$$\hat{Y}^\dagger(n/N)\tilde{\Phi}_n(x) = y\tilde{\Phi}_n(y), \qquad \hat{Q}^\dagger(n/N)\tilde{\Phi}_n(y) = \tfrac{1}{N}\tfrac{\partial}{\partial y}\tilde{\Phi}_n(y), \tag{II.71}$$

where the conjugation is defined as $(\hat{\omega}a(s))^\dagger = a(s)\hat{\omega}^{-1}$. With these definitions the relations (II.65) become

$$\hat{P}(s) = \hat{Y}(s) - V'(\hat{X}(s)), \qquad \hat{Q}(s) = \hat{X}(s) - \tilde{V}'(\hat{Y}(s)). \tag{II.72}$$

Substitution of the expansions (II.68) and (II.69) into (II.72) gives a system of finite-difference algebraic equations, which are obtained comparing the coefficients in front of powers of $\hat{\omega}$. Actually, one can restrict the attention only to negative powers. Then the reduced system contains only equations on the functions X_k and Y_k, because the left hand side of (II.72) is expanded only in positive powers of $\hat{\omega}$. This is a triangular system and for each given potential it can be solved by a recursive procedure. The free energy can be reproduced from the function $R_{n+1} \stackrel{\text{def}}{=} X_{-1}^2(n/N) = Y_{-1}^2(n/N)$. The representation (II.62) implies

$$Z = C'_N N! \prod_{k=0}^{N-1} R_k^{N-k}. \tag{II.73}$$

This gives the solution for all genera.

In fact, to find the solution of (II.72), one has to use the perturbative expansion in $1/N$. Then the problem is reduced to a hierarchy of differential equations. The hierarchy appearing here is Toda lattice hierarchy which will be described in detail in the next section. The use of methods of Toda theory simplifies the problem and allows to find explicit differential, and even algebraic equations directly for the free energy.

4.4 Critical behaviour

We saw in section 3.4 that the multicritical points of the one-matrix model correspond to a one-parameter family of the minimal conformal theories. The two-matrix model is more general than the one-matrix model. Therefore, it can encompass a larger class of continuous models. It turns out that all minimal models (II.43) can be obtained by appropriately adjusting the matrix model potentials [71, 72].

There is an infinite set of critical potentials for each (p,q) point. The simplest one is when one of the potentials has degree p and the other has degree q. Their explicit form has been constructed in [72]. The key fact of the construction is that the momentum function $P(\omega,s)$ (up to an analytical piece) is identified with the resolvent (II.36) of the corresponding matrix X. Comparing with the one-matrix case, one concludes that the (p,q) point is obtained when the resolvent behaves near its singularity as $(x-x_c)^{p/q}$. Thus, it is sufficient to take the operators with the following behaviour at $\omega \to 1$

$$X - X_c \sim (\log\omega)^q, \qquad P - P_c \sim (\log\omega)^p. \tag{II.74}$$

Note that the singular asymptotics of Y and Q follow from (II.72) and lead to the dual (q,p) point. Then one can take some fixed $X(w)$ and $Y(w)$ with the necessary asymptotics at $\omega \to 1$ and $\omega \to \infty$ and solve the equations (II.72) with respect to the potentials $V(X)$ and $\tilde{V}(Y)$. The resulting explicit formulae can be found in [72].

4.5 Complex curve

As in the one-matrix model, the solution of 2MM in the large N limit can be represented in terms of a complex curve [73, 74]. However, there is a difference between these two cases. Whereas in the former case the curve coincides with the Riemann surface of the resolvent, in the latter case the origin of the curve is different. To illuminate it, let us consider how the solution of the model in the large N limit arises.

In this approximation one can apply the saddle point approach described in the previous section. It leads to the following two equations on the resolvents of matrices X and Y

$$y = V'(x) + \omega(x), \qquad x = \tilde{V}'(y) + \tilde{\omega}(y). \tag{II.75}$$

In fact, these equations are nothing else but the classical limit of the relations (II.72) obtained using the orthogonal polynomials. They coincide due to the identification mentioned above of the momentum operators P and Q with the resolvents ω and $\tilde{\omega}$, respectively.

The equations (II.75) can be considered as definitions of the multivalued analytical functions $y(x)$ and $x(y)$. It is clear that they must be mutually inverse. This is actually a

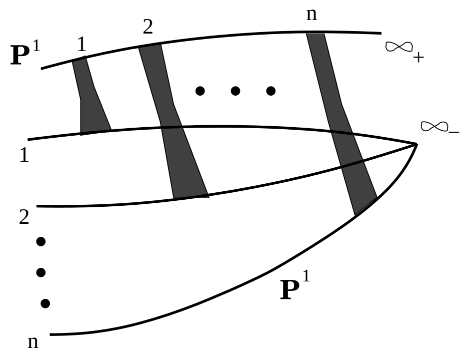


Fig. II.6: Generic curve of 2MM as a cover of x-plane. Each fat line consists from $\tilde{n}$ cuts.

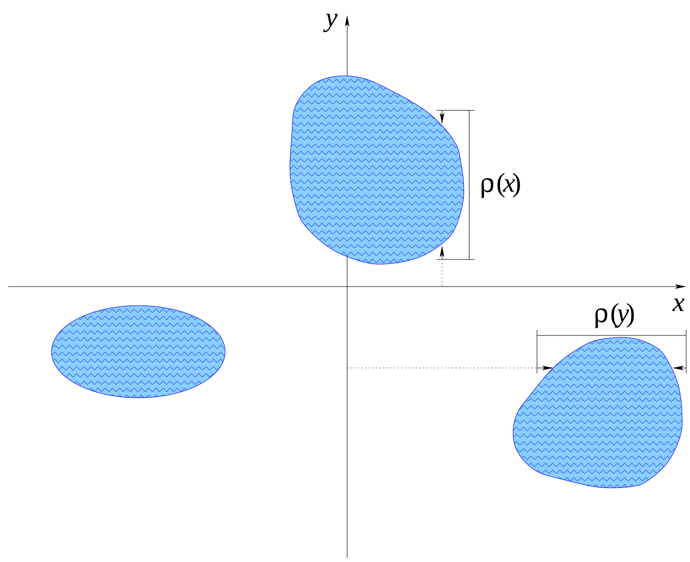


Fig. II.7: Eigenvalue plane of 2MM. The eigenvalues fill several spots which contain information about the density.

non-trivial restriction which, together with the asymptotic condition (II.40), fixes the resolvents and gives the solution. The complex curve, one should deal with, is the Riemann surface of one of these functions.

The general structure of this complex curve was studied in [74]. It was established that if the potentials are of degree $n+1$ and $\tilde{n}+1$, the maximum genus of the curve is $n\tilde{n}-1$. In this most general case the Riemann surface is represented by $n+1$ sheets. One of them is the "physical sheet" glued with each "unphysical" one along $\tilde{n}$ cuts and all "unphysical sheets" join at infinity by the nth order branch point (see fig. II.6).

One can prove the analog of the formulae (II.47) and (II.48) [74]. We still integrate around two conjugated sets of independent cycles A_k and B_k on the curve. The role of the holomorphic differential is played again by $y(z)dz$ where $y(x)$ is a solution of (II.75).

As earlier, the cycles A_k surround the cuts of the resolvent on the physical sheet. This is the place where the eigenvalues of the matrix X live. However, in this picture it is not clear how to describe the distribution of eigenvalues of Y. Of course, it is enough to invert the function $y(x)$ to find it. But the arising picture is nevertheless non-symmetric with respect

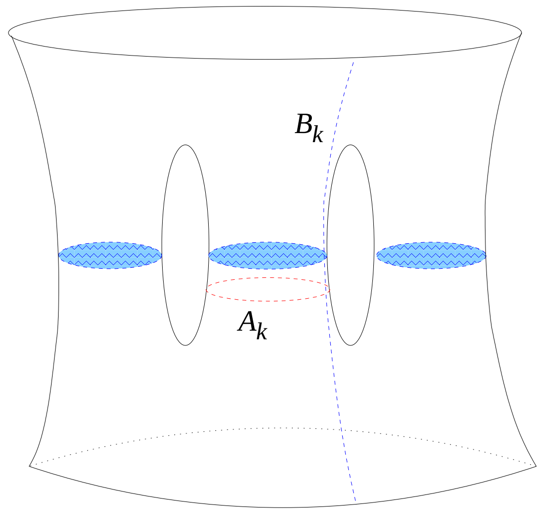


Fig. II.8: Complex curve associated with the solution of 2MM viewed as a "double" of (x, y)-plane.

to the exchange of x and y.

A more symmetric picture can be obtained considering the so called "double" [74]. To introduce this notion, note that since the two matrices X and Y are of the same size, with each eigenvalue x_i one can associate an eigenvalue y_i of the second matrix. Thus, one obtains N pairs (x_i, y_i) which can be put on a plane. In the large N limit, the eigenvalues are distributed continuously so that on the plane (x, y) their distribution appears as several disconnected regions. Each region corresponds to a cut of the resolvent. Its width in the y direction at a given point x is nothing else but the density $\rho(x)$ and *vice versa*. Thus, one arrives at the two-dimensional picture shown on fig. II.7. From this point of view, the functions $y(x)$ and $x(y)$, which are solutions of (II.75), determine the boundaries of the spots of eigenvalues. The fact that they are inverse means that they define the same boundary.

Now to define the double, we take two copies of the (x, y) plane, cut off the spots, and glue the two planes along the boundaries. The resulting surface shown in fig. II.8 is smooth and can be considered as a genus $n_c - 1$ surface with two punctures (corresponding to two infinities) where n_c is the number of spots on the initial surface equal to the number of cuts of the resolvent $\omega(x)$.

All cycles A_k and B_k exist also in this picture and the integration formulae (II.47) and (II.48) along them hold as well. Note that the integrals along A_k cycles can be rewritten as two-dimensional integrals over the (x, y) plane with the density equal to 1 inside the spots and vanishing outside them. Such behaviour of the density is characteristic for fermionic systems. And indeed, 2MM can be interpreted as a system of free fermions.

4.6 Free fermion representation

Let us identify the functions

$$\psi_n(x) = \Phi_n(x) e^{NV(x)}, \qquad \tilde{\psi}_n(y) = \tilde{\Phi}_n(y) e^{N\tilde{V}(y)}. \tag{II.76}$$

with fermionic wave functions. The functions ψ_n and $\tilde{\psi}_n$ can be considered as two representations of the same state similarly to the coordinate and momentum representations. The two representations are related by a kind of Fourier transform and the scalar product between functions of different representations is given by the following integral

$$\left\langle \tilde{\psi} | \psi \right\rangle = \int dx\, dy\, e^{-Nxy} \tilde{\psi}(y) \psi(x). \tag{II.77}$$

The second-quantized fermionic fields are defined as

$$\psi(x) = \sum_{n=0}^{\infty} a_n \psi_n(x), \qquad \tilde{\psi}(y) = \sum_{n=0}^{\infty} a_n^\dagger \tilde{\psi}_n(y). \tag{II.78}$$

Due to the orthonormality of the wave functions (II.76) with respect to the scalar product (II.77), the creation and annihilation operators satisfy the following anticommutation relations

$$\{a_n, a_m^\dagger\} = \delta_{n,m}. \tag{II.79}$$

The fundamental state of N fermions is defined by

$$a_n |N\rangle = 0, \quad n \geq N, \qquad a_n^\dagger |N\rangle = 0, \quad n < N. \tag{II.80}$$

Its wave function can be represented by the Slater determinant. For example, in the x-representation it looks as

$$\Psi_N(x_1, \dots, x_N) = \det_{i,j} \psi_i(x_j). \tag{II.81}$$

The key observation which establishes the equivalence of the two systems is that the correlators of matrix operators coincide with expectation values of the corresponding fermionic operators in the fundamental state (II.81):

$$\left\langle \prod_j \operatorname{tr} B^{m_j} \prod_i \operatorname{tr} A^{n_i} \right\rangle = \left\langle N | \prod_j \hat{y}^{m_j} \prod_i \hat{x}^{n_i} | N \right\rangle. \tag{II.82}$$

Here a second-quantized operator $\hat{\mathcal{O}}(x,y)$ is defined as follows

$$\hat{\mathcal{O}}(x,y) = \int dx\, dy\, e^{-Nxy} \tilde{\psi}(y) \mathcal{O}(x,y) \psi(x). \tag{II.83}$$

The proof of the statement (II.82) relies on the properties of the orthogonal polynomials. For example, for the one-point correlator we have

$$\begin{aligned}
\langle \operatorname{tr} A^n \rangle &\equiv Z^{-1} \int dA\, dB\, \operatorname{tr} A^n\, e^{-N \operatorname{tr}\left(AB - V(A) - \tilde{V}(B)\right)} \\
&= \left(N! \prod_{k=0}^{N-1} h_k \right)^{-1} \int \prod_{k=1}^{N} d\mu(x_k, y_k)\, \Delta(x) \Delta(y) \sum_{i=1}^{N} x_i^n \\
&= \frac{1}{(N-1)!} \int \prod_{k=1}^{N} d\mu(x_k, y_k)\ \det_{ij} \left(\Phi_{j-1}(x_i)\right) \det_{ij} \left(\tilde{\Phi}_{j-1}(y_i)\right) x_1^n \\
&= \sum_{j=0}^{N-1} \int d\mu(x,y)\, \Phi_j(x) \tilde{\Phi}_j(y) x^n = \langle N | \hat{x}^n | N \rangle\,.
\end{aligned} \tag{II.84}$$

This equivalence supplies us with a powerful technique for calculations. For example, all correlators of the type (II.82) can be expressed through the two-point function

$$K_N(x,y) = \left\langle N|\tilde{\psi}(y)\psi(x)|N\right\rangle \tag{II.85}$$

and it is sufficient to study this quantity. In general, the existence of the representation in terms of free fermions indicates that the system is integrable. This, in turn, is often related to the possibility to introduce orthogonal polynomials.

We will see that the similar structures appear in matrix quantum mechanics. Although MQM has much richer physics, it turns out to be quite similar to 2MM.

5 Toda lattice hierarchy

5.1 Integrable systems

One and two-matrix models considered above are examples of *integrable systems*. There exists a general theory of such systems. A system is considered as integrable if it has an infinite number of commuting Hamiltonians. Each Hamiltonian generates an evolution along some direction in the parameter space of the model. Their commutativity means that there is an infinite number of conserved quantities associated with them and, at least in principle, it is possible to describe any point in the parameter space.

Usually, the integrability implies the existence of a *hierarchy* of equations on some specific quantities characterizing the system. The hierarchical structure means that the equations can be solved one by one, so that the solution of the first equation should be substituted into the second one and *etc.* This recursive procedure allows to reproduce all information about the system.

The equations to be solved are most often of the finite-difference type. In other words, they describe a system on a lattice. Introducing a parameter measuring the spacing between nodes of the lattice, one can organize a perturbative expansion in this parameter. Then the finite-difference equations are replaced by an infinite set of partial differential equations. They also form a hierarchy and can be solved recursively. The equations appearing at the first level, corresponding to the vanishing spacing, describe a closed system which is considered as a continuum or classical limit of the initial one. In turn, starting with the classical system describing by a hierarchy of differential equations, one can construct its quantum deformation arriving at the full hierarchy.

The integrable systems can be classified according to the type of hierarchy which appears in their description. In this section we consider the so called *Toda hierarchy* [75]. It is general enough to include all integrable matrix models relevant for our work. In particular, it describes 2MM and some restriction of MQM, whereas 1MM corresponds to its certain reduction.

5.2 Lax formalism

There are several ways to introduce the Toda hierarchy. The most convenient for us is to use the so called *Lax formalism*.

Take two semi-infinite series[1]

$$L = r(s)\hat{\omega} + \sum_{k=0}^{\infty} u_k(s)\hat{\omega}^{-k}, \qquad \bar{L} = \hat{\omega}^{-1}r(s) + \sum_{k=0}^{\infty} \hat{\omega}^{k}\bar{u}_k(s), \tag{II.86}$$

where s is a discrete variable labeling the nodes of an infinite lattice and $\hat{\omega} = e^{\hbar\partial/\partial s}$ is the shift operator in s. The Planck constant $\hbar$ plays the role of the spacing parameter. The

[1] In fact, the first coefficients in the expansions (II.86) can be chosen arbitrarily. Only their product has a sense and how it is distributed between the two operators can be considered as a choice of some gauge. In particular, often one uses the gauge where one of the coefficients equals 1 [76]. We use the symmetric gauge which agrees with the choice of the orthogonal polynomials (II.56) normalized to the Kronecker symbol in 2MM.

operators (II.86) are called Lax operators. The coefficients r, u_k and $\bar{u}_k$ are also functions of two infinite sets of "times" $\{t_{\pm k}\}_{k=1}^{\infty}$. Each time variable gives rise to an evolution along its direction. This system represents Toda hierarchy if the evolution associated with each $t_{\pm k}$ is generated by Hamiltonians $H_{\pm k}$

$$\begin{array}{rclrcl} \hbar\frac{\partial L}{\partial t_k} & = & [H_k, L], & \hbar\frac{\partial \bar{L}}{\partial t_k} & = & [H_k, \bar{L}], \\ \hbar\frac{\partial L}{\partial t_{-k}} & = & [H_{-k}, L], & \hbar\frac{\partial \bar{L}}{\partial t_{-k}} & = & [H_{-k}, \bar{L}], \end{array} \tag{II.87}$$

which are expressed through the Lax operators (II.86) as follows[2]

$$H_k = (L^k)_{>} + \frac{1}{2}(L^k)_0, \qquad H_{-k} = (\bar{L}^k)_{<} + \frac{1}{2}(\bar{L}^k)_0, \tag{II.88}$$

where the symbol $(\ )_{\substack{> \\ <}}$ means the positive (negative) part of the series in the shift operator $\hat{\omega}$ and $(\ )_0$ denotes the constant part. Thus, Toda hierarchy is a collection of non-linear equations of the finite-difference type in s and differential with respect to t_k for the coefficients $r(s,t)$, $u_k(s,t)$ and $\bar{u}_k(s,t)$.

From the commutativity of the second derivatives, it is easy to obtain that the Lax–Sato equations (II.87) are equivalent to the zero-curvature condition for the Hamiltonians

$$\hbar\frac{\partial H_k}{\partial t_l} - \hbar\frac{\partial H_l}{\partial t_k} + [H_k, H_l] = 0. \tag{II.89}$$

It shows that the system possesses an infinite set of commutative flows $\hbar\frac{\partial}{\partial t_k} - H_k$ and, therefore, Toda hierarchy is integrable.

One can get another equivalent formulation if one considers the following eigenvalue problem

$$x\Psi = L\Psi(x;s), \qquad \hbar\frac{\partial \Psi}{\partial t_k} = H_k\Psi(x;s), \qquad \hbar\frac{\partial \Psi}{\partial t_{-k}} = H_{-k}\Psi(x;s). \tag{II.90}$$

The previous equations (II.87) and (II.89) appear as the integrability condition for (II.90). Indeed, differentiating the first equation, one reproduces the evolution law of the Lax operators (II.87) and the second and third equations lead to the zero-curvature condition (II.89). The eigenfunction Ψ is known as Baker–Akhiezer function. It is clear that it contains all information about the system.

Note that equations of Toda hierarchy allow a representation in terms of semi-infinite matrices. Then the Baker–Akhiezer function is a vector whose elements correspond to different values of the discrete variable s. The positive/negative/constant parts of the series in $\hat{\omega}$ are mapped to upper/lower/diagonal triangular parts of matrices.

The equations to be solved are either the equations (II.90) on Ψ or the Lax–Sato equations (II.87) on the coefficients of the Lax operators. Their hierarchic structure is reflected in the fact that one obtains a closed equation on the first coefficient $r(s,t)$ and its solution

[2] Sometimes the second set of Hamiltonians is defined with the opposite sign. This corresponds to the change of sign of t_{-k}. Doing both these replacements, one can establish the full correspondence with the works using this sign convention.

provides the necessary information for the following equations. This first equation is derived considering the Lax–Sato equations (II.87) for $k=1$. They give

$$\hbar\tfrac{\partial \log r^2(s)}{\partial t_1} = u_0(s+\hbar) - u_0(s), \qquad \hbar\tfrac{\partial \log r^2(s)}{\partial t_{-1}} = \bar{u}_0(s) - \bar{u}_0(s+\hbar), \tag{II.91}$$

$$\hbar\tfrac{\partial \bar{u}_0(s)}{\partial t_1} = r^2(s) - r^2(s-\hbar), \qquad \hbar\tfrac{\partial u_0(s)}{\partial t_{-1}} = r^2(s-\hbar) - r^2(s). \tag{II.92}$$

Combining these relations, one finds the so called Toda equation

$$\hbar^2\frac{\partial^2 \log r^2(s)}{\partial t_1 \partial_{-1}} = 2r^2(s) - r^2(s+\hbar) - r^2(s-\hbar). \tag{II.93}$$

Often it is convenient to introduce the following Orlov–Shulman operators [77]

$$\begin{array}{rcl} M & = & \sum\limits_{k\ge 1} k t_k L^k + s + \sum\limits_{k\ge 1} v_k L^{-k}, \\ \bar{M} & = & -\sum\limits_{k\ge 1} k t_{-k} \bar{L}^k + s - \sum\limits_{k\ge 1} v_{-k} \bar{L}^{-k}. \end{array} \tag{II.94}$$

The coefficients $v_{\pm k}$ are fixed by the condition on their commutators with the Lax operators

$$[L, M] = \hbar L, \qquad [\bar{L}, \bar{M}] = -\hbar \bar{L}. \tag{II.95}$$

The main application of these operators is that they can be considered as perturbations of the simple operators of multiplication by the discrete variable s. Indeed, if one requires that $v_{\pm k}$ vanish when all $t_{\pm k}=0$, then in this limit $M=\bar{M}=s$. Similarly the Lax operators reduce to the shift operator. The perturbation leading to the general expansions (II.86) and (II.94) can be described by the dressing operators $\mathcal{W}$ and $\bar{\mathcal{W}}$

$$\begin{array}{ll} L = \mathcal{W}\hat{\omega}\mathcal{W}^{-1}, & \qquad M = \mathcal{W} s \mathcal{W}^{-1}, \\ \bar{L} = \bar{\mathcal{W}}\hat{\omega}^{-1}\bar{\mathcal{W}}^{-1}, & \qquad \bar{M} = \bar{\mathcal{W}} s \bar{\mathcal{W}}^{-1}. \end{array} \tag{II.96}$$

The commutation relations (II.95) are nothing else but the dressed version of the evident relation

$$[\hat{\omega}, s] = \hbar\hat{\omega}. \tag{II.97}$$

To produce the expansions (II.86) and (II.94), the dressing operators should have the following general form

$$\begin{array}{rcl} \mathcal{W} & = & e^{\frac{1}{2\hbar}\phi}\left(1 + \sum\limits_{k\ge 1} w_k \hat{\omega}^{-k}\right)\exp\left(\frac{1}{\hbar}\sum\limits_{k\ge 1} t_k \hat{\omega}^k\right), \\ \bar{\mathcal{W}} & = & e^{-\frac{1}{2\hbar}\phi}\left(1 + \sum\limits_{k\ge 1} \bar{w}_k \hat{\omega}^{k}\right)\exp\left(\frac{1}{\hbar}\sum\limits_{k\ge 1} t_{-k} \hat{\omega}^{-k}\right), \end{array} \tag{II.98}$$

where the zero mode $\phi(s)$ is related to the coefficient $r(s)$ as

$$r^2(s) = e^{\frac{1}{\hbar}(\phi(s) - \phi(s+\hbar))}. \tag{II.99}$$

However, the coefficients in this expansion are not arbitrary and the dressing operators should be subject of some additional condition. It can be understood considering evolution along the times $t_{\pm k}$. Differentiating (II.96) with respect to $t_{\pm k}$, one finds the following expression of the Hamiltonians in terms of the dressing operators

$$\begin{array}{ll} H_k = \hbar(\partial_{t_k}\mathcal{W})\mathcal{W}^{-1}, & H_{-k} = \hbar(\partial_{t_{-k}}\mathcal{W})\mathcal{W}^{-1}, \\ \bar{H}_k = \hbar(\partial_{t_k}\bar{\mathcal{W}})\bar{\mathcal{W}}^{-1}, & \bar{H}_{-k} = \hbar(\partial_{t_{-k}}\bar{\mathcal{W}})\bar{\mathcal{W}}^{-1}, \end{array} \tag{II.100}$$

Here $H_{\pm k}$ generate evolution of L and $\bar{H}_{\pm k}$ are Hamiltonians for $\bar{L}$. However, (II.87) implies that for both operators one should use the same Hamiltonian. This imposes the condition that $H_{\pm k} = \bar{H}_{\pm k}$ which relates two dressing operators. This condition can be rewritten in a more explicit way. Namely, it is equivalent to the requirement that $\mathcal{W}^{-1}\bar{\mathcal{W}}$ does not depend on times $t_{\pm k}$ [75, 78].

Studying the evolution laws of the Orlov–Shulman operators, one can find that [76]

$$\frac{\partial v_k}{\partial t_l} = \frac{\partial v_l}{\partial t_k}. \tag{II.101}$$

It means that there exists a generating function $\tau_s[t]$ of all coefficients $v_{\pm k}$

$$v_k(s) = \hbar^2\,\frac{\partial \log \tau_s[t]}{\partial t_k}. \tag{II.102}$$

It is called *τ-function* of Toda hierarchy. It also allows to reproduce the zero mode ϕ and, consequently, the first coefficient in the expansion of the Lax operators

$$e^{\frac{1}{\hbar}\phi(s)} = \frac{\tau_s}{\tau_{s+\hbar}}, \qquad r^2(s-\hbar) = \frac{\tau_{s+\hbar}\tau_{s-\hbar}}{\tau_s^2}. \tag{II.103}$$

The τ-function is usually the main subject of interest in the systems described by Toda hierarchy. The reason is that it coincides with the partition function of the model. Then the coefficients v_k are the one-point correlators of the operators generating the commuting flows H_k. We will show a concrete realization of these ideas in the end of this section and in the next chapters.

5.3 Free fermion and boson representations

To establish an explicit connection with physical systems, it is sometimes convenient to use a representation of Toda hierarchy in terms of second-quantized free chiral fermions or bosons. The two representations are related by the usual bosonization procedure.

Fermionic picture

To define the fermionic representation, let us consider the chiral fermionic fields with the following expansion

$$\psi(z) = \sum_{r\in\mathbf{Z}+\frac12} \psi_r z^{-r-\frac12}, \qquad \psi^*(z) = \sum_{r\in\mathbf{Z}+\frac12} \psi^*_{-r} z^{-r-\frac12} \tag{II.104}$$

Their two-point function

$$\langle l|\psi(z)\psi^*(z')|l\rangle = \frac{(z'/z)^l}{z-z'} \tag{II.105}$$

leads to the following commutation relations for the modes

$$\{\psi_r, \psi_s^*\} = \delta_{r,s}. \tag{II.106}$$

The fermionic vacuum of charge l is defined by

$$\psi_r|l\rangle = 0, \quad r > l, \qquad \psi_r^*|l\rangle = 0, \quad r < l. \tag{II.107}$$

Also we need to introduce the current

$$J(z) = \psi^*(z)\psi(z) = \hat{p}z^{-1} + \sum_{n\neq 0} H_n z^{-n-1} \tag{II.108}$$

whose components H_n are associated with the Hamiltonians generating the Toda flows. In terms of the fermionic modes they are represented as follows

$$H_n = \sum_{r\in\mathbf{Z}+\frac{1}{2}} \psi_{r-n}^*\psi_r \tag{II.109}$$

and for any l they satisfy

$$H_n|l\rangle = \langle l|H_{-n} = 0, \quad n > 0. \tag{II.110}$$

Finally, we introduce an operator of $GL(\infty)$ rotation

$$\mathbf{g} = \exp\left(\frac{1}{\hbar}\sum_{r,s\in\mathbf{Z}+\frac{1}{2}} A_{rs}\psi_r\psi_s^*\right). \tag{II.111}$$

With these definitions the τ-function of Toda hierarchy is represented as the following vacuum expectation value

$$\tau_{l\hbar}[t] = \left\langle l|e^{\frac{1}{\hbar}H_+[t]}\mathbf{g}e^{-\frac{1}{\hbar}H_-[t]}|l\right\rangle, \tag{II.112}$$

where

$$H_+[t] = \sum_{k>0} t_k H_k, \qquad H_-[t] = \sum_{k<0} t_k H_k. \tag{II.113}$$

It is clear that each solution of Toda hierarchy is characterized in the unique way by the choice of the matrix A_{rs}.

Bosonic picture

The bosonic representation now follows from the bosonization formulae

$$\psi(z) =: e^{\varphi(z)} :, \qquad \psi^*(z) =: e^{-\varphi(z)} :, \qquad \partial\varphi(z) =: \psi^*(z)\psi(z) : . \tag{II.114}$$

Since the last expression in (II.114) is the current (II.108), the Hamiltonians H_n appear now as the coefficients in the mode expansion of the free bosonic field

$$\varphi(z) = \hat{q} + \hat{p}\log z + \sum_{n\neq 0} \frac{1}{n} H_n z^{-n}. \tag{II.115}$$

From (II.109) and (II.106) one finds the following commutation relations

$$[\hat{p}, \hat{q}] = 1, \qquad [H_n, H_m] = n\delta_{m+n,0}, \tag{II.116}$$

which lead to the two-point function of the free boson

$$\langle \varphi(z)\varphi(z') \rangle = \log(z - z'). \tag{II.117}$$

The bosonic vacuum is defined by the Hamiltonians and characterized by the quantum number s, which is the eigenvalue of the momentum operator,

$$\hat{p}|s\rangle = s|s\rangle, \qquad H_n|s\rangle = 0, \quad (n > 0). \tag{II.118}$$

To rewrite the operator $\mathbf{g}$ (II.111) in the bosonic terms, we introduce the vertex operators

$$V_\alpha(z) =: e^{\alpha\varphi(z)} : . \tag{II.119}$$

Here the normal ordering is defined by putting all H_n, $n > 0$ to the right and H_n, $n < 0$ to the left. Besides $: \hat{q}\hat{p} :=: \hat{p}\hat{q} := \hat{q}\hat{p}$. Then the operator of $GL(\infty)$ rotation is given by

$$\mathbf{g} = \exp\left(-\frac{1}{4\pi^2\hbar} \oint dz \oint dw\, A(z,w) V_1(z) V_{-1}(w) \right), \tag{II.120}$$

where

$$A(z,w) = \sum_{r,s} A_{rs} z^{r-\frac{1}{2}} w^{-s-\frac{1}{2}}. \tag{II.121}$$

The substitution of (II.120) into (II.112) and replacing the fermionic vacuum $|l\rangle$ by the bosonic one $|s\rangle$ gives the bosonic representation of the τ-function $\tau_s[t]$.

Connection with the Lax formalism

The relation of these two representations with the objects considered in paragraph 5.2 is based on the realization of the Baker–Akhiezer function $\Psi(z; s)$ as the expectation value of the one-fermion field

$$\Psi(z; s) = \tau_s^{-1}[t] \left\langle \hbar^{-1} s | e^{\frac{1}{\hbar} H_+[t]} \psi(z) \mathbf{g} e^{-\frac{1}{\hbar} H_-[t]} | \hbar^{-1} s \right\rangle . \tag{II.122}$$

One can show that it does satisfy the relations (II.90) with H_k defined as in (II.88) and the Lax operators having the form (II.86).

5.4 Hirota equations

The most explicit manifestation of the hierarchic structure of the Toda system is a set of equations on the τ-function, which can be obtained from the fermionic representation introduced above. One can show [79] that the ensemble of the τ-functions of the Toda hierarchy with different charges satisfies a set of bilinear equations. They are known as Hirota equations and can be written in a combined way as follows

$$\oint_{C_\infty} dz\, z^{l-l'} \exp\left(\frac{1}{\hbar} \sum_{k>0} (t_k - t'_k) z^k \right) \tau_l[t - \tilde{\zeta}_+] \tau_{l'}[t' + \tilde{\zeta}_+] =$$

$$\oint_{C_0} dz\, z^{l-l'} \exp\left(\frac{1}{\hbar} \sum_{k<0} (t_k - t'_k) z^k \right) \tau_{l+1}[t - \tilde{\zeta}_-] \tau_{l'-1}[t' + \tilde{\zeta}_-], \tag{II.123}$$

where

$$\tilde{\zeta}_+/\hbar = (\dots, 0, 0, z^{-1}, z^{-2}/2, z^{-3}/3, \dots), \qquad \tilde{\zeta}_-/\hbar = (\dots, z^3/3, z^2/2, z, 0, 0, \dots) \tag{II.124}$$

and we omitted $\hbar$ in the index of the τ-function. The proof of (II.123) relies on the representation (II.112) of the τ-function with $\mathbf{g}$ taken from (II.111). The starting point is the fact that the following operator

$$\mathcal{C} = \sum_{r\in\mathbf{Z}+\frac{1}{2}} \psi^*_r \otimes \psi_r = \oint \frac{dz}{2\pi i}\, \psi^*(z) \otimes \psi(z) \tag{II.125}$$

plays the role of the Casimir operator for the diagonal subgroup of $GL(\infty)\otimes GL(\infty)$. This means that it commutes with the tensor product of two $\mathbf{g}$ operators

$$\mathcal{C}\,(\mathbf{g}\otimes\mathbf{g}) = (\mathbf{g}\otimes\mathbf{g})\,\mathcal{C}. \tag{II.126}$$

Multiplying this relation by $\langle l+1|e^{\frac{1}{\hbar}H_+[t]} \otimes \langle l'-1|e^{\frac{1}{\hbar}H_+[t']}$ from the left and by $e^{-\frac{1}{\hbar}H_-[t]}|l\rangle \otimes e^{-\frac{1}{\hbar}H_-[t']}|l'\rangle$ from the right, one can commute the fermion operators until they hit the left (right) vacuum. The final result is obtained using the following relations

$$\langle l+1|e^{\frac{1}{\hbar}H_+[t]}\psi^*(z)\mathbf{g}e^{-\frac{1}{\hbar}H_-[t]}|l\rangle = z^l \exp\left(\frac{1}{\hbar}\sum_{n>0} t_n z^n\right)\left\langle l|e^{\frac{1}{\hbar}H_+[t-\tilde{\zeta}_+]}\mathbf{g}e^{-\frac{1}{\hbar}H_-[t]}|l\right\rangle, \tag{II.127}$$

$$\langle l+1|e^{\frac{1}{\hbar}H_+[t]}\mathbf{g}\psi^*(z)e^{-\frac{1}{\hbar}H_-[t]}|l\rangle = z^l \exp\left(\frac{1}{\hbar}\sum_{n<0} t_n z^n\right)\left\langle l+1|e^{\frac{1}{\hbar}H_+[t]}\mathbf{g}e^{-\frac{1}{\hbar}H_-[t-\tilde{\zeta}_-]}|l+1\right\rangle \tag{II.128}$$

together with the similar relations for $\psi(z)$. They can be proven in two steps. First, one commutes the fermionic fields with the perturbing operators

$$\begin{aligned} e^{-\frac{1}{\hbar}H_\pm[t]}\psi^*(z)e^{\frac{1}{\hbar}H_\pm[t]} &= \exp\left(-\frac{1}{\hbar}\sum_{n>0} t_{\pm n} z^{\pm n}\right)\psi^*(z), \\ e^{\frac{1}{\hbar}H_\pm[t]}\psi(z)e^{-\frac{1}{\hbar}H_\pm[t]} &= \exp\left(-\frac{1}{\hbar}\sum_{n>0} t_{\pm n} z^{\pm n}\right)\psi(z). \end{aligned} \tag{II.129}$$

After this it remains to show that, for example, $\psi^*(z)|l\rangle = z^l e^{\frac{1}{\hbar}H_-[\tilde{\zeta}_-]}|l+1\rangle$. The easiest way to do it is to use the bosonization formulae (II.114) and (II.115).

The identities (II.123) can be rewritten in a more explicit form. For this we introduce the Schur polynomials p_j defined by

$$\sum_{k=0}^{\infty} p_k[t]x^k = \exp\left(\sum_{k=1}^{\infty} t_n x^n\right) \tag{II.130}$$

and the following notations

$$\begin{aligned} y_\pm &= (y_{\pm 1}, y_{\pm 2}, y_{\pm 3}, \dots), && \text{(II.131)} \\ \tilde{D}_\pm &= (D_{\pm 1}, D_{\pm 2}/2, D_{\pm 3}/3, \dots), && \text{(II.132)} \end{aligned}$$

where $D_{\pm n}$ represent the Hirota's bilinear operators

$$D_n f[t] \cdot g[t] = \frac{\partial}{\partial x} f(t_n + x) g(t_n - x) \Big|_{x=0}. \qquad \text{(II.133)}$$

Then identifying $y_n = \frac{1}{2\hbar}(t'_n - t_n)$, one obtains a hierarchy of partial differential equations

$$\sum_{j=0}^{\infty} p_{j+i}(-2y_+) p_j(\hbar \tilde{D}_+) \exp\left(\hbar \sum_{k\neq 0} y_k D_k\right) \tau_{l+i+1}[t] \cdot \tau_l[t] =$$
$$\sum_{j=0}^{\infty} p_{j-i}(-2y_-) p_j(\hbar \tilde{D}_-) \exp\left(\hbar \sum_{k\neq 0} y_k D_k\right) \tau_{l+i}[t] \cdot \tau_{l+1}[t]. \qquad \text{(II.134)}$$

The Hirota equations lead to a triangular system of nonlinear difference-differential equations for the τ-function. Since the derivatives of the τ-function are identified with correlators, the Hirota equations are also equations for the correlators of operators generating the Toda flows. The first equation of the hierarchy is obtained by taking $i = -1$ and extracting the coefficient in front of y_{-1}

$$\hbar^2 \, \tau_l \frac{\partial^2 \tau_l}{\partial t_1 \partial t_{-1}} - \hbar^2 \frac{\partial \tau_l}{\partial t_1} \frac{\partial \tau_l}{\partial t_{-1}} + \tau_{l+1}\tau_{l-1} = 0. \qquad \text{(II.135)}$$

Rewriting this equations as

$$\hbar^2 \frac{\partial^2 \log \tau_l}{\partial t_1 \partial t_{-1}} + \frac{\tau_{l+1}\tau_{l-1}}{\tau_l^2} = 0, \qquad \text{(II.136)}$$

one can recognize the Toda equation (II.93) if one takes into account the identification (II.103).

5.5 String equation

Above we considered the general structure of the Toda hierarchy. However, the equations of the hierarchy, for example, the Hirota equations (II.134), have many solutions. A particular solution is characterized by initial condition. The role of such condition can be played by the partition function of a non-perturbed system. If one requires that it should be equal to the τ-function at vanishing times and coincides with the full τ-function after the perturbation, the perturbed partition function can be found by means of the hierarchy equations with the given initial condition.

However, the Toda equations involve partial differential equations of high orders and require to know not only the τ-function at vanishing times but also its derivatives. Therefore, it is not always clear whether the non-perturbed function provides a sufficient initial information. Fortunately, there is another way to select a unique solution of Toda hierarchy. It uses some equations on the operators, usually, the Lax and Orlov–Shulman operators. The corresponding equations are called *string equations*.

The string equations cannot be arbitrary because they should preserve the structure of Toda hierarchy. For example, if they are given by two equations of the following type

$$\bar{L} = f(L, M), \qquad \bar{M} = g(L, M), \qquad \text{(II.137)}$$

the operators defined by the functions f and g must satisfy

$$[f(\hat{\omega}, s), g(\hat{\omega}, s)] = -\hbar f. \tag{II.138}$$

This condition appears since $\bar{L}$ and $\bar{M}$ commute in the same way.

The advantage of use of string equations is that they represent, in a sense, already a partially integrated version of the hierarchy equations. For example, as we will see, instead of differential equations of the second order, they produce algebraic and first order differential equations and make the problem of finding the τ-function much simpler.

5.6 Dispersionless limit

The classical limit of Toda hierarchy is obtained in the limit where the parameter measuring the lattice spacing vanishes. In our notations this parameter is the Planck constant $\hbar$. Putting it to zero, as usual, one replaces all operators by functions. In particular, as in the usual quantum mechanical systems, the classical limit of the derivative operator is a variable conjugated to the variable with respect to which one differentiates. In other words, one should consider the phase space consisting from s and ω, which is the classical limit of the shift operator. The Poisson structure on this phase space is defined by the Poisson bracket induced from (II.97)

$$\{\omega, s\} = \omega. \tag{II.139}$$

All operators now become functions of s and ω and commutators are replaced by the corresponding Poisson brackets defined through (II.139).

All equations including the Lax–Sato equations (II.87), zero curvature condition (II.89), commutators with Orlov–Shulman operators (II.95) can be rewritten in the new terms. Thus, the general structure of Toda hierarchy is preserved although it becomes much simpler. The resulting structure is called *dispersionless Toda hierarchy* and the classical limit is also known as *dispersionless limit*.

