

# 2C2 Multivariate Calculus

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# Chapter 1

## Introduction

It is the nature of things that every syllabus grows. Everyone who teaches it wants to put his favourite bits in, every client department wants their precious fragment included.

This syllabus is more stuffed than most. The recipes must be given; the reasons why they work usually cannot, because there isn't time. I dislike this myself because I like understanding things, and usually forget recipes unless I can see why they work.

I shall try, as far as possible, to indicate by "Proofs by arm-waving" how one would go about understanding why the recipes work, and apologise in advance to any of you with a taste for real mathematics for the crammed course. Real mathematicians like understanding things, pretend mathematicians like knowing tricks. Courses like this one are hard for real mathematicians, easy for bad ones who can remember any old gibberish whether it makes sense or not.

I recommend that you go to the Mathematics Computer Lab and do the following:

Under the Apple icon on the top line select **Graphing Calculator** and double click on it.

When it comes up, click on **demos** and select the full demo.

Sit and watch it for a while.

When bored press the <tab> key to get the next demo. Press <shift><tab> to go backwards.

When you get to the graphing in three dimensions of functions with two variables  $x$  and  $y$ , click on the example text and press <return/enter>.

This will allow you to edit the functions so you can graph your own.

Try it and see what you get for functions like

$z = x^2 + y^2$	case 1
$z = x^2 - y^2$	case 2
$z = xy$	case 3
$z = 4 - x^2 - y^2$	case 4
$z = xy - x^3 - y^2 + 4$	case 5

If you get a question in a practice class asking you about maxima or minima or saddle points, nick off to the lab at some convenient time and draw the picture. It is worth 1000 words. At least.

I also warmly recommend you to run Mathematica or MATLAB and try the DEMO's there. They are a lot of fun. You could learn a lot of Mathematics just by reading the documentation and playing with the examples in either program.

I don't recommend this activity because it will make you better and purer people (though it might.) I recommend it because it is good fun and beats watching television.

I use the symbol  $\square$  to denote the end of a proof and  $P \triangleq \langle expression \rangle$  when  $P$  is defined to be  $\langle expression \rangle$

# Chapter 2

## Optimisation

### 2.1 The Second Derivative Test

I shall work only with functions

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow f \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[\text{eg. } f \begin{bmatrix} x \\ y \end{bmatrix} = z = xy - x^5/5 - y^3/3 + 4]$$

This has as its graph a surface always, see figure 2.1

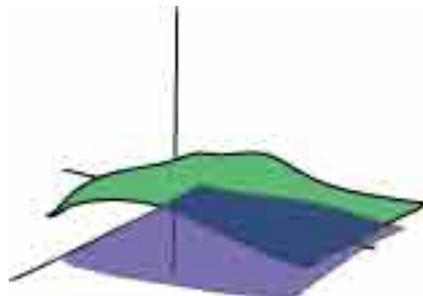
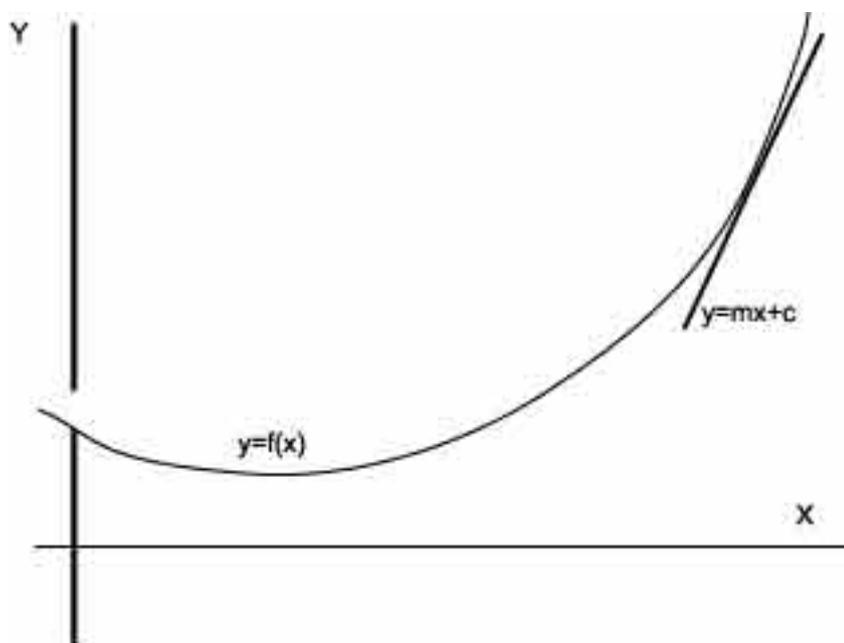


Figure 2.1: Graph of a function from  $\mathbb{R}^2$  to  $\mathbb{R}$



Figure 2.2: Graph of a function from  $\mathbb{R}$  to  $\mathbb{R}$ 

**Definition 2.1.** If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable and  $\begin{bmatrix} a \\ b \end{bmatrix}$  is a critical point of  $f$  then

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{matrix} a \\ b \end{matrix} = [0, 0].$$

**Remark 2.1.1.** I deal with maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  when  $n = 2$ , but generalising to larger  $n$  is quite trivial. We would have that  $f$  is differentiable at  $\mathbf{a} \in \mathbb{R}^n$  if and only if there is a unique affine (linear plus a shift) map from  $\mathbb{R}^n$  to  $\mathbb{R}$  tangent to  $f$  at  $\mathbf{a}$ . The linear part of this then has a (row) matrix representing it

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{a}}$$

**Remark 2.1.2.** We would like to carry the old second derivative test through from one dimension to two (at least) to distinguish between maxima, minima and saddle points. This remark will make more sense if you play with the DEMO's program on the Graphing Calculator and plot the five cases mentioned in the introduction.

I sure hope you can draw the surfaces  $x^2 + y^2$  and  $x^2 - y^2$ , because if not you are DEAD MEAT.

**Definition 2.2.** A **quadratic form on  $\mathbb{R}^2$**  is a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  which is a sum of terms  $x^p y^q$  where  $p, q \in \mathbb{N}$  (the natural numbers:  $0, 1, 2, 3, \dots$ ) and  $p + q \leq 2$  and at least one term has  $p + q = 2$ .

**Definition 2.3. [alternative 1:]** A quadratic form on  $\mathbb{R}^2$  is a function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

which can be written

$$f \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + bxy + cy^2 + dx + ey + g$$

for some numbers  $a, b, c, d, e, g$  **and** not all of  $a, b, c$  are zero.

**Definition 2.4. [alternative 2:]** A quadratic form on  $\mathbb{R}^2$  is a function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

which can be written

$$f \begin{bmatrix} x \\ y \end{bmatrix} = [x - \alpha, y - \beta] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x - \alpha \\ y - \beta \end{bmatrix} + c$$

for real numbers  $\alpha, \beta, c, a_{ij}$   $1 \leq i, j \leq 2$  and with  $a_{12} = a_{21}$

**Remark 2.1.3.** You might want to check that all these 3 definitions are equivalent. Notice that this is just a polynomial function of degree two in two variables.

**Definition 2.5.** If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is twice differentiable at

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

the second derivative is the matrix in the quadratic form

$$[x - a, y - b] \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

**Remark 2.1.4.** When the first derivative is zero, it is the “best fitting quadratic” to  $f$  at  $\begin{bmatrix} a \\ b \end{bmatrix}$ , although we need to add in a constant to lift it up so that it is “more than tangent” to the surface which is the graph of  $f$  at  $\begin{bmatrix} a \\ b \end{bmatrix}$ . You met this in first semester in Taylor’s theorem for functions of two variables.

**Theorem 2.1.** *If the determinant of the second derivative is positive at  $\begin{bmatrix} a \\ b \end{bmatrix}$  for a continuously differentiable function  $f$  having first derivative zero at  $\begin{bmatrix} a \\ b \end{bmatrix}$ , then in a neighbourhood of  $\begin{bmatrix} a \\ b \end{bmatrix}$ ,  $f$  has either a maximum or a minimum, whereas if the determinant is negative then in a neighbourhood of  $\begin{bmatrix} a \\ b \end{bmatrix}$ ,  $f$  is a saddle point. If the determinant is zero, the test is uninformative.*

**“Proof” by arm-waving:** *We have that if the first derivative is zero, the second derivative at  $\begin{bmatrix} a \\ b \end{bmatrix}$  of  $f$  is the approximating quadratic form from Taylor’s theorem, so we can work with this (second order) approximation to  $f$  in order to decide what shape (approximately) the graph of  $f$  has.*

*The quadratic approximation is just a symmetric matrix, and all the information about the shape of the surface is contained in it. Because it is symmetric, it can be diagonalised by an orthogonal matrix, (ie we can rotate the surface until the quadratic form matrix is just*

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

*We can now rescale the new  $x$  and  $y$  axes by dividing the  $x$  by  $|a|$  and the  $y$  by  $|b|$ . This won’t change the shape of the surface in any essential way.*

*This means all quadratic forms are, up to shifting, rotating and stretching*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

*ie.  $x^2 + y^2$                        $-x^2 - y^2$                        $x^2 - y^2$                        $-x^2 + y^2$  since*

$$[x, y] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$$

*et cetera. We do not have to actually do the diagonalisation. We simply note that the determinant in the first two cases is positive and the determinant is not changed by rotations, nor is the sign of the determinant changed by scalings.  $\square$*

**Proposition 2.1.1.** *If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function which has*

$$Df \begin{bmatrix} a \\ b \end{bmatrix} = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \begin{matrix} a \\ b \end{matrix}$$

zero and if

$$D^2 f \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{matrix} a \\ b \end{matrix}$$

is continuous on a neighbourhood of  $\begin{bmatrix} a \\ b \end{bmatrix}$

and if  $\det D^2 f \begin{bmatrix} a \\ b \end{bmatrix} > 0$

then if

$$\frac{\partial^2 f}{\partial x^2} \Big|_{\begin{matrix} a \\ b \end{matrix}} > 0$$

$f$  has a local minimum at  $\begin{bmatrix} a \\ b \end{bmatrix}$

and if

$$\frac{\partial^2 f}{\partial x^2} \Big|_{\begin{matrix} a \\ b \end{matrix}} < 0$$

$f$  has a local maximum at  $\begin{bmatrix} a \\ b \end{bmatrix}$ .

**“Proof”** The trace of a matrix is the sum of the diagonal terms and this is also unchanged by rotations, and the sign of it is unchanged by scalings. So again we reduce to the four possible basic quadratic forms

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x^2 + y^2 \quad -x^2 - y^2 \quad x^2 - y^2 \quad -x^2 + y^2$$

and the trace distinguishes between the first two, being positive at a minimum and negative at a maximum. Since the two diagonal terms have the same sign we need only look at the sign of the first.  $\square$

**Example 2.1.1.** Find and classify all critical points of

$$f \begin{bmatrix} x \\ y \end{bmatrix} = xy - x^4 - y^2 + 2.$$

**Solution**

$$\left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] = [y - 4x^3, x - 2y]$$

at a critical point, this is the zero matrix  $[0, 0]$

so  $y = 4x^3$  and  $y = \frac{1}{2}x$

so  $\frac{1}{2}x = 4x^3 \Rightarrow x = 0$  or  $x^2 = \frac{1}{8}$

so  $x = 0$  or  $x = \frac{1}{\sqrt{8}}$  or  $x = \frac{-1}{\sqrt{8}}$

when  $y = 0$ ,  $y = \frac{1}{2\sqrt{8}}$   $y = \frac{-1}{2\sqrt{8}}$

and there are three critical points  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$   $\begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{2\sqrt{8}} \end{pmatrix}$   $\begin{pmatrix} \frac{-1}{\sqrt{8}} \\ \frac{-1}{2\sqrt{8}} \end{pmatrix}$ .

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{bmatrix} -12x^2 & 1 \\ 1 & -2 \end{bmatrix}$$

$D^2 f \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$  and  $\det = -1$  so  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  is a saddle point.

$D^2 f \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{1}{2\sqrt{8}} \end{pmatrix} = \begin{bmatrix} \frac{-12}{8} & 1 \\ 1 & -2 \end{bmatrix}$  and  $\det = 3 - 1 = 2$  so the point is either a maximum or a minimum.

$D^2 f \begin{pmatrix} \frac{-1}{\sqrt{8}} \\ \frac{-1}{2\sqrt{8}} \end{pmatrix}$  is the same. Since the trace is  $-3\frac{1}{2}$  both are maxima.  $\square$

**Remark 2.1.5.** Only a wild optimist would believe I have got this all correct without making a slip somewhere. So I recommend strongly that you try checking it on a computer with *Mathematica*, or by using a graphics calculator (or the software on the Mac).



# Chapter 3

## Constrained Optimisation

### 3.1 Lagrangian Multipliers

A function may be given not on  $\mathbb{R}^2$  but on some curve in  $\mathbb{R}^2$ . (More generally on some hypersurface in  $\mathbb{R}^n$ ).

Finding critical points in this case is rather different.

**Example 3.1.1.** Define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow x^2 - y^2$ .

Find the maxima and minima of  $f$  on the unit circle

$$S^1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}.$$

Plotting a few points we see  $f$  is zero at  $x = \pm y$ , negative and a minimum at  $x = 0, y = \pm 1$ , positive and a maximum at  $x = \pm 1, y = 0$ . Better yet we use Mathematica and type in:

```
ParametricPlot3D[{Cos[t],Sin[t],s*Sin[2t]},{t,0,2π},{s,0,1},PlotPoints->50]
```

It is obvious that  $Df = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$  is zero only at the origin - which is not much help since the origin isn't on the circle.

**Q 3.1.2** How can we solve this problem? (Other than drawing the graph)

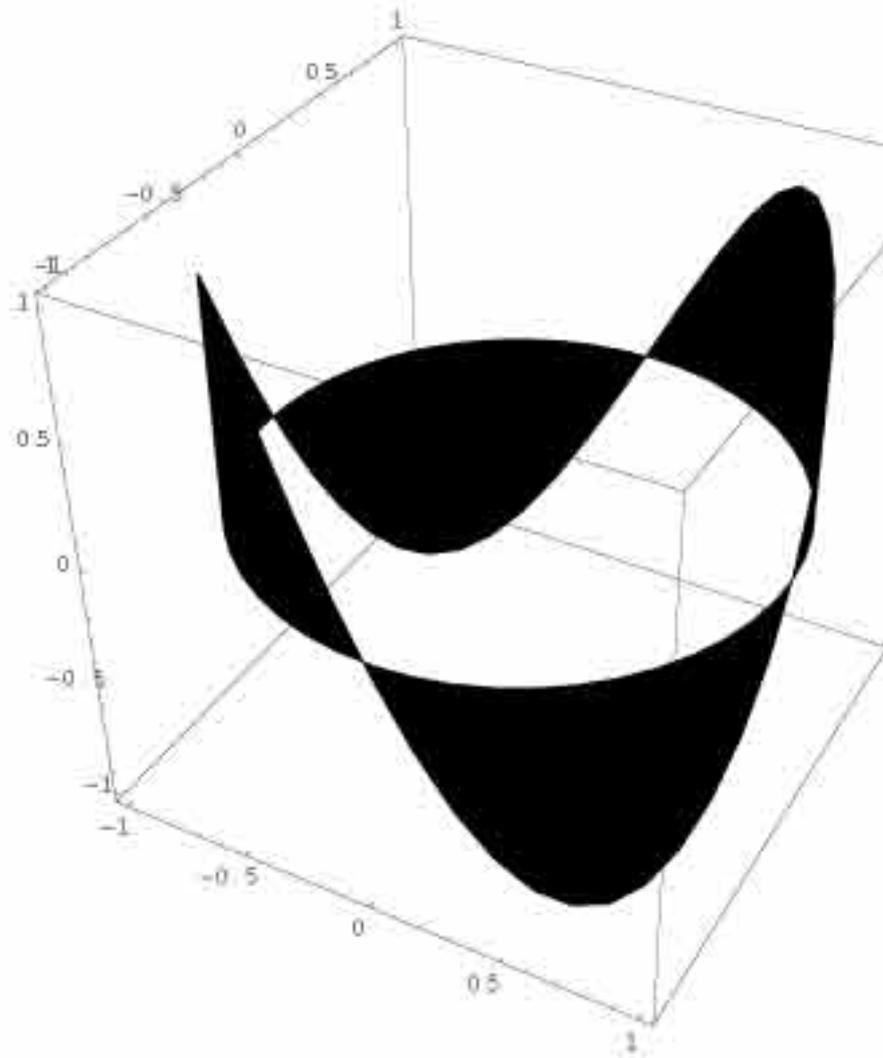


Figure 3.1: Graph of  $x^2 - y^2$  over unit circle

**Answer 1** Express the circle parametrically by 'drawing' the curve

$$\text{eg } \left. \begin{array}{l} x = \cos t \\ y = \sin t \end{array} \right\} t \in [0, 2\pi)$$

Then  $f$  composed with the map

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} : [0, 2\pi) &\longrightarrow \mathbb{R}^2 \\ t &\rightsquigarrow \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \end{aligned}$$

is

$$\begin{aligned} f \circ \begin{bmatrix} x \\ y \end{bmatrix} (t) &= \cos^2 t - \sin^2 t \\ &= \cos 2t \end{aligned}$$

which has maxima at  $2t = 0$  and  $2t = 2\pi$  i.e. at  $t = 0, \pi$  and minima at  $2t = \pi, 3\pi$  i.e.  $t = \pi/2, 3\pi/2$ .

It is interesting to observe that the function  $x^2 - y^2$  restricted to the circle is just  $\cos 2t$ .

### Answer 2 (Lagrange Multipliers)

Observe that if  $f$  has a critical point on the circle, then

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} (= (Df)^T)$$

which points in the direction where  $f$  is increasing most rapidly, must be normal to the circle. (Since if there was a component tangent to the circle,  $f$  couldn't have a critical point there, it would be increasing or decreasing in their direction.)

Now if  $g \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2 - 1$  we have  $\nabla g \begin{bmatrix} x \\ y \end{bmatrix}$  is normal to the circle from first semester.

So for  $f$  to have a critical point on  $S^1$ ,  $\nabla f \begin{bmatrix} x \\ y \end{bmatrix}$  and  $\nabla g \begin{bmatrix} x \\ y \end{bmatrix}$  must be pointing in the same direction (or maybe the exactly opposite direction), ie

$$\exists \lambda \in \mathbb{R}, \quad \nabla f \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \nabla g \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\text{Well, } \nabla g \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\text{and } \nabla f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

$$\text{So } \nabla f \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \left[ \nabla g \begin{bmatrix} x \\ y \end{bmatrix} \right] \quad \exists \lambda \in \mathbb{R}$$

$$\Leftrightarrow \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ -2y \end{bmatrix} \quad \exists \lambda \in \mathbb{R}$$

if  $x = 0$ ,  $\lambda = -1$  is possible and on the unit circle,  $x = 0 \Rightarrow y = \pm 1$  so

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

are two possibilities.

If  $x \neq 0$  then  $2x = 2\lambda x \Rightarrow \lambda = 1$  whereupon  $y = -y \Rightarrow y = 0$  and on  $S^1$ ,  $y = 0 \Rightarrow x = \pm 1$ . So

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

are the other two.

It is easy to work out which are the maxima and which the minima by plugging in values.

**Remark 3.1.1.** The same idea generalises for maps

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

and constraints

$$\begin{aligned} g_1 : \mathbb{R}^n &\rightarrow \mathbb{R} \\ g_2 : \mathbb{R}^n &\rightarrow \mathbb{R} \\ &\vdots \\ g_n : \mathbb{R}^n &\rightarrow \mathbb{R} \end{aligned}$$

with the problem:

Find the critical points of  $f$  subject to

$$g_i(\mathbf{x}) = 0 \quad \forall i : 1 \leq i \leq k$$

We have two ways of proceeding: one is to parametrise the hypersurface given by the  $\{g_i\}$  by finding

$$M : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^n$$

such that every  $\mathbf{y} \in \mathbb{R}^n$  such that  $g_i(\mathbf{y}) = 0$  for all  $i$  comes from some  $\mathbf{p} \in \mathbb{R}^{n-k}$ , preferably only one.

Then  $f \circ M : \mathbb{R}^{n-k} \rightarrow \mathbb{R}$  can have its maxima and minima found as usual by setting

$$D(f \circ M) = \underbrace{[0, 0, \dots, 0]}_{n-k}$$

Or we can find  $\nabla f(\mathbf{x})$  and  $\nabla g_i(\mathbf{x})$ .

Then:

**Proposition 3.1.1.**  $\mathbf{x}$  is a critical point of  $f$  restricted to the set

$$\{g_i(x) = 0; 1 \leq i \leq k\}$$

provided  $\exists \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$

$$\nabla f(\mathbf{x}) = \sum_{i=1, K}^1 \lambda_i \nabla g_i(\mathbf{x})$$

**“Proof”**  $\nabla g_i(x)$  is normal to the  $i$ th constraint and the hypersurface is given by  $k$  constraints which (we hope) are all independent. If you think of  $k$  tangents at the point  $\mathbf{x}$ , you can see them as intersecting hyperplanes, the dimension of the intersection being, therefore,  $n - k$ .

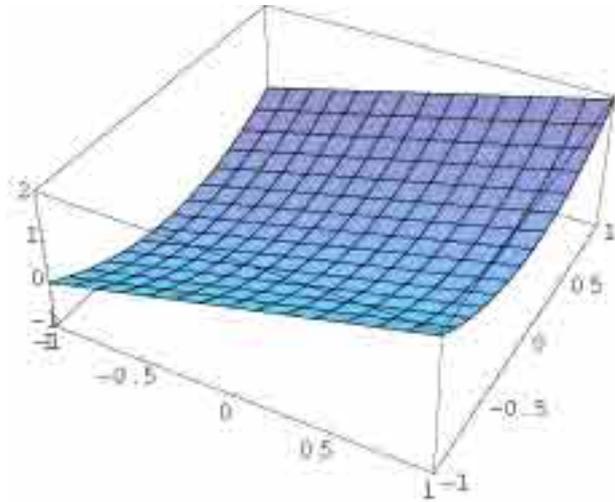
$\nabla f$  cannot have a component along any of the hyperplanes at a critical point, ie  $\nabla f(\mathbf{x})$  is in the span of the  $k$  normal vectors  $\nabla g_i(\mathbf{x}) : 1 \leq i \leq k$ .  $\square$

**Example 3.1.2.** Find the critical points of  $f \begin{bmatrix} x \\ y \end{bmatrix} = x + y^2$  subject to the constraint  $x^2 + y^2 = 4$ .

**Solution** We have  $\nabla f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 \\ 2y \end{bmatrix}$

and

$$g : \mathbb{R}^2 \rightarrow \mathbb{R} \\ \begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow x^2 + y^2 - 4$$

Figure 3.2: Graph of  $x + y^2$ 

has  $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : g \begin{bmatrix} x \\ y \end{bmatrix} = 0 \right\}$  as the constrained set, with

$$\nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Then  $\nabla f \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \nabla g \begin{bmatrix} x \\ y \end{bmatrix} \quad \exists \lambda \in \mathbb{R}$

$$\Rightarrow \exists \lambda \in \mathbb{R}, \begin{bmatrix} 1 \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\Rightarrow \lambda = 1 \text{ and } x = 1/2 \Rightarrow y = \pm \sqrt{\frac{15}{4}}$$

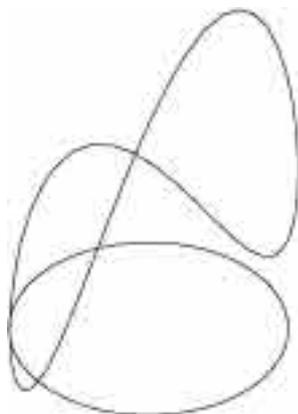
$$\text{or } y = 0 \text{ and } x = \pm 2 \text{ and } \lambda = \pm \frac{1}{2}$$

so the critical points are at

$$\begin{bmatrix} 1/2 \\ \frac{\sqrt{15}}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ -\frac{\sqrt{15}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

If you think about  $f \begin{bmatrix} x \\ y \end{bmatrix} = x + y^2$  you see it is a parabolic trough, as in figure 3.2.

Looking at the part over the circle  $x^2 + y^2 = 4$  it looks like figure 3.3. Both pictures were drawn by Mathematica.

Figure 3.3: Graph of  $x + y^2$  over unit circle

So a minimum occurs at  $\begin{matrix} x = -2 \\ y = 0 \end{matrix}$  a local minimum at  $\begin{matrix} x = 2 \\ y = 0 \end{matrix}$  and maxima  
at  $\begin{bmatrix} x = 1/4 \\ y = \frac{\sqrt{15}}{2} \end{bmatrix}$ . □



# Chapter 4

## Fields and Forms

### 4.1 Definitions Galore

You may have noticed that I have been careful to write vectors in  $\mathbb{R}^n$  as vertical columns

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and in particular  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$  and not  $(x, y) \in \mathbb{R}^2$ .

You will have been, I hope, curious to know why. The reason is that I want to distinguish between elements of two very similar vector spaces,  $\mathbb{R}^2$  and  $\mathbb{R}^{2*}$ .

#### Definition 4.1. Dual Space

$\forall n \in \mathbb{Z}^+$ ,

$$\mathbb{R}^{n*} \triangleq L(\mathbb{R}^n, \mathbb{R})$$

is the (real) vector space of linear maps from  $\mathbb{R}^n$  to  $\mathbb{R}$ , under the usual addition and scaling of functions.

**Remark 4.1.1.** You need to do some checking of axioms here, which is soothing and mostly mechanical.

**Remark 4.1.2.** Recall from Linear Algebra the idea of an *isomorphism* of vector spaces. Intuitively, if spaces are isomorphic, they are pretty much the same space, but the names have been changed to protect the guilty.

**Remark 4.1.3.** The space  $\mathbb{R}^{n*}$  is isomorphic to  $\mathbb{R}^n$  and hence the dimension of  $\mathbb{R}^{n*}$  is  $n$ . I shall show that this works for  $\mathbb{R}^{2*}$  and leave you to verify the general result.

**Proposition 4.1.1.**  $\mathbb{R}^{2*}$  is isomorphic to  $\mathbb{R}^2$

**Proof** For any  $f \in \mathbb{R}^{2*}$ , that is any linear map from  $\mathbb{R}^2$  to  $\mathbb{R}$ , put

$$a \triangleq f \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$b \triangleq f \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then  $f$  is completely specified by these two numbers since

$$\forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2, f \begin{bmatrix} x \\ y \end{bmatrix} = f \left( x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Since  $f$  is linear this is:

$$xf \begin{bmatrix} 1 \\ 0 \end{bmatrix} + yf \begin{bmatrix} 0 \\ 1 \end{bmatrix} = ax + by$$

This gives a map

$$\begin{aligned} \alpha : L(\mathbb{R}^2, \mathbb{R}) &\longrightarrow \mathbb{R}^2 \\ f &\rightsquigarrow \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

where  $a$  and  $b$  are as defined above.

This map is easily confirmed to be linear. It is also one-one and onto and has inverse the map

$$\begin{aligned} \beta : \mathbb{R}^2 &\longrightarrow L(\mathbb{R}^2, \mathbb{R}) \\ \begin{bmatrix} a \\ b \end{bmatrix} &\rightsquigarrow ax + by \end{aligned}$$

Consequently  $\alpha$  is an isomorphism and the dimension of  $\mathbb{R}^{2*}$  must be the same as the dimension of  $\mathbb{R}^2$  which is 2.  $\square$

**Remark 4.1.4.** There is not a whole lot of difference between  $\mathbb{R}^n$  and  $\mathbb{R}^{n*}$ , but your life will be a little cleaner if we agree that they are in fact different things.

I shall write  $[a, b] \in \mathbb{R}^{2*}$  for the linear map

$$[a, b] : \quad \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{R}$$

$$\quad \quad \quad \begin{bmatrix} x \\ y \end{bmatrix} \quad \rightsquigarrow \quad ax + by$$

Then it is natural to look for a basis for  $\mathbb{R}^{2*}$  and the obvious candidate is

$$([1, 0], [0, 1])$$

The first of these sends

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow x$$

which is often called the projection on  $x$ . The other is the projection on  $y$ . It is obvious that

$$[a, b] = a[1, 0] + b[0, 1]$$

and so these two maps span the space  $L(\mathbb{R}^2, \mathbb{R})$ . Since they are easily shown to be linearly independent they *are* a basis. Later on I shall use  $dx$  for the map  $[1, 0]$  and  $dy$  for the map  $[0, 1]$ . The reason for this strange notation will also be explained.

**Remark 4.1.5.** The conclusion is that although different, the two spaces are almost the same, and we use the convention of writing the linear maps as row matrices and the vectors as column matrices to remind us that (a) they are different and (b) not very different.

**Remark 4.1.6.** Some modern books on calculus insist on writing  $(x, y)^T$  for a vector in  $\mathbb{R}^2$ .  $T$  is just the matrix transpose operation. This is quite intelligent of them. They do it because just occasionally, distinguishing between  $(x, y)$  and  $(x, y)^T$  matters.

**Definition 4.2.** An element of  $\mathbb{R}^{n*}$  is called a **covector**. The space  $\mathbb{R}^{n*}$  is called the **dual space** to  $\mathbb{R}^n$ , as was hinted at in Definition 4.1.

**Remark 4.1.7.** Generally, if  $\mathcal{V}$  is any real vector space,  $\mathcal{V}^*$  makes sense.

If  $\mathcal{V}$  is finite dimensional,  $\mathcal{V}$  and  $\mathcal{V}^*$  are isomorphic (and if  $\mathcal{V}$  is **not** finite dimensional,  $\mathcal{V}$  and  $\mathcal{V}^*$  are **NOT** isomorphic. Which is one reason for caring about which one we are in.)

**Definition 4.3. Vector Field** A *vector field* on  $\mathbb{R}^n$  is a map

$$V : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

A continuous vector field is one where the map is continuous, a differentiable vector field one where it is differentiable, and a *smooth* vector field is one where the map is infinitely differentiable, that is, where it has partial derivatives of all orders.

**Remark 4.1.8.** We are being sloppy here because it is traditional. If we were going to be as lucid and clear as we ought to, we would define a space of tangents to  $\mathbb{R}^n$  at each point, and a vector field would be something more than a map which seems to be going to itself. It is important to at least grasp intuitively that the domain and range of  $V$  are different spaces, even if they are isomorphic and given the same name. The domain of  $V$  is a space of places and the codomain of  $V$  (sometimes called the range) is a space of “arrows”.

**Definition 4.4. Differential 1-Form** A *differential 1-form* on  $\mathbb{R}^n$  or *covector field* on  $\mathbb{R}^n$  is a map

$$\omega : \mathbb{R}^n \rightarrow \mathbb{R}^{*n}$$

It is *smooth* when the map is infinitely differentiable.

**Remark 4.1.9.** Unless otherwise stated, we assume that all vector fields and forms are infinitely differentiable.

**Remark 4.1.10.** We think of a vector field on  $\mathbb{R}^2$  as a whole stack of little arrows, stuck on the space. By taking the transpose, we can think of a differential 1-form in the same way.

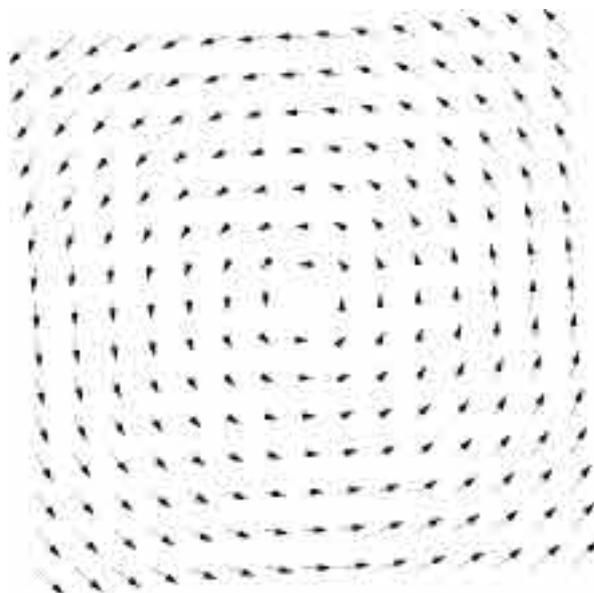
If  $V : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a vector field, we think of  $V \begin{bmatrix} a \\ b \end{bmatrix}$  as a little arrow going from  $\begin{bmatrix} x \\ y \end{bmatrix}$  to  $\begin{bmatrix} x+a \\ y+b \end{bmatrix}$ .

**Example 4.1.1.** Sketch the vector field on  $\mathbb{R}^2$  given by

$$V \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Because the arrows tend to get in each others way, we often scale them down in length. This gives a better picture, figure 4.1. You might reasonably look at this and think that it looks like what you would get if you rotated  $\mathbb{R}^2$  anticlockwise about the origin, froze it instantaneously and put the velocity vector at each point of the space. This is 100% correct.

**Remark 4.1.11.** We can think of a differential 1-form on  $\mathbb{R}^2$  in exactly the same way: we just represent the covector by attaching its transpose. In fact

Figure 4.1: The vector field  $[-y, x]^T$ 

covector fields or 1-forms are not usually distinguished from vector fields as long as we stay on  $\mathbb{R}^n$  (which we will mostly do in this course). Actually, the algebra is simpler if we stick to 1-forms.

So the above is equally a handy way to think of the 1-form

$$\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}^{2*}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow (-y, x)$$

**Definition 4.5.**  $\mathbf{i} \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\mathbf{j} \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  when they are used in  $\mathbb{R}^2$ .

**Definition 4.6.**  $\mathbf{i} \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\mathbf{j} \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  and  $\mathbf{k} \triangleq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  when they are used in  $\mathbb{R}^3$ .

This means we can write a vector field in  $\mathbb{R}^2$  as

$$P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

and similarly in  $\mathbb{R}^3$ :

$$P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

**Definition 4.7.**  $dx$  denotes the projection map  $\mathbb{R}^2 \rightarrow \mathbb{R}$  which sends  $\begin{bmatrix} x \\ y \end{bmatrix}$  to  $x$ .  $dy$  denotes the projection map  $\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow y$ . The same symbols  $dx$ ,  $dy$  are used for the projections from  $\mathbb{R}^3$  along with  $dz$ . In general we have  $dx^i$  from  $\mathbb{R}^n$  to  $\mathbb{R}$  sends

$$\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \rightsquigarrow x^i$$

**Remark 4.1.12.** It now makes sense to write the above vector field as

$$-y \mathbf{i} + x \mathbf{j}$$

or the corresponding differential 1-form as

$$-y dx + x dy$$

This is more or less the classical notation.

Why do we bother with having two things that are barely distinguishable? It is clear that if we have a physical entity such as a force field, we could cheerfully use either a vector field or a differential 1-form to represent it. One part of the answer is given next:

**Definition 4.8.** A smooth 0-form on  $\mathbb{R}^n$  is any infinitely differentiable map (function)

$$f : \mathbb{R}^n \rightarrow \mathbb{R}.$$

**Remark 4.1.13.** This is, of course, just jargon, but it is convenient. The reason is that we are used to differentiating  $f$  and if we do we get

$$Df : \mathbb{R}^n \rightarrow L(\mathbb{R}^n, \mathbb{R})$$

$$\mathbf{x} \rightsquigarrow Df(\mathbf{x}) = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]_{(\mathbf{x})}$$

This I shall write as

$$df : \mathbb{R}^n \rightarrow \mathbb{R}^{n*}$$

and, lo and behold, when  $f$  is a 0-form on  $\mathbb{R}^n$   $df$  is a differential 1-form on  $\mathbb{R}^n$ . When  $n = 2$  we can take  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  as the 0-form and write

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

which was something the classical mathematicians felt happy about, the  $dx$  and the  $dy$  being “infinitesimal quantities”. Some modern mathematicians feel that this is immoral, but it can be made intellectually respectable.

**Remark 4.1.14.** The old-timers used to write, and come to think of it still do,

$$\nabla f(x) \triangleq \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

and call the resulting vector field the **gradient field of  $f$** .

This is just the transpose of the derivative of the 0-form of course.

**Remark 4.1.15.** For much of what goes on here we can use either notation, and it won't matter whether we use vector fields or 1-forms. There will be a few places where life is **much** easier with 1-forms. In particular we shall repeat the differentiating process to get 2-forms, 3-forms and so on.

Anyway, if you think of 1-forms as just vector fields, certainly as far as visualising them is concerned, no harm will come.

**Remark 4.1.16.** A question which might cross your mind is, are all 1-forms obtained by differentiating 0-forms, or in other words, are all vector fields gradient fields? Obviously it would be nice if they were, but they are not. In particular,

$$V \begin{bmatrix} x \\ y \end{bmatrix} \triangleq \begin{bmatrix} -y \\ x \end{bmatrix}$$

is not the gradient field  $\nabla f$  for any  $f$  at all. If you ran around the origin in a circle in this vector field, you would have the force against you all the way. If I ran out of my front door, along Broadway, left up Elizabeth Street, and kept going, the force field of gravity is the vector field I am working with. Or against. The force is the negative of the gradient of the hill I am running up. You would not however, believe that, after completing a circuit and arriving home out of breath, I had been running uphill all the way. Although it might feel like it.

**Definition 4.9.** A vector field that is the gradient field of a function (scalar field) is called **conservative**.

**Definition 4.10.** A 1-form which is the derivative of a 0-form is said to be **exact**.

**Remark 4.1.17.** Two bits of jargon for what is almost the same thing is a pain and I apologise for it. Unfortunately, if you read modern books on, say theoretical physics, they use the terminology of exact 1-forms, while the old fashioned books talk about conservative vector fields, and there is no solution except to know both lots of jargon. Technically they **are** different, but they are often confused.

**Definition 4.11.** Any 1-form on  $\mathbb{R}^2$  will be written

$$\omega \triangleq P(x, y) dx + Q(x, y) dy.$$

The functions  $P(x, y), Q(x, y)$  will be smooth when  $\omega$  is, since this is what it means for  $\omega$  to be smooth. So they will have partial derivatives of all orders.

## 4.2 Integrating 1-forms (vector fields) over curves.

**Definition 4.12.**  $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

**Definition 4.13.** A smooth curve in  $\mathbb{R}^n$  is the image of a map

$$c : I \longrightarrow \mathbb{R}^n$$

that is infinitely differentiable everywhere.

It is *piecewise smooth* if it is continuous and fails to be smooth at only a finite set of points.

**Definition 4.14.** A smooth curve is *oriented* by giving the direction in which  $t$  is increasing for  $t \in I \subset \mathbb{R}$

**Remark 4.2.1.** If you decided to use some interval other than the unit interval,  $I$ , it would not make a whole lot of difference, so feel free to use, for example, the interval of points between 0 and  $2\pi$  if you wish. After all, I can always map  $I$  into your interval if I feel obsessive about it.

**Remark 4.2.2. [Motivation]** Suppose the wind is blowing in a rather erratic manner, over the great gromboolian plain ( $\mathbb{R}^2$ ). In figure 4.2 you can see the path taken by me on my bike together with some vectors showing the wind force.



Figure 4.2: Bicycling over the Great Gromboolian Plain

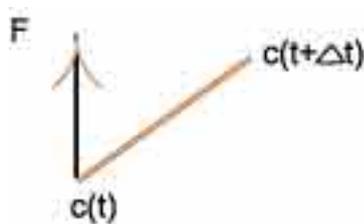


Figure 4.3: An infinitesimal part of the ride

We represent the wind as a vector field  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . I cycle along a curved path, and at time 0 I start out at  $c(0)$  and at time  $t$  I am at  $c(t)$ . I stop at time  $t = 1$ .

I am interested in the effect of the wind on me as I cycle. At  $t = 0$  the wind is pushing me along and helping me. At  $t = 1$  it is against me. In between it is partly with me, partly at right angles to me and sometimes partly against me.

I am not interested in what the wind is doing anywhere else.

I suppose that if the wind is at right angles to my path it has no effect (although it might blow me off the bike in real life).

The question is, how much net help or hindrance is the wind on my journey?

I solve this by chopping my path up into little bits which are (almost) straight line segments.

$F$  is the wind at where I am time  $t$ .

My path is, nearly, the straight line obtained by differentiating my path at time  $t$ ,  $c'(t)$ . This gives the “infinitesimal length” as well as its direction. Note

that this could be defined without reference to a parametrisation.

The component in my direction, multiplied by the length of the path is

$$F \circ c(t) \cdot c'(t) \text{ for time } \Delta t$$

(approximately.)  $\cdot$  denotes the inner or dot product.

(The component in my direction is the projection on my direction, which is the inner product of the Force with the unit vector in my direction. Using  $c'$  includes the “speed” and hence gives me the distance covered as a term.)

I add up all these values to get the net ‘assist’ given by  $F$ .

Taking limits, the net assist is

$$\int_{t=0}^{t=1} F(c(t)) \cdot c'(t) dt$$

**Example 4.2.1.** The vector field  $\mathbf{F}$  is given by

$$\mathbf{F} \begin{bmatrix} x \\ y \end{bmatrix} \triangleq \begin{bmatrix} -y \\ x \end{bmatrix}$$

(I am in a hurricane or cyclone)

I choose to cycle in the unit (quarter) circle from  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  to  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  my path (draw it)  $c$  is

$$c(t) \triangleq \begin{bmatrix} \cos \frac{\pi}{2} t \\ \sin \frac{\pi}{2} t \end{bmatrix}$$

for  $t \in [0, 1]$

Differentiating  $c$  we get:

$$c'(t) = \begin{bmatrix} -\frac{\pi}{2} \sin \frac{\pi}{2} t \\ \frac{\pi}{2} \cos \frac{\pi}{2} t \end{bmatrix}$$

My ‘assist’ is therefore

$$\begin{aligned} & \int_{t=0}^{t=1} \begin{bmatrix} -\sin \frac{\pi}{2} t \\ \cos \frac{\pi}{2} t \end{bmatrix} \cdot \begin{bmatrix} -\frac{\pi}{2} \sin \frac{\pi}{2} t \\ \frac{\pi}{2} \cos \frac{\pi}{2} t \end{bmatrix} dt \\ &= \frac{\pi}{2} \int_{t=0}^{t=1} \left[ \sin^2 \frac{\pi}{2} t + \cos^2 \frac{\pi}{2} t \right] dt \\ &= \frac{\pi}{2} \end{aligned}$$

This is positive which is sensible since the wind is pushing me all the way. Now I consider a different path from  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  to  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . My path is first to go

along the X axis to the origin, then to proceed along the Y-axis finishing at  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . I am going to do this path in two separate stages and then add up the answers. This has to give the right answer from the definition of what a path integral is. So my first stage has

$$c(t) \triangleq \begin{bmatrix} -t \\ 0 \end{bmatrix}$$

and

$$c'(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(which means that I am travelling at uniform speed in the negative x direction.)

The 'assist' for this stage is

$$\int_0^1 \begin{bmatrix} 0 \\ -t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

which is zero. This is not to surprising if we look at the path and the vector field.

