Topics in Mathematical Physics

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Contents

Chapter 1

Differential equations of Mathematical Physics

1.1 Differential equations of elliptic type

Let X be an Euclidean space of dimension n with a coordinate system $x_1, ..., x_n$.

- The Laplace equation is

  $$\Delta u = 0,\ \Delta \doteq \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$$

  Δ is called the Laplace operator. A solution in a domain $\Omega \subset X$ is called *harmonic* function in Ω. It describes a stable membrane, electrostatic or gravity field.

- The Helmholtz equation

  $$\left(\Delta + \omega^2\right) u = 0$$

  For $n = 1$ it is called the equation of harmonic oscillator. A solution is a time-harmonic wave in homogeneous space.

- Let σ be a function in Ω; the equation

  $$\langle \nabla, \sigma \nabla \rangle u = f,\ \nabla \doteq \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$$

is the electrostatic equation with the conductivity σ. We have $\langle \nabla, \sigma\nabla \rangle u = \sigma\Delta u + \langle \nabla\sigma, \nabla u \rangle$.

- Stationary Schrödinger equation

$$\left(-\frac{h^2}{2m}\Delta + V(x)\right)\psi = \mathsf{E}\psi$$

E is the energy of a particle.

1.2 Diffusion equations

- The equation

$$\frac{\partial u(x,t)}{\partial t} - d^2\Delta_x u(x,t) = f$$

in $X \times \mathbb{R}$ describes propagation of heat in X with the source density f.

- The equation

$$\rho\frac{\partial u}{\partial t} - \langle \nabla, p\nabla \rangle u - qu = f$$

describes diffusion of small particles.

- The Fick equation

$$\frac{\partial}{\partial t}c + \operatorname{div}(wc) = D\Delta c + f$$

for convective diffusion accompanied by a chemical reaction; c is the concentration, f is the production of a specie, w is the volume velocity, D is the diffusion coefficient.

- The Schrödinger equation

$$\left(\imath h\frac{\partial}{\partial t} + \frac{h^2}{2m}\Delta - V(x)\right)\psi(x,t) = 0$$

where $h = 1.054... \times 10^{-27}$ erg $\cdot$ sec is the Plank constant. The wave function ψ describes motion of a particle of mass m in the exterior field with the potential V. The density $|\psi(x,t)|^2 dx$ is the probability to find the particle in the point x at the time t.

1.3 Wave equations

1.3.1 The case $\dim X = 1$

- The equation
$$\left(\frac{\partial^2}{\partial t^2} - v^2\left(x\right)\frac{\partial^2}{\partial x^2}\right)u\left(x,t\right) = 0$$
is called D'Alembert equation or the wave equation for one spacial variable x and velocity v.

- The telegraph equations
$$\frac{\partial V}{\partial x} + L\frac{\partial I}{\partial t} + R\frac{\partial I}{\partial x} = 0,\ \frac{\partial I}{\partial x} + C\frac{\partial V}{\partial t} + GV = 0$$
V, I are voltage and current in a conducting line, L, C, R, G are inductivity, capacity, resistivity and leakage conductivity of the line.

- The equation of oscillation of a slab
$$\frac{\partial^2 u}{\partial t^2} + \gamma^2\frac{\partial^4 u}{\partial x^4} = 0$$

1.3.2 The case $\dim\ X\ =\ 2, 3$

- The wave equation in an isotropic medium (membrane equation):
$$\left(\frac{\partial^2}{\partial t^2} - v^2\left(x\right)\Delta\right)u\left(x,t\right) = 0$$

- The acoustic equation
$$\frac{\partial^2 u}{\partial t^2} - \left\langle\nabla, v^2\nabla\right\rangle u = 0,\ \nabla = \left(\partial_1, ..., \partial_n\right)$$

- Wave equation in an anisotropic medium:
$$\left(\frac{\partial^2}{\partial t^2} - \sum a_{ij}\left(x\right)\frac{\partial^2}{\partial x_i \partial x_j} - \sum b_i\left(x\right)\frac{\partial u}{\partial x_i}\right)u\left(x,t\right) = f\left(x,t\right)$$

- The transport equation

$$\frac{\partial u}{\partial t}+\theta\frac{\partial u}{\partial x}+a\left(x\right)u-b\left(x\right)\int_{S(X)}\eta\left(\left\langle\theta,\theta'\right\rangle,x\right)u\left(x,\theta',t\right)d\theta'=q$$

  It describes the density $u=u\left(x,\theta,t\right)$ of particles at a point $\left(x,t\right)$ of space-time moving in direction θ.

- The Klein-Gordon-Fock equation

$$\left(\frac{\partial^2}{\partial t^2}-c^2\Delta+m^2\right)u\left(x,t\right)=0$$

  where c is the light speed. A relativistic scalar particle of the mass m.

1.4 Systems

- The Maxwell system:

$$\begin{aligned}\operatorname{div}\left(\mu H\right)=0,&\quad \operatorname{rot}E=-\frac{1}{c}\frac{\partial}{\partial t}\left(\mu H\right),\\ \operatorname{div}\left(\varepsilon E\right)=4\pi\rho,&\quad \operatorname{rot}H=\frac{1}{c}\frac{\partial}{\partial t}\left(\varepsilon E\right)+\frac{4\pi}{c}I,\end{aligned}$$

  E and H are the electric and magnetic fields, ρ is the electric charge and I is the current; ε,μ are electric permittivity and magnetic permeability, respectively, $v^2=c^2/\varepsilon\mu$. In a non-isotropic medium ε,μ are symmetric positively defined matrices.

- The elasticity system

$$\rho\frac{\partial}{\partial t}u_i=\sum\frac{\partial}{\partial x_j}v_{ij}$$

  where $U\left(x,t\right)=\left(u_1,u_2,u_3\right)$ is the displacement evaluated in the tangent bundle $T\left(X\right)$ and $\left\{v_{ij}\right\}$ is the stress tensor:

$$v_{ij}=\lambda\delta_{ij}\frac{\partial}{\partial x_k}u_k+\mu\left(\frac{\partial}{\partial x_j}u_i+\frac{\partial}{\partial x_i}u_j\right),\ i,j=1,2,3$$

  ρ is the density of the elastic medium in a domain $\Omega\subset X$; λ,μ are the Lamé coefficients (isotropic case).

1.5 Nonlinear equations

1.5.1 $\dim X = 1$

- The equation of shock waves

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

- Burgers equation for shock waves with dispersion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - b\frac{\partial^2 u}{\partial x^2} = 0$$

- The Korteweg-de-Vries (shallow water) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

- Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 6u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} = 0$$

1.5.2 $\dim X = 2, 3$

- The nonlinear Schrödinger equation

$$\imath h\frac{\partial u}{\partial t} + \frac{h^2}{2m}\Delta u \pm |u|^2 u = 0$$

- Nonlinear wave equation

$$\left(\frac{\partial^2}{\partial t^2} - v^2\Delta\right) u + f(u) = 0$$

  where f is a nonlinear function, f.e. $f(u) = \pm u^3$ or $\sin u$.

- The system of hydrodynamics (gas dynamic)

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) &= f \\ \frac{\partial v}{\partial t} + \langle v, \operatorname{grad} v\rangle + \frac{1}{\rho}\operatorname{grad} p &= F \\ \Phi(p, \rho) &= 0 \end{aligned}$$

for the velocity vector $v = (v_1, v_2, v_3)$, the density function ρ and the pressure p of the liquid. They are called continuity, Euler and the state equation, respectively.

- The Navier-Stokes system

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V) &= f \\ \frac{\partial v}{\partial t} + \langle v, \operatorname{grad} v \rangle + \alpha \Delta v + \frac{1}{\rho} \operatorname{grad} p &= F \\ \Phi(p, \rho) &= 0 \end{aligned}$$

where α is the viscosity coefficient.

- The system of magnetic hydrodynamics

$$\begin{aligned} \operatorname{div} B = 0,\ & \frac{\partial B}{\partial t} - \operatorname{rot}(u \times B) = 0 \\ \rho \frac{\partial u}{\partial t} + \rho \langle u, \nabla \rangle u + \operatorname{grad} p - \mu^{-1} \operatorname{rot} B \times B = 0,\ & \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \end{aligned}$$

where u is the velocity, ρ the density of the liquid, $B = \mu H$ is the magnetic induction, μ is the magnetic permeability.

1.6 Hamilton-Jacobi theory

- The Hamilton-Jacobi (Eikonal) equation

$$a^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = v^{-2}(x)$$

- Hamilton-Jacobi system

$$\frac{\partial x}{\partial \tau} = H'_\xi(x, \xi), \ \frac{\partial \xi}{\partial \tau} = -H'_x(x, \xi)$$

where H is called the Hamiltonian function.

- Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$$

where $L = L(t, x, \dot{x})$, $x = (x_1, ..., x_n)$ is the Lagrange function.

1.7 Relativistic field theory

1.7.1 $\dim X = 3$

- The Schrödinger equation in a magnetic field

$$\imath h \frac{\partial u}{\partial t} + \frac{h^2}{2m}\left(\partial_j - \frac{e}{c}A_j\right)^2 \psi - eV\psi = 0$$

- The Dirac equation

$$\left(\imath \sum_0^3 \gamma^\mu \partial_\mu - mI\right)\psi = 0$$

where $\partial_0 = \partial/\partial t,\ \partial_k = \partial/\partial x_k,\ k = 1,2,3$ and $\gamma^k,\ k = 0,1,2,3$ are 4×4 matrices (Dirac matrices):

$$\begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

and

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. The wave function ψ describes a free relativistic particle of mass m and spin 1/2, like electron, proton, neutron, neutrino. We have

$$\left(\imath \sum \gamma^\mu \partial_\mu - mI\right)\left(-\imath \sum \gamma^\mu \partial_\mu - mI\right) = \left(\Box + m^2\right) I,\ \Box \doteq \frac{\partial^2}{\partial t^2} - c^2 \Delta$$

i.e. the Dirac system is a factorization of the vector Klein-Gordon-Fock equation.

- The general relativistic form of the Maxwell system

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0,\ \partial_\nu F^{\mu\nu} = 4\pi J^\mu$$
$$\text{or } F = dA,\ d * dA = 4\pi J$$

where J is the 4-vector, $J^0 = \rho$ is the charge density, $J^* = j$ is the current, and A is a 4-potential.

- Maxwell-Dirac system

$$\partial_\mu F_{\mu\nu} = J_\mu,\ \left(\imath\gamma_\mu\partial_\mu + eA_\mu - m\right)\psi = 0$$

describes interaction of electromagnetic field A and electron-positron field ψ.

- Yang-Mills equation for the Lie algebra $\mathfrak{g}$ of a group G

$$F = \nabla\nabla,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\left[A_\mu, A_\nu\right];$$
$$\nabla * F = J,\ \nabla_\mu F_{\mu\nu} = J_\nu;\ \nabla_\mu = \partial_\mu - gA_\mu,$$

where $A_\mu(x) \in \mathfrak{g}$, $\mu = 0,1,2,3$ are gauge fields, ∇_μ is considered as a connection in a vector bundle with the group G.

- Einstein equation for a 4-metric tensor $g_{\mu\nu} = g_{\mu\nu}(x), x = (x_0, x_1, x_2, x_3); \mu, \nu = 0, ..., 3$

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = Y^{\mu\nu},$$

where $R^{\mu\nu}$ is the Ricci tensor

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}$$
$$\Gamma_{\mu\nu\alpha} \doteq \frac{1}{2}\left(g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} - g_{\nu\alpha,\mu}\right)$$

1.8 Classification of linear differential operators

For an arbitrary linear differential operator in a vector space X

$$a(x,D) \doteq \sum_{|j|\le m} a_j(x)\,D^j = \sum_{j_1+...+j_n\le m} a_{j_1,...,j_n}(x)\,\frac{\partial^{j_1+...+j_n}}{\partial x_1^{j_1}...\partial x_n^{j_n}}$$

of order m the sum

$$a_m(x,D) = \sum_{|j|=m} a_j(x)\,D^j,\ \ |j| = j_1 + ... + j_n$$

is called the principal part. If we make the formal substitution $D \mapsto \imath\xi$, $\xi \in X^*$, we get the function

$$a(x,\imath\xi) = \exp(-\imath\xi x)\,a(x,D)\exp(\imath\xi x)$$

This is a polynomial of order m with respect to ξ.