As in the full theory, a solution of the dispersionless Toda hierarchy is completely characterized by a dispersionless τ-function. In fact, one should consider the free energy since it is the logarithm of the full τ-function that can be represented as a series in $\hbar$

$$\log \tau = \sum_{n\geq 0} \hbar^{-2+2n} F_n. \tag{II.140}$$

Thus, the dispersionless limit is extracted as follows

$$F_0 = \lim_{\hbar\to 0} \hbar^2 \log \tau. \tag{II.141}$$

The dispersionless free energy F_0 satisfies the classical limit of Hirota equations (II.134) and selected from all solutions by the same string equations (II.137) as in the quantum case.

Since the evolution along the times t_k is now generated by the Hamiltonians through the Poisson brackets, it can be seen as a canonical transformation in the phase space defined above. This fact is reflected also in the commutation relations (II.95). Since the Lax and Orlov–Shulman operators are dressed versions of ω and s, correspondingly, and have the same Poisson brackets, one can say that the dispersionless Toda hierarchy describe a canonical transformation from the canonical pair (ω, s) to (L, M). The free energy F_0 plays the role of the generating function of this transformation.

5.7 2MM as τ-function of Toda hierarchy

In this paragraph we show how all abstract ideas described above get a realization in the two-matrix model. Namely, we identify the partition function (II.54) with a particular τ-function of Toda hierarchy. This can be done in two ways using either the fermionic representation or the Lax formalism and its connection with the orthogonal polynomials. However, the fermionic representation which arises in this case is not exactly the same as in paragraph 5.3, although it still gives a τ-function of Toda hierarchy. The difference is that one should use two types of fermions [80]. In fact, they can be reduced to the fermions appearing in the fermionic representation of 2MM presented in section 4.6. They differ only by the basis which is used in the mode expansions (II.104) and (II.78).

We will use the approach based on the Lax formalism. Following this way, one should identify the Lax operators in the matrix model and prove that they satisfy the Lax-Sato equations (II.87). Equivalently, one can obtain the Baker–Akhiezer function satisfying (II.90) where the Hamiltonians H_k are related to the Lax operators through (II.88).

First of all, the Lax operators coincide with the operators $\hat{X}$ and $\hat{Y}$ defined in (II.68) and (II.69). Due to the first equation in (II.66), the operators have the same expansion as required in (II.86) where the first coefficient is

$$r(n\hbar) = \sqrt{h_{n+1}/h_n}. \tag{II.142}$$

The Baker–Akhiezer function is obtained as a semi-infinite vector constructed from the functions $\psi_n(x)$ (II.76)

$$\Psi(x; n\hbar) = \Phi_n(x) e^{NV(x)}. \tag{II.143}$$

Due to (II.70), $L\Psi = x\Psi$. Thus, it remains to consider the evolution of Ψ in the coupling constants. We fix their normalization choosing the potentials as follows

$$V(x) = \sum_{n>0} t_n x^n, \qquad \tilde{V}(x) = -\sum_{n>0} t_{-n} y^n. \tag{II.144}$$

Then the differentiation of the orthogonality condition (II.56) with respect to the coupling constants leads to the following evolution laws

$$\begin{aligned} \frac{1}{N}\frac{\partial \Phi_n(x)}{\partial t_k} &= -\sum_{m=0}^{n-1} (X^k)_{nm}\Phi_m(x) - \frac{1}{2}(X^k)_{nn}\Phi_n(x), \\ \frac{1}{N}\frac{\partial \Phi_n(x)}{\partial t_{-k}} &= \sum_{m=0}^{n-1} (Y^k)_{nm}\Phi_m(x) + \frac{1}{2}(Y^k)_{nn}\Phi_n(x). \end{aligned} \tag{II.145}$$

Using these relations one finds

$$\frac{1}{N}\frac{\partial \Psi}{\partial t_k} = H_k \Psi, \qquad \frac{1}{N}\frac{\partial \Psi}{\partial t_{-k}} = H_{-k}\Psi, \tag{II.146}$$

where $H_{\pm k}$ are defined as in (II.88). As a result, if one identifies $1/N$ with $\hbar$, one reproduces all equations for the Baker–Akhiezer function. This means that the dynamics of 2MM with respect to the coupling constants is governed by Toda hierarchy.

Combining (II.103) and (II.142), one finds

$$h_n = \frac{\tau_{n+1}}{\tau_n}. \tag{II.147}$$

Then the representation (II.62) implies

$$Z(N) \sim \frac{\tau_N}{\tau_0}. \tag{II.148}$$

The factor τ_0 does not depend on N and appears as a non-universal contribution to the free energy. Therefore, it can be neglected and, choosing the appropriate normalization, one can identify the partition function of 2MM with the τ-function of Toda hierarchy

$$Z(N) = \tau_N. \tag{II.149}$$

Moreover, one can find string equations uniquely characterizing the τ-function. First, we note that the Orlov–Shulman operators (II.94) are given by

$$M = \hat{X}\left(V'(\hat{X}) + \hat{P}\right) - \hbar, \qquad \bar{M} = \hat{Y}\left(\tilde{V}'(\hat{Y}) + \hat{Q}\right). \tag{II.150}$$

Then the relations (II.72) imply

$$L\bar{L} = M + \hbar, \qquad \bar{L}L = \bar{M}. \tag{II.151}$$

Multiplying the first equation by L^{-1} from the left and by L from the right and taking into account (II.95), one obtains that

$$M = \bar{M}. \tag{II.152}$$

This result together with the second equation in (II.151) gives one possible form of the string equations. It leads to the following functions f and g from (II.137)

$$f(\hat{\omega}, s) = s\hat{\omega}^{-1}, \qquad g(\hat{\omega}, s) = s. \tag{II.153}$$

It is easy to check that they satisfy the condition (II.138). Combining (II.151) and (II.152), one arrives at another very popular form of the string equation

$$[L, \bar{L}] = \hbar. \tag{II.154}$$

The identification (II.148) allows to use the powerful machinery of Toda hierarchy to find the partition function of 2MM. For example, one can write the Toda equation (II.136) which, together with some initial condition, gives the dependence of $Z(N)$ on the first times $t_{\pm 1}$. In the dispersionless limit this equation simplifies to a partial differential equation and sometimes it becomes even an ordinary differential equation (for example, when it is known that the partition function depends only on the product of the coupling constants $t_1 t_{-1}$). Finally, the string equation (II.154) can replace the initial condition for the differential equations of the hierarchy and produce equations of lower orders.

Chapter III

Matrix Quantum Mechanics

Now we approach the main subject of the thesis which is *Matrix Quantum Mechanics*. This chapter is devoted to the introduction to this model and combines the ideas discussed in the previous two chapters. The reader will see how the technique of matrix models allows to solve difficult problems related to string theory.

1 Definition of the model and its interpretation

Matrix Quantum Mechanics is a natural generalization of the matrix chain model presented in section II.2.2. It is defined as an integral over hermitian $N \times N$ matrices whose components are functions of one real variable which is interpreted as "time". Thus, it represents the path integral formulation of a quantum mechanical system with N^2 degrees of freedom. We will choose the time to be Euclidean so that the matrix integral takes the following form

$$Z_N(g) = \int \mathcal{D}M(t) \exp\left[-N \operatorname{tr} \int dt \left(\frac{1}{2}\dot{M}^2 + V(M)\right)\right], \tag{III.1}$$

where the potential $V(M)$ has the form as in (II.2). As it was required for all (hermitian) matrix models, this integral is invariant under the global unitary transformations

$$M(t) \longrightarrow \Omega^\dagger M(t) \Omega \quad (\Omega^\dagger \Omega = I). \tag{III.2}$$

The range of integration over the time variable in (III.1) can be arbitrary depending on the problem we are interested in. In particular, it can be finite or infinite, and the possibility of a special interest is the case when the time is compact so that one considers MQM on a circle. The latter choice will be important later and now we will concentrate on the simplest case of the infinite time interval.

In section II.2 it was shown that the free energy of matrix models gives a sum over discretized two-dimensional surfaces. In particular, its special *double scaling limit* corresponds to the continuum limit for the discretization and reproduces the sum over continuous surfaces, which is the path integral for two-dimensional quantum gravity. For the case of the simple one-matrix integral (if we do not tune the potential to a multicritical point), the surfaces did not carry any additional structure, whereas we argued that the multi-matrix case should correspond to quantum gravity coupled to matter. Since the MQM integral goes over a continuous set of matrices, we expect to obtain quantum gravity coupled with one scalar field. In turn, such a system can be interpreted as the sum over surfaces (or strings) embedded into one dimension [49].

Let us show how it works. As in section II.2, one can construct a Feynman expansion of the integral (III.1). It is the same as in the one-matrix case except that the propagator becomes time-dependent.

$$t\,{}^{j}_{i} \quad \begin{matrix}\longrightarrow \\ \longleftarrow\end{matrix} \quad {}^{k}_{l}\, t' \quad = \frac{1}{Ng_2}\delta_{il}\delta_{jk}e^{-|t-t'|}$$

Then the expansion (II.10) is generalized to

$$F = \sum_{g=0}^{\infty} N^{2-2g} \sum_{\substack{\text{genus } g \text{ connected} \\ \text{diagrams}}} g_2^{-E} \prod_k (-g_k)^{n_k} \prod_{i=1}^{V} \int_{-\infty}^{\infty} dt_i \prod_{\langle ij \rangle} G(t_i - t_j), \tag{III.3}$$

where $\langle ij \rangle$ denotes the edge connecting ith and jth vertices and $G(t) = e^{-|t|}$. As usual, each Feynman diagram is dual to some discretized surface and the sum (III.3) is interpreted as the sum over all discretizations. The new feature is the appearance of integrals over real variables t_i living at the vertices of the Feynman diagrams or at the centers of the faces of triangulated surfaces. They represent a discretization of the functional integral over a scalar field $t(\sigma)$. The action for the scalar field can be restored from the propagator $G(t)$. Its discretized version is given by

$$-\sum_{\langle ij \rangle} \log G(t_i - t_j). \tag{III.4}$$

Taking into account the exact form of the propagator, in the continuum limit, where the lattice spacing goes to zero, one finds that the action becomes

$$\int d^2\sigma \sqrt{h}\, |h^{ab}\partial_a t \partial_b t|^{1/2}. \tag{III.5}$$

It is not the standard action for a scalar field in two dimensions. The usual one would be obtained if we took the Gaussian propagator $G(t) = e^{-t^2}$. Does it mean that MQM does not describe two-dimensional gravity coupled with $c = 1$ matter?

In [49] it was argued that it does describe by the following reason. The usual scalar field propagator in the momentum space has the Gaussian form $G^{-1}(p) \sim e^{p^2}$. Its leading small momentum behaviour coincides with $G^{-1}(p) \sim 1 + p^2$ which is the momentum representation of the propagator for MQM. Thus, the replacement of one propagator by another affects only the short distance physics which is non-universal. The critical properties of the model surviving in the continuum limit should not depend on this choice. This suggestion was confirmed by a great number of calculations which showed full agreement of the results obtained by the CFT methods and in the framework of MQM.

Finally, we note that 2D gravity coupled with $c = 1$ matter can be interpreted as a non-critical string embedded in one dimension. The latter is equivalent to 2D critical string theory in the linear dilaton background. Thus Matrix Quantum Mechanics is an alternative description of 2D string theory. As it will be shown, it allows to manifest the integrability of this model and to address many questions inaccessible in the usual CFT formulation.

2 Singlet sector and free fermions

In this section we review the general structure of Matrix Quantum Mechanics and present its solution in the so called *singlet sector* of the Hilbert space [65]. The solution is relied on the interpretation of MQM as a quantum mechanical system of fermions. Therefore, we will consider t as a real Minkowskian time to have a good quantum mechanical description.

2.1 Hamiltonian analysis

To analyze the dynamics of MQM, as in the one-matrix case we change the variables from the matrix elements $M_{ij}(t)$ to the eigenvalues and the angular degrees of freedom

$$M(t) = \Omega^\dagger(t)x(t)\Omega(t), \qquad x = \mathrm{diag}(x_1, \dots, x_N), \ \Omega^\dagger\Omega = I. \tag{III.6}$$

Since the unitary matrix depends on time it is not canceled in the action of MQM. The kinetic term gives rise to an additional term

$$\mathrm{tr}\, \dot{M}^2 = \mathrm{tr}\, \dot{x}^2 + \mathrm{tr}\, [x, \dot{\Omega}\Omega^\dagger]^2. \tag{III.7}$$

The matrix $\dot{\Omega}\Omega^\dagger$ is anti-hermitian and can be considered as an element of the $su(n)$ algebra. Therefore, it can be decomposed in terms of the SU(N) generators

$$\dot{\Omega}\Omega^\dagger = \sum_{i=1}^{N-1} \dot{\alpha}_i H_i + \frac{i}{\sqrt{2}} \sum_{i<j} \left(\dot{\beta}_{ij} T_{ij} + \dot{\gamma}_{ij} \tilde{T}_{ij} \right), \tag{III.8}$$

where H_i are the diagonal generators of the Cartan subalgebra and the other generators are represented by the following matrices: $(T_{ij})_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ and $(\tilde{T}_{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$. The MQM degrees of freedom are described now by x_i, α_i, β_{ij} and γ_{ij}. The Minkowskian action in terms of these variables takes the form

$$S_{\rm MQM} = \int dt \, \left[\sum_{i=1}^{N} \left(\frac{1}{2}\dot{x}_i^2 - V(x_i) \right) + \frac{1}{2} \sum_{i<j} (x_i - x_j)^2 (\dot{\beta}_{ij}^2 + \dot{\gamma}_{ij}^2) \right]. \tag{III.9}$$

We did not included the overall multiplier N into the action. It plays the role of the Planck constant so that in the following we denote $\hbar = 1/N$.

To understand the structure of the corresponding quantum theory we pass to the Hamiltonian formulation (see review [26]). It is clear that the Hamiltonian is given by

$$H_{\rm MQM} = \sum_{i=1}^{N} \left(\frac{1}{2} p_i^2 + V(x_i) \right) + \frac{1}{2} \sum_{i<j} \frac{\Pi_{ij}^2 + \tilde{\Pi}_{ij}^2}{(x_i - x_j)^2}, \tag{III.10}$$

where p_i, Π_{ij} and $\tilde{\Pi}_{ij}$ are momenta conjugated to x_i, β_{ij} and γ_{ij}, respectively. Besides, since the action (III.9) does not depend on α_i, we have the constraint that its momentum should vanish $\Pi_i = 0$.

In quantum mechanics all these quantities should be realized as operators. If we work in the coordinate representation, Π_{ij} and $\tilde{\Pi}_{ij}$ are the usual derivatives. But to find $\hat{p}_i$, one should take into account the Jacobian appearing in the path integral measure after the change of variables (III.6). The Jacobian is the same as in (II.27). To see how it affects the momentum operator, we consider the scalar product in the Hilbert space of MQM. The measure of the scalar product in the coordinate representation coincides with the path integral measure and contains the same Jacobian. It can be easy understood because the change (III.6) can be done directly in the scalar product so that

$$\langle\Phi|\Phi'\rangle = \int dM\, \overline{\Phi(M)}\Phi'(M) = \int d\Omega \int \prod_{i=1}^{N} dx_i\, \Delta^2(x)\overline{\Phi(x,\Omega)}\Phi'(x,\Omega). \tag{III.11}$$

Due to this the map to the momentum representation, where the measure is trivial, is given by

$$\Phi(p,\Omega) = \int \prod_{i=1}^{N} \left(dx_i\, e^{-\frac{i}{\hbar}p_i x_i}\right) \Delta(x)\Phi(x,\Omega). \tag{III.12}$$

Then in the coordinate representation the momentum is realized as the following operator

$$\hat{p}_i = \frac{-i\hbar}{\Delta(x)}\frac{\partial}{\partial x_i}\Delta(x). \tag{III.13}$$

As a result, we obtain that the Hamiltonian (III.10) is represented by

$$\hat{H}_{\rm MQM} = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2\Delta(x)}\frac{\partial^2}{\partial x_i^2}\Delta(x) + NV(x_i)\right) + \frac{1}{2}\sum_{i<j}\frac{\hat{\Pi}_{ij}^2 + \hat{\tilde{\Pi}}_{ij}^2}{(x_i-x_j)^2}. \tag{III.14}$$

The wave functions are characterized by the Schrödinger and constraint equations

$$i\hbar\frac{\partial\Phi(x,\Omega)}{\partial t} = \hat{H}_{\rm MQM}\Phi(x,\Omega), \qquad \hat{\Pi}_i\Phi(x,\Omega) = 0. \tag{III.15}$$

2.2 Reduction to the singlet sector

Using the Hamiltonian derived in the previous paragraph, the partition function (III.1) can be rewritten as follows

$$Z_N = \operatorname{Tr} e^{-T\hbar^{-1}\hat{H}_{\rm MQM}}, \tag{III.16}$$

where T is the time interval we are interested in. If one considers the sum over surfaces embedded in the infinite real line, the interval should also be infinite. In this limit only the ground state of the Hamiltonian contributes to the partition function and we have

$$F = \lim_{T\to\infty}\frac{\log Z_N}{T} = -E_0/\hbar. \tag{III.17}$$

Thus, we should look for an eigenfunction of the Hamiltonian (III.14) which realizes its minimum. It is clear that the last term representing the angular degrees of freedom is positive definite and should annihilate this eigenfunction. To understand the sense of this condition, let us note that the angular argument Ω of the wave functions belongs to SU(N). Hence, the wave functions are functions on the group and can be decomposed in its irreducible representations

$$\Phi(x,\Omega) = \sum_r \sum_{a,b=1}^{d_r} D_{ba}^{(r)}(\Omega)\Phi_{ab}^{(r)}(x), \tag{III.18}$$

where r denotes an irreducible representation, d_r is its dimension and $D^{(r)}_{ba}(\Omega)$ is the representation matrix of an element $\Omega \in \mathrm{SU(N)}$ in the representation r. The coefficients are functions of only the eigenvalues $x_i(t)$. On the other hand, the operators $\hat{\Pi}_{ij}$ and $\hat{\tilde{\Pi}}_{ij}$ are generators of the left rotations $\Omega \to U\Omega$. It is clear that in the sum (III.18) the only term remaining invariant under this transformation corresponds to the trivial, or *singlet*, representation. Thus, the condition $\hat{\Pi}_{ij}\Phi = \hat{\tilde{\Pi}}_{ij}\Phi = 0$ restricts us to the sector of the Hilbert space where the wave functions do not depend on the angular degrees of freedom $\Phi(x,\Omega) = \Phi^{(\mathrm{sing})}(x)$.

In this singlet sector the Hamiltonian reduces to

$$\hat{H}^{(\mathrm{sing})}_{\mathrm{MQM}} = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2\Delta(x)}\frac{\partial^2}{\partial x_i^2}\Delta(x) + V(x_i)\right). \tag{III.19}$$

Its form indicates that it is convenient to redefine the wave functions

$$\Psi^{(\mathrm{sing})}(x) = \Delta(x)\Phi^{(\mathrm{sing})}(x). \tag{III.20}$$

In terms of these functions the Hamiltonian becomes the sum of the one particle Hamiltonians

$$\hat{H}^{(\mathrm{sing})}_{\mathrm{MQM}} = \sum_{i=1}^{N}\hat{h}_i, \qquad \hat{h}_i = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x_i^2} + V(x_i). \tag{III.21}$$

Moreover, since a permutation of eigenvalues is also a unitary transformation, the singlet wave function $\Phi^{(\mathrm{sing})}(x)$ should not change under such permutations and, therefore, it is symmetric. Then the redefined wave function $\Psi^{(\mathrm{sing})}(x)$ is completely antisymmetric. Taking into account the result (III.21), we conclude that the problem involving N^2 bosonic degrees of freedom has been reduced to a system of N non-relativistic free fermions moving in the potential $V(x)$ [65]. This fact is at the heart of the integrability of MQM and represents an interesting and still not well understood equivalence between 2D critical string theory and free fermions.

2.3 Solution in the planar limit

According to the formula (III.17) and the fermionic interpretation found in the previous paragraph, one needs to find the ground state energy of the system of N non-interacting fermions. All states of such system are described by Slater determinants and characterized by the filled energy levels of the one-particle Hamiltonian $\hat{h}$ (III.21)

$$\Psi_{n_1,\dots,n_N}(x) = \frac{1}{\sqrt{N!}}\det_{k,l}\,\psi_{n_k}(x_l), \tag{III.22}$$

where $\psi_n(x)$ is the eigenfunction at the nth level

$$\hat{h}\psi_n(x) = \epsilon_n\psi_n(x). \tag{III.23}$$

The ground state is obtained by filling the lowest N levels so that the corresponding energy is given by

$$E_0 = \sum_{n=1}^{N}\epsilon_n. \tag{III.24}$$

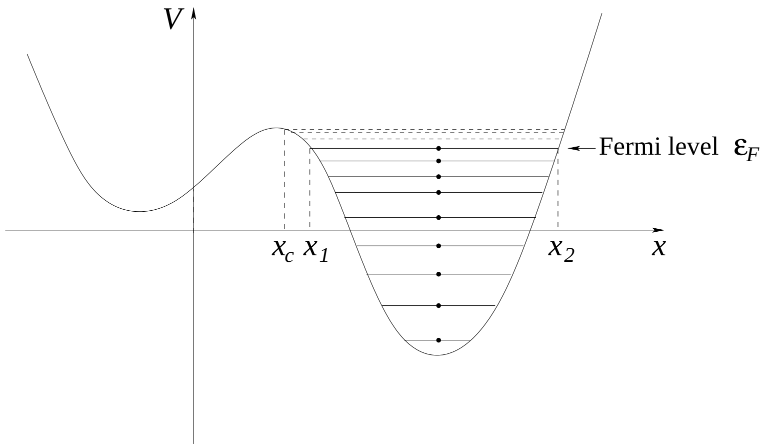


Fig. III.1: The ground state of the free fermionic system. The fermions fill the first N levels up to the Fermi energy. At the critical point where the Fermi level touches the top of the potential the energy levels condensate and the density diverges.

Note, that if the cubic potential is chosen, the system is non-stable. The same conclusion can be made for any unbounded potential. Therefore, strictly speaking, there is no ground state in such situation. However, we are interested only in the perturbative expansion in $1/N$, which corresponds to the expansion in the Planck constant. On the other hand, the amplitudes of tunneling from an unstable vacuum are exponentially suppressed as $\sim e^{-1/\hbar}$. Thus, these effects are not seen in the perturbation theory and we can simplify the life considering even unstable potentials forgetting about the instabilities. All what we need is to separate the perturbative effects from the non-perturbative ones.

As a result, we get the picture presented in fig. III.1. Let us consider this system in the large N limit. Since we identified $1/N$ with the Planck constant $\hbar$, $N \to \infty$ corresponds to the classical limit. In this approximation the energy becomes continuous and particles are characterized by their coordinates in the phase space. In our case the phase space is two-dimensional and each particle occupies the area $2\pi\hbar$. Moreover, due to the fermionic nature, two particles cannot take the same place. Thus, the total area occupied by N particles is 2π. Due to the Liouville theorem it is preserved in the time evolution. Therefore, the classical description of N free fermions is the same as that of an incompressible liquid.

For us it is important now only that the ground state corresponds to a configuration where the liquid fills a connected region (Fermi sea) with the boundary given by the following equation

$$h(x,p) = \frac{1}{2}p^2 + V(x) = \epsilon_F, \tag{III.25}$$

where $\epsilon_F = \epsilon_N$ is the energy at the Fermi level. Then one can write

$$N = \iint \frac{dxdp}{2\pi\hbar}\theta(\epsilon_F - h(x,p)), \tag{III.26}$$

$$E_0 = \iint \frac{dxdp}{2\pi\hbar}h(x,p)\theta(\epsilon_F - h(x,p)). \tag{III.27}$$

Differentiation with respect to ϵ_F gives

$$\hbar\frac{\partial N}{\partial \epsilon_F} \stackrel{def}{=} \rho(\epsilon_F) = \iint \frac{dxdp}{2\pi}\delta(\epsilon_F - h(x,p)) = \frac{1}{\pi}\int_{x_1}^{x_2} \frac{dx}{\sqrt{2(\epsilon_F - V(x))}}, \qquad \text{(III.28)}$$

$$\hbar\frac{\partial E_0}{\partial \epsilon_F} = \iint \frac{dxdp}{2\pi} h(x,p)\delta(\epsilon_F - h(x,p)) = \epsilon_F \rho(\epsilon_F), \qquad \text{(III.29)}$$

where x_1 and x_2 are turning points of the classical trajectory at the Fermi level. These equations determine the energy in an inexplicit way.

To find the energy in terms of the coupling constants, one should exclude the Fermi level ϵ_F by means of the normalization condition (III.26). For some simple potentials the integral in (III.26) can be calculated explicitly, but in general this cannot be done. However, the universal information related to the sum over continuous surfaces and 2D string theory is contained only in the singular part of the free energy. The singularity appears when the Fermi level reaches the top of the potential similarly to the one-matrix case (cf. figs. III.1 and II.4). Near this point the density diverges and shows together with the energy a non-analytical behaviour.

From this it is clear that the singular contribution to the integral (III.28) comes from the region of integration around the maxima of the potential. Generically, the maxima are of the quadratic type. Thus, up to analytical terms we have

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{2(\epsilon_F - V(x))}} \sim -\frac{1}{2}\log(\epsilon_c - \epsilon_F), \qquad \text{(III.30)}$$

where ϵ_c is the critical value of the Fermi level. Denoting $\epsilon = \epsilon_c - \epsilon_F$, one finds [49]

$$\rho(\epsilon) = -\frac{1}{2\pi}\log(\epsilon/\Lambda), \qquad F_0 = \frac{1}{4\pi\hbar^2}\epsilon^2\log(\epsilon/\Lambda), \qquad \text{(III.31)}$$

where we introduced a cut-off Λ related to the non-universal contributions.

2.4 Double scaling limit

In the previous paragraph we reproduced the free energy and the density of states in the planar limit. To find them in all orders in the genus expansion, one needs to consider the double scaling limit as it was explained in section II.2.5. For this one should correlate the large N limit with the limit where the coupling constants approach their critical values. In our case this means that one should introduce coordinates describing the region near the top of the potential. Let x_c is the coordinate of the maximum and $y = \frac{1}{\sqrt{\hbar}}(x - x_c)$. Then the potential takes the form

$$V(y) = \epsilon_c - \frac{\hbar}{2}y^2 + \frac{\hbar^{3/2}\lambda}{3}y^3 + \cdots, \qquad \text{(III.32)}$$

where the dots denote the terms of higher orders in $\hbar$. The Schrödinger equation for the eigenfunction at the Fermi level can be rewritten as follows

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial y^2} - \frac{1}{2}y^2 + \frac{\hbar^{1/2}\lambda}{3}y^3 + \cdots\right)\psi_N(y) = -\hbar^{-1}\epsilon\psi_N(y). \qquad \text{(III.33)}$$

It shows that it is natural to define the rescaled energy variable

$$\mu = \hbar^{-1}(\epsilon_c - \epsilon_F). \tag{III.34}$$

This relation defines the double scaling limit of MQM, which is obtained as $N = \hbar^{-1} \to \infty$, $\epsilon_F \to \epsilon_c$ and keeping μ to be fixed [81, 82, 83, 84].

Note that the double scaling limit (III.34) differs from the naive limit expected from 1MM where we kept fixed the product of N and some power of $\lambda_c - \lambda$ (II.21). In our case the latter is renormalized in a non-trivial way. To get this renormalization, note that writing the relation (III.28) we actually decoupled N and $\hbar$. This means that in fact we rescaled the argument of the potential so that we moved the coupling constant from the potential to the coefficient in front of the action. For example, we can do this with the cubic coupling constant λ by rescaling $x \to x/\lambda$. In this normalization the overall coefficient should be multiplied by λ^{-2} what changes the relation between N and the Planck constant to $\hbar = \lambda^2/N$. Then for $\Delta = \frac{2\pi}{\hbar}(\lambda_c^2 - \lambda^2)$, (III.28) and (III.34) imply

$$\frac{\partial \Delta}{\partial \mu} = 2\pi \rho(\mu). \tag{III.35}$$

Integrating this equation, one finds a complicated relation between two scaling variables. In the planar limit this relation reads

$$\Delta = -\mu \log(\mu/\Lambda). \tag{III.36}$$

The remarkable property of the double scaling limit (III.34) is that it reduces the problem to the investigation of free fermions in the inverse oscillator potential $V_{\rm ds}(x) = -\frac{1}{2}x^2$

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + x^2\right)\psi_\epsilon(x) = \epsilon \psi_\epsilon(x), \tag{III.37}$$

where we returned to the notations x and ϵ for already rescaled matrix eigenvalues and energy. All details of the initial potential disappear in this limit because after the rescaling the cubic and higher terms suppressed by positive powers of $\hbar$. This fact is the manifestation of the universality of MQM showing the independence of its results of the form of the potential.

The equation (III.37) for eigenfunctions has an explicit solution in terms of the parabolic cylinder functions. They have a complicated form and we do not give their explicit expressions. However, the density of states at the Fermi level can be calculated knowing only their asymptotics at large x. The density follows from the WKB quantization condition

$$\left(\Phi_{\epsilon_{n+1}} - \Phi_{\epsilon_n}\right)\Big|_{\sqrt{\Lambda}} = 2\pi, \tag{III.38}$$

where Φ_ϵ is the phase of the wave function $\psi_\epsilon(x) = \frac{C}{\sqrt{x}} e^{i\Phi_\epsilon(x)}$ and the difference is calculated at the cut-off $x \sim \sqrt{\Lambda} \sim \sqrt{N}$. The asymptotic form of the phase is [85]

$$\Phi_\epsilon(x) \approx \frac{1}{2}x^2 + \epsilon \log x - \phi(\epsilon), \tag{III.39}$$

$$\phi(\epsilon) = \frac{\pi}{4} - \frac{i}{2}\log\frac{\Gamma(\frac{1}{2} + i\epsilon)}{\Gamma(\frac{1}{2} - i\epsilon)}. \tag{III.40}$$

In the WKB approximation the index n becomes continuous variable and the density of states is defined as its derivative

$$\rho(\epsilon) \stackrel{\mathrm{def}}{=} \frac{\partial n}{\partial \epsilon} = \frac{1}{2\pi}\log\Lambda - \frac{1}{2\pi}\frac{d\phi}{d\epsilon} = \frac{1}{2\pi}\log\Lambda - \frac{1}{2\pi}\operatorname{Re}\psi(\frac{1}{2}+i\epsilon), \qquad \text{(III.41)}$$

where $\psi(\epsilon) = \frac{d}{d\epsilon}\log\Gamma(\epsilon)$. Neglecting the cut-off dependent term and expanding the digamma function in $1/\mu$ ($\mu = -\epsilon$), one finds the following result

$$\rho(\mu) = \frac{1}{2\pi}\left(-\log\mu + \sum_{n=1}^{\infty}(2^{2n-1}-1)\frac{|B_{2n}|}{n}(2\mu)^{-2n}\right), \qquad \text{(III.42)}$$

where B_{2n} are Bernoulli numbers. Integrating (III.29), one obtains the expansion of the free energy

$$F(\mu) = \frac{1}{4\pi}\left(\mu^2\log\mu - \frac{1}{12}\log\mu + \sum_{n=1}^{\infty}\frac{(2^{2n+1}-1)|B_{2n+2}|}{4n(n+1)}(2\mu)^{-2n}\right). \qquad \text{(III.43)}$$

In fact, to compare this result with the genus expansion of the partition function of 2D string theory, one should reexpand (III.43) in terms of the renormalized string coupling. Its role, as usual, is played by $\kappa = \Delta^{-1}$ and its relation to μ is determined by (III.35). With $\rho(\mu)$ taken from (III.42), one can solve this equation with respect to $\Delta(\mu)$. Then it is sufficient to make substitution into (III.43) to get the following answer

$$F(\Delta) = \frac{1}{4\pi}\left(\frac{\Delta^2}{\log\Delta} - \frac{1}{12}\log\Delta + \sum_{n=1}^{\infty}\frac{(2^{2n+1}-1)|B_{2n+2}|}{4n(n+1)(2n+1)}\left(\frac{2\Delta}{\log\Delta}\right)^{-2n}\right), \qquad \text{(III.44)}$$

where terms $O(\log^{-1}\Delta)$ were neglected because they contain the cut-off and vanish in the double scaling limit.

Some remarks related to the expansion (III.44) are in order. First, the coefficients associated with genus g grow as $(2g)!$. This behaviour is characteristic for closed string theories where the sum over genus-g surfaces exhibits the same growth. Besides, we observe a new feature in comparison with the one-matrix model. Although the couplings were renormalized, the sums over spherical and toroidal surfaces logarithmically diverge. This also can be explained in the context of the CFT approach. Finally, comparing (III.44) with (II.19), we find that the string susceptibility for MQM vanishes $\gamma_{\mathrm{str}} = 0$. This is again in the excellent agreement with the continuum prediction of [86].

Thus, all results concerning the free energy of MQM coincide with the corresponding results for the partition function of 2D string theory. Therefore, it is tempting to claim that they are indeed equivalent theories. But we know that 2D string theory possesses dynamical degrees of freedom: the tachyon, winding modes and discrete states. To justify the equivalence further, we should show how all of them are realized in Matrix Quantum Mechanics.

3 Das–Jevicki collective field theory

The fermionic representation presented in the previous section gives a microscopic description of 2D string theory. As usual, the macroscopic description, which has a direct interpretation in terms of the target space fields of 2D string theory, is obtained as a theory of effective degrees of freedom. These degrees of freedom are collective excitations of the fermions of the singlet sector of MQM and identified with the tachyonic modes of 2D strings. Their dynamics is governed by a *collective field theory* [87], which in the given case was developed by Das and Jevicki [88]. This theory encodes all interactions of strings in two dimensions and, therefore, it gives an example of string field theory formulated directly in the target space.

3.1 Effective action for the collective field

A natural collective field in MQM is the density of eigenvalues

$$\varphi(x,t) = \operatorname{tr} \delta\left(x - M(t)\right). \tag{III.45}$$

In the double scaling limit, which implies $N \to \infty$, it becomes a continuous field. Its dynamics can be derived directly from the MQM action with the inverse oscillator potential [88]

$$S = \frac{1}{2}\int dt \ \operatorname{tr}\left(\dot{M}^2 + M^2\right). \tag{III.46}$$

However, it is much easier to use the Hamiltonian formulation. The Hamiltonian of MQM in the singlet representation is given by the energy of the Fermi sea similarly to the ground state energy (III.27). The difference is that now the Fermi sea can have an arbitrary profile which can differ from the trajectory of one fermion (III.25). Besides, to the expression (III.27) one should add a term which fixes the Fermi level and allows to vary the number of fermions. Otherwise the density would be subject of some normalization condition. Thus, the full double scaled Hamiltonian reads

$$H_{\rm coll} = \iint\limits_{\substack{\rm Fermi \\ \rm sea}} \frac{dxdp}{2\pi}(h(x,p) + \mu), \tag{III.47}$$

where

$$h(x,p) = \frac{1}{2}p^2 + V(x), \quad V(x) = -\frac{1}{2}x^2. \tag{III.48}$$

We restrict ourselves to the case where the boundary of the Fermi sea can be represented by two functions, say $p_+(x,t)$ and $p_-(x,t)$ satisfying the boundary condition $p_+(x_*,t) = p_-(x_*,t)$, where x_* is the leftmost point of the sea (fig. III.2a). It means that we forbid the situations shown on fig, III.2b. In the fermionic picture they do not cause any problems, but in the bosonic description they require a special attention.

In this restricted situation one can take the integral over the momentum in (III.47). The result is

$$H_{\rm coll} = \int \frac{dx}{2\pi}\left(\frac{1}{6}(p_+^3 - p_-^3) + (V(x)+\mu)(p_+ - p_-)\right). \tag{III.49}$$

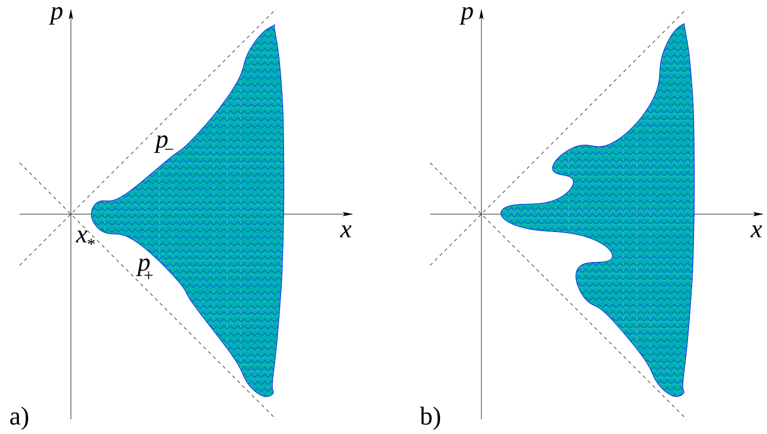


Fig. III.2: The Fermi sea of the singlet sector of MQM. The first picture shows the situation where the profile of the Fermi sea can be described by a two-valued function. The second picture presents a more general configuration.

It is clear that the difference of p_+ and p_- coincides with the density (III.45), whereas their sum plays the role of a conjugate variable. The right identification is the following [89]:

$$p_\pm(x,t) = \partial_x \Pi \pm \pi\varphi(x,t), \tag{III.50}$$

where the equal-time Poisson brackets are defined as

$$\{\varphi(x), \Pi(y)\} = \delta(x-y). \tag{III.51}$$

Substitution of (III.50) into (III.49) gives

$$H_{\rm coll} = \int dx \left(\frac{1}{2}\varphi(\partial_x\Pi)^2 + \frac{\pi^2}{6}\varphi^3 + (V(x)+\mu)\varphi\right). \tag{III.52}$$

One can exclude the momentum $\Pi(x,t)$ by means of the equation of motion

$$-\partial_x\Pi = \frac{1}{\varphi}\int dx \partial_t\varphi \tag{III.53}$$

what leads to the following collective field theory action

$$S_{\rm coll} = \int dt \int dx \left(\frac{1}{2\varphi}\left(\int dx\partial_t\varphi\right)^2 - \frac{\pi^2}{6}\varphi^3 - (V(x)+\mu)\varphi\right). \tag{III.54}$$

This action can be considered as a background independent formulation of string field theory. It contains a cubic interaction and a linear tadpole term. The former describes the effect of splitting and joining strings and the latter represents a process of string annihilation into the vacuum. The important point is that the dynamical field $\varphi(x,t)$ is two-dimensional. The dimension additional to the time t appeared from the matrix eigenvalues. This shows that the target space of the corresponding string theory is also two-dimensional in agreement with our previous conclusion.

Another observation is that whereas the initial matrix model in the inverse oscillator potential was simple with the linear equations of motion

$$\ddot{M}(t) - M(t) = 0, \tag{III.55}$$

the resulting collective field theory is non-linear. Thus, MQM provides a solution of a complicated non-linear theory through the transformation of variables (III.45). Nevertheless, the integrability of MQM is present also in the effective theory (III.54). Indeed, consider the equations of motion for the fields p_+ and p_-. The equations (III.50) and (III.51) imply the following Poisson brackets

$$\{p_\pm(x), p_\pm(y)\} = \mp 2\pi \partial_x \delta(x-y) \tag{III.56}$$

so that the Hamiltonian (III.49) gives

$$\partial_t p_\pm + p_\pm \partial_x p_\pm + \partial_x V(x) = 0. \tag{III.57}$$

This equation is a KdV type equation which is integrable. This indicates that the whole theory is also exactly solvable. In fact, one can write an infinite set of conserved commuting quantities [89]

$$H_n = \iint\limits_{\substack{\text{Fermi}\\ \text{sea}}} \frac{dxdp}{2\pi}(p^2 - x^2)^n. \tag{III.58}$$

It is easy to check that up to surface terms they satisfy

$$\{H_n, H_m\} = 0 \quad \text{and} \quad \frac{d}{dt}H_n = 0. \tag{III.59}$$

The quantities H_n can be considered as Hamiltonians generating some perturbations. Since all of them are commuting, according to the definition given in section II.5.1, we conclude that the system is integrable.

3.2 Identification with the linear dilaton background

Now we choose a particular background of string theory. This will allow to identify the tachyon field and target space coordinates with the corresponding quantities of the collective field theory. In terms of the collective theory the choice of a background means to consider the perturbation theory around some classical solution of (III.54). We choose the solution describing the ground state of MQM. It is obtained from (III.25) and can be written as

$$\pi\varphi_0 = \pm p_\pm = \sqrt{x^2 - 2\mu}. \tag{III.60}$$

This solution is distinguished by the fact that it is static and that the boundary of the Fermi sea coincides with a trajectory of one fermion.