The next stage has a new  $c(t)$ :

$$c(t) \triangleq \begin{bmatrix} 0 \\ t \end{bmatrix}$$

which goes from the origin up a distance of one unit.

$$\begin{aligned} \int_0^1 F(c(t)) \cdot c' dt &= \int \begin{bmatrix} -t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt \\ &= \int 0 dt \end{aligned}$$

so I get no net assist.

This makes sense because the wind is always orthogonal to my path and has no net effect.

Note that the path integral between two points depends on the path, which is not surprising.

**Remark 4.2.3.** The only difficulty in doing these sums is that I might describe the path and fail to give it parametrically - this can be your job. Oh, and the integrals could be truly awful. But that's why Mathematica was invented.

### 4.3 Independence of Parametrisation

Suppose the path is the quarter circle from  $[1, 0]^T$  to  $[0, 1]^T$ , the circle being centred on the origin. One student might write

$$c : [0, 1] \longrightarrow \mathbb{R}^2$$

$$c(t) \triangleq \begin{bmatrix} \cos(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) \end{bmatrix}$$

Another might take

$$c : [0, \pi/2] \longrightarrow \mathbb{R}^2$$

$$c(t) \triangleq \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

Would these two different parametrisations of the same path give the same answer? It is easy to see that they would get the same answer. (Check if you doubt this!) They really *ought* to, since the original definition of the path integral was by chopping the path up into little bits and approximating each by a line segment. We used an actual parametrisation only to make it easier to evaluate it.

**Remark 4.3.1.** For any continuous vector field  $F$  on  $\mathbb{R}^n$  and any differentiable curve  $c$ , the value of the integral of  $F$  over  $c$  does not depend on the choice of parametrisation of  $c$ . I shall prove this soon.

**Example 4.3.1.** For the vector field  $\mathbf{F}$  on  $\mathbb{R}^2$  given by

$$\mathbf{F} \begin{bmatrix} x \\ y \end{bmatrix} \triangleq \begin{bmatrix} -y \\ x \end{bmatrix}$$

evaluate the integral of  $\mathbf{F}$  along the straight line joining  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  to  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

**Parametrisation 1**

$$\begin{aligned}
c : [0, 1] &\longrightarrow \mathbb{R}^2 \\
t &\rightsquigarrow (1-t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1-t \\ t \end{bmatrix} \\
c' &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{aligned}$$

Then the integral is

$$\begin{aligned}
&\int_{t=0}^{t=1} \begin{bmatrix} -t \\ 1-t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} dt \\
&= \int_0^1 t + 1 - t dt = \int_0^1 1 dt = t \Big|_0^1 = 1
\end{aligned}$$

This looks reasonable enough.

**Parametrisation 2**

Now next move along the same curve but at a different speed:

$$c(t) \triangleq \begin{bmatrix} 1 - \sin t \\ \sin t \end{bmatrix}, \quad t \in [0, \pi/2].$$

notice that  $x(c) = 1 - y(c)$  so the ‘curve’ is still along  $y = 1 - x$ .

Here however we have  $c' = \begin{bmatrix} -\cos t \\ \cos t \end{bmatrix}$   $t \in [0, \frac{\pi}{2}]$  and

$$F(c(t)) = \begin{bmatrix} -\sin t \\ 1 - \sin t \end{bmatrix}$$

So the integral is

$$\begin{aligned}
&\int_{t=0}^{t=\pi/2} \begin{bmatrix} -\sin t \\ 1 - \sin t \end{bmatrix} \cdot \begin{bmatrix} -\cos t \\ \cos t \end{bmatrix} dt \\
&= \int_0^{\pi/2} \sin t \cos t + \cos t - \sin t \cos t dt \\
&= \int_0^{\pi/2} \cos t dt = \sin t \Big|_0^{\pi/2} = 1
\end{aligned}$$

So we got the same result although we moved along the line segment at a different speed. (starting off quite fast and slowing down to zero speed on arrival)

Does it always happen? Why does it happen? You need to think about this until it joins the collection of things that are obvious.

In the next proposition,  $[a, b] \triangleq \{x \in \mathbb{R} : a \leq x \leq b\}$

**Proposition 4.3.1.** If  $c : [u, v] \rightarrow \mathbb{R}^n$  is differentiable and  $\varphi : [a, b] \rightarrow [u, v]$  is a differentiable monotone function with  $\varphi(a) = u$  and  $\varphi(b) = v$  and  $e : [a, b] \rightarrow \mathbb{R}^n$  is defined by  $e \triangleq c \circ \varphi$ , then for any continuous vector field  $\mathbf{V}$  on  $\mathbb{R}^n$ ,

$$\int_u^v \mathbf{V}(c(t)) \cdot c'(t) dt = \int_a^b \mathbf{V}(e(t)) \cdot e'(t) dt.$$

**Proof** By the change of variable formula

$$\begin{aligned} \int \mathbf{V}(c(t)) \cdot c'(t) dt &= \int \mathbf{V}(c(\varphi(t))) \cdot c'(\varphi(t)) \varphi'(t) dt \\ &= \mathbf{V}(e(t)) \cdot e'(t) dt \text{ (chain rule)} \end{aligned}$$

□

**Remark 4.3.2.** You should be able to see that this covers the case of the two different parametrisations of the line segment and extends to any likely choice of parametrisations you might think of. So if half the class thinks of one parametrisation of a curve and the other half thinks of a different one, you will still all get the same result for the path integral provided they **do** trace the same path.

**Remark 4.3.3.** Conversely, if you go by different paths between the same end points you will generally get a different answer.

**Remark 4.3.4. Awful Warning** The parametrisation doesn't matter but the orientation does. If you go backwards, you get the negative of the result you get going forwards. After all, you can get the reverse path for any parametrisation by just swapping the limits of the integral. And you know what that does.

**Example 4.3.2.** You travel from  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  to  $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$  in the vector field

$$\mathbf{V} \begin{bmatrix} x \\ y \end{bmatrix} \triangleq \begin{bmatrix} -y \\ x \end{bmatrix}$$

1. by going around the semi circle (positive half)
2. by going in a straight line
3. by going around the semi circle (negative half)

It is obvious that the answer to these three cases are different. (b) obviously gives zero, (a) gives a positive answer and (c) the negative of it.

(I don't need to do any sums but I suggest you do.) (It's very easy!)

## 4.4 Conservative Fields/Exact Forms

**Theorem 4.4.1.** If  $\mathbf{V}$  is a **conservative** vector field on  $\mathbb{R}^n$ , ie  $\mathbf{V} = \nabla\varphi$  for  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable, and if  $c : [a, b] \rightarrow \mathbb{R}$  is any smooth curve, then

$$\begin{aligned}\int_c \mathbf{V} &= \int_a^b \mathbf{V}(c(t)) \cdot c'(t) dt \\ &= \varphi(c(b)) - \varphi(c(a))\end{aligned}$$

**Proof**

$$\int_c \mathbf{V} = \int_{r=a}^b \mathbf{V}(c(t)) \cdot c'(t) dt$$

write

$$c(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Then:

$$\begin{aligned}
\int_c \mathbf{V} &= \int \begin{bmatrix} \frac{\partial \varphi}{\partial x_1}(c(t)) \\ \frac{\partial \varphi}{\partial x_2}(c(t)) \\ \vdots \\ \frac{\partial \varphi}{\partial x_n}(c(t)) \end{bmatrix} \cdot \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}_t dt \\
&= \int \left( \sum \frac{\partial \varphi_i}{\partial x_i} \right)_{c(t)} \left( \frac{dx_i}{dt} \right)_t dt \\
&= \int_{t=a}^{t=b} \frac{d}{dt}(\varphi \circ c) dt \quad (\text{chain rule}) \\
&= \int_{t=c}^{t=b} d(\varphi \circ c) \\
&= \varphi(c(a)) - \varphi(c(b))
\end{aligned}$$

□

In other words, it's the chain rule.

**Corollary 4.4.1.1.** For a conservative vector field,  $\mathbf{V}$ , the integral over a curve  $c$   $\int_c \mathbf{V}$  depends only on the end points of the curve and not on the path. □

**Remark 4.4.1.** It is possible to ask an innocent young student to tackle a thoroughly appalling path integral question, which the student struggles for days with. If the result in fact doesn't depend on the path, there could be an easier way.

**Example 4.4.1.**

$$\mathbf{V} \begin{bmatrix} x \\ y \end{bmatrix} \triangleq \begin{bmatrix} 2x \cos(x^2 + y^2) \\ 2y \cos(x^2 + y^2) \end{bmatrix}$$

Let  $c$  be the path from  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  to  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  that follows the curve shown in figure 4.4, a quarter of a circle with centre at  $[1, 1]^T$ .

Find  $\int_c \mathbf{V}$

The innocent student finds  $c(t) = \begin{bmatrix} 1 - \sin t \\ 1 - \cos t \end{bmatrix}$   $t \in [0, \pi/2]$  and tries to evaluate

$$\int_{t=0}^{\pi/2} \begin{bmatrix} 2(1 - \sin t) \cos((1 - \sin t)^2 + (1 - \cos t)^2) \\ 2(1 - \cos t) \cos((1 - \sin t)^2 + (1 - \cos t)^2) \end{bmatrix} \cdot \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} dt$$



Figure 4.4: quarter-circular path

The lazy but thoughtful student notes that  $\mathbf{V} = \nabla\varphi$  for  $\varphi = \sin(x^2 + y^2)$  and writes down  $\varphi \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \varphi \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$  since  $1^2 + 0^2 = 0^2 + 1^2 = 1$ .

Which saves a lot of ink.

**Remark 4.4.2.** It is obviously a good idea to be able to tell if a vector field is conservative:

**Remark 4.4.3.** If  $\mathbf{V} = \nabla f$  for  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  we have

$$\mathbf{V} \begin{bmatrix} x \\ y \end{bmatrix} \triangleq P \begin{bmatrix} x \\ y \end{bmatrix} dx + Q \begin{bmatrix} x \\ y \end{bmatrix} dy$$

is the corresponding 1-form where

$$P = \frac{\partial f}{\partial x} \quad Q = \frac{\partial f}{\partial y}.$$

In which case

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

So if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

then there is hope.

**Definition 4.15.** A 1-form on  $\mathbb{R}^2$  is said to be *closed* iff

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$$

**Remark 4.4.4.** Then the above argument shows that every exact 1-form is closed. We want the converse, but at least it is easy to check if there is hope.

**Example 4.4.2.**

$$\mathbf{V} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \cos(x^2 + y^2) \\ 2y \cos(x^2 + y^2) \end{bmatrix}$$

has

$$P = 2x \cos(x^2 + y^2), \quad Q = 2y \cos(x^2 + y^2)$$

and

$$\frac{\partial P}{\partial y} = (-4xy \sin(x^2 + y^2)) = \frac{\partial Q}{\partial x}$$

So there is hope, and indeed the field is conservative: integrate  $P$  with respect to  $x$  to get, say,  $f$  and check that

$$\frac{\partial f}{\partial y} = Q$$

**Example 4.4.3. NOT!**

$$\omega = -ydx + xdy$$

has

$$\frac{\partial P}{\partial y} = -1$$

but

$$\frac{\partial Q}{\partial x} = +1$$

So there is no hope that the field is conservative, something our physical intuitions should have told us.

## 4.5 Closed Loops and Conservatism

**Definition 4.16.**  $c : I \rightarrow \mathbb{R}^n$ , a (piecewise) differentiable and continuous function is called a **loop** iff

$$c(0) = c(1)$$

**Remark 4.5.1.** If  $\mathbf{V} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is conservative and  $c$  is any loop in  $\mathbb{R}^n$ ,

$$\int_c \mathbf{V} = 0$$

This is obvious since  $\int_c \mathbf{V} = \varphi(c(0)) - \varphi(c(1))$  and  $c(0) = c(1)$ .

**Proposition 4.5.1.** If  $\mathbf{V}$  is a continuous vector field on  $\mathbb{R}^n$  and for every loop  $\ell$  in  $\mathbb{R}^n$ ,

$$\int_{\ell} \mathbf{V} = 0$$

Then for every path  $c$ ,  $\int_c \mathbf{V}$  depends only on the endpoints of  $c$  and is independent of the path.

**Proof:** If there were two paths,  $c_1$  and  $c_2$  between the same end points and

$$\int_{c_1} \mathbf{V} \neq \int_{c_2} \mathbf{V}$$

then we could construct a loop by going out by  $c_1$  and back by  $c_2$  and  $\int_{c_1 \star c_2} \mathbf{V}$  would be nonzero, contradiction.  $\square$

**Remark 4.5.2.** This uses the fact that the path integral along any path in one direction is the negative of the reversed path. This is easy to prove. Try it. (Change of variable formula again)

**Proposition 4.5.2.** If  $\mathbf{V} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous on a **connected open** set  $\mathcal{D} \subseteq \mathbb{R}^n$

**And** if  $\int_c \mathbf{V}$  is independent of the path

Then  $\mathbf{V}$  is conservative on  $\mathcal{D}$

**Proof:** Let  $\mathbf{0}$  be any point. I shall keep it fixed in what follows and define

$$\varphi(\mathbf{0}) \triangleq 0$$

For any other point  $P \triangleq \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in D$ , we take a path from  $\mathbf{0}$  to  $P$  which, in

some ball centred on  $P$ , comes in to  $P$  changing only  $x^i$ , the  $i^{\text{th}}$  component. In the diagram in  $\mathbb{R}^2$ , figure 4.5, I come in along the x-axis.

In fact we choose  $P' = \begin{pmatrix} x_1 - a \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$  for some positive real number  $a$ , and the

path goes from  $\mathbf{0}$  to  $P'$ , then in the straight line from  $P'$  to  $P$

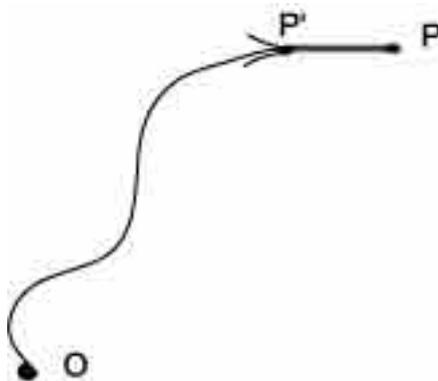


Figure 4.5: Sneaking Home Along the X axis

We have  $\mathbf{V} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ \vdots \\ \mathbf{V}_n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \end{pmatrix}$  for each  $\mathbf{V}_i$  a continuous function

$\mathbb{R}^n$  to  $\mathbb{R}$ , and

$$\int_c \mathbf{V} = \int_0^{P'} \mathbf{V} + \int_{P'}^P \mathbf{V}$$

Where I have specified the endpoints only since  $\mathbf{V}$  has the independence of path property.

For every point  $P \in \mathcal{D}$  I define  $\varphi(P)$  to be  $\int_0^P \mathbf{V}$ , and I can rewrite the above equation as

$$\begin{aligned} \varphi(P) &= \varphi(P') + \int_{P'}^P \mathbf{V} \\ &= \varphi(P') + \int_{t=0}^1 \begin{pmatrix} \mathbf{V}_1(c(t)) \\ \vdots \\ \mathbf{V}_n(c(t)) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ U \end{pmatrix} dt \end{aligned}$$

where  $c(t) = \begin{pmatrix} x_1 - a + t \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$  for  $0 \leq t \leq a$ .

Since the integration is just along the  $x_1$  line we can write

$$\varphi(P) = \varphi(P') + \int_{x=x_0}^{x=x_1} \mathbf{V}_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} dx$$

Differentiating with respect to  $x_1$

$$\frac{\partial \varphi}{\partial x_1} = 0 + \frac{\partial}{\partial x_1} \int_{x=x_0}^{x=x_1} \mathbf{V}_1 \begin{pmatrix} x \\ x_2 \\ \vdots \\ x_n \end{pmatrix} dx$$

Recall the Fundamental theorem of calculus here:

$$\left( \frac{d}{dx} \int_{t=0}^{t=x} f(t) dt = f(x) \right)$$

to conclude that

$$\frac{\partial \varphi}{\partial x_1} = \mathbf{V}_1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Similarly for

$$\frac{\partial \varphi}{\partial x_i}$$

for all  $i \in [1 \dots n]$

In other words,  $\mathbf{V} = \nabla \varphi$  as claimed.  $\square$

**Remark 4.5.3.** We need  $\mathcal{D}$  (the “domain”) to be open so we could guarantee the existence of a little ball around it so we could get to each point from all the  $n$ -directions.

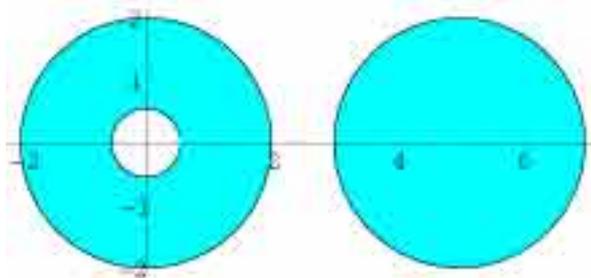


Figure 4.6: A Hole and a non-hole

**Remark 4.5.4.** So far I have cheerfully assumed that

$$\mathbf{V} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is a continuous vector field and

$$c : I \rightarrow \mathbb{R}^n$$

is a piecewise differentiable curve.

There was one place where I was sneaky and defined  $\mathbf{V}$  on  $\mathbb{R}^2$  by

$$\mathbf{V} \begin{pmatrix} x \\ y \end{pmatrix} \triangleq \frac{-1}{(x^2 + y^2)^{3/2}} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is not defined at the origin. Lots of vector fields in Physics are like this. You might think that one point missing is of no consequence. Wrong! One of the problems is that we can have the integral along a loop is zero **provided the loop does not circle the origin**, but loops around the origin have non-zero integrals.

For this reason we often want to restrict the vector field to be continuous (and defined) over some region which has no holes in it.

It is intuitively easy enough to see what this means:

in figure 4.6, the left region has a big hole in it, the right hand one does not.

Saying this is algebra is a little bit trickier.

**Definition 4.17.** For any sets  $X, Y$ ,  $f : X \rightarrow Y$  is 1-1.

$$\forall a, b \in X, f(a) = f(b) \Rightarrow a = b.$$

**Definition 4.18.** For any subset  $U \subseteq \mathbb{R}^n$ ,  $\partial U$  is the *boundary* of  $U$  and is defined to be the subset of  $\mathbb{R}^n$  of points  $\mathbf{p}$  having the property that every open ball containing  $\mathbf{p}$  contains points of  $U$  and points not in  $U$ .

**Definition 4.19.** The unit square is

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1 \right\}$$

**Remark 4.5.5.**  $\partial I^2$  is the four edges of the square. It is easy to prove that there is a 1-1 continuous map from  $\partial I^2$  to  $S^1$ , the unit circle, which has a continuous inverse.

**Exercise 4.5.1.** Prove the above claim

**Definition 4.20.**  $\mathcal{D}$  is *simply connected* iff every continuous map  $f : \partial I^2 \rightarrow \mathcal{D}$  extends to a continuous 1-1 map  $\tilde{f} : I^2 \rightarrow \mathcal{D}$ , i.e.  $\tilde{f}|_{\partial I^2} = f$

**Remark 4.5.6.** You should be able to see that it looks very unlikely that if we have a hole in  $\mathcal{D}$  and the map from  $\partial$  to  $\mathcal{D}$  circles the hole, that we could have a continuous extension to  $I^2$ . This sort of thing requires proof but is too hard for this course. It is usually done by algebraic topology in the Honours year.

**Proposition 4.5.3.** If  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$  is a vector field on  $\mathcal{D} \in \mathbb{R}^2$  and  $\mathcal{D}$  is open, connected and simply connected, then if

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

on  $\mathcal{D}$ , there is a “potential function”  $f : \mathcal{D} \rightarrow \mathbb{R}$  such that  $\mathbf{F} = \nabla f$ , that is,  $\mathbf{F}$  is conservative.

**Proof** No Proof. Too hard for you at present.



# Chapter 5

## Green's Theorem

### 5.1 Motivation

#### 5.1.1 Functions as transformations

I shall discuss maps  $f : I \rightarrow \mathbb{R}^2$  and describe a geometric way of visualising them, much used by topologists, which involves thinking about  $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$  as if it were a piece of chewing gum<sup>1</sup>. We can use the same way of thinking about functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . You are used to thinking of such functions geometrically by visualising the graph, which is another way of getting an intuitive grip on functions. Thinking of stretching and deforming the domain and putting it in the codomain has the advantage that it generalises to maps from  $\mathbb{R}$  to  $\mathbb{R}^n$ , from  $\mathbb{R}^2$  to  $\mathbb{R}^n$  and from  $\mathbb{R}^3$  to  $\mathbb{R}^n$  for  $n = 1, 2$  or  $3$ . It is just a way of visualising what is going on, and although Topologists do this, they don't usually talk about it. So I may be breaking the Topologist's code of silence here.

**Example 5.1.1.**  $f(x) \triangleq 2x$  can be thought of as taking the real line, regarded as a ruler made of chewing gum and stretching it uniformly and moving it across to a second ruler made of, let's say, wood. See figure 5.1 for a rather bad drawing of this.

**Example 5.1.2.**  $f(x) \triangleq -x$  just turns the chewing gum ruler upside down and doesn't stretch it (in the conventional sense) at all. The stretch factor

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<sup>1</sup>If, like the government of Singapore, you don't like chewing gum, substitute putty or plasticene. It just needs to be something that can be stretched and won't spring back when you let go

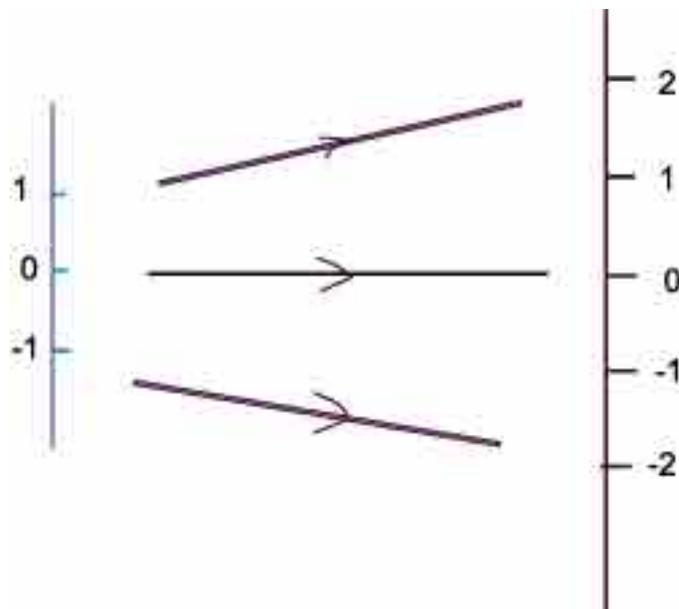


Figure 5.1: The function  $f(x) \triangleq 2x$  thought of as a stretching.

is  $-1$ .

**Example 5.1.3.**  $f(x) \triangleq x + 1$  just shifts the chewing gum ruler up one unit and again doesn't do any stretching, or alternatively the stretch factor is 1. Draw your own pictures.

**Example 5.1.4.**  $f(x) \triangleq |x|$  folds the chewing gum ruler about the origin so that the negative half fits over the positive half; the stretch factor is 1 when  $x > 0$  and  $-1$  when  $x < 0$ . See figure 5.2 for a picture of this map in chewing gum language.

**Example 5.1.5.**  $f(x) \triangleq x^2$  This function is more interesting because the amount of stretch is not uniform. Near zero it is a compression: the interval between 0 and 0.5 gets sent to the interval from 0 to 0.25 so the stretch is one half over this interval. Whereas the interval between 1 and 2 is sent to the interval between 1 and 4, so the stretch factor over the interval is three. I won't try to draw it, but it is not hard.

**Remark 5.1.1.** Now I want to show you, with a particular example, how quite a lot of mathematical ideas are generated. It is reasonable to say of the function  $f(x) \triangleq x^2$  that the amount of stretch depends on where you are. So I would like to define the amount of stretch a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  does at a point. I shall call this  $S(f, a)$  where  $a \in \mathbb{R}$ .



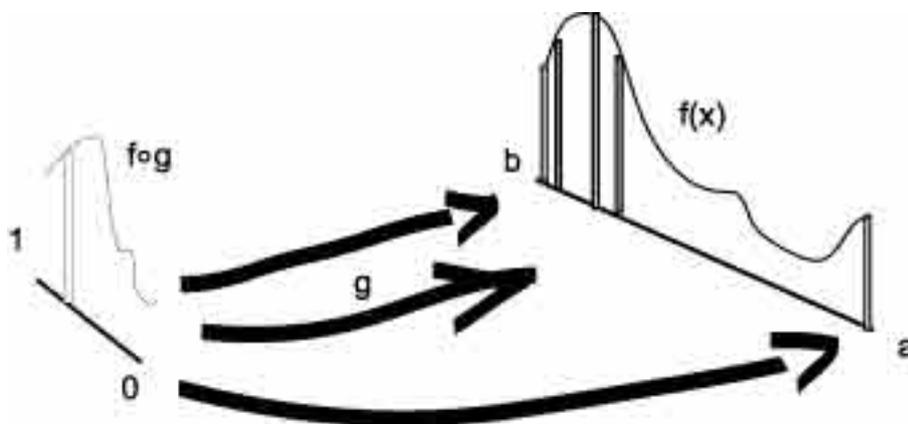


Figure 5.3: The change of variable formula.

Note that this takes account of the sign without our having to bother about it explicitly.

**Remark 5.1.4.** You may recognise this as the chain rule and  $S(f, a)$  as being the derivative of  $f$  at  $a$ . So you now have another way of thinking about derivatives. The more ways you have of thinking about something the better: some problems are very easy if you think about them the right way, the hard part is finding it.

### 5.1.2 Change of Variables in Integration

**Remark 5.1.5.** This way of thinking makes sense of the change of variables formula in integration, something which you may have merely memorised. Suppose we have the problem of integrating some function  $f : U \rightarrow \mathbb{R}$  where  $U$  is an interval. I shall write  $[a, b]$  for the interval  $\{x \in \mathbb{R} : a \leq x \leq b\}$

So the problem is to calculate

$$\int_a^b f(x) dx$$

Now suppose I have a function  $g : I \rightarrow [a, b]$  which is differentiable and, to make things simpler, 1-1 and takes 0 to  $a$  and 1 to  $b$ . This is indicated in the diagram figure 5.3

The integral is defined to be the limit of the sum of the areas of little boxes sitting on the segment  $[a, b]$ . I have shown some of the boxes. We can pull back the function  $f$  (expressed via its graph) to  $f \circ g$  which is in a sense the “same” function—well, it has got itself compressed, in general by different amounts at different places, because  $g$  stretches  $I$  to the (longer in the picture) interval  $[a, b]$ .

Now the integral of  $f \circ g$  over  $I$  is obviously related to the integral of  $f$  over  $[a, b]$ . If we have a little box at  $t \in I$ , the height of the function  $f \circ g$  at  $t$  is exactly the same as the height of  $f$  over  $g(t)$ . But if the *width* of the box at  $t$  is  $\Delta t$ , it gets stretched by an amount which is approximately  $g'(t)$  in going to  $[a, b]$ . So the area of the box on  $g(t)$ , which is what  $g$  does to the box at  $t$ , is approximately the area of the box at  $t$  multiplied by  $g'(t)$ . And since this holds for all the little boxes no matter how small, and the approximation gets better as the boxes get thinner, we deduce that it holds for the integral:

$$\int_0^1 f \circ g(t) g'(t) dt = \int_a^b f(x) dx$$

This is the change of variable formula. It actually works even when  $g$  is not 1-1, since if  $g$  retraces its path and then goes forward again, the backward bit is negative and cancels out the first forward bit. Of course,  $g$  has to be differentiable or the formula makes no sense.<sup>2</sup>

When you do the integral

$$\int_0^{\pi/2} \sin(t) \cos(t) dt$$

by substituting  $x = \sin(t)$ ,  $dx = \cos(t) dt$  to get

$$\int_0^1 x dx$$

you are doing exactly this “stretch factor” trick. In this case  $g$  is the function that takes  $t$  to  $x = \sin(t)$ ; it takes the interval from 0 to  $\pi/2$  to the interval  $I$  (thus compressing it) and the function  $y = x$  pulls back to the function  $y = \sin(t)$  over  $[0, \pi/2]$ . The stretching factor is  $dx = \cos(t) dt$  and is taken care of by the differentials. We shall see an awful lot of this later on in the course.

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<sup>2</sup> $g$  could fail to be differentiable at a finite number of points; we could cut the path up into little bits over which the formula makes sense and works. At the points where it doesn't, well, we just ignore them, because after all, how much are they going to contribute to the integral? Zero is the answer to that, so the hell with them.

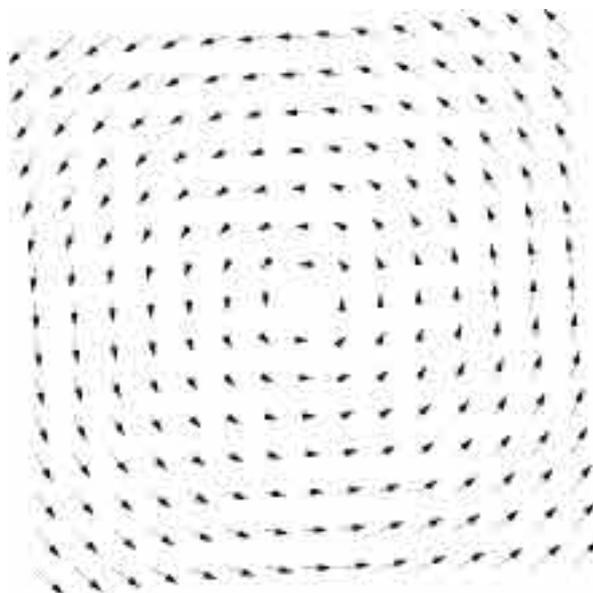


Figure 5.4: A vector field or 1-form with positive “twist”.

### 5.1.3 Spin Fields

**Remark 5.1.6.** I hope that you can see that thinking about functions as stretching intervals and having an amount of stretch at a point is useful: it helps us understand otherwise magical formulae. Now I am ready to use the kind of thinking that we went through in defining the amount of stretch of a function at a point. Instead, I shall be looking at differential 1-forms or  $\mathbb{R}^2$  and looking at the amount of “twist” the 1-form may have at a point of  $\mathbb{R}^2$ . The 1-form  $-ydx + xdy$  clearly has some, see figure 5.4

**Remark 5.1.7.** Think about a vector field or differential 1-form on  $\mathbb{R}^2$  and imagine it is the velocity field of a moving fluid. Now stick a tiny paddle wheel in at a point so as to measure the “rotation” or “twist” or “spin” at a point.

This idea, like the amount of stretch of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is vague, but we can try to make it precise by saying it in algebra.

**Remark 5.1.8.** If

$$V \begin{bmatrix} x \\ y \end{bmatrix} \triangleq P(x, y) dx + Q(x, y) dy$$

is the 1-form, look first at  $Q(x, y)$  along the horizontal line through the point  $\begin{bmatrix} a \\ b \end{bmatrix}$

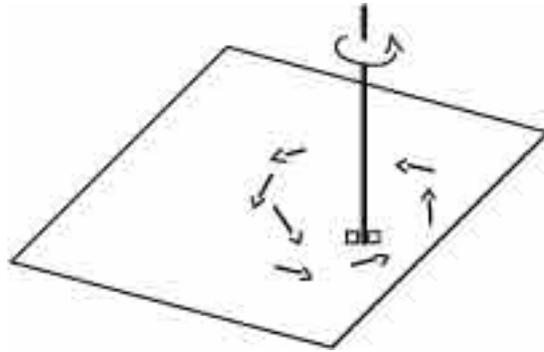
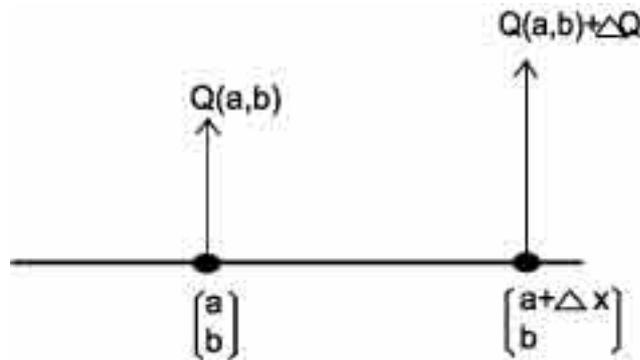


Figure 5.5: The amount of rotation of a vector field

Figure 5.6:  $\Delta Q \approx \frac{\partial Q}{\partial x} \Delta x$ 

If  $\Delta x$  is small, the  $Q$  component to the right of  $\begin{bmatrix} a \\ b \end{bmatrix}$  is

$$Q \begin{bmatrix} a \\ b \end{bmatrix} + \frac{\partial Q}{\partial x} \Delta x$$

and to the left is

$$Q \begin{bmatrix} a \\ b \end{bmatrix} - \frac{\partial Q}{\partial x} \Delta x$$

and the spin per unit length about  $\begin{bmatrix} a \\ b \end{bmatrix}$  in the positive direction is

$$\frac{\partial Q}{\partial x}$$

Similarly there is a tendency to twist in the opposite direction given by

$$\frac{\partial P}{\partial y}$$

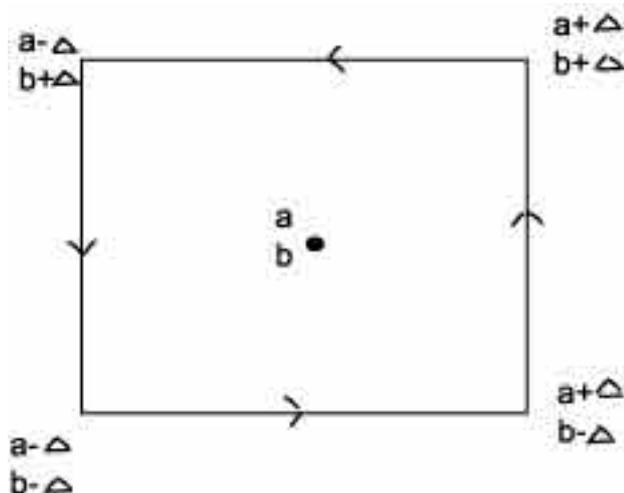


Figure 5.7: Path integral around a small square

So the total spin can be defined as

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

This is a function from  $\mathbb{R}^2$  to  $\mathbb{R}$ .

**Example 5.1.6.**

$$\begin{aligned}\omega &\triangleq x^2y \, dx + 3y^2x \, dy \\ \text{spin}(\omega) &= 3y^2 - 2xy\end{aligned}$$

**Example 5.1.7.**

$$\begin{aligned}\omega &\triangleq -y \, dx + x \, dy \\ \text{spin}(\omega) &= 2\end{aligned}$$

Where 2 is the constant function.

**Remark 5.1.9.** Another way of making the idea of ‘twist’ at a point precise would be to take the integral around a little square centred on the point and divide by the area of the square.

If the square, figure 5.7 has side  $2\Delta$  we go around each side.

From  $\begin{bmatrix} a + \Delta \\ b - \Delta \end{bmatrix}$  to  $\begin{bmatrix} a + \Delta \\ b + \Delta \end{bmatrix}$  we need consider only the  $Q$ -component which at the midpoint is approximately

$$Q \begin{bmatrix} a \\ b \end{bmatrix} + \frac{\partial Q}{\partial x} \Delta$$

and so the path integral for this side is approximately

$$\left[ Q \begin{bmatrix} a \\ b \end{bmatrix} + \left[ \frac{\partial Q}{\partial x} \right] \Delta \right] 2\Delta.$$

The side from  $\begin{bmatrix} a + \Delta \\ b + \Delta \end{bmatrix}$  to  $\begin{bmatrix} a - \Delta \\ b + \Delta \end{bmatrix}$  is affected only by the  $P$  component and the path integral for this part is approximately

$$\left[ P \begin{bmatrix} a \\ b \end{bmatrix} + \frac{\partial P}{\partial y} \Delta \right] (-2\Delta)$$

with the minus sign because it is going in the negative direction.

Adding up the contribution from the other two sides we get

$$\left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] 4\Delta^2$$

and dividing by the area ( $4\Delta^2$ ) we get

$$\text{spin}(\mathbf{V}) \triangleq \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right]$$

again.

## 5.2 Green's Theorem (Classical Version)

**Theorem 5.2.1. Green's Theorem** Let  $U \subset \mathbb{R}^2$  be connected and simply connected (has no holes in it), and has boundary a simple closed curve, that is a loop which does not intersect itself, say  $\ell$ .

Let  $\mathbf{V}$  be a smooth vector field

$$\mathbf{V} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

defined on a region which is open and contains  $U$  and its boundary loop.

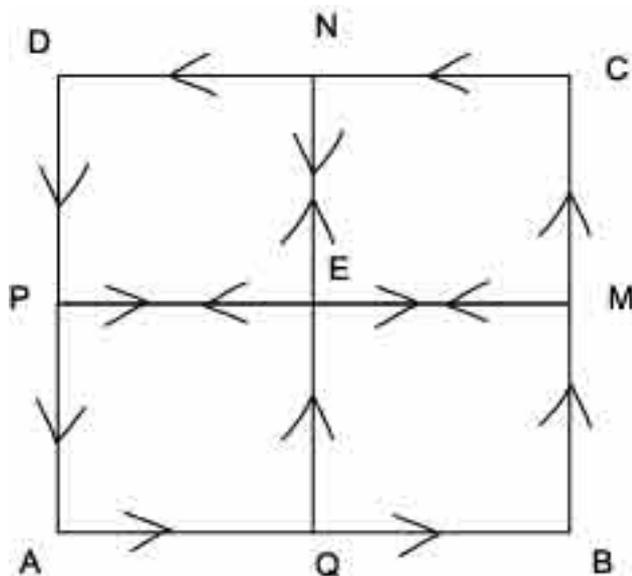


Figure 5.8: Adding paths around four sub-squares

Then

$$\int_{\ell} \mathbf{V} = \iint_U \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

where the loop  $\ell$  is traversed in the positive (anticlockwise) sense.

**“Proof”** I shall prove it for the particular case where  $U$  is a square, figure 5.8. If  $ABCD$  is a square,  $Q$  is the midpoint of  $AB$ ,  $M$  of  $BC$ ,  $N$  of  $CD$ , and  $P$  of  $DA$ , and if  $E$  is the centre of the square, then

$$\int_{ABCD} \mathbf{V} = \int_{AQEP} \mathbf{V} + \int_{QBME} \mathbf{V} + \int_{MCNE} \mathbf{V} + \int_{NDPE} \mathbf{V}$$

This is trivial, since we get the integral around each subsquare by adding up the integral around each edge; the inner lines are traversed twice in opposite directions and so cancel out.

We can continue subdividing the squares as finely as we like, and the sum of the path integral around all the little squares is still going to be the path integral around the big one.

But the path integral around a very small square can be approximated by

$$\left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right]$$

evaluated at the centre of the square and multiplied by its area, as we saw in the last section. And the limit of this sum is precisely the definition of the Riemann integral of

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

over the region enclosed by the square.  $\square$

**Remark 5.2.1.** To do it for more general regions we might hope the boundary is reasonable and fill it with squares. This is not terribly convincing, but we can reason that other regions also have path integral over the boundary approximated by

$$\left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \times (\text{area enclosed by shape})$$

**Remark 5.2.2.** The result for a larger collection of shapes will be proved later.

**Exercise 5.2.1.** Try to prove it for a triangular region, say a right-angled triangle, by chopping the triangle up into smaller triangles.

**Example 5.2.1. of Green's Theorem in use:** Evaluate  $\int_c \sin(x^3)dx + xy + 6 dy$  where  $c$  is the triangular path starting at  $\mathbf{i}$ , going by a straight line to  $\mathbf{j}$ , then down to the origin, then back to  $\mathbf{i}$ .

**Solution:** It is not too hard to do this the long way, but Green's Theorem tells us that the result is the same as

$$\iint_U \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where  $U$  is the inside of the triangle,  $P(x, y) = \sin(x^3)$  and  $Q(x, y) = xy + 6$ . This gives

$$\int_0^1 \int_0^{1-x} (y - 0) dy dx$$

which I leave you to verify is  $1/6$ .

**Remark 5.2.3.** While we are talking about integrals around simple loops, there is some old fashioned notation for such integrals, they often used to write

$$\oint_c f(t) dt$$

The little circle in the integral told you  $c$  was supposed to be a simple loop. Sometimes they had an arrow on the circle.

The only known use for these signs is to impress first year students with how clever you are, and it doesn't work too well.

**Example 5.2.2.** Find:

$$\oint_{S^1} (\log_e(x^6 + 152) + 17y) dx + (\sqrt{1 + y^{58}} + x) dy$$

You must admit this looks horrible regarded as a path integral. It is easily seen however to be

$$\int_{D^2} (1 - 17)$$

this is  $-16$  times the area of the unit disc which is of course  $\pi$  So we get  $-16\pi$

**Remark 5.2.4.** So Green's Theorem can be used to scare the pants off people who have just learnt to do line integrals. This is obviously extremely useful.

### 5.3 Spin fields and Differential 2-forms

The idea of a vector field having an amount of twist or spin at each point turned out to make sense. Now I want to consider something a bit wilder. Suppose I have a physical system which has, for each point in the plane, an amount of twist or spin associated with it. We do not need to assume this comes from a vector field, although it might. I could call such a thing a 'twist field' or 'spin field' on  $\mathbb{R}^2$ . If I did I would be the only person doing so, but there is a proper name for the idea, it is called<sup>3</sup> a *differential 2-form*. To signal the fact that there is a number associated with each point of the plane and it matters which orientation the plane has, we write

$$R(x, y) dx \wedge dy$$

for this 'spin field'. The  $dx \wedge dy$  tells us the positive direction, from  $x$  to  $y$ . If we reverse the order we reverse the sign:

$$dx \wedge dy = -dy \wedge dx$$

---

<sup>3</sup> A differential 2-form exists to represent *anything* that is associated with an oriented plane in a space, not necessarily spin. We could use it for describing pressure in a fluid. But I can almost imagine a spin field, and so I shall pretend that these are the things for which 2-forms were invented, which is close.

**Remark 5.3.1.** The idea makes sense, and we now have a spin field and we can ask, does it come from a vector field? Or more simply, we have a differential 2-form and we would like to know if it comes from a differential 1-form.

Given  $R(x, y) dx \wedge dy$ , is there always a  $P dx + Q dy$  that gives

$$R(x, y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} ?$$

It is easy to see that there are lots of them.

**Example 5.3.1.** Let

$$\psi = xy \sin(y) dx \wedge dy$$

be a “spin field” on  $\mathbb{R}^2$ . Is it derived from some 1-form  $\omega = P dx + Q dy$ ?

**Solution:** Yes, put  $P = 0$  and  $Q = x^2 y \sin(y)/2$ . Then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = xy \sin(y)$$

All I did was to set  $P$  to zero and integrate  $Q$  with respect to  $x$ . This is a bit too easy to be interesting. It stops being so silly if we do it on  $\mathbb{R}^3$ , as we shall see later.