Definition. The functions $\sigma(x,\xi) \doteq a(x,\imath\xi)$ and $\sigma_m(x,\xi) \doteq a_m(x,\imath\xi)$ in $X \times X^*$ are called the *symbol* and *principal symbol* of the operator a. The symbol of a linear differential operator a on a manifold X is a function on the cotangent bundle $T^*(X)$.

If a is a matrix differential operator, then the symbol is a matrix function in $X \times X^*$.

1.8.1 Operators of elliptic type

Definition. An operator a is called *elliptic* in a domain $D \subset X$, if

(*) the principle symbol $\sigma_m(x,\xi)$ does not vanish for $x \in D,\ \xi \in X^* \backslash \{0\}$.

For a $s \times s$-matrix operator a we take $\det \sigma_m$ instead of σ_m in this definition.

Examples. The operators listed in Sec.1 are elliptic. Also

- the Cauchy-Riemann operator

$$a\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} \\ \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} \end{pmatrix}^t$$

  is elliptic, since

$$\sigma_1 = \sigma = \imath \begin{pmatrix} \xi & -\eta \\ \eta & \xi \end{pmatrix},\ \det \sigma_1 = -\xi^2 - \eta^2$$

1.8.2 Operators of hyperbolic type

We consider the product space $V = X \times \mathbb{R}$ and denote the coordinates by x and t respectively. We have then $V^* = X^* \times \mathbb{R}^*$; the corresponding coordinates are denoted by ξ and τ. Write the principal symbol of an operator $a(x,t;D_x,D_t)$ in the form

$$\sigma_m(x,t;\xi,\tau) = a_m(x,t;\imath\xi,\imath\tau) = \alpha(x,t)\left[\tau - \lambda_1(x,t;\xi)\right]...\left[\tau - \lambda_m(x,t;\xi)\right]$$

Definition. We assume that in a domain $D \subset V$

(i) $\alpha(x,t) \neq 0$, i.e. the time direction $\tau \sim dt$ is not characteristic,

(ii) the roots $\lambda_1, ..., \lambda_m$ are real for all $\xi \in X^*$,

(iii) we have $\lambda_1 < ... < \lambda_m$ for $\xi \in X^* \backslash \{0\}$.

The operator a is called *strictly t-hyperbolic (*strictly hyperbolic in variable t), if (i,ii,iii) are fulfilled. It is called *weakly hyperbolic,* if (i) and (ii) are satisfied. It is called *t-hyperbolic,* if there exists a number $\rho_0 < 0$ such that

$$\sigma\left(x,t;\xi+\imath\rho\tau\right)\neq 0,\ \text{for}\ \xi\in V^*,\ \rho<\rho_0$$

The strict hyperbolicity property implies hyperbolicity which, in its turn, implies weak hyperbolicity. Any of these properties implies the same property for $-t$ instead of t.

Example 1. The operator

$$\Box=\frac{\partial^2}{\partial t^2}-v^2\Delta$$

is hyperbolic with respect to the splitting (x,t) since

$$\sigma_2=-\tau^2+v^2\left(x\right)\left|\xi\right|^2=-\left[\tau-v\left(x\right)\left|\xi\right|\right]\left[\tau+v\left(x\right)\left|\xi\right|\right]$$

i.e. $\lambda_1=-v\left|\xi\right|, \lambda_2=v\left|\xi\right|$. It is strictly hyperbolic, if $v\left(x\right)>0$.

Example 2. The Klein-Gordon-Fock operator $\Box+m^2$ is strictly t-hyperbolic.

Example 3. The Maxwell, Dirac systems are weakly hyperbolic, but not strictly hyperbolic.

Example 4. The elasticity system is weakly hyperbolic, but not strictly hyperbolic, since the polynomial $\det\sigma_2$ is of degree 6 and has 4 real roots with respect to τ, two of them of multiplicity 2.

1.8.3 Operators of parabolic type

Definition. An operator $a\left(x,t;D_x,D_t\right)$ is called *t-parabolic* in a domain $U\subset X\times\mathbb{R}$ if the symbol has the form $\sigma=\alpha\left(x,t\right)\left(\tau-\tau_1\right)...\left(\tau-\tau_p\right)$ where $\alpha\neq 0$, and the roots fulfil the condition

(iv) $\operatorname{Im}\tau_j\left(x,t;\xi\right)\geq b\left|\xi\right|^q-c$ for some positive constants q,b,c.

This implies that $p<m$.

Examples. The heat operator and the diffusion operator are parabolic. For the heat operator we have $\sigma=\imath\tau+d^2\left(x,t\right)\left|\xi\right|^2$. It follows that $p=1$, $\tau_1=\imath d^2\left|\xi\right|^2$ and (iv) is fulfilled for $q=2$.

1.8.4 Out of classification

The linear Schrödinger operator does not belong to either of the above classes.

- Tricomi operator
$$a(x, y, D) = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2}$$
is elliptic in the halfplane $\{x > 0\}$ and is strictly hyperbolic in $\{x < 0\}$. It does not belong to either class in the axes $\{x = 0\}$.

1.9 Initial and boundary value problems

1.9.1 Boundary value problems for elliptic equations.

For a second order elliptic equation

$$a(x, D)\, u = f$$

in a domain $D \subset X$ the boundary conditions are: the Dirichlet condition:

$$u|\partial D = v_0$$

or the Neumann condition:
$$\frac{\partial u}{\partial \nu}|\partial D = v_1$$
or the mixed (Robin) condition:

$$\left(\frac{\partial u}{\partial \nu} + bu\right)|\partial D = v$$

1.9.2 The Cauchy problem

$$u(x, 0) = u_0$$

for a diffusion equation
$$a(x, t; D)\, u = f$$
For a second order equation the Cauchy conditions are

$$u(x, 0) = u_0,\ \partial_t u(x, 0) = u_1$$

1.9.3 Goursat problem

$$u\,(x,0) = u_0,\ u\,(0,t) = v\,(t)$$

1.10 Inverse problems

To determine some coefficients of an equation from boundary measurements

Examples

1. The sound speed v to be determined from scattering data of the acoustic equation.
2. The potential V in the Schrödinger equation
3. The conductivity σ in the Poisson equation
and so on.

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Chapter 2

Elementary methods

2.1 Change of variables

Let V be an Euclidean space of dimension n with a coordinate system $x_1, ..., x_n$. If we introduce another coordinate system, say $y_1, ..., y_n$, then we have the system of equations

$$dy_j = \frac{\partial y_j}{\partial x_1} dx_1 + ... + \frac{\partial y_j}{\partial x_n} dx_n,\ j = 1, ..., n$$

If we write the covector $dx = (dx_1, ..., dx_n)$ as a column, this system can be written in the compact form

$$dy = J dx$$

where $J \doteq \{\partial y_j / \partial x_i\}$ is the Jacobi matrix. For the rows of derivatives $D_x = (\partial/\partial x_1, ..., \partial/\partial x_n)\,,\ D_y = (\partial/\partial y_1, ..., \partial/\partial y_n)$ we have

$$D_x = D_y J$$

since the covector dx is bidual to the vector D_x. Therefore for an arbitrary linear differential operator a we have

$$a\left(x, D_x\right) = a\left(x\left(y\right), D_y J\right)$$

hence the symbol of a in y coordinates is equal $\sigma\left(x\left(y\right), \eta J\right)$, where $\sigma\left(x, \xi\right)$ is symbol in x-coordinates.

Example 1. An arbitrary operator with constant coefficients is invariant with respect to arbitrary translation transformation $T_h : x \mapsto x + h,\ h \in V$. Translations form the group that is isomorphic to V.

Example 2. D'Alembert operator

$$\Box = \frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2},\ \sigma = \tau^2 - v^2 \xi^2$$

with constant speed v can be written in the form

$$\Box = -4v^2 \frac{\partial^2}{\partial y \partial z}$$

where $y = x - vt,\ z = x + vt$, since $2\partial/\partial y = \partial/\partial x - v^{-1}\partial/\partial t$, $2\partial/\partial z = \partial/\partial x + v^{-1}\partial/\partial t$.

This implies that an arbitrary solution $u \in C^2$ of the equation

$$\Box u = 0$$

can be represented, at least, locally in the form

$$u(x,t) = f(x - vt) + g(x + vt) \tag{2.1}$$

for continuous functions f, g. At the other hand, if f, g are arbitrary continuous functions, the sum (1) need not to be a C^2-function. Then u is a *generalized* solution of the wave equation.

Example 3. The Laplace operator Δ keeps its form under arbitrary linear orthogonal transformation $y = Lx$. We have $J = L$ and $\sigma(\xi) = -|\xi|^2$. Therefore $\sigma(\eta) = -|\eta L|^2 = -|\eta|^2$. All the orthogonal transformations L form a group $O(n)$. Also the Helmholtz equation is invariant with respect to $O(n)$.

Example 4. The relativistic wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - c^2 \Delta$$

is invariant with respect to arbitrary linear orthogonal transformation in X-space. In fact there is a larger invariance group, called the *Lorentz* group L_n. This is the group of linear operators in V^* that preserves the symbol

$$\sigma(\xi, \tau) = -\tau^2 + c^2 |\xi|^2$$

This is a quadratic form of signature $(n, 1)$. The Lorentz group contains the orthogonal group $O(n)$ and also transformations called boosts:

$$t' = t\cosh\alpha + c^{-1}x_j \sinh\alpha,\ x'_j = ct\sinh\alpha + x_j\cosh\alpha,\ j = 1, ...$$

Dimension d of the Lorentz group is equal to $n\,(n+1)\,/2$, in particular, $d = 6$ for the space dimension $n = 3$. The group generated by all translations and Lorentz transformations is called Poincaré group. The dimension fo the Poincaré group is equal 10.

2.2 Bilinear integrals

Suppose that V is an Euclidean space, $\dim V = n$. The volume form $dV \doteq dx_1 \wedge ... \wedge dx_n$ is uniquely defined; let $L_2\,(V)$ be the space of square integrable functions in V. For a differential operator a we consider the integral form

$$\langle a\phi, \psi\rangle = \int_V \overline{\psi}\,(x)\, a\,(x, D)\,\phi\,(x)\,dV$$

It is linear with respect to the argument ϕ and is additive with respect to ψ whereas

$$\langle a\phi, \lambda\psi\rangle = \overline{\lambda}\,\langle a\phi, \psi\rangle$$

for arbitrary complex constant λ. A form with such properties is called *sesquilinear*. It is bilinear with respect to multiplication by real constants. We suppose that the arguments ϕ, ψ are smooth (i.e. $\phi, \psi \in C^\infty$) funtcions with compact supports. We can integrate this form by parts up to m times, where m is the order of a. The boundary terms vanish, since of the assumption, and we come to the equation

$$\langle a\phi, \psi\rangle = \langle \phi, a^*\psi\rangle \tag{2.2}$$

where a^* is again a linear differential operator of order m. It is called (formally) conjugate operator. The operation $a \mapsto a^*$ is additive and $(\lambda a)^* = \overline{\lambda} a^*$, obviously $a^{**} = a$.

An operator a is called (formally) *selfadjoint* if $a^* = a$.

Example 1. For an arbitrary operator a with constant coefficients we have $a^*\,(D) = a\,(-D)$.

Example 2. A tangent field $b = \sum b_i\,(x)\,\partial/\partial x_i$ is a differential operator of order 1. We have

$$b^* = -b - \operatorname{div} b,\ \operatorname{div} b \doteq \sum \partial b_i/\partial x_i$$

This is no more a tangent field unless the divergence vanishes.

Example 3. The Poisson operator

$$a(x,D) = \sum \partial/\partial x_i \left(a_{ij}(x)\,\partial/\partial x_j\right)$$

is selfadjoint if the matrix $\{a_{ij}\}$ is Hermitian. In particular, the Laplace operator is selfadjoint. Moreover the quadratic Hermitian form

$$\langle -\Delta\phi, \phi\rangle = \sum \left\langle \frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle = \|\nabla \phi\|^2$$

is always nonnegative. This property helps, f.e. to solve the Dirichlet problem in a bounded domain. Note that the symbol of $-\Delta$ is also nonnegative: $|\xi|^2 \geq 0$. In general these two properties are related in much more general operators.