Taking (III.60) as a background, we are interested in small fluctuations of the collective field around this background solution

$$\varphi(x,t) = \varphi_0(x) + \frac{1}{\sqrt{\pi}}\partial_x \eta(x,t). \tag{III.61}$$

The dynamics of these fluctuations is described by the following action obtained by substitution of (III.61) into (III.54)

$$S_{\rm coll} = \frac{1}{2}\int dt \int dx \left(\frac{(\partial_t \eta)^2}{(\pi\varphi_0 + \sqrt{\pi}\partial_x \eta)} - \pi\varphi_0(\partial_x\eta)^2 - \frac{\sqrt{\pi}}{3}(\partial_x\eta)^3 \right). \tag{III.62}$$

The expansion of the denominator in the first term gives rise to an infinite number of vertices of increasing order with the field η. A compact version of this interacting theory would be obtained if we worked with the Hamiltonian instead of the action. Then only cubic interaction terms would appear.

Let us consider the quadratic part of the action (III.62). It is given by

$$S_{(2)} = \frac{1}{2}\int dt \int dx \left(\frac{(\partial_t \eta)^2}{\pi\varphi_0} - (\pi\varphi_0)(\partial_x\eta)^2 \right). \tag{III.63}$$

Thus, $\eta(x,t)$ can be interpreted as a massless field propagating in the background metric

$$g^{(0)}_{\mu\nu} = \begin{pmatrix} -\pi\varphi_0 & 0 \\ 0 & (\pi\varphi_0)^{-1} \end{pmatrix}. \tag{III.64}$$

However, the non-trivial metric can be removed by a coordinate transformation. It is enough to introduce the time-of-flight coordinate

$$q(x) = \int^x \frac{dx}{\pi\varphi_0(x)}. \tag{III.65}$$

The change of coordinate (III.65) brings the action to the form

$$S_{\rm coll} = \frac{1}{2}\int dt \int dq \left((\partial_t\eta)^2 - (\partial_q\eta)^2 - \frac{1}{3\pi\sqrt{\pi}\varphi_0^2}\left((\partial_q\eta)^3 + 3(\partial_q\eta)(\partial_t\eta)^2 \right) + \cdots \right), \tag{III.66}$$

where we omitted the terms of higher orders in η. The action (III.66) describes a massless field in the flat Minkowski spacetime with a spatially dependent coupling constant

$$g_{\rm str}(q) = \frac{1}{(\pi\varphi_0(q))^2}. \tag{III.67}$$

Using the explicit formula (III.60) for the background solution, one can obtain

$$x(q) = \sqrt{2\mu}\cosh q, \qquad p(q) = \sqrt{2\mu}\sinh q. \tag{III.68}$$

Thus, the coupling constant behaves as

$$g_{\rm str}(q) = \frac{1}{2\mu\sinh^2 q} \sim \frac{1}{\mu}e^{-2q}, \tag{III.69}$$

where the asymptotics is given for $q \to \infty$.

Comparing (III.69) with (I.25), we see that the collective field theory action (III.66) describes 2D string theory in the linear dilaton background. In the asymptotic region of large q the flat coordinates (t, q) can be identified with the coordinates of the target space of string theory coming from the $c = 1$ matter X and the Liouville field ϕ on the world sheet. The identification reads as follows

$$it \leftrightarrow X, \qquad q \leftrightarrow \phi. \tag{III.70}$$

Thus, the time of MQM and the time-of-flight coordinate, which is a function of the matrix eigenvalue variable, form the flat target space of the linear dilaton background. It is also clear that the two-dimensional massless collective field $\eta(q, t)$ coincides with the redefined tachyon $\eta = e^{2\phi}T$.

In fact, the above identification is valid only asymptotically. When we go to the region of small q, one should include into account the Liouville exponent $\mu e^{-2\phi}$ in the CFT action (I.39). The tachyon field can be considered as a wave function describing the lowest eigenstate of the Hamiltonian of this CFT. Therefore, instead of Klein–Gordon equation, it should satisfy the Liouville equation

$$\left(\partial_X^2 + \partial_\phi^2 + 4\partial_\phi + 4 - \mu e^{-2\phi}\right) T(\phi, X) = 0. \tag{III.71}$$

Hence the Liouville mode does not coincide exactly with the collective field coordinate q. The correct identification is obtained as follows [90]. We return to the eigenvalue variable x and consider its conjugated momentum $p = -i\partial/\partial x$. The Fourier transform of the Klein–Gordon equation written in the metric (III.64) gives

$$\left(\partial_t^2 - \sqrt{x^2 - 2\mu}\,\partial_x \sqrt{x^2 - 2\mu}\,\partial_x\right)\eta(x, t) = 0 \Rightarrow \left(\partial_t^2 - (p\partial_p)^2 - 2p\partial_p - 1 - 2\mu p^2\right)\tilde{\eta}(p, t) = 0. \tag{III.72}$$

where $\tilde{\eta}$ is the Fourier image of η. Finally, the change of variables

$$it = X, \qquad ip = \frac{1}{\sqrt{2}}e^{-\phi} \tag{III.73}$$

together with $T(\phi, X) \sim p^3\tilde{\eta}(p, t)$ brings (III.72) to the Liouville equation (III.71). This shows that, more precisely, the Liouville coordinate is identified with the logarithm of the momentum conjugated to the matrix eigenvalue.

The meaning of this rule becomes more clear after realizing that the Fourier transform with an imaginary momentum of the collective field φ is the Wilson loop operator

$$W(l, t) = \operatorname{tr}\left(e^{-lM}\right) = \int dx\, e^{-lx}\varphi(x, t). \tag{III.74}$$

This operator inserts a loop of the length l into the world sheet. Therefore, it has a direct geometrical interpretation and its parameter l is related to the scale of the metric, which is governed by the Liouville mode ϕ. Thus, it is quite natural that l and ϕ are identified through (III.73) where one should take $ip = l$. The substitution of the expansion (III.61) gives

$$W(l, t) = W_0 + \frac{1}{\sqrt{\pi}}\int dx\, e^{-lx}\partial_x\eta(x, t) = W_0 + \frac{l}{\sqrt{\pi}}\tilde{\eta}(-il, t), \tag{III.75}$$

where

$$W_0 = \frac{\sqrt{2\mu}}{l}K_1(\sqrt{2\mu}l) \tag{III.76}$$

is the genus zero one-point function of the density. Using this representation, it is easy to check that the Wilson loop operator satisfies the following Wheeler–DeWitt equation [91]

$$\left(\partial_t^2 - (l\partial_l)^2 + 2\mu l^2\right) W(l,t) = 0. \tag{III.77}$$

Thus, in the l-representation it is the Wilson loop operator that is the analog of the free field for which the derivative terms have standard form. Therefore, we should identify the field η from section I.5.1 with W rather than with $\tilde{\eta}$ defined in (III.72). The precise relation between the tachyon field and the Wilson loop operator is the following

$$T(\phi,X) = e^{-2\phi}W(l(\phi),-iX) = e^{-2\phi}W_0 + e^{-2\phi}\int\limits_0^\infty dq\, \exp\left[-\sqrt{\mu}e^{-\phi}\cosh q\right]\partial_q\eta. \tag{III.78}$$

The integral transformation (III.78) expresses solutions of the non-linear Liouville equation through solutions of the Klein–Gordon equation. This reduces the problem of calculating the tachyon scattering amplitudes in the linear dilaton background to calculation of the S-matrix for the collective field theory of the Klein–Gordon field η. As we saw above, this theory is integrable. Therefore, the scattering problem in 2D string theory can be exactly solved. Before to show that, we should introduce the operators creating the asymptotic states, *i.e.*, the tachyon vertex operators.

3.3 Vertex operators and correlation functions

The vertex operators of the tachyon field were constructed in section I.5.1. Their Minkowskian form is given by (I.44) and describes the left and right movers. Note that the representation for the operators was written only in the asymptotic region $\phi \to \infty$ where the Liouville potential can be ignored. Therefore, we can use the simple identification (III.70) to relate the matrix model quantities with the target space objects. Then one should find operators that behave as left and right movers in the space of t and q.

First, the t-dependence of a matrix model operator is completely determined by the inverse oscillator potential, which leads to the following simple Heisenberg equation

$$\frac{\partial}{\partial t}\hat{A}(t) = i\left[\frac{1}{2}(\hat{p}^2 - \hat{x}^2), \hat{A}(t)\right]. \tag{III.79}$$

Its solution is conveniently represented in the basis of the chiral operators

$$\hat{x}_\pm(t) \stackrel{\text{def}}{=} \frac{\hat{x}(t) \pm \hat{p}(t)}{\sqrt{2}} = \hat{x}_\pm(0)e^{\pm t}. \tag{III.80}$$

Therefore, the time-independent operators, which should be used in the Schrödinger representation, are $e^{\mp t}\hat{x}_\pm(t)$. This suggests that the vertex operators can be constructed from powers of $x_\pm$. Indeed, it was argued [89] that their matrix model realization is given by

$$T_n^\pm = e^{\pm nt}\,\mathrm{tr}\,(M \mp P)^n. \tag{III.81}$$

To justify further this choice, let us consider the collective field representation of the operators (III.81)

$$T_n^\pm = e^{\pm nt}\int \frac{dx}{2\pi}\int\limits_{p_-(x,t)}^{p_+(x,t)} dp\,(x\mp p)^n = \frac{e^{\pm nt}}{n+1}\int\frac{dx}{2\pi}(x\mp p)^{n+1}\Big|_{p_-}^{p_+}, \qquad \text{(III.82)}$$

where $p_\pm$ are second quantized fields satisfying equations (III.57). We shift these fields by the classical solution

$$p_\pm(x,t) = \pm\pi\varphi_0(x) + \frac{\alpha_\mp(x,t)}{\pi\varphi_0(x)}. \qquad \text{(III.83)}$$

Then the linearized equations of motion for the quantum corrections $\alpha_\pm$ coincide with the conditions for chiral fields

$$(\partial_t \mp \partial_q)\alpha_\pm = 0 \Rightarrow \alpha_\pm = \alpha_\pm(t\pm q). \qquad \text{(III.84)}$$

In the asymptotics $q\to\infty$, $\pi\varphi_0 \approx x \approx \frac{1}{2}e^q$. Therefore, in the leading approximation the operators (III.82) read

$$T_n^\pm \approx \int \frac{dq}{2\pi} e^{n(q\pm t)}\alpha_\pm. \qquad \text{(III.85)}$$

The integral extracts the operator creating the component of the chiral field which behaves as $e^{n(q\pm t)}$. This shows that the matrix operators (III.81) do possess the necessary properties. The factor $e^{-2\phi}$, which is present in the definition of the vertex operators (I.44) and absent in the matrix case, can be seen as coming from the measure of integration over the world sheet or, equivalently, as a result of the redefinition of the tachyon field (III.78). In fact, it is automatically restored by the matrix model.

The matrix operators (III.81) correspond to the Minkowskian vertex operators with imaginary momenta $k = in$. After the continuation of time to the Euclidean region $t\to -iX$, they also can be considered as vertex operators with Euclidean momenta $p = \mp n$. To realize other momenta, one should analytically continue from this discrete set to the whole complex plane. In particular, the vertex operators of Minkowskian real momenta are obtained as

$$V_k^\pm \sim T_{-ik}^\pm = e^{\mp ikt}\,\mathrm{tr}\,(M\mp P)^{-ik}. \qquad \text{(III.86)}$$

Using these operators, one can construct and calculate scattering amplitudes of tachyons. The result has been obtained from both the collective field theory formalism [92, 93, 94, 95] and the fermionic representation [26, 96, 97, 98, 85]. Moreover, the generating functional for all S-matrix elements has been constructed [98]. It takes an especially transparent form when the S-matrix is represented as a composition of three processes: fermionization of incoming tachyon modes, scattering in the free fermion theory and reverse bosonization of the scattered fermions

$$S_{TT} = \iota_{f\to b}\circ S_{FF}\circ \iota_{b\to f}. \qquad \text{(III.87)}$$

The fermionic S-matrix S_{FF} was explicitly calculated from the properties of the parabolic cylinder functions [97]. We do not give more details since these results will be reproduced in much simpler way from the formalism which we develop in the next chapters.

We restrict ourselves to two remarks. The first one is that in those cases where the scattering amplitudes in 2D string theory can be calculated by the CFT methods, the results coincided with the corresponding calculations in MQM [99]. The only thing to be done to ensure the complete agreement is a local redefinition of the vertex operators. It turns out that the exact relation between the tachyon operators (I.44) and their matrix model realization (III.81) include the so called *leg-factors* [100]

$$V_k^+ = \frac{\Gamma(-ik)}{\Gamma(ik)} T_{-ik}^+, \qquad V_k^- = \frac{\Gamma(ik)}{\Gamma(-ik)} T_{-ik}^-. \tag{III.88}$$

This redefinition is not surprising because the matrix model gives only a discrete approximation to the local vertex operators and in the continuum limit the operators can be renormalized. Therefore, one should expect the appearance of such leg-factors in any matrix/string correspondence. Note that the Minkowskian leg-factors (III.88) are pure phases. Thus, they represent a unitary transformation and do not affect the amplitudes. However, they are relevant for the correct spacetime physics, in particular, for the gravitational scattering of tachyons [95]. In fact, the leg-factors can be associated with a field redefinition given by the integral transformation (III.78). Written in the momentum space for q, it gives rise to additional factors for the left and right components whose ratio produces the leg-factor.

The second remark is that in the case when the Euclidean momenta of the incoming and outgoing tachyons belong to an equally spaced lattice (as in a compactified theory), the generating functional for S-matrix elements has been shown to coincide with a τ-function of Toda hierarchy [98]. However, this fact has not been used to address other problems like scattering in presence of a tachyon condensate. We will show that with some additional information added, it allows to solve many interesting questions related to 2D string theory in non-trivial backgrounds.

3.4 Discrete states and chiral ring

Finally, we show how the discrete states of 2D string theory appear in MQM. They are created by a natural generalization of the matrix operators (III.81) [101]

$$T_{n,\bar{n}} = e^{(\bar{n}-n)t} \operatorname{tr} \left((M+P)^n (M-P)^{\bar{n}}\right) \tag{III.89}$$

which have the following Euclidean momenta

$$p_X = i(n-\bar{n}), \qquad p_\phi = n+\bar{n}-2. \tag{III.90}$$

Comparing with the momenta of the discrete states (I.49), one concludes that

$$m = \frac{n-\bar{n}}{2}, \qquad j = \frac{n+\bar{n}}{2}. \tag{III.91}$$

Taking into account that n and $\bar{n}$ are integers, one finds that the so defined pair (j,m) spans all discrete states.

It is remarkable that the collective field theory approach allows to unveil the presence of a large symmetry group [101, 90, 25]. Indeed, the operators (III.89) are realized as

$$T_{j,m} = e^{-2mt} \int \frac{dx}{2\pi} \int\limits_{p_-}^{p_+} dp\, (x+p)^{j+m} (x-p)^{j-m}, \tag{III.92}$$

where we changed indices of the operator from $n, \bar{n}$ to j, m. One can check that they obey the commutation relations of w_∞ algebra

$$\{T_{j_1,m_1}, T_{j_2,m_2}\} = 4i\,(j_1 m_2 - j_2 m_1)\, T_{j_1+j_2-1,m_1+m_2}. \tag{III.93}$$

In particular, the operators $T_{j,m}$ are eigenstates of the Hamiltonian $H = -\frac{1}{2}T_{1,0}$

$$\{H, T_{j,m}\} = -2imT_{j,m} \tag{III.94}$$

what means that they generate its spectrum. When we replace the Poisson brackets of the classical collective field theory by the quantum commutators, the w_∞ algebra is promoted to a $W_{1+\infty}$ algebra.

Each generator (III.89) gives rise to an element of the ground ring of the $c = 1$ CFT [25] which plays an important role in many physical problems. First, we introduce the so called *chiral ground ring*. It consists of chiral ghost number zero, conformal spin zero operators $\mathcal{O}_{JM}$ which are closed under the operator product $\mathcal{O} \cdot \mathcal{O}' \sim \mathcal{O}''$ up to BRST commutators. The entire chiral ring can be generated from the basic operators

$$\begin{aligned} \mathcal{O}_{0,0} &= 1, \\ y \stackrel{\text{def}}{=} \mathcal{O}_{\frac{1}{2},\frac{1}{2}} &= (cb + i\partial X - \partial\phi)\, e^{iX+\phi}, \\ w \stackrel{\text{def}}{=} \mathcal{O}_{\frac{1}{2},-\frac{1}{2}} &= (cb - i\partial X - \partial\phi)\, e^{-iX+\phi}. \end{aligned} \tag{III.95}$$

The *ground ring* is constructed from products of the chiral and antichiral operators. We consider the case of the theory compactified at the self-dual radius $R = 1$ in the absence of the cosmological constant μ. Then the ground ring contains the following operators

$$\mathcal{V}_{j,m,\bar{m}} = \mathcal{O}_{j,m}\bar{\mathcal{O}}_{j,\bar{m}}. \tag{III.96}$$

The ring has four generators

$$a_1 = y\bar{y}, \quad a_2 = w\bar{w}, \quad a_3 = y\bar{w}, \quad a_4 = w\bar{y}. \tag{III.97}$$

These generators obey one obvious relation which determines the ground ring of the $c = 1$ theory

$$a_1 a_2 - a_3 a_4 = 0. \tag{III.98}$$

It has been shown [25] that the symmetry algebra mentioned above is realized on this ground ring as the algebra of diffeomorphisms of the three dimensional cone (III.98) preserving the volume form

$$\Theta = \frac{da_1 da_2 da_3}{a_3}. \tag{III.99}$$

Furthermore, it was argued that the inclusion of perturbations by marginal operators deforms the ground ring to

$$a_1 a_2 - a_3 a_4 = M(a_1, a_2), \tag{III.100}$$

where M is an arbitrary function. In particular, to introduce the cosmological constant, one should take M to be constant. As a result, one removes the conic singularity and obtains a smooth manifold

$$a_1 a_2 - a_3 a_4 = \mu. \tag{III.101}$$

In the limit of uncompactified theory only the generators a_1 and a_2 survive. The symmetry of volume preserving diffeomorphisms is reduced to some abelian transformations plus area preserving diffeomorphisms of the plane (a_1, a_2) that leave fixed the curve

$$a_1 a_2 = \mu \tag{III.102}$$

or its deformation according to (III.100). The suggestion was to identify the plane (a_1, a_2) with the eigenvalue phase space of MQM. Namely, one has the following relations

$$a_1 = x + p, \qquad a_2 = x - p. \tag{III.103}$$

Then it is clear that equation (III.102) corresponds to the Fermi surface of MQM. The mentioned abelian transformations are associated with time translations. And the operators (III.89) are identified with $a_1^n a_2^{\bar{n}}$.

The sense of the operators a_3 and a_4 existing in the compactified theory is also known. They correspond to the winding modes of strings which we are going to consider in the next section. The relation (III.101) satisfied by these operators is very important because it shows how to describe the theory containing both the tachyon and winding modes. However, it has not been yet understood how to obtain this relation directly from MQM.

4 Compact target space and winding modes in MQM

In the previous section we showed that the modes of 2D string theory in the linear dilaton background are described by the collective excitations of the singlet sector of Matrix Quantum Mechanics. If we compactify the target space of string theory, then there appear additional states — windings of strings around the compactified dimension. In this section we demonstrate how to describe these modes in the matrix language.

4.1 Circle embedding and duality

In matrix models the role of the target space of non-critical string theory is played by the parameter space of matrices. Therefore, to describe 2D string theory with one compactified dimension one should consider MQM on a circle. The partition function of such matrix model is given by the integral (III.1) where the integration is over a finite Euclidean time interval $t \in [0, \beta]$ with the two ends identified. The length of the interval is $\beta = 2\pi R$ where R is the radius of the circle. The identification $t \sim t + \beta$ requires to impose the periodic boundary condition $M(0) = M(\beta)$. Thus, one gets

$$Z_N(R,g) = \int\limits_{M(0)=M(\beta)} \mathcal{D}M(t) \exp\left[-N \,\mathrm{tr} \int\limits_0^\beta dt \left(\frac{1}{2}\dot{M}^2 + V(M)\right)\right]. \tag{III.104}$$

As usual, one can write the Feynman expansion of this integral. It is the same as in (III.3) with the only difference that the propagator should be replaced by the periodic one

$$G(t) = \sum_{m=-\infty}^{\infty} e^{-|t+m\beta|}. \tag{III.105}$$

We see that for large β the term $m = 0$ dominates and we return to the uncompactified case. But for finite β the sum should be retained and this leads to important phenomena related with the appearance of vortices on the discretized world sheet [102, 103, 104, 105].

We mentioned in section I.5.3 that the $c = 1$ string theory compactified at radius R is T-dual to the same theory at radius $1/R$. Does this duality appear in the sum over discretized surfaces regularizing the sum over continuous geometries? To answer this question, we perform the duality transformation of the Feynman expansion of the matrix integral (III.104)

$$F = \sum_{g=0}^{\infty} N^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \Gamma_g}} \lambda^V \prod_{i=1}^{V} \int\limits_0^\beta dt_i \prod_{\langle ij \rangle} \sum_{m_{ij}=-\infty}^{\infty} e^{-|t_i - t_j + \beta m_{ij}|}, \tag{III.106}$$

where we have chosen for simplicity the cubic potential (II.15). The duality transformation is obtained applying the Poisson formula to the propagator (III.105)

$$G(t_i - t_j) = \frac{1}{\beta} \sum_{k_{ij}=-\infty}^{\infty} e^{i\frac{2\pi}{\beta}k_{ij}(t_i-t_j)} \tilde{G}(k_{ij}) = \frac{1}{\beta} \sum_{k_{ij}=-\infty}^{\infty} e^{i\frac{2\pi}{\beta}k_{ij}(t_i-t_j)} \frac{2}{1+\left(\frac{2\pi}{\beta}k_{ij}\right)^2}. \tag{III.107}$$

The substitution of (III.107) into (III.106) allows to integrate over t_i which gives the momentum conservation constraint at each vertex

$$k_{ij_1} + k_{ij_2} + k_{ij_3} = 0. \tag{III.108}$$

This reduces the number of independent variables from E to $E - V + 1$. (One additional degree of freedom appears due to the zero mode which is canceled in all $t_i - t_j$.) By virtue of the Euler theorem this equals to $L - 1 + 2g$. According to this, we attach a momentum p_I to each elementary loop (face) of the graph (one of p_I is fixed) and define remaining $2g$ variables as momenta l_a running along independent non-contractable loops. Thus, one arrives at the following representation

$$F = \sum_{g=0}^{\infty} \left(\frac{N\beta}{\lambda^2}\right)^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \tilde{\Gamma}}} \left(\frac{\lambda^2}{\beta}\right)^L \left(\prod_{I=1}^{L-1} \sum_{p_I=-\infty}^{\infty}\right) \left(\prod_{a=1}^{2g} \sum_{l_a=-\infty}^{\infty}\right) \prod_{\langle IJ \rangle} \tilde{G}\left(p_I - p_J + \sum_{a=1}^{2g} l_a \epsilon_{IJ}^a\right), \tag{III.109}$$

where the sum goes over the dual graphs (triangulations) with L dual vertices and we introduced the matrix ϵ_{IJ}^a equal ± 1 when a dual edge $\langle IJ \rangle$ crosses an edge belonging to ath non-contractable cycle (the sign depends on the mutual orientation) and zero otherwise.

The transformation (III.107) changes $R \to 1/R$ which is seen from the form of the propagators. But the result (III.109) does not seem to be dual to the original representation (III.106). Actually, at the spherical level, instead of describing a compact target space of the inverse radius, it corresponds to the embedding into the discretized real line with lattice spacing $1/R$ [26]. This is natural because the variables p_I live in the momentum space of the initial theory which is discrete.

Thus, even in the continuum limit, the sum over discretized surfaces embedded in a circle cannot be identical to its continuum analog. The reason is that it possesses additional degrees of freedom which are ignored in the naive continuum limit. These are the vortex configurations. Indeed, in the continuum geometry the simplest vortex of winding number n is described by the field $X(\theta) = nR\theta$ where θ is the azimuth angle. However, this is a singular configuration and should be disregarded. In contrast, on a lattice the singularity is absent and such configurations are included into the statistical sum. For example, in the notations of (III.106) the number of vortices associated with a face I is given by

$$w_I = \sum_{\langle ij \rangle \in I} m_{ij}. \tag{III.110}$$

It is clear that it coincides with the number of times the string is wrapped around the circle. In other words, the vortices are world sheet realizations of windings in the target space. Thus, MQM with compactified time intrinsically contains winding string configurations. But just due to this fact, it fails to reproduce the partition function of compactified 2D string theory.

It is clear that to obtain the sum over continuous surfaces possessing selfduality one should somehow exclude the vortices. This can be done restricting the sum over m_{ij} in (III.106). The distribution m_{ij} can be seen as an abelian gauge field defined on links of a graph. Then the quantity (III.110) is its field strength. We want that this strength vanishes. With this condition only the "pure gauge" configurations of m_{ij} are admissible. They are represented as

$$m_{ij} = m_i - m_j + \sum_{a=1}^{2g} \tilde{\epsilon}_{ij}^a \tilde{l}_a, \tag{III.111}$$

where integers $\tilde{l}_a$ are associated with non-contractable loops of the dual graph and $\tilde{\epsilon}_{ij}^a$ is the matrix dual to ϵ_{IJ}^a. If we change the sum over all m_{ij} by the sum over these pure gauge configurations, the free energy (III.106) is rewritten as follows

$$\tilde{F} = \beta \sum_{g=0}^{\infty} N^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \Gamma_g}} \lambda^V \prod_{i=1}^{V-1} \int_{-\infty}^{\infty} dt_i \left(\prod_{a=1}^{2g} \sum_{\tilde{l}_a=-\infty}^{\infty} \right) \prod_{\langle ij \rangle} \exp \left[-|t_i - t_j + \beta \sum_{a=1}^{2g} \tilde{\epsilon}_{ij}^a \tilde{l}_a| \right], \tag{III.112}$$

where the overall factor β arises from the integration over the zero mode. The sum over m_i resulted in the extension of the integrals over t_i to the whole line. Due to this the dual transformation gives rise to integrals over momenta rather than discrete sums. Repeating the steps which led to (III.109) and renormalizing the momenta, one obtains

$$\tilde{F} = \frac{2\pi}{R} \sum_{g=0}^{\infty} \left(\frac{N\beta}{\lambda^2} \right)^{2-2g} \sum_{\substack{\text{connected} \\ \text{diagrams } \tilde{\Gamma}}} \left(\frac{\lambda^2}{4\pi^2} \right)^L \left(\prod_{I=1}^{L-1} \int_{-\infty}^{\infty} dp_I \right) \times$$
$$\times \left(\prod_{a=1}^{2g} \sum_{l_a=-\infty}^{\infty} \right) \prod_{\langle IJ \rangle} \frac{2}{1 + \frac{1}{4\pi^2} \left(p_I - p_J + \frac{2\pi}{R} \sum_{a=1}^{2g} l_a \epsilon_{IJ}^a \right)^2}. \tag{III.113}$$

The only essential difference between two representation (III.112) and (III.113) is the propagator. However, the universality of the continuum limit implies that the results in the macroscopic scale do not depend on it. Moreover, if we choose the Gaussian propagator, which follows from the usual Polyakov action, its Fourier transform coincides with the original one. Due to this one can neglect this discrepancy. Then the two representations are dual to each other with the following matching of the arguments

$$R \to 1/R, \qquad N \to RN. \tag{III.114}$$

Thus, the exclusion of vortices allowed to make the sum over discretized surfaces selfdual and they are those degrees of freedom that are responsible for the breaking of this duality in the full matrix integral.

We succeeded to identify and eliminate the vortices in the sum over discretized surfaces. How can this be done directly in the compactified Matrix Quantum Mechanics defined by the integral (III.104)? In other words, what matrix degrees of freedom describe the vortices? Let us see how MQM in the Hamiltonian formulation changes after compactification. The partition function is represented by the trace in the Hilbert space of the theory of the evolution operator as in (III.16)

$$Z_N(R) = \operatorname{Tr} e^{-\frac{\beta}{\hbar} \hat{H}_{\rm MQM}}. \tag{III.115}$$

Now the time interval coincides with β which can also be considered as the inverse temperature. It is finite so that one should consider the finite temperature partition function. This fact drastically complicates the problem because one should take into account the contributions of all states and not only the ground state. In particular, all representations of the SU(N) global symmetry group come to the game. The states associated with these representations are those additional states which appear in the compactified theory. Therefore, it is natural to expect that they correspond to vortices on the discretized world sheet or windings of the string [105, 106].

The first check of this expectation which can be done is to verify the duality (III.114). Since we expect vortices only in the non-singlet representations of SU(N), to exclude them one should restrict oneself to the singlet sector as in the uncompactified theory. Then we still have a powerful description in terms of free fermions. In the double scaling limit, we are interested in, the fermions move in the inverse oscillator potential. Thus we return to the problem stated by equation (III.37). However, the presence of a finite temperature gives rise to a big difference with the previous situation. Due to the thermal fluctuations, the Fermi surface cannot be defined in the ensemble with fixed number of particles N. The solution is to pass to the grand canonical ensemble with the following partition function

$$\mathcal{Z}(\mu, R) = \sum_{N=0}^{\infty} e^{-\beta\mu N} Z_N(R). \tag{III.116}$$

The chemical potential μ is exactly the Fermi level which is considered now as the basic variable while N becomes an operator. The grand canonical free energy $\mathcal{F} = \log \mathcal{Z}$ can be expressed through the density of states. In the singlet sector the relation reads as follows

$$\mathcal{F}^{(\mathrm{sing})} = \int_{-\infty}^{\infty} d\epsilon\, \rho(\epsilon) \log\left(1 + e^{-\beta(\epsilon+\mu)}\right), \tag{III.117}$$

where the energy ϵ is rescaled by the Planck constant to be of the same order as μ. The compactification does not affect the density which is therefore given by equation (III.41). There is a nice integral representation of this formula

$$\rho(\epsilon) = \frac{1}{2\pi} \mathrm{Re} \int_{\Lambda^{-1}}^{\infty} d\tau\, \frac{e^{i\epsilon\tau}}{2\sinh\frac{\tau}{2}} \Rightarrow \frac{\partial\rho(\epsilon)}{\partial\epsilon} = -\frac{1}{2\pi} \mathrm{Im} \int_0^{\infty} d\tau\, e^{i\epsilon\tau}\, \frac{\tau/2}{\sinh\tau/2}. \tag{III.118}$$

Taking the first derivative makes the integral well defined and allows to remove the cut-off. By the same reason, let us consider the third derivative of the free energy (III.117) with respect to μ. It can be written as

$$\frac{\partial^3\mathcal{F}^{(\mathrm{sing})}}{\partial\mu^3} = -\beta \int_{-\infty}^{\infty} d\epsilon\, \frac{\partial^2\rho}{\partial\epsilon^2} \frac{1}{1+e^{\beta(\epsilon+\mu)}}. \tag{III.119}$$

Then the substitution of (III.118) and taking the integral over ϵ by residues closing the contour in the upper half plane gives [105]

$$\frac{\partial^3\mathcal{F}^{(\mathrm{sing})}}{\partial\mu^3} = \frac{\beta}{2\pi} \mathrm{Im} \int_0^{\infty} d\tau\, e^{-i\mu\tau} \frac{\frac{\tau}{2}}{\sinh\frac{\tau}{2}}\, \frac{\frac{\pi\tau}{\beta}}{\sinh\frac{\pi\tau}{\beta}}. \tag{III.120}$$

This representation possesses the explicit duality symmetry

$$R \to 1/R, \qquad \mu \to R\mu, \tag{III.121}$$

where one should take into account that $\partial_\mu^3 \mathcal{F}^{(\rm sing)} \to R^{-3}\partial_\mu^3 \mathcal{F}^{(\rm sing)}$. Thus, the grand canonical free energy of the singlet sector of MQM compactified on a circle is indeed selfdual. This fact can be verified also from the expansion of the free energy in $1/\mu$. The result reads [105]

$$\mathcal{F}^{(\rm sing)}(\mu, R) = -\frac{R}{2}\mu^2 \log \mu - \frac{1}{24}\left(R + \frac{1}{R}\right)\log \mu + \sum_{n=1}^{\infty} f_{n+1}^{(\rm sing)}(R)(\mu\sqrt{R})^{-2n}, \tag{III.122}$$

where the coefficients are selfdual finite series in R

$$f_n^{(\rm sing)}(R) = \frac{2^{-2n-1}(2n-2)!}{n-1}\sum_{k=0}^{n} |2^{2k}-2||2^{2(n-k)}-2|\frac{|B_{2k}||B_{2(n-k)}|}{(2k)![2(n-k)]!}R^{n-2k}. \tag{III.123}$$

To return to the canonical ensemble, one should take the Laplace transform of $\mathcal{F}(\mu)$. But already analyzing equation (III.116), one can conclude that the canonical free energy is also selfdual. This is because $\beta N\mu$ is selfdual under the simultaneous change of R, μ and N according to (III.121) and (III.114). The same result can be obtained by the direct calculation which shows that the expansion of the canonical free energy in $\frac{1}{\Delta}\log\Delta$ differs from the expansion of the grand canonical one in $1/\mu$ only by the sign of the first term [105, 26]. These results confirm the expectation that the singlet sector of MQM does not contain vortices and that the latters are described by higher SU(N) representations.

4.2 MQM in arbitrary representation: Hamiltonian analysis

We succeeded to describe the partition function of the compactified MQM in the singlet sector which does not contain winding excitations of the corresponding string theory. Also it is possible to calculate correlation functions of the tachyon modes in this case [107]. However, if we want to understand the dynamics of windings, we should study MQM in the non-trivial SU(N) representations [108].

The dynamics in the sector of the Hilbert space corresponding to an irreducible representation r is described by the projection of the Hamiltonian (III.14) on this subspace. As in the case of the singlet representation, it is convenient to redefine the wave function, which is now represented as a matrix, by the Vandermonde determinant

$$\Psi_{ab}^{(r)}(x) = \Delta(x)\Phi_{ab}^{(r)}(x). \tag{III.124}$$

Then the action of the Hamiltonian $\hat{H}_{\rm MQM}$ on the wave functions $\Psi_{ab}^{(r)}$ is given by the following matrix-differential operator

$$\hat{H}_{ab}^{(r)} = \sum_{c,d=1}^{d_r} P_{ac}^{(r)}\left[\delta_{cd}\sum_{i=1}^{N}\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial {x_i}^2} + V(x_i)\right) + \frac{\hbar^2}{4}\sum_{i\ne j}\frac{\left(Q_{ij}^{(r)}\right)_{cd}}{(x_i-x_j)^2}\right]P_{db}^{(r)}, \tag{III.125}$$

where

$$Q_{ij}^{(r)} \equiv \tau_{ij}^{(r)}\tau_{ji}^{(r)} + \tau_{ji}^{(r)}\tau_{ij}^{(r)} \tag{III.126}$$

and we introduced the representation matrix of $u(N)$ generators $\tau_{ij} \in u(N)$: $\left(\tau^{(r)}_{ij}\right)_{ab} = D^R_{ab}(\tau_{ij})$, which satisfies

$$[\tau_{ij}, \tau_{kl}] = \delta_{jk}\tau_{il} - \delta_{il}\tau_{kj}. \tag{III.127}$$

$P^{(r)}$ is the projector to the subspace consisting of the wave functions that satisfy

$$\left(\tau^{(r)}_{mm}\Psi^{(r)}\right)_{ab} = \sum_{c=1}^{d_r}\left(\tau^{(r)}_{mm}\right)_{ac}\Psi^{(r)}_{cb} = 0, \quad m = 1,\dots,N. \tag{III.128}$$

The explicit form of $P^{(r)}$ is given by

$$P^{(r)} = \int\limits_0^{2\pi}\prod_{m=1}^{N}\frac{d\theta_m}{2\pi}\, e^{i\sum\limits_{m=1}^{N}\theta_m\tau^{(r)}_{mm}}. \tag{III.129}$$

The Hilbert structure is induced by the scalar product (III.11) which has the following decomposition

$$\langle\Psi|\Psi'\rangle = \sum_r\frac{1}{d_r}\int\prod_{i=1}^{N}dx_i\sum_{a,b=1}^{d_r}\overline{\Psi^{(r)}_{ab}(x)}\Psi'^{(r)}_{ab}(x). \tag{III.130}$$

This shows that the full Hilbert space is indeed a direct sum of the Hilbert spaces corresponding to irreducible representations of SU(N). In the each subspace the scalar product is given by the corresponding term in the sum (III.130).

Note that in the Schrödinger equation

$$i\hbar\frac{\partial\Psi^{(r)}_{ab}}{\partial t} = \sum_{c=1}^{d_r}\hat{H}^{(r)}_{ac}\Psi^{(r)}_{cb}. \tag{III.131}$$

the last index b is totally free and thus can be neglected. In other words, one should consider the eigenvalue problem given by the equation (III.131) with the constraint (III.128) where the matrix $\Psi^{(r)}_{ab}$ is replaced by the vector $\Psi^{(r)}_a$ and each solution is degenerate with multiplicity d_r.

Example: adjoint representation

Let us consider how the above construction works on the simplest non-trivial example of the adjoint representation. The representation space in this case is spanned by $|\tau_{ij}\rangle$. The $u(N)$ generators act on these states as

$$\tau_{ij}|\tau_{mn}\rangle \equiv |[\tau_{ij}, \tau_{mn}]\rangle = \delta_{jm}|\tau_{in}\rangle - \delta_{in}|\tau_{mj}\rangle \tag{III.132}$$

what means that their representation matrices are given by

$$\left(\tau^{(\mathrm{adj})}_{ij}\right)_{kl,mn} = \delta_{ik}\delta_{jm}\delta_{ln} - \delta_{in}\delta_{jl}\delta_{km}. \tag{III.133}$$

The operator $Q^{(\mathrm{adj})}_{ij}$ and the projector $P^{(\mathrm{adj})}$ are found to be

$$\left(Q^{(\mathrm{adj})}_{ij}\right)_{kl,mn} = (\delta_{ik}+\delta_{jl})\delta_{km}\delta_{ln} - (\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk})\delta_{kl}\delta_{mn} + (i\leftrightarrow j), \tag{III.134}$$

$$P^{(\mathrm{adj})}_{kl,mn} = \delta_{kl}\delta_{nk}\delta_{ln}. \tag{III.135}$$

The projector (III.135) leads to that only the diagonal components of the adjoint wave function survive

$$\Psi_{kl}^{(\mathrm{adj})} = \sum_{m,n=1}^{N} P_{kl,mn}^{(\mathrm{adj})} \Psi_{mn}^{(\mathrm{adj})} = \delta_{kl} \Psi_{kk}^{(\mathrm{adj})}. \tag{III.136}$$

Due to this it is natural to introduce the functions $\psi_k^{(\mathrm{adj})} = \Psi_{kk}^{(\mathrm{adj})}$ on which the operator $Q_{ij}^{(\mathrm{adj})}$ simplifies further

$$\left(Q_{ij}^{(\mathrm{adj})}\right)_{kk,mm} = 2\left[(\delta_{ik}+\delta_{jk})\delta_{km} - (\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk})\right]. \tag{III.137}$$

As a result, one obtains a set of N coupled equations on $\psi_k^{(\mathrm{adj})}(x,t)$

$$\left\{ i\frac{\partial}{\partial t} + \sum_{i=1}^{N}\left(\frac{1}{2N}\frac{\partial^2}{\partial {x_i}^2} - NV(x_i)\right)\right\}\psi_k^{(\mathrm{adj})}(x,t) - \frac{1}{N}\sum_{l(\neq k)} \frac{\psi_k^{(\mathrm{adj})} - \psi_l^{(\mathrm{adj})}}{(x_k-x_l)^2} = 0. \tag{III.138}$$

This shows that instead by a system of free fermions the adjoint representation is described by an interacting system and we lose the integrability that allows to solve exactly the singlet case.

4.3 MQM in arbitrary representation: partition function

Since the full Hilbert space is decomposed into the direct sum, the partition function (III.115) of the compactified MQM can be represented as the sum of contributions from different representations

$$Z_N(R) = \sum_r d_r Z_N^{(r)}(R) = \sum_r d_r \,\mathrm{Tr}_{(r)}\, e^{-\frac{\beta}{\hbar}\hat{H}^{(r)}}, \tag{III.139}$$

where the Hamiltonian is defined in (III.125) and the trace is over the subspace of the rth irreducible representation. An approach to calculate the partition functions $Z_N^{(r)}$ was developed in [108]. It is based on the introduction of a new object, the so called *twisted partition function*. It is obtained by rotating the final state by a unitary transformation with respect to the initial one

$$Z_N(\Omega) = \mathrm{Tr}\left(e^{-\frac{\beta}{\hbar}\hat{H}_{\mathrm{MQM}}}\hat{\Theta}(\Omega)\right), \tag{III.140}$$

where $\hat{\Theta}(\Omega)$ is the rotation operator. The partition functions in a given SU(N) representation can be obtained by projecting the twisted partition function with help of the corresponding character $\chi^{(r)}(\Omega)$

$$Z_N^{(r)} = \int [d\Omega]_{SU(N)} \chi^{(r)}(\Omega) Z_N(\Omega). \tag{III.141}$$

The characters for different representations are orthogonal to each other

$$\int [d\Omega]_{SU(N)} \chi^{(r_1)}(\Omega^\dagger)\chi^{(r_2)}(\Omega\cdot U) = \delta_{r_1,r_2}\chi^{(r_1)}(U) \tag{III.142}$$

and are given by the Weyl formula

$$\chi^{(r)}(\Omega) \stackrel{\mathrm{def}}{=} \mathrm{tr}\left[D^{(r)}(\Omega)\right] = \frac{\det\limits_{i,j}\left(e^{il_i\theta_j}\right)}{\Delta\left(e^{i\theta}\right)}, \tag{III.143}$$

where $z_i = e^{i\theta_i}$ are eigenvalues of Ω, Δ is the Vandermonde determinant and the ordered set of integers $l_1 > l_2 > \cdots > l_N$ is defined in terms of the components of the highest weight $\{m_k\}$: $l_i = m_i + N - i$ of the given representation.