**Remark 5.3.2.** It should be obvious that just as we had the derivative taking 0-forms to 1-forms, so we have a process for getting 2-forms from 1-forms. The process is called the *exterior derivative*, written  $d$  and on  $\mathbb{R}^2$  is defined by:

**Definition 5.1.** If  $\omega \triangleq P dx + Q dy$  is a 1-form on  $\mathbb{R}^2$  the exterior derivative of  $\omega$ ,  $d\omega$ , is defined by

$$d\omega \triangleq \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

**Remark 5.3.3.** Although I have been doing all this on  $\mathbb{R}^2$ , it all goes over to  $\mathbb{R}^3$  and indeed  $\mathbb{R}^n$  for any larger  $n$ . It is particularly important in  $\mathbb{R}^3$ , so I shall go through this case separately.

**Remark 5.3.4.** If you can believe in a ‘spin field’ in  $\mathbb{R}^2$  you can probably believe in one on  $\mathbb{R}^3$ . Again, you can see that a little paddle wheel in a vector field flow on  $\mathbb{R}^3$  could turn around as a result of different amounts of push on different sides. Now this time the paddle wheel could be chosen to be in

any plane through the point, and the amount of twist would depend on the point and the plane chosen. If you think of time as a fourth dimension, you can see that it makes just as much sense to have a spin field on  $\mathbb{R}^4$ . In both cases, there is a point and a preferred plane and there has to be a number associated with the point and the plane. After all, twists occur in planes. This is exactly why 2-forms were invented. Another thing about them: if you kept the point fixed and varied the plane continuously until you had the same plane, only upside down, you would get the negative of the answer you got with the plane the other way up. In  $\mathbb{R}^3$  the paddle wheel stick would be pointing in the opposite direction.

We can in fact specify the amount of spin in three separate planes, the  $x - y$  plane, the  $x - z$  plane, and the  $y - z$  plane, and this is enough to be able to calculate it for any plane. This looks as though we are really doing Linear Algebra, and indeed we are.

**Definition 5.2. 2-forms on  $\mathbb{R}^3$**  A smooth differential 2-form on  $\mathbb{R}^3$  is written

$$\psi \triangleq E(x, y, z) dx \wedge dy + F(x, y, z) dx \wedge dz + G(x, y, z) dy \wedge dz$$

where the functions  $E, F, G$  are all smooth.

**Remark 5.3.5.** If you think of this as a spin field on  $\mathbb{R}^3$  with  $E(x, y, z)$  giving the amount of twist in the  $x - y$  plane, and similarly for  $F, G$ , you won't go wrong. This is a useful way to visualise a differential 2-form on  $\mathbb{R}^3$ .

**Remark 5.3.6.** It might occur to you that I have told you how we write a differential 2-form, and I have indicated that it can be used for talking about spin fields, and told you how to visualise the spin fields and hence differential 2-forms. What I have not done is to give a formal definition of what one *is*. Patience, I'm coming to this.

**Example 5.3.2.** Suppose the plane  $x + y + z = 0$  in  $\mathbb{R}^3$  is being rotated in the positive direction when viewed from the point  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , at a constant rate of one unit. Express the rotation in terms of its projection on the  $x - y$ ,  $x - z$  and  $y - z$  planes.

**Solution** If you imagine the plane rotating away and casting a shadow on the  $x - y$  ( $z = 0$ , but remember the orientation!) plane, clearly there would be some 'shadow' twist, but not the full quantity. Likewise the projections on the other planes.

Take a basis for the plane consisting of two orthogonal vectors in the plane of length one. There are an infinite number of choices: I pick

$$\mathbf{u} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

(I got these by noticing that  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  was in the plane and then I took the cross product with the normal to the plane to get

$$\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

finally I scaled them to have length one.)

Now I write

$$du = \frac{dx}{\sqrt{2}} - \frac{dz}{\sqrt{2}}$$

$$dv = \frac{-1}{\sqrt{6}} dx + \frac{2}{\sqrt{6}} dy - \frac{1}{\sqrt{6}} dz$$

which I got by telling myself that the projection onto the basis vector  $\mathbf{u}$  should be called  $du$ , and that it would in fact send  $[x, y, z]^T$  to  $1/\sqrt{2} \mathbf{i} - 1/\sqrt{2} \mathbf{k}$  which is the mixture

$$\frac{dx}{\sqrt{2}} - \frac{dz}{\sqrt{2}}$$

And similarly for the expression for  $dv$ .

Last, I write the spin as

$$1 \, du \wedge dv = \left( \frac{1}{\sqrt{2}} dx - \frac{1}{\sqrt{2}} dz \right) \wedge \left( \frac{-1}{\sqrt{6}} dx + \frac{2}{\sqrt{6}} dy - \frac{1}{\sqrt{6}} dz \right)$$

Expanding this and using  $dx \wedge dx = 0$  and  $dz \wedge dx = -dx \wedge dz$  I get

$$\frac{2}{\sqrt{12}} dx \wedge dy - \frac{2}{\sqrt{12}} dx \wedge dz + \frac{2}{\sqrt{12}} dy \wedge dz$$

$$= \frac{1}{\sqrt{3}} dx \wedge dy - \frac{1}{\sqrt{3}} dx \wedge dz + \frac{1}{\sqrt{3}} dy \wedge dz$$

This shows equal amounts of spin on each plane, and a negative twist on the  $x - z$  plane, which is right. (Think about it!)

Note that the sum of the squares of the coefficients is 1. This is the amount of spin we started with. Note also that the sums are easy although they introduce the  $\wedge$  as if it is a sort of multiplication. I shall not try to justify this here. At this point I shall feel happy if you are in good shape to do the sums we have coming up.

**Remark 5.3.7.** It would actually make good sense to write it out using  $dz \wedge dx$  instead of  $dx \wedge dz$ . Then the plane  $x + y + z = 0$  with the positive orientation can be written as

$$\frac{1}{\sqrt{3}} dx \wedge dy + \frac{1}{\sqrt{3}} dz \wedge dx + \frac{1}{\sqrt{3}} dy \wedge dz$$

In this form it is rather strikingly similar to the unit normal vector to the plane. Putting  $dx \wedge dy = \mathbf{k}$  and so on, is rather tempting. It is a temptation to which physicists have succumbed rather often.

**Example 5.3.3.** The spin field  $2 dx \wedge dy + 3dz \wedge dx + 4 dy \wedge dz$  on  $\mathbb{R}^3$  is examined by inserting a probe at the origin so that the oriented plane is again  $x + y + z = 0$  with positive orientation seen from the point  $\mathbf{i} + \mathbf{j} + \mathbf{k}$ . What is the amount of spin in this plane?

**Solution 1** Project the vector  $2 dx \wedge dy + 3dz \wedge dx + 4 dy \wedge dz$  on the vector

$$\frac{1}{\sqrt{3}} dx \wedge dy + \frac{1}{\sqrt{3}} dz \wedge dx + \frac{1}{\sqrt{3}} dy \wedge dz$$

to get

$$\begin{aligned} \frac{9}{\sqrt{3}} \left( \frac{1}{\sqrt{3}} dx \wedge dy + \frac{1}{\sqrt{3}} dz \wedge dx + \frac{1}{\sqrt{3}} dy \wedge dz \right) \\ = 3 dx \wedge dy + 3 dz \wedge dx + 3 dy \wedge dz \\ = 3\sqrt{3} du \wedge dv \end{aligned}$$

**Solution 2** (Physicist's solution) Write the spin as a vector  $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  and the normal to the oriented plane as

$$\frac{1}{\sqrt{3}} \mathbf{i} + \frac{1}{\sqrt{3}} \mathbf{j} + \frac{1}{\sqrt{3}} \mathbf{k}$$

Now take the dot product to get the length of the spin (pseudo)vector:  $9/\sqrt{3} = 3\sqrt{3}$ . The whole vector is therefore

$$3 \mathbf{i} + 3 \mathbf{j} + 3 \mathbf{k}$$

This is  $3\sqrt{3}$  times the unit normal to the plane  $x + y + z = 0$ .

**Remark 5.3.8.** It should be clear that in  $\mathbb{R}^3$  it is largely a matter of taste as to which system, vectors or 2-forms, you use. The advantage of using 2-forms is partly that they generalise to higher dimensions. You could solve a problem similar to the above but in four dimensions if you used 2-forms, while the physicist would be stuck. So differential forms have been used in the past for terrorising physicists, which takes a bit of doing. The modern physicists are, of course, quite comfortable with them.

**Remark 5.3.9.** Richard Feynman in his famous “Lecture Notes in Physics” points out that vectors used in this way, to represent rotations, are not really the same as ordinary vectors such as are used to describe force fields. He calls them ‘pseudovectors’ and makes the observation that we can only confuse them with vectors because we live in a three dimensional space, and in  $\mathbb{R}^4$  we would have *six* kinds of rotation.

**Exercise 5.3.1.** Confirm that Feynman knows what he is talking about and that six is indeed the right number.

### 5.3.1 The Exterior Derivative

**Remark 5.3.10.** Now I tell you how to do the exterior derivative from 1-forms to 2-forms on  $\mathbb{R}^3$ . Watch carefully!

**Definition 5.3.** If

$$\omega \triangleq P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

is a smooth 1-form on  $\mathbb{R}^3$  then the exterior derivative applied to it gives the 2-form:

$$d\omega \triangleq \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) dx \wedge dz + \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz$$

**Remark 5.3.11.** This is not so hard to remember as you might think and I will now give some simple rules for working it out. Just to make sure you can do it on  $\mathbb{R}^4$  I give it in horrible generality. (Actually I have a better reason than this which will emerge later.)

**Definition 5.4.** If

$$\omega \triangleq P_1 dx^1 + P_2 dx^2 + \cdots + P_n dx^n$$

is a differential 1-form on  $\mathbb{R}^n$ , the *exterior derivative* of  $\omega$ ,  $d\omega$  is the differential 2-form

$$\begin{aligned} & \left( \frac{\partial P_2}{\partial x^1} - \frac{\partial P_1}{\partial x^2} \right) dx^1 \wedge dx^2 + \left( \frac{\partial P_3}{\partial x^1} - \frac{\partial P_1}{\partial x^3} \right) dx^1 \wedge dx^3 + \dots \\ & \dots + \left( \frac{\partial P_3}{\partial x^2} - \frac{\partial P_2}{\partial x^3} \right) dx^2 \wedge dx^3 + \dots + \left( \frac{\partial P_n}{\partial x^{n-1}} - \frac{\partial P_{n-1}}{\partial x^n} \right) dx^{n-1} \wedge dx^n \end{aligned}$$

This looks frightful but is actually easily worked out:

**Rule 1:** Partially differentiate every function  $P_j$  by every variable  $x^i$ . This gives  $n^2$  terms.

**Rule 2** When you differentiate  $P_j dx^j$  with respect to  $x^i$ , write the new differential bit as:

$$\frac{\partial P_j}{\partial x^i} dx^i \wedge dx^j$$

**Rule 3:** Remember that  $dx^i \wedge dx^j = -dx^j \wedge dx^i$ . Hence  $dx^i \wedge dx^i = 0$  So we throw away  $n$  terms leaving  $n(n-1)$ , and collect them in matching pairs.

**Rule 4** Bearing in mind Rule 3, collect up terms in increasing alphabetical order so if  $i < j$ , we get a term for  $dx^i \wedge dx^j$ .

**Remark 5.3.12.** It is obvious that if you have a 1-form on  $\mathbb{R}^n$ , the derived 2-form has  $n(n-1)/2$  terms in it.

**Proposition 5.3.1.** When  $n = 2$ , this gives

$$d\omega = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

**Proof**

Start with:

$$\omega \triangleq P dx + Q dy$$

Following rule 1 we differentiate everything in sight and put  $du \wedge$  in front of the differential already there when we differentiate with respect to  $u$ , where

$u$  is  $x$  or  $y$ . This gives us:

$$d\omega = \frac{\partial P}{\partial x} dx \wedge dx + \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial y} dy \wedge dy$$

Now we apply rule 3 and throw out the first and last term to get

$$d\omega = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy$$

and finally we apply rules 3 and 4 which has  $dx \wedge dy$  as the preferred (alphabetic) ordering so we get:

$$d\omega = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

as required. □

**Example 5.3.4.** Now I do it for  $\mathbb{R}^3$  and you can see how easy it is to get the complicated expression for  $d\omega$  there:

I shall take the exterior derivative of

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$$

which is a 1-form on  $\mathbb{R}^3$ :

**Rules 1, 2** give me

$$\frac{\partial P}{\partial x} dx \wedge dx + \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx$$

when I do the  $P$  term.

The  $Q$  term gives me:

$$\frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial y} dy \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy$$

and finally the  $R$  term gives me:

$$\frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz + \frac{\partial R}{\partial z} dz \wedge dz$$

**Rule 3** tells me that of these nine terms, three are zero, the  $dx \wedge dx$ , the  $dy \wedge dy$  and  $dz \wedge dz$  terms. I could have saved a bit of time by not even writing them down.

It also tells me that the remaining six come in pairs. I collect them up in accordance with **Rule 4** to get:

$$\left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) dx \wedge dz + \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz$$

**Remark 5.3.13.** Not so terrible, was it? All you have to remember really is to put  $du \wedge$  in front of the old differential when you partially differentiate with respect to  $u$ , and do it for every term and every variable. Then remember  $du \wedge dv = -dv \wedge du$  and so  $du \wedge du = 0$  and collect up the matching pairs. After some practice you can do them as fast as you can write them down.

**Remark 5.3.14.** Just as in two dimensions, the exterior derivative applied to a 1-form or vector field gives us a 2-form or spin-field. Only now it has three components, which is reasonable.

**Remark 5.3.15.** If you are an old fashioned physicist who is frightened of 2-forms, you will want to pretend  $dy \wedge dz = \mathbf{i}$ ,  $dz \wedge dx = \mathbf{j}$  and  $dx \wedge dy = \mathbf{k}$ .

Which means that, as pointed out earlier, spin fields on  $\mathbb{R}^3$  can be confused with vector fields by representing a spin in a plane by a vector orthogonal to the plane of the spin and having length the amount of the spin.

This doesn't work on  $\mathbb{R}^4$ .

You will therefore write that if

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$$

is a vector field on  $\mathbb{R}^3$ , there is a derived vector field which measures the spin of  $\mathbf{F}$ . It is called the *curl* and is defined by:

**Definition 5.5.**

$$\text{curl}(\mathbf{F}) \triangleq \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}$$

**Remark 5.3.16.** This is just our formula for the exterior derivative with  $dy \wedge dx$  put equal to  $\mathbf{i}$ ,  $dx \wedge dz$  put equal to  $-\mathbf{j}$  and  $dx \wedge dy$  put equal to  $\mathbf{k}$ .

It is a problem to remember this, since the old fashioned physicists couldn't easily work it out, so instead of remembering the simple rules for the exterior derivative, they wrote:

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

This they pretended was a vector in  $\mathbb{R}^3$  and then :

**Definition 5.6.**

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

where  $\times$  is the cross product in  $\mathbb{R}^3$ .

**Exercise 5.3.2.** Confirm that the two notations are equivalent.

**Example 5.3.5.** Find the 2-form  $d\omega$  where

$$\omega \triangleq xz \, dx + xyz \, dy - y^2 \, dz$$

**Solution:**

Just writing down the terms following the four rules but skipping the zero terms I get:

$$(yz) \, dx \wedge dy + (-x) \, dx \wedge dz + (-2y - xy) \, dy \wedge dz$$

Doing it using the curl rule I get:

$$\text{curl}(\omega) = \nabla \times \begin{bmatrix} xz \\ xyz \\ -y^2 \end{bmatrix} = \begin{bmatrix} -2y - xy \\ x \\ yz \end{bmatrix}$$

These are obviously the same under the convention of writing  $dx \wedge dy = \mathbf{k}$  and so on.

I know which I prefer, but you can make up your own minds. Be sure you can use both systems with reasonable skill.

**Remark 5.3.17.** Again, the old fashioned physicists had no satisfactory way of handling problems in  $\mathbb{R}^n$  for  $n > 3$ . The exterior derivative  $d\omega$  looks a fairly straightforward thing compared with  $\nabla \times \mathbf{F}$ . Of course, you can get used to any notation if you use it often enough. Some of the more inflexible minds, having spent a lot of time mastering one notation scream in horror at the thought of having to learn another. The question of which is better doesn't come into it. The well known equation

$$\mathbf{NEW} = \mathbf{EVIL}$$

is invoked.

**Remark 5.3.18.** We can generalise 2-forms to  $\mathbb{R}^n$  with minimal fuss:

**Definition 5.7. Differential 2-forms on  $\mathbb{R}^n$** 

A smooth differential 2-form on  $\mathbb{R}^n$  is written as

$$\sum_{1 \leq i < j \leq n} F_{i,j}(\mathbf{x}) dx^i \wedge dx^j$$

where the  $F_{i,j}$  are smooth functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

**Remark 5.3.19.** We can again ask, this time for  $\mathbb{R}^3$ , the question, is any spin field (differential 2-form) on  $\mathbb{R}^3$  in fact the exterior derivative (derived spin field) of a vector field (differential 1-form)? Algebraically, if

$$\omega = E(\mathbf{x}) dx \wedge dy + F(\mathbf{x}) dx \wedge dz + G(\mathbf{x}) dy \wedge dz$$

is there a 1-form  $\psi$

$$\psi = P dx + Q dy + R dz$$

such that

$$d\psi = \omega?$$

that is, can we find  $P, Q, R$  such that

$$\left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = E$$

and

$$\left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) = F$$

and

$$\left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) = G$$

**Remark 5.3.20.** You might like to try this out for simple cases. Experiment, explore, it is a more interesting world if you do. I shall come back to this later.

**Remark 5.3.21.** Another question we might ask (based on simple curiosity and trying to push things from two dimensions to three) is: If  $\omega$  is a smooth 1-form on  $\mathbb{R}^3$  and  $d\omega = 0$ , is it the case that  $\omega = df$  for some 0-form  $f$ ?

Or, saying it in the old fashioned language, if the curl of a smooth vector field is zero, is it conservative? Note that if the curl of a vector field exists, it *has* to be a vector field on at least a subset of  $\mathbb{R}^3$ , since this is the only place where we have a cross product to be able to compute the curl.

The answer to both questions is yes. On the other hand it can fail to be true if the vector field/1-form is defined on only a subset of  $\mathbb{R}^3$  which has holes in it.

**Exercise 5.3.3.** Define “has no holes in it” for subsets of  $\mathbb{R}^3$

**Proposition 5.3.2.** If  $\text{curl}\mathbf{F} = \mathbf{0}$  on all of  $\mathbb{R}^3$ , then  $\mathbf{F}$  is conservative.

Recall:

**Definition 5.8. Exact** A 1-form  $\omega$  is *exact* iff it there is a 0-form  $f$  such that  $\omega = df$ .

**Remark 5.3.22.** This was stated for forms on  $\mathbb{R}^2$  but since the dimension isn't mentioned, it must work for  $\mathbb{R}^3$  as well.

**Proposition 5.3.3.** If  $\omega$  is a 1-form and  $d\omega = \mathbf{0}$  then  $\omega$  is exact.

**Proof:** Later.

**Example 5.3.6.**

$$\omega = 2xy^3z^4 dx + 3x^2y^2z^4 dy + 4x^2y^3z^3 dz$$

Is this 1-form exact? That is, is  $\omega = df$  for some 0-form  $f$  on  $\mathbb{R}^3$ ?

Applying the claim (as yet unproven) above:

$$\begin{aligned} d\omega &= (6xy^2z^4 - 6xy^2z^4) dx \wedge dy + (8xy^3z^3 - 8xy^3z^3) dx \wedge dz \\ &\quad + (12x^2y^2z^3 - 12x^2y^2z^3) dy \wedge dz = \mathbf{0} \end{aligned}$$

This tells us that provided the form is defined on all of  $\mathbb{R}^3$ , which it is, then it must be exact, and the corresponding vector field is conservative. So there must be a 0-form  $f$ . We must have

$$\frac{\partial f}{\partial x} = 2xy^3z^4$$

so integrating with respect to  $x$  we get

$$x^2y^3z^4 + u(y, z)$$

for some unknown function of  $y$  and  $z$  only. Similar integration for the other two functions leads us to the conclusion that  $u = \mathbf{0}$  and

$$f(x, y, z) = x^2y^3z^4$$

**Remark 5.3.23.** There is nothing much new here over the two dimensional case and you could do it for 1-forms on  $\mathbb{R}^4$  in exactly the same way.

### 5.3.2 For the Pure Mathematicians.

**Remark 5.3.24.** You may be feeling uneasy that we have not given a formal definition of a differential 2-form on  $U \subseteq \mathbb{R}^n$ , for any  $n$ . Instead I have just told you how to write them down and how to derive (some of) them from 1-forms. In this respect your experience of them is just like your experience of cats. You know how to recognise one, and you know and what to do with one when you meet it. In Mathematics, if not in real life, it is possible to do better.

I told you that a differential 0-form on  $U$  was a smooth map

$$f : U \longrightarrow \mathbb{R}$$

and that a differential 1-form on  $U \subseteq \mathbb{R}^n$  was a smooth map

$$\omega : U \longrightarrow \mathbb{R}^{n*}$$

In the first case we attached a number to each point of  $U$ , in the second we attach a covector, barely distinguishable from a vector.

You are within your rights to expect me to tell you that a differential 2-form on  $U$  is a map from  $U$  to some vector space of thingies which can be used to represent torques. Certainly these 'thingies' have to have some sort of association with oriented planes at the very least.

We can actually do this: we find it is a map from  $U$  to a vector space of things called *alternating 2-tensors* written  $\Omega^2(\mathbb{R}^n)$  and the  $dx^i \wedge dx^j$  are basis vectors of it.

You can see that the dimension of this space will be  $n(n-1)/2$  because that is how many  $dx^i \wedge dx^j$  there are. So a point in this vector space will be

$$\sum_{1 \leq i < j \leq n} a_{i,j} dx^i \wedge dx^j$$

for some collection of  $n(n-1)/2$  numbers  $a_{i,j}$ .

A differential 2-form on  $U$  attaches such things to points of the space  $U$ : the  $a_{i,j}$  vary as we move about in  $U$ . We are now attaching alternating tensors, which gives us a *tensor field*. A differential form is an example of a tensor field: 1-forms and vector fields are the easiest cases, 2-forms a bit harder.

**Remark 5.3.25.** The actual definition of  $\Omega^2(\mathbb{R}^n)$  may come as a bit of an anticlimax.

I shall do it by defining the basis vectors of the space, the  $dx^i \wedge dx^j$ , for  $i < j$ .

Each  $dx^i$  is a projection from  $\mathbb{R}^n$  to  $\mathbb{R}$  so it is not surprising that  $dx^i \wedge dx^j$  is a map:

$$dx^i \wedge dx^j : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix} \rightsquigarrow x^i y^j - x^j y^i$$

In other words, it is just the determinant of the  $i$  and  $j$  rows. It is obvious that there are  $C_2^n = n(n-1)/2$  ways of picking two rows from  $n$ , and any two choices are different maps. If you reverse the order of the rows you get  $dx^j \wedge dx^i = -dx^i \wedge dx^j$  which is right.

**Definition 5.9.**  $\Omega^2(\mathbb{R}^n)$  is the vector space of maps from  $\mathbb{R}^n \times \mathbb{R}^n$  to  $\mathbb{R}$  spanned by the above maps.

**Remark 5.3.26.** It is now easy to prove that it really is a vector space with the right dimension. This is not the only way to define it, or even the best way, but it is an easy way which is why I have picked it.

**Remark 5.3.27.** So now you know what a differential 2-form on  $U \subseteq \mathbb{R}^n$  really is, and you have met a definition of an alternating tensor field. Crikey, life doesn't get much better than this.

**Remark 5.3.28.** For the record, for possible future needs, but not for your present needs, a differential  $k$ -form on  $U \subseteq \mathbb{R}^n$  is still an alternating tensor field, i.e a differentiable map from  $\mathbb{R}^n$  to  $\Omega^k(\mathbb{R}^n)$ . The space  $\Omega^k(\mathbb{R}^n)$  is defined as a space of maps

$$\underbrace{\mathbb{R}^n \times \mathbb{R}^n \cdots \times \mathbb{R}^n}_{k \text{ copies}} \longrightarrow \mathbb{R}$$

A basis vector in this space makes a choice of  $k$  different rows and calculates the determinant of the resulting  $k \times k$  matrix. There are obviously  $C_k^n$  different basis vectors and the span of them all is  $\Omega^k(\mathbb{R}^n)$ .

**Remark 5.3.29.** The exterior derivative generalises to an exterior derivative  $d$  that takes  $k$ -forms to  $k+1$  forms. If you have a lot of terms of the form

$$F_{i_1, i_2, \dots, i_k} dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_k}$$

you just differentiate all of them with respect to everything as in rule 1.

You put  $du\wedge$  in front of the differential part when you have differentiated the function part with respect to  $u$ , just as for Rule 2.

You remember that if any two terms get swapped among all those

$$dx^{i_1} \wedge dx^{i_2} \wedge \dots$$

the sign is changed, so if any are the same you put the term to be zero.

Then you collect up in alphabetic order. And that does it.

**Remark 5.3.30.** It requires, perhaps, rather more (multi-)linear algebra than you have met so far for you to feel altogether happy about this. If you are prepared to take my word for it that I can write any oriented plane in  $\mathbb{R}^3$  as so much  $dx \wedge dy$  plus some amount of  $dx \wedge dz$  added to a quantity of  $dy \wedge dz$ , then you can proceed without further worry. If you feel insecure without formal definitions, they are in the appendix.

### 5.3.3 Return to the (relatively) mundane.

**Remark 5.3.31.** It is easy to see how to write differential 2-forms on  $\mathbb{R}^4$ :

On  $\mathbb{R}^4$  with  $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$  specifying the four components.

We would have

$$Pdx \wedge dy + Qdx \wedge dz + Rdx \wedge dw + Sdy \wedge dz + Tdy \wedge dw + Udz \wedge dw$$

as a differential 2-form. This agrees with Richard Feynman which is cheering.

**Remark 5.3.32.** You will perhaps be even more cheered to note that we don't go beyond 3-forms on  $\mathbb{R}^3$ .

**Proposition 5.3.4.** We have 0-forms  $\xrightarrow{d}$  1-forms  $\xrightarrow{d}$  2-forms on  $\mathbb{R}^2$

and  $d^2 = 0$ .

**Proof:** If  $f$  is the 0-form,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is the 1-form and

$$d^2f = \left[ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right] dx \wedge dy = 0$$

is the derived 2-form.  $\square$

**Remark 5.3.33.** This also holds on  $\mathbb{R}^3$ :

**Proposition 5.3.5.** For 0-forms on  $\mathbb{R}^3$ ,  $d^2 = \mathbf{0}$ .

Proof: Given  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a 0-form,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

and

$$\begin{aligned} d^2f &= \\ \left( \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} \right) dx \wedge dy &+ \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) dx \wedge dz + \left( \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z} \right) dy \wedge dz \\ &= \mathbf{0} \end{aligned}$$

$\square$

**Remark 5.3.34.** We can turn this into old-fashioned language by writing:

$$\text{curl}(\nabla(f)) = \mathbf{0}$$

or

$$\nabla \times \nabla(f) = \mathbf{0}$$

**Remark 5.3.35.** Since there is no new idea in this and it is just the preceding subject rewritten in the old fashioned notation, I leave it to you to verify it.

## 5.4 More on Differential Stretching

**Remark 5.4.1.** Recall my discussing the change of variable formula for integration of functions of a single variable. I explained how it is useful to think of the derivative as a *differential length stretching* term. I elaborated on this because it works also for higher dimensions.

**Remark 5.4.2.** It works in particular for cases where the curve is in  $\mathbb{R}^2$  or  $\mathbb{R}^3$

**Example 5.4.1.** Find an expression for the arc length of the graph of  $y = x^2$  from the origin to the point  $[1, 1]^T$ . Using Mathematica or otherwise, find the length of the curve.

**Solution** We can write the problem as  $\int_c d\ell$  where  $d\ell$  is an ‘infinitesimal’ bit of the curve. It is reasonable to write

$$d\ell = \sqrt{(dx)^2 + (dy)^2}$$

Parametrise the curve by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

Then note that the quantity  $d\ell$  becomes just the norm of the differential term so we get:

$$\begin{aligned} & \int_0^1 \left\| \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} \right\| dt \\ &= \int_0^1 \sqrt{1 + 4t^2} dt \end{aligned}$$

Writing

`Integrate[Sqrt[1 + 4t^2], {t, 0, 1}]`

in Mathematica (or by making the substitution  $2t = \sinh(x)$ ) we get:

$$\frac{1}{4} \left( 2\sqrt{5} + \sinh^{-1}(2) \right)$$

Writing

`N[%]`

or

`NIntegrate[Sqrt[1 + 4t^2], {t, 0, 1}]`

to get a numerical evaluation, we get 1.47894 which looks believable.

Note that all arc length problems, in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , can be seen as

$$\int_I \|\dot{\mathbf{x}}\| dt$$

where

$$\begin{aligned} \mathbf{x} : I &\longrightarrow \mathbb{R}^n \\ t &\rightsquigarrow \mathbf{x}(t) \end{aligned}$$

is the parametrisation of the curve.

**Exercise 5.4.1.** Find the length of the arc of  $y = \cosh(x)$  between  $x = -1$  and  $x = 1$ .

**Remark 5.4.3.** As well as doing it for curves, it also works just as well for parametrisation of *surfaces*.

Let  $g : I^2 \longrightarrow \mathbb{R}^2$  be a map of the unit square into  $\mathbb{R}^2$ . Recall:

**Definition 5.10.**

$$I^2 \triangleq \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : 0 \leq x \leq 1, 0 \leq y \leq 1 \right\}$$

**Remark 5.4.4.** The picture corresponding to figure 5.3 which did it for curves is figure 5.9

I have drawn a cubical box at the front over  $g(I^2)$  and pulled it back to a box over  $I^2$  to show what happens when you integrate the (blue coloured) function  $f$ . I have chosen  $g$  to be 1-1 and smooth.

**Remark 5.4.5.** It is plain that the idea of the one-dimensional case still works here, but the cubical boxes standing on a tiny square base in  $I^2$ , will be taken by  $g$  to cubical boxes standing on deformed squares in  $g(I^2)$ . We get these deformed squares in the limit by taking the derivative of  $g$  at the point in  $I^2$  to see what the deformation is, and in particular what it does to the area of those tiny squares. The area stretching factor will then go into the formula for changing the variable by composing with  $g$ .

The derivative of  $g$  at a point  $\mathbf{a}$  in  $I^2$  is going to be  $2 \times 2$  matrix

$$\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

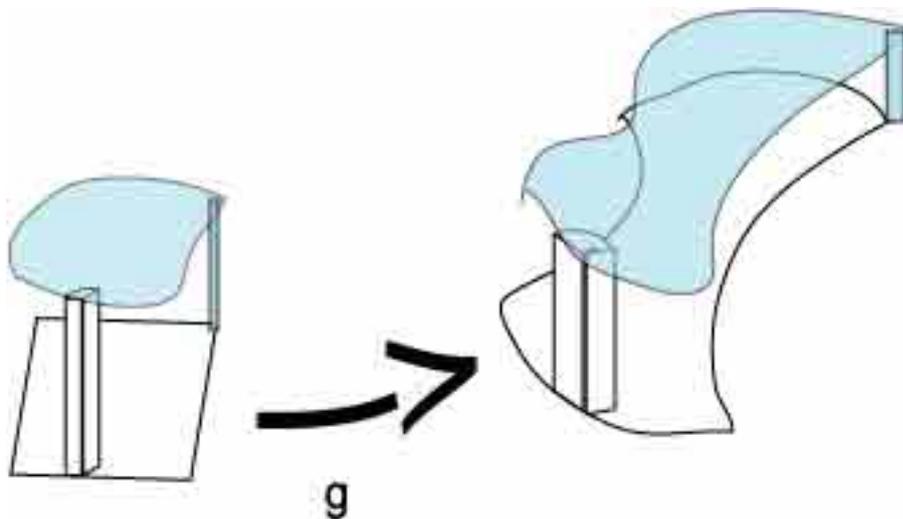


Figure 5.9: Change of variables in Integration for a function of two variables

where

$$g = \begin{bmatrix} x(s, t) \\ y(s, t) \end{bmatrix}$$

When I evaluate those partial derivatives at the point  $\mathbf{a}$  I shall get a matrix of numbers which is the linear map which best approximates  $g$  at the point  $\mathbf{a}$

**Remark 5.4.6.** We need to see what linear maps from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  do to area. I have drawn in figure 5.10 the image of the unit square by the linear map given by the matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since the area of the unit square is one, and since a linear map will take smaller squares to proportionately smaller parallelograms, the area stretching factor for the linear map is simply the area of the parallelogram with vertices at the origin,

$$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \text{ and } \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

Now we can calculate the area of this parallelogram by chopping off the triangle at the top and moving it down to the bottom. This now gives a new parallelogram which is the image of  $I^2$  by the matrix

$$\begin{bmatrix} a - bc/d & c \\ 0 & d \end{bmatrix}$$

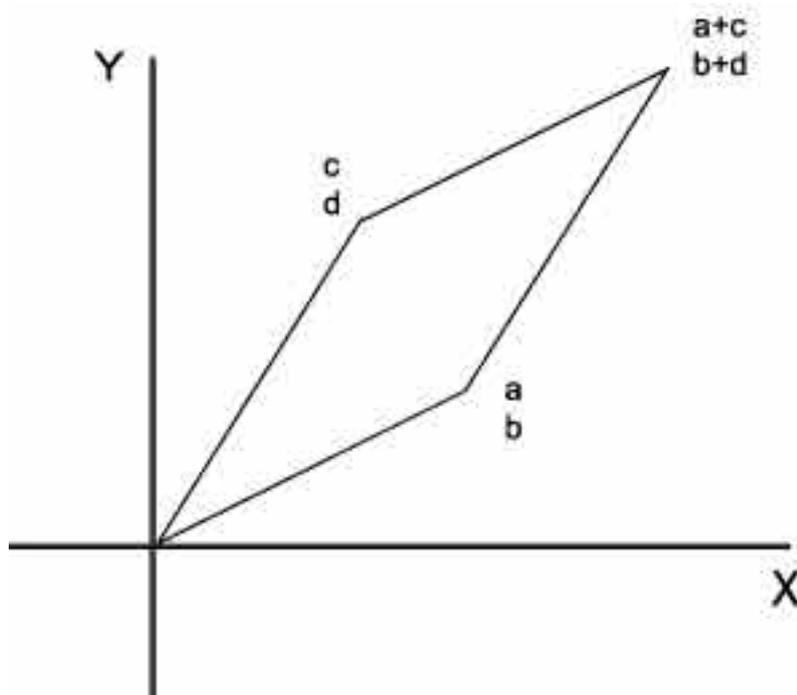


Figure 5.10: Image by a linear map of the unit square

These two maps have the same area stretching factor. I hope you can see that I merely subtracted the proper multiple of

$$\begin{bmatrix} c \\ d \end{bmatrix} \text{ from } \begin{bmatrix} a \\ b \end{bmatrix}$$

to get the second matrix.

The diagram in figure 5.11 shows the new parallelogram. It can also be skewed, without changing the area, (chop out a triangle and move it) so that it is rectangular, and has vertices at the origin, at

$$\begin{bmatrix} a - \frac{bc}{d} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d \end{bmatrix} \text{ and } \begin{bmatrix} a - \frac{bc}{d} \\ d \end{bmatrix}$$

This rectangle would arise from the linear map represented by the matrix:

$$\begin{bmatrix} \frac{ad-bc}{d} & 0 \\ 0 & d \end{bmatrix}$$

which has the same area stretch as the original matrix, and takes the unit square to the rectangle. It is easy to see that the area of this rectangle is

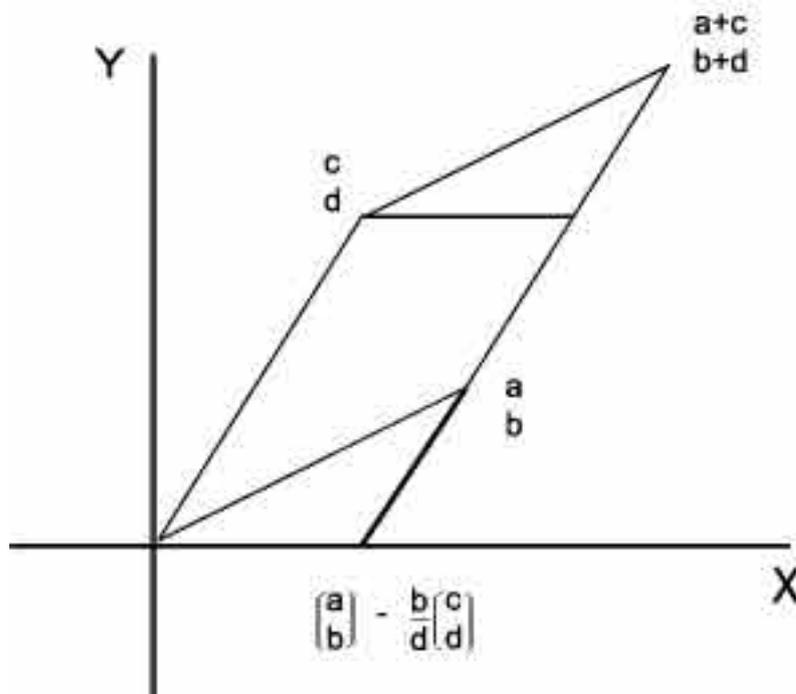


Figure 5.11: Image by a linear map of the unit square

$ad - bc$ . Since none of the transformations have changed the area, this is the area of the original parallelogram and is the area stretching factor of the linear map. Note that it is negative when the image of  $\mathbf{i}$  has been moved past the image of  $\mathbf{j}$  so that the original square has been turned upside down. So we actually get the *oriented* area stretching term out of what is the determinant of the matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

**Remark 5.4.7.** It is a good idea to take  $g$  to be 1-1 and smooth and to have a smooth inverse except perhaps on a set of area zero.<sup>4</sup> In this case, the *Jacobian Determinant*

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

is either always positive or always negative since it cannot be zero. We get the right answer for the area stretch either way if we take its absolute value.

<sup>4</sup> I was rather dismissive of curves which failed to be differentiable at only a finite set of points, and I propose to be equally dismissive of functions from  $I^2$  which fail to be smooth or 1-1 at a finite number of lines; the reason is the same, they won't make any difference to the integral.

**Proposition 5.4.1.** The change of variable formula for integrating a function over a surface is:

$$I^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

$$\int_{g(I^2)} f = \int_{I^2} (f \circ g) |\det(D(g))|$$

where  $D(g)$  is the derivative of  $g$ . You should be able to see that  $|\det D(g)|$  is the “differential area stretching factor”.

**Proof:** We have done all the hard work in messing about with parallelograms. We note then that we can get an approximation to a Riemann double sum over little regions in  $g(I^2)$  by taking squares in  $I^2$  and their images by  $g$ . In the neighbourhood of a point of  $I^2$ ,  $g$  can be approximated by the (linear) derivative (together with a shift to put the image in the right place). The height of the function  $f \circ g$  over a point  $\mathbf{a}$  is, by definition, the height of  $f$  over  $g(\mathbf{a})$  and the base of the cuboidal box over  $g(\mathbf{a})$  has had its area stretched by an amount

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}$$

So the volume of the little box has been stretched by the same amount, and so the Riemann Sum of the little cuboids over  $g(I^2)$  is the sum of the corresponding cuboids over  $I^2$  with the area stretching factor taken into account. The approximation improves as the boxes are made smaller and so the formula comes out of the limit of the Riemann sums.  $\square$

**Remark 5.4.8.** This lacks the careful rigour that Pure Mathematicians prefer, but making it rigorous is not very difficult. The idea is the main thing. The guys who invented these ideas were happy with proofs like this one.

**Example 5.4.2.** Find the area enclosed by the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Typing:

```
<< Graphics'ImplicitPlot'
ImplicitPlot[x^2/9 + y^2/4 == 1, {x, -4, 4}]
```

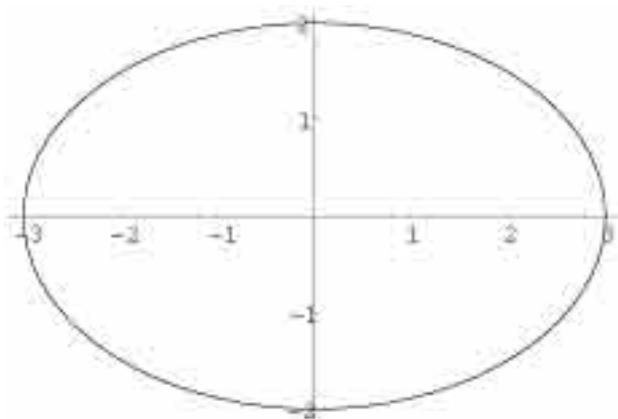


Figure 5.12: The ellipse  $x^2/9 + y^2/4 = 1$

into Mathematica and running it gives the picture of figure 5.12

Now we write the ellipse as a transform of the unit circle:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3x \\ 2y \end{bmatrix}$$

It is easy to see that this stretches the circle by a factor of 3 in the  $x$  direction and 2 in the  $y$  direction. The derivative of this map is the diagonal matrix with diagonal entries 3 and 2 and determinant 6. So the area enclosed by the ellipse is six times the area of the unit disc, that is,  $6\pi$ .

**Remark 5.4.9.** A little thought suggests that if you take a map  $g : I^2 \longrightarrow \mathbb{R}^3$  then there ought to be a formula for the area stretching in this case. This would be nice, for then we could calculate areas of tori and spheres and ellipsoids.

**Remark 5.4.10.** All of these things are taken care of automatically by using differential forms.

This strikes me as a good argument in favour of them. I shall now show how easy it is to take care of the differential area stretch without having to do much.

**Example 5.4.3.** Let  $I^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$  be a pair of smooth maps.

I shall regard  $f$  as a 2-form

$$Pdx \wedge dy$$

I write  $g$  as  $\begin{bmatrix} s \\ t \end{bmatrix} \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix}$  so  $g = \begin{bmatrix} x(s, t) \\ y(s, t) \end{bmatrix}$

I want  $\int_{g(I^2)} f$ , which means  $\int_{g(I^2)} P dx \wedge dy$

I want to transform this to be  $\int_{I^2} f \circ g \, ds \wedge dt$ . I write down the chain rule:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} ds \\ dt \end{bmatrix}$$

so

$$\begin{aligned} dx &= \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \\ dy &= \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \end{aligned}$$

so

$$\begin{aligned} dx \wedge dy &= \left[ \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \right] \wedge \left[ \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \right] \\ &= \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} ds \wedge dt + \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} dt \wedge ds \\ &= \left[ \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \right] ds \wedge dt \\ &= \det(Dg) ds \wedge dt. \end{aligned}$$

**Remark 5.4.11.** And out comes the differential area stretch without me having to do any sweating. The good news is that this works for maps which take the square and parametrise some surface in  $\mathbb{R}^3$ .