Let a, b be arbitrary functions in a domain $D \subset V$ that are smooth up to the boundary $\Gamma = \partial D$. They need not to vanish in Γ. Then the integration by parts brings boundary terms to the righthand side of (2). In particular, for the Laplace operator we get the equation

$$\int_D \bar{b}\Delta a dV = -\int_D \sum \frac{\partial a}{\partial x_i}\frac{\partial \bar{b}}{\partial x_i} dV + \int_\Gamma \bar{b} \sum \mathbf{n}_i \frac{\partial a}{\partial x_i} dS$$

where $\mathbf{n} = (\mathbf{n}_1, ..., \mathbf{n}_n)$ is the unit outward normal field in Γ and dS is the Euclidean surface measure. The sum of the terms $\mathbf{n}_i \partial a/\partial x_i$ is equal to the normal derivative $\partial a/\partial \mathbf{n}$. Integrating by parts in the first term, we get finally

$$\int_D \bar{b}\Delta a dV = \int_D \overline{\Delta b} a dV + \int_\Gamma \bar{b}\partial a/\partial \mathbf{n} dS - \int_\Gamma \partial \bar{b}/\partial \mathbf{n} a dS \qquad (2.3)$$

This is a *Green* formula.

2.3 Conservation laws

For some hyperbolic equations and system one can prove that the "energy" is conservated, i.e. it does not depend on time. Consider for simplicity the selfadjoint wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x_i} v^2 \frac{\partial}{\partial x_i}\right) u(x,t) = 0$$

in a space-time $V = X \times \mathbb{R}$. Suppose that a solution u decreases as $|x| \to \infty$ for any fixed t and ∇u stays bounded. Then we can integrate by parts in the X-integral

$$-\sum \left\langle \frac{\partial}{\partial x_i} v^2 \left(\frac{\partial}{\partial x_i} u \right), u \right\rangle = \sum \left\langle v^2 \left(\frac{\partial u}{\partial x_i} \right), \frac{\partial u}{\partial x_i} \right\rangle = \|v \nabla u\|^2$$

Take time derivative of the lefthand side:

$$-\frac{\partial}{\partial t} \left\langle \frac{\partial^2 u}{\partial t^2}, u \right\rangle = \frac{\partial}{\partial t} \left\| \frac{\partial u}{\partial t} \right\|^2$$

At the other hand from the equation

$$-\frac{\partial}{\partial t} \left\langle \frac{\partial^2 u}{\partial t^2}, u \right\rangle = -\frac{\partial}{\partial t} \sum \left\langle \frac{\partial}{\partial x_i} \left(v^2 \frac{\partial u}{\partial x_i} \right), u \right\rangle = \frac{\partial}{\partial t} \|v \nabla u\|^2$$

and

$$\frac{\partial}{\partial t} \left(\left\| \frac{\partial u}{\partial t} \right\|^2 + \|v \nabla u\|^2 \right) = 0$$

Integrating this equation from 0 to t, we get the equation

$$\int \left(\left\| \frac{\partial u(x,t)}{\partial t} \right\|^2 + \|v \nabla u(x,t)\|^2 \right) dx = \int \left(\left\| \frac{\partial u(x,0)}{\partial t} \right\|^2 + \|v \nabla u(x,0)\|^2 \right) dx$$

The left side has the meaning is the energy of the wave u at the time t.

2.4 Method of plane waves

Let again V be a real vector space of dimension $n < \infty$ and λ be a nonzero linear functional on V. A function u in V is called a λ-plane-wave, if $u(x) = f(\lambda(x))$ for a function $f : \mathbb{R} \to \mathbb{C}$. The function f is called the *profile* of u. The meaning of the term is that any u is constant on each hyperplane $\lambda = \text{const}$.

For example both the terms in (1) are plane waves for the covectors $\lambda = (1, -v)$ and $\lambda = (1, v)$ respectively. In general, if we look for a plane-wave solution of a partial differential equation, we get an ordinary differential equation for its profile.

Example 1. For an arbitrary linear equation with constant coefficients

$$a\,(D)\,u = 0$$

the exponential function $\exp\left(\imath\xi x\right)$ is a solution if and only if the covector ξ satisfies the characteristic equation $\sigma\left(\xi\right) = 0$.

Example 2. For the Korteweg & de-Vries equation

$$u_t + 6uu_x + u_{xxx} = 0$$

in $\mathbb{R} \times \mathbb{R}$ and arbitrary $a > 0$ there exists a plane-wave solution for the covector $\lambda = (1, -a)$:

$$u\,(x,t) = \frac{a}{2\cosh^2\left(2^{-1}a^{1/2}\,(x - at - x_0)\right)}$$

It decreases fast out of the line $x - at = \lambda_0$. A solution of this kind is called *soliton.*

Example 3. Consider the Liouville equation

$$u_{tt} - u_{xx} = g\exp\left(u\right)$$

where g is a constant. For any a, $0 \leq a < 1$ there exists a plane-wave solution

$$u\,(x,t) = \ln\frac{a^2\left(1-a^2\right)}{2g\cosh\left(2^{-1}a\,(x - at - x_0)\right)}$$

Example 4. For the "Sine-Gordon" equation

$$u_{tt} - u_{xx} = -g^2\sin u$$

the function

$$u\,(x,t) = 4\arctan\left(\exp\left(\pm g\left(1-a^2\right)^{1/2}(x - at - x_0)\right)\right)$$

is a plane-wave solution.

Example 5. The Burgers equation

$$u_t + uu_x = \nu u_{xx},\ \ \nu \neq 0$$

It has the following solution for arbitrary c_1, c_2

$$u = c_1 + \frac{c_2 - c_1}{1 + \exp\left((2\nu)^{-1}\left(c_2 - c_1\right)(x - at)\right)},\ \ 2a = c_1 + c_2$$

2.5 Fourier transform

Consider ordinary linear equation with constant coefficients

$$a\left(D\right)u=\left(a_m\frac{d^m}{dx^m}+a_{m-1}\frac{d^{m-1}}{dx^{m-1}}+...+a_0\right)u=w \tag{2.4}$$

To solve this equation, we asume that $w\in L_2$ and write it by means of the Fourier integral

$$w\left(x\right)=\int\exp\left(\imath\xi x\right)\widehat{w}\left(\xi\right)d\xi$$

and try to solve the equation (4) for $w\left(x\right)=\exp\left(\imath\xi x\right)$ for any ξ. Write a solution in the form $u_\xi=\exp\left(\imath\xi x\right)\widetilde{u}\left(\xi\right)$ and have

$$\widehat{w}\left(\xi\right)\exp\left(\imath\xi x\right)=a\left(D\right)u_\xi=a\left(D\right)\exp\left(\imath\xi x\right)\widetilde{u}\left(\xi\right)=\sigma\left(\xi\right)\widetilde{u}\left(\xi\right)\exp\left(\imath\xi x\right)$$

or $\sigma\left(\xi\right)\widetilde{u}\left(\xi\right)=\widehat{w}\left(\xi\right)$. A solution can be found in the form:

$$\widetilde{u}\left(\xi\right)=\sigma^{-1}\left(\xi\right)\widehat{w}\left(\xi\right)$$

if the symbol does not vanish. We can set

$$u\left(x\right)=\frac{1}{2\pi}\int_{\mathbb{R}^*}\frac{\widehat{w}\left(\xi\right)}{\sigma\left(\xi\right)}\exp\left(\imath\xi x\right)d\xi$$

Example 1. The symbol of the ordinary operator $a_-=D^2-k^2$ is equal to $\sigma=-\xi^2-k^2\neq0$. It does not vanish.

Proposition 1 *If $m>0$ and w has compact support, we can find a solution of (4) in the form*

$$u\left(x\right)=\frac{1}{2\pi}\int_{\lneqq}\frac{\widehat{w}\left(\zeta\right)}{\sigma\left(\zeta\right)}\exp\left(\imath\zeta x\right)d\zeta \tag{2.5}$$

where $\Gamma\subset\mathbb{C}\setminus\{\sigma=0\}$ is a cycle that is homologically equivalent to $\mathbb{R}$ in $\mathbb{C}$.

PROOF. The function $\widehat{w}\left(\xi\right)$ has the unique analytic continuation $\widehat{w}\left(\zeta\right)$ at $\mathbb{C}$ according to Paley-Wiener theorem. The integral (4) converges at infinity, since Γ coincides with $\mathbb{R}$ in the complement to a disc, the function $\widehat{w}\left(\xi\right)$ belongs to L_2 and $\left|\sigma\left(\xi\right)\right|\geq c\left|\xi\right|$ for $\left|\xi\right|>A$ for sufficiently big A.

Example 2. The symbol of the Helmholtz operator $a_+=D^2+k^2$ vanishes for $\xi=\pm k$. Take $\Gamma_+\subset\mathbb{C}_+=\left\{\operatorname{Im}\zeta\geq0\right\}$. Then the solution (4) vanishes at any ray $\left\{x>x_0\right\}$, where w vanishes. If we take $\Gamma=\Gamma_-\subset\mathbb{C}_-$, these rays will be replaced by $\left\{x<x_0\right\}$.

2.6 Theory of distributions

See Lecture notes FI3.

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Chapter 3

Fundamental solutions

3.1 Basic definition and properties

Definition. Let $a(x, D)$ be a linear partial differential operator in a vector space V and U is an open subset of V. A family of distributions $F_y \in D'(V)$, $y \in U$ is called a *fundamental solution* (or Green function, source function, potential, propagator), if

$$a(x, D) F_y(x) = \delta_y(x) \, dx$$

This means that for an arbitrary test function $\phi \in D(V)$ we have

$$F_y(a'(x, D) \phi(x)) = \phi(y)$$

Fix a system of coordinates $x = (x_1, ..., x_n)$ in V; the volume form $dx = dx_1 ... dx_n$ is a translation invariant. We can write a fundamental solution (f.s.) in the form $F_y(x) = E_y(x) \, dx$, where E is a generalized function. The difference between F_y and E_y is the behavior under coordinate changes: $E_y(x') \, dx' = E_y(x) \, dx$, where $x' = x'(x)$, hence $E_y(x') = E_y(x) |\det \partial x / \partial x'|$. The function E_y for a fixed y is called a source function with the source point y.

If $a(D)$ is an operator with constant coefficients in V and $E_0(x) = E(x)$ is a source function that satisfies $a(D) E_0 = \delta_0$, then $E_y(x) \, dx \doteq E(x - y) \, dx$ is a f.s in $U = V$. Later on we call E *source* function; we shall use the same notation E_y for a f.s. and corresponding source function, if we do not expect a confusion.

Proposition 1 *Let E be a f.s. for $U \subset V$. If w is a function (or a distribution) with compact support $K \subset U$, then the function (distribution)*

$$u(x) \doteq \int E_y(x) \, w(y) \, dy$$

is a solution of the equation $a(x, D) u = w$.

PROOF for functions E and w

$$a(x,D)\,u = \int a(x,D)\,E_y(x)\,w(y)\,dy = \int \delta_y(x)\,w(y)\,dy = w(x)$$

The same arguments for distributions E, w and u:

$$a(x,D)\,u(\phi) = u(a'(x,D)\,\phi) = w(F_y(a'(x,D)\,\phi)) = w(\phi)$$

$\boxdot$

If E is a source function for an operator $a(D)$ with constant coefficients, this formula is simplified to $u = E * w$ and $a(D)(E * w) = a(D)\,E * w = \delta_0 * w = w$.

Reminder. For arbitrary distribution f and a distribution g with compact support in V the convolution is the distribution

$$f * g(\phi) = f \times g(\phi(x+y)),\ \phi \in D(V)$$

Here $f \times g$ is a distribution in the space $V_x \times V_y$,where both factors are isomorphic to V, x and y are corresponding coordinates.

If the order of a is positive, there are many fundamental solutions. If E is a f.s. and U fulfils $a(D)\,U = 0$, then $E' \doteq E + U$ is a f.s. too.

Theorem 2 *An arbitrary differential operator $a \neq 0$ with constant coefficients possesses a f.s.*

Problem. Prove this theorem. Hint: modify the method of Sec.4.

3.2 Fundamental solutions for elliptic operators

Definition. A linear differential operator $a(x,D)$ is called *elliptic* in an open set $W \subset V$, if the principal symbol $\sigma_m(x,\xi)$ of a does not vanish for $\xi \in V^* \backslash \{0\}$, $x \in W$.

Now we construct fundamental solutions for some simple elliptic operators with constant coefficients.

Example 1. For the ordinary operator $a(D) = D^2 - k^2$ we can find a f.s. by means of a formula (5) of Ch.2, where $w = \delta_0$ and $\widehat{w} = 1$:

$$E(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\exp(\imath\zeta x)}{\zeta^2 + k^2} d\zeta = -(2k)^{-1} \exp(-k|x|)$$

Example 2. For the Helmholtz operator $a(D) = D^2 + k^2$ we can find a f.s. by means of (5), Ch.2, where $w = \delta_0$ and $\widehat{w} = 1$:

$$E(x) = \frac{1}{2\pi}\int_{\lessgtr_{\mp}} \frac{\exp(\imath\zeta x)}{k^2 - \zeta^2} d\zeta$$

If we take $\Gamma = 1/2\,(\Gamma_+ + \Gamma_-)$, then $E(x) = (2k)^{-1}\sin(k\,|x|)$.