Thus, the twisted partition function plays the role of the generating functional for the set of partition functions $Z_N^{(r)}$. The characters for the unitary group are well known objects. Therefore, the main problem is to find $Z_N(\Omega)$. It was done in the double scaling limit where the potential $V(x)$ becomes the potential of the inverse oscillator. Then it can be related by analytical continuation to the usual harmonic oscillator potential where the twisted partition function can be trivially found. The derivation is especially simple when one uses the matrix Green function defined by the following initial value problem

$$\left\{\frac{\partial}{\partial\beta} - \frac{1}{2}\,\mathrm{tr}\left(\frac{\partial^2}{\partial M^2} - \omega^2 M^2\right)\right\} G(\beta, M, M') = 0, \quad G(0, M, M') = \delta^{(N^2)}(M - M'). \tag{III.144}$$

The solution is well known to be

$$G(\beta, M, M') = \left(\frac{\omega}{2\pi\sinh(\omega\beta)}\right)^{\frac{1}{2}N^2} \exp\left[-\frac{\omega}{2}\coth(\omega\beta)\,\mathrm{tr}\,(M^2 + M'^2) + \frac{\omega}{\sinh(\omega\beta)}\,\mathrm{tr}\,(MM')\right]. \tag{III.145}$$

It is clear that the twisted partition function is obtained from this Green function as follows

$$Z_N(\Omega) = \int dM\, G(\beta, M, \Omega^\dagger M \Omega). \tag{III.146}$$

One can easily perform the simple Gaussian integration over M and find the following result

$$Z_N(\Omega;\omega) = 2^{-\frac{1}{2}N^2}\left(2\sinh^2\frac{\omega\beta}{2}\right)^{-N/2} \prod_{i>j}\frac{1}{\cosh(\omega\beta) - \cos(\theta_i - \theta_j)}. \tag{III.147}$$

The answer for the inverse oscillator is obtained by the analytical continuation to the imaginary frequency $\omega \to i$. It can be also represented (up to $(-1)^{N/2}$) in the following form

$$Z_N(\Omega) = q^{\frac{1}{2}N^2}\prod_{i,j=1}^{N}\frac{1}{1 - q e^{i(\theta_i - \theta_j)}}. \tag{III.148}$$

where $q = e^{i\beta}$. Remarkably, the partition function (III.148) depends only on the eigenvalues of the twisting matrix Ω. Due to this, the integral (III.141) is rewritten as follows

$$Z_N^{(r)} = \frac{1}{N!}\int_0^{2\pi}\prod_{k=1}^{N}\frac{d\theta_k}{2\pi}\left|\Delta(e^{i\theta})\right|^2 \chi^{(r)}(e^{i\theta}) Z_N(\theta). \tag{III.149}$$

In fact, it is not evident that the analytical continuation of the results obtained for the usual oscillator gives the correct answers for the inverse oscillator. The latter is complicated by the necessity to introduce a cut-off since otherwise it would represent an unstable system. Because of that the analytical continuation should be performed in a way that avoids the problems related with arising divergences. In this respect, the presented derivation is not rigorous. However, the validity of the final result (III.148) was confirmed by the reasonable physical conclusions which were derived relying on it. Besides, in [108] an alternative derivation based on the density of states, which are eigenstates of the inverse oscillator Hamiltonian with the twisted boundary conditions, was presented. It led to the same formula as in (III.148).

4.4 Non-trivial SU(N) representations and windings

The technique developed in the previous paragraph allows to study the compactified MQM in the non-trivial representations in detail. In particular, it was shown [108] that the partition function associated with some representation of SU(N) corresponds to the sum over surfaces in the presence of pairs of vortices and anti-vortices of charge defined by this representation. In terms of 2D string theory this means that $\log Z_N^{(r)}$ gives the partition function of strings among which there are strings wrapped around the compactified dimension n times in one direction and the same number of strings wrapped n times in the opposite one.

This result can be established considering the diagrammatic expansion of $Z_N^{(r)}$. The expansion is found using a very important fact that the non-trivial representations are associated with correlators of matrix operators which are traces of matrices taken in different moments of time. For example, the two-point correlators describe the propagation of states belonging to the adjoint representation

$$\left\langle \mathrm{tr}\,\left(e^{\alpha_1 M(0)} e^{\alpha_2 M(\beta)}\right)\right\rangle = \sum_{i,j=1}^{N} \left\langle 0|e^{\alpha_1 x_i}\left(e^{-\frac{\beta}{\hbar}\hat{H}^{(\mathrm{adj})}}\right)_{ij} e^{\alpha_2 x_j}|0\right\rangle. \tag{III.150}$$

The diagrammatic expansion for the correlators is known and it gives an expansion for the partition functions $Z_N^{(r)}$ after a suitable identification of the legs of the Feynman graphs.

Another physical consequence of the previous analysis is that there is a large energy gap between the singlet and the adjoint representations. It was found to be [106, 108]

$$\delta = \mathcal{F}^{(\mathrm{adj})} - \mathcal{F}^{(\mathrm{sing})} \underset{\mu\to\infty}{\sim} -\frac{\beta}{2\pi}\log(\mu/\Lambda). \tag{III.151}$$

Due to this gap the contribution of few vortices to the partition function is negligible and they seem to be suppressed. However, the vortices have a large entropy related to the degeneracy factor d_r in the sum (III.139). For the adjoint representation it equals $d_{(\mathrm{adj})} = N^2 - 1$. Therefore, there is a competition between the two factors. It leads to the existence of a phase transition when the radius of compactification becomes sufficiently small [106]. Indeed, from (III.139) one finds

$$Z_N(R) \approx Z_N^{(\mathrm{sing})}(R)\left(1 + d_{(\mathrm{adj})}e^{-\delta} + \cdots\right) \approx \exp\left[F^{(\mathrm{sing})} + \mathrm{const}\cdot N^2\,(\mu/\Lambda)^R\right]. \tag{III.152}$$

Since $\Lambda \sim N$, we see that for large radii the second term in the exponent is very small and is irrelevant with respect to the first one. However, at $R_c = 2$ the situation changes and now the contribution of entropy dominates. Physically this means that the vortex-antivortex pairs become dynamically more preferable and populate densely the string world sheet. This effect is called the vortex condensation and the change of behaviour at R_c is known as the Berezinski–Kosterlitz–Thouless phase transition [102, 103]. For radii $R < R_c$ MQM does not describe anymore the $c = 1$ CFT. Instead it describes $c = 0$ theory corresponding to the pure two-dimensional gravity. This fact can be easily understood from the MQM point of view because at very small radii we expect the usual dimension reduction. The dimension reduction of MQM is the simple one-matrix model which is known to describe pure gravity as it was shown in section II.2.

Of course, this phase transition is seen also in the continuum formalism. There it is related to the fact whether the operator creating vortex-antivortex pairs is relevant or not. These operators were introduced in (I.53). The simplest such operator has the form

$$\int d^2\sigma \, e^{(R-2)\phi} \cos(R\tilde{X}). \tag{III.153}$$

It is relevant if it decreases in the asymptotics $\phi \to \infty$. This happens when $R < 2$ exactly as it was predicted from the matrix model. This gives one more evidence that we correctly identified the winding modes of string theory with the states of MQM arising in the non-trivial SU(N) representations.

* * *

We conclude that Matrix Quantum Mechanics successfully describes 2D string theory in the linear dilaton background. All excitations of 2D string theory were identified with appropriate degrees of freedom of MQM and all continuum results were reproduced by the matrix model technique. Moreover, MQM in the singlet sector represents an integrable system which allowed to exactly solve the corresponding (tachyon) sector of string theory.

Once the string physics in the linear dilaton background has been understood and solved, it is natural to turn our attention to other backgrounds. We have in our hands two main tools to obtain new backgrounds: to consider either a tachyon or winding condensation since their vertex operators are well known. But the most interesting problem is string theory in curved backgrounds. We know that 2D string theory does possess such a background which describes the two-dimensional dilatonic black hole. Therefore, we expect that the Matrix Quantum Mechanics is also able to describe it and may be to provide its exact solution.

Chapter IV

Winding perturbations of MQM

1 Introduction of winding modes

1.1 The role of the twisted partition function

In this chapter we consider one of the two possibilities to change the string background in Matrix Quantum Mechanics. We introduce non-perturbative sources of windings which perturb the theory and give rise to a winding condensation. From the previous chapter we know that the winding modes of string theory are described in MQM by the non-trivial SU(N) representations. However, the problem is that we do not have a control on them. Namely, we do not know how to introduce a portion of windings of charge 1, another portion of windings of charge 2, *etc.* In other words one should have an analog of the couplings $\tilde{t}_n$ which are associated with the perturbations by the vortex operators in the CFT (I.63). Such couplings would allow to construct a generating functional for all correlators of windings.

Actually, we already encountered one generating functional of quantities related to windings. This was the twisted partition function (III.140) which generated the partition functions of MQM in different representations. It turns out that this is the object we are looking for because it can be considered as the generating functional of vortex operators with couplings being the moments of the twisting matrix. In the following two sections we review the work [109] where this fact has been established and exploited to get a matrix model for 2D string theory on the black hole background.

By definition the twisted partition function describes MQM with twisted boundary condition. Therefore, it can be represented by the following matrix integral

$$Z_N(\Omega) = \int\limits_{M(\beta)=\Omega^\dagger M(0)\Omega} \mathcal{D}M(t) \exp\left[-N \operatorname{tr} \int\limits_0^\beta dt \left(\frac{1}{2}\dot{M}^2 + V(M)\right)\right]. \tag{IV.1}$$

Hence it has the usual representation as the sum over Feynman diagrams of the matrix model or as the sum over discretized two-dimensional surfaces embedded in one-dimensional space. Since the target space is compactified, we expect to obtain something like the expansion given in (III.106). However, the presence of the twisting matrix introduces new ingredients.

The only thing which can change is the propagator

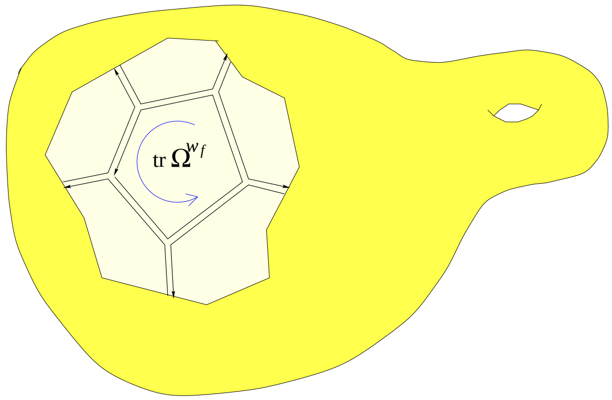


Fig. IV.1: A piece of a discretized world sheet. The twisted boundary condition associates with each loop of the dual graph a moment of the twisting matrix corresponding to the winding number of the loop.

$$G_{ij,kl}(t,t') = \quad t\, {}^{j}_{i} \; \overset{\longrightarrow}{\underset{\longleftarrow}{}} \; {}^{k}_{l}\, t' \; = \langle M_{ij}(t) M_{kl}(t') \rangle_0 .$$

As it is seen from its definition through the two-point correlator, it should satisfy the periodic boundary condition

$$G_{ij,kl}(t+\beta,t') = {\Omega^\dagger}_{ii'}\, G_{i'j',kl}(t,t')\, \Omega_{j'j}. \tag{IV.2}$$

Due to this, the propagator is given by a generalization of the simple periodic solution (III.105). In contrast to previous cases, its index structure is not anymore described by the Kronecker symbols but by the twisting matrix

$$G_{ij,kl}(t,t') = \frac{1}{N} \sum_{m=-\infty}^{\infty} (\Omega^m)_{il} (\Omega^{-m})_{jk} e^{-|t-t'+m\beta|}. \tag{IV.3}$$

As a result, the contraction of indices along each loop in a given graph gives the factor $\mathrm{tr}\,\Omega^{w_I}$ where the integer field w_I was defined in (III.110). It is equal to the winding number of the loop around the target space circle (see fig. IV.1). We will call this field *vorticity*.

Thus, the only effect of the introduction of the twisted boundary condition is that the factors N, which were earlier associated with each loop, are replaced by the factors $\mathrm{tr}\,\Omega^{w_I}$. The rest remains the same as in (III.106). In particular, one has the sum over distributions m_{ij}. This sum can be splitted into the sum over vorticities w_I and the sum over the "pure gauge" configurations (III.111). The latter can be removed at the cost of extending the integrals over time to the whole real axis. We conclude that each term in the resulting sum is characterized by a graph with particular distributions of times t_i at vertices and vorticity at faces (loops) w_I. It enters the sum with the following coefficient

$$N^{2-2g} \lambda^V \prod_{I=1}^{L} \frac{tr\Omega^{w_I}}{N} \prod_{\langle ij \rangle} e^{-|t_i - t_j + \beta m_{ij}|}, \tag{IV.4}$$

As usual, in the double scaling limit the sum over discretizations becomes the sum over continuous geometries and the twisted partition function (IV.1) can be interpreted as the partition function of 2D string theory including the sum over vortex insertions. The vortices of charge m are coupled to the mth moment of the twisting matrix so that the moments control the probability to find vortices of a given vorticity.

To use this interpretation to extract some information concerning a particular vortex configuration (for instance, to study the dependence of the theory perturbed by vortices of charge 1 of the coupling constant), one should express the twisted partition function as a function of moments $s_m = \operatorname{tr}\Omega^m$. However, it turns out to be a quite difficult problem. Therefore, one should find an alternative way to describe the perturbed system.

1.2 Vortex couplings in MQM

The main problem with the twisted partition function is that its natural argument is a matrix. Moreover, in the large N limit its size goes to infinity. (Although $Z_N(\Omega)$ depends actually only on the eigenvalues, as it was shown in section III.4.3, this does not help much.) On the other hand, usually one integrates over matrices of a large size. For example, the partition functions of MQM in different representations were represented as integrals over the twisting matrix (III.141). The measure of the integration was given by the characters of irreducible representations. But the characters are not related to the coupling constants of vortices in an explicit way.

We can generalize this construction and integrate the twisted partition function with an arbitrary measure. Then the vortex coupling constants will be associated with some parameters of the measure. Thus, the problem can be reformulated as follows: what choice of the measure gives the most convenient parameterization of the generating functional of vortices?

The answer is as simple as it can be. Indeed, as usual, we require from the measure the invariance under the unitary transformations

$$\Omega \to U^\dagger \Omega U \quad (U^\dagger U = I). \tag{IV.5}$$

Then, as in the one-matrix model, the most natural choice of the measure is given by the exponential of a potential. Since the matrix is unitary, in contrast to 1MM, now both positive and negative powers of the twisting matrix are allowed. Thus, we define the following functional [109]

$$Z_N[\lambda] = \int [d\Omega]_{SU(N)} \exp\left(\sum_{n\neq 0} \lambda_n \operatorname{tr}\Omega^n\right) Z_N(\Omega). \tag{IV.6}$$

Note that the parameters λ_n are coupled exactly to the moments s_n of the twisting matrix playing the role of fugacities of vortices. Therefore, the functional (IV.6) is nothing else but the Legendre transform of the twisted partition function considered as a function of the moments.

This statement can be formulated more rigorously with help of the following identity

$$\int [d\Omega]_{SU(N)} \exp\left(\sum_{n\neq 0} \lambda_n \operatorname{tr}\Omega^n\right) = \exp\left(\sum_{n>0} n\lambda_n\lambda_{-n}\right), \tag{IV.7}$$

which is valid up to non-perturbative terms $O(e^{-N})$ provided the couplings do not grow linearly in N. This property of integrals over the unitary groups shows that in the large N

limit the moments s_n can be considered as independent variables and the measure $[d\Omega]_{SU(N)}$ is expressed in a very simple way through them. As a result, the generating function (IV.6) is written as

$$Z_N[\lambda] = \int\limits_{-\infty}^{\infty} \prod_{n\neq 0} \frac{ds_n}{\sqrt{\pi}}\, e^{\sum\limits_{n\neq 0}\left(\lambda_n s_n - \frac{1}{2|n|} s_n s_{-n}\right)} Z_N[s]. \tag{IV.8}$$

The relation (IV.7) is a generating equation for integrals of products of moments. Among them the following relation is most important for us

$$\int [d\Omega]_{SU(N)} \operatorname{tr}\Omega^n \ \operatorname{tr}\Omega^m = |n|\delta_{n+m,0}. \tag{IV.9}$$

It helps to elucidate the sense of the couplings λ_n which is hidden in the diagrammatic representation of $Z_N[\lambda]$. This representation follows from the expansion of the twisted partition function if to perform the integration over Ω. From (IV.9) one concludes that there will be three kinds of contributions. The first one is a trivial factor given by the r.h.s of (IV.7). It comes from the coupling of two moments from the measure in (IV.6). The second contribution arises when a moment from the measure is coupled with the factor $\frac{1}{N}\operatorname{tr}\Omega^{w_I}$ in (IV.4) associated with a vortex of vorticity w_I. It results in substitution of the coupling λ_{w_I} in place of the trace of the twisting matrix. Thus, whenever it appears, λ_n is always associated with a vortex of winding number n. Hence, it plays the role of the coupling constant of the operator creating the vortices. Finally, there is the third contribution related with the coupling of two moments from (IV.4). However, it was argued that it vanishes in the double scaling limit [109].

To summarize, the double scaling limit of the free energy of (IV.6) coincides with the partition function of the $c = 1$ theory perturbed by vortex operators with the coupling constants proportional to λ_n. Thus, we obtain a matrix model realization of the CFT (I.63) with $t_n = 0$. Since the couplings λ_n are explicitly introduced from the very beginning, it is much easier to work with $Z_N[\lambda]$ than with the twisted partition function. Moreover, it turns out that in terms of λ's the system becomes integrable.

1.3 The partition function as τ-function of Toda hierarchy

To reveal the relation of the partition function (IV.6) to integrable systems, one should do two things. First, one should pass to the grand canonical ensemble

$$\mathcal{Z}_\mu[\lambda] = \sum_{N=0}^{\infty} e^{-\beta\mu N} Z_N[\lambda]. \tag{IV.10}$$

One can observe that $-\beta\mu$ plays the role of the "zero time" λ_0 which appears if one includes $n = 0$ into the sum in (IV.6). Therefore, the necessity to use the grand canonical ensemble goes in parallel with the change from Ω to λ's and it is natural in this context.

Second, we use the result (III.148) for the twisted partition function found in the double scaling limit [108]. Combining (III.148), (IV.6) and (IV.10) and integrating out the angular part of the twisting matrix, one obtains

$$\mathcal{Z}_\mu[\lambda] = \sum_{N=0}^{\infty} \frac{e^{-\beta\mu N}}{N!} \oint \prod_{k=1}^{N} \left(\frac{dz_k}{2\pi i z_k} \frac{e^{u(z_k)}}{q^{1/2} - q^{-1/2}}\right) \prod_{i\neq j} \frac{z_i - z_j}{q^{1/2} z_i - q^{-1/2} z_j}, \tag{IV.11}$$

where $q = e^{i\beta}$, z_k are eigenvalues of Ω, and $u(z) = \sum\limits_n \lambda_n z^n$ is a potential associated with the perturbation. Initially, the eigenvalues belonged to the unit circle. But due to the holomorphicity, the integrals in (IV.11) can be understood as contour integrals around $z = 0$. Finally, using the Cauchy identity

$$\frac{\Delta(x)\Delta(y)}{\prod\limits_{i,j}(x_i - y_j)} = \det_{i,j}\left(\frac{1}{x_i - y_j}\right), \tag{IV.12}$$

we rewrite the product of different factors as a determinant what gives the following representation for the grand canonical partition function of 2D string theory perturbed by vortices

$$\mathcal{Z}_\mu[\lambda] = \sum_{N=0}^{\infty} \frac{e^{-\beta\mu N}}{N!} \oint \prod_{k=1}^{N} \frac{dz_k}{2\pi i} \det_{i,j}\left(\frac{\exp\left[\frac{1}{2}(u(z_i) + u(z_j))\right]}{q^{1/2}z_i - q^{-1/2}z_j}\right). \tag{IV.13}$$

Relying on this representation one can prove that the grand canonical partition function coincides with a τ-function of Toda hierarchy [109]. In this case it is convenient to establish the equivalence with the fermionic representation of τ-function (II.111). We claim that if one chooses the matrix determining the operator of $GL(\infty)$ rotation as follows

$$A_{rs} = \delta_{r,s} q^{i\mu + r}, \tag{IV.14}$$

the τ-function is given by

$$\tau_l[t] = e^{-\sum\limits_{n>0} n t_n t_{-n}} \mathcal{Z}_{\mu - il}[\lambda], \tag{IV.15}$$

where the coupling constants are related to the Toda times through

$$\lambda_n = 2i \sin(\pi n R)\, t_n. \tag{IV.16}$$

Indeed, with the matrix (IV.14) the operator $\mathbf{g}$ is written as

$$\mathbf{g} \equiv \exp(q^{i\mu}\hat{A}) = \exp\left(e^{-\beta\mu} \oint \frac{dz}{2\pi i} \psi(q^{-1/2}z)\psi^*(q^{1/2}z)\right). \tag{IV.17}$$

The expansion of the exponent gives the sum over N and factors $e^{-\beta\mu N}$ in (IV.13). Then in the Nth term of the expansion one should commute the exponents $e^{H_\pm[t]}$ associated with perturbations between each other and with $\hat{A}^N$. The former commutation gives rise to the trivial factor appearing in (IV.15). It is to be compared with the similar contribution to $Z_N[\lambda]$ coming from (IV.7). The commutator with $\hat{A}^N$ is found using the relations (II.129). As a result, one obtains

$$\oint \prod_{k=1}^{N} \left(\frac{dz_k}{2\pi i} \exp\left[\sum_{n\neq 0} (q^{n/2} - q^{-n/2}) t_n z_k^n\right]\right) \left\langle l \left| \prod_{k=1}^{N} \psi(q^{-1/2}z_k)\psi^*(q^{1/2}z_k) \right| l \right\rangle. \tag{IV.18}$$

Comparing this expression with (IV.13) we see the necessity to redefine the coupling constants according to (IV.16) to match the potentials. Finally, the quantum average in the vacuum of charge l produces the same determinant as in (IV.13) and additional factor q^{lN} (see (II.105)). The latter leads to the shift of μ shown in (IV.15).

Actually, this result is not unexpected because, as we mentioned in the end of section III.3.3, a similar result has been obtained for the generating functional of tachyon correlators. The tachyon and winding perturbations are related by T-duality. Therefore, both the tachyon and winding perturbations of 2D string theory should be described by the same τ-function with T-dual parameters.

Nevertheless, it is remarkable that one can obtain an explicit matrix representation of this τ-function which can be directly interpreted in terms of discretized surfaces with vortices. Therefore, one can use the powerful matrix technique to solve some problems which may be inaccessible even by methods of integrable systems. For example, while the tachyon and winding perturbations are integrable when they are introduced separately, the integrability disappears as only both of them are present. In such situation the Toda hierarchy does not work anymore, but the matrix description is still valid.

The Toda description can be used exploiting its hierarchy of equations. To characterize their unique solution we should provide either a string equation or an initial condition. The string equation can be found in principle [110] (and we will show how it appears in the dual picture of tachyon perturbations), but it is not so evident. In contrast, it is clear that the initial condition is given by the partition function with vanishing coupling constants, *i.e.*, without vortices. It corresponds to the partition function of the compactified MQM in the singlet sector. It is well known and its expansion is given by (III.122). Thus, we have the necessary information to use the equations of Toda hierarchy.

These equations are of the finite-difference type. Therefore, usually one represents them as a series of partial differential equations. We associated this expansion in section II.5 with an expansion in the Planck constant which is the parameter measuring the lattice spacing. What is this parameter in our case? On the string theory side one has the genus expansion. The only possibility is to identify these two expansions. From (III.122) we see that the parameter playing the role of the string coupling constant, which is the parameter of the genus expansion, is $g_{\text{cl}} \sim \mu^{-1}$. Thus, one concludes that the role of the spacing parameter is played by μ^{-1} and to get the dispersionless limit of Toda hierarchy one should investigate the limit of large μ. Note that this conclusion is in the complete agreement with the consequence of (IV.15) that μ is associated with the discrete charge of τ-function.

2 Matrix model of a black hole

2.1 Black hole background from windings

Due to the result of the previous section, the Toda hierarchy provides us with equations on the free energy as a function of μ and the coupling constants. Any result found for finite couplings λ_n would already correspond to some result in string theory with a non-vanishing condensate of winding modes. In section I.6.2 we discussed that such winding condensates do not have a local target space interpretation. In other words, there is no special field in the string spectrum describing them. Therefore, the effect of winding condensation should be seen in another characteristics of the background: dilaton and metric. Thus, it is likely that considering MQM with non-vanishing λ_n, we actually describe 2D string theory in a curved background.

Let us consider the simplest case when only $\lambda_{\pm 1} \neq 0$. Without lack of generality one can take them equal $\lambda_1 = \lambda_{-1} \sim \lambda$. Then the corresponding string theory is described by the so called *Sine–Liouville CFT*

$$S_{\rm SL} = \frac{1}{4\pi} \int d^2\sigma \left[(\partial X)^2 + (\partial\phi)^2 - Q\hat{\mathcal{R}}\phi + \mu e^{\gamma\phi} + \lambda e^{\rho\phi} \cos(R\tilde{X}) \right], \qquad \text{(IV.19)}$$

where $\tilde{X}$ is T-dual to the field $X(\sigma)$ which is compactified at radius R. The requirement that the perturbations are given by marginal operators, leads to the following conditions on the parameters

$$\gamma = -Q + \sqrt{Q^2 - 4}, \qquad \rho = -Q + \sqrt{R^2 + Q^2 - 4}. \qquad \text{(IV.20)}$$

The central charge of this theory is $c = 2 + 6Q^2$. Therefore, to get $c = 26$, as always in matrix models, one should take $Q = 2$.

As we discussed above, the theory (IV.19) with the vanishing cosmological constant μ was suggested to be dual to 2D string theory in the black hole background described in section I.6.1. The exact statement of this conjecture was presented in section I.6.3. In particular, it was shown that the parameters of the model should be identified with the level k of the gauge group as follows

$$R = \sqrt{k}, \qquad Q = \frac{1}{\sqrt{k-2}}. \qquad \text{(IV.21)}$$

The condition on R comes from the matching of the asymptotic radii of the cigar geometry describing the Euclidean black hole and of the cylindrical target space of the Sine–Liouville CFT. The value of Q is fixed by matching the central charges.

Due to these restrictions on the parameters, there is only one point in the parameter space of the two models where they intersect. It corresponds to the following choice

$$Q = 2, \qquad R = 3/2, \qquad \mu = 0. \qquad \text{(IV.22)}$$

Since for these values of the parameters the Sine–Liouville CFT can be obtained as a matrix model constructed in the previous section, on the one hand, and is dual to the coset CFT, on the other hand, at this point we have a matrix model description of string theory in the black hole background [109]. The string partition function is given by the free energy of (IV.6) or its grand canonical counterpart (IV.10) where one puts μ and all couplings except $\lambda_{\pm 1}$ to zero as well as $R = 3/2$.

This remarkable correspondence opens the possibility to study the black hole physics using the matrix model methods. Of course, the most interesting questions are related to the thermodynamics of black holes. In particular, any theory of quantum gravity should be able to explain the microscopic origin of the black hole entropy and to resolve the information paradox. One might hope that the matrix model will allow to identify the fundamental degrees of freedom of this system to solve both these problems.

2.2 Results for the free energy

Before to address the question about the entropy, it is much more easy to get some information about another thermodynamical quantity — the free energy. The grand canonical free energy is given by the logarithm of the partition function (IV.10) and, hence, by the logarithm of the τ-function. Thus, one can use the integrable structure of Toda hierarchy to find it.

The problem is that we cannot work directly in the black hole point of the parameter space. Indeed, it puts μ to zero whereas the dispersionless limit of Toda hierarchy, which allows to write differential equations on the free energy, requires to consider large μ. The solution is to study the theory with a large non-vanishing μ treating λ as a perturbation. Then, one should try to make an analytical continuation to the opposite region of small μ. In the end, one should also fix the radius of the compactification.

In fact, in the matrix model it is very natural to turn on the cosmological constant μ and to consider an arbitrary radius R. The values (IV.22) (except $Q = 2$) are not distinguished anyhow. Even $\mu = 0$ is not the most preferable choice because there is another value which is associated with a critical point where the theory acquires some special properties (see below). Moreover, to analyze thermodynamical issues, one should be able to vary the temperature of the system what means for the black hole to vary the radius R. Thus, it is strange that only at the values (IV.22) MQM describes a black hole. What do other values correspond to? It is not clear at the moment, but it would be quite natural that they describe some deformation of the initial black hole background. Therefore, we will keep μ and R arbitrary in the most of calculations.

Let us use the Toda integrable structure to find the free energy $\mathcal{F}(\mu, \lambda) = \log \mathcal{Z}_\mu(\lambda)$ where $t_1 t_{-1} = \lambda^2$ and all other couplings vanish. Due to the winding number conservation, it depends only on the product of two couplings and not on them separately. The identification (IV.15) allows to conclude that the evolution along the first times is governed by the Toda equation (II.136). Since the shift of the discrete charge l is equivalent to an imaginary shift of μ, in terms of the free energy the Toda equation becomes

$$\frac{\partial^2 \mathcal{F}(\mu, \lambda)}{\partial t_1 \partial t_{-1}} + \exp\left[\mathcal{F}(\mu + i, \lambda) + \mathcal{F}(\mu - i, \lambda) - 2\mathcal{F}(\mu, \lambda)\right] = 1. \tag{IV.23}$$

Rewriting the finite shifts of μ as the result of action of a differential operator, one obtains

$$\frac{1}{4}\lambda^{-1}\partial_\lambda \lambda \partial_\lambda \mathcal{F}(\mu, \lambda) + \exp\left[-4\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\mathcal{F}(\mu, \lambda)\right] = 1. \tag{IV.24}$$

The main feature of this equation is that it is compatible with the scaling

$$\lambda \sim \mu^{\frac{2-R}{2}} \tag{IV.25}$$

which can be read off from the Sine–Liouville action (IV.19). Due to this, the free energy can be found order by order in its genus expansion which has the usual form of the expansion in μ^{-2}

$$\mathcal{F}(\mu,\lambda)=\lambda^2+\mu^2\left[-\frac{R}{2}\log\mu+\tilde{f}_0(\zeta)\right]+\left[-\frac{R+R^{-1}}{24}\log\mu+\tilde{f}_1(\zeta)\right]+\sum_{g=2}^{\infty}\mu^{2-2g}\tilde{f}_g(\zeta), \quad \text{(IV.26)}$$

where $\zeta=(R-1)\lambda^2\mu^{R-2}$ is a dimensionless parameter. The first term is not universal and can be ignored. It is intended to cancel 1 in the r.h.s. of (IV.24). The coefficients $\tilde{f}_g(\zeta)$ are smooth functions near $\zeta=0$. The initial condition given by (III.122) fixes them at the origin: $\tilde{f}_0(0)=\tilde{f}_1(0)=0$ and $\tilde{f}_g(0)=R^{1-g}f_g^{(\mathrm{sing})}(R)$ with $f_g^{(\mathrm{sing})}(R)$ from (III.123).

It is clear that one can redefine the coefficients in such a way that the genus expansion will be associated with an expansion in λ. More precisely, if one introduces the following scaling variables

$$w=\mu\xi, \qquad \xi=(\lambda\sqrt{R-1})^{-\frac{2}{2-R}}, \quad \text{(IV.27)}$$

the genus expansion of the free energy reads

$$\mathcal{F}(\mu,\lambda)=\lambda^2+\xi^{-2}\left[\frac{R}{2}w^2\log\xi+f_0(w)\right]+\left[\frac{R+R^{-1}}{24}\log\xi+f_1(w)\right]+\sum_{g=2}^{\infty}\xi^{2g-2}f_g(w). \quad \text{(IV.28)}$$

Thus, the string coupling constant is identified as $g_{\mathrm{cl}}\sim\xi$. This is a simple consequence of the scaling (IV.25). It is clear that the dimensionless parameters ζ and w are inverse to each other: $\zeta=w^{R-2}$. We included the factor $(R-1)$ in the definition of the scaling variables for convenience as it will become clear from the following formulae. This implies that $R>1$. It is the region we are interested in because it contains the black hole radius. However, the final result can be presented in the form avoiding this restriction.

Plugging (IV.28) into (IV.24), one obtains a system of ordinary differential equations for $f_g(w)$. Each equation is associated with a definite genus. At the spherical level there is a closed non-linear equation for $f_0(w)$

$$\frac{R-1}{(2-R)^2}\left(w\partial_w-2\right)^2 f_0(w)+e^{-\partial_w^2 f_0(w)}=0. \quad \text{(IV.29)}$$

Its solution is formulated as a non-linear algebraic equation for $X_0=\partial_w^2 f_0$ [109]

$$w=e^{-\frac{1}{R}X_0}-e^{-\frac{R-1}{R}X_0}. \quad \text{(IV.30)}$$

In terms of the solution of this equation, the spherical free energy itself is represented as

$$\mathcal{F}_0(\mu,\lambda)=\frac{1}{2}\mu^2\left(R\log\xi+X_0\right)+\xi^{-2}\left(\frac{3}{4}Re^{-\frac{2}{R}X_0}-\frac{R^2-R+1}{R-1}e^{-X_0}+\frac{3}{4}\frac{R}{R-1}e^{-2\frac{R-1}{R}X_0}\right). \quad \text{(IV.31)}$$

Let us rewrite the equation (IV.30) in terms of the susceptibility $\chi=\partial_\mu^2\mathcal{F}$, more precisely, in terms of its spherical part

$$\chi_0=R\log\xi+X_0. \quad \text{(IV.32)}$$

The result can be written in the following form

$$\mu e^{\frac{1}{R}\chi_0}+(R-1)\lambda^2 e^{\frac{2-R}{R}\chi_0}=1. \tag{IV.33}$$

From this it is already clear why we included the factor $(R-1)$ in the scaling variables. Note that this form of the answer is not restricted to $R>1$ and it is valid for all radii. However, it shows that the limit $\mu\to 0$ exists only for $R>1$. Otherwise the susceptibility becomes imaginary. For $R>1$, a critical point where the equation (IV.33) does not have real solutions anymore also exists and it is given by

$$\mu_c=-(2-R)(R-1)^{\frac{R}{2-R}}\lambda^{\frac{2}{2-R}}. \tag{IV.34}$$

This critical value of the cosmological constant was found previously by Hsu and Kutasov [111]. The result (IV.34) shows that the vanishing value of μ is inaccessible also for $R>2$. Actually, in this region the situation is even more dramatic because ξ becomes an increasing function of λ and the genus expansion breaks down in the limit of large λ. This is related to the fact that the vortex perturbation in (IV.19) is not marginal for $R>2$ and is non-renormalizable because it grows in the weak coupling region. Thus, the analytical continuation to the black hole point $\mu=0$ is possible only in the finite interval of radii $1<R<2$. Fortunately, the needed value $R=3/2$ belongs to this interval and the proposal survives this possible obstruction.

The equation (IV.33) can be used to extract expansion of the spherical free energy either in λ or in μ. In particular, the former expansion reproduces the $2n$-point correlators of vortex operators

$$\left\langle \tilde V_R^n \tilde V_{-R}^n\right\rangle_0=-n!\mu^2 R^{2n+1}\left((1-R)\mu^{R-2}\right)^n\frac{\Gamma(n(2-R)-2)}{\Gamma(n(1-R)+1)}. \tag{IV.35}$$

For small values of n they have been found and for other values conjectured by Moore in [112]. These correlators should coincide with the coefficients in the λ-expansion of $\mathcal{F}_0$ because they can be organized into the partition function as follows

$$\mathcal{F}(\mu,\lambda)=\left\langle e^{\tilde\lambda\tilde V_R+\tilde\lambda\tilde V_{-R}}\right\rangle_{\text{gr.c.}}=\mathcal{F}(\mu,0)+\sum_{n=1}^{\infty}\frac{\tilde\lambda^{2n}}{(n!)^2}\left\langle \tilde V_R^n\tilde V_{-R}^n\right\rangle_{\text{gr.c.}}, \tag{IV.36}$$

where the expectation value is evaluated in the grand canonical ensemble. The comparison shows that the correlators do coincide if one identifies $\lambda=R\tilde\lambda$.[1] This indicates that the correct relation between the Toda times t_n and the CFT coupling constants $\tilde t_n$ in (I.63) is the following

$$t_n=iR\tilde t_n,\qquad t_{-n}=-iR\tilde t_{-n},\quad (n>0). \tag{IV.37}$$

The appearance of the factor R will be clear when we consider the dual system with tachyon perturbations.

The black hole limit $\mu=0$ corresponds to $X_0=0$. Then the free energy (IV.31) becomes

$$\mathcal{F}_0(0,\lambda)=-\frac{(2-R)^2}{4(R-1)}(\sqrt{R-1}\lambda)^{\frac{4}{2-R}}. \tag{IV.38}$$

[1] In fact, the correlators (IV.35) differ by sign from the coefficients in the expansion of $\mathcal{F}_0$. This is related to that $\mathcal{F}_0$ is the grand canonical free energy whereas the paper [112] considered the canonical ensemble.

Note that at the point $R = 3/2$ the free energy is proportional to an integer power of the coupling constant $\sim \lambda^8$. Therefore, the spherical contribution seems to be non-universal. However, for general R it is not so and this is crucial for thermodynamical issues.

At the next levels the equations obtained from (IV.24) form a triangular system so that the equation for $f_g(w)$ is linear with respect to this function and contains all functions of lower genera as a necessary input [109]

$$\left(\frac{R-1}{(2-R)^2}\left(w\partial_w+2g-2\right)^2 - e^{-X_0}\partial_w^2\right) f_g = -\left[\xi^{2-2g}\exp\left(-4\sin^2\left(\frac{\xi}{2}\partial_w\right)\sum_{k=0}^{g-1}\xi^{2k-2}f_k\right)\right]_0 , \tag{IV.39}$$

where $[\cdots]_0$ means the terms of zero order in ξ-expansion. Up to now, only the solution for the genus $g=1$ has been obtained [109]

$$\mathcal{F}_1(\mu,\lambda) = \frac{R+R^{-1}}{24}\left(\log\xi + \frac{1}{R}X_0\right) - \frac{1}{24}\log\left(1-(R-1)e^{\frac{2-R}{R}X_0}\right). \tag{IV.40}$$

For the genus $g=2$ the differential operator of the second order contains 4 singular points and the solution cannot be presented in terms of hypergeometric functions [113].

2.3 Thermodynamical issues

An attempt to analyze thermodynamics of the black hole relying on the result (IV.38) was done in [109] and [34]. However, no definite conclusions have been obtained. First of all, it is not clear whether the free energy of the black hole vanishes or not. The "old" analysis in the framework of dilaton gravity predicts that it should vanish [29, 30]. However, the matrix model leads to the opposite conclusion. One can argue [109] that since for $R = 3/2$ the leading term is non-universal, it can be thrown away giving the vanishing free energy. But if the matrix model realizes string theory in a black hole background for any R, this would be quite unnatural.

Moreover, even in the framework of the dilaton gravity the issue is not clear. The value of the free energy depends on a subtraction procedure which is to be done to regularize diverging answers. There is a natural reparameterization invariant procedure which leads to a non-vanishing free energy [34] in contradiction with the previous results. However, in this case it is not clear how to get the correct expressions for the mass and entropy.

The related problem which prevents to clarify the situation is what quantity should be associated with the temperature. At the first glance this is the inverse radius. In particular, if one follows this idea and uses the reparameterization invariant subtraction procedure, one arrives at reasonable results but the mass of the black hole differs by factor 2 from the standard expression (I.56) [114]. Note that the possibility to get this additional factor was emphasized in [115]. It is related to the definition of energy in dilaton gravity.

However, from the string point of view the radius is always fixed. Therefore, in the analysis of the black hole thermodynamics, the actual variations of the temperature were associated with the position of a "wall" which is introduced to define the subtraction [29, 30, 34]. But there is no corresponding quantity in the matrix model.

In the next chapter, relying on the analysis of a dual system, we will argue that it is R^{-1} that should be considered as the temperature. Also we will shed some light on the puzzle with the free energy.