**Remark 5.4.12.** I shall refer to this process as ‘composing with  $g$  on the functional part and composing with  $g'$  on the differential part of the form.’

**Example 5.4.4.** Parametrise  $S^2$  in  $\mathbb{R}^3$  and integrate the function

1

over  $S^2$  to obtain the area of the sphere

**Solution:**

Recall that

$$\begin{aligned} g : [0, 2\pi] \times [-1, 1] &\longrightarrow \mathbb{R}^3 \\ (s, t) &\rightsquigarrow \begin{bmatrix} \sqrt{1-t^2} \cos(s) \\ \sqrt{1-t^2} \sin(s) \\ t \end{bmatrix} \end{aligned}$$

parametrised the 2-sphere by wrapping the rectangle around the sphere in a cylinder and then pushing the cylinder in horizontally. (Check that  $x^2 + y^2 + z^2 = 1$ , which shows that we do finish up on the sphere, then check that for the point on the sphere with cylindrical coordinates  $(1, \theta, z)$  there is a point which gets sent to it.) Writing

$$g(s, t) = \begin{bmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{bmatrix}$$

The derivative of  $g$  is therefore:

$$g'(s, t) = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix}$$

and

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} ds \\ dt \end{bmatrix}$$

This leads to:

$$dx \wedge dy = \left( \left( \frac{\partial x}{\partial s} ds \right) + \left( \frac{\partial x}{\partial t} dt \right) \right) \wedge \left( \left( \frac{\partial y}{\partial s} ds \right) + \left( \frac{\partial y}{\partial t} dt \right) \right)$$

that is:

$$dx \wedge dy = \left( \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \right) ds \wedge dt$$

and similarly:

$$dx \wedge dz = \left( \frac{\partial x}{\partial s} \frac{\partial z}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial z}{\partial s} \right) ds \wedge dt$$

and

$$dy \wedge dz = \left( \frac{\partial y}{\partial s} \frac{\partial z}{\partial t} - \frac{\partial y}{\partial t} \frac{\partial z}{\partial s} \right) ds \wedge dt$$

This gives the transformation on the differential part using  $g'$ .

Evaluating them for my map  $g$  I get

$$dx \wedge dy = (t \sin^2(s) + t \cos^2(s)) ds \wedge dt = t ds \wedge dt$$

and

$$dx \wedge dz = -\sqrt{1-t^2} \sin(s) ds \wedge dt$$

and

$$dy \wedge dz = \sqrt{1-t^2} \cos(s) ds \wedge dt$$

Finding the area in  $\mathbb{R}^3$  is like finding the length of a curve in  $\mathbb{R}^2$ ; there, recall, we had

$$\int_I \|\dot{\mathbf{x}}\|$$

for the length. Similarly, here we have

$$\int_{[0,2\pi] \times [-1,1]} \left\| \begin{array}{l} dx \wedge dy \\ dx \wedge dz \\ dy \wedge dz \end{array} \right\|$$

The differential area-stretching factor is the norm of the 2-form

$$1(dx \wedge dy + dx \wedge dz + dy \wedge dz)$$

with a corresponding result for maps  $g$  embedding  $I^2$  (or any other rectangle or two dimensional region) in  $\mathbb{R}^n$  for any  $n > 2$ .

This gives the area of the sphere as

$$\begin{aligned} & \int_{[0,2\pi] \times [-1,1]} \left\| \begin{array}{l} t \\ \sqrt{1-t^2} \sin(s) \\ \sqrt{1-t^2} \cos(s) \end{array} \right\| ds \wedge dt \\ &= \int_{-1}^1 \int_0^{2\pi} 1 ds \wedge dt \end{aligned}$$

Now we have it in the final form we can leave the wedge out and the answer is  $4\pi$ . Note that this is the same as the area of the circumscribing cylinder: the projection onto the sphere does not change the area.

**Remark 5.4.13.** You would naturally like to know what the formula is in Old Fashioned Language. The answer is rather natural.

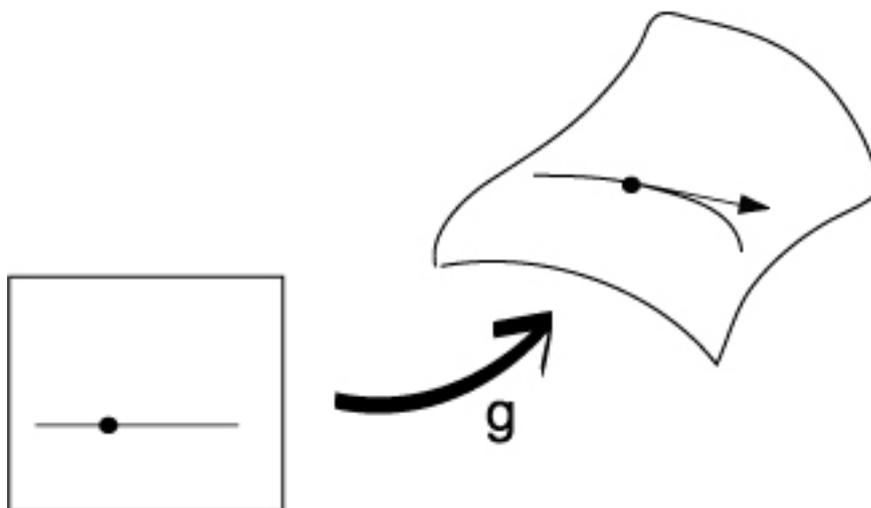


Figure 5.13: Curves and tangents

**Proposition 5.4.2.** If  $g : I^2 \rightarrow \mathbb{R}^3$  is a smooth 1-1 map with smooth inverse, and

$$g(s, t) = \begin{bmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{bmatrix}$$

then

$$\frac{\partial g}{\partial s} = \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial s} \end{bmatrix} \quad \text{and} \quad \frac{\partial g}{\partial t} = \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{bmatrix}$$

are two tangents to curves in  $g(I^2)$  and are linearly independent in  $\mathbb{R}^3$ .

Figure 5.13 shows one tangent to a curve.

**Proof:**

Going back to the definition of the partial derivative at a point of  $I^2$  we see that if we keep  $t = b$  and look at the curve in  $\mathbb{R}^3$  obtained by letting  $s$  vary, then the first partial derivative vector is just the tangent to this curve at the point  $(a, b)^T$  when we evaluate it at that point. Similarly for the other. If the derivative of  $g$  at the point is non singular then the two vectors in  $I^2$  are taken to independent vectors in  $\mathbb{R}^3$ . But the derivative of  $g$  is never singular because we have a smooth inverse.  $\square$

**Remark 5.4.14.** In  $\mathbb{R}^3$ , the cross product of two vectors is orthogonal to the pair and has length the product of the two lengths times the sine of the angle

between them. This length is in fact the area of the parallelogram defined by the two vectors, the origin and their sum.

The area stretch done by  $g$  at a neighbourhood of a point is going to be given by the length stretch in the  $s$  direction multiplied by the length stretch in the  $t$  direction, multiplied by the sine of the angle between the image of the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$  by the derivative. In other words, the cross product of the above partial derivatives.

We have therefore the change of variables and area stretching formula:

$$\int_{g(I^2)} f = \int_{I^2} f \circ g \left\| \frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t} \right\|$$

This is equivalent to the formula obtained from calculating the differential part of the 2-form. As it had better be.

I keep using the following idea:

**Definition 5.11. Smooth embedding** A map  $g : I^k \rightarrow \mathbb{R}^n$  is said to be a *smooth embedding* of  $I^k$  in  $\mathbb{R}^n$  if it is a map which is smooth, 1-1, and has a smooth inverse from the image.

**Definition 5.12. Smooth Embedding a.e** A map  $g : I^k \rightarrow \mathbb{R}^n$  is said to be a *smooth embedding almost everywhere (a.e.)* of  $I^k$  in  $\mathbb{R}^n$  if it is continuous and is a smooth embedding except on a subset of  $I^k$  having Lebesgue measure zero. Lebesgue measure is the natural generalisation of length in  $\mathbb{R}$ , area in  $\mathbb{R}^2$  and volume in  $\mathbb{R}^3$ . In particular the measure of the cube  $I^k$  is one, whereas its boundary has measure zero in  $\mathbb{R}^k$ .

**Remark 5.4.15.** Since we are doing a certain amount of integration, we can usually be dismissive about things going wrong on sets of zero length, area, volume, whatever. So  $g$  can fail to be smooth or 1-1 on such negligible sets and we can neglect them.

**Remark 5.4.16.**  $I^k$  is a subset of  $\mathbb{R}^k$  so differentiability makes sense. You can only embed  $I^k$  in  $\mathbb{R}^n$  if  $n \geq k$ .

**Proposition 5.4.3.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a map that is integrable and  $g : I^2 \rightarrow \mathbb{R}^n$  is a map which is a smooth embedding of  $I^2$  in  $\mathbb{R}^n$  almost everywhere, then

$$\int_{g(I^2)} f = \int_{I^2} (f \circ g)(s, t) \cdot \|\omega\|$$

where  $\omega$  is the constant 2-form

$$\sum_{1 \leq i < j \leq n} dx^i \wedge dx^j$$

and

$$g(t) = \begin{bmatrix} x^1(s, t) \\ x^2(s, t) \\ \vdots \\ x^n(s, t) \end{bmatrix}$$

on  $\mathbb{R}^n$ .

**Proof:** I shan't prove it, it requires more (multi-)linear algebra than you have covered. The result should be intuitively appealing if you think about it. It is obviously consistent with my claim about the differential area stretching  $\square$

**Remark 5.4.17.** This is the general change of variable formula for maps from  $I^2$  into  $\mathbb{R}^n$  and we are now integrating  $f$  over some surface sitting in  $\mathbb{R}^n$ .

**Exercise 5.4.2.** Write down what you feel ought to be the formula for finding the length of a curve embedded in  $\mathbb{R}^n$ . Test it out on particular curves where you can make some estimate of the result.

**Example 5.4.5.** The region  $T^2 \subset \mathbb{R}^4$  is defined by:

$$T^2 = \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in \mathbb{R}^4 : w^2 + x^2 = 1 \text{ and } y^2 + z^2 = 1 \right\}$$

Find its area.

**Solution** Parametrise  $T^2$  by

$$g : [0, 2\pi] \times [0, 2\pi] \longrightarrow \mathbb{R}^4$$

$$(s, t) \rightsquigarrow \begin{bmatrix} w(s, t) = \cos(s) \\ x(s, t) = \sin(s) \\ y(s, t) = \cos(t) \\ z(s, t) = \sin(t) \end{bmatrix}$$

Then

$$dw = \frac{\partial w}{\partial s} ds + \frac{\partial w}{\partial t} dt = -\sin(s) ds$$

Similarly,  $dx = \cos(s) ds$ ,  $dy = -\sin(t) dt$ ,  $dz = \cos(t) dt$  Hence

$$dw \wedge dx = (-\sin(s) \cos(s)) ds \wedge ds = 0$$

Similarly

$$dw \wedge dy = \sin(s) \sin(t) ds \wedge dt, dw \wedge dz = \sin(s) \sin(t) ds \wedge dt,$$

$$dw \wedge dz = -\sin(s) \cos(t) ds \wedge dt, dx \wedge dy = \cos(s)(-\sin(t)) ds \wedge dt$$

and

$$dx \wedge dz = \cos(s) \cos(t) ds \wedge dt, dy \wedge dz = 0$$

The norm of this vector of six components is:

$$\begin{aligned} &0^2 + (\sin(s) \sin(t))^2 + (-\sin(s) \cos(t))^2 \\ &+ ((\cos(s)(-\sin(t)))^2 + (\cos(s) \cos(t))^2 + 0^2 \end{aligned}$$

This is just 1. So the area of  $T^2$  is

$$\int_0^{2\pi} \int_0^{2\pi} 1 ds dt = 4\pi^2$$

**Remark 5.4.18.** Most of the old fashioned guys wouldn't have the faintest idea how to start on this. A modern mathematician is someone who can do this in his head in a few minutes. An old fashioned mathematician is someone who can't see any reason why  $T^2$  should *have* an area, let alone know how to compute it.

## 5.5 Green's Theorem Again

**Remark 5.5.1.** Now I have explored all the ideas on differential length and area stretching, I can prove Green's Theorem for regions which are the images of squares by maps which are smooth embeddings (except perhaps on sets over which the integral of any function will be zero, where they only have to be continuous). The ideas here will generalise considerably.

**Exercise 5.5.1.** Show that a disc can be obtained as the image of a square by a map  $g$  which is differentiable and has a differentiable inverse at every point except the top and bottom of the square.

**Remark 5.5.2.** Suppose we have a differential 1-form on  $\mathbb{R}^2$  and a curve in  $\mathbb{R}^2$  defined by a smooth embedding  $g : I \rightarrow \mathbb{R}^2$ .

Then we can pull back the differential 1-form in a similar way to the way we pulled back a function:

**Definition 5.13.** If  $g : I \rightarrow U \subseteq \mathbb{R}^2$  is a smooth embedding and if  $\omega$  is a differential 1-form on  $U$ ,  $g^*\omega$  is the differential 1-form on  $I$  defined by

$$g^*\omega \triangleq P \circ g \frac{dx}{dt} dt + Q \circ g \frac{dy}{dt} dt$$

where

$$\omega \triangleq P dx + Q dy$$

and

$$g(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

**Remark 5.5.3.** If you draw a picture of this you will see that we are turning the vector field on  $\mathbb{R}^2$  into one along the curve by just looking at the component tangent to the curve and pulling this back to  $I$ .

**Remark 5.5.4.** This works for 1-forms on  $\mathbb{R}^n$ :

If

$$\omega \triangleq P_1 dx^1 + P_2 dx^2 + \cdots + P_n dx^n$$

$$g^* dx^i \triangleq \frac{dx^i}{dt} dt$$

where

$$g(t) = \begin{bmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^n(t) \end{bmatrix}$$

and  $g^*P_i = P_i \circ g$

**Remark 5.5.5.** We can say that we use composition with  $g$  on the function part, each  $P_i$  goes to  $P_i \circ g$ , and we use composition with  $g'$  on the differential part, to get  $g^*\omega$ .

**Remark 5.5.6.** In particular we recover the case where  $n = 2$  and

$$g(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$g^* dx = \frac{dx}{dt} dt; \quad g^* dy = \frac{dy}{dt} dt.$$

So if  $\omega = Pdx + Qdy$

$$\begin{aligned} g^*\omega &= (P \circ g) \left[ \frac{dx}{dt} \right] dt + (Q \circ g) \left[ \frac{dy}{dt} \right] dt \\ &= \left[ P \circ g \left[ \frac{dx}{dt} \right] + Q \circ g \left[ \frac{dy}{dt} \right] \right] dt. \end{aligned}$$

**Remark 5.5.7.** We can do the same thing with maps of  $I^2$ , the unit square, into  $\mathbb{R}^n$ , and differential 2-forms on  $\mathbb{R}^n$  getting pulled back to  $I^2$ :

**Definition 5.14.** If  $g : I^2 \rightarrow U \subset \mathbb{R}^2$  is a smooth embedding, and if  $\omega$  is a 2-form on  $U$ ,

$$\begin{aligned} \omega &\triangleq Pdx \wedge dy \\ g^*\omega &\triangleq (P \circ g) dx \wedge dy \end{aligned}$$

and

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} ds \\ dt \end{bmatrix}$$

where

$$\begin{array}{ccc} I^2 & \xrightarrow{g} & U \\ \begin{bmatrix} s \\ t \end{bmatrix} & \rightsquigarrow & \begin{bmatrix} x(s, t) \\ y(s, t) \end{bmatrix} \end{array}$$

allows us to calculate  $dx \wedge dy$  in terms of  $ds \wedge dt$ .

This gives:

$$\begin{aligned} dx \wedge dy &= \left( \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \right) \wedge \left( \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \right) \\ &= \left( \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \right) ds \wedge dt \end{aligned}$$

So

$$g^*(\omega) = P \circ g \left( \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \right) ds \wedge dt$$

**Remark 5.5.8.** We again use composition with  $g$  on the function part, and with its derivative on the differential part.

**Proposition 5.5.1.** If  $g : I^2 \rightarrow U \subset \mathbb{R}^2$  is a 1 – 1 smooth embedding a.e, and if  $\omega$  is a differential 2-form on  $U$

$$\int_{I^2} g^* \omega = \int_{g(I^2)} \omega$$

“**Proof**” the result is to automatically give the usual change of variable formula.  $\square$

**Remark 5.5.9.** This generalises to the case where

$$g : I^m \rightarrow U \subseteq \mathbb{R}^n$$

is a smooth embedding a.e. Then  $g^*$  takes  $k$ -forms on  $U$  to  $k$ -forms on  $I^m$  by (1) composition with  $g$  to get the function part and (2) composition with  $Dg$  to get the  $dx^i$  turned into  $dt^j$  and for  $\omega$  any  $k$ -form

$$\int_{I^k} g^* \omega = \int_{g(I^k)} \omega$$

I am not going to prove the claim in general, it is basically the change of variable formula. Note that there is no need to take special account of the sign or to take absolute values of numbers, since this is taken care of by the  $dx \wedge dy$  terms. It is a good idea to get the thing into standard shape before actually integrating however, or you can get the sign wrong. Fubini's theorem needs some changes before it works for integrating 2-forms.

**Proposition 5.5.2.** If  $\omega$  is a smooth 0-form on  $\mathbb{R}^2$  and

$$c : I \rightarrow \mathbb{R}^2$$

is an embedding then:

$$d(c^*(\omega)) = c^*(d\omega)$$

**Proof:** A 0-form is just a function and I shall call it  $f$  to make it more friendly sounding for those of you made nervous by greek letters. If we write

$$c : I \rightarrow \mathbb{R}^2$$

$$t \rightsquigarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

then since  $f$ , being a 0-form, has no differentials to bother about,  $c^*(f) = f \circ c$  and

$$d(f \circ c) = \frac{d(f \circ c)}{dt} dt$$

is the derived 1-form.

This is, using the chain rule:

$$\left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) dt$$

It might be useful to remember where we are and rewrite this as:

$$\left( \left[ \frac{\partial f}{\partial x} \right]_{c(t)} \left[ \frac{dx}{dt} \right]_t + \left[ \frac{\partial f}{\partial y} \right]_{c(t)} \left[ \frac{dy}{dt} \right]_t \right) dt$$

Now over  $U$ ,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Now applying  $c^*$  to this we evaluate the function part at  $c(t)$  and fix up the differentials using the derivative of  $c$ , which gives us the line preceding.  $\square$

**Remark 5.5.10.** We can go up a dimension and do this for maps which embed squares in  $\mathbb{R}^2$ . The argument is almost the same

**Proposition 5.5.3.** If  $\omega$  is a differential 0-form on  $U \subset \mathbb{R}^2$  and  $c : I^2 \rightarrow U$  is a smooth embedding,

$$d(c^*(\omega)) = c^*(d\omega)$$

**Proof** A 0-form is just a function, call it  $f$

$$\begin{aligned} d(c^*f) &= D(f \circ c) && \text{(definition of } c^*) \\ &= Df \circ Dc && \text{(chain rule)} \\ &= c^*df && \text{(definition of } c^*) \end{aligned}$$

$\square$

**Remark 5.5.11.** The notation used here is very condensed and it is probably a good idea to write it out in old fashioned terms so I give the proof again:

**Proposition 5.5.4. Repeat:** If  $f : U \rightarrow \mathbb{R}$  is a function defined on some set  $U \subseteq \mathbb{R}^2$  and if  $c : I^2 \rightarrow U$  is a smooth embedding of the unit square in  $U$ , then

$$d(c^*(f)) = c^*(df)$$

**Proof** Since  $f$  is a 0-form there are no differentials to bother about, and  $c^*(f) = f \circ c$  which is another 0-form, this time on  $I^2$ .

The exterior derivative applied to 0-forms is just the ordinary derivative and for  $f \circ c$  is, if we write:

$$c : I^2 \longrightarrow U \\ \begin{bmatrix} s \\ t \end{bmatrix} \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

just

$$\frac{\partial(f \circ c)}{\partial s} ds + \frac{\partial(f \circ c)}{\partial t} dt$$

which we shall write out explicitly as

$$\frac{\partial f}{\partial x} \left( \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \right) + \frac{\partial f}{\partial y} \left( \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \right)$$

Using the chain rule. Now

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

and applying  $c^*$  to this gives us the preceding line.  $\square$

**Remark 5.5.12.** This works also for differential 1-forms:

**Proposition 5.5.5.** If  $\omega$  is a differential 1-form on  $U \subseteq \mathbb{R}^2$  and  $c : I^2 \longrightarrow U$  is a smooth embedding then

$$d(c^*\omega) = c^*d\omega$$

**Proof** If

$$\omega \triangleq Pdx + Qdy \\ d\omega = \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx \wedge dy.$$

define

$$c : I^2 \longrightarrow U \\ \begin{bmatrix} s \\ t \end{bmatrix} \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

then

$$Dc = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

and

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} ds \\ dt \end{bmatrix}$$

$$\begin{aligned} c^*\omega &= P \left[ c \begin{bmatrix} s \\ t \end{bmatrix} \right] \underbrace{\left[ \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \right]}_{dx} + Q \left[ c \begin{bmatrix} s \\ t \end{bmatrix} \right] \underbrace{\left[ \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \right]}_{dy} \\ &= \left[ P \circ c \begin{bmatrix} s \\ t \end{bmatrix} \right] \frac{\partial x}{\partial s} + Q \circ c \begin{bmatrix} s \\ t \end{bmatrix} \frac{\partial y}{\partial s} \Big] ds + \left[ P \circ c \begin{bmatrix} s \\ t \end{bmatrix} \right] \left[ \frac{\partial x}{\partial t} + Q \circ c \begin{bmatrix} s \\ t \end{bmatrix} \frac{\partial y}{\partial t} \right] dt \end{aligned}$$

$$\begin{aligned} d(c^*\omega) &= \left[ \frac{\partial}{\partial s} \left[ (P \circ c) \frac{\partial x}{\partial t} + (Q \circ c) \frac{\partial y}{\partial t} \right] - \frac{\partial}{\partial t} \left[ (P \circ c) \frac{\partial x}{\partial s} + (Q \circ c) \frac{\partial y}{\partial s} \right] \right] ds \wedge dt \\ &= \left[ (P \circ c) \frac{\partial^2 x}{\partial s \partial t} + \frac{\partial x}{\partial t} \left( \frac{\partial (P \circ c)}{\partial s} \right) + (Q \circ c) \frac{\partial^2 y}{\partial s \partial t} + \frac{\partial y}{\partial t} \left( \frac{\partial (Q \circ c)}{\partial s} \right) \right. \\ &\quad \left. - (P \circ c) \frac{\partial^2 x}{\partial s \partial t} - \frac{\partial x}{\partial s} \left( \frac{\partial (P \circ c)}{\partial t} \right) - (Q \circ c) \frac{\partial^2 y}{\partial t \partial s} - \frac{\partial y}{\partial s} \left( \frac{\partial (Q \circ c)}{\partial t} \right) \right] ds \wedge dt \end{aligned}$$

Notice that of these eight terms, the first and fifth cancel and the third and seventh cancel.

Using

$$\left[ \frac{\partial (P \circ c)}{\partial s}, \frac{\partial (P \circ c)}{\partial t} \right] = \left[ \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \right] \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

(chain rule) and likewise for  $Q \circ c$ ,

$$\begin{aligned} d(c^*\omega) &= \left[ \begin{aligned} &\frac{\partial x}{\partial t} \left[ \frac{\partial P}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial s} \right] - \frac{\partial x}{\partial s} \left[ \frac{\partial P}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial t} \right] \\ &+ \frac{\partial y}{\partial t} \left[ \frac{\partial Q}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial Q}{\partial y} \frac{\partial y}{\partial s} \right] - \frac{\partial y}{\partial s} \left[ \frac{\partial Q}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial Q}{\partial y} \frac{\partial y}{\partial t} \right] \end{aligned} \right] ds \wedge dt \\ &= \left[ \begin{aligned} &\frac{\partial Q}{\partial x} \left[ \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial y}{\partial s} \frac{\partial x}{\partial t} \right] - \frac{\partial P}{\partial y} \left[ \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \right] \\ &+ \frac{\partial Q}{\partial y} \left[ \frac{\partial y}{\partial t} \frac{\partial y}{\partial s} - \frac{\partial y}{\partial t} \frac{\partial y}{\partial s} \right] + \frac{\partial P}{\partial x} \left[ \frac{\partial x}{\partial t} \frac{\partial x}{\partial s} - \frac{\partial x}{\partial t} \frac{\partial x}{\partial s} \right] \end{aligned} \right] ds \wedge dt \end{aligned}$$

The last two terms are zero so this reduces to:

$$\begin{aligned} dc^*(\omega) &= \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \left[ \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial y}{\partial s} \frac{\partial x}{\partial t} \right] ds \wedge dt \\ &= c^*d\omega \end{aligned}$$

□

**Remark 5.5.13.** It works just as well on  $\mathbb{R}^n$  but there are more terms. It also works for differential  $k$ -forms for any  $k < n$  on  $U \subset \mathbb{R}^n$ . As it stands it is a rather tedious but straightforward calculation: the sort of thing that makes you feel like a real mathematician at relatively low cost. You probably get the general idea by now.

**Remark 5.5.14.** after that moderately painful part the rest is easy:

**Definition 5.15. boundary operator** If  $U \subset \mathbb{R}^n$  is any set, a *boundary point* of  $U$  is a point such that every open ball on it intersects both  $U$  and the set complement of  $U$ ,  $\mathbb{R}^n \setminus U$ . The set of all boundary points of  $U$  is written  $\partial U$  and  $\partial$  is called the *boundary operator*.

**Remark 5.5.15.** Now I pull the rabbit out of the hat:

**Proposition 5.5.6. Green's Theorem** Let  $\omega$  be a differential 1-form on  $U \subset \mathbb{R}^2$  ( $U$  open) and let  $\mathcal{D} \subset U$  be any region which is parametrised by a smooth embedding a.e.  $c : I^2 \rightarrow U$ .

Then

$$\int_{\partial \mathcal{D}} \omega = \int_{\mathcal{D}} d\omega.$$

**Proof**

$$\begin{aligned} \int_{\partial \mathcal{D}} \omega &= \int_{\partial(c(I^2))} \omega && \text{(definition of } \mathcal{D} \text{)} \\ &= \int_{\partial I^2} c^* \omega && \text{(by the change of variables} \\ &&& \text{formula and adding four curves.)} \\ &= \int_{I^2} dc^* \omega && \text{(by Green's Theorem for a square)} \\ &= \int_{I^2} c^* d\omega && \text{(by Proposition 5.5.5)} \\ &= \int_{c(I^2)} d\omega && \text{(by the change of variables formula)} \\ &= \int_{\mathcal{D}} d\omega && \text{(definition of } \mathcal{D} \text{)} \end{aligned}$$

□

**Remark 5.5.16.** Now I bow deeply and you clap and throw money (notes only).

**Remark 5.5.17.** This gives Greens Theorem for quite a lot of shapes in  $\mathbb{R}^2$ . We can actually note that  $c$  does not have to be smooth everywhere: if  $c$  is continuous and invertible and is smooth except at a finite set of points, with inverse smooth except at a finite set of points, this will not change any integrals.

So Green's Theorem also works on  $D^2$ , by an exercise I gave a while back.

**Remark 5.5.18.** The results given can be strengthened considerably. But the present form serves our purposes.

**Remark 5.5.19.** We can almost prove the result that was stated to be too hard at the end of the last chapter. I state it again but in modern language:

**Proposition 5.5.7.** If  $\omega$  is a smooth 1-form on  $U \subseteq \mathbb{R}^2$  which is closed, and if  $U$  is connected and simply connected, then  $\omega$  is exact.

(or in translation into old-fashioned language, if  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$  is a vector field on  $\mathbb{R}^2$  and  $\partial Q/\partial x - \partial P/\partial y = 0$  and  $U$  is connected and has no holes in it, then  $\mathbf{F}$  is conservative.)

### Almost Proof

Take any continuous simple (1-1) loop in  $U$ ; then this can be expressed as a map from  $\partial I^2$  to  $U$ . Since  $U$  is simply connected we can extend this to a continuous 1-1 map  $\tilde{f}$  from  $I^2$  to  $U$ .

If this were smooth almost everywhere we could apply Green's Theorem to the interior and since  $d\omega = 0$  we can conclude that the integral around the loop must be zero. This would be enough to conclude that every path integral depends only on its endpoints, which would give us the required result.

Unfortunately we have no guarantee that  $\tilde{f}$  is smooth. To get around this we could rather laboriously prove that every continuous map can be approximated by a smooth map, and argue that the line integrals along the non-smooth arcs are approximated by the line integrals around the smooth approximation, and likewise for the surface integrals. This can be done, but it is a lot of work and we don't have time for it. Too bad. We conclude therefore that the result looks plausible, but is a hard one to prove.  $\square$



# Chapter 6

## Stokes' Theorem (Classical and Modern)

### 6.1 Classical

**Theorem 6.1. Stokes** *Let  $S$  be a piecewise-smooth, orientable surface in  $\mathbb{R}^3$  with boundary  $\partial S$ . Let  $\mathbf{n}$  be any normal vector to the surface, and let this induce the derived orientation on  $\partial S$ .*

*Let  $\mathbf{F}$  be a smooth vector field on  $\mathbb{R}^3$ .*

*Then*

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

*where  $d\mathbf{S}$  is the normal vector to the surface element with the same orientation as  $\mathbf{n}$  and  $d\mathbf{r}$  is the tangent to the length element having the derived orientation.*

**Remark 6.1.1.** This is the standard form of Stokes' Theorem and is in a form which Stokes might almost have recognised. The next job is to explain what some of the words mean.

**Remark 6.1.2.** If  $U$  is a smooth surface in  $\mathbb{R}^3$  and if  $\hat{\mathbf{n}}(\mathbf{x})$  is a unit length normal vector to  $\mathbf{x} \in U$  then there is a map

$$\begin{aligned} \hat{\mathbf{n}} &: U \longrightarrow \mathbb{R}^3 \\ \mathbf{x} &\rightsquigarrow \hat{\mathbf{n}}(\mathbf{x}) \end{aligned}$$

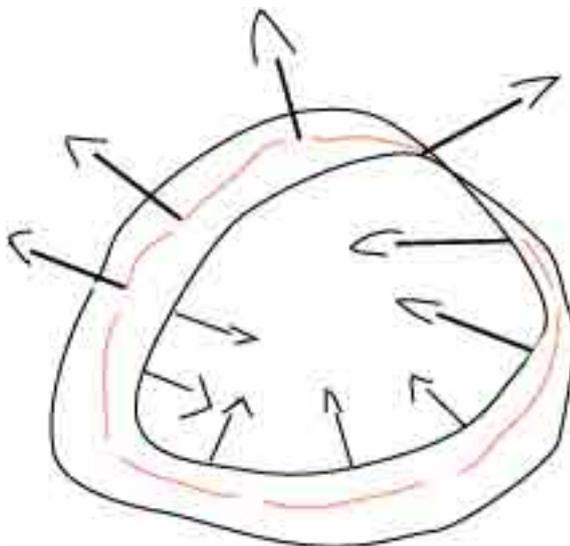


Figure 6.1: A non-orientable surface

Well, there is always a choice of directions; there are always *two* unit normal vectors to a smooth surface at a point. One is the negative of the other. Picking one of them is equivalent to deciding which way is *up*.

This ought to depend on the surface being smooth; if it looked like the roof of a house then there would be ridges without a normal vector. If however there were two linearly independent tangent vectors to the surface at a point, the cross product would give me a normal vector. So if the surface were the graph of a differentiable function at a point there certainly ought to be a normal vector, in fact a normal line.

Suppose we make a choice of which of two unit normals to take at some particular point  $\mathbf{x} \in S$ .

Now I take a path in  $U$ . I can ensure that I make “the same” choice of unit normal along the path. What I mean by this is that I ensure that the function  $\hat{\mathbf{n}}$  is continuous. Small changes in the position  $\mathbf{x}$  on the surface will make small changes in  $\hat{\mathbf{n}}(\mathbf{x})$ . This makes  $\hat{\mathbf{n}} : U \rightarrow \mathbb{R}^3$  continuous, the vectors  $\hat{\mathbf{n}}(\mathbf{x})$  will change as we move about, but not too drastically if  $U$  is smooth.

Now you have agreed to this as blindingly obvious, look at the Möbius strip of figure 6.1. You can carry the normal vector all the way around a loop and on returning to your starting point, the vector is pointing in the opposite direction.

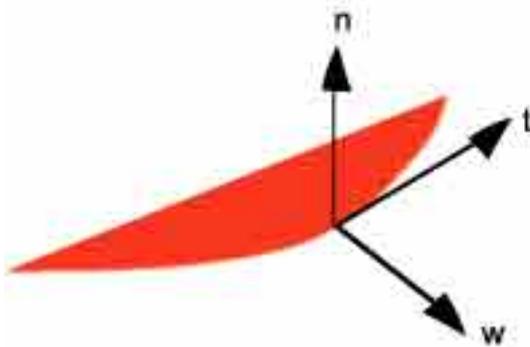


Figure 6.2: The derived orientation

Oops.

We get around this by calling the surfaces where this doesn't happen **orientable**, and things like the mobius strip are **non-orientable**. Then we simply ignore the non-orientable ones.

**Definition 6.1. Induced Orientation** If we have an orientable surface, we choose an orientation which is equivalent to making a choice of normal vector,  $\mathbf{n}$ . Now I *can* move this about over the surface in such a way that it changes continuously with  $\mathbf{x}$

Move it towards the boundary of the surface (if it has one).

At the boundary, take a vector normal to the boundary and also to the vector  $\mathbf{n}$  pointing away from the surface call it  $\mathbf{w}$ . This is shown in figure 6.2

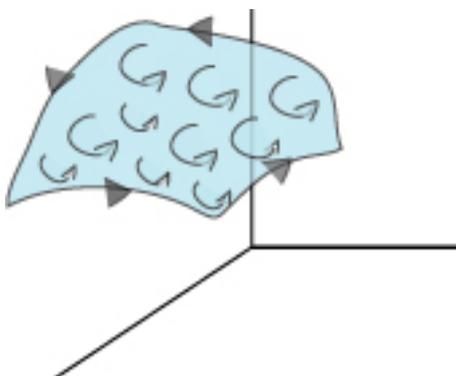
Then there are two tangent vectors. Choose  $\mathbf{t}$  so that  $(\mathbf{w}, \mathbf{t}, \mathbf{n})$  is a right-handed system like

$$\left[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right], \text{ otherwise known as } (\mathbf{i}, \mathbf{j}, \mathbf{k})$$

Then the direction of  $\mathbf{t}$  is called the *induced orientation* on the boundary.

**Remark 6.1.3.** It may not have been too obvious from the statement of Stokes' Theorem, but the idea is the same as Green's Theorem. All we do is to go from a vector field in  $\mathbb{R}^2$  to one in  $\mathbb{R}^3$ , and instead of having a flat surface sitting in  $\mathbb{R}^2$  it is still a bit of surface, sitting curved in  $\mathbb{R}^3$ . Then if you integrate the spin of the vector field over the surface, you must get the same result as if you take the path integral around the boundary.

In figure 6.3 I sketch the picture this should evoke:

Figure 6.3: Green's Theorem in  $\mathbb{R}^3$ 

**Remark 6.1.4.** The only problem is to get clear the business of integrating the curl of  $\mathbf{F}$  over the surface. We do this by taking the curl of the vector field to be, as in the last chapter, a vector normal to the plane of rotation of the vector field  $\mathbf{F}$ . We take the vector  $d\mathbf{S}$  to be normal to the surface and of length the 'infinitesimal area element'. The dot product of these two vectors gives the amount of spin in the tangent to the surface. Integrating this dot product over the surface ought to give us the same result as it does in Green's Theorem, and for the same reason.

**“Proof” of Stokes Theorem (Classical Version)** (For the case where  $S$  is  $g(I^2)$  for  $g$  a piecewise smooth embedding.)

Partition  $I^2$  into little rectangles side  $\Delta u, \Delta v$  and map them into  $g(I^2)$  by

$$g : I^2 \longrightarrow \mathbb{R}^3$$

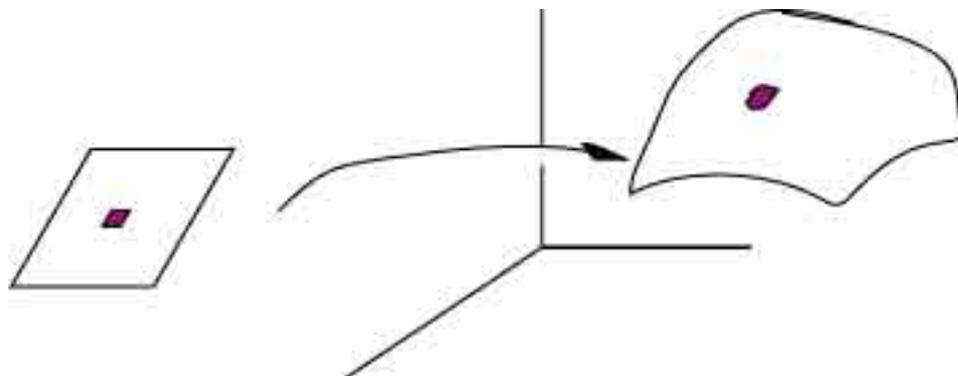
$$\begin{bmatrix} u \\ v \end{bmatrix} \rightsquigarrow \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}$$

I show a picture of this in figure 6.4

Let  $\begin{bmatrix} a \\ b \end{bmatrix}$  be the centre of one such square. The curl of  $\mathbf{F}$  at the point  $g \begin{bmatrix} a \\ b \end{bmatrix}$  is the vector  $\nabla \times \mathbf{F}$ .

It should be clear that

$$\frac{\partial g}{\partial u} \Big|_{\begin{matrix} a \\ b \end{matrix}}$$

Figure 6.4: Parametrising a surface in  $\mathbb{R}^3$ 

is a tangent curve in the surface  $g(I^2)$  at the point  $g \begin{bmatrix} a \\ b \end{bmatrix}$ , and that

$$\left. \frac{\partial g}{\partial v} \right|_{\begin{matrix} a \\ b \end{matrix}}$$

is another one. (After all if you keep one one of two variables fixed you are putting a curve in  $\mathbb{R}^3$  which lies in the surface.)

Then a normal to the surface at  $g \begin{bmatrix} a \\ b \end{bmatrix}$  is

$$\mathbf{n}(a, b) = \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix}$$

all partial derivatives being evaluated at  $\begin{bmatrix} a \\ b \end{bmatrix}$ . This is because the cross product is always orthogonal to the other two vectors, both of which are tangent to the surface.

The amount of "twist" in the plane tangent to  $U$  at  $g \begin{bmatrix} a \\ b \end{bmatrix}$  is the dot product  $[\nabla \times \mathbf{F}] \cdot \hat{\mathbf{n}}$  which is the projection on  $\hat{\mathbf{n}}$  where

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|}$$

is the unit normal.

The “area element”  $d\mathcal{S}$  is the vector  $\hat{\mathbf{n}}$  multiplied by the area of the ‘infinitesimal’ area that  $\Delta u \Delta v$  has become after being stretched by  $g'$ . This turns out by what would look slightly miraculous if one were disposed to think that way, to be the length of the vector

$$\begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix}$$

So the amount of twist that  $\text{curl}(\mathbf{F})$  exerts on the surface at the point  $g \begin{bmatrix} a \\ b \end{bmatrix}$  multiplied by the infinitesimal area element is

$$[\nabla \times \mathbf{F}] \cdot \left( \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix} \right) du dv$$

where  $[\nabla \times \mathbf{F}]$  is evaluated at  $g \begin{bmatrix} a \\ b \end{bmatrix}$  and the partial derivatives are all evaluated at  $\begin{bmatrix} a \\ b \end{bmatrix}$

Integrating this over the surface gives the spin part of Stokes Theorem.

But this just pulls back the integral to  $I^2$ ; taking limits as  $\Delta u \rightarrow 0$   $\Delta v \rightarrow 0$  we get

$$\int_{u=0}^1 \int_{v=0}^1 ([\nabla \times \mathbf{F}]) \cdot \left[ \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right] du dv$$

Now

$$[\nabla \times \mathbf{F}] \cdot \left[ \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right]$$

at  $g \begin{bmatrix} a \\ b \end{bmatrix}$  defines a spin field on  $I^2$  multiplied by the area stretch  $g'$  does, and by Green's Theorem for a square this is equal to the integral of the corresponding vector field around the boundary:

$$\int_{\partial I^2} \mathbf{F} \circ g$$

if we again take the component tangent to the boundary. Which is what we get if we take

$$\int_{\partial g(I^2)} \mathbf{F} \cdot g' = \int_{\partial g(I^2)} \mathbf{F} \cdot d\mathbf{r}$$

□

**Remark 6.1.5.** In other words, Stokes' theorem is just Green's theorem after we pull it back to  $I^2$  properly.

**Remark 6.1.6.** Stokes' Theorem (updated version) (dates from about 1870) and was not proved by Stokes. In fact the original statement is in a letter from Sir William Thomson, later known as Lord Kelvin, to Stokes dated 1850. Stokes set the problem of proving it in an examination set for the top mathematics students at Cambridge in 1854, but we don't know if anyone proved it. Probably not. Kelvin himself proved it. Maxwell proved it in *Electricity and Magnetism*. Stokes was rather lucky to have his name perpetuated by a theorem he may never have proved.

## 6.2 Modern

**Theorem 6.2.** Let  $g : I^2 \rightarrow U \subseteq \mathbb{R}^3$  be a smooth embedding a.e. and  $\omega$  a smooth differential 1-form on  $U$ .

Then

$$\int_{\partial g(I^2)} \omega = \int_{g(I^2)} d\omega.$$

**Remark 6.2.1.** This looks like Greens Theorem with only the dimension of  $U$  changed from 2 to 3. This is right. The proof is the same too.