Example 3. For the Cauchy-Riemann operator $a = \overline{\partial}_z \doteq 1/2\,(\partial_x + \imath\partial_y)$ the function

$$E = \frac{1}{\pi z},\ z = x + iy$$

is a f.s. To prove this fact we need to show that

$$\int \frac{a'\phi dxdy}{\pi z} = \phi(0)$$

for any $\phi \in D(\mathbb{R}^2)$, where $a' = -\overline{\partial}_z$. The kernel z^{-1} is locally integrable, hence the integral is equal to the limit of integrals over the set $U(\varepsilon) = \{|z| \geq \varepsilon\}$ as $\varepsilon \to 0$. We have $a' = -\overline{\partial}_z$; integrating by parts yields

$$-\int_{U(\varepsilon)} \frac{\overline{\partial}_z\phi dxdy}{\pi z} = \pi^{-1}\int_{U(\varepsilon)} \phi\overline{\partial}_z\left(\frac{1}{z}\right)dxdy - (2\pi)^{-1}\int_{\partial U(\varepsilon)} \phi\frac{dy - \imath dx}{z}$$

The first term vanishes since the function z^{-1} is analytic in $U(\varepsilon)$. The boundary $\partial U(\varepsilon)$ is the circle $|z| = \varepsilon$ with opposite orientation. Take the parametrization by $x = \varepsilon\cos\alpha$, $y = \varepsilon\sin\alpha$ and calculate the second term:

$$-(2\pi)^{-1}\int_{\partial U(\varepsilon)} \phi\frac{dy - \imath dx}{z} = (2\pi)^{-1}\int_0^{2\pi} \phi(\varepsilon\cos\alpha, \varepsilon\sin\alpha)\,d\alpha$$
$$\to (2\pi)^{-1}\phi(0)\int_0^{2\pi} d\alpha = \phi(0)$$

Example 4. For the Laplace operator Δ in the plane $V = \mathbb{R}^2$ the function

$$E = -(2\pi)^{-1}\ln|x|$$

is a f.s. To check it we apply the Green formula to the domain $D = U(\varepsilon)$ (see Ch.2)

$$E(\Delta\phi) = \lim_{\varepsilon\to 0}\int_D E\Delta\phi dV = \lim_{\varepsilon\to 0}\left(\int_D \Delta E\phi dV + \int_\Gamma E\partial\phi/\partial\mathbf{n}dS - \int_\Gamma \partial E/\partial\mathbf{n}\phi dS\right)$$

Here $\Delta E = 0$ in D, $\Gamma = \partial U(\varepsilon)$, $\partial/\partial\mathbf{n} = -\partial/\partial r$, $\partial E/\partial\mathbf{n} = (2\pi r)^{-1}$, $r = |x|$ and

$$E(\Delta\phi) = (2\pi)^{-1}\left[-\int_{r=\varepsilon} \ln r\ \partial\phi/\partial r dS + \int_{r=\varepsilon} r^{-1}\phi dS\right]$$

The first integral tends to zero as $\varepsilon \to 0$, since the function $\partial\phi/\partial r$ is bounded and $r\ln r \to 0$. In the second one we have $r^{-1}dS = d\alpha$, hence the integral tends to $2\pi\phi(0)$, which implies $E(\Delta\phi) = \phi(0)$. This means that $\Delta E = \delta_0$, Q.E.D.

The function E is invariant with respect to rotations. This is the only rotation invariant f.s. up to a constant term.

Example 5. The function $E(x) = -(4\pi|x|)^{-1}$ is a f.s. for the Laplace operator in $\mathbb{R}^3$.

Problem. To check this fact.

Here are the basic properties of elliptic operators:

Theorem 3 *Let $a(x, D)$ be an elliptic operator with C^∞-coefficients in an open set $W \subset V$. An arbitrary distribution solution to the equation*

$$a(x, D)u = w$$

in W is a C^∞-function, if w is such a function. If the coefficients and the function w are real analytic, so is u.

Corollary 4 *Any source function $E_y(x)$ of an arbitrary elliptic operator $a(x, D)$ with real analytic coefficients is a real analytic function of $x \in V \setminus \{y\}$.*

Problem. Let a be an elliptic operator with constant coefficients. To show that the fundamental solution for a, constructed by the method of Sec.4 is an analytic function in $V \setminus \{0\}$.

3.3 More examples

A hyperbolic operator can not have a fundamental solution which is a C^∞-function in the complement to the source point.

Example 6. For D'Alembert operator $a(D) = \Box_2 \doteq \partial_t^2 - v^2\partial_x^2$ we can find a fundamental solution by means of the coordinate change $y = x - vt$, $z = x + vt$ (see Ch.2): $\Box_2 = -4v^2\partial_y\partial_z$. Introduce the function (Heaviside's function)

$$\theta(x) = 1, \text{ for } x \geq 0, \; \theta(x) = 0 \text{ for } x < 0$$

We have $\partial_x\theta(\pm x) = \pm\delta_0$, hence we can take

$$E(y, z) = \left(4v^2\right)^{-1}\theta(-y)\,\theta(z) \tag{3.1}$$

Returning to the space-time coordinate we get

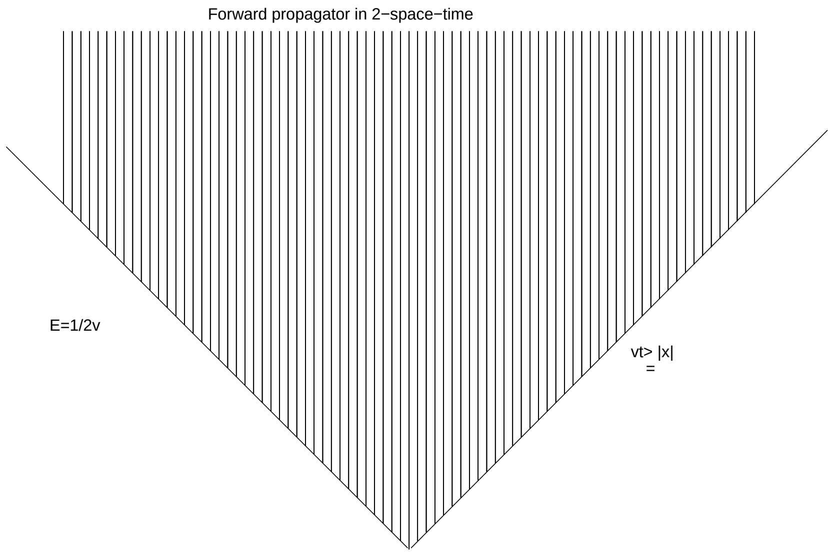
$$E(x, t) = (2v)^{-1}\theta(vt - |x|)$$

i.e. $E(x, t) = (2v)^{-1}$ if $vt \geq |x|$ and $E(x, t) = 0$ otherwise. The coefficient $2v$ appears because of E is a density, (or a distribution), not a function.

We can replace $\theta(-y)$ to $-\theta(y)$ and $\theta(z)$ to $-\theta(-z)$ in (1) and get three more options for a f.s.

Example 7. Consider the first order operator $a(D) = \sum a_j\partial_j$, with constant coefficients $a_j \in \mathbb{R}$ and introduce the variables $y_1, ..., y_n$ such that $a(D)y_1 = 1$, $a(D)y_j = 0$, $j = 2, ..., n$ and $\det \partial y/\partial x = 1$. Then the generalized function $E(x) = \theta(y_1)\,\delta_0(y_2, ..., y_n)$ is a fundamental solution for a.

Problem. To check this statement.



3.4 Hyperbolic polynomials and source functions

Definition. Let $p(\xi)$ be a polynomial in V^* with complex coefficients. It is called *hyperbolic* with respect to a vector $\eta \in V^* \setminus \{0\}$, if there exists a number $\rho_\eta < 0$ such that

$$p(\xi + \imath \rho \eta) \neq 0, \text{ for } \xi \in V^*, \ \rho < \rho_\eta$$

Let p_m be the principal part of p. It is called *strictly hyperbolic,* if the equation $\pi(\lambda) = p_m(\xi + \lambda \eta) = 0$ has only real zeros for real ξ and these zeros are simple for $\xi \neq 0$. If p_m is strictly hyperbolic, then p is hyperbolic for arbitrary lower order terms.

Definition. Let $a(x, D)$ be a differential operator in V and t be a linear function in V, called the time variable. The operator a is called *hyperbolic* with respect to t (or t-hyperbolic) in $U \subset V$, if the symbol $\sigma(x, \xi)$ is a hyperbolic polynomial in ξ with respect to the covector $\eta(x) = t$ for any point $x \in U$.

Let p be a hyperbolic polynomial with respect to η. Consider the cone $V^* \setminus \{p_m(\xi) = 0\}$ and take the connected component $\Gamma(p, \eta)$ of this cone that contains η. This is a convex cone and p is hyperbolic with respect to any $g \in \Gamma(p, \eta)$ and $g \in -\Gamma(p, \eta)$. The *dual* cone is defined as follows

$$\Gamma^*(p, \eta) = \{x \in V, \xi(x) \geq 0, \ \forall \xi \in \Gamma(p, \eta)\}$$

The dual cone is closed, convex and proper, (i.e. it does not contain a line).

Theorem 5 *Let $a(D)$ be a hyperbolic operator with constant coefficients with respect to a covector η_0. Then there exists a f.s. E of a such that*

$$\operatorname{supp} E \subset \Gamma^*(p, \eta_0)$$

where p is the principal symbol of a.

Proof. Fix a covector $\eta \in \Gamma(a, \eta_0)$, $|\eta| = 1$ and set

$$E(x) = (2\pi)^{-n} \lim_{\varepsilon \to 0} E_{\rho,\varepsilon}(x) \tag{3.2}$$

$$E_{\rho,\varepsilon}(x) = \int_{V^*} \frac{\exp(\imath \langle \xi + \imath\rho\eta, x\rangle) \exp\left(-\varepsilon (\xi + \imath\rho\eta)^2\right)}{p(\xi + \imath\rho\eta)} d\xi$$

where $n = \dim V$ and p is the symbol of a. The dominator $p(\xi + \imath\rho\eta)$ does not vanish as $\xi \in V^*, \eta \in \Gamma(p, \eta_0)$, $\rho < \rho_\eta$, since a is η_0-hyperbolic. The integral converges at infinity since of the decreasing factor $\exp(-\varepsilon\xi^2)$ and commutes with any partial derivative. The integrand can be extended to $\mathbb{C}^n$ as a meromorphic differential form

$$\omega = p^{-1}(\zeta) \exp\left(\imath\zeta x - \varepsilon\zeta^2\right) d\zeta$$

that is holomorphic in the cone $V^* + \imath\Gamma(a, \eta_0) \subset \mathbb{C}^n$. The integral of ω does not depend on ρ in virtue of the Cauchy-Poincaré theorem (the special case of the general Stokes theorem). Check that it is a fundamental solution:

$$\begin{aligned}
a(D) E_{\varepsilon,\rho}(x,t) &= \imath^{-m} \int_{V^*} \frac{p(\xi + \imath\rho\eta) \exp(\imath \langle \xi + \imath\rho\eta, x\rangle) \exp\left(-\varepsilon (\xi + \imath\rho\eta)^2\right)}{p(\xi + \imath\rho\eta)} d\xi \\
&= \int_{V^*} \exp(\imath \langle \xi + \imath\rho\eta, x\rangle) \exp\left(-\varepsilon (\xi + \imath\rho\eta)^2\right) d\xi \\
&= \int_{V^*} \exp(\imath\xi x) \exp\left(-\varepsilon\xi^2\right) d\xi \\
&= (\pi\varepsilon)^{-n/2} \exp\left(-(4\varepsilon)^{-1}\left(|x|^2\right)\right) \to (2\pi)^n \delta_0 \text{ in } S'
\end{aligned}$$

For the third step we have applied again the Cauchy-Poincaré theorem. This yields $a(D) E = \delta_0$. Estimate this integral in the halfspace $\{\langle \eta, x\rangle < 0\}$:

$$\begin{aligned}
|E_\varepsilon(x)| &= \left| \int_{V^*} \frac{\exp(\imath \langle \xi, x\rangle) \exp(-\rho \langle \eta, x\rangle) \exp\left(-\varepsilon (\xi + \imath\rho\eta)^2\right)}{p(\xi + \imath\rho\eta)} d\xi \right| \\
&\leq \exp(-\rho \langle \eta, x\rangle) \int_{V^*} \frac{\exp(-\varepsilon\xi^2) \exp(\varepsilon\rho^2\eta^2)}{|p(\xi + \imath\rho\eta)|} d\xi \leq C\varepsilon^{-n/2} \exp\left(\varepsilon\rho^2\right) \exp(-\rho \langle \eta, x\rangle)
\end{aligned}$$

for a constant C, since of $|p(\xi + \imath\rho\eta)| \geq c_0 > 0$. We take $\varepsilon = \rho^{-2}$;the right side is then equal to $O(\rho^n \exp(-\rho \langle \eta, x\rangle))$. This quantity tends to zero as $\rho \to -\infty$, since $\langle \eta, x\rangle < 0$. Therefore $E(x) = 0$ in the half-space $H_\eta \doteq \{\langle \eta, x\rangle < 0\}$. Therefore E vanishes in the union of all half-spaces $H_\eta, \eta \in \Gamma(p, \eta_0)$. The complement to this union in V is just $\Gamma^*(p, \eta_0)$. This completes the proof. ⊡

Proposition 6 *Let $W \subset V$ be a closed half-space such that the conormal η is not a zero the symbol a_m. If u a solution to the equation $a(D) u = 0$ supported by W, then $u = 0$.*

PROOF. Take a hyperplane $H \subset V \backslash W$. The distribution U vanishes in a neighborhood of H and by Holmgren uniqueness theorem (see Ch.4) it vanishes everywhere.