3 Correlators of windings

After this long introduction, finally we arrived at the point where we start to discuss the new results of this thesis. The first of these results concerns correlators of winding operators in the presence of a winding condensate. According to the proposal of [109] reviewed in the previous section, they give the correlators of winding modes in the black hole background. The calculation of these correlators represents the next step in exploring the Toda integrable structure describing the winding sector of 2D string theory. For the one- and two-point correlators in the spherical approximation this task has been fulfilled in the work [116].

Due to the identification (IV.15), the generating functional of all correlators of vortices is the τ-function of Toda hierarchy. For the Sine–Liouville theory where only the first couplings $\lambda_{\pm 1}$ are non-vanishing, the correlators are defined as follows

$$\mathcal{K}_{i_1\cdots i_n} = \frac{\partial^n}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} \log\tau_0\Big|_{\lambda_{\pm 2}=\lambda_{\pm 3}=\cdots=0}, \tag{IV.41}$$

where the coupling constants λ_n are related to the Toda times t_n by (IV.16). Whereas to find the free energy it was enough to establish the evolution law along the first times, to find the correlators one should know how the τ-function depends on all Toda times, at least near $t_n = 0$.

The evolution law for the first times $t_{\pm 1}$ was determined by the Toda equation. It is the first equation in the hierarchy of bilinear differential Hirota equations (II.134). The idea of [116] was to use these equations to find the correlators.

3.1 Two-point correlators

The first step was to identify the necessary equations because not the whole hierarchy is relevant for the problem. After that we observe that the extracted equations are of the finite-difference type. To reduce them to differential equations, one should plug in the ansatz (IV.28), where now the coefficients f_g are functions of all dimensionless parameters: w and $s = (s_{\pm 2}, s_{\pm 3}, \ldots)$. The first parameter was defined in (IV.27) and the parameters s_n are related to higher times

$$s_n = i\left(-\frac{t_{-1}}{t_1}\right)^{n/2} \xi^{\Delta[t_n]} t_n, \tag{IV.42}$$

where $\Delta[t_n]$ is the dimension of the coupling with respect to μ

$$\Delta[t_n] = 1 - \frac{R|n|}{2}. \tag{IV.43}$$

The spherical approximation corresponds to the dispersionless limit of the hierarchy. It is obtained as $\xi \to 0$. Thus, extracting the first term in the small ξ expansion, we found the spherical approximation of the initial equations. In principle, they could mix different correlators and of different genera. However, it turned out that in the spherical limit the situation is quite simple. In the equations we have chosen only second derivatives of the spherical part of the free energy survive. We succeeded to rewrite the resulting equations as equations on the generating functions of the two-point correlators. There are two such functions: for correlators with vorticities of the same and opposite signs

$$F^{\pm}(x,y) \;=\; \sum_{n,m=0}^{\infty} x^n y^m \tilde{X}^{\pm}_{\pm n,\pm m}, \tag{IV.44}$$

$$G(x,y) \;=\; \sum_{n,m=1}^{\infty} x^n y^m \tilde{X}_{n,-m}, \tag{IV.45}$$

where

$$\tilde{X}^{\pm}_{0,m} := \mp i \tfrac{1}{|n|} \tfrac{\partial^2}{\partial t_n \partial \mu} \mathcal{F}_0, \quad n \neq 0, \tag{IV.46}$$

$$\tilde{X}^{\pm}_{n,m} := 2 \tfrac{1}{|n||m|} \tfrac{\partial^2}{\partial t_n \partial t_m} \mathcal{F}_0, \quad n,m \neq 0. \tag{IV.47}$$

The equations for $F^{\pm}(x,y)$ and $G(x,y)$, respectively, read

$$\tfrac{x+y}{x-y}\left(e^{\frac12 F^{\pm}(x,x)} - e^{\frac12 F^{\pm}(y,y)}\right) = x\partial_x F^{\pm}(x,y) e^{\frac12 F^{\pm}(y,y)} + y\partial_y F^{\pm}(x,y) e^{\frac12 F^{\pm}(x,x)}, \tag{IV.48}$$

$$A\left[y\partial_y(G(x,y) - 2F^{-}(y,0)) - 2\right] e^{\frac12 F^{+}(x,x)+F^{+}(x,0)} = \tfrac{1}{y}\partial_x G(x,y) e^{\frac12 F^{-}(y,y)-F^{-}(y,0)}, \tag{IV.49}$$

where

$$A = \exp\left(-\partial_\mu^2 \mathcal{F}_0\right) = \xi^{-R} e^{-X_0}. \tag{IV.50}$$

The equations (IV.48) come from the Hirota bilinear identities (II.134) taking $i=0$, extracting the coefficients in front of $y_{\pm n} y_{\pm m}$, $n,m>0$, multiplying by $x^n y^m$ and summing over all n and m. The similar procedure with $i=1$ and $y_n y_{-m}$ gives the equation (IV.49). In fact, the difference between F^+ and F^- is not essential and it disappears when one chooses $t_1 = -t_{-1}$ ($\lambda_1 = \lambda_{-1}$). Therefore, in the following we will omit the inessential sign label in $F(x,y)$.

Since the dependence of μ is completely known and the cosmological constant plays a distinguished role, the quantities $\tilde{X}^{\pm}_{0,m}$ can be actually considered as one-point correlators. We define their generating function as

$$h(x) = F(x,0). \tag{IV.51}$$

The equations (IV.48) and (IV.49) for the functions $F(x,y)$ and $G(x,y)$ have been explicitly solved in terms of this generating function $h(x)$

$$F(x,y) \;=\; \log\left[\frac{4xy}{(x-y)^2}\mathrm{sh}^2\left(\frac12(h(x)-h(y)+\log\frac{x}{y})\right)\right], \tag{IV.52}$$

$$G(x,y) \;=\; 2\log\left(1 - Axye^{h^+(x)+h^-(y)}\right). \tag{IV.53}$$

These solutions are universal in the sense that they are valid for any system describing by Toda hierarchy. In other words, their form does not depend on the potential or another initial input. All dependence of particular characteristics of the model enters through the one-point correlators and the free energy. In a little bit different form these solutions appeared in [73, 117] and resemble the equations for the two-point correlators in 2MM found in [72].

3.2 One-point correlators

The main difficulty is to find the one-point correlators. The Hirota equations are not sufficient to accomplish this task. One needs to provide an additional input. It comes from the fact that we know the dependence of the free energy of the first times $t_{\pm 1}$ and of the cosmological constant μ. Due to this one can write a relation between the one-point correlators entering $h(x)$ and two-point correlators of the kind $\tilde{X}_{n,\pm 1}$. (Roughly speaking, one should integrate over μ and differentiate with respect to $t_{\pm 1}$.) The latter are generated by two functions, $\partial_y F(x,0)$ and $\partial_y G(x,0)$. As a result, we arrive at the following two equations

$$\begin{aligned} \partial_y F(x,0) &= \hat{K}^{(+)} h(x) + \tilde{X}^{+}_{1,0}, && \text{(IV.54)} \\ \partial_y G(x,0) &= \hat{K}^{(-)} h(x), && \text{(IV.55)} \end{aligned}$$

where $\hat{K}^{(\pm)}$ are linear integral-differential operators. These operators are found from the explicit expressions for the free energy and the scaling variables and have the following form

$$\begin{aligned} \hat{K}^{(+)} &= -a \left[w - (1 + (R-1)x \frac{\partial}{\partial x}) \int^{w} dw \right], && \text{(IV.56)} \\ \hat{K}^{(-)} &= a \left[w - (1 + x\frac{\partial}{\partial x}) \int^{w} dw \right], && \text{(IV.57)} \end{aligned}$$

where $a = 2\frac{\sqrt{R-1}}{2-R} \xi^{-\frac{R}{2}}$. On the other hand, the generating functions F and G are known in terms of $h(x)$ from (IV.52) and (IV.53). Substituting them into (IV.54) and (IV.55) and taking the derivative with respect to w, we obtain two equations for one function $h(x)$

$$\begin{aligned} \left[-a \left((R-1)x\frac{\partial}{\partial x} - w\frac{\partial}{\partial w} \right) + \frac{2}{x} e^{-h(x;w)} \frac{\partial}{\partial w} \right] h(x;w) &= 2\frac{\partial}{\partial w} \tilde{X}^{+}_{0,1}, && \text{(IV.58)} \\ \left[a \left(x\frac{\partial}{\partial x} - w\frac{\partial}{\partial w} \right) - 2Axe^{h(x;w)} \frac{\partial}{\partial w} \right] h(x;w) &= 2xe^{h(x;w)} \frac{\partial}{\partial w} A. && \text{(IV.59)} \end{aligned}$$

In [116] we succeeded to solve these differential equations. As it must be, they turned out to be compatible. The solution was represented in terms of the following algebraic equation

$$e^{\frac{1}{R}h} - ze^{h} = 1, \qquad \text{(IV.60)}$$

where $z = x\, \frac{\xi^{-R/2}}{\sqrt{R-1}}\, e^{-\frac{R-1}{R}X_0} = x\lambda\, e^{-\frac{R-1}{R}\chi_0}$. Note that if we take different $t_{\pm 1}$, one would have two equations for $h^{\pm}$ with the parameters $z^{\pm}$ where λ is replaced by $t_{\mp 1}$, respectively.

The equation (IV.60) represents the main result of the work [116]. It was used to find the explicit expressions for the correlators which are given by the coefficients of the expansion of the generating functions $h(x)$, $F(x,y)$ and $G(x,y)$ multiplied by the factors relating t_n with λ_n. The resulting expressions are

$$\mathcal{K}_n = \frac{1}{2\sin \pi n R} \frac{\Gamma(nR+1)}{n!\Gamma(n(R-1)+1)} \frac{\xi^{-\frac{nR+2}{2}}}{(R-1)^{n/2}} \left(\frac{e^{-\frac{n(R-1)+1}{R}X_0}}{n(R-1)+1} - \frac{e^{-(n+1)\frac{R-1}{R}X_0}}{n+1} \right), \qquad \text{(IV.61)}$$

$$\begin{aligned}\mathcal{K}_{n,-m} &= -\frac{\Gamma(nR+1)}{2\sin\pi nR}\frac{\Gamma(mR+1)}{2\sin\pi mR}\frac{\xi^{-(n+m)R/2}}{(R-1)^{(n+m)/2}}e^{-(n+m)\frac{R-1}{R}X_0}\times \\ &\times \sum_{k=1}^{\min(n,m)}\frac{k(R-1)^k e^{k\frac{R-2}{R}X_0}}{(n-k)!(m-k)!\Gamma(n(R-1)+k+1)\Gamma(m(R-1)+k+1)}. \qquad \text{(IV.62)} \\ \mathcal{K}_{n,m} &= -\frac{\Gamma(nR+1)}{2\sin\pi nR}\frac{\Gamma(mR+1)}{2\sin\pi mR}\frac{\xi^{-(n+m)R/2}}{(R-1)^{(n+m)/2}}e^{-(n+m)\frac{R-1}{R}X_0}\times \\ &\times \sum_{k=1}^{n}\frac{k}{(n-k)!(m+k)!\Gamma(n(R-1)+k+1)\Gamma(m(R-1)-k+1)}. \qquad \text{(IV.63)}\end{aligned}$$

The correlators are expressed as functions of X_0 and ξ. The former contains all the non-trivial dependence of the coupling constants μ and λ, whereas the latter provides the correct scaling. The results for the black hole point $\mu = 0$ are obtained when one considers the limit $X_0 = 0$.

We associated the factor $(2\sin(\pi nR))^{-1}$, coming from the change of couplings (IV.16), with each index n of the correlators. Exactly at the black hole radius $R = 3/2$, it becomes singular for even n. Besides, it leads to negative answers and breaks the interpretation of the correlators as probabilities to find vortices of a given vorticity. Most probably, one should not attach these factors to the correlators because they are part of the leg-factors which always appear when comparing the matrix model and CFT results (see section III.3.3). This point of view is supported by the fact that these factors appear as the same wave-function renormalization in all multipoint correlators and disappear if one considers the normalized correlators like $\frac{\mathcal{K}_{n,m}}{\mathcal{K}_n\mathcal{K}_m}$. Besides, one can notice that one does not attach this factor to $\lambda = \sqrt{t_1 t_{-1}}$ and, nevertheless, one finds agreement for the free energy with the results of [112] and [111].

3.3 Comparison with CFT results

Due to the FZZ conjecture (see section I.6.3) we expect that our one-point correlators should contain information about the amplitudes of emission of winding modes by the black hole, whereas the two-point correlators describe the S-matrix of scattering of the winding states from the tip of the cigar (or from the Sine-Liouville wall). In [38] and [118] some two- and three-point correlators were computed in the CFT approach to this theory. It would be interesting to compare their results with the correlators calculated here from the MQM approach. However, there are immediate obstacles to this comparison.

First of all, these authors do not give any results for the one point functions of windings. In the conformal theory such functions are normally zero since the vortex operators have the dimension one. But in the string theory we calculate the averages of a type $\left\langle \int d^2\sigma\, \hat{V}_n(\sigma)\right\rangle$ integrated over the parameterization space. They are already quantities of zero dimension, and the formal integration leads to the ambiguity $0 * \infty$ which should in general give a finite result. Another possible reason for vanishing of the one-point correlators could be the additional infinite W-symmetry found in [119] for the CFT (IV.19) at $R = 3/2,\ \mu = 0$. The generators of this symmetry do not commute with the vortex operators $\hat{V}_n,\ n \neq \pm 1$. Hence its vacuum average should be zero, unless there is a singlet component under this symmetry in it.

Note that the one-point functions were calculated [118] in a non-conformal field theory with the action

$$S = \frac{1}{4\pi}\int d^2\sigma \left[(\partial X)^2 + (\partial\phi)^2 - 2\hat{\mathcal{R}}\phi + m\left(e^{-\frac{1}{2}\phi} + e^{\frac{1}{2}\phi}\right)\cos(\frac{3}{2}X)\right]. \tag{IV.64}$$

This theory coincides with the Sine–Liouville CFT (again at $R = 3/2$ and $\mu = 0$) in the limit $m \to 0$, $\phi_0 \to -\infty$, with $\lambda = me^{-\frac{1}{2}\phi_0}$ fixed, where ϕ_0 is a shift of the zero mode of the Liouville field $\phi(z)$. So we can try to perform this limit in the calculated one-point functions directly. As a result we obtain $\mathcal{K}_n^{(m)} \sim m^2\lambda^{3n-4}$. The coefficient we omitted is given by a complicated integral which we cannot perform explicitly. It is important that it does not depend on the couplings and is purely numerical. We see that, remarkably, the vanishing mass parameter enters in a constant power which is tempting to associate with the measure $d^2\sigma$ of integration. Moreover, the scaling in λ is the same as in (IV.61) (at $R = 3/2$) up to the n-independent factor λ^{-8}. All these n-independent factors disappear if we consider the correlators normalized with respect to $\mathcal{K}_1^{(m)}$ which is definitely nonzero. They behave like $\sim \lambda^{3(n-1)}$ what coincides with the MQM result.

In fact, there is still a possibility for agreement of the matrix model results with the CFT prediction that the one-point correlators should vanish. It involves a mixing of operators where not only primary operators appear. Probably, after the correct identification, one-point correlators will vanish in the matrix model too. But we do not see any reason why this should be the case.

The possibility to introduce leg-factors and the possible mixing of operators make difficult to compare also our results for the two-point correlators with ones obtained in the Sine–Liouville or coset CFT and given in (I.70). There only the two-point correlators of opposite and equal by modulo vorticities have been calculated. They cannot be identified with our quantity $\mathcal{K}_{n,-n}$ while the normalization of the vortex operators in the matrix model with respect to the vortex operators in CFT is not established. This could be done with help of the one-point correlators but they are not known in the CFT approach.

Let us mention that from the CFT side some three-point correlators are also accessible [38]. However, we did not study the corresponding problem in the matrix model yet. The calculation of these correlators would provide already sufficient information to make the comparison between the two theories.

Note that the situation is that complicated because the two theories have only one intersection point, so that the correlators to be compared are just numbers. We would be in a better position if we are able to extend the FZZ correspondence to arbitrary radius or cosmological constant, for example. Such extension may give answers to many questions which are not understood until now.

Chapter V

Tachyon perturbations of MQM

In this chapter we consider the second way to obtain a non-trivial background in 2D string theory, which is to perturb it by tachyon sources. Of course, the results should be T-dual to ones obtained by a winding condensation. However, although expected, the T-duality of tachyons and windings is not evident in Matrix Quantum Mechanics. We saw in chapter III that they appear in a quite different way. And this is a remarkable fact that the two so different pictures do agree. It gives one more evidence that MQM provides the correct description of 2D string theory and their equivalence can be extended to include the perturbations of both types.

Besides, the target space interpretations of the winding and tachyon perturbations are different. Therefore, although they are described by the same mathematical structure (Toda hierarchy), the physics is not the same. In particular, as we will show, it is impossible to get a curved background using tachyon perturbations, whereas the winding condensate was conjectured to correspond to the black hole background. At the same time, the interpretation in terms of free fermions allows to obtain a more detailed information about both the structure of the target space and thermodynamical properties of tachyonic backgrounds.

Since the introduction of tachyon modes does not require a compactification of the time direction, as the winding modes do, we are not forced to work with Euclidean theory. In turn, the free fermionic representation is naturally formulated in spacetime of Minkowskian signature. Therefore, in this chapter we will work with the real Minkowskian time t. Nevertheless, we are especially interested in the case when the tachyonic momenta are restricted to values of the Euclidean theory compactified at radius R. It is this case that should be dual to the situation considered in the previous chapter and we expect to find that it is exactly integrable.

1 Tachyon perturbations as profiles of Fermi sea

First of all, one should understand how to introduce the tachyon sources in Matrix Quantum Mechanics. The first idea is just to follow the CFT approach: to add the vertex operators realized in terms of matrices to the MQM Hamiltonian. However, this idea fails by several reasons. The first one is that although the matrix realization of the vertex operators is well known (III.81), this form of the operators is valid only in the asymptotic region. When approaching the Liouville wall, they are renormalized in a complicated way. Thus, we cannot write a Hamiltonian that determines the dynamics everywhere. The second reason is that such perturbations introduced into the Hamiltonian disappear in the double scaling limit where only the quadratic part of the potential is relevant. This is especially obvious for the spectrum of perturbations corresponding to the self-dual radius of compactification $R = 1$. Then the perturbing terms do not differ from the terms of the usual potential which are all inessential.

The last argument shows that one should do something with the system directly in the double scaling limit. There the system is universal being always described by the inverse oscillator potential. This means that *we should change not the system but its state*. Indeed, we established the correspondence with the linear dilaton background only for the ground state of the singlet sector of MQM. In particular, in the spherical approximation this state is described by the stationary Fermi sea (III.60). The propagation of small perturbations above this ground state was associated with the scattering of tachyons [95]. Therefore, it is natural to expect that a tachyon condensation is obtained when one considers excited states associated with non-perturbative deformations of the Fermi sea. This idea has got a concrete realization in the work [120].

1.1 MQM in the light-cone representation

A state containing a non-perturbative source of particles is, usually, a coherent state obtained by action of the exponent of the operator creating the particles on a ground state. Of course, the easiest way to describe such states is to work in the representation where the creation operator is diagonal. In our case, there are two creation operators of right and left moving tachyons. They are associated with powers of the following matrix operators

$$X_{\pm} = \frac{M \pm P}{\sqrt{2}}. \tag{V.1}$$

Their eigenvalues $x_{\pm}$ can be considered as light-cone like variables but defined in the phase space of the theory rather than in the target space. Thus, we see that to describe the tachyon perturbations, one should work in the light-cone representation of MQM.

Such light-cone representation was constructed in [120]. With X_+ and X_- we associate the right and left Hilbert spaces, respectively, whose elements are functions of one of these variables. The scalar product is defined as

$$\langle \Phi_{\pm} | \Phi'_{\pm} \rangle = \int dX_{\pm}\, \overline{\Phi_{\pm}(X_{\pm})} \Phi'_{\pm}(X_{\pm}). \tag{V.2}$$

Since the matrix operators $X_{\pm}$ obey the canonical commutation relation

$$[(X_+)^i_j, (X_-)^k_l] = -i\ \delta^i_l \delta^k_j, \tag{V.3}$$

the operator of coordinate in the right Hilbert space is the momentum operator in the left one and the wave functions in the two representations are related by a Fourier transform. In the double scaling limit the dynamics of the system is governed by the inverse oscillator matrix Hamiltonian. It is written in the variables (V.1) as

$$H_0 = -\frac{1}{2} \mathrm{Tr}\, (X_+ X_- + X_- X_+) \tag{V.4}$$

so that the second-order Schrödinger equation associated with it in the usual representation becomes a first-order one when written in the ±-representations

$$\partial_t \Phi_\pm(X_\pm, t) = \mp \,\mathrm{Tr}\, \left(X_\pm \frac{\partial}{\partial X_\pm} + \frac{N}{2} \right) \Phi_\pm(X_\pm, t). \tag{V.5}$$

The general solution is

$$\Phi_\pm(X_\pm, t) = e^{\mp \frac{1}{2} N^2 t} \Phi_\pm(e^{\mp t} X_\pm). \tag{V.6}$$

Similarly to the usual representation (III.18), the right and left Hilbert spaces are decomposed into a direct sum of subspaces labeled by irreducible representations of $SU(N)$, and the functions $\vec{\Phi}_\pm^{(r)} = \{\Phi_\pm^{(r,a)}\}_{a=1}^{d_r}$ belonging to given irreducible representation r depend only on the N eigenvalues $x_{\pm,1}, \dots, x_{\pm,N}$. It is important that all these subspaces are invariant under the action of the Hamiltonian (V.4), which means that the decomposition is preserved by dynamics. Remarkably, the Hamiltonian (V.4) reduces on the functions $\vec{\Phi}_\pm^{(r)}(x_\pm)$ to its radial part only

$$H_0 = \mp i \sum_k (x_{\pm,k} \partial_{x_{\pm,k}} + \frac{N}{2}). \tag{V.7}$$

This property is a potential advantage of the light-cone approach in comparison with the usual one where the Hamiltonian does contain an angular part, which induces a Calogero-like interaction.

In the scalar product (V.2), the angular part can also be integrated out, leaving only the trace over the representation indices and the square of the Vandermonde determinant $\Delta(x_\pm)$ as in (III.11). Therefore, we do the usual redefinition to remove this determinant

$$\vec{\Psi}_\pm^{(r)}(x_\pm) = \Delta(x_\pm) \vec{\Phi}_\pm^{(r)}(x_\pm). \tag{V.8}$$

For the new functions $\vec{\Psi}_\pm^{(r)}(x_\pm)$ the scalar product reads

$$\langle \vec{\Psi}_\pm^{(r)} | \vec{\Psi}_\pm'^{(r)} \rangle = \sum_{a=1}^{d_r} \int \prod_{k=1}^{N} dx_{\pm,k} \overline{\Psi_\pm^{(r,a)}(x_\pm)} \Psi_\pm'^{(r,a)}(x_\pm), \tag{V.9}$$

and the Hamiltonian takes the same form as in (V.7), but with a different constant term

$$H_0 = \mp i \sum_k (x_{\pm,k} \partial_{x_{\pm,k}} + 1/2). \tag{V.10}$$

In the singlet representation we reproduce the known result: the wave functions $\Psi_\pm^{(\mathrm{singlet})}$ are completely antisymmetric and the singlet sector describes a system of N free fermions. In what follows we will concentrate on this sector of the Hilbert space. As we know form section III.3, it is sufficient to describe tachyon excitations of 2D string theory. We will start from the properties of the ground state of the model, representing the unperturbed 2D string background and then go over to the perturbed fermionic states describing the (time-dependent) backgrounds characterized by nonzero expectation values of some vertex operators.

1.2 Eigenfunctions and fermionic scattering

A wave function of a free fermionic system is represented by the Slater determinant of one-particle wave functions. Therefore, according to (V.10) it is enough to study the problem of scattering of one fermion described by the one-particle Hamiltonian

$$H_0 = -\frac{1}{2}(\hat{x}_+\hat{x}_- + \hat{x}_-\hat{x}_+). \tag{V.11}$$

As we saw, in the light-cone variables the Hamiltonian (V.11) is represented by a linear differential operator of the first order. Therefore, it is easy to write its eigenfunctions, which are given by the simple power functions

$$\psi^E_\pm(x_\pm) = \frac{1}{\sqrt{2\pi}} x_\pm^{\pm iE - \frac{1}{2}}. \tag{V.12}$$

Given this simple result, one can ask how it is able to incorporate information about the scattering?

To answer this question, note that there are two light-cone representations, right and left. It turns out that *all the non-trivial dynamics of fermions in the inverse oscillator potential is hidden in the relation between these two representations.* This relation is just the usual unitary transformation between two quantum mechanical representations. In the given case it is described by the Fourier transform

$$[\hat{S}\psi_+](x_-) = \int dx_+\, K(x_-, x_+)\psi_+(x_+). \tag{V.13}$$

The exact form of the kernel $K(x_-, x_+)$ depends on the non-perturbative definition of the model. There are two possible definitions (theories of type I and II [97]). In the first model the domain of definition of the wave functions is restricted to the positive half-lines $x_\pm > 0$ and the kernel has one of the two possible forms[1]

$$K(x_-, x_+) = \sqrt{\frac{2}{\pi}}\cos(x_- x_+) \quad \text{or} \quad K(x_-, x_+) = i\sqrt{\frac{2}{\pi}}\sin(x_- x_+). \tag{V.14}$$

In the second model the domain coincides with the whole line and the kernel is $\frac{1}{\sqrt{2\pi}}e^{ix_-x_+}$. The choice of the model corresponds to that either we consider fermions from one side of the inverse oscillator potential or from both sides. The two sides of the potential are connected only by tunneling processes which are non-perturbative in the cosmological constant being proportional to $e^{-2\pi\mu}$. Therefore, the choice of the model does not affect the perturbative (genus) expansion of the free energy. In the following, we prefer to work with the first model avoiding the doubling of fermions and choose the cosine kernel. The description of the light-cone quantization of the second model can be found in appendix of [120].

[1]The fact that there are two choices for the kernel can be explained as follows. In order to define the theory of type I for the original second-order Hamiltonian in the usual representation, we should fix the boundary condition at $x = 0$, and there are two linearly independent boundary conditions. The difference between them is seen only at the non-perturbative level.

Calculating the transformation (V.13) on the eigenfunctions (V.12), one finds that it is diagonal

$$[\hat{S}^{\pm 1}\psi_{\pm}^{E}](x_{\mp}) = \mathcal{R}(\pm E)\psi_{\mp}^{E}(x_{\mp}), \tag{V.15}$$

where the coefficient is given by

$$\mathcal{R}(E) = \sqrt{\frac{2}{\pi}}\cosh\left(\frac{\pi}{2}(i/2 - E)\right)\Gamma(iE + 1/2). \tag{V.16}$$

The coefficient is nothing else but the scattering coefficient of fermions off the inverse oscillator potential. It can be seen as fermionic S-matrix in the energy representation. Since $\mathcal{R}(E)$ is a pure phase the S-matrix is unitary. Thus, one gets a non-perturbatively defined formulation of the double scaled MQM or, in other terms, of 2D string theory.

From the above discussion it follows that the scattering amplitude between two arbitrary in and out states is given by the integral with the Fourier kernel (V.14)

$$\begin{aligned} \langle \psi_- | \hat{S}\psi_+ \rangle = \langle \hat{S}^{-1}\psi_- | \psi_+ \rangle = \langle \psi_- | K | \psi_+ \rangle \\ \langle \psi_- | K | \psi_+ \rangle \equiv \textstyle\int_0^\infty dx_+ dx_- \, \overline{\psi_-(x_-)} \, K(x_-, x_+)\psi_+(x_+). \end{aligned} \tag{V.17}$$

The integral (V.17) can be interpreted as a scalar product between the in and out states. Since the in and out eigenfunctions (V.12) form two complete systems of δ-function normalized orthonormal states and the S-matrix is diagonal on them, they satisfy the orthogonality relation

$$\langle \psi_-^{E} | K | \psi_+^{E'} \rangle = \mathcal{R}(E)\delta(E - E'). \tag{V.18}$$

Note that the fermionic S-matrix has been calculated in [97] from properties of the parabolic cylinder functions defined by equation (III.37). In our case it appears from the usual Fourier transformation and it does not involve a solution of complicated differential equations. Thus, the light-cone representation does crucially simplify the problem of scattering in 2D string theory.

1.3 Introduction of tachyon perturbations

But the main advantage of the light-cone representation becomes clear when one considers the tachyon perturbations. As we discussed, they should be introduced as coherent states of tachyons. Following this idea, we consider the one-fermion wave functions of the form

$$\Psi_{\pm}^{E}(x_{\pm}) = e^{\mp i\varphi_{\pm}(x_{\pm};E)}\psi_{\pm}^{E}(x_{\pm}), \tag{V.19}$$

where the expansion of the phase $\varphi_{\pm}(x_{\pm}; E)$ in powers of $x_{\pm}$ in the asymptotics $x_{\pm} \to \infty$ is fixed and gives the spectrum of tachyons. The exact form of $\varphi_{\pm}$ is determined by the condition that S-matrix (V.13) remains diagonal on the perturbed wave functions

$$\hat{S}\ \Psi_{+}^{E} = \Psi_{-}^{E}, \tag{V.20}$$

what means that Ψ_{+}^{E} and Ψ_{-}^{E} are two representations of the same physical state. This condition can also be expressed as the orthonormality of in and out eigenfunctions (V.19)

$$\langle \Psi_-^{E_-} | K | \Psi_+^{E_+} \rangle = \delta(E_+ - E_-) \tag{V.21}$$

with respect to the scalar product (V.17). The normalization to 1 fixes the zero mode of the phase.

Two remarks are in order. First, the wave functions (V.19) are not eigenfunctions of the Hamiltonian (V.11). Nevertheless, they can be promoted to solutions of the Schrödinger equation by replacement $x_\pm \to e^{\mp t} x_\pm$ and multiplying by the overall factor $e^{\mp \frac{1}{2}t}$. As we will show, this leads to a time-dependent Fermi sea and the corresponding string background. But the energy of the whole system remains constant. In principle, one can introduce a perturbed Hamiltonian with respect to which $\Psi_\pm^E$ would be eigenfunctions [120]. In the $\pm$-representations it is defined as solutions of the following equations

$$H_\pm = H_0^\pm + x_\pm \partial\varphi_\pm(x_\pm; H_\pm). \tag{V.22}$$

The orthonormality condition (V.21) can be equivalently rewritten as the condition that the Hamiltonians $H_\pm$ define the action of the same self-adjoint operator H. However, such Hamiltonian has nothing to do with the physical time evolution. In particular, with respect to the time defined by H the Fermi sea is stationary and its profile coincides with the classical trajectory of the fermion with the highest energy. Nevertheless, this Hamiltonian contains all information about the perturbation and may be a useful tool to investigate the perturbed system.

The second remark is that introducing the tachyon perturbations according to (V.19), we change the Hilbert space of the system. Indeed, such states cannot be created by an operator acting in the initial Hilbert space formed by the non-perturbed eigenfunctions (V.12). Roughly speaking, this is so because the scalar product of a perturbed state (V.19) with an eigenfunction (V.12) diverges. Therefore, in contrast to the infinitesimal perturbations considered in [95], the coherent states (V.19) are not elements of the Hilbert space associated with the fermionic ground state. Thus, our perturbation is intrinsically non-perturbative and we arrive at the following picture. With each tachyon background one can associate a Hilbert space. Its elements describe propagation of small tachyon perturbations over this background. But the Hilbert spaces associated with different backgrounds are not related to each other.

An explicit description of these backgrounds can be obtained in the quasiclassical limit $\mu \to \infty$, which is identified with the spherical approximation of string theory. In this limit the state of the system of free fermions is described as an incompressible Fermi liquid and, consequently, it is enough to define the region of the phase space filled by fermions to determine completely the state. Assuming that the filled region is connected, the necessary data are given by one curve representing the boundary of the Fermi sea. In a general case the curve is defined by a multivalued function. For example, for the ground state in the coordinates (x, p) it is given by the two-valued function $p(x)$ (III.60).

In [120] we found the equations determining the profile of the Fermi sea for the perturbation (V.19). They arise as saddle point equations for the double integral in the left hand side of (V.21). Each of the two equations defines a curve in the phase space on which the corresponding integral is localized. To produce δ-function these curves should coincide. As a result, we arrive at the condition that the following two equations should be compatible at $E_+ = E_- = -\mu$

$$x_+ x_- = M_\pm(x_\pm) \equiv \mu + x_\pm \partial\varphi_\pm(x_\pm; -\mu). \tag{V.23}$$

One of these equations is most naturally written as $x_+ = x_+(x_-)$ where the function $x_+(x_-)$ is single-valued in the asymptotic region $x_- \to \infty$. The other, in turn, is determined by the function $x_-(x_+)$ with the same properties in the asymptotics $x_+ \to \infty$. Their compatibility means that the functions $x_+(x_-)$ and $x_-(x_+)$ are mutually inverse. This condition imposes a restriction on the perturbing phases $\varphi_\pm$. It allows to restore the full phases from their asymptotics at infinity. The resulting curve coincides with the boundary of the Fermi sea.

Thus, we see that the tachyon perturbations are associated with changes of the asymptotic form of the Fermi sea of free fermions of the singlet sector of MQM. Given the asymptotics, the exact form can be found with help of equations (V.23) which express the matching condition of in-coming and out-going tachyons. Note that the replacement $x_\pm \to e^{\mp t}x_\pm$ does lead to a time-dependent Fermi sea.

1.4 Toda description of tachyon perturbations

Up to now, we considered perturbations of tachyons of arbitrary momenta. Let us restrict ourselves to the case which is the most interesting for us: when the momenta are imaginary and form an equally spaced lattice as in the compactified Euclidean theory or as in the presence of a finite temperature. Thus, the perturbations to be studied are given by the phases

$$\varphi_\pm(x_\pm; E) = V_\pm(x_\pm) + \frac{1}{2}\phi(E) + v_\pm(x_\pm; E), \tag{V.24}$$

where the asymptotic part has the following form

$$V_\pm(x_\pm) = \sum_{k\geq 1} t_{\pm k} x_\pm^{k/R}. \tag{V.25}$$

The rest contains the zero mode $\phi(E)$ and the part $v_\pm$ vanishing at infinity. They are to be found from the compatibility condition (V.20) (or (V.23)) and expressed through the parameters of the potentials (V.25). These parameters are the parameter R measuring the spacing of the momentum lattice, which plays the role of the compactification radius in the corresponding Euclidean theory, and the coupling constants $t_{\pm n}$ of the tachyons.

In the work [120] we demonstrated that with each coupling t_n one can associate a flow generated by some operator H_n. These operators are commuting in the sense of (II.89). Moreover, they have the same structure as the Hamiltonians from the Lax formalism of Toda hierarchy. Namely, one can introduce the analogs of the two Lax operators $L_\pm$ and the Hamiltonians H_n are expressed through them similarly to (II.88)

$$H_{\pm n} = \pm (L_\pm^{n/R})_{\gtrless} \pm \frac{1}{2}(L_\pm^{n/R})_0, \qquad n > 0. \tag{V.26}$$

Thus, the perturbations generated by (V.25) are integrable and described by Toda hierarchy.

This result has been proven by explicit construction of the representation of all operators of the Lax formalism of section II.5.2. The crucial fact for this construction is that in the basis of the non-perturbed functions (V.12) the operators of multiplication by $x_\pm$ coincide with the energy shift operator

$$\hat{x}_\pm \psi_\pm^E(x_\pm) = \hat{\omega}^{\pm 1}\psi_\pm^E(x_\pm), \qquad \hat{\omega} = e^{-i\partial_E}. \tag{V.27}$$

Due to this property the perturbed wave functions (V.19) can be obtained from the non-perturbed ones by action of some operators $\mathcal{W}_\pm$ in the energy space

$$\Psi^E_\pm \equiv e^{\mp i\varphi_\pm(x_\pm)}\psi^E_\pm = \mathcal{W}_\pm \psi^E_\pm. \tag{V.28}$$

The operators $\mathcal{W}_\pm$ are constructed from $\hat\omega$ and can be represented as series in $\hat\omega^{1/R}$

$$\mathcal{W}_\pm = e^{\mp i\phi/2} \left(1+\sum_{k\ge 1} w_{\pm k}\hat\omega^{\mp k/R}\right)\exp\left(\mp i\sum_{k\ge 1} t_{\pm k}\hat\omega^{\pm k/R}\right). \tag{V.29}$$

This shows that if one starts from a wave function of a given energy, for instance $E=-\mu$, then the perturbed function is a linear combination of states with energies $-\mu+in/R$. Of course, they do not belong to the initial Hilbert space, but this is not important for the construction. The important fact is that only a discrete set of energies appears. Therefore, one can identify this set of imaginary energies (shifted by $-\mu$ which plays the role of the Fermi level) with the discrete lattice $\hbar s$ of the Lax formalism.

It is easy to recognize the operators $\mathcal{W}_\pm$ as the dressing operators (II.98). The coupling constants $t_{\pm n}$ play the role of the Toda times. And the wave function $\Psi^E_\mp(x_+)$ appears as the Baker–Akhiezer function (II.90). Since the Lax operators act on the Baker–Akhiezer function as the simple multiplication operators, in our case they are just represented by $\hat x_\pm$. Their expansion in terms of $\hat\omega$ is given by the representation of $\hat x_\pm$ in the basis of $\Psi_\pm^{-\mu+in/R}$ and can be obtained by dressing $\hat\omega^{\pm 1}$. Similarly, the Orlov–Shulman operators (II.94) are the dressed version of the energy operator $-\hat E$

$$\begin{aligned} L_\pm &= \mathcal{W}_\pm\,\hat\omega^{\pm 1}\,\mathcal{W}_\pm^{-1} = e^{\mp i\phi/2}\hat\omega^{\pm 1}e^{\pm i\phi/2}\left(1+\sum_{k\ge 1} a_{\pm k}\hat\omega^{\mp k/R}\right), && \text{(V.30)}\\ M_\pm &= -\mathcal{W}_\pm\,\hat E\,\mathcal{W}_\pm^{-1} = \frac{1}{R}\sum_{k\ge 1} k t_{\pm k}L_\pm^{k/R} - \hat E + \frac{1}{R}\sum_{k\ge 1} v_{\pm k}L_\pm^{-k/R}. && \text{(V.31)}\end{aligned}$$

All the relations determining the Toda structure can be easily established. The only difficult place is the connection between the right and left representations. In section II.5.2 we mentioned that to define Toda hierarchy, the dressing operators must be subject of some condition. Namely, $\mathcal{W}_+^{-1}\mathcal{W}_-$ should not depend on the Toda times $t_{\pm n}$. It turns out that this condition is exactly equivalent to the requirement (V.20) which we imposed on the perturbations. Indeed, in terms of the dressing operators it is written as

$$\mathcal{W}_- = \mathcal{W}_+\hat{\mathcal{R}}, \tag{V.32}$$

where $\hat{\mathcal{R}}$ is the operator corresponding to (V.16). This operator is independent of the couplings, so that the necessary condition is fulfilled.

Besides, this framework provides us with the string equations in a very easy way. They are just consequences of the trivial relations[2]

$$[\hat x_+,\hat x_-] = -i, \qquad \hat x_+\hat x_- + \hat x_-\hat x_+ = -E. \tag{V.33}$$

[2]We neglect the subtleties related with the necessity to insert the S-matrix operator passing from the right to left representation and back. The exact formulae can be found in [120].

Relying on the defining equation (V.32), it was shown that one obtains the following string equations

$$L_+L_- = M_+ + i/2, \qquad L_-L_+ = M_- - i/2, \qquad M_- = M_+. \tag{V.34}$$

They resemble the string equations of the two-matrix model (II.151) and (II.152). Similarly to that case, only two of them are independent.

Actually, we should say that $L_\pm$ and $M_\pm$ are not exactly the operators that appear in the Lax formalism of Toda hierarchy. The reason is that they are series in $\hat{\omega}^{1/R}$ whereas the standard definition of the Lax operators (II.86) involves series in $\hat{\omega}$. This is because our shift operator is Rth power of the standard one what follows from the relation between s and the energy E. Therefore, the standard Lax and Orlov–Shulman operators should be defined as follows

$$L = L_+^{1/R}, \quad \bar{L} = L_-^{1/R}, \quad M = RM_+, \quad \bar{M} = RM_-. \tag{V.35}$$

In terms of these Lax operators the string equation (V.34) is rewritten as

$$[L^R, \bar{L}^R] = i. \tag{V.36}$$

This equation was first derived in [110] for the dual Toda hierarchy, where $R \to 1/R$, describing the winding perturbations of MQM considered in the previous chapter.[3] Thus, indeed there is a duality between windings and tachyons: both perturbations are described by the same integrable structure with dual parameters. Finally, the standard Toda times are related with the coupling constants as $t_{\pm n} \to \pm t_{\pm n}$ (see footnote 2 on page 54) and the Planck constant is pure imaginary $\hbar = i$.