Recall that we had for

$$g : I^2 \rightarrow U \subset \mathbb{R}^2 \\ \begin{bmatrix} u \\ v \end{bmatrix} \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

and if  $\omega$  is a 1-form on  $U$ ,  $\omega = Pdx + Qdy$ ,  $g^*\omega$  was the 1-form on  $I^2$  defined by

$$(g^*\omega) \begin{bmatrix} u \\ v \end{bmatrix} = P \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] dx + Q \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] dy$$

and

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = g' \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$$

And if  $\omega = Pdx \wedge dy$  is a 2-form on  $\mathbb{R}^2$ ,  $g^*\omega$  is the 2-form on  $I^2$  defined by

$$(g^*\omega) \begin{bmatrix} u \\ v \end{bmatrix} = P \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] dx \wedge dy$$

where  $dx \wedge dy$  is obtained by

$$\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \end{aligned}$$

so

$$\begin{aligned} dx \wedge dy &= \left[ \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right] \wedge \left[ \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right] \\ &= \left[ \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right] du \wedge dv. \end{aligned}$$

**Remark 6.2.2.** This extends without fuss to  $g : I^2 \longrightarrow U \subseteq \mathbb{R}^3$ .

**Definition 6.2.** If  $\omega$  is a differential 1-form on  $U$ ,  $\omega \triangleq Pdx + Qdy + Rdz$  on  $U$

$$g^*\omega \begin{bmatrix} u \\ v \end{bmatrix} \triangleq P \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] dx + Q \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] dy + R \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] dz$$

and

$$g' = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

so

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$$

ie.

$$\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \\ dz &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \end{aligned}$$

So

$$\begin{aligned}
g^*(\omega) &= (P \circ g) \frac{\partial x}{\partial u} du + (P \circ g) \frac{\partial x}{\partial v} dv \\
&+ (Q \circ g) \frac{\partial y}{\partial u} du + (Q \circ g) \frac{\partial y}{\partial v} dv \\
&+ (R \circ g) \frac{\partial z}{\partial u} du + (R \circ g) \frac{\partial z}{\partial v} dv \\
&= \left[ (P \circ g) \frac{\partial x}{\partial u} + (Q \circ g) \frac{\partial y}{\partial u} + (R \circ g) \frac{\partial z}{\partial u} \right] du \\
&+ \left[ (P \circ g) \frac{\partial x}{\partial v} + (Q \circ g) \frac{\partial y}{\partial v} + (R \circ g) \frac{\partial z}{\partial v} \right] dv
\end{aligned}$$

**Remark 6.2.3.** The idea is the same: evaluate the pullback to  $I^2$  by using composition with  $g$  to get the value and composition with  $g'$  to get the differential part.

**Remark 6.2.4.** Now we do it for a 2-form on  $U \subseteq \mathbb{R}^3$ .

**Definition 6.3.** If  $\omega \triangleq Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$  on  $U \subseteq \mathbb{R}^3$  and  $g : I^2 \rightarrow U$  is smooth,  $g^*(\omega)$  is the 2-form on  $I^2$  given by:

$$P \circ g \begin{bmatrix} u \\ v \end{bmatrix} dx \wedge dy + Q \circ g \begin{bmatrix} u \\ v \end{bmatrix} dx \wedge dz + R \circ g \begin{bmatrix} u \\ v \end{bmatrix} dy \wedge dz$$

and

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, \quad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

so

$$\begin{aligned}
dx \wedge dy &= \left[ \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right] \wedge \left[ \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right] \\
&= \left[ \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right] du \wedge dv \\
dx \wedge dz &= \left[ \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial z}{\partial u} \right] du \wedge dv \\
dy \wedge dz &= \left[ \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right] du \wedge dv
\end{aligned}$$

which gives  $g^*\omega$  as a differential 2-form on  $U \subseteq \mathbb{R}^3$ . We get:

$$\begin{aligned}
g^*(\omega) &= \left[ (P \circ g) \begin{bmatrix} u \\ v \end{bmatrix} \cdot \left[ \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right] + (Q \circ g) \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial z}{\partial u} \right] \right. \\
&\quad \left. + (R \circ g) \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right] \right] du \wedge dv \\
&= \left\{ \begin{bmatrix} R \circ g \\ -Q \circ g \\ P \circ g \end{bmatrix} \cdot \left[ \left[ \frac{\partial g}{\partial u} \right] \times \left[ \frac{\partial g}{\partial v} \right] \right] \right\} du \wedge dv
\end{aligned}$$

You will recognise the second term as part of the “area stretching factor” for  $g : I^2 \rightarrow U \subseteq \mathbb{R}^3$  at a point.

Note again that this comes out of the computation of the induced 2-form quite automatically.

**Proposition 6.2.1.** If  $\omega$  is a smooth differential 1-form on  $U \subseteq \mathbb{R}^3$  and  $g : I^2 \rightarrow U$  is a smooth embedding a.e.,

$$\int_{\partial g(I^2)} \omega = \int_{\partial I^2} g^* \omega$$

**Proof** We have for each edge of  $(I^2)$ ,  $g$  defines a parametric curve in  $U \subseteq \mathbb{R}^3$  which is part of the boundary of  $g(I^2)$  and  $g^* \omega$  is constructed to take care of the length stretch automatically:

$$\begin{aligned}
\int_{g(I)} \omega &= \int_0^1 \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt \\
&= \int_I g^* \omega
\end{aligned}$$

□

**Proposition 6.2.2.** If  $\omega$  is a 1-form on  $I^2$ ,

$$\int_{\partial I^2} \omega = \int_{I^2} d\omega$$

**Proof** This is just Green’s Theorem for  $I^2$ . □

**Proposition 6.2.3.** If  $\omega$  is a smooth differential 1-form on  $U \subseteq \mathbb{R}^3$  and  $g : I^2 \rightarrow U$  is a smooth embedding,

then  $dg^* \omega = g^* d\omega$ .

**Proof** If  $\omega = Pdx + Qdy + Rdz$  we have from definition 6.2

$$g^*\omega = \left[ P \circ g \frac{\partial x}{\partial u} + Q \circ g \frac{\partial y}{\partial u} + R \circ g \frac{\partial z}{\partial u} \right] du + \left[ P \circ g \frac{\partial x}{\partial v} + Q \circ g \frac{\partial y}{\partial v} + R \circ g \frac{\partial z}{\partial v} \right] dv$$

so

$$dg^*\omega = \left[ \frac{\partial B}{\partial u} - \frac{\partial A}{\partial v} \right] du \wedge dv$$

where

$$\begin{aligned} A &= \left[ (P \circ g) \frac{\partial x}{\partial u} + (Q \circ g) \frac{\partial y}{\partial u} + (R \circ g) \frac{\partial z}{\partial u} \right] \\ B &= \left[ (P \circ g) \frac{\partial x}{\partial v} + (Q \circ g) \frac{\partial y}{\partial v} + (R \circ g) \frac{\partial z}{\partial v} \right] \end{aligned}$$

ie.

$$\begin{aligned} dg^*\omega &= \left\{ \left[ \frac{\partial(P \circ g)}{\partial u} \frac{\partial x}{\partial v} + (P \circ g) \frac{\partial^2 x}{\partial u \partial v} + \frac{\partial(Q \circ g)}{\partial u} \frac{\partial y}{\partial v} + (Q \circ g) \frac{\partial^2 y}{\partial u \partial v} \right. \right. \\ &\quad \left. \left. + \frac{\partial(R \circ g)}{\partial u} \frac{\partial z}{\partial v} + (R \circ g) \left[ \frac{\partial^2 z}{\partial u \partial v} \right] \right] \right. \\ &\quad \left. - \left[ \frac{\partial(P \circ g)}{\partial v} \frac{\partial x}{\partial u} + (P \circ g) \frac{\partial^2 x}{\partial v \partial u} + \frac{\partial(Q \circ g)}{\partial v} \frac{\partial y}{\partial u} + (Q \circ g) \frac{\partial^2 y}{\partial v \partial u} \right. \right. \\ &\quad \left. \left. + \frac{\partial(R \circ g)}{\partial v} \frac{\partial z}{\partial u} + (R \circ g) \frac{\partial^2 z}{\partial z \partial u} \right] \right\} du \wedge dv \\ &= \left\{ \left[ \frac{\partial(P \circ g)}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial(P \circ g)}{\partial v} \frac{\partial x}{\partial u} \right] + \left[ \frac{\partial(Q \circ g)}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial(Q \circ g)}{\partial v} \frac{\partial y}{\partial u} \right] \right. \\ &\quad \left. \left[ \frac{\partial(R \circ g)}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial(R \circ g)}{\partial v} \frac{\partial z}{\partial u} \right] \right\} du \wedge dv \end{aligned}$$

but

$$\frac{\partial(P \circ g)}{\partial u} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial u}$$

and similarly for  $Q \circ g$  and  $R \circ g$ .

So

$$\begin{aligned} dg^*\omega &= \left\{ \left[ \begin{array}{l} \frac{\partial P}{\partial x} \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} \\ - \frac{\partial P}{\partial x} \frac{\partial x}{\partial v} \frac{\partial x}{\partial u} - \frac{\partial P}{\partial y} \frac{\partial y}{\partial v} \frac{\partial x}{\partial u} - \frac{\partial P}{\partial z} \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \end{array} \right] \right. \\ &\quad \left. + \text{similar terms for } Q \text{ and } R \right\} du \wedge dv \\ &= \left\{ \left[ \frac{\partial P}{\partial y} \left[ \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial x}{\partial u} \right] + \frac{\partial P}{\partial z} \left[ \frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial u} \right] \right] \right. \\ &\quad \left. + \text{similar terms for } Q \text{ and } R \right\} du \wedge dv \end{aligned}$$

From definition 6.3,  $g^*d\omega$  is

$$\left\{ \begin{aligned} & \left[ \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \circ g \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right] \right] \\ & + \left[ \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right] \circ g \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial z}{\partial u} \right] \right] \\ & + \left[ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right] \circ g \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right] \right] \end{aligned} \right\} du \wedge dv.$$

The  $P$  terms are

$$\begin{aligned} & \frac{\partial P}{\partial y} \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] \left[ \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} - \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} \right] \\ & + \frac{\partial P}{\partial z} \left[ g \begin{bmatrix} u \\ v \end{bmatrix} \right] \left[ \frac{\partial x}{\partial v} \frac{\partial z}{\partial u} - \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} \right] \end{aligned}$$

and the  $Q$  and  $R$  terms can be collected in the same way, to establish that  $dg^*\omega = g^*d\omega$   $\square$ .

**Remark 6.2.5.** It has to be said that by your standards this is a nasty calculation, but all it requires of you is lots of partial differentiating.

**Proposition 6.2.4.** If  $\omega$  is a smooth differential 2-form on  $U \subseteq \mathbb{R}^3$  and  $g : I^2 \rightarrow U$  is a smooth embedding a.e.,

$$\int_{g(I^2)} \omega = \int_{I^2} g^*\omega$$

**Proof**  $g$  parametrises the surface in  $U$  and if  $\omega = Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$  then  $g^*\omega$  has  $P \circ g$ ,  $Q \circ g$  and  $R \circ g$  for the function part and  $Dg$  acts on  $du$  and  $dv$  to give us  $du \wedge dv$  on the differential part. We have again

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$$

We need to verify that we get the correct “area stretching” formula out.

From definition 6.3 we had, recall,

$$\begin{aligned} g^*(\omega) = & \left\{ P \circ g \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right] + Q \circ g \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial x}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial z}{\partial u} \right] \right. \\ & \left. + R \circ g \begin{bmatrix} u \\ v \end{bmatrix} \left[ \frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial z}{\partial u} \right] \right\} du \wedge dv \end{aligned}$$

This is

$$\left[ \begin{bmatrix} R \circ g \\ -Q \circ g \\ P \circ g \end{bmatrix} \cdot \left[ \left[ \frac{\partial g}{\partial u} \right] \times \left[ \frac{\partial g}{\partial v} \right] \right] \right] du \wedge dv$$

(The permutation of the  $P, Q, R$  (and the minus sign) come from the way the  $dx \wedge dy$  acts on a piece of surface normal to the  $(d)z$  direction.)

We can rewrite this as

$$\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| \left[ \begin{bmatrix} R \circ g \\ -Q \circ g \\ P \circ g \end{bmatrix} \cdot \hat{\mathbf{n}}[u, v] \right] du \wedge dv$$

where  $\hat{\mathbf{n}}[u, v]$  is the unit normal to the surface at  $g \begin{bmatrix} u \\ v \end{bmatrix}$ , and

$$\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|$$

is the “area stretching factor”.

We have that

$$\int_{g(I^2)} \omega$$

is the limit of the sums of values of  $\omega$  on small elements of the surface  $g(I^2)$ . Suppose  $g$  takes a rectangle  $\Delta u \times \Delta v$  in  $I^2$  to a (small) piece of the surface.

$\omega$  at  $g \begin{bmatrix} u \\ v \end{bmatrix}$  is, say,

$$Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$$

and the unit normal to the surface is  $\hat{\mathbf{n}}[u, v]$  (located at  $g \begin{bmatrix} u \\ v \end{bmatrix}$ ).

Write  $\hat{\mathbf{n}}[u, v]$  as

$$\begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix}$$

The  $dx \wedge dy$  of the 2-form affects only the  $\hat{n}_z$  component and does so linearly, likewise the  $dx \wedge dz$  is a rotation in the plane orthogonal to  $\hat{n}_y$ . The sum of

the three components is easily seen to be.

$$\begin{bmatrix} R \circ g \begin{bmatrix} u \\ v \end{bmatrix} \\ -Q \circ g \begin{bmatrix} u \\ v \end{bmatrix} \\ P \circ g \begin{bmatrix} u \\ v \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} \hat{n}x \\ \hat{n}y \\ \hat{n}z \end{bmatrix} = \begin{bmatrix} R \circ g \\ -Q \circ g \\ P \circ g \end{bmatrix} \cdot \hat{n}$$

multiplying by the “area stretching factor” we obtain

$$\int_{g(I^2)} \omega = \int_{I^2} g^* \omega$$

□

**Theorem 6.3. [Stokes]** *Let  $g : I^2 \rightarrow U \subseteq \mathbb{R}^3$  be a smooth embedding a.e. and  $\omega$  a smooth differentiable 1-form on  $U$ . Then*

$$\int_{\partial g(I^2)} \omega = \int_{g(I^2)} d\omega$$

**Proof** *By taking the pieces separately and summing the results we can assume without loss of generality that  $g$  is smooth. Then*

$$\begin{aligned} \int_{\partial g(I^2)} \omega &= \int_{\partial I^2} g^* \omega && \text{by proposition 6.2.1} \\ &= \int_{I^2} dg^* \omega && \text{by Green's Theorem for a square} \\ &= \int_{I^2} g^* d\omega && \text{by proposition 6.2.3} \\ &= \int_{g(I^2)} d\omega && \text{by proposition 6.2.4} \end{aligned}$$

□

**Remark 6.2.6.** It looks believable that if

$$g : I^m \rightarrow U \subseteq \mathbb{R}^n$$

is a smooth embedding a.e. (hence  $m \leq n$ ) and  $\omega$  is a smooth  $m$ -form on  $\mathbb{R}^n$ , then

$$\int_{gI^m} \omega = \int_{I^m} g^* \omega$$

and this is indeed the case. This is the general ‘differential measure-stretching’ change of variables rule.

It looks also believable that on  $I^m$ ,  $m > 1$  if  $\omega$  is an  $m - 1$  form, and  $g : I^m \rightarrow U \subseteq \mathbb{R}^n$  is a smooth embedding,

$$dg^* \omega = g^* d\omega$$

which is also the case.

It also looks plausible that for any  $m - 1$  form  $\omega$  on  $I^m$ ,

$$\int_{\partial I^m} \omega = \int_{I^m} d\omega$$

This is also the case.

Hence we have for any dimension, by copying out the proof of Theorem 6.3, if  $\omega$  is a smooth  $k - 1$  form on  $\mathbb{R}^n$  and  $g : I^k \rightarrow U \subset \mathbb{R}^n$  is a smooth embedding a.e.,

$$\int_{\partial g(I^k)} \omega = \int_{g(I^k)} d\omega$$

In this or more general forms, this is now known (to the well informed) as Stokes’ Theorem. It is fair to say that Stokes would have needed to do some work to recognise it.

It includes the case when  $n = 1, k = 0$  when it says:

$$\int_a^b f(x) dx = F(b) - F(a)$$

when

$$f = \frac{dF}{dx}$$

For this reason Stokes’ Theorem is sometimes known as the Fundamental Theorem of Calculus. Please note that this is not as your textbook author appears to think an analogy, it is simply a consequence of correct generalisation.

**Remark 6.2.7.** Since we are integrating, we can neglect the failure of  $g$  to be smooth or 1-1 on sets of measure (length, area, volume  $\dots$ ) zero. So Stokes Theorem works for lots of sets including spheres and things which have no boundary. In the exercises you will find that this can be pushed a lot further than such restricted cases.

**Example 6.2.1.** Let  $U$  be the hemisphere

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0 \right\}$$

And

$$\omega = xyz \, dx + x \, dy + y \, dz$$

be a differential 1-form on  $\mathbb{R}^3$ .

Calculate

$$\int_U d\omega$$

and

$$\oint_{\partial U} \omega$$

and show they are the same.

**Solution 1:**

The line integral looks easier so I do that first. I parametrise  $S^1 = \partial U$  by

$$x = \cos(t), \quad y = \sin(t), \quad z = 0; \quad 0 \leq t \leq 2\pi$$

Then I want

$$\int_0^{2\pi} \cos(t) \sin(t) 0 \, dx + \cos(t) \, dy + \sin(t) 0$$

since  $z=0$  and hence  $dz = 0$  Substituting for  $dy$  ( $dy = dy/dt dt$ ), this becomes

$$\begin{aligned} & \int_0^{2\pi} \cos(t) \cos(t) dt \\ &= \int_0^{2\pi} \frac{\cos(2t) + 1}{2} dt = \pi \end{aligned}$$

Now for the surface integral over the hemisphere. First I parametrise the hemisphere with

$$x = \sqrt{1-v^2} \cos(u), \quad y = \sqrt{1-v^2} \sin(u), \quad z = v$$

This gives:

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

and similarly for  $dy$  and  $dz$ . Working it out:

$$dx = -\sin(u)\sqrt{1-v^2} du - \frac{v \cos(u)}{\sqrt{1-v^2}} dv$$

$$dy = +\cos(u)\sqrt{1-v^2} du - \frac{v \sin(u)}{\sqrt{1-v^2}} dv$$

$$dz = 0 du + 1 dv$$

Now we calculate  $d\omega$  where:

$$\omega = xyz dx + x dy + y dz$$

$$d\omega = (1 - xz) dx \wedge dy + (0 - xy) dx \wedge dz + (1 - 0) dy \wedge dz$$

This gives:

$$d\omega = (1 - v\sqrt{1-v^2}\cos(u)) dx \wedge dy - (1 - v^2)\cos(u)\sin(u) dx \wedge dz + dy \wedge dz$$

Now we have to calculate the  $\wedge$  terms:  $dx \wedge dy =$

$$\left(-\sin(u)\sqrt{1-v^2} du - \frac{v \cos(u)}{\sqrt{1-v^2}} dv\right) \wedge \left(+\cos(u)\sqrt{1-v^2} du - \frac{v \sin(u)}{\sqrt{1-v^2}} dv\right)$$

$$= v \sin^2(u) - (-v \cos^2(u)) du \wedge dv$$

$$= v du \wedge dv$$

And

$$dx \wedge dz = \left(-\sin(u)\sqrt{1-v^2} du - \frac{v \cos(u)}{\sqrt{1-v^2}} dv\right) \wedge (0du + 1dv)$$

$$= -\sin(u)\sqrt{1-v^2} du \wedge dv$$

and finally:

$$dy \wedge dz = \cos(u)\sqrt{1-v^2} du \wedge dv$$

Substituting in the integral gives:

$$\int_U \left( (1 - v\sqrt{1-v^2} \cos(u))(v) + \right. \\ \left. ((1 - v^2) \sin(u) \cos(u))(\sin(u)\sqrt{1-v^2}) + 1(\cos(u)\sqrt{1-v^2}) \right) du \wedge dv$$

This is

$$\iint \left\{ (v - v^2\sqrt{1-v^2} \cos(u) + (1 - v^2)^{3/2} \cos(u) \sin^2(u) + \cos(u)\sqrt{1-v^2}) \right\} du \wedge dv$$

All that remains is to ensure that the orientation is correct: at  $u = 0, v = 0$  we ought to get the outward normal: it is enough to verify that  $dy \wedge dz = \mathbf{k}$  is pointing out. This is  $\cos(u)\sqrt{1-v^2}$  which is  $+1$  which is correct.

Now we can leave the wedge out and perform the double integral. Using the mathematica expression:

```
Integrate[(v - v^2*Sqrt[1 - v^2]*Cos[u]) +
          Cos[u]*(Sin[u])^2*Sqrt[1 - v^2]*(1 - v^2) +
          Cos[u]*Sqrt[1 - v^2], {u, 0, 2Pi}, {v, 0, 1}]
```

we get the result  $\pi$  again.

**Solution 2** Now I do it all again but using the old fashioned physicist's notation.

Now we write the 1-form as a vector field:

$$\mathbf{F} = xyz \mathbf{i} + x \mathbf{j} + y \mathbf{k}$$

and

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = 1 \mathbf{i} + xy \mathbf{j} + (1 - xz) \mathbf{k}$$

For the path integral around the unit circle at  $z = 0$  we have again

$$x = \cos(t), \quad y = \sin(t), \quad z = 0$$

so the path integral becomes

$$\oint 0 + \cos(t) \frac{dy}{dt} + \sin(t) \frac{dz}{dt} dt \\ = \oint \cos^2(t) dt$$

as before.

For the integral over the surface we again need a parametrisation  $g$ :

$$x = \sqrt{1 - v^2} \cos(u), \quad y = \sqrt{1 - v^2} \sin(u), \quad z = v$$

for

$$0 \leq v \leq 1, \quad 0 \leq u \leq 2\pi$$

and I need to calculate

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix}$$

This is:

$$\begin{bmatrix} -\sin(u)\sqrt{1-v^2} \\ \cos(u)\sqrt{1-v^2} \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{\cos(u)(-v)}{\sqrt{1-v^2}} \\ \frac{\sin(u)(-v)}{\sqrt{1-v^2}} \\ 1 \end{bmatrix}$$

Evaluating the cross product gives:

$$d\mathbf{S} = \begin{bmatrix} \cos(u)\sqrt{1-v^2} \\ \sin(u)\sqrt{1-v^2} \\ v \end{bmatrix}$$

We have

$$\text{curl}\mathbf{F} = \begin{bmatrix} 1 \\ xy \\ 1 - xz \end{bmatrix} = \begin{bmatrix} 1 \\ (1 - v^2) \sin(u) \cos(u) \\ 1 - v\sqrt{1 - v^2} \cos(u) \end{bmatrix}$$

Now we integrate

$$\begin{bmatrix} 1 \\ (1 - v^2) \sin(u) \cos(u) \\ 1 - v\sqrt{1 - v^2} \cos(u) \end{bmatrix} \cdot \begin{bmatrix} \cos(u)\sqrt{1 - v^2} \\ \sin(u)\sqrt{1 - v^2} \\ v \end{bmatrix}$$

with respect to  $u, v$  over the region  $0 \leq v \leq 1, \quad 0 \leq u \leq 2\pi$

This comes out as

$$\iint \left\{ (v - v^2\sqrt{1 - v^2} \cos(u) + (1 - v^2)^{3/2} \cos(u) \sin^2(u) + \cos(u)\sqrt{1 - v^2}) \right\} dudv$$

as before.

**Remark 6.2.8.** As you can see, it is worth being able to decide if a 2-form is derived from a 1-form: in the above case the double integral is much harder than the single one around the circle. Given the double integral of  $d\omega$  it would have been worth looking for an  $\omega$  to integrate around the boundary.

## 6.3 Divergence

**Remark 6.3.1.** It will have occurred to the more reflective of you that we ought to be able to take the exterior derivative of a 2-form on  $\mathbb{R}^3$  and get a differential 3-form.

**Definition 6.4.** If

$$\psi \triangleq E(x, y, z) dx \wedge dy + F(x, y, z) dx \wedge dz + G(x, y, z) dy \wedge dz$$

is a smooth differential 2-form on  $\mathbb{R}^3$ , the exterior derivative applied to it gives:

$$\frac{\partial E}{\partial z} dz \wedge dx \wedge dy + \frac{\partial F}{\partial y} dy \wedge dx \wedge dz + \frac{\partial G}{\partial x} dx \wedge dy \wedge dz$$

Where I followed the same rules as before and didn't bother to get the zero terms. Collecting up:

$$d\psi = \left( \frac{\partial E}{\partial x} - \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} \right) dx \wedge dy \wedge dz$$

**Definition 6.5. Classical Notation** If

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

is a smooth vector field on  $\mathbb{R}^3$  then

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

We memorise this by writing:

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$$

**Definition 6.6.**

$$I^3 \triangleq \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \right\}$$

**Theorem 6.4. Divergence: Classical Form** *If  $V$  is a subset of  $\mathbb{R}^3$  that is the image of a smooth embedding a.e. of  $I^3$  and  $\mathbf{F}$  is a smooth vector field on an open neighbourhood of  $V$  then*

$$\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \operatorname{div} \mathbf{F}$$

where  $d\mathbf{S}$  is the area element times the unit outward normal to the surface  $\partial V$ .

**Theorem 6.5. Modern Form** *If  $V$  is a subset of  $\mathbb{R}^3$  that is the image of a smooth embedding a.e. of  $I^3$  and  $\omega$  is a differential 2-form on an open neighbourhood of  $V$  then*

$$\int_{\partial V} \omega = \int_V d\omega$$

**Proof:** I shall prove it for the special case of the cube  $I^3$ . The case for the embedding of a cube then follows by the usual argument, the only complicated bit being the load of partial derivatives in the higher dimensional part showing

$$g^*d\omega = dg^*\omega$$

which is just another computation.

To add variety I shall prove it for the cube using the classical notation and pretending the 2-form is a vector field.

Look at one face of the cube and observe that over this face,

$$\int \mathbf{F} \cdot d\mathbf{S}$$

has meaning the amount of flow (flux was used in the seventeenth century and still is in some quarters) coming out of the surface. See figure 6.5. If we subdivide the cube into six subcubes and add up  $\int \mathbf{F} \cdot d\mathbf{S}$  for each subcube we must get the same result as

$$\int \mathbf{F} \cdot d\mathbf{S}$$

for the whole cube. This is because the flow out of any one interior cube face is counted twice, one in each direction so cancels out in the sum. The same process can be repeated indefinitely over progressively small subcubes.

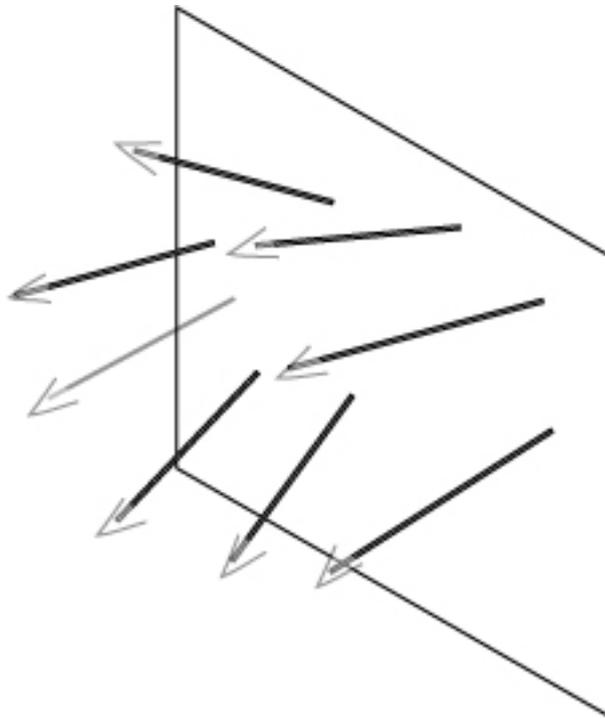


Figure 6.5: Flow from a face

Now look at a very small subcube centred at a point

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and having edge  $2\Delta$

The flow out of the face with centre

$$\begin{bmatrix} a + \Delta \\ b \\ c \end{bmatrix}$$

is only in the  $\mathbf{i}$  direction so is approximately

$$\left( P + \frac{\partial P}{\partial x} \Delta \right) \times 4\Delta^2$$

(which is the field at the centre of the face in the  $\mathbf{i}$  direction multiplied by the surface area.)

The flow out of the opposite face is

$$-\left(P - \frac{\partial P}{\partial x} \Delta\right) \times 4\Delta^2$$

with a minus sign because the normal is pointing in the opposite direction.

For the other four faces of the cube we get the corresponding terms with

$$\left(Q + \frac{\partial Y}{\partial y} \Delta\right) \times 4\Delta^2$$

and its opposite and

$$\left(R + \frac{\partial R}{\partial z} \Delta\right) \times 4\Delta^2$$

and its opposite. Adding these up we get

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \times 8\Delta^3$$

which is  $\operatorname{div} \mathbf{F}$  multiplied by the volume of the little cube.

Integrating this function over the cube is done by approximating by Riemann sums and taking limits so we get, for the cube,

$$\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \operatorname{div} \mathbf{F}$$

Saying this in new-fangled language we get

$$\int_{\partial(I^3)} \omega = \int_{I^3} d\omega$$

The rest is a straightforward calculation to make it work for regions which are the images of smooth embeddings a.e. of cubes in  $\mathbb{R}^3$ , and also images by maps which are smooth embeddings except on a set of area zero.  $\square$

**Remark 6.3.2.** It is good clean fun to show that the sets which are smooth embeddings a.e. of a cube includes the unit ball,

$$D^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \right\}$$

Try it. It will make you better and purer people. More like me.

**Example 6.3.1.** Find the amount of flow of the vector field  $z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$  through the unit sphere  $S^2$ .

**Solution** The normal vector to a point

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\} \in S^2$$

is just

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

The projection of the pseudovector  $z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$  on this is  $(xz + xy + yz)$  which is a straightforward function and we want to integrate this over  $S^2$ . This we know how to do, we parametrise by

$$\begin{aligned} g : [0, 2\pi] \times [-1, 1] &\longrightarrow \mathbb{R}^3 \\ u, v &\rightsquigarrow \begin{bmatrix} \sqrt{1-v^2} \cos(u) \\ \sqrt{1-v^2} \sin(u) \\ v \end{bmatrix} \end{aligned}$$

And we have to put in the area stretching term which is

$$\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|$$

to get

$$\int_{v=-1}^1 \int_{u=0}^{2\pi} \left( v\sqrt{1-v^2} \cos(u) + (1-v^2) \cos(u) \sin(u) + v\sqrt{1-v^2} \sin(u) \right) \times (\text{the area stretching factor}) du dv$$

This is clearly doable but would make the bravest heart sink a bit. Even typing it correctly into Mathematica would take a while.

We have Gauss galloping to the rescue this time (he invented the Divergence Theorem) telling us that the result is the same as if we integrate  $\text{div } \mathbf{F}$  over the interior of the ball.

Now  $\text{div } \mathbf{F}$  is:

$$\nabla \cdot \mathbf{F} = \frac{\partial z}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z} = 0$$

Integrating the zero function over the solid ball  $D^3$  gives zero. You have to allow that this is the easy way to do it.

Knowing the right answer, you can see that this is correct because the sphere has eight octants which are moved into each other by reversing the sign of one or more axes, and the symmetry in the field causes the total to cancel out. But this is being wise after the event.

**Example 6.3.2.** Prove that for any region  $U$  which is the image by a smooth embedding a.e. of a cube in  $\mathbb{R}^3$  and any vector field  $\mathbf{F}$

$$\int_{\partial U} \text{curl } \mathbf{F} = 0$$

**Proof:** I claim that  $d^2 = 0$ . Hence  $\text{div}(\text{curl}(\mathbf{F})) = 0$ . So the integral over  $U$  of  $\text{div } \text{curl}$  is zero, so the surface integral of  $\text{curl}(\mathbf{F})$  is zero.

All I have to do is a simple calculation to show that going from 1-forms to 2-forms by  $d$  and then from 2-forms to three forms by  $d$  gives me zero. Starting with

$$P dx + Q dy + R dz$$

I get

$$\left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy + \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) dx \wedge dz + \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz$$

and taking the exterior derivative of this I get:

$$\left( \frac{\partial}{\partial x} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \right) dz \wedge dy \wedge dz$$

You can see that the terms cancel pairwise to give zero, so the result is proved.

**Remark 6.3.3.** There are a large number of applications of these ideas in electromagnetism and fluid mechanics. Alas, I have no time to cover them but you should be in a good position to understand them when they are used in Physics.



# Chapter 7

## Fourier Theory

### 7.1 Various Kinds of Spaces

**Remark 7.1.1.**  $\mathcal{C}[-\pi, \pi]$  is defined to be the set of continuous maps from the interval  $\{x \in \mathbb{R} : -\pi \leq x \leq \pi\}$  to  $\mathbb{R}$ . It is easy to verify that  $\mathcal{C}[-\pi, \pi]$  is a vector (linear) space, where we add maps and scale them pointwise:

$$\forall f, g \in \mathcal{C}[-\pi, \pi], \forall x \in [-\pi, \pi] (f + g)(x) \triangleq f(x) + g(x)$$

and

$$\forall f \in \mathcal{C}[-\pi, \pi], \forall t \in \mathbb{R}, \forall x \in [-\pi, \pi] \quad (tf)(x) \triangleq tf(x)$$

**Exercise 7.1.1.** Prove that with these operations,  $\mathcal{C}[-\pi, \pi]$  is a vector space. (A soothing exercise in axiom bashing.)

**Remark 7.1.2.** It is rather well known that  $\mathbb{R}^n$  is a vector space for each positive integer  $n$  and has dimension  $n$ . It is obvious that  $\mathcal{C}[-\pi, \pi]$  does not have dimension  $n$  for any positive integer  $n$ .

**Exercise 7.1.2.** Prove this claim

**Remark 7.1.3.** One of the reasons for studying abstract vector spaces is so that we can transfer intuitions about  $\mathbb{R}^2$  and  $\mathbb{R}^3$  to function spaces. We can carry this further:

**Definition 7.1.1. Normed Vector Space** A *norm* on a vector space  $\mathcal{V}$  is a map

$$\begin{aligned} \|\cdot\| : \mathcal{V} &\longrightarrow \mathbb{R} \\ \mathbf{x} &\rightsquigarrow \|\mathbf{x}\| \end{aligned}$$

such that:

$$\forall \mathbf{x} \in \mathcal{V}, \quad \|\mathbf{x}\| \geq 0 \quad \text{and} \quad \|\mathbf{x}\| = 0 \Rightarrow \mathbf{x} = \mathbf{0} \quad (7.1)$$

$$\forall \mathbf{x} \in \mathcal{V}, \forall t \in \mathbb{R}, \quad \|t\mathbf{x}\| = |t|\|\mathbf{x}\| \quad (7.2)$$

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V}, \quad \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad (7.3)$$

A pair  $(\mathcal{V}, \|\cdot\|)$  where  $\|\cdot\|$  is a norm is called a *normed vector space*.

**Remark 7.1.4.** In  $\mathbb{R}^n$

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

is a norm.

**Exercise 7.1.3.** prove the above claim.

**Definition 7.1.2. Inner Product Space** A vector space  $\mathcal{V}$  has an *inner product*  $\langle \cdot, \cdot \rangle$  iff

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} &\longrightarrow \mathbb{R} \\ \mathbf{x}, \mathbf{y} &\rightsquigarrow \langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

is such that:

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V} \quad \langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle \quad (7.4)$$

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \forall t \in \mathbb{R} \quad \langle \mathbf{x}, t\mathbf{y} \rangle = t \langle \mathbf{x}, \mathbf{y} \rangle \quad (7.5)$$

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V} \quad \langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \quad (7.6)$$

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \forall t \in \mathbb{R} \quad \langle t\mathbf{x}, \mathbf{y} \rangle = t \langle \mathbf{x}, \mathbf{y} \rangle \quad (7.7)$$

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \quad \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle \quad (7.8)$$

$$\forall \mathbf{x} \in \mathcal{V} \quad \langle \mathbf{x}, \mathbf{x} \rangle \geq 0 \quad \text{and} \quad \langle \mathbf{x}, \mathbf{x} \rangle = 0 \Rightarrow \mathbf{x} = \mathbf{0}. \quad (7.9)$$

The pair  $(\mathcal{V}, \langle \cdot, \cdot \rangle)$  is called an *inner product space*

**Remark 7.1.5.** lines 7.4 to 7.7 are summarised by saying that the inner product is *bilinear* line 7.8 says it is symmetric, and line 7.9 says it is *positive definite*.

**Example 7.1.1.** The good old dot product on  $\mathbb{R}^n$  is an inner product.

**Exercise 7.1.4.** Prove the last claim. (This sort of thing is called axiom bashing and is supposed to be good for the soul. If it is, that would explain why I am so saintly.)

**Proposition 7.1.1. Schwartz Inequality** If  $(\mathcal{V}, \langle, \rangle)$  is an inner product space then:

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V}, (\langle \mathbf{x}, \mathbf{y} \rangle)^2 \leq (\langle \mathbf{x}, \mathbf{x} \rangle)(\langle \mathbf{y}, \mathbf{y} \rangle)$$

**Proof**

$$\begin{aligned} \forall s, t \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \quad & \langle s\mathbf{x} - t\mathbf{y}, s\mathbf{x} - t\mathbf{y} \rangle \geq 0 \\ \Rightarrow & \langle s\mathbf{x}, s\mathbf{x} \rangle - \langle s\mathbf{x}, t\mathbf{y} \rangle - \langle t\mathbf{y}, s\mathbf{x} \rangle + \langle t\mathbf{y}, t\mathbf{y} \rangle \geq 0 \\ \Rightarrow & s^2 \langle \mathbf{x}, \mathbf{x} \rangle - 2st \langle \mathbf{x}, \mathbf{y} \rangle + t^2 \langle \mathbf{y}, \mathbf{y} \rangle \geq 0 \\ \Rightarrow & 2st \langle \mathbf{x}, \mathbf{y} \rangle \leq s^2 \langle \mathbf{x}, \mathbf{x} \rangle + t^2 \langle \mathbf{y}, \mathbf{y} \rangle \end{aligned}$$

In particular, putting  $s = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}$  and  $t = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$  we get

$$2\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{x}, \mathbf{y} \rangle \leq 2(\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle)$$

from which we deduce that

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

If we put  $s = -\sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}$  and  $t = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$  we deduce

$$-\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

Putting the two deductions together we obtain

$$(\langle \mathbf{x}, \mathbf{y} \rangle)^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle$$

□

**Remark 7.1.6.** If we define  $\cos(\theta)$  for the angle between two vectors by the rule:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\theta)$$

then the above inequality tells us that  $-1 \leq \cos(\theta) \leq 1$  which is nice to know. Alternatively we could use that fact to remember the Schwartz inequality.

**Definition 7.1.3. Metric Space** A *metric* on a set  $X$  is a map

$$d : X \times X \longrightarrow \mathbb{R}$$

such that

$$\forall x, y \in X \quad d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \Rightarrow x = y \quad (7.10)$$

$$\text{and} \quad \forall x, y \in X \quad d(x, y) = d(y, x) \quad (7.11)$$

$$\text{and} \quad \forall x, y, z \in X \quad d(x, z) \leq d(x, y) + d(y, z) \quad (7.12)$$

A pair  $(X, d)$  where  $d$  is a metric on  $X$  is called a *metric space*.

**Remark 7.1.7.** This abstracts our idea of what a distance function has to be like. The first says that the distance between places is always positive except when the places are the same, when it is zero. The second says that the distance between here and the pub is the same as the distance between the pub and here. And the third, the triangle inequality, says that going to the pub via some other place can never make for a shorter distance all up.

This clearly works for pubs.

There is nothing in here that says that  $X$  has to be a vector space and no reason why it should be. It is fairly easy to think of a sensible metric on  $S^1$ , the unit circle.

**Exercise 7.1.5.** Go on then.

**Remark 7.1.8.** We can define continuity for maps between metric spaces:

**Definition 7.1.4.** If  $f : (X, d) \longrightarrow (Y, e)$  is a map between metric spaces, then  $f$  is *continuous at*  $a \in X$  iff

$$\forall \varepsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ : \forall x \in X, d(x, a) < \delta \Rightarrow e(f(x), f(a)) < \varepsilon$$

**Definition 7.1.5.** If  $f : (X, d) \longrightarrow (Y, e)$  is a map between metric spaces, then  $f$  is *continuous* iff  $\forall a \in X$ ,  $f$  is continuous at  $a$ .

**Remark 7.1.9.** Note that this is equivalent to the usual definition of continuity on the old familiar spaces  $\mathbb{R}^n$  but that it now makes sense to say when a function  $f : S^1 \longrightarrow S^1$  is continuous provided we specify a way to say what a distance is for points in  $S^1$ . Easiest is to just give the distance in  $\mathbb{R}^2$ .

**Remark 7.1.10.** Recall that inner product spaces and normed spaces have to be vector spaces.

**Proposition 7.1.2.** If  $(X, \langle \cdot, \cdot \rangle)$  is an inner product space then it is also a normed vector space with:

$$\forall \mathbf{x} \in X \quad \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

**Proof:** Calling something a norm doesn't make it one, so we need to show that with  $\|\mathbf{x}\|$  defined as above, the axioms for a norm hold.

$\forall \mathbf{x} \in X, \|\mathbf{x}\| \geq 0$  since  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$  and

$$\|\mathbf{x}\| = 0 \Rightarrow \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = 0 \Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle = 0 \Rightarrow \mathbf{x} = \mathbf{0}$$

and

$$\forall t \in \mathbb{R}, \forall \mathbf{x} \in X, \|t\mathbf{x}\| = \sqrt{\langle t\mathbf{x}, t\mathbf{x} \rangle} = \sqrt{t^2} \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = |t| \|\mathbf{x}\|$$

Finally,

$$\begin{aligned} \forall \mathbf{x}, \mathbf{y} \in X \quad \|\mathbf{x} + \mathbf{y}\| &= \sqrt{\langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle} \\ &\Rightarrow \|(\mathbf{x} + \mathbf{y})\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 \\ &\Rightarrow \|(\mathbf{x} + \mathbf{y})\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\ &\Rightarrow \|(\mathbf{x} + \mathbf{y})\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\ &\Rightarrow \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \end{aligned}$$

□

**Remark 7.1.11.** Note the use of the Schwartz inequality in the last part.

**Proposition 7.1.3.** If  $(\mathcal{V}, \|\cdot\|)$  is a normed vector space then there is a metric  $d$  derived from the norm by

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V}, \quad d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

**Proof** It is immediate that  $\forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \quad d(\mathbf{x}, \mathbf{y}) \geq 0$  and

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{V} \quad d(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow \|\mathbf{x} - \mathbf{y}\| = 0 \Rightarrow \mathbf{x} - \mathbf{y} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{y}$$

Further:

$$\begin{aligned} \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}, \quad d(\mathbf{x}, \mathbf{z}) &= \|\mathbf{x} - \mathbf{z}\| \\ \text{and } d(\mathbf{x}, \mathbf{y}) &= \|\mathbf{x} - \mathbf{y}\| \\ \text{and } d(\mathbf{y}, \mathbf{z}) &= \|\mathbf{y} - \mathbf{z}\| \end{aligned}$$

and from the last property of a norm

$$\begin{aligned}\|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\| &\leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| \\ \Rightarrow \|\mathbf{x} - \mathbf{z}\| &\leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|\end{aligned}$$

□

**Remark 7.1.12.** This means that in the case of  $\mathbb{R}^n$  we have been using a metric without knowing it. Like Moliere's hero who was very impressed to discover he had been speaking prose for years and years.