Corollary 7 *If a a hyperbolic operator with constant coefficients with respect to a covector η, then there exists only one fundamental solution supported by the set $\{\langle \eta, x \rangle \leq 0\}$.*

3.5 Wave propagators

The wave operator

$$\square_n \doteq \frac{\partial^2}{\partial t^2} - v^2 \Delta_x$$

in $V = X \times \mathbb{R}$ with a positive velocity $v = v(x)$ is hyperbolic with respect to the time variable t. The symbol $\sigma(\xi, \tau) = -\tau^2 + v^2 |\xi|^2$ is strictly hyperbolic with respect to the covector $\eta_0 = (0, 1)$. Now we assume that the velocity v is constant; the wave operator is hyperbolic and with respect to any covector $(\eta, 1)$ such that $v|\eta| < 1$ and the union of these covectors is just the cone $\Gamma(\sigma, \eta_0)$. The dual cone is

$$\Gamma^*(\sigma, \eta_0) = \{(x, t), v|x| \geq t\}$$

The f.s. supported by this cone is called the *forward propagator*. For the opposite cone $-\Gamma^* = \{v|x| \leq t\}$ the corresponding f.s. is called the *backward propagator*. Both fundamental solutions are uniquely defined.

For the case $\dim V = 2$ both propagators were constructed in Example 6 in the previous section.

Proposition 8 *The forward propagator for $\square_4$ is*

$$E_4(x, t) = \frac{1}{4\pi v^2 t} \delta(|x| - vt) \tag{3.3}$$

This f.s. acts on test functions $\phi \in D(V)$as follows

$$E_4(\phi) = \left(4\pi v^2\right)^{-1} \int_0^\infty t^{-1} \int_{|x|=vt} \phi(x, t)\, dS dt$$

PROOF. Write (2) for the vector $\eta_0 = (0, -1)$ in the form

$$E(x, t) = (2\pi)^{-4} \lim_{\varepsilon \to 0} E_\varepsilon(x, t) \tag{3.4}$$

$$E_\varepsilon(x, t) = -\int_{X^*} \int_{-\infty}^{\infty} \frac{\exp(\imath \xi x + \imath(\tau - \imath \rho) t) \exp(-\varepsilon \xi^2)}{(\tau - \imath \rho)^2 - |v\xi|^2} d\tau d\xi$$

where ξ, τ are coordinates dual to x, t, respectively and we write ξx instead of $\langle \xi, x \rangle$. The interior integral converges, hence we need not the decreasing factor $\exp(-\varepsilon\tau^2)$. We know that E vanishes for $t < 0$; assume that $t > 0$. The backward propagator vanishes, hence we can take the difference as follows:

$$E_\varepsilon(x,t) = -\int_{X^*} \exp\left(-\varepsilon\xi^2\right) d\xi \times$$
$$\int_{-\infty}^{\infty} \left[\frac{\exp(\imath\xi x + \imath(\tau - \imath\rho)t)}{(\tau - \imath\rho)^2 - |v\xi|^2} - \frac{\exp(\imath\xi x + \imath(\tau + \imath\rho)t)}{(\tau + \imath\rho)^2 - |v\xi|^2} \right] d\tau$$

The interior integral is equal to the integral of the meromorphic form $\omega = \left(\zeta^2 - |v\xi|^2\right)^{-1} \exp(\imath\langle\xi, x\rangle + \imath\zeta t)$ over the chain $\{\operatorname{Im}\zeta = -\rho\} - \{\operatorname{Im}\zeta = \rho\}$, which is equivalent to the union of circles $\{|\zeta - |v\xi|| = \rho\} \cup \{|\zeta + |v\xi|| = \rho\}$. By the residue theorem we find

$$\int ...d\tau = 2\pi\imath \exp(\imath\xi x) \frac{\exp(\imath vt|\xi|) - \exp(-\imath vt|\xi|)}{2v|\xi|} = -2\pi \exp(\imath\xi x) \frac{\sin(vt|\xi|)}{v|\xi|}$$

consequently

$$E_\varepsilon(x,t) = 2\pi v^{-1} \int_{X^*} \exp(\imath\xi x) \frac{\sin(vt|\xi|)}{|\xi|} \exp\left(-\varepsilon\xi^2\right) d\xi$$
$$= 2\pi v^{-1} F^* \left(\frac{\sin(vt|\xi|)}{|\xi|} \exp\left(-\varepsilon\xi^2\right) \right)$$

Lemma 9 *We have for an arbitrary $a > 0$*

$$F\left(\delta_{S(a)}\right) = 4\pi a \frac{\sin(a|\xi|)}{|\xi|} \tag{3.5}$$

where $\delta_{S(a)}$ denotes the delta-density on the sphere $S(a)$ of radius a :

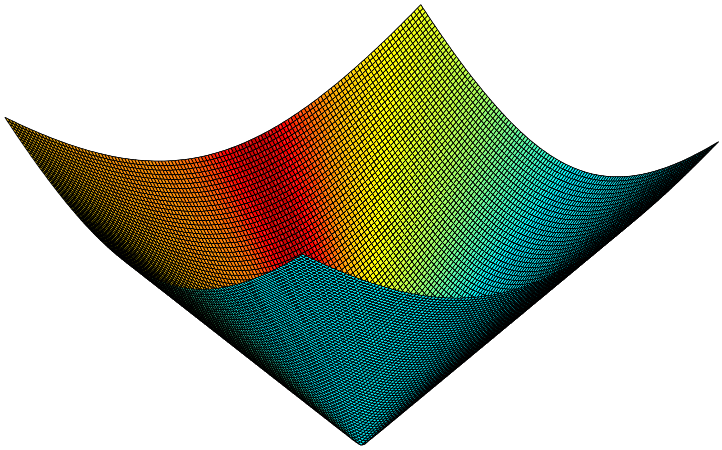
$$\delta_{S(a)}(\phi) = \int_{S(a)} \phi dS$$

Proof of Lemma. For an arbitrary test function $\psi \in D(X^*)$ we have

$$F\left(\delta_{S(a)}\right)(\psi) = \delta_{S(a)}(F(\psi)) = \delta_{S(a)}\left(\int \exp(-\imath x\xi)\psi(\xi) d\xi \right)$$
$$= \int \psi(\xi)\delta_{S(a)}(\exp(-\imath x\xi)) d\xi$$

The functional $\delta_{S(a)}$ has compact support and therefore is well defined on the smooth function $\exp(-\imath x\xi)$. Calculate the value:

$$\delta_{S(a)}(\exp(-\imath x\xi)) = \int_{S(a)} \exp(-\imath x\xi) dS = a^2 \int_0^{2\pi} \int_0^{\pi} \exp(-\imath a|\xi|\cos\theta) \sin\theta d\theta d\varphi$$
$$= -2\pi\imath a^2 \frac{\exp(\imath a|\xi|) - \exp(-\imath a|\xi|)}{a|\xi|} = 4\pi a \frac{\sin(a|\xi|)}{|\xi|}$$



This implies (5).

We have $F^{*}F=(2\pi)^{3}I$, where I stands for the identity operator, hence by (4)

$$F^{*}\left(\frac{\sin\left(a\left|\xi\right|\right)}{\left|\xi\right|}\right)=2\pi^{2}a^{-1}\delta_{S(a)}$$

We find from (5)

$$E_{\varepsilon}\left(x,t\right)=2\pi v^{-1}F^{*}\left(\frac{\sin\left(vt\left|\xi\right|\right)}{\left|\xi\right|}\exp\left(-\varepsilon\xi^{2}\right)\right)\to4\pi^{3}v^{-2}t^{-1}\delta_{S(vt)}$$

This implies (3) in virtue of (4). $\square$

Problem. Calculate the backward propagator for $\square_4$.

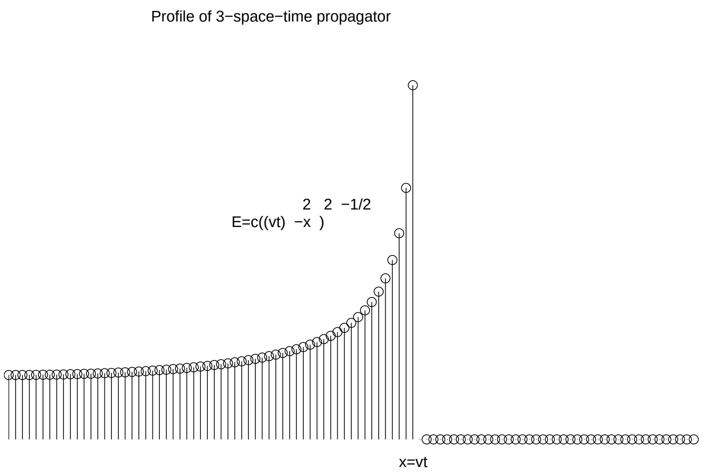
Note that the support of E_4 is the conic 3-surface $S=\{vt=|x|\}$, see the picture

Proposition 10 *For $n=3$ the forward propagator is equal to*

$$E_{3}\left(x,t\right)=\frac{\theta\left(vt-\left|x\right|\right)}{2\pi v\sqrt{v^{2}t^{2}-\left|x\right|^{2}}}\tag{3.6}$$

Replacing $\theta\left(vt-|x|\right)$ by $-\theta\left(-vt-|x|\right)$, we obtain the backward propagator.