1.5 Dispersionless limit and interpretation of the Lax formalism

Especially simple and transparent formulation is obtained in the dispersionless limit of Toda hierarchy. In this limit the state of the fermionic system is described by the Fermi sea in the phase space of free fermions. Therefore, we expect that the Toda hierarchy governs the dynamics of this sea.

Indeed, as we know from section II.5.6, the shift operator becomes a classical variable and together with the lattice parameter defines the symplectic form

$$\{\omega, E\} = \omega. \tag{V.37}$$

Remembering that the Lax operators $L_\pm$ coincide with the light-cone variables $x_\pm$, the classical limit of the string equations reads

$$\{x_-, x_+\} = 1, \tag{V.38}$$

$$x_+x_- = M_\pm(x_\pm) = \frac{1}{R}\sum_{k\geq 1} kt_{\pm k}\, x_\pm^{k/R} + \mu + \frac{1}{R}\sum_{k\geq 1} v_{\pm k}\, x_\pm^{-k/R}. \tag{V.39}$$

The first equation is nothing else but the usual symplectic form on the phase space of MQM. The second equation coincides with the compatibility condition (V.23) where the explicit form of the perturbing phase was substituted. Thus, this is the equation that determines the

[3] Before the work [110], only the string equation with integer R appeared in the literature [121, 122].

exact form of the Fermi sea. We note also that for the self-dual case $R=1$, the deformation described by (V.39) is similar to the deformation of the ground ring (III.100) suggested by Witten in [25].

We conclude that for the tachyon perturbations all the ingredients of the Lax formalism have a clear interpretation in terms of free fermions:

- the discrete lattice on which the Toda hierarchy is defined is the set of energies given by the sum of the Fermi level $-\mu$ and Euclidean momenta of the compactified theory;
- the Lax operators are the light-cone coordinates in the phase space of the free fermions;
- the Orlov–Shulman operators define asymptotics of the profile equation describing deformations of the Fermi level;
- the Baker–Akhiezer function is the perturbed one-fermion function;
- the first string equation describes the canonical transformation form the light-cone coordinates $x_\pm$ to the energy E and $\log\omega$.
- the second string equation is the equation for the profile of the Fermi sea.

1.6 Exact solution of the Sine–Liouville theory

As in the case of the winding perturbations, the Toda integrable structure can be applied to find the exact solution of the Sine–Liouville theory dual to the one considered in (IV.19). But in the present case we possess a more powerful tool to extract the solution: the string equation. In the dispersionless limit, it allows to avoid any differential equations and gives the solution quite directly.

In fact, the procedure leading to the solution is quite general and works for any potential of a finite degree. This procedure was suggested in [110] and can be summarized as follows. Let all $t_{\pm k}$ with $k>n$ vanish. Then in the dispersionless limit the representation (V.30) implies

$$x_\pm(\omega,E)=e^{-\frac{1}{2R}\chi}\omega^{\pm 1}\left(1+\sum_{k=1}^{n}a_{\pm k}(E)\,\omega^{\mp k/R}\right), \tag{V.40}$$

where $\chi=\partial_E^2\log\tau$.[4] The problem is to find the coefficients $a_{\pm k}$. For this it is enough to substitute the expressions (V.40), with $E=-\mu$, in the profile equations (V.39) and compare the coefficients in front of $\omega^{\pm k/R}$. Thus, the problem reduces to a finite triangular system of algebraic equations.

For the case of the Sine–Liouville theory, when there are only the first couplings $t_{\pm 1}$, the result of this procedure is the following [120]

$$x_\pm=e^{-\frac{1}{2R}\chi}\omega^{\pm 1}(1+a_\pm\omega^{\mp\frac{1}{R}}), \tag{V.41}$$

$$\mu e^{\frac{1}{R}\chi}-\frac{1}{R^2}\left(1-\frac{1}{R}\right)t_1t_{-1}e^{\frac{2R-1}{R^2}\chi}=1, \qquad a_\pm=\frac{t_{\mp 1}}{R}\,e^{\frac{R-1/2}{R^2}\chi}. \tag{V.42}$$

[4]From now on, we omit the index 0 indicating the spherical approximation.

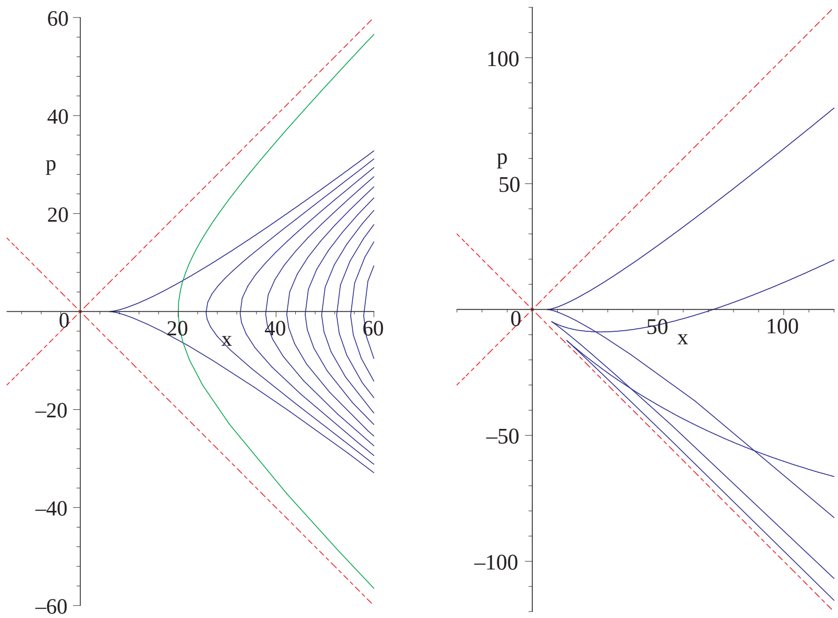


Fig. V.1: Profiles of the Fermi sea ($x_\pm = x \pm p$) in the theory of type I at $R = 2/3$. The first picture contains several profiles corresponding to $t_1 = t_{-1} = 2$ and values of μ starting from $\mu_c = -1$ with step 40. For comparison, the unperturbed profile for $\mu = 100$ is also drawn. The second picture shows three moments of the time evolution of the critical profile at $\mu = -1$.

This solution reproduces both the free energy and the one-point correlators. The former is expressed trough the solution of the first equation in (V.42) and the latter is contained in (V.41) where one should identify

$$\omega(x_\pm) = e^{\frac{1}{2R}\chi} x_\pm e^{-Rh_\pm(x_\pm)}. \tag{V.43}$$

Here $h_\pm(x_\pm)$ are the generating functions of the one-point correlators similar to (IV.51).

Comparing (V.42) and (V.41) with (IV.33) and (IV.60), respectively, one finds that the former equations are obtained from the latters by the following duality transformation[5]

$$R \to 1/R, \qquad \mu \to R\mu, \qquad t_n \to R^{-nR/2} t_n. \tag{V.44}$$

Note that it would be more natural to transform the couplings as $t_n \to R^{1-nR/2} t_n$ adding additional factor R. Then the scaling parameter $\lambda^2 \mu^{R-2}$ would not change. Moreover, in the CFT formulation this factor is naturally associated with the Liouville factor $e^{-2\phi}$ of each marginal operator. Its absence in our case is related to that we identified the winding couplings in incorrect way. The correct couplings $\tilde{t}_n$ are given in (IV.37). Thus, whereas the couplings of the tachyon perturbations are exactly the Toda times, for the winding perturbations they differ by factor R.

The explicit solution (V.41) determines the boundary of the Fermi sea, describing a condensate of tachyons of momenta $\pm 1/R$, in a parametric form. For each set of parameters

[5]Performing this transformation, one should take into account that the susceptibility transforms as $\chi \to R^{-2}\chi - R^{-1}\log R$. This result can be established, for example, from (IV.32).

one can draw the corresponding curve. The general situation is shown on fig. V.1. The unperturbed profile corresponds to the hyperbola asymptotically approaching the $x_\pm$ axis, whereas the perturbed curves deviate from the axes by a power law. We see that there is a critical value of μ, where the contour forms a spike. It coincides with (T-dual of) μ_c given by (IV.34). At this point the quasiclassical description breaks down and our results provide a geometric interpretation for this. On the second picture the physical time evolution of the critical profile is demonstrated.

It is interesting that only the case $t_{\pm 1} > 0$ has a good interpretation. In all other cases at some moment of time the Fermi sea begins to penetrate into the region $x < 0$. This corresponds to the transfusion of the fermions through the top of the potential to the other side. Such processes are forbidden at the perturbative level.

This can be understood for $t_1 t_{-1} < 0$ because the corresponding CFT is not unitary and one can expect some problems. On the other hand, the case $t_{\pm 1} < 0$ is well defined from the Euclidean CFT point of view, as is the case of positive couplings, because they differ just by a shift of the Euclidean time. It is likely that the problem is intrinsically related to the Minkowskian signature. In terms of the Minkowskian time the potential is given by $\cosh(t/R)$ and it is crucial with which sign it appears in the action. The two possibilities lead to quite different pictures as it is clear for our solution.

In the same way one can find the solution in the classical limit of the theory of type II. In this case one can introduce two pairs of perturbing potentials describing the asymptotics of the wave functions at $x_\pm \to \infty$ and $x_\pm \to -\infty$. For sufficiently large μ the Fermi sea consists of two connected components and the theory decomposes into two theories of type I. However, in contrast to the previous case, there are no restrictions on the signs of the coupling constants. When μ decreases, the two Fermi seas merge together at some critical value μ^*. This leads to interesting (for example, from the point of view of the Hall effect) phenomena. Here we will only mention that, depending on the choice of couplings, it can happen that for some interval of μ around the point μ^*, the Toda description is not applicable.

2 Thermodynamics of tachyon perturbations

2.1 MQM partition function as τ-function

In the previous section we showed that the tachyon perturbations with momenta as in the compactified Euclidean theory are described by the constrained Toda hierarchy. Hence, they are characterized by a τ-function. What is the physical interpretation of this τ-function? In [98] it was identified as the generating functional for scattering amplitudes of tachyons. On the other hand, since the tachyon spectrum coincides with that of the theory at finite temperature, it is tempting to think that the theory possesses a thermodynamical interpretation. Then the τ-function could be seen as the partition function of the model.

Although expected, the existence of the thermodynamical interpretation is not guaranteed because the system is formulated in the Minkowskian time and except the coincidence of the spectra there is no reference to a temperature. However, we will show that it does exist at least for the case of the Sine-Liouville perturbation [123]. In the following we will accept this point of view and will show that the τ-function is indeed the grand canonical partition function at temperature $T = 1/\beta$. In the spherical limit this was done in [120, 123] and we will present that derivation in the next paragraph. Here we prove the statement to all orders in perturbation theory following the work [124] which will be discussed in detail in chapter VI.

The grand canonical partition function is defined as follows

$$\mathcal{Z}(\mu, t) = \exp\left[\int_{-\infty}^{\infty} dE\, \rho(E) \log\left(1 + e^{-\frac{1}{\hbar}\beta(\mu+E)}\right)\right], \tag{V.45}$$

where $\rho(E)$ is the density of states. It can be found by confining the system in a box of size $\sim \sqrt{\Lambda}$ similarly as it was done in section III.2.4. The difference is that now we work in the light cone representation. Therefore, one should generalize the quantization condition (III.38). The generalization is given by

$$[\hat{S}\Psi](\sqrt{\Lambda}) = \Psi(\sqrt{\Lambda}) \tag{V.46}$$

so that one identifies the scattered state with the initial one at the wall. Then from the explicit form of the perturbed wave function (V.19) with (V.24) one finds [124]

$$\rho(E) = \frac{\log \Lambda}{2\pi} - \frac{1}{2\pi}\frac{d\phi(E)}{dE}. \tag{V.47}$$

Dropping out the Λ-dependent non-universal contribution, integrating by parts in (V.45), closing the contour in the upper half plane and taking the integral by residues of the thermal factor, one obtains[6] [124]

$$\mathcal{Z}(\mu, t) = \prod_{n\geq 0} \exp\left[\frac{i}{\hbar}\phi\left(i\hbar\frac{n+\frac{1}{2}}{R} - \mu\right)\right]. \tag{V.48}$$

[6]Actually, the function $\phi(E)$ has logarithmic cuts which contribute to the integral. However, their contribution reduces to the sum of integrals along these cuts of the form $\frac{\beta}{2\pi}\int_\Delta dE\, \frac{\phi(E+\epsilon)-\phi(E-\epsilon)}{1+e^{\beta(\mu+E)}}$. The discontinuity on the cut of $\phi(E)$ is constant so that we remain only with the integral of the thermal factor. As a result the nth cut gives the contribution $\sim \log\left(1 + e^{-\beta\left(\mu+i(n+\frac{1}{2})\right)}\right) = O(e^{-\beta\mu})$, which is non-perturbative and can be neglected in our consideration.

On the other hand, the zero mode of the perturbing phase is actually equal to the zero mode of the dressing operators (II.98). Hence it is expressed through the τ-function as in (II.103). Since the shift in the discrete parameter s is equivalent to an imaginary shift of the chemical potential μ, this formula reads

$$e^{\frac{i}{\hbar}\phi(-\mu)} = \frac{\tau_0\left(\mu + i\frac{\hbar}{2R}\right)}{\tau_0\left(\mu - i\frac{\hbar}{2R}\right)}. \tag{V.49}$$

Comparing this result with (V.48), one concludes that

$$\mathcal{Z}(\mu, t) = \tau_0(\mu, t). \tag{V.50}$$

2.2 Integration over the Fermi sea: free energy and energy

As always, in the dispersionless limit one can give to all formulae a clear geometrical interpretation. Moreover, this limit allows to obtain additional information about the system. Namely, in this paragraph we show how such thermodynamical quantities as the free energy and the energy of 2D string theory perturbed by tachyon sources are restored from the Fermi sea of the singlet sector of MQM. This was done in the work [123].

Since in the classical limit the profile of the Fermi sea uniquely determines the state of the free fermion system, it encodes all the interesting information. Indeed, in the full quantum theory the expectation value of an operator is given by its quantum average in the given state. Its classical counterpart is the integral over the phase space of the corresponding function multiplied by the density of states. For the free fermions the density of states equals 1 or 0 depending on either the phase space point is occupied or not. As a result, the classical value of an observable $\mathcal{O}$ is represented by its integral over the Fermi sea

$$< \mathcal{O} >= \frac{1}{2\pi} \int\int_{\text{Fermi sea}} dx_+ dx_- \, \mathcal{O}(x_+, x_-). \tag{V.51}$$

Now the description in terms of the dispersionless Toda hierarchy comes into the game. We saw that the tachyon perturbation gives rise to the canonical transformation from $x_\pm$ to the set of variables $(E, \log\omega)$. The explicit map is given by (V.40). Due to this the measure in (V.51) can be rewritten in terms of E and ω. As a result, one arrives at the following formula

$$\partial_\mu < \mathcal{O} >= -\frac{1}{2\pi} \int\limits_{\omega_-(\mu)}^{\omega_+(\mu)} \frac{d\omega}{\omega} \, \mathcal{O}(x_+(\omega, \mu), x_-(\omega, \mu)). \tag{V.52}$$

The limits of integration are defined by the cut-off. We choose it as two walls at $x_+ = \sqrt{\Lambda}$ and $x_- = \sqrt{\Lambda}$. Then the limits can be found from the equations

$$x_\pm(\omega_\pm(\mu), -\mu) = \sqrt{\Lambda}. \tag{V.53}$$

In the following we will need the solution of these equations to the second order in the cut-off. From (V.40) one can obtain the following result

$$\omega_\pm \approx \omega_{(0)}^{\pm 1}(1 \mp a_\pm \omega_{(0)}^{-1/R}), \qquad \omega_{(0)} = \sqrt{\Lambda e^{\chi/R}}. \tag{V.54}$$

The result (V.52) is rather general. It is valid for any observable. We applied it to the main two observables which are the number of particles and the energy. They were already defined in (III.27) and (III.26) for the case of the ground state. In a more general case they are written as integrals (V.51) with $\mathcal{O}=1$ and $\mathcal{O}=-x_+x_-$, respectively.

In the first case the integral (V.52) is easily calculated for the perturbing potential of any degree. For the energy, in principle, it can also be calculated for any finite n. However, the expressions become quite complicated and involve higher order terms of the expansion of the limits $\omega_\pm$ in Λ. Therefore, we restrict the calculation of the energy to the case of the Sine–Liouville perturbation. Then, taking into account (V.54), one finds [123]

$$\partial_\mu N = -\frac{1}{2\pi}\log\Lambda - \frac{1}{2\pi R}\chi, \qquad \text{(V.55)}$$

$$\partial_\mu E = \frac{1}{2\pi}e^{-\frac{1}{R}\chi}\left\{(1+a_+a_-)\left(\log\Lambda+\frac{1}{R}\chi\right)-2a_+a_-\right\}+\frac{R}{2\pi}(t_1+t_{-1})\Lambda^{1/2R}, \qquad \text{(V.56)}$$

where the first equation is valid for any $t_{\pm k}$. Using (V.42), which implies $a_+a_- = \frac{R}{1-R}e^{\frac{2R-1}{R^2}X}$, one can integrate (V.56) and obtain the following expression for the energy of the system

$$\begin{aligned} 2\pi E &= \xi^{-2}\left(\frac{1}{2R}e^{-\frac{2}{R}X}+\frac{2R-1}{R(1-R)}e^{-\frac{1}{R^2}X}-\frac{1}{2(1-R)}e^{-2\frac{1-R}{R^2}X}\right)(\chi+R\log\Lambda) \\ &+ \xi^{-2}\left(\frac{1}{4}e^{-\frac{2}{R}X}-\frac{R}{1-R}e^{-\frac{1}{R^2}X}+\frac{R(4-5R)}{4(1-R)^2}e^{-2\frac{1-R}{R^2}X}\right)+R(t_1+t_{-1})\mu\Lambda^{1/2R} \end{aligned} \qquad \text{(V.57)}$$

We observe that the last term is non-universal since it does not contain a singularity at $\mu=0$. However, $\log\Lambda$ enters in a non-trivial way. It is combined with a non-trivial function of μ and ξ.

To reproduce the free energy, note that in the grand canonical ensemble it is related to the number of particles through

$$N=\partial\mathcal{F}/\partial\mu, \qquad \text{(V.58)}$$

where we changed our notations assuming the usual thermodynamical definition of the free energy at the temperature T

$$\mathcal{F}=-T\log\mathcal{Z}. \qquad \text{(V.59)}$$

Comparing (V.58) and (V.55), one obtains the expected result

$$\mathcal{F}=-\frac{1}{\beta}\log\tau. \qquad \text{(V.60)}$$

In particular, the temperature is associated with $1/\beta$. Integrating the equation (V.42) for the susceptibility χ, one can get an explicit representation for the free energy. For example, in the Sine–Liouville case one obtains

$$2\pi\mathcal{F}=-\frac{1}{2R}\mu^2(\chi+R\log\Lambda)-\xi^{-2}\left(\frac{3}{4}e^{-\frac{2}{R}X}-\frac{R^2-R+1}{1-R}e^{-\frac{1}{R^2}X}+\frac{3R}{4(1-R)}e^{-2\frac{1-R}{R^2}X}\right). \qquad \text{(V.61)}$$

This result coincides with the T-dual transform of (IV.31).

2.3 Thermodynamical interpretation

In [123] we proved that the derived expressions for the energy and free energy allow a thermodynamical interpretation. This is not trivial because there are two definitions of the macroscopic energy. One of them is the sum of microscopic energies of the individual particles which is expressed as an integral over the phase space. We used this definition in the previous paragraph. Another definition follows from the first law of thermodynamics which relates the energy, free energy and entropy

$$S = \beta(E - F). \tag{V.62}$$

It allows to express the energy and entropy as derivatives of the free energy with respect to the temperature

$$E = \frac{\partial(\beta F)}{\partial \beta}, \qquad S = -\frac{\partial F}{\partial T}. \tag{V.63}$$

To have a consistent thermodynamical interpretation, the two definitions must give the same result. It is a very non-trivial check, although very simple from the technical point of view. All that we need is to differentiate the free energy, which is given in (V.61), and check that the result coincides with (V.57).

However, there are two important subtleties. The first one is that the first law is formulated in terms of the canonical free energy F rather than for its grand canonical counterpart $\mathcal{F}$. This is in agreement with the fact that it is the canonical free energy that is interpreted as the partition function of string theory. Thus, it is F that carries an information about properties of the string background and should be differentiated. Due to this one should pass to the canonical ensemble

$$F = \mathcal{F} - \mu \frac{\partial \mathcal{F}}{\partial \mu}. \tag{V.64}$$

Then from (V.61), (V.57) and (V.62), one finds the final expressions for the canonical free energy and entropy to be used in the thermodynamical formulae (V.63)

$$\begin{aligned} 2\pi F &= \frac{1}{R}\int^{\mu} s\chi(s)ds = \frac{1}{2R}\mu^2(\chi + R\log\Lambda) \\ &+ \xi^{-2}\left(\frac{1}{4}e^{-\frac{2}{R}X} - Re^{-\frac{1}{R^2}X} + \frac{R}{4(1-R)}e^{-2\frac{1-R}{R^2}X}\right), \end{aligned} \tag{V.65}$$

$$\begin{aligned} S &= \xi^{-2}\left(\frac{R}{1-R}e^{-\frac{1}{R^2}X} - \frac{1}{2(1-R)}e^{-2\frac{1-R}{R^2}X}\right)(\chi + R\log\Lambda) \\ &+ \xi^{-2}\left(-\frac{R^3}{1-R}e^{-\frac{1}{R^2}X} + \frac{R^2(3-4R)}{4(1-R)^2}e^{-2\frac{1-R}{R^2}X}\right) + \frac{R^2}{2\pi}(t_1 + t_{-1})\mu\Lambda^{1/2R}. \end{aligned} \tag{V.66}$$

If the first subtlety answers to the question *"what to differentiate?"*, the second one concerns the problem *"how to differentiate?"*. The problem is what parameters should be held fixed when one differentiates with respect to the temperature. First, this may be either μ or N. It is important to make the correct choice because they are non-trivial functions of each other. Since we are working in the canonical ensemble, it is natural to take N as independent variable. Besides, one should correctly identify the coupling λ and the cut-off Λ. Their definition can involve R and, thus, contribute to the result.

It turns out that the coupling and the cut-off that we have chosen are already the correct ones. In [123] it was shown by direct calculation that the thermodynamical relations (V.63), where the derivatives are taken with N, λ and Λ fixed, are indeed fulfilled. Thus, the 2D string theory perturbed by tachyons of momenta $\pm i/R$ in Minkowskian spacetime has a consistent interpretation as a thermodynamical system at temperature $T = 1/(2\pi R)$.

This result also answers to the question risen in section IV.2.3: what variable should be associated with the temperature? Our analysis definitely says that this is the parameter R. We do not need to introduce such notion as "temperature at the wall" [29, 30, 34]. The differentiation is done directly with respect to the compactification radius. Also this supports the idea that in the dual picture a black hole background should exist for any compactification radius, at least in the interval $1 < R < 2$.

We calculated entropy (V.66) using the standard thermodynamical relations. It vanishes in the absence of perturbations when $\lambda = 0$ but it is a complicated function in general case. It would be quite interesting to understand the microscopic origin of this entropy. In other words, we would like to find the microscopic degrees of freedom giving rise to the non-vanishing entropy. However, we have not found the solution yet. The problem is that a state of the system is uniquely characterized by the profile of the Fermi sea and there is only one profile described by our solution for each state. The only possibility which we found to obtain different microscopic states is to associate them with different positions in time of the same Fermi sea. Although the Fermi sea is time-dependent, all macroscopic thermodynamical quantities do not depend on time. Thus, different microscopic states would define the same macroscopic state. This idea is supported also by the fact that the entropy vanishes only if the Fermi sea is stationary. However, we have not succeeded to get the correct result for the entropy from this picture.

It is tempting to claim that the obtained thermodynamical quantities describe after the duality transformation (V.44) the thermodynamics of winding perturbations and their string backgrounds. This is, of course, true for the free energy. However, it is not clear whether the energy and entropy are dual in the two systems. For example, it is not understood even how to define the energy of a winding condensate. Nevertheless, if we assume that our results can be related to the backgrounds generated by windings, this gives a plausible picture. For example, the non-vanishing entropy is compatible with the existence of a black hole.

Our results concern arbitrary radius R and cosmological constant μ. When the latter goes to zero, which corresponds to the black hole point according to the FZZ conjecture (section I.6.3), the situation becomes little bit special. In this limit we find

$$\begin{aligned}
2\pi F &= \frac{(2R-1)^2}{4(1-R)}\tilde{\lambda}^{\frac{4R}{2R-1}}, \\
2\pi E &= \frac{2R-1}{2R(1-R)}\left(\log(\Lambda\xi) - \frac{R}{2(1-R)}\right)\tilde{\lambda}^{\frac{4R}{2R-1}}, \\
S &= \frac{2R-1}{2(1-R)}\left(\log(\Lambda\xi) - \frac{R^2(3-2R)}{2(1-R)}\right)\tilde{\lambda}^{\frac{4R}{2R-1}},
\end{aligned} \tag{V.67}$$

where $\tilde{\lambda} = \left(\frac{1-R}{R^3}\right)^{1/2}\lambda$. One observes that the logarithmic term $\xi^{-2}\log(\Lambda\xi)$ in the free energy disappears, whereas it is present in the energy and the entropy. This term is the leading one. Therefore, both the energy and the entropy are much larger than the free energy. This can explain the puzzle that, on the one hand, the dilaton gravity predicts the vanishing free energy and, on the other hand, the matrix model gives a non-vanishing result. Our approach shows that it does not vanish but it is negligible in comparison with other quantities so that in the main approximation the law $S = \beta E$ is valid.

3 String backgrounds from matrix solution

3.1 Collective field description of perturbed solutions

We introduced the tachyon perturbations as one of the ways to change the background of 2D string theory. In particular, we expect that the perturbations change the value of the tachyon condensate. To confirm this expectation, one should extract a target space picture from the matrix model solution describing the perturbed system. As we saw in the unperturbed case, for this purpose it is convenient to use the collective field theory of Das and Jevicki (see section III.3). Whereas the fermionic representation of section 1 is suitable for the solution of many problems, the relation to the target space phenomena is hidden in this formulation and it becomes clear in the collective field approach. We tried to understand it relying on the Das–Jevicki theory in the work [125].

We restrict ourselves to the spherical approximation. Then the string background is uniquely determined by the profile of the Fermi sea made of free fermions of the singlet sector of MQM. Thus, we should find the background using a given profile as a starting point. This was already done for the simplest case of the linear dilaton background which was obtained from the ground state of the fermionic system. In particular, we related the tachyon field to fluctuations of the density of matrix eigenvalues around the ground state. We expect that this identification holds to be true also in more complicated cases. All that we need to obtain another background is to replace the background value of the density by a new function determined by the exact form of the deformed profile of the Fermi sea. In this way one can find an effective action for the field describing the density fluctuations.

It appears after the substitution (III.61) into the background independent effective action of Das and Jevicki (III.54). However, in contrast to the previous case, the background value of the density φ_0 now depends on time t, so we write

$$\varphi(x,t)=\frac{1}{\pi}\varphi_0(x,t)+\frac{1}{\sqrt{\pi}}\partial_x\eta(x,t). \tag{V.68}$$

This time-dependence leads to additional terms. Extracting only the quadratic term of the expansion in the fluctuations η, one finds the following result

$$S_{(2)}=\frac{1}{2}\int dt\int\frac{dx}{\varphi_0}\left[(\partial_t\eta)^2-2\frac{\int dx\partial_t\varphi_0}{\varphi_0}\partial_t\eta\partial_x\eta-\left(\varphi_0^2-\left(\frac{\int dx\partial_t\varphi_0}{\varphi_0}\right)^2\right)(\partial_x\eta)^2\right]. \tag{V.69}$$

The crucial property of this action is that for any $\varphi_0(x,t)$ the determinant of the matrix coupled to the derivatives of η equals -1. Besides, there are no terms without derivatives. As a result, this quadratic part can be represented as the usual action for a massless scalar field in a curved metric $g_{\mu\nu}$.

$$S_{(2)}=-\frac{1}{2}\int dt\int dx\,\sqrt{-g}g^{\mu\nu}\partial_\mu\eta\partial_\nu\eta. \tag{V.70}$$

In a more general case we would have to introduce a dilaton dependent factor coupled to the kinetic term. The metric $g_{\mu\nu}$ in the coordinates (t,x) is fixed by (V.69) up to a conformal factor. For example, we can choose it to coincide with the matrix we were talking about, so that $\det g=-1$.

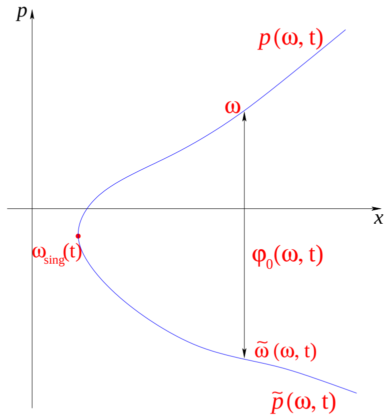


Fig. V.2: The Fermi sea of the perturbed MQM. Its boundary is defined by a two-valued function with two branches parameterized by $p(\omega,t)$ and $\tilde{p}(\omega,t)$. The background field φ_0 coincides with the width of the Fermi sea.

The action (V.70) is conformal invariant. Therefore, one can always redefine the coordinates to bring the metric to the usual Minkowski form $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$. We are able to find the transformation to such flat coordinates for a large class of functions $\varphi_0(x,t)$, which are solutions of the classical equations following from (III.54). In particular, it includes the integrable perturbations generated by the potential (V.25).

First, let us extract the background value of the density φ_0 from an MQM solution, which is usually formulated in terms of the exact form of the Fermi sea of the MQM singlet sector. The form of the sea can be described by the two chiral fields $p_\pm(x,t)$ introduced in (III.50). They are two branches of the function $p(x,t)$ representing the boundary and coincide with its upper and lower components.

Let ω be a parameter along the boundary. Then the position of the boundary in the phase space is given in the parametric form by two functions, $x(\omega,t)$ and $p(\omega,t)$. From the equation of motion (III.57) with the inverse oscillator potential, it is easy to derive that these functions should satisfy

$$\frac{\partial x}{\partial \omega}\left(\frac{\partial p}{\partial t} - x\right) = \frac{\partial p}{\partial \omega}\left(\frac{\partial x}{\partial t} - p\right). \qquad \text{(V.71)}$$

It is convenient also to introduce the“mirror” parameter $\tilde{\omega}(\omega,t)$ such that (see fig. V.2).

$$x(\tilde{\omega}(\omega,t),t) = x(\omega,t), \quad \tilde{\omega} \neq \omega. \qquad \text{(V.72)}$$

Then p_+ can be identified with $p(\omega,t)$ and p_- with $p(\tilde{\omega}(\omega,t),t)$. The solution for the background field is given by their difference and therefore it is represented in the parametric form

$$\varphi_0(\omega,t) = \frac{1}{2}(p(\omega,t) - \tilde{p}(\omega,t)), \qquad \text{(V.73)}$$

where we denoted $\tilde{p}(\omega,t) = p(\tilde{\omega}(\omega,t),t)$. Due to (III.53), (III.50) and (V.73), the effective action (V.69) can now be rewritten as

$$S_{(2)} = \int dt \int \frac{dx}{p-\tilde{p}} \left[(\partial_t\eta)^2 + (p+\tilde{p})\partial_t\eta\partial_x\eta + p\tilde{p}(\partial_x\eta)^2\right]. \qquad \text{(V.74)}$$

A solution of (V.71) can be easily constructed if one takes

$$\frac{\partial x}{\partial \omega} = p - \frac{\partial x}{\partial t}. \tag{V.75}$$

Then the equation (V.71) implies

$$\frac{\partial p}{\partial \omega} = x - \frac{\partial p}{\partial t}. \tag{V.76}$$

An evident solution of these two equations is

$$\begin{array}{rcl} p(\omega,t) & = & \sum\limits_{k=0}^{\infty} a_k \sinh\left[(1-b_k)\,\omega + b_k t + \alpha_k\right], \\ x(\omega,t) & = & \sum\limits_{k=0}^{\infty} a_k \cosh\left[(1-b_k)\,\omega + b_k t + \alpha_k\right], \end{array} \tag{V.77}$$

for any set of a_k, b_k and α_k. In principle, the solution can also contain a continuous spectrum.

Relying only on the property (V.75) and the relations following from the definition of the mirror parameter

$$\frac{\partial \tilde\omega}{\partial \omega} = \frac{\partial x/\partial \omega}{\tilde p - \tilde x_t}, \qquad \frac{\partial \tilde\omega}{\partial t} = \frac{p - \tilde x_t}{\tilde p - \tilde x_t} - \frac{\partial \tilde\omega}{\partial \omega}, \tag{V.78}$$

where $\tilde x_t \equiv \partial_t x(\tilde\omega, t)$, we showed [125] that the following coordinate transformation brings the action (V.74) to the standard form with the kinetic term written in the Minkowski metric $\eta_{\mu\nu}$

$$\tau = t - \frac{\omega+\tilde\omega}{2}, \qquad q = \frac{\omega - \tilde\omega}{2}. \tag{V.79}$$

The change of coordinates (V.79) is remarkably simple and has a transparent interpretation. It associates the light-cone coordinates $\tau \pm q$ with the parameters of incoming and outgoing tachyons, $t-\omega$ and $t-\tilde\omega$. For the ground state given by the solution (III.45), the mirror parameter is $\tilde\omega = -\omega$ so that we return to the simple situation described in section III.3.2.

In a particular case when $b_k = k/R$, the solution (V.77) reproduces the profile of the Fermi sea corresponding to the tachyon perturbations of the two previous sections. Indeed, in that case the description in terms of Toda hierarchy implies the representation (V.40). Returning from the light-cone to the (x,p) coordinates, one obtains (V.77), where ω is now the logarithm of the shift operator from the previous sections.

3.2 Global properties

The field η in coordinates (τ, q) satisfies the simple Klein–Gordon equation. Thus, the transformation (V.79) trivializes the dynamics and makes the integrability explicit. It is interesting that the spacetime in coordinates (τ, q) can still be non-trivial. Although the metric is flat and have the Minkowski form, we still have a possibility to have a non-trivial global structure because the image of the initial (t,x)-plane (or, more precisely, (t,ω)-plane, on which the initial solution is defined) under the coordinate transformation (V.79) can cover only a subspace of the plane of the new coordinates. If we identify this subspace as the physical region to be considered, the global structure of this space will be non-trivial. Depending on boundary conditions, either boundaries will appear or a compactification will take place.

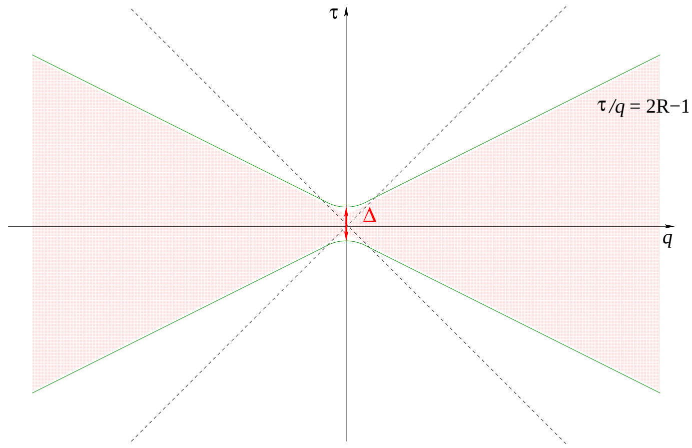


Fig. V.3: Flat spacetime of the perturbed theory for the case $R < 1$.

The explicit form of the transformation (V.79) allows to study the exact form of the resulting spacetime. This was done in [125] for the simplest case of the Sine–Liouville perturbation corresponding to the following parameters in (V.77)

$$\begin{array}{llllllll} a_0 &=& \sqrt{2}e^{-\frac{1}{2R}\chi}, & a_1 &=& \frac{\sqrt{2}\lambda}{R}e^{\frac{R-1}{2R^2}\chi}, & a_k &=& 0, \quad k>1, \\ b_0 &=& 0, & b_1 &=& 1/R, & \alpha_k &=& 0. \end{array} \quad \text{(V.80)}$$

The result crucially depends on the parameter R playing the role of the compactification radius. There are 3 different cases.

When $R \geq 1$ nothing special happens and the image of the (t, ω)-plane coincides with the whole plane of τ and q.

For $1/2 < R < 1$ the resulting space is deformed so that asymptotically it can be considered as two conic regions $|\tau/q| < 2R - 1$ (fig. V.3). When one approaches the origin, one finds that the two cones are smoothly glued along a finite interval of length

$$\Delta = 2R \log\left[\frac{R}{1-R}\frac{a_0}{a_1}\right]. \quad \text{(V.81)}$$

From this result it is easy to understand what happens when we switch off the perturbation. This corresponds to the limit $a_1 \to 0$. Then the interval Δ (the minimal distance between the upper and lower boundaries) logarithmically diverges and the boundaries go away to infinity. In this way we recover the entire Minkowski space. The similar picture emerges in the limit $R \to 1$ when we return to the case $R \geq 1$.

For $R < 1/2$ the picture is similar to the previous case, but the time and space coordinates are exchanged so that the picture should be rotated by $90°$.

For special value $R = 1/2$, the space is also deformed and has the form of a strip.

It is interesting that the deformed spacetime shown on fig. V.3 exhibits the same singularity as the one we found in the analysis of the free energy and the perturbed Fermi sea. The singularity appears when the length Δ vanishes so that the two conic regions separate from each other. The corresponding critical value of μ is exactly the same as (IV.34) after the duality transformation (V.44).

The obtained conic spacetimes possess boundaries. Therefore, some boundary conditions should be imposed on the fields propagating there. The most natural ones are vanishing and periodic boundary conditions. However, we have not been able to make a definite choice which conditions are relevant.

Another unsolved problem is related to the thermodynamical interpretation. We saw that the perturbed 2D string theory in the Minkowskian spacetime can be considered as a theory at finite temperature $T = 1/(2\pi R)$. It would be very interesting to reproduce this result from the analysis of quantum field theory in the spacetime obtained above. It is well known that moving boundaries or a compactification with varying radius can give rise to a thermal spectrum of observed particles [126]. Therefore, our picture seems to be reasonable. However, the problem is technically difficult due to the complicated form of the boundary near the origin.

3.3 Relation to string background

According to the analysis of section III.3.2, a massless scalar field of the collective theory in the flat coordinates can be identified with the tachyon field redefined by a dilaton factor. This was shown for the case of the linear dilaton background, but it is valid also in the perturbed case. Indeed, let us compare the action (V.70) to the low-energy effective action for the tachyon field (I.27) restricted to two dimensions. They coincide if one makes the usual identification $T = e^{\Phi}\eta$ and requires the following property

$$m_\eta^2 = (\nabla\Phi)^2 - \nabla^2\Phi - 4\alpha'^{-1} = 0, \tag{V.82}$$

which ensures that η is a massless field. This condition appears as an additional constraint for the equations of motion on the background fields. Its appearance is directly related to the restriction on the form of the action coming from the Das–Jevicki formalism. In particular, if the determinant of the matrix in (V.69) was arbitrary, we would not have this condition.

In [125] we argued that the constraint (V.82) selects a unique solution of the equations of motion similarly to an initial condition. It ensures that the metric and dilaton are fixed as in the usual linear dilaton background (I.23). As we know from section I.3.4, the tachyon can not be fixed by equations written in the leading order in α'. We expect that it does not vanish and modified by perturbations from the Liouville form (I.29). However, we cannot expect that the dilaton or the metric are modified too. Thus, the introduction of arbitrary tachyon perturbation cannot change the local structure of the target space: it always remains flat.

Note that this result illustrates that the T-dual theories on the world sheet are not the same in the target space. Whereas the CFT perturbed by windings looks similar to that of perturbed by tachyons, the former was supposed to correspond to the black hole background and the latter lives always in the target space of the vanishing curvature.

We conclude that the field η representing the fluctuations of the density of the matrix model and the coordinates (V.79) can be seen as the tachyon and coordinates of the string target space, respectively. However, as it was discussed in the end of section III.3.2, this identification is valid only in the free asymptotic region where one can neglect the Liouville and other interactions. In particular, in this region the change of coordinates (V.79) becomes inessential.

The exact relation between the collective field theory and the target space of string theory requires identification of the tachyon with the loop operator rather than with the density (see (III.78)). The former is the Laplace transform of the latter, so that the relation between the density and the string tachyon is non-local. For the simplest case of the ground state, it was shown that this integral transform maps the Klein–Gordon equation to the Liouville equation. In our case we expect to obtain the Liouville equation perturbed by the higher vertex operators. Let us check whether this is the case.