**Exercise 7.1.6.** Prove that Pythagoras' Theorem holds in any inner product space

**Remark 7.1.13.** Why am I doing this abstraction? Because we want to work in function spaces. If we can say that a space has an inner product we can go around doing what we were doing in  $\mathbb{R}^n$  with a good conscience, all the stuff about lines and hyperplanes and projections and orthogonality makes sense in a function space. This is a way of pushing intuitions based on the plane and the space we live in up to infinite dimensional function spaces, quite a smart trick.

## 7.2 Function Spaces

**Remark 7.2.1.**  $\mathcal{C}[a, b]$  has a natural inner product which is a generalisation of the standard inner product on  $\mathbb{R}^n$ . To see this, contemplate the geometrical picture of the 'dot' product in figure 7.1

It should be clear that we could think of the functions from  $[a, b]$  to  $\mathbb{R}$  as vectors with a component for every  $t \in [a, b]$  and if  $f, g$  are two such functions, the equivalent of the dot product would be the infinite sum of  $f(t)g(t)$ . It should come as no surprise that we have:

$$\langle f, g \rangle \triangleq \int_a^b f(t)g(t) dt$$

**Proposition 7.2.1.**  $\langle f, g \rangle \triangleq \int_a^b f(t)g(t) dt$  is an inner product on  $\mathcal{C}[a, b]$ .

**Proof:**

$$\langle f, - \rangle: \mathcal{C}[a, b] \longrightarrow \mathbb{R}$$

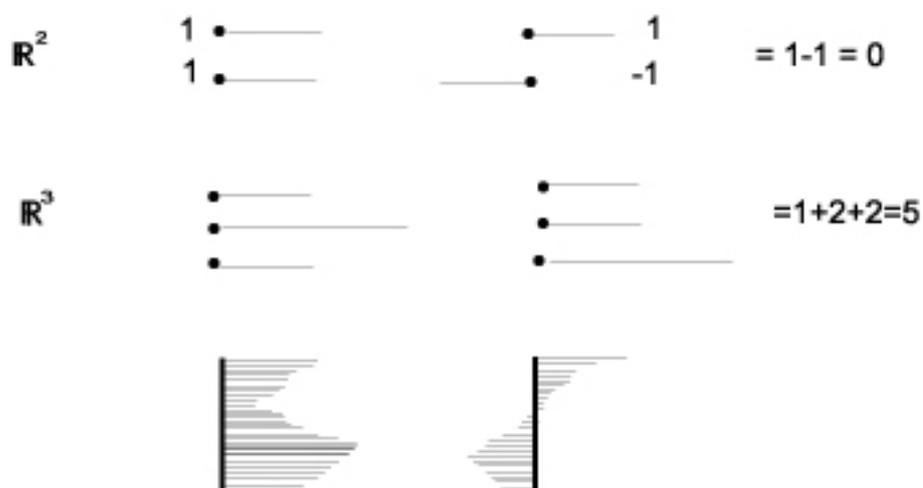


Figure 7.1: Inner Products

$$g \rightsquigarrow \int_a^b f(t)g(t) dt$$

is clearly linear since

$$\int (f(t)(g+h)(t)) dt = \int f(t)g(t) dt + \int f(t)h(t) dt$$

and

$$\forall \alpha \in \mathbb{R}, \int f(t)(\alpha g)(t) dt = \alpha \int f(t)g(t) dt$$

and  $\forall f, g \in \mathcal{C}[a, b]$ ,  $\langle f, g \rangle = \langle g, f \rangle$  so  $\langle, \rangle$  is bilinear.

Finally, it is trivial that

$$\langle f, f \rangle \geq 0$$

and

$$\langle f, f \rangle = 0 \Rightarrow \int_a^b (f(t))^2 dt = 0$$

Now if  $\exists t \in [a, b] : f(t) \neq 0$ , then  $(f(t))^2 > 0$  and there is a neighbourhood of  $t$  for which  $f^2 > 0$  by continuity. Hence  $\int_a^b f^2 > 0$ . The contrapositive of this is the proposition

$$\langle f, f \rangle = 0 \Rightarrow f = \mathbf{0}$$

□

**Remark 7.2.2.** It follows that we have a derived metric on  $\mathcal{C}[a, b]$  given by

$$d(f, g) = \sqrt{\int (f(t) - g(t))^2 dt}$$

and a norm given by

$$\|f\| = \sqrt{\int_a^b (f(t))^2 dt}$$

**Remark 7.2.3.** This is called the  $\mathcal{L}^2$  norm on the space. There are others, and other metrics. The idea of having different notions of distance on the same set is a bit strange at first, but one gets used to it. Compare:

$$d(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|$$

and

$$\|f\| = \sup_{t \in [a, b]} |f(t)|$$

**Remark 7.2.4.** This gives another and different sense of the “distance” between two functions and the “size” of a function. So don’t talk or think of “the” distance between functions. These two are different and there are others.

**Remark 7.2.5.** We need to use the idea of a distance between functions when we are making precise the idea of a sequence of functions approximating to a function. For example we might approximate some function by a sequence of polynomials. It is important to be clear about the idea of a sequence converging, but this makes sense in any metric space: which metric the convergence occurs in is of some practical importance. For example, we might be converging in the sense of the last metric, but wind up with a very bad approximation to the derivative which got steadily worse as we converge to the values of the function.

**Remark 7.2.6.** We can now say when two functions in  $\mathcal{C}[a, b]$  are orthogonal; it is when the inner product is zero.

**Proposition 7.2.2.**  $\cos$  and  $\sin$  are orthogonal in  $\mathcal{C}[-\pi, \pi]$

**Proof:**

$$\int_{-\pi}^{\pi} \sin(t) \cos(t) dt = \frac{1}{2} \int_{-\pi}^{\pi} \sin(2t) dt = \left. \frac{-1}{4} \cos(2t) \right]_{-\pi}^{\pi} = 0$$

□

This can be strengthened:

**Proposition 7.2.3.**  $\sin(nt)$ ,  $\cos(mt)$  are orthogonal on  $\mathcal{C}[-\pi, \pi]$  for any integers  $n, m$ .

**Proof:** If  $n = 0$  then  $\sin(nt)$  is just the zero function so the inner product (integral) is certainly zero. If  $m = 0$  then we have the constant function 1 and the claim is

$$\int_{-\pi}^{\pi} \sin(nt) = 0$$

but

$$\int_{-\pi}^{\pi} \sin(nt) = \left. \frac{-1}{n} \cos(nt) \right]_{-\pi}^{\pi} = 0$$

Finally if neither  $n$  nor  $m$  is zero we note that  $\cos$  is an even function and  $\sin$  is an odd function, so the resulting function is odd, and for every positive term there is a corresponding negative one in the integral, so the integral is zero.  $\square$

Moreover:

**Proposition 7.2.4.** If  $n \neq m$ ,  $\sin(nt)$  and  $\sin(mt)$  are orthogonal on  $\mathcal{C}[-\pi, \pi]$

**Proof:** Recall:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

and bearing in mind that  $\sin$  is odd and  $\cos$  is even

$$2 \sin A \cos B = \cos(A + B) + \cos(A - B)$$

from which it follows that

$$\sin(nt) \sin(mt) = \frac{1}{2} [\cos(n - m)t + \cos(n + m)t]$$

So

$$\int_{-\pi}^{\pi} \sin(nt) \sin(mt) = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n - m)t + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n + m)t$$

both terms being zero when  $n \neq m$   $\square$

Finally:

**Proposition 7.2.5.** If  $n \neq m$ ,  $\cos(nt)$  and  $\cos(mt)$  are orthogonal on

$$\mathcal{C}[-\pi, \pi]$$

**Proof:** Left as an easy exercise.  $\square$

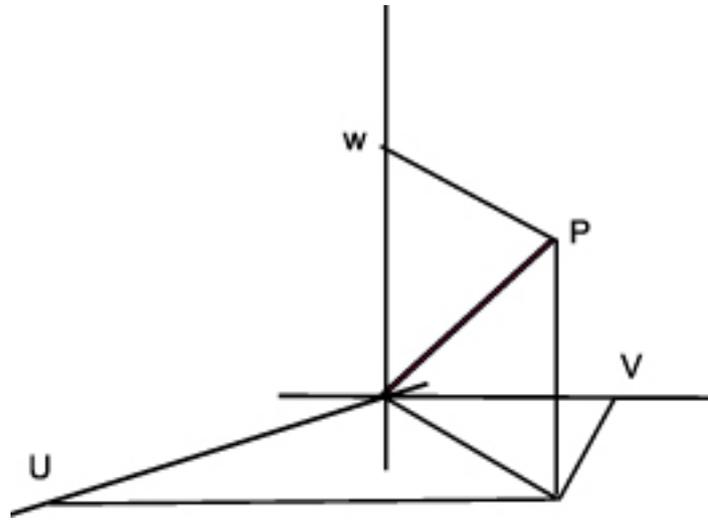


Figure 7.2: Projection on an orthonormal set:  $P = U + V + W$

**Remark 7.2.7.** Note that the underlying interval is crucial here. It all collapses in  $\mathcal{C}[-1, 1]$  for instance— although we could always change the functions to be  $\sin(n\pi x)$  and  $\cos(n\pi x)$ , and this would work.

### 7.3 Applications

**Remark 7.3.1.** Why are we interested in orthogonal functions? Well, we have something very like an orthogonal basis for a space of functions. I could certainly project some other function in  $\mathcal{C}[-\pi, \pi]$  onto these orthogonal functions.

If we project a vector  $P$  in  $\mathbb{R}^3$  onto three orthogonal axes, the projections  $U, V, W$  have the useful property that they sum to  $P$ . This is indicated by figure 7.2. The result is quite general:

**Proposition 7.3.1.** In any inner product space, if  $P$  is projected onto a finite set  $\{\mathbf{v}_j\}, j \in J$  of orthogonal vectors by

$$P_{\text{proj } \mathbf{v}_j} = \frac{\langle P, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle} \mathbf{v}_j = \mathbf{u}_j$$

then

$$P = \sum_{j \in J} \mathbf{u}_j$$

whenever  $P$  is in the span of the  $\mathbf{v}_j$

**Proof:**

Since  $P$  is in the span of the  $\{\mathbf{v}_j\}$  certainly there are numbers  $t_j$  such that

$$P = \sum_{j \in J} t_j \mathbf{v}_j$$

Then by bilinearity of the inner product we have

$$\forall i \in J \quad \langle P, \mathbf{v}_i \rangle = \langle \sum_{j \in J} t_j \mathbf{v}_j, \mathbf{v}_i \rangle = \sum_{j \in J} t_j \langle \mathbf{v}_j, \mathbf{v}_i \rangle$$

But since the different  $\mathbf{v}_j$  are orthogonal,  $\langle \mathbf{v}_j, \mathbf{v}_i \rangle = 0$  for  $i \neq j$ . Hence  $\langle P, \mathbf{v}_i \rangle = t_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle$  and so

$$t_i = \frac{\langle P, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}$$

and the result follows.  $\square$

**Example 7.3.1.** Take the function

$$\text{sign}(x) = \frac{x}{|x|}$$

which is not defined at the origin, or make it zero there if you feel a need. Now we calculate the projection down onto the vector  $\sin(nt)$  in  $\mathcal{C}[-\pi, \pi]$  by using the inner product. Remember that the projection of  $\mathbf{v}$  on  $\mathbf{u}$  in any inner product space is

$$\frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

The coefficient is

$$\int_{-\pi}^{\pi} \text{sign}(x) \sin(nt) dt = 2 \int_0^{\pi} \sin(nt) dt = \frac{2}{n}(1 - \cos(n\pi))$$

divided by

$$\int_{-\pi}^{\pi} \sin^2(nt) dt = \int_{-\pi}^{\pi} \frac{1 - \cos(2nt)}{2} dt = \pi$$

So the projection of  $\text{sign}(x)$  on  $\sin(nt)$  is

$$\frac{4}{n\pi} \sin(nt)$$

when  $n$  is odd and 0 when  $n$  is even.

The following Mathematica program is useful:

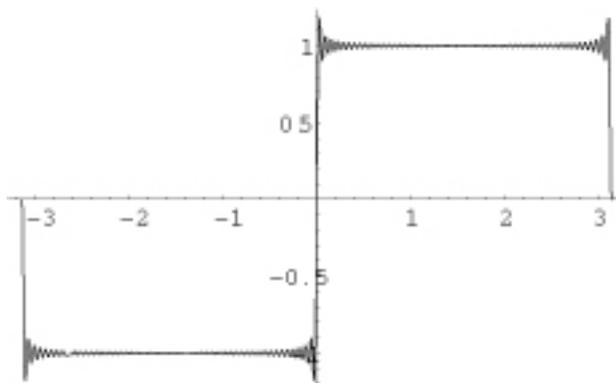


Figure 7.3: Approximating a squarewave

```
Plot[(4/Pi)*(Sum[Sin[n*t]/n, {n, 1, 101, 2}]), {t, -Pi, Pi}]
```

This gives the sum from  $n = 1$  to 101 in steps of 2 that is, taking only odd terms, of the sum of the functions  $\sin(nt)$  with the coefficients

$$\frac{4}{n\pi}$$

You can see the output in figure 7.3 and it is clearly an approximation to the original discontinuous function. Certain things have happened to it in being transmogrified to its new shape: It is now continuous and periodic. This is because the sum of continuous component functions is continuous, and each component function is defined over all  $\mathbb{R}$  and is periodic over  $[-\pi, \pi]$ , so the sum must be also. If we plot the function over a larger range we get a “square wave”, or a passing approximation to it. Try using the mathematica notebook “squarewave” and playing with it. In particular try it for one term, two terms, and so on, in the sum, to see how it builds up.

**Exercise 7.3.1.** Try to get a sawtooth function this way by taking the function

$$\text{tri}(x) = \text{sign}(x) - x \quad \text{on the interval } [-\pi, \pi]$$

**Remark 7.3.2.** It is fairly easy, given Mathematica, to explore this idea for lots of functions. Try it. See parabola.nb in the mathematica notebooks for a painless expansion of the parabola  $y = x^2$ , but you should try it yourself first.

**Remark 7.3.3.** One point to note is that I have done this for functions not in  $\mathcal{C}[-\pi, \pi]$ , the one in the example and the one in the exercise are not continuous. This needs thinking about.

**Remark 7.3.4.** Another point to observe, we may have a sequence which converges in the  $\mathcal{L}^2$  metric but not in the supremum metric. Whether we have convergence of any sort needs to be examined. Mucking about with the sum of more and more terms (it's called 'experimental method') certainly gives the impression of some sort of convergence.

**Remark 7.3.5.** Another point to note is that the series of trig functions does not appear to converge very smoothly to the sign function. It overshoots and then oscillates. This overshoot is called the *Gibbs Phenomenon* and can be investigated.

## 7.4 Fiddly Things

**Remark 7.4.1.** This section is about niggling little matters of principle and detail. Basically, we want to know when we can trust sequences of functions to converge to something. And it would be a good idea to know what the words mean.

It makes sense to say that a sequence of functions in  $\mathcal{C}[a, b]$  converges in the metric derived from the inner product:

**Definition 7.4.1. Natural Numbers:**  $\mathbb{N}$  is the set of natural numbers,  $\{0, 1, 2, \dots\}$

**Definition 7.4.2. Integers:**  $\mathbb{Z}$  is the set of *integers*:  $\{\dots, -1, 0, 1, 2, \dots\}$ . We write  $\mathbb{Z}^+$  for the set of positive integers  $\{1, 2, 3, 4, \dots\}$ .

**Definition 7.4.3. Rational Numbers:** We write  $\mathbb{Q}$  for the set of rational numbers, that is numbers which can be expressed in the form  $a/b$  for  $a, b \in \mathbb{Z}$  and  $b \neq 0$ .

**Remark 7.4.2.** You probably think that you know what the natural numbers and integers are, and who am I to shake your certainties? Actually, you are merely familiar with them. The above 'definitions' are actually nothing of the sort. If you are the kind of person who wants to know what a natural number is, or a real number, then I am afraid I shan't be telling you in this course. You know enough about their properties to be able to catch me out in any of the more obvious lies, and this will be good enough to pass the examination. Whether it will be enough for your intellectual life is entirely up to you. Mathematicians got on very well for several centuries without knowing what a real number is, why shouldn't you? And when some of them

did find out, they got very uncomfortable. You have to decide if you are the sort of person who has to *know* or whether you accept whatever you are told by someone in authority. Crikey, that's me in this case! Who'd have thought it.

**Definition 7.4.4.** If  $\{f_n : n \in \mathbb{N}\}$  is a sequence of points in a metric space, then the limit of the sequence is  $f$  iff

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \in \mathbb{N}, \quad n > N \Rightarrow d(f, f_n) < \varepsilon$$

**Remark 7.4.3.** I am thinking of the case where the  $f_n$  and  $f$  are functions; thinking of a function as a point in a space may seem strange, but that was implied by our taking projections.

**Definition 7.4.5. Cauchy sequences** A sequence of points  $f_n$  in a metric space is a *Cauchy sequence* iff

$$\forall \varepsilon \in \mathbb{R}^+ \exists N \in \mathbb{N} \forall n, m \in \mathbb{N}, \quad n, m > N \Rightarrow d(f_n, f_m) < \varepsilon$$

**Remark 7.4.4.** The sequence of ‘points’ are getting closer together. It is easy to see that if a sequence  $f_n$  converges to  $f$  (Written  $f_n \rightarrow f$ ) then the sequence is a Cauchy sequence. (Exercise: prove this claim.) We would expect that the converse is true, if a sequence is cauchy then it converges to something. After all, we can picture a succession of little balls of radius  $\varepsilon$  getting progressively smaller. The balls are inside each other if we choose them sensibly, so they seem to be homing in on something. Unfortunately the space can have holes in it.

**Example 7.4.1.** The rational numbers,  $\mathbb{Q}$  were defined above to be those numbers which can be written as  $a/b$  where  $a$  and  $b$  are integers. It is well known that  $\sqrt{2}$ ,  $e$  and  $\pi$  are all irrational. So the sequence

$$1, 1.4, 1.41, 1.414, \dots$$

consisting of the finite approximations to  $\sqrt{2}$  of increasing precision is a Cauchy sequence in  $\mathbb{Q}$  which does not converge to anything in the space  $\mathbb{Q}$ .

**Definition 7.4.6.** A metric space is *complete* when every Cauchy sequence in it converges to an element of the space.

Then the problem is that  $\mathbb{Q}$  is not complete. It has holes in it. (Rather a lot of holes. In fact more holes than points. But that is another story.)

The same thing can easily happen in function spaces.

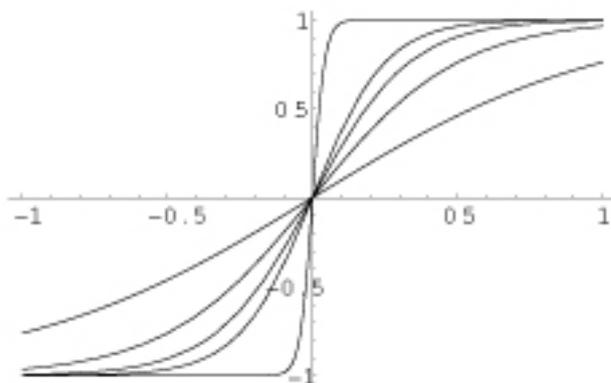


Figure 7.4: Approximating a discontinuous function

**Exercise 7.4.1.** Define

$$\tanh(x) \triangleq \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and } f_n(x) \triangleq \tanh(nx)$$

Show that  $f_n$  is a Cauchy sequence in  $\mathcal{C}(\mathbb{R})$  but that the limit function is  $\text{sign}(x)$  which is not in the space.

**Remark 7.4.5.** It is worth plotting these functions in Mathematica: see figure 7.4

I have shown  $f_1, f_2, f_3, f_4$  and  $f_{24}$ . Verifying that the sequence is Cauchy is a useful exercise in getting things clear in your mind. Verifying that it converges to something not in the space is also good for keeping your ideas in order.

**Remark 7.4.6.** The problem is that we started out with a rather limited class of functions, the continuous ones, and we only need to be able to integrate them and products of them with other functions. So the first step is to say that instead of working in  $\mathcal{C}[a, b]$  we would do well to work in a space of functions which is large enough to contain discontinuous functions which can still be integrated.

Which functions can be integrated? You might suppose they all can be; this is because you have only met nice friendly functions, not the mean, evil functions which resist integration.

**Example 7.4.2.** Let evil  $f : I \rightarrow \mathbb{R}$  be defined as  $f(x) = -1$  if  $x \in \mathbb{Q} \cap I$  and  $f(x) = +1$  if  $x \in I \setminus \mathbb{Q}$ . Now it is easy to see that between every two distinct rational numbers there is another different rational number. It is

not too hard to prove that between every distinct rational numbers there is an irrational number. (Given  $a < b \in \mathbb{Q}$  add  $\sqrt{2}(b-a)/2$  to  $a$ . It is easy to show this must be between  $a$  and  $b$  and also to show it is irrational.)

It is also easy to show that between any two distinct irrational numbers there is a rational number. Take the decimal expansions of the two numbers, go down the line until they differ and one digit is bigger than the corresponding place in the other number. Now choose the larger digit and follow it by zeros for ever. The result is clearly between the two numbers and is also obviously rational.

It follows that evil  $f$  as defined above is not Riemann integrable, since any partition of  $I$  will have for each partition interval  $[a, b]$  points of both types. So the supremum of  $f$  on the interval will be 1 and the infimum will be  $-1$  and this won't get any better as we make the partition finer. Since the integral is defined only if the limit of the suprema tends to the limit of the infima over the partition intervals, we have a non (Riemann) integrable function.

Note that  $f^2$  is the constant function 1 and is rather easily integrated.

**Remark 7.4.7.** To get out of the difficulty I shall work with the space of *piecewise continuous* functions. These functions are certainly integrable as are their products and sums, which are also piecewise integrable, so they form a vector space which includes things like the sign function. They do not include the evil function defined above.

**Definition 7.4.7.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be *piecewise continuous* iff  $f$  is continuous on  $[a, b]$  except at a finite set of points and the limits

$$\lim_{h \uparrow 0} f(x_0 + h), \quad \lim_{h \downarrow 0} f(x_0 + h)$$

exist for all interior points, and the appropriate limit exists for the end points.

**Remark 7.4.8.**  $\lim_{h \uparrow 0} f(x_0 + h)$  means that  $h$  approaches 0 from below, i.e. that  $h$  is negative but gets less so. Contrariwise for  $\lim_{h \downarrow 0} f(x_0 + h)$

**Exercise 7.4.2.** Draw the graphs of some functions in the class of piecewise continuous functions, and some not.

**Proposition 7.4.1.** The set  $\mathcal{PC}[a, b]$  of piecewise continuous functions on  $[a, b]$  is a vector space.

**Proof** We merely have to note that it is closed under addition and scalar multiplication. The latter is trivial. The former will usually require us to take the intersection of intervals on which both functions are continuous.  $\square$

**Proposition 7.4.2.** The product of two functions in  $\mathcal{PC}[a, b]$  is also in  $\mathcal{PC}[a, b]$ .

**Proof:** This again requires us to find intervals on which both functions are continuous and rely on the result that the product of continuous functions is continuous.  $\square$

**Remark 7.4.9.** It is obvious from the definition that any continuous function is in  $\mathcal{PC}[a, b]$ .

**Remark 7.4.10.** It is also obvious that all functions in  $\mathcal{PC}[a, b]$  are Riemann integrable. After all, continuous functions are, and the piecewise continuous fail to be continuous at only a finite set of points, and we can forget about what happens at a finite set of points because they have length zero and will not affect any integral. So to integrate one of them, calculate the integrals of the intervals over which they are continuous and add up the answers.

**Remark 7.4.11.** This sounds like a good space to work in. We can still do all the projection onto orthogonal functions that we want. There is however a slight catch:

**Example 7.4.3. Not!  $\mathbf{0}$**  is the zero function from  $I$  to  $\mathbb{R}$  which sends every number to 0. I define  $\mathbf{o} : I \rightarrow \mathbb{R}$  to be zero except at 1 where it takes the value 1. Now these are different functions (not very, but that's the point) but in the  $\mathcal{L}^2$  metric, the distance between them is zero. As far as the integral of the square of the difference is concerned, the difference is actually just  $\mathbf{o} - \mathbf{0} = \mathbf{o}$  and  $\mathbf{o}^2 = \mathbf{o}$ . And the integral of this over  $I$  is zero. So  $d(\mathbf{0}, \mathbf{o}) = 0$  but  $\mathbf{0} \neq \mathbf{o}$

Bummer. This can't be allowed in a metric space. It contradicts the first axiom.

**Remark 7.4.12.** We get around this problem by simply declaring that we shall deal with not the functions, but classes of functions. And two functions are in the same class precisely when the integral of the square of their difference is zero. So in particular,  $\mathbf{0}$  and  $\mathbf{o}$  are equivalent. They differ only on a finite set of points, so we shall regard them as the same. Then on the *classes* we have that the distance between things is zero only if they are the same.

**Remark 7.4.13.** This looks messy but everything works out. Trust me. Or better yet, don't trust me. Check up on everything I do from here on to make sure it is not a swindle. Even better, go back over everything I have done and make sure I haven't pulled some trick on you. I am but indifferent honest. In particular, if student one uses  $f_1$  and  $g_1$  where student two uses  $f_2$

and  $g_2$  and they calculate  $\langle f, g \rangle$  for the inner product between the classes, do they get the same result when  $f_1, f_2 \in f, g_1, g_2 \in g$ ? If not, all bets are off.

**Exercise 7.4.3.** Confirm that the inner product between classes of functions obtained by choosing any member of one class and any member of the second class and calculating

$$\int_a^b f_i(t)g_i(t)$$

for choices  $f_i, g_i$  is well defined, that is, it does not depend on the choices.

**Remark 7.4.14.** In view of the last exercise, it is reasonable to talk about  $\mathcal{PC}[a, b]$  as if the elements are functions, even though they are not. When I say something involving a function, you can mentally replace it by the class of functions which differ from the one I mentioned only on finitely many points. Technically however:

**Definition 7.4.8.**  $\mathcal{PC}[a, b]$  is the set of equivalence classes of piecewise continuous functions from  $[a, b]$  to  $\mathbb{R}$ , where two functions are equivalent iff they differ only on a finite set of points.

**Proposition 7.4.3.** With addition of classes defined by addition of their elements, scaling defined likewise, the set  $\mathcal{PC}[a, b]$  is a vector space. With  $\langle [f], [g] \rangle$  defined on the classes  $[f], [g]$  by

$$\langle [f], [g] \rangle = \int_a^b f(t)g(t)dt$$

where  $f \in [f]$  is an element of the equivalence class and similarly for  $g$ ,  $\mathcal{PC}[a, b]$  is an inner product space.

**Proof:**

Exercise

□

**Remark 7.4.15.** While it is necessary to ensure that we are not gibbering when we do things like this, and that treating equivalence classes of functions as though they are just functions is logically OK, one should not make too much of it. Either using equivalence classes is going to work because it doesn't really matter in any serious application whether we use a function  $f$  or another  $g$  which is different from the first but not enough to change any outcomes, or it will turn out that we get into terrible trouble pulling swifties like this one. Verifying that we don't in this case is good for your intellectual integrity and also very easy. To some people, alas, all of Mathematics is just inscrutable bafflegab anyway. Let's be kind to them, but not too kind.

**Remark 7.4.16.** If you have verified the last proposition, you will feel comfortable about sloppy usage like ‘taking a function  $f$  in  $\mathcal{PC}[a, b]$ ’. You will note that it is sloppy and that it really means we take a function and use it to specify the equivalence class in  $\mathcal{PC}[a, b]$ . After a while you may find yourself slipping into this no doubt deplorable usage yourself. Shortly after that you will find yourself dismissive of people who insist on using the terms ‘equivalence class of functions’, classifying them as finicky pedants.

It happens to the best of us.

**Remark 7.4.17.** Now we ask the obvious question: Is  $\mathcal{PC}[a, b]$  complete? Or does it still have holes in it? I hate to say this after all the fuss about going to PC spaces which if fashion were the arbiter would certainly be politically correct, but the answer is still a resounding **NO!** The space  $\mathcal{PC}[a, b]$  is still shot full of holes.

Remember evil  $f$ ? We can get a sequence of functions in  $\mathcal{PC}[a, b]$  which converges to evil  $f$ . This is true because the rational numbers can be counted, that is, put into 1-1 correspondence with the natural numbers. So although each member of the sequence is a *bona fide* member of  $\mathcal{PC}[a, b]$ , the limit is not. Basically, the  $n^{\text{th}}$  term in the sequence fails to be continuous at  $n$  points and the limit is not continuous at any of them. So every term in the sequence is actually in the same equivalence class, but the limiting function is not!

Bummer squared.

**Remark 7.4.18.** There *is* a way of coping with this; we define a new integral called the Lebesgue integral. If we can express a function as a limit of Riemann integrable functions, then we can define the Lebesgue integral of the function as the limit of the Riemann integrals. This is not the usual definition of the Lebesgue integral but is equivalent to it. So the function evil  $f$  which was +1 except on the rational numbers, is the limit of functions  $f_n$  which are +1 except on  $n$  distinct rationals, and each of which therefore has integral

$$\int_0^1 f_n = 1$$

So the Lebesgue integral of evil  $f$  is also 1.

From the definition you can see that if a function is Riemann integrable then it is Lebesgue integrable and the integrals are the same.

The space of equivalence classes of functions from  $[a, b]$  to  $\mathbb{R}$  which have Lebesgue integrable squares, with the inner product defined via the integral,

is a complete inner product space. It contains evil  $f$  in the same class as the constant function 1.

**Remark 7.4.19.** Since I don't wish to get involved with any more niceties and I do not want to prove the last claim, I shall stick in practice to the Piecewise Continuous functions  $\mathcal{PC}[a, b]$  and forget about the possibility of taking sequences of functions that converge to things like evil  $f$ .

## 7.5 Odd and Even Functions

**Remark 7.5.1.** We shall also be limiting ourselves to functions defined over intervals which are symmetric about the origin, i.e.  $\mathcal{PC}[-a, a]$  for positive  $a$ . This has the consequence that we can note some convenient properties of even and odd functions.

**Definition 7.5.1.**  $f : [-a, a] \rightarrow \mathbb{R}$  is *even* iff

$$\forall x \in [-a, a], f(-x) = f(x)$$

**Definition 7.5.2.**  $f : [-a, a] \rightarrow \mathbb{R}$  is *odd* iff

$$\forall x \in [-a, a], f(-x) = -f(x)$$

Some more or less obvious remarks:

**Proposition 7.5.1.** The product of even functions is even, of odd functions even, and the product of an even function with an odd function is odd.

**Proof:**

Go on. □

**Proposition 7.5.2.** The integral of an odd function over  $[-a, a]$  is zero. □

**Proposition 7.5.3.** The integral of an even function over  $[-a, a]$  is twice the integral over  $[0, a]$ . □

**Proposition 7.5.4.** If  $f$  is even and differentiable then  $f'$  is odd.

**Proof:**

Compose  $f$  with the map  $-x$  and apply the chain rule. □

**Proposition 7.5.5.** If  $f$  is odd and differentiable then  $f'$  is even.

**Proof:**

Same argument. □

**Proposition 7.5.6.** The functions  $\cos(nx)$  are even for all  $n \in \mathbb{Z}$  and the functions  $\sin(nx)$  are odd for all  $n \in \mathbb{Z}$  on the interval  $[-\pi, \pi]$  □

**Remark 7.5.2.** This may explain why I projected the sign function down only on the  $\sin(nx)$  terms. Projecting on the  $\cos$  terms would have given me zero.

**Exercise 7.5.1.** Go over all the proofs sketched above and fill in all the gaps until you are satisfied that you believe the claims made or that your scepticism is unappeasable.

## 7.6 Fourier Series

Suppose we have some set of pairwise orthogonal vectors  $\{\mathbf{v}_j : j \in J\}$  and a vector  $P$  not in the span of the  $\{\mathbf{v}_j\}$ . We can still take the projections of  $P$  on the  $\mathbf{v}_j$  and sum them. This time we don't get back to  $P$ , but we get as close as we could get and still be in the span of the  $\{\mathbf{v}_j\}$ :

**Proposition 7.6.1.** In an inner product space, if

$$\forall j \in J, \quad \mathbf{u}_j = \frac{\langle P, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle} \mathbf{v}_j$$

and  $V = \text{span}\{\mathbf{v}_j : j \in J\}$  then

$$Q = \sum_{j \in J} \mathbf{u}_j$$

is the closest point of  $V$  to  $P$ .

**Proof:**

It is easy to see by writing out

$$\langle (P - Q), \mathbf{v}_i \rangle$$

that  $P - Q$  is orthogonal to each of the  $\mathbf{v}_j$  and so  $P - Q$  is orthogonal to the subspace  $V$ . By Pythagoras theorem, the distance from  $P$  to any other point of  $V$  is greater than the length of  $P - Q$ . □

**Remark 7.6.1.** This tells us that when we write the set of projections for a given function  $f$  on the sine and cosine functions  $\sin(nx)$ ,  $\cos(mx)$  for  $n, m \in \mathbb{Z}^+$ , that as we take increasingly large integers we are getting closer to  $f$ . Now the question comes up: how close do we actually get in the limit?

**Definition 7.6.1.** A set of vectors  $\mathcal{B}$  in an inner product space  $\mathcal{V}$  is a *Topological Basis* for  $\mathcal{V}$  iff

$$\forall \mathbf{v} \in \mathcal{V}, \exists \{\mathbf{v}_j : j \in \mathbb{Z}^+\} \subset \mathcal{B}, \exists \{t_j : j \in \mathbb{Z}^+\} \subset \mathbb{R} :$$

$$\mathbf{v} = \sum_1^{\infty} t_j \mathbf{v}_j \triangleq \lim_{n \rightarrow \infty} \sum_1^n t_j \mathbf{v}_j$$

where the limit is in the metric derived from the inner product, and

$$\forall n \in \mathbb{Z}^+, \forall t_j \in \mathbb{R}, \forall \mathbf{v}_j \in \mathcal{B},$$

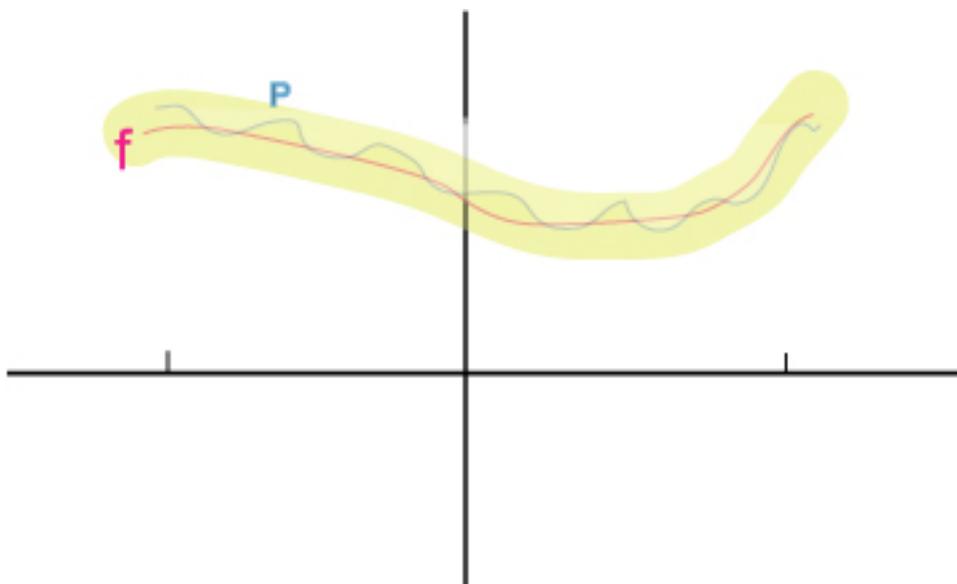
$$\sum t_j \mathbf{v}_j = \mathbf{0} \Rightarrow \forall j \in [1 \cdots n], t_j = 0$$

**Remark 7.6.2.** There is another sort of basis which we shall not deal with, so I shall just drop the word ‘topological’ and refer to a basis.

**Exercise 7.6.1.** Show that if the elements of  $\mathcal{B}$  are pairwise orthogonal, that is if the inner product for any two distinct elements is zero and if  $\mathcal{B}$  does not contain the vector  $\mathbf{0}$ , then the second (independence) condition is satisfied.

**Remark 7.6.3.** I want to show that the trigonometric functions form a basis for the space  $\mathcal{PC}[-\pi, \pi]$ . First let’s be clear that what we mean here is that (a) the above set of functions is an orthogonal set and since it does not contain the zero function must be independent by the last exercise, and (b) any piecewise continuous function is the limit *in the mean* of scaled sums of functions in this set. *in the mean* means that we have convergence in the  $\mathcal{L}^2$  metric.

The argument depends on two subsidiary propositions, one of which is a well known theorem called the Weierstrass Approximation Theorem which is too hard for the course. It states that any *continuous* function can be approximated by a sequence of trigonometric functions *uniformly*. I shall explain precisely what this means soon. The other is that any piecewise continuous function on a closed interval can be approximated by a continuous function in the mean. I shall prove this shortly. First the statement of the Weierstrass Theorem:

Figure 7.5: Uniform convergence to  $f$ 

**Theorem 7.1. Weierstrass** *For any continuous function*

$$f : [-\pi, \pi] \longrightarrow \mathbb{R}$$

with  $f(\pi) = f(-\pi)$  and for any  $\varepsilon \in \mathbb{R}^+$ , there exists a trigonometric polynomial

$$P(x) = a_0 + \sum_{j=1}^{j=N} a_j \cos(jx) + b_j \sin(jx)$$

such that

$$\forall x \in [-\pi, \pi], \quad |P(x) - f(x)| < \varepsilon$$

**Remark 7.6.4.** Note that  $N$  will depend rather a lot on  $\varepsilon$  and will be bigger the smaller  $\varepsilon$  is. Note also that  $N$  does not depend on  $x$ . This is a strong sort of convergence called *uniform convergence*. You can see that it is telling us that we can find  $P$  that is wholly contained in a tubular region around the graph of  $f$ , as in figure 7.5

The tube has height  $2\varepsilon$  of course.

**Remark 7.6.5.** You will find a proof of this theorem as a special case of the Stone-Weierstrass Theorem in George Simmons' nice little book *Introduction to Topology and Modern Analysis* although you need to be told that the topology in it is point set topology, not proper topology. There is a direct

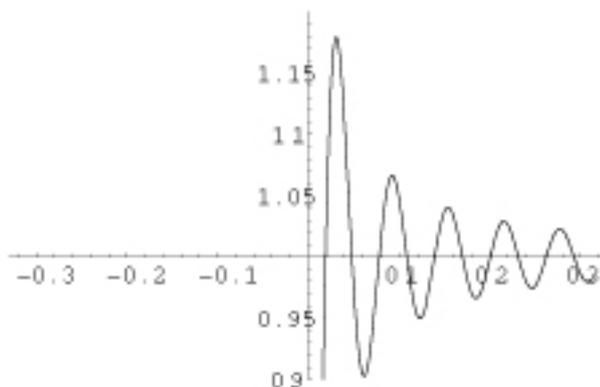


Figure 7.6: The Gibbs Effect

proof in the excellent *An Introduction to Linear Analysis* by Kreider, Kuller, Ostberg and Perkins.

**Remark 7.6.6.** The latter has a nice analysis of the Gibbs Phenomenon, showing that the ‘overshoot’ is computable. In fact they compute the overshoot as a multiple of the actual value: it turns out to be:

$$\frac{2}{\pi} \int_0^{\pi} \frac{\sin(t)}{t} dt$$

Mathematica gives this as approximately) 1.178979744472167270232029, an overshoot of nearly 18% which is quite big. Unfortunately there is an error in the computation for my edition of KKOP, they give 1.089490, just under 9% overshoot, half the Mathematica value.

The graph for the square wave close to height 1 is shown in figure 7.6: this is just the result of applying a magnifying glass to figure 7.3.

Note that the overshoot does not change much as we add more terms, it just gets closer to the point of discontinuity. The overshoot may be seen as a legitimate protest from a nicely behaved, properly brought up, smooth function being forced to approximate a nasty, rough discontinuous one.

**Remark 7.6.7.** Mathematica sure makes finding errors easy. Except when they are bugs in Mathematica of course. In this case it is hard to doubt that there is an error in the text. My edition is dated from 1966, back in the stone age when sums like this were a lot of work.

**Remark 7.6.8.** This leaves the matter of being able to approximate a piecewise continuous function by a continuous one. I deal with the case of one

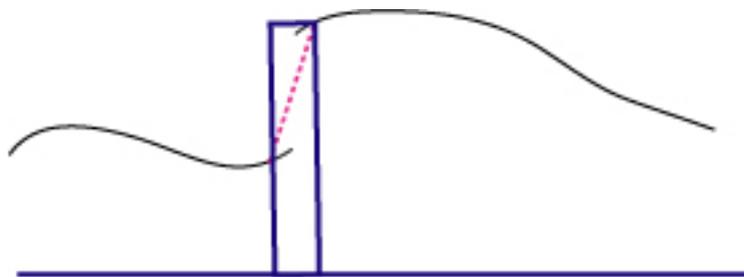


Figure 7.7: Approximating a discontinuity

discontinuity and leave the case of several as an exercise for the sceptical. The argument is contained entirely in figure 7.7 where the step discontinuity is bridged by a continuous approximation; the replacement function goes along the dotted line.

**Proposition 7.6.2.** If  $f$  is a function with a step discontinuity, it can be approximated as closely as may be required in the  $\mathcal{L}^2$  metric by a continuous function.

**Proof:**

The distance between the two functions in the  $\mathcal{L}^2$  metric is the integral of the square of the difference function, which is zero outside the box and is roughly the area of the region between the curves inside the box. We can make this as small as we like by making the box thinner, that is by making the dashed line more and more nearly vertical.  $\square$

**Proposition 7.6.3.** The set of functions  $\{\cos(nt), n \in \mathbb{N}\} \cup \{\sin(nt) : n \in \mathbb{Z}^+\}$  is a basis for  $\mathcal{PC}[-\pi, \pi]$ .

**Proof:**

For any  $\varepsilon \in \mathbb{R}^+$ , and for any  $f \in \mathcal{PC}[-\pi, \pi]$ , we can find a continuous function  $\tilde{f}$  which differs from  $f$  in the  $\mathcal{L}^2$  metric by less than  $\varepsilon/2$  by the above argument applied to all the points of discontinuity of  $f$ . We can make sure that  $\tilde{f}(\pi) = \tilde{f}(-\pi)$  by the same method.

We can find, according to Weierstrass, a trigonometric polynomial  $P$  which differs from  $\tilde{f}$  by less than  $\varepsilon/\sqrt{8\pi}$  at every point of  $[-\pi, \pi]$ , and which therefore is of distance less than  $\varepsilon/2$  in the  $\mathcal{L}^2$  metric. The triangle inequality ensures that the distance between  $P$  and  $f$  in the  $\mathcal{L}^2$  metric is less than  $\varepsilon$ . This shows that we can find a sequence of trigonometric polynomials which converges to  $f$  in the space  $\mathcal{PC}[-\pi, \pi]$  with the  $\mathcal{L}^2$  metric. So the set of trigonometric polynomials spans  $\mathcal{PC}[-\pi, \pi]$ . We have already seen that the

orthogonality property means the set of trigonometric polynomials is linearly independent, so they are a basis.  $\square$

**Remark 7.6.9.** This shows where a bit of linear algebra can get you. The power of abstraction is allowing us to do things in infinite dimensional function spaces that are extensions of what we can do in two and three dimensions. This is very cool.