The support of the convex cone $\{vt\geq|x|\}$ in E_3. The profile of the function E_3 is shown in the picture



PROOF. We apply the "dimension descent" method. Write $x = (x_1, x_2, x_3)$ in (3) and integrate this function for fixed $y = (x_1, x_2)$ against the density dx_3 from $-\infty$ to ∞. The line $(x_1, x_2) = y$ meets the surface S only if $t \geq v^{-1} |y|$. Therefore the function

$$E_3(y,t) \doteq \int E_4(x,t)\, dx_3 = (2\pi v)^{-1} \int \delta(vt - |x|)\, dx_3$$

is supported by the cone $K_3 \doteq \{vt \geq |y|\}$. Apply this equation to a test function:

$$E_3(\psi) = E_4(\psi \times e) = \left(4\pi v^2 t\right)^{-1} \delta(vt - |x|)(\psi \times e) = \left(4\pi v^2 t\right)^{-1} \int_{y^2 + x_3^2 = (vt)^2} \psi(y,t)\, dS$$

where $e = e(x_3) = 1$. Consider the projection $p : \mathbb{R}^3 \to \mathbb{R}^2,\ x \mapsto y = (x_1, x_2)$. The mapping $p : S \to K$ covers the cone K twice and we have $\mathbf{n}_3 dS = dx_1 dx_2$, where $\mathbf{n}$ is the normal unit field to S and $\mathbf{n}_3 = (vt)^{-1} \left((vt)^2 - |y|^2\right)^{-1/2}$. It follows

$$dS = \frac{vt dx_1 dx_2}{\sqrt{v^2 t^2 - |y|^2}}$$

and

$$E_3(\psi) = \frac{1}{2\pi v} \int \frac{\psi(y)\, dx_1 dx_2}{\sqrt{v^2 t^2 - |y|^2}}$$

which coincides with (6). We need only to check that E_3 is the forward propagator for the operator $\square_3$. It is supported by the proper convex cone K and

$$\square_3 E_3 = \int \square_4 E_4 dx_3 = \int \delta_0(x,t)\, dx_3 = \delta_0(y,t)$$

since $\int \partial_3^2 E_4 dx_3 = 0$. $\boxdot$

3.6 Inhomogeneous hyperbolic operators

Example 7. The forward propagator for the Klein-Gordon-Fock operator $\square_4 + m^2$ is equal to

$$D(x,t) = \frac{\theta(t)}{4\pi c^2 t}\delta(ct - |x|) - \frac{m}{4\pi}\theta(ct - |x|)\frac{J_1\left(m\sqrt{c^2t^2 - |x|^2}\right)}{\sqrt{c^2t^2 - |x|^2}} \tag{3.7}$$

where J_1 is a Bessel function. Recall that the Bessel function of order ν can be given by the formula

$$J_\nu(z) = \sum_0^\infty \frac{(-1)^k}{k!\Gamma(\nu + k + 1)}\left(\frac{z}{2}\right)^{\nu+2k}$$

Remark. The generalized functions in (3), (6), and (7) can be written as pullbacks of some functions under the mapping

$$X \times \mathbb{R} \to \mathbb{R}^2,\ (x,t) \to q = v^2t^2 - |x|^2,\ \theta(t) \tag{3.8}$$

It is obvious for (6) since $\theta(vt - |x|) = \theta(q)\theta(t)$. Fix the coordinates (x_0, x) in V, where $x_0 = vt$. In formulae (3) and (7) we can write $(ct)^{-1}\delta(ct - |x|) = \delta(q)$. Indeed, we have by definition

$$\delta(q)(\alpha) = \int_{q=0}\frac{\alpha}{dq} = \sqrt{1+v^2}\int_0^\infty\int_{q=0}\frac{\phi}{|\nabla q|}dSdt$$

where $\alpha = \phi dx dx_0$ is a test density, i.e. $\phi \in D(X \times \mathbb{R})$; dS is the area in the 2-surface $q = 0,\ t = \mathrm{const}$. We have $\nabla q = (2x, 2v^2t) = (2x, 2v^2t)$, $|\nabla q| = 2\sqrt{1+v^2}vt$ and

$$\int_{q=0}\frac{\phi}{|\nabla q|}dS = \left(2\sqrt{1+v^2}vt\right)^{-1}\int\phi dS$$

Then

$$E_4 = \frac{\theta(t)}{4\pi v^2 t}\delta(|x| - vt) = \frac{\theta(t)}{2\pi v}\delta(q)$$
$$D = \theta(t)\left[\frac{1}{2\pi c}\delta(q) - \frac{m}{4\pi}\theta(q)\frac{J_1(m\sqrt{q})}{\sqrt{q}}\right]$$

This fact has the following explanation. The wave operator and the Klein-Gordon-Fock operator are invariant with respect to the Lorentz group $L_3 = O(3,1)$. This is the group of linear transformations in V that preserve the quadratic form q; the dual transformations in V^* preserve the dual form

$q^*(\xi,\tau) = \tau^2 - c^2 |\xi|^2$. Any Lorentz transformation preserves the volume form $dV = dxdt$ too. Therefore the variety of all source functions is invariant with respect to this group. The forward propagator is uniquely define. Therefore it is invariant with respect to the *orthochronic* Lorentz group $L_{3\uparrow}$, i.e. to group of transformations $A \in L_3$ that preserves the time direction. The functions q and $\operatorname{sgn} t$ are invariant of the orthochronic Lorentz group and any other invariant function (even a generalized function) is a function of these two. We see that is the fact for the forward propagators (as well as for backward propagators).

Example 8. The function

$$D^c(x,t) \doteq -(2\pi)^{-4} \int_{X^*} \int_{\mathbb{R}^*} \frac{\exp(\imath\tau t + \imath(\xi,x))\, d\tau d\xi}{\tau^2 - \xi^2 + \varepsilon\imath}$$

is also a fundamental solution for the wave operator $\Box_4$. The integral must be regularized at infinity by introducing a factor like $\exp(-\varepsilon'\xi^2)$; it does not depend on $\varepsilon > 0$ since the dominator has no zeros in $V^* = X^* \times \mathbb{R}^*$. This function is called *causal* propagator and plays fundamental role in the techniques of Feynman diagrams ? It is invariant with respect to the complete Lorentz group L_3. The causal propagator vanishes in no open set, hence it is not equal to a linear combination of the forward and backward propagators. The causal propagator for the Klein-Gordon-Fock operator is defined in by the same formula with the extra term m^2 in the dominator.

3.7 Riesz groups

This construction provides an elegant and uniform method for explicit construction of forward propagators for powers of the wave operator in arbitrary space dimension.

Let V be a space of dimension n with the coordinates $(x_1, ..., x_n)$; set $q(x) = x_1^2 - x_2^2 - ... - x_n^2$. The set $K \doteq \{x_1 \geq 0, q(x) \geq 0\}$ is a proper convex cone in V. Consider the family of distributions

$$q_+^\lambda(\phi) = \int_K q^\lambda(x)\, \phi(x)\, dx,\ \phi \in D(V),\ \lambda \in \mathbb{C}$$

This family is well-defined in the halfplane $\{\operatorname{Re}\lambda > 0\}$ and is analytic, i.e. $q_+^\lambda(\phi)$ is an analytic function of λ for any ϕ. The family has a meromorphic continuation to whole plane $\mathbb{C}$ with poles at the points

$$\lambda = 0, -1, -2, ...;\ \lambda = \frac{n}{2} - 1, \frac{n}{2} - 2, ...$$

and after normalization

$$Z_\lambda \doteq \frac{q_+^{\lambda - n/2} dx}{\pi^{(n-2)/2} 2^{2\lambda - 1} \Gamma(\lambda)\, \Gamma(\lambda + 1 - n/2)}, \frac{x_+^{\lambda-1} dx}{\Gamma(\lambda)} \tag{3.9}$$

becomes an entire function of λ with values in the space of tempered distributions. We have always $\operatorname{supp} Z_\lambda \subset K$, hence Z_λ is an element of the algebra A_K of tempered distributions with support in the convex closed cone K. The convolution is well-defined in this algebra; it is associative and commutative. The following important formula is due to Marcel Riesz :

$$Z_\lambda * Z_\mu = Z_{\lambda+\mu} \tag{3.10}$$

The points $\lambda = 0, -1, -2, ...$ are poles of the numerator and denominator in (9) and the value of Z_λ at these points can be found as a ratio of residues:

$$Z_0 = \delta_0 dx, \ Z_{-k} = \Box^k Z_0 \tag{3.11}$$

where $\Box = \partial_1^2 - \partial_2^2 - ... - \partial_n^2$ is the differential operator dual to the quadratic form q. In particular, the convolution $\phi \mapsto Z_0 * \phi$ is the identity operator; this together with (10) means that the family of convolution operators $\{Z_\lambda *\}$ is a commutative group, which is isomorphic to the additive group of $\mathbb{C}$. It is called the Riesz group. From (11) we see that

$$\Box^k Z_k = Z_{-k} * Z_k = Z_0 = \delta_0 dx$$

This means that Z_k is a fundamental solution for the hyperbolic operator $\Box^k$ (which is not strictly hyperbolic for $k > 1$). Moreover it is a forward propagator, since $\operatorname{supp} Z_k \subset K$.

If the dimension n is even, the point $\lambda = k = n/2 - 1$ is again a pole of the numerator and denominator in (9), as a consequence of which the support of Z_k is contained in the boundary ∂K. This fact is an expression of the strong Huyghens principle: for even dimension the wave initiated by a local source has back front, whereas the forward front exists for arbitrary dimension. This is just the case for $n = 4$, $k = 1$.

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Chapter 4

The Cauchy problem

4.1 Definitions

Let $a(x, D)$ be a linear differential operator of order m with smooth coefficients in the space V^n and W be an open set in V. Let t be a smooth function in W such that $dt \neq 0$ (called time variable) and f, g be some functions in W. The Cauchy problem for "time" variable t for the data f, g is to find a solution u to the equation

$$a(x, D)\, u = f \tag{4.1}$$

in W that fulfils the initial condition

$$u - g = O(t^m)$$

in a neighborhood of $W_0 \doteq \{x \in W,\ t(x) = 0\}$.

First, assume that the right-hand side f and g are smooth. Introduce the coordinates $x' = (x_1, ..., x_{n-1})$ (space variables) such that (x', t) is a coordinate system in V. Write the equation in the form

$$a(x, D)\, u = \alpha_0 \partial_t^m u + \alpha_1 \partial_t^{m-1} u + ... + \alpha_m u = f \tag{4.2}$$

where $\alpha_j,\ j = 0, 1, ..., m$ is a differential operator of order $\leq j$ which does not contain time derivatives. In particular, α_0 is a function. The initial condition can be written in the form

$$u|_{t=0} = g_0,\ \partial_t u|_{t=0} = g_1, ..., \partial_t^{m-1} u|_{t=0} = g_{m-1}$$

where $g_j = \partial_t^j g|_{t=0}$, $j = 0, ..., m-1$ are known functions in W_0. Set $t = 0$ in (2) and find

$$\alpha_0 \partial_t^m u|_{t=0} = \left(f - \alpha_1 \partial_t^{m-1} u - ... - \alpha_m u\right)|_{t=0} = f|_{t=0} - \alpha_1 g_{m-1} - ... - \alpha_m g_0$$

from this equation we can find the function $\partial_t^m u|W_0$, if $\alpha_0|W_0 \neq 0$. Take t-derivative of both sides of (2) and apply the above arguments to determine $\partial_t^{m+1} u|W_0$ and so on.

Definition. The hypersurface W_0 is called *non-characteristic* for the operator a at a point $x \in W_0$, if $\alpha_0(x) \neq 0$. Note that $\alpha_0(x) = \sigma_m(x, dt(x))$, where $\sigma_m|W \times V^*$ is the principal symbol of a and $\eta \in V^*,\ \eta(x) = t$. An arbitrary smooth hypersurface $H \subset V$

is non-characteristic at a point x, if $\sigma_m(x,\eta) \neq 0$, where η is the conormal vector to H at x.

The necessary condition for solvability of the Cauchy for arbitrary data is that the hypersurface W_0 is everywhere non-characteristic. This condition is not sufficient. For elliptic operator a an arbitrary hypersurface is non-characteristic, but the Cauchy problem can be solved only for a narrow class of initial functions $g_0, ..., g_{m-1}$.

Example 1. For the equation

$$\frac{\partial^2 u}{\partial t \partial x} = 0$$

the variable t as well as x is characteristic, $n = 2$; $\sigma_2 = \tau\xi$. $dt = (0,1)\,; \sigma_2(0,1) = 0$.

Example 2. For the heat equation

$$\partial_t u - \Delta_{x'} u = 0$$

the variable t is characteristic, but the space variables are not. $u|t = 0 = u_0$.

Example 3. The Poisson equation $\Delta u = 0$ is elliptic, but the Cauchy problem

$$u|W_0 = g_0, \partial_t u|W_0 = g_1$$

has no solution in W, unless g_0 and g_1 are analytic functions. In fact, it has no solution in the half-space W_+, if g_0, g_1 are in $L_2(W_0)$, unless these functions satisfy a strong consistency condition.