The loop operator $W(l,t)$ is related to the density fluctuations $\eta(x,t)$ by (III.75). Therefore, to derive an equation on $W(l,t)$, we should make the Laplace transform of the equation of motion following from (V.74). It is convenient to rewrite this equation in the following form

$$\partial_t^2\eta + \partial_x\left(p_+p_-\partial_x\eta\right) + x\partial_x\eta + \partial_x\left((p_+ + p_-)\partial_t\eta\right) = 0. \tag{V.83}$$

To obtain this result we used the equation (III.57) on $p_\pm(x,t)$ and the condition (V.72). Now we differentiate (V.83) with respect to x and substitute

$$\partial_x\eta(x,t) \longrightarrow \int\limits_{-i\infty}^{i\infty} dl\, e^{lx} W(l,t). \tag{V.84}$$

The resulting equation reads

$$\left[\partial_t^2 + l^2 p_+(-\partial_l,t)p_-(-\partial_l,t) - l\partial_l + l^2\left(p_+(-\partial_l,t) + p_-(-\partial_l,t)\right)\frac{1}{l}\partial_t\right] W(l,t) = 0. \tag{V.85}$$

Since, in general, $p_+(x,t)$ and $p_-(x,t)$ are very complicated functions, the derived equation does not look like the Liouville equation perturbed by vertex operators. Moreover, the operator in the left hand side is pseudodifferential. This becomes clear if we consider a limit of (V.85) where it takes a more explicit form.

Let us study the case of the Sine–Liouville perturbation. Then the function $p(x,t)$ is represented in the parametric form (V.77) with (V.80). When $\lambda = 0$, (V.85) reduces to the Liouville equation and one returns to the case of the linear dilaton background. We are interested in the first λ correction to this result. This is equivalent to the linear approximation in a_1. To this order, one has

$$\tilde\omega(\omega,t) \approx -\omega - \frac{2a_1}{a_0}\frac{\sinh\frac{t}{R}\sinh\left(\left(1-\frac{1}{R}\right)\omega\right)}{\sinh\omega}, \tag{V.86}$$

$$x(\omega,t) \approx a_0\cosh\omega + a_1\cosh\left(\left(1-\frac{1}{R}\right)\omega + \frac{t}{R}\right), \tag{V.87}$$

$$p(\omega,t) \approx a_0\sinh\omega + a_1\sinh\left(\left(1-\frac{1}{R}\right)\omega + \frac{t}{R}\right), \tag{V.88}$$

$$\tilde p(\omega,t) \approx -a_0\sinh\omega - a_1\left[\cosh\left(\left(1-\frac{1}{R}\right)\omega - \frac{t}{R}\right) + \frac{2\sinh\frac{t}{R}\sinh\left(\left(1-\frac{1}{R}\right)\omega\right)}{\tanh\omega}\right] \tag{V.89}$$

Substituting these expansions into (V.85), one obtains

$$\left[p_t^2 - (l\partial_l)^2 + a_0^2l^2 + 2a_1l^2\left\{a_0\cosh\frac{t}{R}\ \cosh\frac{\hat\omega}{R} + \sinh\frac{t}{R}\ \frac{\sinh\frac{\hat\omega}{R}}{\sinh\hat\omega}\frac{1}{l}\partial_t\right\}\right] W(l,t) = 0, \tag{V.90}$$

where $\hat{\omega}$ is considered as a differential operator $\hat{\omega} = \mathrm{arccosh}\left(-\frac{1}{a_0}\partial_l\right)$. This shows that even the first term in the λ-expansion of the equation on the loop operator of the matrix model does not have a simple form.

The usual identification of the Liouville coordinate ϕ suggests that $l = e^{-\phi}/\sqrt{2}$. This does not give any simplifications in the equation (V.90). May be they appear at least in the limit $\phi \to \infty$? Then $\hat{\omega} \sim \phi + \log \partial_\phi$ and (V.90) reduces to

$$\left[p_t^2 - (\partial_\phi)^2 + \frac{a_0^2}{2}\, e^{-2\phi} + \frac{a_0 a_1}{2}\, e^{-2\phi} \left(\frac{\sqrt{2}}{a_0} e^{\phi} \partial_\phi\right)^{\frac{1}{R}} \left\{\cosh\frac{t}{R} + \sinh\frac{t}{R}\, (\partial_\phi)^{-1}\, \partial_t\right\}\right] W(l(\phi), t) = 0. \tag{V.91}$$

We see that although the last term scales as $e^{\left(\frac{1}{R}-2\right)\phi}$, similarly to the Sine–Liouville term, it contains also pseudodifferential operators like $(\partial_\phi)^{1/R}$. As a result, the loop operator does not satisfy any equations of the Liouville type even in the asymptotic region and in the weak coupling regime.

This result suggests that the tachyon field of string theory is related to the collective field of MQM in a more complicated way than by the transformation (III.78). The exact relation can require the Laplace transform with respect to a different variable and a more complicated relation between the momentum and the Liouville coordinate. We have not succeeded to find it. Moreover, it seems that the analysis of the leading order in η is not enough to find the string background. For example, we showed how to relate a matrix model solution to a solution of the Klein–Gordon equation. As we know from section III.3.2, the latter can be transformed to a solution of the Liouville equation. Therefore, one could conclude that even a perturbed solution of the matrix model corresponds to the linear dilaton background, what is, of course, not true. Thus, one should involve an additional information to uniquely fix the background.

Note that all these results concern 2D string theory with tachyon condensation, but they are not related (at least directly) to the most interesting problem of 2D string theory in curved backgrounds. The latter should be obtained by winding perturbations. Although from the CFT point of view they are not much different from the tachyon perturbations, in the MQM framework their description is much more complicated. For the tachyon modes there is a powerful free fermion representation and, as a consequence, the Das–Jevicki collective field theory. It is this formulation that allows to get some information about the string target space. For windings there is no analog of this formalism and how to show, for example, that the winding perturbations of MQM correspond to the black hole background is still an open problem.

Chapter VI

MQM and Normal Matrix Model

In this chapter we present the next work included in this thesis [124]. It stays a little bit aside the main line of our investigation. Nevertheless it opens one more aspect of 2D string theory in non-trivial backgrounds and its Matrix Quantum Mechanical formulation, in particular. This work establishes an equivalence of the tachyonic perturbations of MQM described in the previous chapter with the so called Normal Matrix Model (NMM). This model appears in the study of various physical and mathematical problems. Therefore, before to discuss our results, we will briefly describe the main features of NMM and the related issues.

1 Normal matrix model and its applications

1.1 Definition of the model

The Normal Matrix Model was first introduced in [127, 128]. It is a statistical model of random complex matrices which commute with their conjugates

$$[Z, Z^\dagger] = 0. \tag{VI.1}$$

As usual, its partition function is represented as the matrix integral

$$Z_N = \int d\nu(Z, Z^\dagger)\, \exp\left[-\frac{1}{\hbar} W(Z, Z^\dagger)\right], \tag{VI.2}$$

where the measure $d\nu$ is a restriction of the usual measure on the space of all $N \times N$ complex matrices to those that satisfy the relation (VI.1). For our purposes we introduced explicitly the Planck constant at the place of N. They are supposed to be related in the large N limit which is obtained as $N \to \infty$, $\hbar \to 0$ with $\hbar N$ fixed.

The potentials which can be considered are quite general. We will be interested especially in the following type of potentials

$$W_R(Z, Z^\dagger) = \operatorname{tr}(ZZ^\dagger)^R - \hbar\gamma \operatorname{tr}\log(ZZ^\dagger) - \operatorname{tr} V(Z) - \operatorname{tr}\bar{V}(Z^\dagger). \tag{VI.3}$$

where

$$V(Z) = \sum_{k\geq 1} t_k Z^k, \qquad \bar{V} = \sum_{k\geq 1} t_{-k} {Z^\dagger}^k. \tag{VI.4}$$

The probability measure with such a potential depends on several parameters. These are R, γ and two sets of t_n and t_{-n}. The latter are considered as coupling constants because the dependence on them of the partition function contains an information about correlators of the matrix operators $\mathrm{tr}\, Z^k$. The other two parameters R and γ are just real numbers characterizing the particular model.

The partition function (VI.2) resembles the two-matrix model studied in section II.4. Similarly to that model, one can do the reduction to eigenvalues. The difference with respect to 2MM is that now the eigenvalues are complex numbers and the measure takes the form

$$d\nu(Z, Z^\dagger) = \frac{1}{N!}[d\Omega]_{SU(N)} \prod_{k=1}^{N} d^2 z_k \, |\Delta(z)|^2. \tag{VI.5}$$

Thus, instead of two integrals over real lines one has one integral over the complex plane. Despite this difference, one can still introduce the orthogonal polynomials and the related fermionic representation. Then, repeating the arguments of section II.5.7, it is easy to prove that the partition function (VI.2) is a τ-function of Toda hierarchy as well. For the case of $R = 1$ this was demonstrated in [73] and for generic R the proof can be found in appendix A of [124]. In fact, for the particular case $R = 1$ (and $\gamma = 0$) it was proven [74] that NMM and 2MM coincide in the sense that they possess the same free energy as function of the coupling constants. Nevertheless, their interpretation remains different.

The eigenvalue distribution of NMM in the large N limit is also similar to the picture arising in 2MM and shown on fig. II.7. The eigenvalues fill some compact spots on a two-dimensional plane. The only difference is that earlier this was the plane formed by real eigenvalues of the two matrices, and now this is the complex z-plane. Therefore, the width of the spots does not have anymore a direct interpretation in terms of densities. Instead, the density inside the spots for a generic potential can be non-trivial. For example, for the potential (VI.3) it is given by

$$\rho(z, \bar{z}) = \frac{1}{\pi}\partial_z \partial_{\bar{z}} W_R(z, \bar{z}) = \frac{R^2}{\pi}(z\bar{z})^{R-1}. \tag{VI.6}$$

For $R = 1$ where NMM reduces to 2MM, we return to the constant density.

1.2 Applications

Recently, the Normal Matrix Model found various physical applications [129, 130]. Most remarkably, it describes phenomena whose characteristic scale differs by a factor of 10^9. Whereas some of these phenomena are purely classical, another ones are purely quantum.

Quantum Hall effect

First, we mention the relation of NMM to the Quantum Hall effect [131]. There one considers electrons on a plane in a strong magnetic field B. The spectrum of such system consists from Landau levels. Even if the magnetic field is not uniform, the lowest level is highly degenerate. The degeneracy is given by the integer part of the total magnetic flux $\frac{1}{2\pi\hbar}\int B(z) d^2z$, and the one-particle wave functions at this level have the following form

$$\psi_n(z) = P_n(z) \exp\left(-\frac{W(z)}{2\hbar}\right). \tag{VI.7}$$

Here $W(z)$ is related to the magnetic field through $B(z) = \frac{1}{2}\Delta W(z)$ and $P_n(z)$ are holomorphic polynomials of degree n with the first coefficient normalized to 1.

Usually, one is interested in situations when all states at the lowest level are occupied. Then the wave function of N electrons is the Slater determinant of the one-particle wave functions (VI.7). Hence, it can be represented as

$$\Psi_N(z_1,\dots,z_N) = \frac{1}{\sqrt{N!}}\Delta(z)\exp\left(-\frac{1}{2\hbar}\sum_{k=1}^{N} W(z_k)\right). \tag{VI.8}$$

Its norm coincides with the probability measure of NMM. Therefore, the partition function (VI.2) appears in this picture as a normalization factor of the N-particle wave function

$$Z_N = \int \prod_{k=1}^{N} d^2 z_k \, |\Psi_N(z_1,\dots,z_N)|^2. \tag{VI.9}$$

Similarly, the density of electrons can be identified with the eigenvalue density and the same is true for their correlation functions.

Due to this identification, in the quasiclassical limit the study of eigenvalue spots is equivalent to the study of electronic droplets. In particular, varying the matrix model potential one can investigate how the shape of the droplets changes with the magnetic field. At the same time, varying the parameter $\hbar N$ one examines its evolution with increasing the number of electrons.

Note that although we discussed the semiclassical regime, the system remains intrinsically quantum. The reason is that all electrons under consideration occupy the same lowest level, whereas the usual classical limit implies that higher energy levels are most important.

Laplacian growth and interface dynamics

It was shown [131] that when one increases the number of electrons the semiclassical droplets from the previous paragraph evolve according to the so called Darcy's law which is also known as Laplacian growth. It states that the normal velocity of the boundary of a droplet occupying a simply connected domain $\mathcal{D}$ is proportional to the gradient of a scalar function

$$\frac{1}{\hbar}\frac{\delta\vec{n}}{\delta N} \sim \vec{\nabla}\varphi(z), \tag{VI.10}$$

which is harmonic outside the droplet and vanishes at its boundary

$$\begin{array}{rcll} \Delta\varphi(z,\bar z) & = & 0, & z\in \mathbf{C}\backslash\mathcal{D}, \\ \varphi(z,\bar z) & = & 0, & z\in\partial\mathcal{D}. \end{array} \tag{VI.11}$$

In the matrix model this function appears as the following correlator

$$\varphi(z,\bar z) = \hbar\left\langle \operatorname{tr}\left(\log(z-Z)(\bar z - Z^\dagger)\right)\right\rangle. \tag{VI.12}$$

It turns out that exactly the same law governs the dynamics of viscous flows. This phenomenon appears when an incompressible fluid with negligible viscosity is injected into a viscous fluid. In this case the harmonic function φ has a concrete physical meaning. It is identified with the pressure in the viscous fluid $\varphi = -P$. In fact, the Darcy's law is only an approximation to a real evolution. Whereas the condition $P = 0$ in the incompressible fluid is reasonable, the vanishing of the pressure at the interface is valid only when the surface tension can be neglected. This approximation fails to be true when the curvature of the boundary becomes large. Then the dynamics is unstable and the incompressible fluid develops many fingers so that its shape looks as a fractal. This is known as the Saffman–Taylor fingering.

NMM provides a mathematical framework for the description of this phenomenon. From the previous discussion it is clear that it describes the interface dynamics in the large N limit. In this approximation the singularity corresponding to the described instabilities arises when the eigenvalue droplet forms a spike which we encountered already in the study of the Fermi sea of MQM (section V.1.6). We know that at this point the quasiclassical approximation is not valid anymore. But the full quantum description still exists. Therefore, it is natural to expect that the fingering, which is a feature of the Laplacian growth, can be captured by including next orders of the $1/N$ expansion of NMM.

Complex analysis

The Darcy's law (VI.10) shows that there is a relation between NMM and several problems of complex analysis. Indeed, on the one hand, it can be derived from NMM as the evolution law of eigenvalue droplets and, on the other hand, it gives rise to a problem to find a harmonic function given by the domain in the complex plane. The latter problem appears in different contexts such as the conformal mapping problem, the Dirichlet boundary problem, and the 2D inverse potential problem [117]. For instance, if we fix the asymptotics of $\varphi(z,\bar z)$ requiring that

$$\varphi(z,\bar z) \underset{z\to\infty}{\sim} \log|z|, \tag{VI.13}$$

the solution of (VI.11) is unique and given by the holomorphic function $\omega(z)$

$$\varphi(z,\bar z) = \log|\omega(z)|, \tag{VI.14}$$

which maps the domain $\mathcal{D}$ onto the exterior of the unit circle and has infinity as a fixed point. To find such a function is the content of the conformal mapping problem.

Further, the Dirichlet boundary problem, which is to find a harmonic function $f(z,\bar z)$ in the exterior domain given a function $g(z)$ on the boundary of $\mathcal{D}$, is solved in terms of the above defined $\omega(z)$. The solution is given by

$$f(z,\bar z) = -\frac{1}{\pi i}\oint_{\partial\mathcal{D}} g(\zeta)\partial_\zeta G(z,\zeta)d\zeta, \tag{VI.15}$$

where the Green function is

$$G(z,\zeta) = \log\left|\frac{\omega(z)-\omega(\zeta)}{\omega(z)\overline{\omega(\zeta)}-1}\right|. \tag{VI.16}$$

In turn, $\omega(z)$ is obtained as the holomorphic part of $G(z,\infty)$.

Finally, the inverse potential problem can be formulated as follows. Let the domain $\mathcal{D}$ is filled by a charge spread with some density. The charge creates an electrostatic potential which is characterized by two functions φ_+ and φ_- defined in the interior and exterior domains, respectively. They and their derivatives are continuous at the boundary $\partial\mathcal{D}$. The problem is to restore the form of the charged domain given one of these functions. At the same time, when both of them are known the task can be trivially accomplished. Therefore, the problem is equivalent to the question how to restore φ_- from φ_+. Its relation to the Dirichlet boundary problem becomes now evident because, since φ_- is harmonic, it is given (up to a logarithmic singularity at infinity) by the formula (VI.15) with $g = \varphi_+$.

Thus, we see that the Normal Matrix Model provides a unified description for all these mathematical and physical problems. The main lesson which we learn from this is that all of them possess a hidden integrable structure revealed in NMM as Toda integrable hierarchy.

2 Dual formulation of compactified MQM

2.1 Tachyon perturbations of MQM as Normal Matrix Model

In section V.1 we showed how to introduce tachyon perturbations into the Matrix Quantum Mechanical description of 2D string theory. Although it is still the matrix model framework, we have done it in an unusual way. Instead to deform the matrix model potential, we have reduced MQM to the singlet sector and changed there the one-fermion wave functions. One can ask: can the resulting partition function be represented directly as a matrix integral?

In fact, the tachyon perturbations of MQM are quite similar to the perturbations of the two-matrix model. First, they are both described by Toda hierarchy. Second, the phase space of MQM looks as the eigenvalue (x, y) plane of 2MM. In the former case the fermions associated with the time dependent eigenvalues fill the non-compact Fermi sea, whereas in the latter case they fill some spots. If the 2MM potential is unstable like the inverse oscillator potential $-x_+x_-$, the spots will be non-compact as well. Thus, it is tempting to identify the two pictures.

Of course, MQM is much richer theory than 2MM and one may wonder how such different theories could be equivalent. The answer is that we have restricted ourselves just to a little sector of MQM. First, we use the restriction to the singlet sector and, second, we are interested only in the scattering processes. This explains why only two matrices appear. They are associated with in-coming and out-going states or, in other words, with $M(-\infty)$ and $M(\infty)$.

But it is easy to guess that the idea to identify MQM perturbed by tachyons with 2MM does not work. Indeed, the partition function of 2MM is a τ-function of Toda hierarchy when it is considered in the canonical ensemble. Therefore, one should find a representation of the grand canonical partition function of MQM which has the form of a canonical one. Such a representation does exist and is given by (V.48). But it implies a discrete equally spaced energy spectrum. It is evident for the system without perturbations where the partition function is a product of $\mathcal{R}$-factors (V.16) corresponding to $E_n = -\mu + i\hbar\frac{n+\frac{1}{2}}{R}$. However, it is difficult to obtain such a spectrum from two hermitian matrices. Moreover, one can show that their diagonalization would produce Vandermonde determinants of monomials of incorrect powers.

All these problems are resolved if one considers another model of two matrices which is the NMM. In the work [124] we proved that the grand canonical partition function of MQM with tachyon perturbations coincides with a certain analytical continuation of the canonical partition function of NMM. Thus, NMM can be regarded as a new realization of 2D string theory perturbed by tachyons.

Our proof is based on the fact that the two partition functions are τ-functions of Toda hierarchy. Therefore, it is enough to show that they are actually the same τ-function. This fact follows from the coincidence of either string equations or the initial conditions given by the non-perturbed partition functions. Finally, one should correctly identify the parameters of the two models. We suggested two ways to identify the parameters.

Model I

First, let us consider NMM given by the integral (VI.2) with the potential (VI.3) where $\gamma = \frac{1}{2}(R-1) + \frac{\alpha}{\hbar}$. We denote its partition function by $\mathcal{Z}_{\hbar}^{\text{NMM}}(N, t, \alpha)$. Then there is the following identification

$$\mathcal{Z}_{\hbar}^{\text{MQM}}(\mu, t) = \lim_{N\to\infty} \mathcal{Z}_{i\hbar}^{\text{NMM}}(N, t, R\mu - i\hbar N). \tag{VI.17}$$

We proved this result by direct comparison of the two non-perturbed partition functions. Then the coincidence (VI.17) follows from the fact that both $\mathcal{Z}^{\text{MQM}}$ and $\mathcal{Z}^{\text{NMM}}$ are τ-functions.

According to (V.48), the non-perturbed partition function of MQM is given by

$$\mathcal{Z}_{\hbar}^{\text{MQM}}(\mu, 0) = \prod_{n\geq 0} \mathcal{R}\left(i\hbar\frac{n+\frac{1}{2}}{R} - \mu\right), \tag{VI.18}$$

where $\mathcal{R}$ is the reflection coefficient (V.16). (Recall that $\mathcal{R}$ is related to the zero mode ϕ of the perturbing phase through $\log\mathcal{R}(E) = \frac{i}{\hbar}\phi(E)|_{t_n=0}$.) On the other hand, the partition function of NMM similarly to 2MM can be represented as a product (II.62) of the normalization coefficients of orthogonal polynomials. When all $t_{\pm k} = 0$, the orthogonal polynomials are simple monomials and the normalization factors h_n are given by

$$h_n(\alpha) = \frac{1}{2\pi i}\int_{\mathbf{C}} d^2z\, e^{-\frac{1}{\hbar}(z\bar{z})^R}(z\bar{z})^{(R-1)/2+\frac{\alpha}{\hbar}+n}. \tag{VI.19}$$

Up to some inessential factors, the integral produces the same Γ-function which appears in the expression for the reflection coefficient $\mathcal{R}$. Therefore, up to non-perturbative corrections, one can identify

$$h_n(\alpha) \sim \Gamma\left(\frac{\alpha}{\hbar R} + \frac{n+\frac{1}{2}}{R} + \frac{1}{2}\right) \sim \mathcal{R}\left(-i\hbar\frac{n+\frac{1}{2}}{R} - i\frac{\alpha}{R}\right). \tag{VI.20}$$

Given this fact, it is trivial to establish the relation (VI.17) in the non-perturbed case:

$$\begin{aligned}\lim_{N\to\infty}\mathcal{Z}_{i\hbar}^{\text{NMM}}(N, 0, R\mu - i\hbar N) &= \lim_{N\to\infty}\prod_{n=0}^{N-1} h_n(R\mu - \hbar N)|_{\hbar\to i\hbar} \\ = \lim_{N\to\infty}\prod_{n=0}^{N-1}\mathcal{R}\left(i\hbar\tfrac{N-n-\frac{1}{2}}{R} - \mu\right) &= \prod_{n=0}^{\infty}\mathcal{R}\left(i\hbar\tfrac{n+\frac{1}{2}}{R} - \mu\right) = \mathcal{Z}_{\hbar}^{\text{MQM}}(\mu, 0).\end{aligned} \tag{VI.21}$$

Besides, it is easy to show that a shift of the discrete charge s of the τ-function associated with $\mathcal{Z}^{\text{NMM}}$ is equivalent to an imaginary shift of μ:

$$\tau_s(\mu, t) = \tau_0\left(\mu + i\hbar\frac{s}{R}, t\right). \tag{VI.22}$$

As we know, this is the characteristic property of the τ-function of MQM. This completes the proof of (VI.17).

Note that although the relation (VI.17) involves the large N limit, it is valid to all orders in the genus expansion. This is because N enters non-trivially through the parameter α. Actually, N appears always in the combination with μ like in (VI.17). This can be understood from the fact that the discrete charge of the τ-function is identified, on the one hand, with N (see (II.149) for the 2MM case) and, on the other hand, with $-\frac{i}{\hbar}R\mu$ (see (VI.22)).

Model II

This fact hints that there should exist another way to match the two models where these parameters are directly identified with each other

$$N = -\frac{i}{\hbar} R\mu. \tag{VI.23}$$

This gives the second model proposed in [124], which relates the two partition functions as follows

$$\mathcal{Z}_{\hbar}^{\mathrm{MQM}}(\mu, t) = \mathcal{Z}_{i\hbar}^{\mathrm{NMM}}(-\frac{i}{\hbar}R\mu, t, 0). \tag{VI.24}$$

This second model is simpler than the first one because it does not involve the large N limit and allows to compare the $1/N$ expansion of NMM directly with the $1/\mu$ expansion of MQM.

The equivalence of the two partition functions is proven in the same way as above. As for the first model, they are both given by τ-functions of Toda hierarchy. After the identification (VI.23), the charges of these τ-functions are identical. Therefore, it only remains to show that without the perturbation $\mathcal{Z}_{i\hbar}^{\mathrm{NMM}}(N, 0)$ is equal to the unperturbed partition function (VI.18). In this case the method of orthogonal polynomials together with (VI.20) gives

$$\mathcal{Z}_{i\hbar}^{\mathrm{NMM}}(N, 0) = \prod_{n=0}^{N-1} \mathcal{R}\left(-i\hbar \frac{n+\frac{1}{2}}{R}\right). \tag{VI.25}$$

Then we represent the finite product as a ratio of two infinite products

$$\mathcal{Z}_{i\hbar}^{\mathrm{NMM}}(N, 0) = \Xi(0)/\Xi(N), \qquad \text{where} \quad \Xi(N) = \prod_{n=N}^{\infty} \mathcal{R}\left(-i\hbar(n+\frac{1}{2})/R\right). \tag{VI.26}$$

$\Xi(0)$ is a constant and can be neglected, whereas $\Xi(N)$ can be rewritten as

$$\Xi(N) = \prod_{n=0}^{\infty} \mathcal{R}\left(-i\hbar N/R - i\hbar(n+\frac{1}{2})/R\right). \tag{VI.27}$$

Taking into account the unitarity of the $\mathcal{R}$-factor,

$$\overline{\mathcal{R}(E)}\mathcal{R}(E) = \mathcal{R}(-E)\mathcal{R}(E) = 1, \tag{VI.28}$$

and substituting N from (VI.23), we obtain

$$\mathcal{Z}_{i\hbar}^{\mathrm{NMM}}(-\frac{i}{\hbar}R\mu, 0) \sim \Xi^{-1}(-\frac{i}{\hbar}R\mu) = \prod_{n=0}^{\infty} \mathcal{R}\left(\mu + i\hbar(n+\frac{1}{2})/R\right) = \mathcal{Z}_{\hbar}^{\mathrm{MQM}}(\mu, 0). \tag{VI.29}$$

Note that the difference in the sign of μ from (VI.18) does not matter since the partition function is an even function of μ (up non-universal terms). Since the two partition functions are both solutions of the Toda hierarchy, the fact that they coincide at $t_k = 0$ implies that they coincide for arbitrary perturbation.

2.2 Geometrical description in the classical limit and duality

The relation of the perturbed MQM and NMM is a kind of duality. Most explicitly, this is seen in the classical limit where the both models have a geometrical description in terms of incompressible liquids. In the case of MQM, it describes the Fermi sea in the phase space parameterized by two real coordinates $x_\pm$, whereas in the case of NMM the liquid corresponds to the compact eigenvalue spots on the complex z-plane. Thus, the first conclusion is that the variables of one model are obtained from the variables of the other by an analytical continuation. The exact relation is the following

$$x_+ \leftrightarrow z^R, \qquad x_- \leftrightarrow \bar{z}^R. \tag{VI.30}$$

It relates all correlators in the two models if simultaneously one substitutes $\hbar \to i\hbar$ and $N = -iR\mu/\hbar$.

In particular, the analytical continuation (VI.30) replaces the non-compact Fermi sea of MQM by a compact eigenvalue spot of NMM. We have already discussed in the context of MQM that the profile of the Fermi sea is determined by the solution $x_-(x_+)$ (or or its inverse $x_+(x_-)$ depending on what asymptotics is considered) of the string equation (V.39). The same is true for the boundary of the NMM spot. Since the two models coincide, the string equations are also the same up to the change (VI.30). Nevertheless, they define different profiles. The MQM equation gives a non-compact curve and the NMM equation leads to a compact one. For example, when all $t_n = 0$ the two equations read

$$x_+ x_- = \mu, \qquad (z\bar{z})^R = \hbar N/R \tag{VI.31}$$

and describe a hyperbola and a circle, correspondingly. This explicitly shows how the analytical continuation relates the Fermi seas of the two models.

A more transparent way to present this relation is to consider a complex curve associated with the solution in the classical limit. We showed in section II.4.5 how to construct such a curve for the two-matrix model. NMM does not differ from 2MM in this respect and the construction can be repeated in our case. The only problem is that for generic R the potential (VI.3) involves infinite branch singular point. Therefore, the curve given by the Riemann surface of the function $\bar{z}(z)$, which describes the shape of the eigenvalue spots, is not rational anymore and the results of [74] cannot be applied. However, the complex curve constructed as a "double" still exists and has the same structure as for the simple $R = 1$ case shown in fig. II.8.

In the work [124] we considered the situation when there is only one simply connected domain filled by eigenvalues of the normal matrix. This restriction corresponds to the fact that the dual Fermi sea of MQM is simply connected. If we give up this restriction, it would correspond to excitations of MQM which break the Fermi sea to several components. They represent a very interesting issue to study but we have not considered them yet.

When there is only one spot, one gets the curve shown in fig. VI.1. It is convenient to think about it as a curve embedded into $\mathbf{C}^2$. Let the coordinates of $\mathbf{C}^2$ are parameterized by z and $\bar{z}$. Then the embedding is defined by the function $\bar{z}(z)$ (or its inverse $z(\bar{z})$) which is a solution of the string equation. But we know that when z and $\bar{z}$ are considered as complex conjugated this function defines also the boundary of the eigenvalue spot. At the same time if one takes z and $\bar{z}$ to be real, the same function gives the profile of the Fermi sea of MQM.

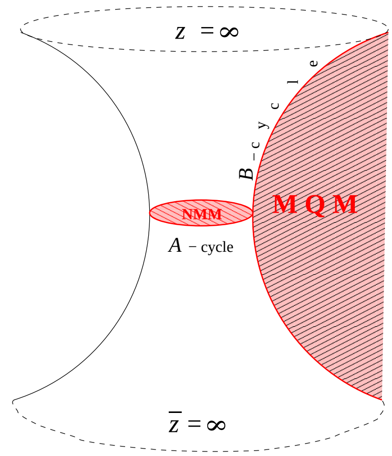


Fig. VI.1: Complex curve associated with both models, NMM and MQM. The regions filled by eigenvalues of the two models coincide with two real sections of the curve. The duality exchanges the A and B cycles which bound the filled regions.

Thus, the two models are associated with two real sections of the same complex curve: its intersection with the planes

$$z^* = \bar{z} \quad \text{and} \quad z^* = z, \ \bar{z}^* = \bar{z} \tag{VI.32}$$

coincides with the boundary of the region filled by eigenvalues of either NMM or MQM, respectively. It is clear that these sections can be thought also as the pair of non-contractible cycles A and B on the curve.

Moreover, the integrals over these cycles, which give the moduli of the curve, also exhibit duality. To discuss them, one should define a holomorphic differential to be integrated along the cycles. In [124] we derived it using the interpretation of the large N limit of NMM in terms of the inverse potential problem discussed in the previous section. This interpretation supplies us with the notion of electrostatic potential of a charged domain, which appears to be very natural in our context.

Such potential is a harmonic function outside the domain $\mathcal{D}$ and it is a solution of the Laplace equation with the density (VI.6) inside the domain

$$\varphi(z,\bar{z}) = \begin{cases} \varphi(z) + \bar{\varphi}(\bar{z}), & z \notin \mathcal{D}, \\ (z\bar{z})^R, & z \in \mathcal{D}. \end{cases} \tag{VI.33}$$

To fix the potential completely, we should also impose some asymptotic condition at infinity. This asymptotics is determined by the coupling constants $t_{\pm n}, n = 1, 2, \ldots$ and can be considered as the result of placing a dipole, quadrupole etc. charges at infinity.

The solution of this electrostatic problem is obtained as follows. The continuity of the potential $\varphi(z,\bar{z})$ and its first derivatives leads to the following conditions to be satisfied on the boundary $\gamma = \partial\mathcal{D}$

$$\varphi(z) + \bar{\varphi}(\bar{z}) = (z\bar{z})^R, \tag{VI.34}$$

$$z\partial_z\varphi(z) = \bar{z}\partial_{\bar{z}}\bar{\varphi}(\bar{z}) = Rz^R\bar{z}^R. \tag{VI.35}$$

Each of two equations (VI.35) can be interpreted as an equation for the contour γ. Since we obtain two equations for one curve, they should be compatible. It is clear that these equations are nothing else but the string equation. Therefore, solutions for the chiral fields φ and $\bar{\varphi}$ can be found comparing (VI.35) with (V.39). In this way we have

$$\begin{aligned} \varphi(z) &= \hbar N \log z + \tfrac{1}{2}\phi + \sum_{k\geq 1} t_k z^k - \sum_{k\geq 1} \tfrac{1}{k} v_k z^{-k}, \\ \bar{\varphi}(\bar{z}) &= \hbar N \log \bar{z} + \tfrac{1}{2}\phi + \sum_{k\geq 1} t_{-k} \bar{z}^k - \sum_{k\geq 1} \tfrac{1}{k} v_{-k} \bar{z}^{-k}. \end{aligned} \tag{VI.36}$$

The zero mode ϕ is fixed by the condition (VI.34). However, we already know its relation to the τ-function given by (II.103). Thus, in the dispersionless limit $\hbar \to 0$, we find

$$\phi = -\hbar \frac{\partial}{\partial N} \log \mathcal{Z}_{\hbar}^{\text{NMM}}. \tag{VI.37}$$

Now we can construct the holomorphic differential on the curve described above. It is given by

$$\Phi \overset{\text{def}}{=} \begin{cases} \Phi_+(z) = \varphi(z) - \frac{1}{2}(z\bar{z}(z))^R & \text{in the north hemisphere,} \\ \Phi_-(\bar{z}) = -\bar{\varphi}(\bar{z}) + \frac{1}{2}(z(\bar{z})\bar{z})^R & \text{in the south hemisphere.} \end{cases} \tag{VI.38}$$

The field Φ gives rise to a closed (but not exact) holomorphic 1-form $d\Phi$, globally defined on the complex curve. Its analyticity follows from equation (VI.34), which now holds on the entire curve, since z and $\bar{z}$ are no more considered as conjugated to each other. With this definition and using the relation of the zero mode of the electrostatic potential to the τ-function, it is easy to calculate the integrals of $d\Phi$ around the cycles:

$$\frac{1}{2\pi i} \oint_A d\Phi = \hbar N = R\mu, \qquad \int_B d\Phi = \hbar \frac{\partial}{\partial N} \log \mathcal{Z}_{\hbar}^{\text{NMM}} = -\frac{1}{R}\frac{\partial \mathcal{F}}{\partial \mu}. \tag{VI.39}$$

Here the first integral is obtained by picking up the pole and the second integral is given by the zero mode ϕ and the diverging non-universal contribution, which we neglected. This result can also be found by means of the procedure of transfer of an eigenvalue from a point belonging to the boundary γ of the spot to infinity. This procedure was first applied to 2MM in [74] and then generalized to the present case in [124].

As the relations (VI.39) are written, the duality between MQM and NMM is not seen. It becomes evident if to remember that the free energies are taken in different ensembles. If one changes the ensemble, the cycles are exchanged. For example, for the canonical free energy of MQM defined as $F = \mathcal{F} + \hbar R \mu M$, where $M = -\frac{1}{\hbar R}\frac{\partial \mathcal{F}}{\partial \mu}$ is the number of eigenvalues, the relations (VI.39) take the following form

$$\frac{1}{2\pi i} \oint_A d\Phi = \frac{1}{\hbar}\frac{\partial F}{\partial M}, \qquad \int_B d\Phi = \hbar M. \tag{VI.40}$$

Thus, the duality between the two models is interpreted as the duality with respect to the exchange of the conjugated cycles on the complex curve.

We see that the fact that one compares the grand canonical partition function of one model with the canonical one of the other is very crucial. Actually, such kind of dualities can be interpreted as an electric-magnetic duality which replaces a gauge coupling constant by its inverse [132]. This idea may get a concrete realization in supersymmetric gauge theories. Their relation to matrix models was established recently in [51, 133]. This may open a new window for application of matrix models already in critical string theories.

Chapter VII

Non-perturbative effects in matrix models and D-branes

1 Non-perturbative effects in non-critical strings

In all previous chapters we considered matrix models as a tool to produce perturbative expansions of two-dimensional gravity coupled to matter and of string theory in various backgrounds. However, it is well known that these theories exhibit non-perturbative effects which play a very important role. In particular, they are responsible for the appearance of D-branes and dualities between different string theories (see section I.2.3). Are matrix models able to capture such phenomena?

We saw that in the continuum limit matrix models can have several non-perturbative completions. For example, in MQM one can place a wall either at the top of the potential or symmetrically at large distances from both sides of it. In any case, the non-perturbative definition is always related to particularities of the regularization which is done putting a cut-off. Therefore, it is non-universal.

Nevertheless, it turns out that the non-perturbative completion of the perturbative results of matrix models is highly restricted. Actually, we expect that it is determined, roughly speaking, up to a coefficient. For example, if we consider the perturbative expansion of the free energy

$$F_{\rm pert}=\sum_{g=0}^{\infty} g_{\rm str}^{2g-2} f_g, \tag{VII.1}$$

the general form of the leading non-perturbative corrections is the following

$$F_{\rm non-pert}\sim C g_{\rm str}^{f_A} e^{-\frac{f_D}{g_{\rm str}}}, \tag{VII.2}$$

where f_A and f_D can be found from the asymptotic behaviour of the coefficients f_g as the genus grows. The overall constant C is undetermined and reflects the non-universality of the non-perturbative effects.

On the other hand, the matrix model free energy should reproduce the partition function of the corresponding string theory. More precisely, its perturbative part describes the partition function of closed strings. At the same time, the non-perturbative corrections are associated with open strings with ends living on a D-brane. A particular source of non-perturbative terms of order $e^{-1/g_{\rm str}}$ was identified with D-instantons — D-branes localized in spacetime. It was shown [134] that the leading term in the exponent of (VII.2) should be given by a string diagram with one hole, which is a disk for the spherical approximation [134]. The boundary of the disk lives on a D-instanton implying the Dirichlet boundary conditions for all string coordinates.

Thus, in the two approaches one has more or less clear qualitative picture of how the non-perturbative corrections to the partition function arise. However, to find them explicitly, one should understand which D-instantons are to be taken into account and how to calculate the corrections using these D-instantons. This is not a hard problem in the critical string theory, whereas it remained unsolved for a long time for non-critical strings. The obstacle was that the D-instantons can be localized in the Liouville direction only in the strong coupling region, because this is the region where the minimum of the energy of the brane, which goes like $1/g_{\rm str}$, is realized. As a result, the perturbative expansion breaks down together with the description of these D-instantons.

A clue came with the work of A. and Al. Zamolodchikov [135] where the necessary D-branes were constructed in Liouville field theory. This opened the possibility to study non-perturbative effects in non-critical strings and to compare them with the matrix model results. In fact, such results existed only for some class of minimal ($c < 1$) models [136] described in the matrix approach, for example, using 2MM (see section II.4).

In the work [137] we extended the matrix model results on non-perturbative corrections to the case of the $c = 1$ string theory perturbed by windings. This was done using the Matrix Quantum Mechanical description of section IV.2. Namely, the leading non-perturbative contribution described by f_D in (VII.2) was calculated. In our case this coefficient is already a function of a dimensionless parameter composed from μ and the Sine–Liouville coupling λ. Also it was verified that in all cases, including both the minimal unitary models and the small coupling limit of the considered $c = 1$ string, these results can be reproduced from conformal field theory calculations where some set of D-branes is appropriately chosen.

In the following we describe the matrix models results on the non-perturbative corrections in non-critical string theories and then we reproduce some of them in the CFT framework.

2 Matrix model results

2.1 Unitary minimal models

To understand how the non-perturbative corrections to the partition function appear in matrix models, let us start with the simplest case, of pure gravity. It corresponds to the unitary minimal (p, q) model with $p = 2$, $q = 3$. The partition sum of the model is given by the solution of the Painleve-I equation for the matrix model free energy $F(\mu)$, where μ is the cosmological constant. More precisely, the equation is written for its second derivative $u(\mu) = -\partial_\mu^2 F(\mu)$ and reads as follows

$$u^2(\mu) - \frac{1}{6}u''(\mu) = \mu. \tag{VII.3}$$

In the considered case, string perturbation theory is an expansion in even powers of $g_{\rm str} = \mu^{-5/4}$:

$$u(\mu) = \mu^{1/2} \sum_{h=0}^{\infty} c_h \mu^{-5h/2}, \tag{VII.4}$$

where $c_0 = 1, c_1 = -1/48, \ldots$, and $c_h \underset{h\to\infty}{\sim} -a^{2h}\Gamma(2h - 1/2)$, with $a = 5/8\sqrt{3}$. The series (VII.4) is asymptotic, and hence non-perturbatively ambiguous. The size of the leading non-perturbative ambiguities can be estimated as follows. Suppose u and $\tilde{u}$ are two solutions of (VII.3) which share the asymptotic behavior (VII.4). Then, the difference between them, $\varepsilon = \tilde{u} - u$, is exponentially small in the limit $\mu \to \infty$, and we can treat it perturbatively. Plugging $\tilde{u} = u + \varepsilon$ into (VII.3), and expanding to first order in ε, we find that

$$\varepsilon'' = 12u\varepsilon \tag{VII.5}$$

which can be written for large μ as

$$\frac{\varepsilon'}{\varepsilon} = r\sqrt{u} + b\frac{u'}{u} + \cdots \tag{VII.6}$$

with $r = -2\sqrt{3}$, $b = -1/4$. Using (VII.4), $u = \sqrt{\mu} + \cdots$, one finds that

$$\varepsilon \propto \mu^{-\frac{1}{8}} e^{-\frac{8\sqrt{3}}{5}\mu^{\frac{5}{4}}}. \tag{VII.7}$$

As we mentioned, the constant of proportionality in (VII.7) is a free parameter of the solution and cannot be determined from the string equation (VII.3) without further physical input.