**Definition 7.6.2.** The basis described in the above proposition is called the *Fourier basis*.

**Definition 7.6.3.** The values of the projection coefficients onto the Fourier basis vectors of any function  $f$  are called the *Fourier coefficients* for  $f$ .

**Definition 7.6.4.** The trigonometric series for  $f$  is called the *Fourier Series* for  $f$ , or its *Fourier Expansion*.

**Remark 7.6.10.** Notations vary: in ours the function  $\cos(0t) = 1$  has coefficient

$$a_0 = \frac{\int_{-\pi}^{\pi} f(t) dt}{\int 1 dt}$$

Other authors have an  $a_0$  twice this. The reason is that the square of the norm of each vector  $\sin(kt), \cos(kt)$  is  $\pi$  for  $k \geq 1$  and  $2\pi$  for the function 1.

Some authors normalise the basis so that instead of working with  $\cos(kt)$  we work with  $1/\sqrt{\pi} \cos(kt)$  and similarly for  $\sin(kt)$  and the constant function  $1/\sqrt{2\pi}$ . I shan't do this, instead I have:

$$\forall f \in \mathcal{PC}[-\pi, \pi], f = a_0 + \sum_{j=1}^{\infty} a_j \cos(jt) + b_j \sin(jt)$$

where

$$a_0 = \frac{\int_{-\pi}^{\pi} f(t) dt}{2\pi}; \quad a_j = \frac{\int_{-\pi}^{\pi} f(t) \cos(jt) dt}{\pi}, \quad b_j = \frac{\int_{-\pi}^{\pi} f(t) \sin(jt) dt}{\pi}$$

for  $j \in \mathbb{Z}^+$ .

**Corollary 7.2. Parseval's Equality** For every  $f \in \mathcal{PC}[-\pi, \pi]$ ,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t))^2 dt = 2a_0^2 + \sum_{j=1}^{\infty} (a_j^2 + b_j^2)$$

where the  $a_j$  and  $b_j$  are the Fourier coefficients for  $f$ .

**Proof:**

Given

$$f = a_0 + \sum_{j=1}^{\infty} a_j \cos(jt) + b_j \sin(jt)$$

we have

$$\begin{aligned} \langle f, f \rangle &= \int_{-\pi}^{\pi} (f(t))^2 dt \\ &= \int_{-\pi}^{\pi} \left( a_0 + \sum_{j=1}^{\infty} a_j \cos(jt) + b_j \sin(jt) \right) \left( a_0 + \sum_{j=1}^{\infty} a_j \cos(jt) + b_j \sin(jt) \right) dt \\ &= \int_{-\pi}^{\pi} a_0^2 + \sum_{j=1}^{\infty} a_j^2 \cos^2(jt) + b_j^2 \sin^2(jt) \\ &= 2\pi a_0^2 + \pi \sum_{j=1}^{\infty} a_j^2 + b_j^2 \end{aligned}$$

which gives the required result when we divide by  $\pi$ .  $\square$

**Remark 7.6.11.** Again, be warned that this can have different forms if the basis functions are normalised. Engineers who do signal and image processing will hear their lecturers talk about the energy in the signal being preserved by the transformation.

A very strong form of convergence occurs when the function  $f$  is piecewise differentiable. The following theorem gives lots of fascinating results. Unfortunately I don't have the time to prove it:

**Proposition 7.6.4.** If  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  is piecewise differentiable, then the Fourier series converges pointwise to  $f(x)$  on every interval on which  $f$  is differentiable, and when  $\lim_{x \uparrow a} f(x) \neq \lim_{x \downarrow a} f(x)$ , the Fourier series converges to

$$\frac{1}{2} \left( \lim_{x \uparrow a} f(x) + \lim_{x \downarrow a} f(x) \right)$$

**Remark 7.6.12.** To show where this gets you, remember the series for  $\text{sign}(x)$  and note that

$$\frac{4}{\pi} \left( \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots \right)$$

converges to 0 at  $0, \pm\pi$  and to  $+1$  for  $x \in (0, \pi)$ . So when  $x = \pi/2$  we deduce that

$$1 = \frac{4}{\pi} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right)$$

or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

which is not otherwise particularly obvious.

**Exercise 7.6.2.** By finding the Fourier series for  $y = |x|$  and evaluating it at the origin, obtain a series for  $\pi^2$ .

**Exercise 7.6.3.** By evaluating the Fourier series for  $y = x^2$  at the origin, obtain another series for  $\pi^2$ .

**Remark 7.6.13.** Sometimes we have a function defined on some other interval. It is simple to transform the domain back to  $[-\pi, \pi]$  by an affine map and needs no new ideas.

**Exercise 7.6.4.** Find the right functions to do Fourier Theory for piecewise continuous functions defined for the interval  $[1, \sqrt{5}]$ .

## 7.7 Differentiation and Integration of Fourier Series

**Theorem 7.3. The Differentiation Theorem** *If  $f$  is continuous and has a piecewise continuous derivative  $f'$  on  $[-\pi, \pi]$ , then the Fourier series for  $f'$  is the series obtained by term by term differentiating of the Fourier Series for  $f$ . If the second derivative  $f''$  exists then the convergence is pointwise to  $f'$ .*

**Theorem 7.4. The Integration Theorem** *For any  $f \in \mathcal{PC}[-\pi, \pi]$  with Fourier series*

$$a_0 + \sum_{j=1}^{\infty} (a_j \cos(jt) + b_j \sin(jt))$$

*the function*

$$\int_0^x f(t) dt$$

*has a series expansion*

$$a_0 x + \sum \frac{a_j \sin(jx)}{j} - \frac{b_j \cos(jx)}{j} + K$$

where  $K$  is a constant. By putting  $x = 0$  we find

$$K = \sum_1^{\infty} \frac{b_j}{j}$$

Using the series expansion for  $a_0 x$  we obtain the Fourier series

$$\sum_{j=1}^{\infty} \frac{b_j}{j} + \sum_{j=1}^{\infty} \frac{-b_j \cos(jx) + (a_j + (-1)^{k+1} a_0) \sin(jx)}{j}$$

I shan't prove either of these theorems, you will find them in *KKOP*.

They enable us to calculate some Fourier series relatively quickly and painlessly.

## 7.8 Functions of several variables

**Remark 7.8.1.** Fourier expansions of functions from  $[-\pi, \pi]$  are simple enough conceptually and easy to compute given current machines. This is the start of the Fourier Transform which has been used extensively for the analysis of signals in Engineering, particularly Electrical Engineering. But these days we are often confronted with 2-dimensional signals. They are sometimes called *images*. So two dimensional transforms are important. Also, it will turn out to be helpful to use Fourier Theory in finding solutions to the partial differential equations which arise in the study of diffusion and wave propagation. These are frequently defined on three dimensional regions of space, so it will be necessary to know how to do an analysis of functions from rectangles and cubes.

Fortunately this is very easy.

**Definition 7.8.1.** Let  $Q \subset \mathbb{R}^2$  denote the square  $[-\pi, \pi] \times [-\pi, \pi]$

**Definition 7.8.2.**  $f : Q \rightarrow \mathbb{R}$  is *rectangularly piecewise continuous* iff there is a decomposition of  $Q$  into rectangles which overlap only on their boundaries, such that  $f$  is continuous on the interiors of the rectangles, and bounded on their boundaries.

**Remark 7.8.2.** This is more restrictive than necessary but makes life easier.

**Definition 7.8.3.**  $\mathcal{RPC}(Q)$  denotes the set of equivalence classes of rectangularly piecewise continuous functions from  $Q$ , where two functions are equivalent iff they differ only on a set of area zero.

**Definition 7.8.4.** For all  $f, g \in \mathcal{RPC}(Q)$ ,

$$\langle f, g \rangle = \int_Q f(s, t)g(s, t)$$

is the standard inner product.

**Remark 7.8.3.** Calling it an inner product doesn't make it one:

**Exercise 7.8.1.** Prove this is an inner product on  $\mathcal{RPC}(Q)$

**Proposition 7.8.1.** The space  $\mathcal{RPC}(Q)$  has the functions

$$\{1, \cos(nx) \cos(my), \cos(nx) \sin(my), \\ \sin(nx) \cos(my), \sin(nx) \sin(my) : n, m \in \mathbb{Z}^+\}$$

as an orthogonal basis.

**Proof:**

The orthogonality is very simple: For example

$$\int_Q \cos(nx) \cos(my) \cos(px) \sin(qy)$$

is clearly zero since it is

$$\int_{-\pi}^{\pi} \cos(nx) \cos(px) \int_{-\pi}^{\pi} \cos(my) \sin(qy)$$

and both terms in the product are zero.

The proof of convergence goes the same way as for one dimension. We need a Weierstrass theorem for functions of two variables that says we can approximate uniformly any continuous function on  $Q$  by trigonometric polynomials, and we also need to prove that a piecewise continuous function can be approximated in the  $\mathcal{L}^2$  metric by a continuous one; you can try to draw the picture for a discontinuity between rectangular regions to show we can do this. Take my word for it that there is indeed a Weierstrass theorem for squares, and verify that we can join up two discontinuous patches either side of a line discontinuity.  $\square$

Figure 7.8 shows the problem of patching over a fault line, and although it is a little more complicated it can be done.

This means that it is possible to do approximations to functions of two variables on any rectangular region. By composing with suitable functions

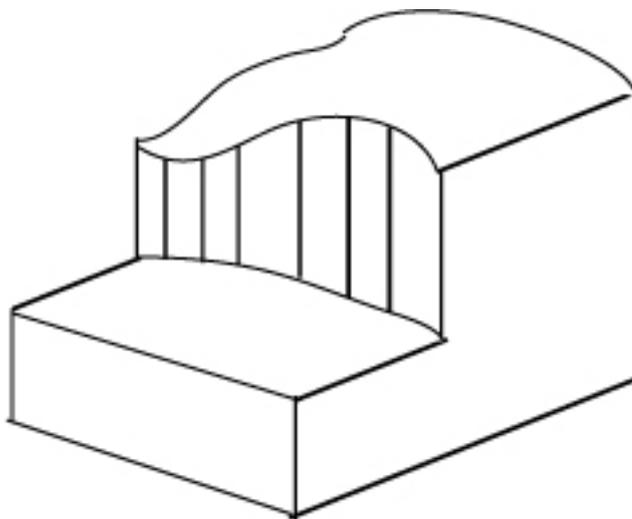


Figure 7.8: A fault line discontinuity

from other regions to rectangles (embeddings almost everywhere) we can get Fourier series on these regions too. For example, discs, using the inverse of the polar coordinate transformation.

Such series are called *double Fourier series*.

And it is a small step to doing it for solid regions of  $\mathbb{R}^3$  with *triple Fourier series*. Again, if  $f : Q \rightarrow \mathbb{R}$  is piecewise smooth, then the Fourier sequence converges to  $f$  pointwise, (not just in the  $\mathcal{L}^2$  metric) in both the two and three dimensional cases.

This is just the start of a big area of Mathematics. The generalisations include doing it for functions defined on all  $\mathbb{R}$  (The Fourier Transform) and using other orthogonal bases besides the Trigonometric functions (generalised Fourier theory). It is not a coincidence that the sine and cosine functions are the solutions to the ordinary differential equation

$$\ddot{x} = -x$$

In fact their orthogonality comes about precisely because they are eigenvectors of a linear operator on an infinite dimensional space. This leads to considering other more complicated linear operators each with their own family of orthogonal functions, although in general we have to take a somewhat different inner product. Check out the Bessel functions in Mathematica.

The Matlab toolkit for doing image processing can be explored by those having access to it.



# Chapter 8

## Partial Differential Equations

### 8.1 Introduction

I shall assume that you have encountered Ordinary Differential Equations or ODE's for short, and that you have grasped that they are central to much of the science and technology that have changed the world in the last few centuries. Now you get to find out why they are called 'Ordinary'.

A crucial feature of setting up an ODE or a system of ODE's is that we have two elements. One is the local law of dynamics which says very generally how things tend to change *locally*. The other is the boundary condition, often an initial value telling where something starts. Putting these together, we get a 'solution' that is, say, the particular time development of a system.

There is an obvious generalisation of ODEs to the situation where instead of something varying in just one dimension, time in many cases, it can vary in two (or more) dimensions. A solution to an ODE is a curve (usually the path in state space of some system as time changes). A solution to the more general problem and to a *Partial Differential Equation*, PDE for short, would be some surface (or higher dimensional manifold) sitting in a space. We expect the crucial two features to remain the same: there will be some local law for the system and some boundary conditions which select a particular solution from an infinite family of them.

**Example 8.1.1.** I take a loop of wire and twist it about a bit. Then I dip it in soap solution and get a nice soap film in the wire. If I hold the wire up in  $\mathbb{R}^3$ , there is a function defined over the 'shadow' of the loop and the film, which tells us the height of the soap film everywhere. More generally, I have

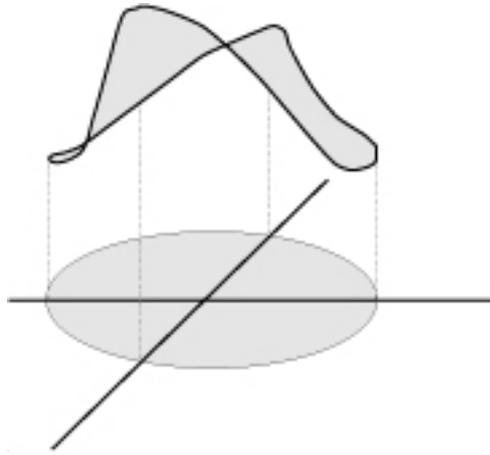


Figure 8.1: Blowing Bubbles.

a function from  $S^1$  into  $\mathbb{R}^3$  which embeds the circle in three-space, and this extends to a function from the unit disk,  $D^2$  into  $\mathbb{R}^3$ .

The illustration of figure 8.1 shows you the possibilities.

The soap film extension is only one among an infinite number of possible extensions (blow on the film to distort it to get some others), the question is, what made the film choose the particular shape it did? The answer is that *surface tension* was busy trying to minimise the area, given the boundary. Now this is a purely local thing, like a vector field, while the surface that you actually get is a global solution. The shape of the boundary wire is the boundary condition. So there ought to be a way of setting up something that is a generalisation of an ODE and finding a way to solve it which would give a solution to the soap film problem.

There is indeed a whole body of Mathematics dedicated to precisely this sort of problem and its higher dimensional analogues, and it is called the study of Partial Differential Equations. Just as ordinary differential equations have differentiation of the time or some other single variable because the solution is a curve, so the PDEs have partial derivatives occurring in them because the solution will be a surface or some higher dimensional manifold. It is more complicated than ODE theory for several reasons, one of them being that the boundary of a curve is just a pair of points (unless the curve is closed, when it doesn't have a boundary), whereas the boundary of a two dimensional thing like a disk is a circle, which is a lot more complicated than a couple of points.

Actually, most PDEs are so hard we don't have the foggiest ideas about how

to solve them<sup>1</sup>, we can only do a few easy ones. But those we can solve are very, very, useful. In the remainder of the course I can only start on the subject, but I shall try to see that you get a feel for the basics.

**Example 8.1.2.** I take a solid ball of iron and sit it on a table. Everything is at room temperature. Now I heat up the table just under the ball by applying a blow-torch, the temperature of which is rather a lot higher than room temperature, say  $1000^{\circ}$ . How does the temperature of the interior point  $(x, y, z)$  of the solid ball change in time? It obviously starts off at room temperature at time zero, and then goes up fairly fast, and the closer  $(x, y, z)$  is to the blow torch, the faster it goes up. It would be nice to have some details: leaving it to the fluffiness of natural language is not good enough for scientists. The answer would be a function of four variables,  $x, y, z$  and  $t$ . If we could obtain such a function and confirm its correctness by experimenting, we should undoubtedly feel we understood a fair bit about heat flow, something which could come in useful.

**Remark 8.1.1.** If I have a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , and if I differentiate it, I get a (row) matrix of partial derivatives,

$$\left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

It makes sense therefore to guess that if there is a (Partial) differential equation the solution to which is a disk or ball mapped into  $\mathbb{R}$ , the the equation itself will have partial derivatives in it. Hence the name.

**Example 8.1.3.** It can be shown that if  $f : D^2 \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is a function which describes the height of a soap film above the  $z = 0$  plane, then provided there are no other forces but surface tension operating, and providing the function  $f$  on the boundary is not too different from a constant function, then  $f$  approximately satisfies the condition

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

or

$$f_{xx} + f_{yy} = 0$$

if this notation is more to your taste.

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<sup>1</sup>Well, closed form or analytic solutions in terms of standard functions are very rare, and even solutions in terms of explicit infinite series are often impracticable. But numerical methods can give us a solution to high accuracy in many cases. Determining whether the numerical solution is a stable, safe one is still under investigation.

If you are told that  $f : S^1 \rightarrow \mathbb{R}$  is a particular function, and that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

then the instruction ‘solve for  $f$ ’ means to find the unique (you hope) function  $f : D^2 \rightarrow \mathbb{R}$  which has this property and is as specified on  $S^1 = \partial D^2$ .

**Remark 8.1.2.** PDE problems where we know relationships such as this locally, and we are given all values of a function on a boundary and want to find the function on the interior, are called *Dirichlet Problems* after the man who made a speciality of tackling them in the nineteenth century. This, incidentally, was the first bloke to work out what a function is. The function evil  $f$  which is 1 on the irrationals and -1 on the rationals was his idea. He was German, not French as the name suggests. He was born in 1805, so this is all recent stuff, only a century and a half or so old<sup>2</sup>.

Another kind of problem, not a Dirichlet problem but related is:

**Example 8.1.4.** If I heat up one half of a copper rod to  $100^\circ$  and keep the other half at  $0^\circ$  while doing so (try not to think about the midway point) and then take away the freezer and the flame, the function of length giving the temperature will start off as a step function and gradually even out until the bar is a nice  $50^\circ$  everywhere, assuming no heat is lost to the outside world. Given information about how heat is conducted through the material, we ought to be able to compute the function at any time after  $t_0$ . We think of time as the positive reals, so we know the value of the function  $\theta(x, t)$  completely at  $t = 0$ , the step function, where  $\theta$  is the temperature, and it is known that the *Heat Equation* must be satisfied:

$$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2}$$

So again we have a partial differential equation.

Partial Differential Equations then occur quite naturally as ways of describing Physical systems. We have two jobs to do:

- From a physical situation, set up the equation which describes the system

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<sup>2</sup>You may reasonably suspect that this is a joke. On the other hand, most of what you did in first year was known to Newton in 1695 when he had more or less given up on Science and Mathematics as less important than Theology. The first artificial satellite had been invented by Newton many years before. It took the Engineers about three hundred years to catch up. Seen from that point of view, you are doing quite well.

- Solve the equation

We next consider some simple cases of the first part, setting up the equation. I shall defer considering how to actually solve them for some time. The cases we look at will be very simple and may give you the completely erroneous impression that mathematicians are interested only in simple things like heat conduction along a rod and vibrating strings. The reason we are looking at simple cases is essentially the same reason as you do not give a three month old baby a nice steak dinner to eat. It has neither the teeth to bite into it nor the digestion to absorb it. So you give it squishy stuff instead that doesn't need big teeth or a strong jaw. Once you have shown you can chomp through the easy cases, you will be ready to chew on the more interesting problems.

## 8.2 The Diffusion Equation

### 8.2.1 Intuitive

In this subsection I am going to give you a loose, intuitive, sloppy approach, as done by all the best engineers and mathematicians of the eighteenth century. In the next subsection I shall do it in a more respectable algebraic manner, so as to guarantee intellectual respectability. Some people worry about both these things.

Imagine, then, a long tube closed at both ends and containing a large number of bees which were put in at one end before it was closed. If  $x$  is used to measure the distance down the tube,  $t$  is the time, let  $f(x, t)$  measure the density of the bees at location  $x$  and time  $t$ . To get the density of the bees at a point, we take a little bit of tube of length  $\Delta x$  centred on  $x$ , count the number of bees, and divide by  $\Delta x$ . Then we take the limit as  $\Delta x$  gets closer to zero. Anyone who objects to anything as silly as this on the grounds that the answer will almost always be zero, and that bees take up some space and aren't points, is simply refusing to enter into the spirit of things and will lose out on some innocent fun.

If we put the origin, 0, at the left hand end of the tube, let  $N(x, t)$  be the number of bees between location 0 and location  $x$  at time  $t$ . Then we have that :

$$f(x, t) = \frac{\partial N(x, t)}{\partial x}$$

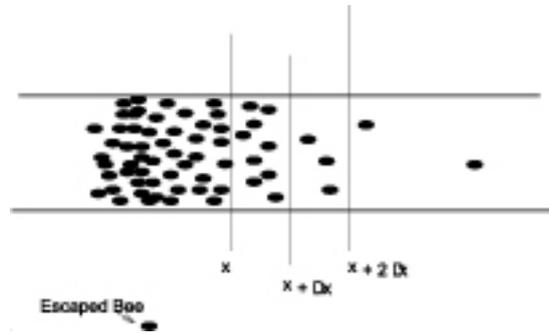


Figure 8.2: Dynamics of a swarm of bees.

Now look at the bees in some such small slab, as shown in figure 8.2. We suppose that the bees move about at random, quite independently except that possibly they may bounce off each other if they collide. They are just as likely to be going one way as another at any time, and they buzz around in the way that bees, atoms and small children at parties are prone to do.

It is fairly plausible that the number of bees going from the slab between  $x$  and  $x + \Delta x$  into the slab to the right of it, from  $x + \Delta x$  to  $x + 2\Delta x$ , over any time interval from  $t$  to  $t + \Delta t$ , is proportional to the difference between the number of bees in the two slabs. The actual number of bees will depend on such things as the mean bee velocity, but if half the bees are going one way and half the bees are going another, then there will be approximately  $\Delta x * f(x)/2$  bees going right across the barrier and  $\Delta x * f(x + \Delta x)/2$  going to the left from the second slab, if the bees are going fast enough.

The rate of flow of bees then past a point  $x$  will be simply proportional to the rate of change of density at  $x$ ,  $\frac{\partial f}{\partial x}$ . If the density is increasing, the bees will tend to go backwards, so  $N$  will tend to increase and we can write:

$$\frac{\partial N}{\partial t} = c^2 \frac{\partial f}{\partial x}$$

where  $N$  is the number of bees between 0 and  $x$ , and  $c^2$  is a positive constant telling us something about the mobility of the bees.

Now we have that  $N$  is of course related to  $f$ , in fact  $f$  is the space derivative of  $N$ ,  $f = \frac{\partial N}{\partial x}$ . We therefore try to use these facts to say something about the change of *density* in time.

Differentiating the last equation partially with respect to  $x$ , we get:

$$\frac{\partial}{\partial x} \frac{\partial N}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2}$$

and reversing the order of partial differentiation, which is OK if the function  $f$  is continuously differentiable, we get

$$\frac{\partial}{\partial t} \frac{\partial N}{\partial x} = c^2 \frac{\partial^2 f}{\partial x^2}$$

and given that we recognise the definition of  $f$  lurking in the equation we can finish up with:

$$\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2} \quad (8.1)$$

This equation, 8.1, is known as the *Diffusion Equation* in one dimension. We can confirm the reasonableness of it as a description of heat, atoms and even, to a crude approximation, bees, by experiment and argument. Experiment is more convincing to everybody except theologians and philosophers, and gives the expected answers. If you are a whiz programmer, you can set up a program where there are a number of slabs next to each other, say  $A, B, C, \dots$  and there are some number of bees at time zero in each slab, say  $N_A, N_B, N_C, \dots$ . The rules are that that there is a jump to time 1 during which each bee makes a random choice between moving into the preceding slab (or vanishing if there isn't a preceding slab), moving into the following slab (or vanishing if there isn't one), or simply staying where it is. The probabilities of going left or right are equal. Now iterate the process for some initial distribution of the bees in the slabs and watch what happens<sup>3</sup>. Eventually all the bees leave. If you want you can make it circular so there is a conservation of bees, or you can treat the end points differently. You are doing a discrete simulation of the diffusion equation, which is pretty reasonable for bees. Note that it ties in with a probabilistic 'random walk' model. Note that the continuous approximation for bees is fundamentally daft but still works, and it works even better for atoms, such as semiconductor dopants diffusing into silicon. Engineers use this in designing transistors<sup>4</sup>.

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<sup>3</sup>This is crying out to be made into a Mathematica animation. I hope someone with more time than me can do it and send me the result.

<sup>4</sup> A transistor is not, as you may have supposed, a kind of radio. It is actually the thing inside it that allows the radio to work. It has been extended to the silicon chip in relatively recent years.

The chain of reasoning I have given is pretty much what the eighteenth century mathematicians did to justify the diffusion of heat along a rod, the main difference being they said it in French and left out the bees <sup>5</sup>.

Bees are reasonably well described by the Diffusion Equation, but so are a lot of other things, including heat conduction (which is largely a matter of vibrating atoms), and hence the diffusion equation is also known as the *Heat Equation*. The diffusion of gases through pipes and atoms of one substance in another, from dyes in water to doping agents in silicon, are also described by the same equation. Bees are easier to visualise, but perhaps not so important in the grand scheme of things as heat conduction or atoms. Much depends on your point of view.

The next stage of development of the argument is to consider a thin planar slab of bees, which can now move in two dimensions instead of being compelled to go either backwards or forwards. And the final stage for most books is to go to the full three dimensional case, where the bees can float free.

In order to treat the two and three dimensional cases it is necessary to consider the space,  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , to be decomposed into little squares or boxes in a manner which is by this time rather familiar to you.

## 8.2.2 Saying it in Algebra

Watch me like a hawk here. This is tricky but cool.

Suppose we have a three dimensional space and that there are bees flying around in it. Let  $T(\mathbf{x}, t)$  denote the density of the bees at location  $\mathbf{x}$  at time  $t$ . Let  $U$  denote a region of the space (think of a solid ball shaped region if you want to visualise this), then by definition the number of bees inside the region  $u$  at time  $t$  is just:

$$\int_U T(\mathbf{x}, t) dV$$

---

<sup>5</sup>It is remarkable that the French did such a lot of the mathematics of this subject, but you don't know the half of it. Most of them weren't mathematicians, they were lawyers, medics, engineers and blokes who, generally speaking, did it for fun in the evenings after a hard day's work. (Gauss, who was not French, was a privy councillor. If you know what a privy is, you are doubtless wondering how you counsel them, but this is your problem.) You have to have a fairly high IQ to think that this sort of thing is entertaining, but it was thought to be the sort of activity which reflective gentlemen should do. In England there weren't any reflective gentlemen, the gentlemen were horsing around killing foxes, dressing up in silly clothes and fancy hair-dos, and gambling. Of course, they didn't have television in those days.

The flow of bees flying into the region  $U$  at time  $t$  is, by definition,

$$\frac{\partial}{\partial t} \int_U T(\mathbf{x}, t) dV$$

Each bee has to fly through the boundary of  $U$  to get into  $U$ . The gradient field of  $T$  gives us the direction in which the density of bees is increasing, bees will fly down the gradient just as in the one dimensional case. So the rate of flow of bees into  $U$  is just

$$\int_{\partial U} c^2 \nabla T \cdot \mathbf{n} dA$$

for some positive constant  $c^2$ . By the Gauss Divergence Theorem, this can be written as:

$$c^2 \int_U \nabla^2 T dV$$

Equating the two expressions for the flow of bees into  $U$  we get:

$$\frac{\partial}{\partial t} \int_U T(\mathbf{x}, t) dV = c^2 \int_U \nabla^2 T dV$$

and interchanging the partial derivative with the integral:

$$\int_U \frac{\partial T}{\partial t} - c^2 \nabla^2 T dV = 0$$

If the integral of a continuous function  $f$  over every region  $U$  is zero, then  $f$  must be zero. Suppose it weren't zero at some point  $\mathbf{x}$ . Then it must be non-zero in some little region around  $\mathbf{x}$ , and if  $f(\mathbf{x}) > 0$ , take  $U$  to be the region around  $\mathbf{x}$  where  $f$  is positive. Then  $\int_U f > 0$ , contradiction. Likewise if  $f(\mathbf{x}) < 0$ . It follows therefore that

$$\frac{\partial T}{\partial t} = c^2 \nabla^2 T \tag{8.2}$$

which is the diffusion equation in three dimensions. Note that the argument works for dimension two with minor changes.

Now we do it for heat. Let  $T(\mathbf{x}, t)$  denote the temperature of a point  $\mathbf{x}$  at time  $t$  in some region of  $\mathbb{R}^3$ . This is a function  $T : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ . It gives rise to a gradient vector field on  $\mathbb{R}^n$  which will change in time. We write this as  $\nabla T$ . It matters, because heat rolls down the temperature hill.

If we fix, again, some definite region  $U$ , the amount of heat in the region  $U$  in  $\mathbb{R}^n$  is given by a simple rule: the specific heat of a solid is the amount of heat it takes to raise a unit mass of the solid by a temperature of  $1^\circ$ , so in a region  $U$  if we assume the specific heat and the density are constants,  $\sigma$  and  $\rho$ , we conclude that the heat in the region  $U$  at time  $t$  is given by

$$H_U = \int_U \sigma \rho T(\mathbf{x}, t) d\mathbf{x}$$

and we can regard the heat flow into  $U$  as

$$\frac{dH_U}{dt} = \int_U \sigma \rho \frac{\partial T(\mathbf{x}, t)}{\partial t} d\mathbf{x}$$

$H_U$  is, for a given box  $U$ , just a function of time <sup>6</sup>.

Heat flows into the box  $U$  down the temperature gradient at a rate proportional to the conductivity of the material,  $K$  say, and the gradient of the temperature,  $\nabla T$ , at some point  $\mathbf{x}$ , is in the opposite direction to the heat flow. If we want to get the vector telling us the rate of flow of heat at the point  $\mathbf{x}$  at time  $t$ , we can call it  $\mathbf{v}$  and write

$$\mathbf{v} = -K\nabla T$$

Now the rate of flow of heat *out of* the box  $U$  is going to be

$$\int_{\partial U} \mathbf{v} \cdot \mathbf{n}$$

where  $\mathbf{n}$  is the outward normal, which is equal, by the Divergence theorem to

$$\int_U \operatorname{div}(\mathbf{v}) = -K \int_U \nabla \cdot \nabla(T)$$

This succession of  $\nabla$ s is written, with a rather shaky excuse, as  $\nabla^2$ , as before. It is clear that  $\nabla^2 f$  is shorthand for, in the case of two variables  $x$  and  $y$ ,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

---

<sup>6</sup>It might be a good idea to think of the amount of heat as the number of bees and the temperature as the bee density, with some constants thrown in.

We may therefore equate the heat flow *into* the box,  $dH_U/dt$  to the temperature  $T$  in two different ways:

$$\frac{dH_U}{dt} = K \int_U \nabla^2 T = \int_U \sigma \rho \frac{\partial T}{\partial t}$$

Or to put it another way,

$$\int_U \left[ K \nabla^2 T - \sigma \rho \frac{\partial T}{\partial t} \right] = 0$$

Since this holds for all  $U$ , the function inside the brackets must be the zero function, and so we get the general heat equation:

$$K \nabla^2 T = \sigma \rho \frac{\partial T}{\partial t}$$

where  $\sigma$  is the specific heat,  $\rho$  is the density of the material and  $K$  is the conductivity of the material. This is just the diffusion equation, but you have some information about the (positive) constants and the properties of materials. Whether you prefer to think of temperature or bee density is entirely optional.

### 8.3 Laplace's Equation

In the case where there is no heat supplied to or leaving the object, and the system is in a steady state, we get the famous equation of Laplace:

$$\nabla^2 T = 0$$

which for a general function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  can be written at greater length as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad (8.3)$$

This equation also applies to a large number of other situations: in two dimensions it applies, to a good approximation, for nearly flat surfaces, to soap films, it also applies to the electric field produced by a set of point charges except at the charged points themselves, to gravitational fields similarly, and hence has importance in dynamics. Wherever there is some sort of minimum energy configuration there is often a function satisfying Laplace's Equation, or some equation approximated by Laplace's Equation, describing the state.

**Example 8.3.1.**

The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = x^2 - y^2$  satisfies Laplace's Equation everywhere. The function with  $f(x, y) = x^2 + y^2$  does not. Any constant function does, and if  $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is linear and invertible, and  $f$  satisfies Laplace's Equation, so does  $f \circ g$ .

**Exercise 8.3.1.**

1. Show that  $f(x, y) = x/(x^2 + y^2)$  satisfies Laplace's Equation where it is defined. Sketch a portion of the graph.
2. Show that  $f(x, y) = \sin(x) \cosh(y)$  satisfies Laplace's Equation. Sketch a portion of the graph.
3. Find two more functions which satisfy Laplace's Equation on some region in  $\mathbb{R}^2$  and two which do not.

It is useful to think of  $\nabla^2$  as an operator which takes a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  to a new function,  $\nabla^2 f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then we want to know, what functions get killed by the Laplacian Operator  $\nabla^2$ ? That is, which functions  $f$  get sent to the zero function by  $\nabla^2$ ?

Functions which satisfy Laplace's Equation are known as *Harmonic Functions* and the study of Harmonic Functions is called *Potential Theory*. Many mathematicians have spent the best years of their lives finding out things about harmonic functions, mostly just from curiosity, but the results are often very handy to engineers, so I shall mention a few of them here.

First, we can see immediately from the divergence theorem that

$$\int_U \nabla \cdot \nabla(f) = \int_{\partial U} \nabla(f) \cdot \mathbf{n}$$

and that  $\nabla(f) \cdot \mathbf{n}$  is just the derivative of  $f$  in the direction  $\mathbf{n}$ . So for a harmonic function, these are always zero. We use the notation

$$\frac{\partial f}{\partial \mathbf{n}}$$

for  $\nabla(f) \cdot \mathbf{n}$

If you think about what this means in dimension 2, you can see that if you draw any simple closed curve in the plane, and integrate the slope of any

harmonic function  $f$  along the outward normal around the curve, you have to get zero. It follows immediately that a function such as  $x^2 + y^2$  is not harmonic, since the unit circle centred at the origin has got  $\partial f / \partial \mathbf{n}$  a positive constant. It suggests that you might be luckier with something like  $xy$  or  $x^2 - y^2$  which at least has the right sort of behaviour at the origin. We can get a little more mileage out of this by making it a little more complicated: if we look at two functions,  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  we can define  $\mathbf{F} = f \nabla g$ . This multiplies the vector field  $\nabla g$  by the value of the function  $f$  at each point. Now by straightforward manipulations:

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot (f \nabla g) = f \nabla^2 g + \nabla f \nabla g$$

and

$$\mathbf{F} \cdot \mathbf{n} = \mathbf{n} \cdot (f \nabla g) = f \frac{\partial g}{\partial \mathbf{n}}$$

for any vector  $\mathbf{n}$ . If we take some region  $U$  and apply the divergence theorem to  $\mathbf{F}$ , we get:

$$\int_U (f \nabla^2 g + \nabla f \nabla g) = \int_{\partial U} f \frac{\partial g}{\partial \mathbf{n}}$$

for  $\mathbf{n}$  the normal to the boundary, which equation is called Green's First Identity.

Repeating this with the functions in the reverse order, to the vector field  $g \nabla f$  we get

$$\int_U (g \nabla^2 f + \nabla f \nabla g) = \int_{\partial U} g \frac{\partial f}{\partial \mathbf{n}}$$

Subtracting this from Green's First Identity we get

$$\int_U (f \nabla^2 g - g \nabla^2 f) = \int_{\partial U} \left( f \frac{\partial g}{\partial \mathbf{n}} - g \frac{\partial f}{\partial \mathbf{n}} \right)$$

This is called Green's Second Identity. These are useful in Fluid Mechanics for those with good memories.

Now putting  $f = g$ , a harmonic function, in the first identity we get

$$\int_U (\nabla f)^2 = \int_{\partial U} f \frac{\partial f}{\partial \mathbf{n}}$$

Now suppose  $f$  is zero on  $\partial U$ . Then the right hand side is zero, and so the left hand side is zero too. Hence  $\nabla f = 0$  and so  $f$  is constant. Since it is zero on the boundary,  $f = 0$ .

This tells us that if  $f$  is harmonic and is zero on the boundary of a region, it is zero throughout the region.

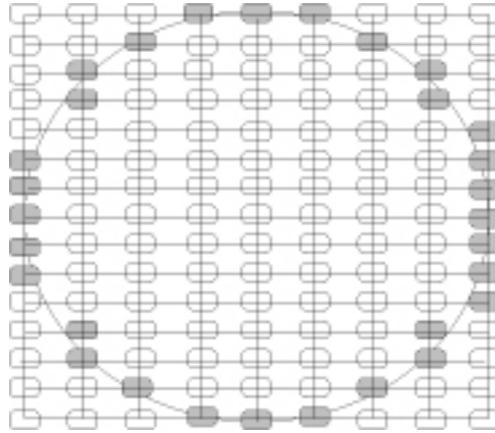


Figure 8.3: Numerical Computation of Laplace's Equation on a Systolic Array

If  $f, g$  are two harmonic functions on  $U$  which agree on  $\partial U$ , then their difference is zero on the boundary, so their difference is zero everywhere, so the functions are equal. In other words:

**Theorem 8.1.**

*The solution to a Dirichlet problem for Laplace's Equation is unique if it exists.* □

A third nice fact about Harmonic functions is that if you take a point  $\mathbf{x}$  and look at the value at  $f(\mathbf{x})$  and then take a sphere centred on  $\mathbf{x}$  and integrate the value of  $f$  over the sphere, then the result is always equal to the value of  $f(\mathbf{x})$  multiplied by the area of the sphere. In two dimensions, it is a circle instead of a sphere, and the perimeter of the circle is what we must multiply by. A simpler way to put this is that the average value on the boundary of a ball is the same as the value at the centre of the ball. This result gives a nice parallel algorithm for finding a solution to the Dirichlet Problem for Laplace's Equation in the plane:

Take a circle and suppose  $f$  is known on the circle and we want to extend it to the disk. Make a grid of processing elements, each joined to its nearest neighbours as in fig. 8.3. The elements which intersect the circle are shaded: put in each element a number giving the value of the function  $f(x, y)$  at that point. Put random numbers in the processing elements inside the circle, and ignore any elements outside the circle.

An iteration of the system takes any element in the interior of the disk and

replaces the value inside it by the average of the values of its four neighbours. Elements on the circle itself have the numbers left unchanged. We simply iterate this process. Eventually the numbers on the inside stop changing, and when they do the grid gives a discrete approximation to a solution to Laplace's Equation. Anyone who likes programming can fake the parallel processing on a PC. If you don't like the precision, do it with more processing elements. If you want to generalise to something more complicated than a circle, the general principle is clear, and if you want to increase the dimension, it is easy to see how to modify the algorithm.

Real hardware parallel machines ('Systolic Arrays') have been built at Stanford and Carnegie-Mellon Universities. Students of Robotics at this University have used the method in software for finding trajectories which avoid obstacles. You can try this for the problems in the next section to get out numerical solutions. They are not as neat as analytic solutions (in terms of functions) of course.

**Exercise 8.3.2.**

Show that the only harmonic functions on the line are affine (linear plus a shift) and confirm the third nice fact for this case.

Confirm the third nice fact by integrating around the circle of radius  $r$  centred on  $(a,b)$ , for the function  $f(x, y) = xy$ .

## 8.4 The Wave Equation

**Remark 8.4.1.** One of the earliest applications of PDEs is to the study of vibrating systems. As with bees and heat, I look at the simplest case first. This is done out of kindness for your soft mathematical gums and not because it is the most interesting.

Suppose we have an elastic string suspended at both ends. If anyone deforms the string and lets it go at time  $t = 0$ , the string will oscillate. At any later time  $t$  there is some function specifying the shape of the string, constrained to take fixed values at the end points. So the displacement vertically of a horizontal string can be written  $f(x, t)$ , for  $x \in [a, b]$ ,  $t \in \mathbb{R}^+$ .

What can we say about the local dynamics of the string? Naturally we shall put in some simplifying assumptions such as zero air resistance and internal damping, so the string will be assumed to be perfectly elastic. It is fair to hope that over short times the flagrant falsity of the assumptions will not

affect the general behaviour significantly. I also assume each point of the string moves only vertically.

Let  $\rho$  denote the density of the string, supposed constant, and let  $T(x, t)$  denote the tension in the string. Let  $H(x, t)$  and  $V(x, t)$  denote the horizontal and vertical components of this tension. The assumption that the string moves only vertically ensures that  $H$  is constant. The acceleration of a small bit of string of length  $\Delta x$  at location  $x$  at time  $t$  is, by definition,

$$\frac{\partial^2 f}{\partial t^2} \triangleq f_{tt}$$

This by Newton's Law is the force divided by the mass,

$$\forall t \in \mathbb{R}^+ \forall x \in [a, b], \quad \rho \Delta x f_{tt} = V(x + \Delta x, t) - V(x, t)$$

or more perspicuously:

$$\forall t \in \mathbb{R}^+ \forall x \in [a, b], \quad \frac{V(x + \Delta x, t) - V(x, t)}{\Delta x} = \rho \frac{\partial^2 f}{\partial t^2}$$

Taking limits we obtain:

$$\forall t \in \mathbb{R}^+ \forall x \in [a, b], \quad \frac{\partial V}{\partial x} = \rho \frac{\partial^2 f}{\partial t^2}$$

Now  $V(x, t) = H(x, t) \tan(\theta)$ , where  $\theta$  is the angle made by the string at  $(x, t)$ . That is

$$\forall t \in \mathbb{R}^+ \forall x \in [a, b], \quad V(x, t) = H \frac{\partial f}{\partial x}$$

where  $H$  is the (constant) horizontal component of  $T$ . Substituting for  $V$  above we obtain:

$$\forall t \in \mathbb{R}^+ \forall x \in [a, b],$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\rho}{H} \frac{\partial^2 f}{\partial t^2} \tag{8.4}$$

Equation 8.4 is the wave equation.

**Remark 8.4.2.** I shall defer actually solving it generally until later, but curiosity might make us want to see if we could solve the problem:

**Example 8.4.1.** Take a steel string  $\pi$  units long and fixed at the end points at height zero. Suppose its density per unit length is  $\rho$ , given, and its tension

is set at some value, say by hanging a kilogram weight off one end prior to fixing it. Twang it in the middle. What pitch would you get?

**Solution:** If you reflect briefly on the fact that you might hope to solve this problem getting an answer in cycles per second having measured the length in metres, the density in kilograms per meter and the tension in kilograms, you will see why Mathematics was known in some quarters as Greek Magic.