4.2 Cauchy problem for distributions

The non-characteristic Cauchy problem can be applied to generalized functions as well. First, we write our space as the direct product $V = X \times \mathbb{R}$ by means of coordinates x' and t. For arbitrary test densities ψ and ρ in X and $\mathbb{R}$, respectively, we can take the product $\phi(x', t) = \psi(x')\rho(t)$. It is a test density in V. Let now u an arbitrary (generalized) function in V, fix ψ and define the function in $\mathbb{R}$ by

$$u_\psi(\rho) = u(\psi\rho) = \int v_\psi \rho, \; v_\psi \in C^\infty$$

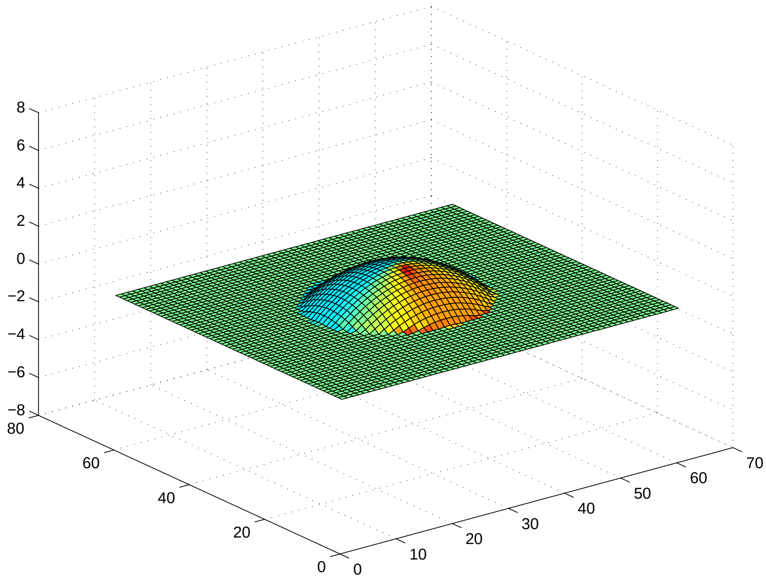
Definition. The function u is called *weakly smooth* in t-variable (or t-smooth), if the functional u_ψ coincides with a smooth function for arbitrary $\psi \in D(X)$.

Any smooth function is obviously weakly smooth in any variable. A weaker sufficient condition can be done in terms of the wave front of u.

If u is weakly smooth in t, then $\partial_t u$ is also weakly smooth in t and the restriction operator $u \mapsto u|_{t=\tau}$ is well defined for arbitrary τ :

$$u|_{t=\tau}(\psi) = \lim_{k\to\infty} u(\psi\rho_k)$$

where $\rho_k \in D(\mathbb{R})$ is an arbitrary sequence of densities that weakly tends to the delta-distribution δ_τ. The limit exists, because of the assumption on u.



Theorem 1 *Suppose that the operator a with smooth coefficients is non-characteristic in t. Any generalized function u that satisfies the equation $a(x,D)u = 0$ is weakly smooth in t variable. The same is true for any solution of the equation $a(x,D)u = f$, where f is an arbitrary weakly t-smooth function.*

It follows that for any solution of the above equation the initial data $\partial_t^j u|_{t=0}$ are well defined, hence the initial conditions (2) is meaningful.

Now we formulate the generalized version of the Holmgren uniqueness theorem:

Theorem 2 *Let $a(x,D)$ be an operator with real analytic coefficients, H is a non-characteristic hypersurface. There exists an open neighborhood W of H in V such that any function that satisfies of $a(x,D)u = 0$ in W that fulfils zero initial conditions in H, vanishes in W.*

4.3 Hyperbolic Cauchy problem

Theorem 3 *Suppose that the operator a with constant coefficients is t-hyperbolic. Then for arbitrary generalized functions $g_0, ..., g_{m-1}$ in $W_0 = \{t = 0\}$ and arbitrary function $f \in D'(V)$ that is weakly smooth in t, there exists a unique solution of the t-Cauchy problem.*

Proof. The uniqueness follows from the Holmgren theorem. Choose linear functions $x = (x_1, ..., x_{n-1})$ such that (x,t) is a coordinate system.

Lemma 4 *The forward propagator E for a possesses the properties*

$$\partial_t^j E|_{t=\varepsilon} \to \frac{\delta_{m-1}^j}{\sigma_m(\eta)} \delta_0(x) \ \text{as } \varepsilon \searrow 0, \ j = 0, ..., m-1 \tag{4.3}$$

Proof of Lemma. Apply the formula (2) of Ch.3

$$E(x,t) = (2\pi)^{-n} \lim_{\varepsilon\to 0} E_{\rho,\varepsilon}(x,t)$$
$$E_{\rho,\varepsilon}(x,t) = \int_{X^*} \int_{-\infty}^{\infty} \frac{\exp((\imath\tau+\rho)t)}{a(\imath\xi, \imath\tau+\rho)} d\tau \exp\left(\imath\xi x - \varepsilon|\xi|^2\right) d\xi, \rho < \rho_\eta$$

We use here the notation $V^* = X^* \times \mathbb{R}$ and the corresponding coordinates (ξ,τ). We assume that $m \geq 2$, therefore the interior integral converges without the auxiliary decreasing factor $\exp(-\varepsilon\tau^2)$. We can write the interior integral as follows

$$-\imath \int_\gamma \frac{\exp(\zeta t)\, d\zeta}{a(\imath\xi,\zeta)}$$

where $\gamma = \{\operatorname{Re}\zeta = \rho\}$. All the zeros of the dominator are to the left of γ. By Cauchy Theorem we can replace γ by a big circle γ' that contains all the zeros, since the numerator is bounded in the halfplane $\{\operatorname{Re}\zeta < \rho\}$. The integral over γ' is equal to the residue of the form $\omega = a^{-1}(\imath\xi,\zeta)\exp(\zeta t)\, d\zeta$ at infinity times the factor $(2\pi\imath)^{-1}$. The residue tends to the residue of the form $a^{-1}(\imath\xi,\zeta)\, d\zeta$ as $t \to 0$. The later is equal to zero since order of a is greater 1. This implies (4) for $j = 0$. Taking the j-th derivative of the propagator, we come to the form $\zeta^j a^{-1}(\imath\xi,\zeta)\, d\zeta$. Its residue at infinity vanishes as far as $j < m-1$. In the case $j = m-1$ the residue at infinity is equal to α_0^{-1}, where α_0 is as in (3). Therefore the $m-1$-th time derivative of the interior integral tends to $2\pi\alpha_0^{-1}$, hence

$$\partial_t^{m-1} E_{\rho,\varepsilon}(x,t) \to 2\pi\alpha_0^{-1} \int_{X^*} \exp(\imath\xi x)\, d\xi = (2\pi)^{n-1} \alpha_0^{-1} \delta_0(x)$$

Taking in account that $\alpha_0 = a_m(\eta)$, we complete the proof. ⊡

Note that for any higher derivative $\partial_t^j E$ the limit as (4) exists and can be found from (4) and the equation $a(D)E = 0$ for $t > 0$.

Proof of Theorem. First we define a solution $\widetilde{u}$ of (1) by

$$\widetilde{u} = E * f$$

The convolution is well defined, since $\operatorname{supp} f \subset H_+$ and $\operatorname{supp} E \subset K$, the cone K is convex and proper. The distribution $\widetilde{u}$ is weakly smooth in t, hence the initial data of it are well defined. Therefore we need now to solve the Cauchy problem for the equation

$$a(D)u = 0 \tag{4.4}$$

with the initial conditions

$$u|_{t=0} = g_0,\ \partial_\eta u|_{t=0} = g_1, ..., \partial^{m-1} u|_{t=0} = g_{m-1} \tag{4.5}$$

where $g_j = u_j - \partial_t^j \widetilde{u}|_{t=0}$, $j = 0, ..., m-1$. Take first the convolution

$$e_0 \doteq \alpha_0 \left(\partial_t^{m-1} E * g_0\right)(t,x) = \alpha_0 \int \partial_t^{m-1} E(t, x-y)\, g_0(y)\, dy \tag{4.6}$$

This is a solution of (5); according to Lemma and $e_0|_{t=0} = g_0$. The derivatives $\partial_t^j e_0|_{t=0}$ can be calculated by differentiating (7), since any time derivative of E has a limit as

$t \to 0$. Therefore we can replace the unknown function u by $u' \doteq u - e_0$. The function u' must satisfies the conditions like (6) with $g_0 = 0$. Then we take the convolution

$$e_1 \doteq \alpha_0 \left(\partial_t^{m-2} E * g_1\right)$$

By Lemma we have $e_1|_{t=0} = 0$ and $\partial_t e_1|_{t=0} = g_1$. Then we replace $u\prime$ by $u'' = u' - e_1$ and so on. $\boxdot$

4.4 Solution of the Cauchy problem for wave equations

Applying the above Theorem to the wave equations with the velocity v, we get the classical formulae:

Case $n = 2$. The D'Alembert formula

$$\begin{aligned} 2vu\left(x,t\right) &= \int_0^t \int_{x-v(t-s)}^{x+v(t-s)} f\left(y,s\right) dyds \\ &+ \int_{x-vt}^{x+vt} g_1\left(y\right) dy + v\left[g_0\left(x+vt\right) + g_0\left(x-vt\right)\right] \end{aligned}$$

Case $n = 3$. The Poisson formula

$$\begin{aligned} 2\pi vu\left(x,t\right) &= \int_0^t \int_{B(x,v(t-s))} \frac{f\left(y,s\right) dyds}{\sqrt{v^2\left(t-s\right)^2 - \left|x-y\right|^2}} \\ &+ \int_{B(x,vt)} \frac{g_1\left(y\right) dy}{\sqrt{v^2 t^2 - \left|x-y\right|^2}} + \partial_t \int_{B(x,vt)} \frac{g_0\left(y\right) dy}{\sqrt{v^2 t^2 - \left|x-y\right|^2}} \end{aligned}$$

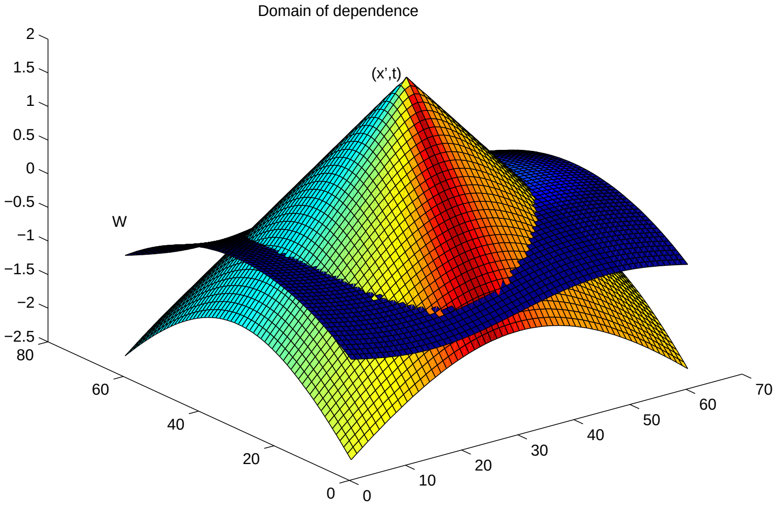
Case $n = 4$. The Kirchhof formula

$$\begin{aligned} 4\pi v^2 u\left(x,t\right) &= \int_{B(x,vt)} \frac{f\left(y, t - v^{-1}\left|x-y\right|\right) dy}{\left|x-y\right|} \\ &+ \int_{S(x,vt)} g_1\left(y\right) dS + \partial_t \left(t^{-1} \int_{S(x,vt)} g_0\left(y\right) dS\right) \end{aligned}$$

Here $B\left(x,r\right)$ denotes the ball with center x, radius r; $S\left(x,r\right)$ is the boundary of this ball.

4.5 Domain of dependence

Assume for simplicity that the right side vanishes: $f = 0$. The solution in a point (x,t) does not depend on the initial data out of the ball $B\left(x,vt\right)$, i.e. a wave that is initiated by the initial functions g_0 and g_1 is propagated with the finite velocity v. This is called the general Huygens principle. In the case $n = 3$ the wave propagating from a compact source has back front (see the picture). This is called the special Huygens principle (Minor premiss).


Domain of dependence
(x',t)
W
2
1.5
1
0.5
0
−0.5
−1
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−2
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80
60
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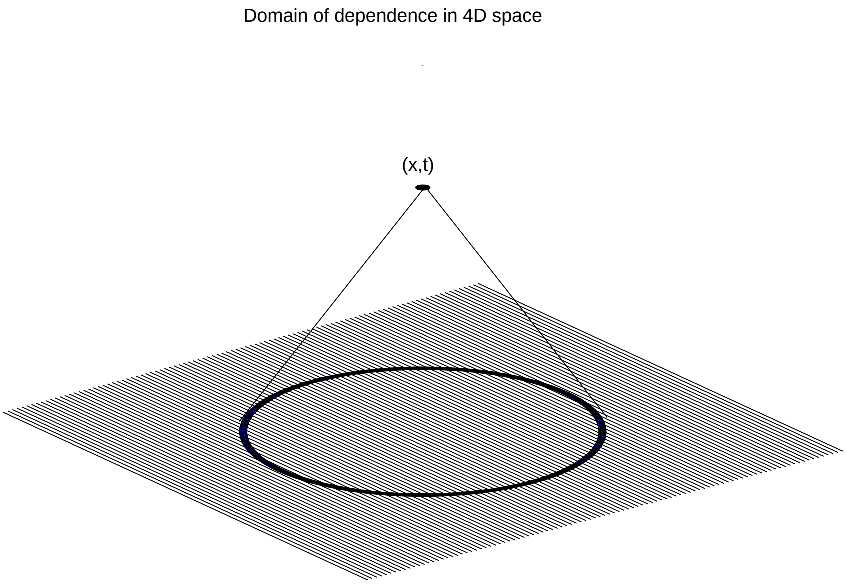
Domain of dependence in 4D space
(x,t)

The special Huygens principle holds for the wave equation with constant velocity in the space of arbitrary *even* dimension $n \geq 4$.