This example demonstrates the general procedure to extract non-perturbative corrections in the matrix model framework. All that we need is to know a differential equation on the free energy (string partition function). Then the leading behaviour of the corrections follows from the expansion around a perturbative solution.

Another lesson is that it is quite convenient to look for the answer in the form of the quantity r defined as in (VII.6):

$$r = \frac{\partial_\mu \log \varepsilon}{\sqrt{u}}, \tag{VII.8}$$

where ε is again the leading non-perturbative ambiguity in $u = -F''$. It is clear that this quantity is directly related to the constant f_A appearing in (VII.2). Its main advantage in comparison with f_A is that it is a pure number and does not depend on normalization of the string coupling $g_{\rm str}$.

In [136] the analysis above was generalized to the case of $(p, p+1)$ minimal models coupled to gravity which correspond to the unitary series. It is not surprising that the authors of [136] found it convenient to parameterize the results in terms of r (VII.8). It was found that for general p there is in fact a whole sequence of different solutions for r labeled by two integers (m, n) which vary over the same range as the Kac indices labeling the degenerate representations of the Virasoro algebra or the primary operators in the minimal models:

$$m = 1, 2, \dots, p-1, \qquad n = 1, 2, \dots, p \quad \text{and} \quad (m, n) \sim (p-m, p+1-n). \tag{VII.9}$$

The result for $r_{m,n}$ was found to be:

$$r_{m,n} = -4 \sin \frac{\pi m}{p} \sin \frac{\pi n}{p+1}. \tag{VII.10}$$

2.2 $c = 1$ string theory with winding perturbation

In the work [137] we performed the similar analysis for the compactified $c = 1$ string theory perturbed by windings with the Sine–Liouville potential. In the CFT framework this theory is described by the action (IV.19).[1] Its matrix counterpart is represented by the model considered in section IV.1. Its solution was presented in section IV.2.2. In particular, it was shown that the Legendre transform $\mathcal{F}$ of the string partition sum F satisfies the Toda differential equation (IV.24). The initial condition for this equation is supplied by the unperturbed $c = 1$ string theory on a circle. The perturbative part of its partition partition function was given in (III.122). The full answer contains also non-perturbative corrections which can be read off its integral representation

$$\mathcal{F}(\mu, 0) = \frac{R}{4} \mathrm{Re} \int_{\Lambda^{-1}}^{\infty} \frac{ds}{s} \frac{e^{-i\mu s}}{\sinh \frac{s}{2} \sinh \frac{s}{2R}} = \mathcal{F}_{\rm pert}(\mu, 0) + O(e^{-2\pi\mu}) + O(e^{-2\pi R\mu}). \tag{VII.11}$$

The result (VII.11) shows that at $\lambda = 0$ there are two types of non-perturbative corrections associated with the poles of the integrand. These occur at $s = 2\pi i k$ and $s = 2\pi R i k$, $k \in \mathbf{Z}$, and give rise to the non-perturbative effects $\exp(-2\pi\mu k)$ and $\exp(-2\pi R\mu k)$, respectively.

At finite λ, the situation is more interesting. In general, the corrections can become dependent on the Sine–Liouville coupling λ. However, the series of non-perturbative corrections

$$\Delta\mathcal{F} = \sum_{n=1}^{\infty} C_n e^{-2\pi n\mu} \tag{VII.12}$$

gives rise to an exact solution of the *full* Toda equation (IV.24). Due to this the corresponding instantons are insensitive to the presence of the Sine–Liouville perturbation. We will return to this fact, and explain its interpretation in Liouville theory, in the next section.

[1] In fact, the CFT couplings can differ from the corresponding matrix model quantities by multiplicative factors. Therefore, we will distinguish between μ and $\mu_{_L}$ for Liouville theory.

The second type of corrections, which starts at $\lambda = 0$ like $\Delta\mathcal{F} = e^{-2\pi R\mu}$, does not solve the full equation (IV.24), and does get λ dependent corrections. To study these corrections, we proceed in a similar way to that described in the previous paragraph. Namely, we expand the differential equation on the free energy of the matrix model, which is in our case the Toda equation (IV.24), around some perturbative solution. The difference with respect to the previous case is that now we have a partial differential equation instead of the ordinary one. As a result, the final equation on a quantity measuring the strength of the non-perturbative correction will be differential, whereas it was algebraic for the $c < 1$ case.

Indeed, the linearization of the Toda equation around some solution $\mathcal{F}$, gives

$$\frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda\varepsilon(\mu,\lambda) - 4e^{-\partial_\mu^2\mathcal{F}_0(\mu,\lambda)}\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\varepsilon(\mu,\lambda) = 0, \qquad \text{(VII.13)}$$

where in the exponential in the second term we approximated

$$4\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\mathcal{F}(\mu,\lambda) \simeq \partial_\mu^2\mathcal{F}_0 = R\log\xi + X(y). \qquad \text{(VII.14)}$$

This is similar to the fact that in the discussion of the Painleve equation, one can replace u in (VII.5) by its spherical limit $\sqrt{\mu}$. This also means that to find the leading correction one should know explicitly only the spherical part of the perturbative expansion. After the change of variables from (λ,μ) to (ξ,y) defined in (IV.27) (we change the notation from w to y to follow the paper [137]), equation (VII.13) can be written as

$$\alpha\xi^2(y\partial_y + \xi\partial_\xi)^2\varepsilon(\xi,y) = 4e^{-X(y)}\sin^2\left(\frac{\xi}{2}\partial_y\right)\varepsilon(\xi,y), \qquad \text{(VII.15)}$$

where $\alpha \equiv \frac{R-1}{(2-R)^2}$. As in section IV.2, we will work only with radii $1 < R < 2$ which include the black hole radius $R = 3/2$.

To proceed further, we should plug in some ansatz for ε into equation (VII.15). We expect that the leading non-perturbative correction has the exponential form (VII.2). Since in the $c = 1$ theory the string coupling is proportional to $1/\mu$, we use the following ansatz

$$\varepsilon(\xi,y) = P(\xi,y)e^{-\mu f(y)}. \qquad \text{(VII.16)}$$

Here $P(\xi,y)$ is a power-like prefactor in $g_{\rm str}$, and $f(y)$ is the function we are interested in (the analogue of r in the minimal models). Substituting (VII.16) into (VII.15) and keeping only the leading terms in the $\xi \to 0$ limit, one finds the following first order differential equation

$$\sqrt{\alpha}\, e^{\frac{1}{2}X(y)}(1 - y\partial_y)g(y) = \pm\sin\left[\partial_y g(y)\right], \qquad \text{(VII.17)}$$

where we introduced

$$g(y) = \frac{1}{2}yf(y). \qquad \text{(VII.18)}$$

The $\pm$ in (VII.17) is due to the fact that one actually finds the square of this equation. Below we will show that the solution with the minus sign is in fact unphysical. But for a while we keep both signs.

Equation (VII.17) is a first order differential equation in y, and to solve it we need to specify boundary conditions. As discussed earlier for the perturbative series, it is natural to specify these boundary conditions at $\lambda \to 0$, or $y \to \infty$. We saw above that there are two solutions, $f(y \to \infty) \to 2\pi$ or $2\pi R$. This implies via (VII.18) that $g(y \to \infty) \simeq \pi y$ or $\pi R y$. We already saw that $g(y) = \pi y$ gives an exact solution, and this is true for (VII.17) as well (as it should be). Thus, to study non-trivial non-perturbative effects, we must take the other boundary condition

$$g(y \to \infty) \simeq \pi R y. \tag{VII.19}$$

Remarkably, the non-linear differential equation (VII.17) is exactly solvable. For the initial condition (VII.19), the solution can be written as [137]

$$g(y) = y\phi(y) \pm \frac{1}{\sqrt{\alpha}} e^{-\frac{1}{2}X(y)} \sin\phi(y), \tag{VII.20}$$

where $\phi(y) = \partial_y g$ satisfies the equation

$$e^{\frac{2-R}{2R}X(y)} = \pm\sqrt{R-1}\,\frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)}. \tag{VII.21}$$

Equations (VII.20), (VII.21) are the main result of this subsection. They provide the leading non-perturbative correction for all couplings μ and λ. As we will see, they contain much more information than it is accessible in the CFT framework. We next discuss some features of the corresponding non-perturbative effects.

Small coupling limits

Consider first the situation for small λ, or large y, when the Sine–Liouville term can be treated perturbatively. The first three terms in the expansion of $\phi(y)$ are

$$\phi(y) \approx \pi R \pm \frac{R\sin(\pi R)}{\sqrt{R-1}}\, y^{-\frac{2-R}{2}} + \frac{R}{2}\sin(2\pi R)\, y^{-(2-R)}. \tag{VII.22}$$

This gives the following result for $f(y)$ (VII.16):

$$\begin{aligned} f(y) &= 2\pi R \pm \frac{4\sin(\pi R)}{\sqrt{R-1}}\, y^{-\frac{2-R}{2}} + \frac{R\sin(2\pi R)}{R-1} y^{-(2-R)} + O(y^{-3(2-R)/2}) \\ &= 2\pi R \pm 4\sin(\pi R)\, \mu^{-\frac{2-R}{2}}\,\lambda + R\sin(2\pi R)\,\mu^{-(2-R)}\,\lambda^2 + O(\lambda^3). \end{aligned} \tag{VII.23}$$

We see that for large y, the expansion parameter is $y^{-\frac{(2-R)}{2}} \sim \lambda$, as one would expect.

Another interesting limit is $\mu \to 0$ at fixed λ, *i.e.* $y \to 0$, which leads to the Sine-Liouville model with $\mu = 0$. In this limit $X \to 0$ and the first two terms in the expansion of ϕ around this point are

$$\phi(y) = \phi_0 + \frac{R}{2}\left((R-1)\cot\left(\frac{R-1}{R}\phi_0\right) - \cot\left(\frac{1}{R}\phi_0\right)\right)^{-1} y + O(y^2), \tag{VII.24}$$

where ϕ_0 is defined by the equation

$$\frac{\sin\left(\frac{1}{R}\phi_0\right)}{\sin\left(\frac{R-1}{R}\phi_0\right)} = \pm\frac{1}{\sqrt{R-1}}. \tag{VII.25}$$

The function $f(y)$ is given in this limit by the expansion

$$f(y) = \pm\frac{2(2-R)}{y\sqrt{R-1}}\sin\phi_0 + 2\phi_0 + O(y). \tag{VII.26}$$

Note that the behavior of f as $y \to 0$, $f \sim 1/y$, leads to a smooth limit as $\mu \to 0$ at fixed λ. The non-perturbative correction (VII.16) goes like $\exp(-\mu f(y))$, so that as $y \to 0$ the argument of the exponential goes like $\mu/y = 1/\xi$, and all dependence on μ disappears.

For $R = 3/2$, which is supposed to correspond to the Euclidean black hole, the equations simplify. One can explicitly find ϕ_0 because (VII.25) gives

$$\cos\frac{\phi_0}{3} = \pm\frac{1}{\sqrt{2}} \Rightarrow \phi_0 = \frac{3\pi}{4} \text{ or } \phi_0 = \frac{9\pi}{4}. \tag{VII.27}$$

As a result, one finds at this value of the radius a simple result

$$\mu f(y) = \pm\frac{\mu}{y} + \frac{(6\mp 3)\pi}{2}\mu + \cdots = \pm\frac{1}{2}\lambda^4 + \frac{(6\mp 3)\pi}{2}\mu + \cdots. \tag{VII.28}$$

Note that the solution with the minus sign leads to a growing exponential, $e^{\frac{1}{2}\lambda^4}$. Therefore, it can not be physical, as mentioned above. In fact, for the particular case $R = 3/2$ one can find the whole function $f(y)$ explicitly. The result is [137]

$$f(y) = 6\ \arccos\left[\pm\left(1+\sqrt{1+4y}\right)^{-1/2}\right] \pm \frac{1}{2y}(1+4y)^{1/4}(3-\sqrt{1+4y})\ . \tag{VII.29}$$

$c = 0$ critical behaviour

A nice consistency check of our solution is to study the RG flow from $c = 1$ to $c = 0$ CFT coupled to gravity. Before coupling to gravity, the Sine-Gordon model associated to (IV.19) describes the following RG flow. In the UV, the Sine-Gordon coupling effectively goes to zero, and one approaches the standard CFT of a compact scalar field. In the IR, the potential given by the Sine-Gordon interaction gives a world sheet mass to X, and the model approaches a trivial $c = 0$ fixed point. This RG flow manifests itself after coupling to gravity in the dependence of the physics on μ. Large μ corresponds to the UV limit; in it, all correlators approach those of the $c = 1$ theory coupled to gravity. Decreasing μ corresponds in this language to the flow to the IR, with the $c = 0$ behavior recovered as μ approaches a critical value μ_c. In fact, this critical value coincides with (IV.34) found studying the partition function obtained from the matrix model.

The non-perturbative contributions to the partition function computed in this section must follow a similar pattern. In particular, $f(y)$ must exhibit a singularity as $y \to y_c$, with

$$y_c = -(2-R)(R-1)^{\frac{R-1}{2-R}} \tag{VII.30}$$

and furthermore, the behavior of f near this singularity should reproduce the non-perturbative effects of the $c = 0$ model coupled to gravity discussed in the previous paragraph. Let us check whether this is the case.

Near the critical point the relation (IV.30) between y and X degenerates:

$$\frac{y_c - y}{y_c} \simeq \frac{R-1}{2R^2}(X - X_c)^2 + O\left((X - X_c)^3\right). \tag{VII.31}$$

Solving it for the critical point, one finds that

$$e^{-\frac{2-R}{2R}X_c} = \sqrt{R-1}. \tag{VII.32}$$

Substituting (VII.32) in (VII.21) we find

$$\frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)} = \frac{1}{R-1}. \tag{VII.33}$$

Thus, the $c = 0$ critical point corresponds to $\phi \to 0$.[2] The first two terms in the expansion of ϕ around the singularity are

$$\phi(y) = \sqrt{3}(X_c - X)^{1/2} - \frac{\sqrt{3}(R^2 - 2R + 2)}{20R^2}(X_c - X)^{3/2} + O\left((X_c - X)^{5/2}\right). \tag{VII.34}$$

Substituting this in (VII.20) one finds

$$g(y) = -y_c\frac{2\sqrt{3}(R-1)}{5R^2}(X_c - X)^{5/2} + O\left((X_c - X)^{5/2}\right) \tag{VII.35}$$

or, using (VII.18):

$$f(y) \approx -\frac{8\sqrt{3}}{5}\left(\frac{2R^2}{R-1}\right)^{1/4}\left(\frac{\mu_c - \mu}{\mu_c}\right)^{5/4}. \tag{VII.36}$$

The power of $\mu - \mu_c$ is precisely right to describe the leading non-perturbative effect in pure gravity. It is interesting to compare also the coefficient in (VII.36) to what is expected in pure gravity. It is most convenient to do this by again computing the quantity r (VII.8) because it does not depend on the relative normalization of the $c = 0$ cosmological constant and the critical parameter in the $c = 1$ theory. u is computed by evaluating the leading singular term as $\mu \to \mu_c$ in $\partial_\mu^2 \mathcal{F}_0 = R\log\xi + X(y)$. One finds

$$r = -2\sqrt{3}\left(\frac{2R^2}{R-1}\right)^{1/4}\left(\frac{\mu_c - \mu}{\mu_c}\right)^{1/4}(X_c - X)^{-1/2} = -2\sqrt{3} \tag{VII.37}$$

in agreement with the result (VII.10) for pure gravity. This provides another non-trivial consistency check of our solution.

[2] Note that if we chose the minus sign in (VII.21), we would find a more complicated solution for ϕ. One can show that it would lead to a wrong critical behavior. This is an additional check of the fact that the physical solution corresponds to the plus sign in (VII.21).

3 Liouville analysis

In this section we study the non-perturbative effects in non-critical strings from the CFT point of view. As we discussed in section 1, they are associated with D-instantons and given by the string disk amplitudes with Dirichlet boundary conditions corresponding to a given instanton. The first question that we need to address is which D-branes should be considered for this analysis? In other words, which D-branes contribute to the leading non-perturbative effects?

If a conformal field theory is a coupling of some matter to Liouville theory, all its correlations functions, and the partition function itself, are factorized to the product of contributions from the matter and from the Liouville part. Due to this property we can discuss the boundary conditions in the two theories independently from each other. The possible boundary conditions in the matter sector will be discussed in the following subsections. Now we will be concentrating on the Dirichlet boundary conditions in Liouville theory discovered by Zamolodchikovs.

In the work [135] they constructed boundary states appearing as quantizations of a classical solution for which the Liouville field ϕ goes to the strong coupling region on the boundary of the world sheet. In fact, it was shown that there is a two-parameter family of consistent quantizations. Thus, one can talk about (m', n') branes of Liouville theory. Which of these branes should be taken in evaluating instanton effects?

The analysis of [135] shows that open strings stretched between the (m', n') and (m'', n'') Liouville branes belong to one of a finite number of degenerate representations of the Virasoro algebra with a given central charge. The precise set of degenerate representations that arises depends on m', n', m'', n''. Degenerate representations at $c > 25$ (the case of the minimal models coupled to Liouville) occur at negative values of world sheet scaling dimension, except for the simplest degenerate operator, 1, whose dimension is zero. One finds [135] that in all sectors of open strings, except those corresponding to $m' = n' = m'' = n'' = 1$ there are negative dimension operators. It is thus natural to conjecture that the only stable D-instantons correspond to the case $(m', n') = (1, 1)$. We will assume this in the analysis below both for the minimal models and for the $c = 1$ string theory.

3.1 Unitary minimal models

First, let us briefly describe the CFT formulation of the minimal models. In the conformal gauge they are represented by the Liouville action

$$S_L = \int \frac{d^2\sigma}{4\pi} \left((\partial\phi)^2 - Q\hat{\mathcal{R}}\phi + \mu_{_L} e^{-2b\phi} \right), \tag{VII.38}$$

where the central charge of the Liouville model is

$$c_L = 1 + 6Q^2 \tag{VII.39}$$

and the parameter b is related to Q via the relation

$$Q = b + \frac{1}{b}. \tag{VII.40}$$

In general, b and Q are determined by the requirement that the total central charge of matter, which is given in (II.43), and Liouville is equal to 26. In our case, (II.43) and (VII.39) imply that

$$b = \sqrt{\frac{p}{q}}. \tag{VII.41}$$

An important class of conformal primaries in Liouville theory corresponds to the operators

$$V_\alpha(\phi) = e^{-2\alpha\phi} \tag{VII.42}$$

whose scaling dimension is given by $\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q-\alpha)$. The Liouville interaction in (VII.38) is $\delta\mathcal{L} = \mu_{{}_L} V_b$. Finally, we mention that the unitary models correspond to the series with $q = p+1$. In the following we restrict our attention to this particular case. In any case, only these models were analyzed in the matrix approach.

Now we turn to the discussion of non-perturbative effects. Minimal model D-branes are well understood. They were constructed and analyzed in [138]. These D-branes are in one to one correspondence with primaries of the Virasoro algebra and, therefore, they are labeled by the indices from the Kac table (VII.9). For our purposes, the main property that will be important is the disk partition sum (or boundary entropy) corresponding to the (m,n) brane, which is given by

$$Z_{m,n} = \left(\frac{8}{p(p+1)}\right)^{1/4} \frac{\sin\frac{\pi m}{p}\sin\frac{\pi n}{p+1}}{\left(\sin\frac{\pi}{p}\sin\frac{\pi}{p+1}\right)^{1/2}}. \tag{VII.43}$$

The minimal model part of the background can be thought of as a finite collection of points. All D-branes corresponding to it are localized and therefore should contribute to the non-perturbative effects. Taking into account that only the (1,1) Liouville D-brane is supposed to contribute, we conclude that the D-instantons to be considered in $c < 1$ minimal models coupled to gravity have the form: $(1,1)$ brane in Liouville $\times$ (m,n) brane in the minimal model. We next show that these D-branes give rise to the correct leading non-perturbative effects (VII.10).

As we explained in the previous section, the quantity r (VII.8) is very convenient to compare results of two theories. It turns out that that it is a natural object to consider in the continuum approach as well. Indeed, from the continuum point of view, r is represented as follows

$$r = \frac{\frac{\partial}{\partial\mu_{{}_L}} Z_{\rm disk}}{\sqrt{-\partial^2_{\mu_{{}_L}} F}}, \tag{VII.44}$$

where in the numerator we used the fact that $\log\varepsilon$ is the disk partition sum corresponding to the D-instanton (see (VII.2)). Thus we see that r is the ratio between the one point function of the cosmological constant operator V_b on the disk, and the square root of its two point function on the sphere. This is a very natural object to consider since it is known in general in CFT that n point functions on the disk behave like the square roots of $2n$ point functions on the sphere. This is actually the reason why we do not have to worry about the multiplicative factor relating μ and $\mu_{{}_L}$, as it drops out in the ratio leaving just a number.

To compute (VII.44), we start with the numerator in (VII.44). We have

$$\frac{\partial}{\partial \mu_L} Z_{\text{disk}} = Z_{m,n} \times \langle V_b \rangle_{(1,1)}, \tag{VII.45}$$

where we used the fact that the contribution of the minimal model is simply the disk partition sum (VII.43), and the second factor is the one point function of the cosmological constant operator (VII.38) on the disk with the boundary conditions corresponding to the $(1,1)$ Liouville D-brane. $Z_{m,n}$ is given by eq. (VII.43), whereas the one point function of V_b can be computed through the boundary wave function constructed in [135]

$$\Psi_{1,1}(P) = \frac{2^{3/4} 2\pi i P [\pi \mu_L \gamma(b^2)]^{-iP/b}}{\Gamma(1-2ibP)\Gamma(1-2iP/b)}, \tag{VII.46}$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{VII.47}$$

The wave function $\Psi_{1,1}(P)$ can be interpreted as an overlap between the $(1,1)$ boundary state and the state with Liouville momentum P. Therefore, $\Psi_{1,1}(P)$ is proportional to the one point function on the disk, with $(1,1)$ boundary conditions, of the Liouville operator V_α, with

$$\alpha = \frac{Q}{2} + iP. \tag{VII.48}$$

The proportionality constant is a pure number (independent of P and Q). We will not attempt to calculate this number precisely. Instead we deduce it by matching any one of the matrix model predictions. Then we can use it in all other calculations. As a result, we obtain

$$\langle V_b \rangle_{(1,1)} = -C \, \frac{2^{1/4}\sqrt{\pi}[\pi \mu_L \gamma(b^2)]^{\frac{1}{2}(1/b^2-1)}}{b\Gamma(1-b^2)\Gamma(1/b^2)}. \tag{VII.49}$$

We next move on to the denominator of (VII.44). This is given by the two point function $\langle V_b V_b \rangle_{\text{sphere}}$. This quantity was calculated in [139, 140]. It is convenient to first compute the three point function $\langle V_b V_b V_b \rangle_{\text{sphere}}$ and then integrate once, to avoid certain subtle questions regarding the fixing of the $\mathrm{SL}(2,\mathbf{C})$ Conformal Killing Group of the sphere. The final result is [137]

$$\langle V_b V_b \rangle_{\text{sphere}} = \frac{1/b^2-1}{\pi b} \left[\pi \mu_L \gamma(b^2)\right]^{1/b^2-1} \gamma(b^2)\gamma(1-1/b^2). \tag{VII.50}$$

We are now ready to compute r. Plugging in (VII.43), (VII.49) and (VII.50) into (VII.44), we find

$$r_{m,n} = -2C \sin\frac{\pi m}{p} \sin\frac{\pi n}{p+1}, \tag{VII.51}$$

which agrees with the matrix model result (VII.10) if we set $C = 2$. Since C is independent of m, n and p, we can fix it by matching to any one case, and then use it in all others. Thus, we conclude that the Liouville analysis gives the same results for the leading non-perturbative corrections as the matrix model one.

3.2 $c = 1$ string theory with winding perturbation

In this subsection we will discuss the Liouville interpretation of the matrix model results presented in section 2.2. Following the lesson of the previous paragraph, we expect that the non-perturbative effects in the $c = 1$ string theory are all associated with the $(1, 1)$ Liouville brane. Thus, it remains to reveal the D-brane content of the matter sector and to perform calculation of the corresponding correlation functions.

First, let us consider the unperturbed $c = 1$ theory corresponding to $\lambda = 0$. As we saw, in the matrix model analysis one finds two different types of leading non-perturbative effects (see (VII.11)), $\exp(-2\pi\mu)$, and $\exp(-2\pi R\mu)$. It is not difficult to guess the origin of these non-perturbative effects from the CFT point of view. The $\exp(-2\pi\mu)$ contribution is due to a Dirichlet brane in the $c = 1$ CFT, *i.e.* a brane located at a point on the circle parameterized by X, whereas the $\exp(-2\pi R\mu)$ term comes from a brane wrapped around the X circle.

This identification can be verified in the same way as we did for the minimal models in the previous paragraph. To avoid normalization issues, one can again calculate the quantity r (VII.44). The matrix model prediction for the Neumann brane[3] is

$$r = -\frac{2\pi\sqrt{R}}{\sqrt{\log\frac{\Delta}{\mu}}}, \tag{VII.52}$$

where we used the sphere partition function $\mathcal{F}_0(\mu, 0)$ of the unperturbed compactified $c = 1$ string given by the first term in (III.122).

The CFT calculation is similar to that performed in the $c < 1$ case. The partition function of $c = 1$ CFT on a disk with Neumann boundary conditions is well known and is given by

$$Z_{\rm Neumann} = 2^{-1/4}\sqrt{R}. \tag{VII.53}$$

The disk amplitude corresponding to the $(1, 1)$ Liouville brane, and the two-point function on the sphere are computed using equations (VII.49) and (VII.50), in the limit $b \to 1$. The limit is actually singular, but computing everything for generic b and taking the limit at the end of the calculation leads to sensible, finite results. The leading behavior of (VII.49) as $b \to 1$ is

$$\langle V_b\rangle_{(1,1)} \approx -\frac{2^{5/4}\sqrt{\pi}}{\Gamma(1-b^2)}, \tag{VII.54}$$

with the constant C in (VII.49) chosen to be equal to 2 according to the minimal model analysis. The two point function (VII.50) approaches

$$\partial^2_{\mu_L}\mathcal{F}_0 \simeq -\frac{\log\mu_L}{\pi\Gamma^2(1-b^2)}. \tag{VII.55}$$

Substituting (VII.53), (VII.54) and (VII.55) into (VII.44) gives precisely the result (VII.52). This provides a non-trivial check of the statement that the constant C is a pure number independent of all the parameters of the model.

[3]Similar formulae can be written for the Dirichlet brane.

The agreement of (VII.52) with the Liouville analysis supports the identification of the Neumann D-branes as the source of the non-perturbative effects $\exp(-2\pi R\mu)$. A similar analysis leads to the same conclusion regarding the relation between the Dirichlet $c = 1$ branes and the non-perturbative effects $\exp(-2\pi\mu)$ (the two kinds of branes are related by T-duality).

Having understood the structure of the unperturbed theory, we next turn to the theory with generic λ. In the matrix model we found that the non-perturbative effects associated with the Dirichlet brane localized on the X circle are in fact independent of λ (see (VII.12) and the subsequent discussion). In the continuum formulation this corresponds to the claim that the disk partition sum with the $(1, 1)$ boundary conditions for Liouville, and Dirichlet boundary conditions for the matter field X is λ-independent. In other words, all n-point functions of the Sine–Liouville operator given by the last term in (IV.19) on the disk vanish

$$\left\langle \left(\int d^2 z e^{(R-2)\phi} \cos(R\tilde{X}) \right)^n \right\rangle_{(1,1)\times\text{Dirichlet}} = 0. \tag{VII.56}$$

Is it reasonable to expect (VII.56) to be valid from the world sheet point of view? For odd n (VII.56) is trivially zero because of winding number conservation. Indeed, the Dirichlet boundary state for X breaks translation invariance, but preserves winding number. The perturbation in (IV.19) carries winding number, and for odd n all terms in (VII.56) have non-zero winding number. Thus, the correlator vanishes.

For even n one has to work harder, but it is still reasonable to expect the amplitude to vanish in this case. Indeed, consider the T-dual statement to (VII.56), that the n point functions of the momentum mode $\cos(X/R)$, on the disk with $(1, 1) \times$ Neumann boundary conditions, vanish. This is reasonable since the operator whose correlation functions are being computed localizes X at the minima of the cosine, while the D-brane on which the string ends is smeared over the whole circle. It might be possible to make this argument precise by using the fact that in this case the D-instanton preserves a different symmetry from that preserved by the perturbed theory, and thus it should not contribute to the non-perturbative effects.

To summarize, the matrix model analysis predicts that (VII.56) is valid. We will not attempt to prove this assertion here from the Liouville point of view (it would be nice to verify it even for the simplest case, $n = 2$), and instead move on to discuss the non-perturbative effects due to the branes wrapped around the X circle.

The solution given by (VII.20) and (VII.21) should correspond from the Liouville point of view to the disk partition sum with the $(1, 1)$ boundary conditions on the Liouville field and Neumann conditions on the X field. The prediction is that

$$\left\langle \left(\int d^2 z e^{(R-2)\phi} \cos(R\tilde{X}) \right)^n \right\rangle_{(1,1)\times\text{Neumann}} \tag{VII.57}$$

are the coefficients in the expansion of $f(y)$ (VII.16) in a power series in λ, the first terms of which are given by (VII.23). It would be very nice to verify this prediction directly using Liouville theory, but in general this seems hard given the present state of the art. A simple check that can be performed using results of [135] is to compare the order λ term in (VII.23) with the $n = 1$ correlator (VII.57).

Like in the other cases studied earlier, to make this comparison it is convenient to define a dimensionless quantity given by the ratio of the one point function on the disk (VII.57) and the square root of the appropriate two point function on the sphere,

$$\rho = \left. \frac{\frac{\partial}{\partial\lambda}\log\varepsilon}{\sqrt{-\partial_\lambda^2\mathcal{F}_0}}\right|_{\lambda=0}. \tag{VII.58}$$

The matrix model result for this quantity is

$$\rho = -\left. \frac{\mu\frac{\partial}{\partial\lambda}f}{\sqrt{-\partial_\lambda^2\mathcal{F}_0}}\right|_{\lambda=0} = -2\sqrt{2}\sin(\pi R). \tag{VII.59}$$

In the Liouville description, ρ is given by

$$\rho = \frac{B_{\mathcal{T}}\langle V_{b-\frac{R}{2}}\rangle_{(1,1)}}{\sqrt{-\langle \mathcal{T}^2\rangle}}, \tag{VII.60}$$

where $\mathcal{T} = V_{b-\frac{R}{2}}\cos(R\tilde{X})$ and $B_{\mathcal{T}}$ is the one point function of $\cos(R\tilde{X})$ on the disk. It has the same value as (VII.53)

$$B_{\mathcal{T}} = 2^{-1/4}\sqrt{R}. \tag{VII.61}$$

The one-point function of the operator of $V_{b-\frac{R}{2}}$ is related to the wave function $\Psi_{1,1}(P)$ (VII.46) with momentum $iP = b - R/2 - Q/2$ and is given by

$$\langle V_{b-\frac{R}{2}}\rangle_{(1,1)} = -\frac{2^{5/4}\sqrt{\pi}[\pi\mu_{_L}\gamma(b^2)]^{\frac{1}{2}(1/b^2-1+R/b)}}{b\Gamma(1-b^2+Rb)\Gamma(1/b^2+R/b)}. \tag{VII.62}$$

The two-point function of $\mathcal{T}$ on the sphere is computed as above from the three point function. One finds [137]

$$\left\langle \mathcal{T}^2\right\rangle = \frac{\left(\frac{1}{b^2}+\frac{R}{b}-1\right)}{2\pi b}\left[\pi\mu_{_L}\gamma(b^2)\right]^{\frac{1}{b^2}+\frac{R}{b}-1}\gamma\left(b^2-Rb\right)\gamma\left(1-\frac{1}{b^2}-\frac{R}{b}\right). \tag{VII.63}$$

Substituting these results into (VII.60) leads, in the limit $b=1$, to (VII.59). We see that the Liouville results are again in complete agreement with the corresponding matrix model calculation.

Thus, we found that whenever it is possible to compare matrix model and CFT predictions for non-perturbative effects they always coincide. All these agreements also support our proposal that only the $(1,1)$ Liouville D-brane is responsible for the leading non-perturbative corrections. Unfortunately, these results say nothing about other (m',n') Zamolodchikov's D-branes. (See, however, the recent work [141] about the role of $(1,n)$ branes in the $c=1$ string theory.)

It seems to be a very remarkable fact that matrix models, whose connection with string theory relies only on a perturbative expansion and even is not completely understood, are able to describe correctly also the non-perturbative physics. This should give a promising direction for future research and for new developments both in matrix models and string theory itself.

Conclusion

We conclude this thesis by summarizing the main results achieved here and giving the list of the main problems, which either were not solved or not addressed at all, although their understanding would shed light on important physical issues.

1 Results of the thesis

- The two- and one-point correlators of winding modes at the spherical level in the compactified Matrix Quantum Mechanics in the presence of a non-vanishing winding condensate (Sine–Liouville perturbation) have been calculated [116].
- It has been shown how the tachyon perturbations can be incorporated into MQM. They are realized by changing the Hilbert space of the one-fermion wave functions of the singlet sector of MQM in such way that the asymptotics of the phases contains the perturbing potential. At the quasiclassical level these perturbations are equivalent to non-perturbative deformations of the Fermi sea which becomes time-dependent. The equation determining the exact form of the Fermi sea has been derived [120].
- When the perturbation contains only tachyons of discrete momenta as in the compactified Euclidean theory, it is integrable and described by the constrained Toda hierarchy. Using the Toda structure, the exact solution of the theory with the Sine-Liouville perturbation has been found [120]. The grand canonical partition function of MQM has been identified as a τ-function of Toda hierarchy [124].
- For the Sine-Liouville perturbation the energy, free energy and entropy have been calculated. It has been shown that they satisfy the standard thermodynamical relations what proves the interpretation of the parameter R of the perturbations in the Minkowski spacetime as temperature of the system [123].
- A relation of the perturbed MQM solution to a free field satisfying the Klein–Gordon equation in the flat spacetime has been established. The global structure of this spacetime and its relation to the string target space were discussed [125].
- MQM with tachyon perturbations with equidistant spectrum has been proven to be equivalent to certain analytical continuation of the Normal Matrix Model. They coincide at the level of the partition functions and all correlators. In the quasiclassical limit this equivalence has been interpreted as a duality which exchanges the conjugated cycles of a complex curve associated with the solution of the two models. Physically this duality is of the electric-magnetic type (S-duality) [124].

- The leading non-perturbative corrections to the partition function of 2D string theory perturbed by a source of winding modes have been found using its MQM description. In particular, from this result some predictions for the non-perturbative effects of string theory in the black hole background have been extracted [137].
- The matrix model results concerning non-perturbative corrections to the partition function of the $c < 1$ unitary minimal models and the $c = 1$ string theory have been verified from the string theory side where they arise from amplitudes of open strings attached to D-instantons. Whenever this check was possible it showed excellent agreement of the matrix model and CFT calculations [137].

2 Unsolved problems

- The first problem is the disagreement of the calculated (non-zero) one-point correlators with the CFT result that they should vanish. The most reasonable scenario is the existence of an operator mixing which includes also some of the discrete states. However, if this is indeed the case, by comparing with the CFT result one can only find the coefficients of this mixing. But it was not yet understood how to check this coincidence independently.

- Whereas we have succeeded to find the correlators of windings in presence of a winding condensate and to describe the T-dual picture of a tachyon condensate, we failed to calculate tachyon correlators in the theory perturbed by windings and *vice versa*. The reason is that the integrability seems to be lost when the two types of perturbations are included. Therefore, the problem is not solvable anymore by the present technique.

- On this way it would be helpful to find a matrix model incorporating both these perturbations. Of course, MQM does this, but we mean to represent them directly in terms of a matrix integral with a deformed potential. Such representation for windings was constructed as a unitary one-matrix integral, whereas for tachyon perturbations this task is accomplished by Normal Matrix Model. However, there is no matrix integral which was proven to describe both perturbations simultaneously.

  Nevertheless, we hope that such matrix model exists. For example, in the CFT framework at the self-dual radius of compactification there is a nice description which includes both winding and tachyon modes. It is realized in terms of a ground ring found by Witten. A similar structure should arise in the matrix model approach.

  In fact, in the end of the paper [120] a 3-matrix model was proposed, which is supposed to incorporate both tachyon and winding perturbations. However, the status of this model is not clear up to now. The reason to believe that it works is based on the expectation that in the case when only one type of the perturbations is present, the matrix integral gives the corresponding τ-function of MQM. This is obvious for windings, but it is difficult to prove this statement for tachyons. It is not clear whether these are technical difficulties or they have a more deep origin.

- Studying the Das–Jevicki collective field theory, we saw that the discrete states are naturally included into the MQM description together with the tachyon modes. However, we realized only how to introduce a non-vanishing condensate of tachyons. We did not address the question how the discrete states can also be incorporated into the picture where they appear as a kind of perturbations of the Fermi sea.

- Also we did not consider seriously how the perturbed Fermi sea consisting from several simply connected components can be analyzed. Although a qualitative picture is clear, the exact mathematical description is not known yet. In particular, it would be interesting to generalize the duality of MQM and NMM to this multicomponent case.

- The next unsolved problem is to find the exact relation between the collective field of MQM and the tachyon of string theory. The solution of this problem can help to understand the correspondence, including possible leg-factors, of the vertex operators of the matrix model to the CFT operators.

- It is not clear whether the non-trivial global structure of the spacetime on which the collective field of MQM is defined has a physical meaning. What are the boundary conditions? What is the physics associated with them? All these questions have no answers up to now.

  Although it seems to be reasonable that the obtained non-trivial global structure can give rise to a finite temperature, this has not been demonstrated explicitly. This is related to a set of technical problems. However, the integrability of the system, which has already led to a number of miraculous coincidences, allows to hope that these problems can be overcome.

- One of the main unsolved problems is how to find the string background obtained by the winding condensation. In particular, one should reproduce the black hole target space metric for the simplest Sine–Liouville perturbation. Unfortunately, this has not been done. In principle, some information about the metric should be contained in the mixed correlators already mentioned here. But they have neither been calculated.

  For the case of tachyon perturbations, the crucial role in establishing the connection with the target space physics is played by the collective field theory of Das and Jevicki. There is no analogous theory for windings. Its construction could lead to a real breakthrough in this problem.

- The thermodynamics represents one of the most interesting issues because we hope to describe the black hole physics. We have succeeded to analyze it in detail for the tachyon perturbations and even to find the entropy. However, we do not know yet how to identify the degrees of freedom giving rise to the entropy. Another way to approach this problem would be to consider the winding perturbations. But it is also unclear how to extract thermodynamical quantities from the dynamics of windings.

- All our results imply that it is very natural to consider the theory where all parameters like μ, λ and R are kept arbitrary. At the same time, from the CFT side a progress has been made only either for $\lambda = 0$ (the $c = 1$ CFT coupled to Liouville theory) or for $\mu = 0,\ R = 3/2$ (the Sine–Gordon theory coupled to gravity at the black hole radius). This is a serious obstacle for the comparison of results of the matrix model and CFT approaches.

  In particular, we observe that from the matrix model point of view the values of the parameters corresponding to the black hole background of string theory are not distinguished anyhow. Therefore, we suppose that for other values the corresponding string background should have a similar structure. But the explicit form of this more general background has not yet been found.

- Finally, it is still a puzzle where the Toda integrable structure is hidden in the CFT corresponding to the perturbed MQM. In this CFT there are some infinite symmetries indicating the presence of such structure. But this happens only at the self-dual radius, whereas MQM does not give any restrictions on R. Probably the answer is in the operator mixing mentioned above because the disagreement in the one-point correlators found by the two approaches cannot be occasional. Until this problem is solved, the understanding of the relation between both approaches will be incomplete.

We see that many unsolved problems wait for their solution. This shows that, in spite of the significant progress, 2D string theory and Matrix Quantum Mechanics continue to be a rich field for future research. Moreover, new unexpected relations with other domains of theoretical physics were recently discovered. And maybe some manifestations of the universal structure that describes these theories are not discovered yet and will appear in the nearest future.

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