To try to solve the above problem, imagine the simplest possible solution of the wave equation. I incline to think that if we could freeze the wire at its maximum amplitude it would look, if we took the origin in the middle, rather like  $\cos(x)$ . This would allow it to be about the least complicated shape that had the ends at  $\pm\pi/2$  fixed at value zero. As time changed, the wave would flatten down to zero then turn upside down. We can get this effect by multiplying by  $\sin(2\pi\omega t)$  where  $\omega$  is the frequency in cycles per second. This suggests a trial solution to be:

$$f(x, t) = \cos(x) \sin(2\pi\omega t)$$

Differentiating partially twice for  $x$  we get

$$f_{xx} = -f(x, t)$$

and differentiating twice partially with respect to  $t$  we get

$$f_{tt} = -f(x, t)(4\pi^2\omega^2)$$

So the wave equation is satisfied with

$$\frac{1}{4\pi^2\omega^2} = \frac{\rho}{H}$$

which gives

$$\omega = \frac{1}{2\pi} \sqrt{\frac{H}{\rho}}$$

If the length of the wire were different, we should simply scale it by changing the units.

**Exercise 8.4.1.** Find the fundamental frequency of vibration of a wire half a metre long with a density of 0.033 kilograms per metre with a tension of ten kilograms.

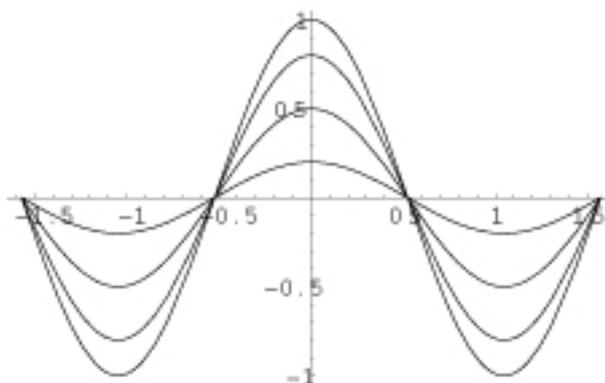


Figure 8.4: Another solution to the vibrating string

**Remark 8.4.3.** You ought to take time off to think about the fact that you can actually compute an answer to this. On the one hand a friend does the experiment. He or she gets some wire and measures the weight of a cut length. They then hang a known weight off one end, then drive a peg in to stop one end moving, the other end already being fixed. Now they twang it. He or she connects up a microphone to a CRT oscilloscope and measures the frequency, or listens to it and compares it with tuning forks. The output is a number to some accuracy.

You, meanwhile, take the numbers they have given you and perform certain acts of writing down squiggly marks on paper. Then you do some arithmetic, and bingo, you too come out with a number and can tell your friend the readings of the oscilloscope. How come this amazing relationship between the squiggled marks on the paper and the physical set-up? That it works is well known. That it is amazing that it works is something you might not have thought about, but surely it is incredible.

**Remark 8.4.4.** It is clear that there are other solutions to how the string can move: if we placed a finger about a third of the way along we might plausibly force a vibrating string to give a solution like figure 8.4 where the vertical amplitude has been shown (much exaggerated) at four different times. Some reflection will show a discrete but infinite set of plausible solutions, looking rather like Fourier terms. The one sketched is

$$\cos(3t) \sin(2\pi\omega t)$$

More generally we can get solutions

$$a_n \cos((2n + 1)t) \sin(2\pi\omega t), \quad n \in \mathbb{Z}$$

It is also the case that the sum of any solutions is also a solution. In other words there is a whole infinite dimensional vector space of solutions, and we have looked only at some basis elements of the space. I leave you to brood on this.

**Exercise 8.4.2.** Prove the claim that the set of solutions is a linear space.

**Remark 8.4.5.** We can imagine the problem of drumming: I take a thin membrane and attach it to a rigid circle or maybe a square. It is a 2-dimensional version of the string. Now I give it a good smack in the middle. What pitch is the resulting sound? In order to work the answer out, I should need to have a two dimensional version of equation 8.4. Can we make a stab at setting up a 2-d wave equation?

**Remark 8.4.6.** I might make a guess at:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{u^2} \frac{\partial^2 f}{\partial t^2} \quad (8.5)$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 f}{\partial t^2} \quad (8.6)$$

in three dimensions, for some constant  $u$ .

It is beyond the scope of the course to deal with these, but you might like to experiment to see if you can persuade yourself that these are plausible equations for describing waves in two and three dimensions.

## 8.5 Schrödinger's Equation

Since this is not on the syllabus I shall just mention that Quantum Physics leads to the study of Schrödinger's Equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - V(x, y, z, t) = k \frac{\partial \psi}{\partial t}$$

where  $V$  is a potential field. This equation gives a description of the state of a single particle. How it was set up remains rather mysterious, but having got it, courtesy of Schrödinger, we can check to see if it works.

Calculating the possible solutions to this equation for a given potential function gives results generally analogous to the distinct wave solutions to the

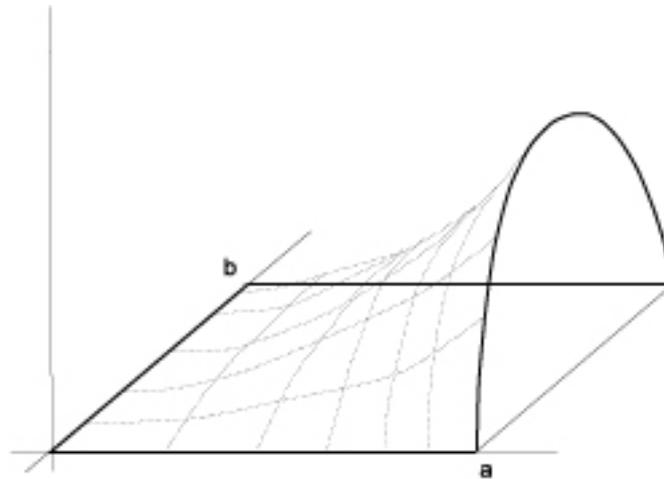


Figure 8.5: Soap Film on a rectangular wire

vibrating string. They form a discrete set which look a little bit like the terms in a Fourier series.

This has a good deal to do with the fact that the energy levels of electrons in an atom take discrete values which in turn has a good deal to do with the periodic table of elements and also with the spectral lines which are seen when looking at diffracted light.

**Remark 8.5.1.** This should give the (correct) impression that partial differential equations are rather important when trying to understand how the universe works.

**Remark 8.5.2.** Having said something about the setting up of the classical PDEs, (and having mentioned one modern one) we turn now to the issue of solving them.

## 8.6 The Dirichlet Problem for Laplace's Equation

**Remark 8.6.1.** We cannot hope to solve the general Dirichlet Problem for Laplace's Equation, but we shall treat a few simple cases.

Suppose we take a rectangle in the plane, and lift up one of the sides of the rectangle by a function. See figure 8.5. It will be useful to think of it as a

wire frame. We are going to find the equation of the soap film which will be formed when the whole thing is dipped in soap<sup>7</sup>.

Formally, we have  $0 \leq x \leq a, 0 \leq y \leq b$  as the region  $U$ , We have that there is some unique function  $f : U \rightarrow \mathbb{R}$  which is unknown, but that we have:

1.  $f(x, 0) = 0, \forall x \in [0, a]$
2.  $f(0, y) = 0, \forall y \in [0, b]$
3.  $f(x, b) = 0, \forall x \in [0, a]$
4.  $f(a, y) = h(y), \forall y \in [0, b]$
- 5.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

for some given function  $h$ . I have illustrated  $h$  in figure 8.5 with a nice parabolic function, but let us keep  $h$  general at the moment. The problem is to find  $f$ , the soap film function. I remind you that this is not being done because we care about soap, but because very much more significant problems can be done using the same methods, and it is useful to have clear pictures of a simple sort in your mind.

The first thing we do is make an assumption which is not immediately justifiable or even reasonable, but which actually works.

### Separation of Variables

**Suppose that the function  $f(x, y)$  can be written as a product of functions of  $x$  and  $y$  separately.**

Write

$$f(x, y) = p(x)q(y)$$

Then differentiating partially with respect to  $x$  gives

$$\frac{\partial f}{\partial x} = q(y) \frac{dp}{dx}, \quad \frac{\partial^2 f}{\partial x^2} = q \frac{d^2p}{dx^2}$$

---

<sup>7</sup> I am simplifying here: the Partial Differential Equation for Soap films or area minimisation is non-linear; the general problem for solving it for given boundary conditions is known as *Plateau's Problem*. The PDE is approximated well by Laplace's Equation provided the non-linear effects are small, which will happen if the function  $f$  is not too different from an affine function, and solving Laplace's Equation for a given boundary condition is often a good start on the Plateau Problem. From now on, I shall cheerfully talk of soap films as if they were exactly solved by Laplace's Equation

and similarly

$$\frac{\partial f}{\partial y} = p(x) \frac{dq}{dy}, \quad \frac{\partial^2 f}{\partial y^2} = p \frac{d^2 q}{dy^2}$$

Now Laplace's Equation gives us

$$q\ddot{p} + p\ddot{q} = 0$$

Or

$$\frac{\ddot{p}}{p} = -\frac{\ddot{q}}{q} = c$$

for some constant  $c$ .

Thus we have reduced the partial differential equation to two simultaneous Ordinary Differential Equations,  $\ddot{p} = c p$  and  $\ddot{q} = -c q$ , which we know how to solve<sup>8</sup>.

The boundary conditions for these can be worked out from the boundary values for the original PDE: we have that  $f(0, y) = p(0)q(y) = 0$ , for all the  $y \in [0, b]$ , and since  $q = 0$  will not be a solution at  $x = a$ , we must have  $p(0) = 0$ . Similarly  $q(0) = 0, q(b) = 0$ . We also deduce that  $p(a)q(y) = h(y)$ . You may be coming to feel that what is going to happen is that we are going to have that nice parabola at  $f(a, y)$  simply scaled down progressively to zero as we reduce  $x$  to zero. This seems physically reasonable.

First we solve

$$\ddot{q} = -c q$$

a familiar old face. We recall that the general solution is

$$q(y) = A \sin(\sqrt{cy}) + B \cos(\sqrt{cy})$$

if we suppose  $c$  is positive, and we just swap  $p, q$  otherwise.

Now

$$q(0) = 0 \Rightarrow B = 0, \text{ and } q(b) = 0 \Rightarrow c = \left(\frac{n\pi}{b}\right)^2, n = 1, 2, \dots$$

---

<sup>8</sup> I have used the notation  $\dot{p}$  and  $\dot{q}$  rather casually; we are differentiating with respect to different variables here. I interpret the dot as 'Differentiate with respect to the (single) variable'. Other, sterner, folk insist that we use Newton's dot notation only when the variable is time. I have tried it in other notations, and it is longer and harder to read.

So we get solutions for  $q$  of the form

$$q(y) = A_n \sin\left(\frac{n\pi y}{b}\right)$$

Going back to the equation for  $p$ , we have that

$$p(x) = C \sinh(\sqrt{c}x) + D \cosh(\sqrt{c}x)$$

is the general solution and we know that  $p(0) = 0$  and hence that  $D = 0$ .

But we now know something about  $c$ , from the boundary conditions on  $q$ , so we can conclude that for every positive integer  $n$ , there is a solution in waiting,

$$f_n(x, y) = c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

These 'solutions in waiting' as I have called them, exist for all positive integers  $n$ , and for all constants  $c_n$ , and the sum of any set of them is also a solution-in-waiting. In order to be a real solution, we have to find a sum of them which also satisfy the final remaining boundary condition, they have to agree with the function  $h$ , the parabola in the figure. We cannot reasonably expect the sum of a finite collection of sine functions to be a parabola, but we can get closer and closer— we are just doing Fourier Theory.

This means that

$$f(a, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = h(y)$$

Each Fourier coefficient is  $c_n \sinh\left(\frac{n\pi x}{b}\right)$ , given by

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b h(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

If we can do the integrals, we can calculate the coefficients as far as we like, and in some happy cases we can get explicit solutions.

We can get a reasonable agreement with figure 8.5 if we put  $b = \pi$  and  $h(y) = \sin(y)$ . We therefore work through the following example:

### Example 8.6.1.

#### Problem

Let the harmonic function  $f : [0, 1] \times [0, \pi]$  satisfy the following boundary conditions:

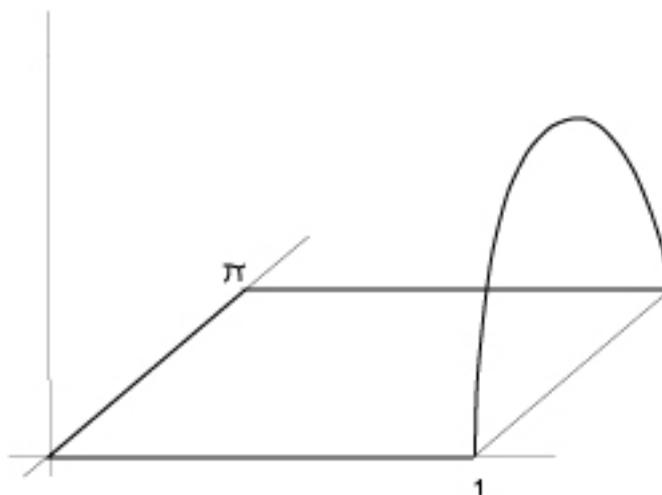


Figure 8.6: Function on the boundary

1.  $f(x, 0) = 0, \forall x \in [0, 1]$
2.  $f(0, y) = 0, \forall y \in [0, \pi]$
3.  $f(1, y) = 0, \forall y \in [0, \pi]$
4.  $f(1, y) = \sin(y), \forall y \in [0, \pi]$

Sketch the graph of  $f$  on the boundary of the rectangle, and calculate the function  $f$  on the interior.

**Solution** The graph is illustrated in fig 8.6.

We suppose that the solution is separable,  $f(x, y) = p(x)q(y)$ .

This tells us that

$$\frac{\ddot{p}}{p} = -\frac{\ddot{q}}{q} = c$$

and we have no way of knowing whether  $c$  is positive or negative until we investigate the boundary conditions, since we could always interchange  $x$  and  $y$  in the problem.

Since we know that at  $x = 1$  we have that  $f(1, y) = \sin(y)$ , we have  $p(1)q(y) = \sin(y)$ , which tells us that  $p(1) = 1$  and

$$q(y) = \sin(y)$$

is a possibility, with  $c = 1$ . In which case,

$$\frac{\ddot{p}}{p} = 1$$

and we have

$$p(x) = A \cosh(x) + B \sinh(x)$$

is a solution in waiting. It, or some (possibly infinite) sum of such solutions, must satisfy the boundary conditions not so far used. These are straightforward: we must have

$$p(0) = 0; p(1) = 1$$

and this immediately tells us that

$$A = 0, B = \frac{1}{\sinh(1)}$$

is a solution. So the final solution is

$$f(x, y) = \frac{\sinh(x)}{\sinh(1)} \sin(y)$$

It is straightforward to verify (1) that the boundary conditions are all satisfied and (2) that the function is harmonic (everywhere). Since we have a uniqueness theorem, we have produced the only possible solution.

### Exercise 8.6.1.

Verify that the given solution satisfies Laplace's Equation.

**Remark 8.6.2.** If you have any soul in you at all, you will now stand up and clap for half an hour at something so wonderful.

### Example 8.6.2.

#### Problem

Let a square of side  $\pi$  units be made of metal, and let three of the four sides be kept at a temperature of  $0^\circ$ . Let the last side have temperature  $\sin(x)$  at distance  $x$  along from one end. Find the temperature at the centre of the plate when the system is in equilibrium.

#### Solution

$$\frac{e^{\pi/2} - e^{-\pi/2}}{e^\pi - e^{-\pi}}$$

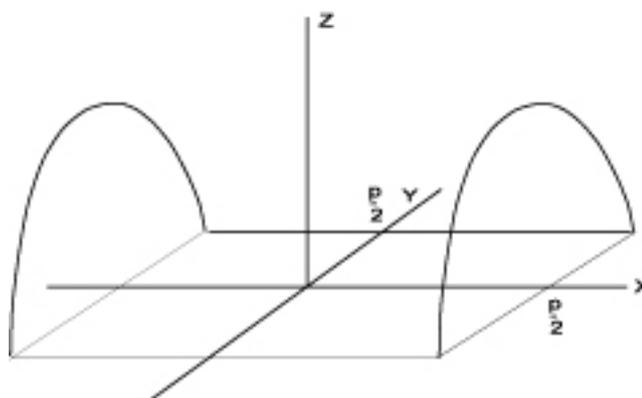


Figure 8.7: A Little Problem

This is essentially the same as the last worked example of course. Only the names have been changed to protect the guilty<sup>9</sup>. There is a scaled version of the last solution because the plate is bigger. I have expanded the sinh function out for those of you who like to see everything in terms of exponentials.

Go on, check it out, *then* clap! It isn't quite as wonderful as Euler's Formula,  $e^{i\pi} = -1$ , but it is pretty damned smart and can be verified by experiment, which is more than can be said for the Euler Formula<sup>10</sup>.

### Exercise 8.6.2.

1. Suppose the function  $f$  of the preceding worked examples had been modified so that it is defined on the interval

$$[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$$

and is zero along two opposite sides and is a cosine function along the other two opposite sides, as in fig 8.7. Find an explicit form for the harmonic function from first principles.

2. You have calculated the Fourier Series for a certain number of functions by now: choose some function where you have the Fourier Series already worked out and use it as an alternative to my function  $h(y) = \sin(y)$  to obtain a Fourier Series solution to your very own soap film problem.

<sup>9</sup>The innocent are not in need of protection, their strength is as the strength of ten, because their hearts are pure.

<sup>10</sup> What is marvellous isn't the answer, it is that somebody of the same species as you was smart enough to figure it out, and you are smart enough to follow the argument. If this doesn't strike you as astonishingly wonderful, you are probably dead but haven't noticed yet.

3. The constraints on the boundary look rather strong, and you might wonder what you could do with a case where one edge was fixed to have height  $h_1(y)$  and the opposite edge was fixed to have height  $h_2(y)$ . Deal with the case where one end of a square of side  $\pi$  is made to have height  $\sin(y)$ , the opposite is made to have height  $-\sin(y)$  and the other two are kept at height zero. Do this by finding solutions to (a) the case where three sides are zero and one side is at height  $\sin(y)$ , which has been done, (b) the case where the opposite side is kept at height  $-\sin(y)$  and all other sides kept at zero and then (c) adding up the answers. After all, if two functions satisfy Laplace's Equation, so does their sum.(!) Of course, if the two functions  $h_1, h_2$  are the same there might be a quicker method, as in our earlier worked example.

## 8.7 Laplace on Disks

Take the map  $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which takes  $(r, \theta)$  to  $(x = r \cos \theta, y = r \sin \theta)$ , otherwise the polar coordinates transformation. Suppose we restrict the map to  $\mathbb{R}^+ \times [0, 2\pi)$  as usual so as to make it one-one onto the plane  $\mathbb{R}^2$  except for a small problem at the origin. If a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies Laplace's Equation, what equation does  $f \circ P$  satisfy? The answer is *Laplace's Equation in Polar coordinates*, and it is worth knowing, because it gives us a chance to do for disks and sectors what we have just done for rectangles.

The map  $P$  has derivative:

$$\begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

We can write:

$$\left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta} \right) \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

Now inverting the matrix (by appealing to the Inverse Function Theorem) we get:

$$\left( \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta} \right) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \begin{bmatrix} \cos \theta & \sin \theta \\ -1/r \sin \theta & 1/r \cos \theta \end{bmatrix}$$

Or more fully:

$$\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta}$$

In particular

$$\begin{aligned} \frac{\partial f}{\partial x} &= \cos \theta \frac{\partial f}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial f}{\partial \theta} \\ \frac{\partial^2 f}{\partial x^2} &= \cos \theta \frac{\partial}{\partial r} \left( \cos \theta \frac{\partial f}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial f}{\partial \theta} \right) \\ &\quad - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial f}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial f}{\partial \theta} \right) \\ \frac{\partial^2 f}{\partial y^2} &= \sin \theta \frac{\partial}{\partial r} \left( \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} \right) \\ &\quad + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} \right) \end{aligned}$$

When these are evaluated and added many terms cancel out and we get:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

Now if the left hand side is zero we get the *Polar Form of Laplace's Equation*:

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} = 0$$

Which you should memorise.

### Exercise 8.7.1.

Complete the above calculation to derive for yourself the Polar form of Laplace's Equation.

Now suppose we have a piece of circular wire bent so that its projection is a circle, as in figure 8.8. The shape indicated can be represented as the graph of a function  $h : S^1 \rightarrow \mathbb{R}$ . We assume again that a function  $f : D^2 \rightarrow \mathbb{R}$  exists with the following properties:

1.  $f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0$
2.  $f(1, \theta) = h(\theta)$

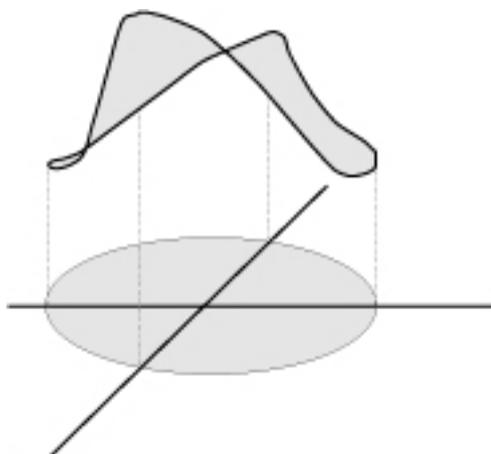


Figure 8.8: Soap film on a circular wire

$$3. f(r, \theta) = p(r)q(\theta)$$

The first is just the Polar form of Laplace's Equation, the second says that we know the value of a function satisfying the equation on the boundary of the disk, and the last says that the variables are separable. (This has the status of pious hope at this stage.) I have used the shorthand notation for the partial derivatives partly because I am cramped off with the  $\text{\TeX}$  expressions for the other form, and partly because it will be good for you to have to practise with it.

Putting the first and the last together, we get

$$\begin{aligned} r^2 q \ddot{p} + r q \dot{p} + p \ddot{q} &= 0 \\ \Rightarrow r^2 q \ddot{p} + r q \dot{p} &= -p \ddot{q} \\ \Rightarrow r^2 \frac{\ddot{p}}{p} + r \frac{\dot{p}}{p} &= k \\ \text{and } -\frac{\ddot{q}}{q} &= k \end{aligned}$$

For some constant  $k$ . We therefore have again reduced the original PDE down to two ODEs,

$$r^2 \ddot{p} + r \dot{p} - kp = 0; \quad \ddot{q} + kq = 0$$

both of which look fairly straightforward.

We consider the possibilities for  $k$ ; it can be negative, positive or zero. If it is zero, we rapidly deduce that  $q(\theta) = m\theta + c$  and this can only mean

that  $q$  is constant, otherwise the function could not be continuous on the boundary. (It has to have  $q(0) = q(2\pi)$ .) But if  $q$  does not depend on  $\theta$ , the height function around the circle would also have to be constant. If the constant were zero, there is a unique solution, the zero function, similarly, if the function is constant on the boundary it has to have the same constant value throughout the interior. This is possible but not exciting enough to contemplate further.

If the constant  $k$  is negative, we get  $\ddot{q} = \lambda^2 q$ , for positive  $\lambda$ , with exponential solution

$$q(\theta) = c_1 e^{\lambda\theta} + c_2 e^{-\lambda\theta}$$

which is impossible to have continuous on the boundary for positive  $\lambda$  except in the thoroughly uninteresting case when  $c_1 = c_2 = 0$ .

Thus we may conclude that  $k > 0$ . This forces solutions to the equation in  $q$  to be periodic:

$$q(\theta) = c_1 \sin(\sqrt{k}\theta) + c_2 \cos(\sqrt{k}\theta)$$

so  $\sqrt{k}$  must be a positive integer  $n$ .

Going now to the equation for  $p$ ,

$$r^2 \ddot{p} + r\dot{p} - \lambda^2 p = 0$$

This is easily seen to have solutions of the form

$$p(r) = Ar^\lambda + Br^{-\lambda}$$

The  $r^{-\lambda}$  terms go off to infinity as  $r \rightarrow 0$ , so we are left with terms which have to be of the form  $r^n$  for positive integers  $n$ .

Thus we conclude that any solution must be a sum of such solutions, so we get:

$$f(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \sin(n\theta) + B_n \cos(n\theta))$$

And in order to get the given function  $h$  on the boundary, we have

$$f(1, \theta) = h(\theta) = A_0 + \sum_{n=1}^{\infty} (A_n \sin(n\theta) + B_n \cos(n\theta))$$

Which means that we have its ordinary Fourier Series, hence

$$A_n = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin(n\theta) d\theta, n = 1, 2, \dots$$

and

$$B_n = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos(n\theta) d\theta, n = 0, 1, 2, \dots$$

The problem is solved, you may now cheer wildly and scream yourselves hoarse in support of something pretty smart.

**Exercise 8.7.2.**

1. Suppose the function defined on  $S^1$  in figure 8.8 is smoother than it looks and is actually just  $\sin 2\theta$ . Find the unique extension to the disk which satisfies Laplace's Equation.
2. Suppose we are given a semicircle,

$$\{r = 1, 0 \leq \theta \leq \pi\} \cup \{\theta = 0, -1 \leq r \leq 1\}$$

Suppose the temperature is maintained at zero on the diameter, and is given by  $h(\theta)$  on the arc. Show how to solve Laplace's Equation for this case.

3. If in the above problem,  $h(\theta) = \sin(4\theta)$ , sketch the solution and calculate it exactly.
4. We are given a unit disk made out of metal. Suppose that the top half of the unit circle on the disk is kept at a temperature of  $100^\circ$  and the bottom half at  $0^\circ$ . Find the steady state temperature in the inside of the disk. Be suitably fluffy about what happens at the points where the two temperatures are adjacent.

## 8.8 Solving the Heat Equation

The heat equation is more general than Laplace's Equation, so a little more complication is to be expected. Recall that we had:

$$c^2 \nabla^2 f = f_t$$

In the case of a one dimensional bar, this comes to

$$c^2 f_{xx} = f_t$$

and we investigate this case first.

Let us suppose that we take a bar of metal of length 1, and heat it up in some way so that the temperature at location  $x$  is  $h(x)$  for  $0 \leq x \leq 1$ . Let us suppose that the ends are kept at zero temperature from the starting time to the end of time. If you have a bar of some different length, just change your units. If you want to know what temperature units I am using, it doesn't much matter, although perhaps it would be a good idea to avoid the Kelvin scale since negative temperatures might be convenient and we would prefer some realism. I shall also assume, again in a spirit of optimism, that the function  $f$  can have its variables separated, i.e. is a product of two functions, one of space and the other of time. Then we have the boundary value problem of a function  $f$  satisfying the following conditions:

1.  $c^2 f_{xx} = f_t$
2.  $f(0, t) = 0$
3.  $f(1, t) = 0$
4.  $f(x, 0) = h(x)$
5.  $f(x, t) = p(x)q(t)$

It is useful to draw the diagram of figure 8.9 in order to see the two dimensions of the problem.

Then we have:

$$f(x, t) = p(x)q(t) \Rightarrow f_{xx} = q\ddot{p} \quad \text{and} \quad f_t = p\dot{q}$$

This gives

$$c^2 q\ddot{p} = p\dot{q}$$

So

$$c^2 \frac{\ddot{p}}{p} = \frac{\dot{q}}{q} = k$$

for some constant  $k$ .

This gives us two ODEs, just as for the case of Laplace's Equation,

$$c^2 \ddot{p} - kp = 0$$

and

$$\dot{q} - kq = 0$$

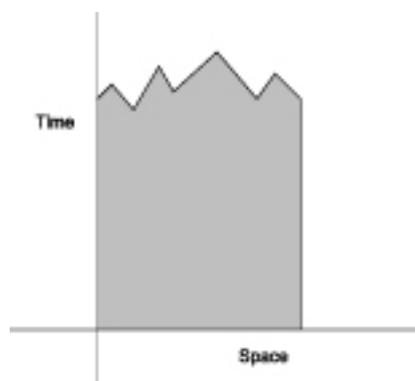


Figure 8.9: The Heat Equation for a rod

The second has solution  $q(t) = Ae^{kt}$ , which tells us that  $k \leq 0$ , since a runaway temperature, for example, is hard to credit. If we put  $k = -\lambda^2$ , the first equation is

$$\ddot{p} = -\left(\frac{\lambda}{c}\right)^2 p$$

which has solution

$$p(x) = A \cos\left(\frac{\lambda}{c}x\right) + B \sin\left(\frac{\lambda}{c}x\right)$$

It is time now to apply the boundary conditions so as to get some information about  $\lambda$ . Since  $f(0, t) = 0$  for all  $t$ , we get  $p(0) = 0$  and hence  $A = 0$ . Since  $f(1, t) = 0$ , we get

$$\frac{\lambda}{c} = n\pi, n = 1, 2, \dots$$

This gives a family of solutions to the Heat Equation:

$$f_n(x, t) = c_n e^{-n^2\pi^2 c^2 t} \sin(n\pi x)$$

The general solution is some infinite sum of these for a choice of  $c_n$  which makes the initial state at  $t = 0$  equal to the given function  $h(x)$ . So we choose to expand the given  $h(x)$  on the interval  $[0, 1]$  in terms of sin functions, which gives us the required  $c_n$ , and we are done. This will require some rescaling, since you have done your Fourier Theory on the interval  $[-\pi, \pi]$  rather than on  $[0, 1]$ , but this is not particularly difficult.

**Exercise 8.8.1.** Write down the affine embedding which sends  $[0, 1]$  to  $[-\pi, \pi]$ . Now work out the Fourier Series for the function  $x(1 - x)$  on  $[0, 1]$ .

**Example 8.8.1.** Suppose the initial value at time  $t = 0$  is  $h(x) = \sin(\pi x)$ . Then  $c_1 = 1$ , and otherwise  $c_n = 0$ . So the complete solution in this case is

$$f(x, t) = e^{-\pi^2 c^2 t} \sin(\pi x)$$

So at time  $t = 1$  the temperature at the middle of the bar is

$$e^{-\pi^2 c^2}$$

I hope you agree that this is pretty clever stuff.

Instead of keeping the ends at any temperature, we can insulate them so that no heat leaves the bar. We can express this in terms of conditions that the derivative of the function  $p(x)$  must be zero at the ends of the bar. This leads to a similar equation but with cosine terms instead of sine terms as the solution. We give an easy example of this kind of problem, worked through from first principles:

**Example 8.8.2.**

**Problem**

Suppose an insulated rod of length  $2\pi$  units is heated to a temperature of  $\pi - |x|$  for a distance  $x$  along the rod from the centre. We do not keep both ends of the rod to be at zero for all time, but instead ensure that no heat leaves the bar anywhere, and we remove the heater at time zero. Given that the density, specific heat and conductivity of the bar are  $\rho, \sigma, K$  respectively, calculate from first principles the temperature of the point one quarter of the way along the bar after 3 time units have elapsed. What happens to the temperature at the ends of the bar?

**Solution**

Note that I have changed the length of the rod and measured from its centre to make the sums easier.

$$K \frac{\partial^2 T}{\partial x^2} = \sigma \rho \frac{\partial T}{\partial t}$$

is the heat equation in one dimension, where  $x$  is the distance along the bar,  $t$  is the time, and  $T(x, t)$  is the temperature at the point  $x$  at time  $t$ .

Writing  $T(x, t) = p(x)q(t)$  on the assumption of separability of variables, This becomes:

$$Kq\ddot{p} = \sigma\rho p\dot{q}$$

After rearranging we obtain:

$$\frac{\ddot{p}}{p} = \frac{\sigma\rho}{K} \frac{\dot{q}}{q} = k$$

for some constant  $k$ . The second equation we may solve immediately to give:

$$q(t) = e^{\left(\frac{Kk}{\sigma\rho}\right)t}$$

and since  $\rho, \sigma, K$  are positive,  $k$  must be negative for a physically intelligible solution. Putting  $k = -\lambda^2$ , the first equation then has solutions of the form

$$p_\lambda(x) = A_\lambda \cos \lambda x + B_\lambda \sin \lambda x$$

and any sum of such solutions for any number of different  $\lambda$  will also be a solution. We wish next to use the boundary conditions to work out what restrictions are implied on the possible  $\lambda$  and the constants.

We have from the fact that the function  $T(x, 0) = p(x)q(0)$  is symmetric, that  $p$  is symmetric, and hence that  $B_\lambda = 0$ , for any value of  $\lambda$ . We also have the condition  $\dot{p}(-\pi) = \dot{p}(\pi) = 0$ , since no heat leaves the rod at the ends. This requires that

$$A_\lambda \lambda \sin \lambda \pi = 0$$

which requires that  $\lambda$  be an integer, which may be zero. There is no particular reason to have negative integers, so we shan't.

We seek next a Fourier expansion of the function  $\pi - |x|$  on the interval  $[-\pi, \pi]$ . This is because we want to know what sum of functions

$$A_0 + \sum_{i=1}^{\infty} A_i \cos ix$$

can be the function  $p(x)$  which is  $T(x, 0) = \pi - |x|$ .

Thus the general solution so far is a sum:

$$\sum_{i=0, \infty} A_i \cos(ix) e^{-\left(\frac{K i^2}{\sigma\rho}\right)t}$$

To find the right sequence of coefficients to satisfy the initial condition for the temperature distribution, that is, to calculate the Fourier Series for  $p$ , we need to calculate

$$\int_{-\pi}^{\pi} \cos nx(\pi - |x|) dx$$

which is most easily accomplished by observing that the function is symmetric about the origin and so is simply

$$\begin{aligned} & 2 \int_0^\pi \cos nx(\pi - x) dx \\ &= 2\pi \int_0^\pi \cos nx dx - 2 \int_0^\pi x \cos nx dx \end{aligned}$$

which for  $n > 0$  is

$$-2 \int_0^\pi x \cos nx dx$$

and integrating by parts we get

$$\begin{aligned} & -2 \left[ \frac{x}{n} \sin nx + \frac{1}{n^2} \cos nx \right]_0^\pi \\ &= \frac{2}{n^2} [1 - \cos n\pi] \end{aligned}$$

which is zero when  $n$  is even and  $4/n^2$  when  $n$  is odd. When  $n = 0$  we have simply the area under the graph of  $\pi - |x|$  between  $-\pi$  and  $\pi$  which is  $\pi^2$ . We therefore have that the Fourier Series for  $\pi - |x|$  on  $[-\pi, \pi]$  is given by:

$$\pi - |x| \approx \frac{\pi}{2} + \frac{4}{\pi} \cos x + \frac{4}{9\pi} \cos 3x + \frac{4}{25\pi} \cos 5x + \dots$$

or if you prefer it more formally:

$$p \approx \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$$

Thus we conclude (almost) by writing down the solution to the heat equation as

$$T(x, t) \approx \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x e^{-\frac{K(2n+1)^2}{\sigma\rho}t}$$

It is easier to verify that the space function  $p$  and the time function  $q$  both satisfy, termwise, the required ODEs than it is to produce them. If you believe that I got the Fourier expansion of  $\pi - |x|$  right, then it follows that

$$\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi}{2}$$

by evaluating at  $x = 0$ . Alternatively, the sum of the reciprocals of the squares of the odd numbers is  $\pi^2/8$ . This doesn't look very likely offhand, and you can check it out by adding up a few thousand terms to see if it is probably right.

The original problem (which you have probably now forgotten, it was so far back in space and time) was to say what the temperature is at a point quarter of the way along the rod at time  $t = 3$ . This means that  $x = \pi/2$  is the point we care about, and we notice that  $\cos(2n + 1)\pi/2 = 0$ . This tells us that the temperature at this point does not actually change in time, and that it started at  $\pi/2$  and will remain so indefinitely. This is reasonable, since what one would expect to happen is that, with no heat leaving or entering the system, the temperature will tend to uniformity. (Imagine all those trapped bees!). And the average temperature is  $\pi/2$  to start out with, so the proposition that the bar will wind up at that temperature everywhere is believable, as is the proposition that the point of average temperature will stay that way. This works because my original function  $p$  is linear over each half of the bar and symmetrical. To have noticed this at the beginning and simply written down the answer would have shown some genuine talent. If you noticed this, congratulations, you are either very practised or very smart.

The temperature at the end points is

$$T(\pi, t) = T(-\pi, t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} e^{-\frac{K(2n+1)^2}{\sigma\rho} t}$$

which tends to  $\pi/2$  from 0 as time goes by. Note that if  $K$  were 0 and no heat flowed, nothing would happen. If it is small, whatever happens, happens slowly. This seems reasonable.

## 8.9 Solving the Wave Equation

The ideas here are essentially the same.

**Example 8.9.1.** A wire is originally horizontal with tension  $H$  going from  $-\pi$  to  $\pi$ , and is held into the triangular shape shown in figure 8.10. Find the shape of the wire at times following its release.

**Solution:** By symmetry we need only worry about terms which are cosine terms in space. In fact the problem is basically silly, but let's plug through it so we can see what is happening and deal with more complicated questions.

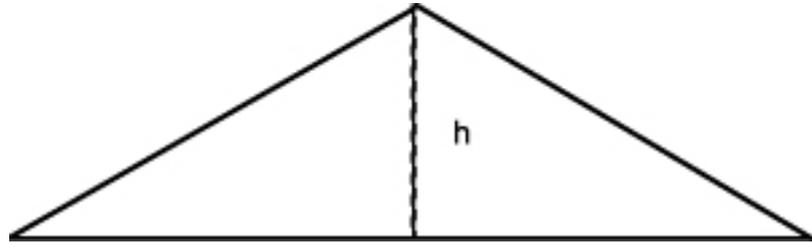


Figure 8.10: a plucked string

First, as before we assume that a function  $f(x, t)$  describing the wire shape is separable into  $f(x, t) = p(x)q(t)$ . Then the wave equation can be written

$$q\ddot{p} = \frac{\rho}{H}p\ddot{q}$$

which can be written:

$$\frac{\rho}{H} \frac{\ddot{q}}{q} = \frac{\ddot{p}}{p} = k$$

for some constant  $k$ . The usual arguments show  $k$  must be negative, I write it therefore as  $-K^2$ . This gives

$$\ddot{q} = -K^2 \frac{H}{\rho} q$$

and

$$\ddot{p} = -K^2 p$$

The former gives

$$q(t) = a_n \cos\left(K \sqrt{\frac{H}{\rho}} t\right) + b_n \sin\left(K \sqrt{\frac{H}{\rho}} t\right)$$

as the general solution and the latter gives

$$p(x) = A_n \cos(Kx) + B_n \sin(Kx)$$

Again we observe that any linear combination of these for different choices of  $a$  will give a solution-in-waiting.

Now we have to match the boundary condition which is that at  $t = 0$  we have the triangular function

$$f(x, 0) = p(x) = h - \left|\frac{hx}{\pi}\right|$$

and this requires that we express  $p$  by its Fourier expansion. Since the function is symmetric we can forget about the sine terms and obtain the expansion in cosine terms only.

This will give us a set of integral values for  $K$  and corresponding  $A_n$ .

The constraints on  $q$  derive from the fact that we started with the string at rest. Thus

$$\left. \frac{\partial f}{\partial t} \right|_0 = 0$$

This gives us

$$p(x)\dot{q}(0) = 0$$

which tells us that  $q$  contains cosine terms only.

We have the Fourier series for  $|x|$  is

$$\frac{\pi}{2} - \frac{4}{\pi} \left( \cos(x) + \frac{\cos(3x)}{9} + \frac{\cos(5x)}{25} + \dots \right)$$

So the Fourier series for

$$\frac{h}{\pi} |x|$$

is

$$\frac{h}{2} - \frac{4h}{\pi^2} \left( \cos(x) + \frac{\cos(3x)}{9} + \frac{\cos(5x)}{25} + \dots \right)$$

and that for

$$h - \frac{h}{\pi} |x|$$

is

$$\frac{h}{2} + \frac{4h}{\pi^2} \left( \cos(x) + \frac{\cos(3x)}{9} + \frac{\cos(5x)}{25} + \dots \right)$$

This gives an expression for  $p(x)$  as a sum of cosine terms and values of  $K$  which are the odd integers.

We have then that the solution is of the form

$$a_0 + \sum_{n=1}^{\infty} f_n(x, t)$$

where

$$f_n(x, t) = a_n \cos(nx) \cos\left(n\sqrt{\frac{H}{\rho}}t\right)$$

with  $a_0 = h/2$  and  $a_n = 0$  when  $n$  is even, and

$$a_n = \frac{4h}{\pi^2 n^2}$$

when  $n$  is odd.

**Remark 8.9.1.** The above calculation really is rather silly. If we assume that the function  $f(x, t)$  is separable,

$$f(x, t) = p(x)q(t)$$

then the answer has to be that the wire preserves its shape indefinitely except that it is scaled by some time varying function. And the time varying function has to be something which starts off at a maximum of 1 and oscillates. The only point of interest is to decide on the form of the time variation, which comes from the expansion for  $p$ . Note that this gives the amplitude of the various harmonics.

**Remark 8.9.2.** Note that if we have the wave equation

$$f_{xx} = \frac{\rho}{H} f_{tt}$$

a solution  $\cos(x) \cos(\sqrt{\frac{H}{\rho}}t)$  can be written:

$$\frac{1}{2} (\cos(x + vt) + \cos(x - vt))$$

which is an average of two waves going in opposite directions with velocity

$$v = \sqrt{\frac{H}{\rho}}$$

This is telling us that the propagation of a transverse wave along the wire will be at a speed which is proportional to the square root of the tension and inversely as the square root of the density. This should not come as too much of a surprise.

## 8.10 And in Conclusion..

I have just got you to stick your toes in the water as far as PDEs are concerned. There are people who make a lifetime's work of solving Laplace's

Equation for progressively more complicated boundary conditions, and they feel their time is well spent. There are applications of PDEs in mining, working out from gravity surveys where the body (the ore body) is buried, in Electromagnetism, in Quantum Physics, in studying waves in every medium you can imagine and a good few you can't, in the twisting and bending of solids, and the list goes on. I finish with a little exercise which should convince you that the material covered in the course has a certain value:

**Exercise 8.10.1. And God Said ...**

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 0 & \operatorname{div} \mathbf{H} &= 0 \\ \operatorname{curl} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} & \operatorname{curl} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

The above are Maxwell's Equations relating the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{H}$ .  $c$  is a constant which depends on the electrical and magnetic properties of free space and which can be measured by fixed physical apparatus.

Show that For any field  $\mathbf{F}$  on  $\mathbb{R}^3$ ,  $\operatorname{curl} \operatorname{curl} \mathbf{F} = \operatorname{grad} \operatorname{div} \mathbf{F} - \nabla^2 \mathbf{F}$ .

Now using Maxwell's Equations show that:

1.

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

2.

$$\nabla \times (\nabla \times \mathbf{H}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

3.

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

4.

$$\nabla^2 \mathbf{H} = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

5. Deduce that there are electromagnetic waves that travel at a speed of  $c$ .

**...And there *was* light!**