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Chapter 5

Helmholtz equation and scattering

5.1 Time-harmonic waves

Let $a(x, D_x, D_t)$ be a linear differential operator of order m with smooth coefficients in the space-time $V = X^n \times \mathbb{R}$ with coordinates (x, t), whose coefficients do not depend on the time variable t. Consider the equation

$$a(x, D_x, D_t)\, U(x, t) = F(x, t) \tag{5.1}$$

A function of the form $F(x, t) = \exp(\imath\omega t) f(x)$ is called time-harmonic of frequency ω, the function f is called the amplitude. If a solution is also time-harmonic function $U(x, t) = \exp(\imath\omega t) u(x)$, we obtain the time-independent equation for the amplitudes

$$a(x, D_x, \imath\omega)\, u(x) = f(x)$$

Example 1. For the Laplace operator in space-time $a(D_x, D_t) = D_t^2 + \Delta_X$ we have

$$a(D_x, \imath\omega) = -\omega^2 + \Delta$$

This is a negative operator. Therefore any solutions of the equation $(\omega^2 - \Delta)\, u = 0$ in X of finite energy i.e. $u \in L_2(X)$ decreases fast at infinity.

Example 2. If $a = D_t^2 - v^2(x)\,\Delta$ is the wave operator and the velocity v does not depend on time, then

$$-a(x, D_x, \imath\omega) = \omega^2 + v^2(x)\,\Delta$$

is the Helmholtz operator. The Helmholtz equation with $f = 0$ is usually written in the form

$$\left(\Delta + \mathbf{n}^2\omega^2\right) u = 0$$

where the function $\mathbf{n} \doteq v^{-1}$ is called the *refraction* coefficient. The Helmholtz operator is not definite; there are many oscillating bounded solutions of the form $u(x) = \exp(\imath\xi x)$, where $\sigma \doteq \mathbf{n}^2\omega^2 - \xi^2 = 0,\ \xi \in \mathbb{R}^n$. This solution is unbounded, when $\xi \in \mathbb{C}^n \backslash \mathbb{R}^n$.

Find a fundamental solution for the time-independent equation:

Proposition 1 *Let $E(x,t)$ be a fundamental solution for (1) that can be represented by means of the Fourier integral*

$$E_y(x,t) = \frac{1}{2\pi}\int \exp(\imath\omega t)\,\hat{E}_y(x,\omega)\,d\omega \tag{5.2}$$

for a tempered (Schwartz) distribution $\widehat{E}$. Then

$$a(x, D_x, \imath\omega)\,\hat{E}_y(x,\omega) = \delta_y(x) \tag{5.3}$$

i.e. $\hat{E}(x,\omega)$ is a source function for the operator $a(x, D_x, \imath\omega)$ for any ω such that $\hat{E}$ is weakly ω-smooth.

Proof. We have

$$\delta_y(x,t) = a(x, D_x, D_t)\,E_y(x,t) = \frac{1}{2\pi}\int \exp(\imath\omega t)\,a(x, D_x, \imath\omega)\,\hat{E}_y(X,\omega)\,d\omega$$

At the other hand

$$\delta_0(t) = \frac{1}{2\pi}\int \exp(\imath\omega t)\,d\omega,\ \ \delta_y(x,t) = \delta(t)\,\delta_y(x) = \frac{1}{2\pi}\int \exp(\imath\omega t)\,d\omega\delta_y(x)$$

Comparing, we get

$$\int \exp(\imath\omega t)\,a(x, D_x, \imath\omega)\,\hat{E}(x,\omega)\,d\omega = \int \exp(\imath\omega t)\,d\omega\delta(x)$$

which implies (3) in the sense of generalized functions in $V \times \mathbb{R}^*$. If $\hat{E}_y(x,\omega)$ is ω-smooth for some ω_0 we can consider the restriction of both sides to the hyperplane $\omega = \omega_0$. Then we obtain (3).

5.2 Source functions for Helmholtz equation

Apply this method to the wave operator with a constant velocity v. Take the forward propagator E. It is supported in $\{t \geq \mathbf{n}\,|x|\}$ and bounded as $t \to \infty$. Therefore it can be represented by means of the Fourier integral (2) for the tempered distribution

$$\hat{E}(x,\omega) = \int_{\mathbf{n}|x|}^{\infty} E(x,t)\exp(-\imath\omega t)\,dt \tag{5.4}$$

The corresponding source function for the Helmholtz operator is equal $F_n(x,\omega) \doteq -v^2\hat{E}_{n+1}(x,-\omega)$. Calculate it:
Case $n = 1$. We have $E_2(x,t) = (2v)^{-1}\,\theta(vt - |x|)$ and

$$F_1(x,\omega) = -\int_{\mathbf{n}|x|}^{\infty} \exp(\imath\omega t)\,dt = \frac{-\imath}{2\omega\mathbf{n}}\exp(\imath\omega\mathbf{n}\,|x|)$$

Case $n=3$. We have

$$F_3(x,\omega) = -\frac{1}{4\pi\mathbf{n}|x|}\int_0^\infty \delta(|x|-vt)\exp(\imath\omega t)\,dt = -\frac{\exp(\imath\omega\mathbf{n}|x|)}{4\pi|x|}$$

Case $n=2$. We have

$$F_2(x,\omega) = \frac{1}{2\pi}\int_{\mathbf{n}|x|}^\infty \frac{\exp(\imath\omega t)}{\sqrt{t^2-\mathbf{n}^2|x|^2}}dt$$

This integral is not an elementary function; it is equal to $c_0 H_0^{(1)}(\omega\mathbf{n}|x|)$, where $H_0^{(1)}$ is a Hankel function. The equation $F_2(x) = -(2\pi)^{-1}\ln|x| + R(x,\omega)$, where R is a C^1-function in a neighbourhood of the origin.

Proposition 2 *The function $F_n(x,\omega)$ is the boundary value at the ray $\{\omega>0\}$ of a function $F_n(x,\zeta)$ that is holomorphic in the half-plane $\{\zeta=\omega+\imath\tau,\ \tau>0\}$.*

PROOF. The integral (4) has holomorphic continuation at the opposite half-plane.

5.3 Radiation condition

Let K be a compact set in X^3 with smooth boundary and connected complement $X\backslash K$. Consider the *exterior* boundary problem

$$\begin{aligned}\left(\Delta+k^2\right)u(x) &= 0,\ x\in X\backslash K,\ k=\omega\mathbf{n} \\ u(x) &= f(x),\ x\in\partial K \text{ (Dirichlet condition)}\end{aligned} \tag{5.5}$$

where f is a function on the boundary. Any solution is a real analytic function $u=u(x)$ since the Helmholtz operator is of elliptic type. A solution is not unique, unless an additional condition is imposed. The *radiation* (Sommerfeld) condition is as follows

$$u_r - \imath k u = o\left(r^{-1}\right) \text{ as } r=|x|\to\infty;\ u_r = \partial u/\partial r \tag{5.6}$$

Theorem 3 *If $k>0$ and $f=0$, there is only trivial solution $u=0$ to the exterior problem satisfying the radiation condition.*

PROOF. Let $S(R)$ denote the sphere $\{|x|=R\}$ in X. We have $S(R)\subset X\backslash K$ for large $R\geq R_0$. Write for an arbitrary solution u :

$$\int_{S(R)} |u_r-\imath k u|^2\,dS = \int_{S(R)}\left(|u_r|^2+k^2|u|^2\right)dS - \imath k\int_{S(R)}(\overline{u}_r u - \overline{u}u_r)\,dS \tag{5.7}$$

By (6) the left side tends to zero as $R\to\infty$. At the other hand, by Green's formula

$$\int_{S(R)}(\overline{u}_r u-\overline{u}u_r)\,dS = \int_{\partial K}(\partial_\nu\overline{u}u - \overline{u}\partial_\nu u)\,dS$$

where ∂_ν stands for the normal derivative on ∂K. The right side vanishes, since $u|\partial K = 0$, hence (7) implies

$$\int_{S(R)} |u|^2 \, dS \to 0, \; R \to \infty \tag{5.8}$$

Lemma 4 *[Rellich] For $k > 0$ any solution u of the Helmholtz equation in $X \backslash K$ satisfying (8) equals identically zero.*

Proof. Consider the integral

$$U(r) \doteq \int_{S^2} u(rs)\,\phi(s)\,ds$$

where ϕ is a continuous function and ds is the Euclidean area density in the unit sphere S^2. We prove that $U(r) = 0$ for $r > R_0$ and for each eigenfunction ϕ of the spherical Laplace operator Δ_S:

$$\Delta_S \phi = \lambda \phi$$

$R(\lambda) \doteq (\Delta_S - \lambda Id)^{-1}$. In view of the formula

$$\Delta = \partial_r^2 + 2r^{-1}\partial_r + r^{-2}\Delta_S$$

it follows that $U(r)$ satisfies the ordinary equation

$$U_{rr} + 2r^{-1}U_r + \left(k^2 + \lambda r^{-2}\right) U = 0, \; r > R_0$$

This differential equation has two solutions of the form

$$U_\pm(r) = C_\pm r^{-1} \exp(\pm \imath k r) + o\left(r^{-1}\right), \; r \to \infty$$

Clearly, no nontrivial linear combination of U_+ and U_- is $o(r^{-1})$. A the other hand the hypothesis implies that $U(r) = o(r^{-1})$; we deduce that $U = 0$. The operator Δ_S is self-adjoint non-positive and the resolvent is compact. The set of eigenfunctions is a complete system in $L_2(S^2)$ by Hilbert's theorem. This implies that $u = 0$ for $|x| > R_0$. The function u is real analytic, consequently it vanishes everywhere in $X \backslash K$. □

Now we state existence of a solution of the problem (5).

Theorem 5 *[Kirchhof-Helmholtz] If the function f is sufficiently smooth, then there exists a solution of (5) satisfying the radiation condition. This solution is of the form*

$$u(x) = \int_{\partial K} \left[f(y)\,\partial_\nu \frac{\exp(\imath k |x-y|)}{|x-y|} - g(y)\,\frac{\exp(\imath k |x-y|)}{|x-y|} \right] dS_y, \; g = \partial_\nu u \tag{5.9}$$

Sketch of a proof. First we replace the Helmholtz operator by $\Delta + (k+\imath\varepsilon)^2$. The function $F_3(x, k+\varepsilon\imath) \doteq -(4\pi|x|)^{-1}\exp((\imath k-\varepsilon)|x|)$ is a fundamental solution which coincides with $F_3(x,k)$ for $\varepsilon = 0$. The symbol equals $\sigma_\varepsilon = -|\xi|^2 + (k+\varepsilon\imath)^2$ and $|\sigma_\varepsilon| \geq \varepsilon^2 > 0$. Therefore there exists a unique function $u_\varepsilon \in L_2(X\backslash K)$ that satisfies the conditions

$$\left(\Delta + (k+\imath\varepsilon)^2\right) u_\varepsilon = 0 \text{ in } X\backslash K$$
$$u_\varepsilon|\partial K = f$$

(This fact follows from standard estimates for solutions of a elliptic boundary value problem.) Moreover the sequence u_ε has a limit u in $X\backslash K$ and on ∂K as $\varepsilon \to 0$ and $\partial_\nu u_\varepsilon \to \partial_\nu u$. This is called *limiting absorption principle.* By Green's formula

$$u_\varepsilon(x) = \int_{\partial K} - \int_{S(R)} \left[f(y)\,\partial_\nu \frac{\exp((\imath k-\varepsilon)|x-y|)}{|x-y|} - g(y)\frac{\exp((\imath k-\varepsilon)|x-y|)}{|x-y|}\right] dS_y$$

for $x \in B(R)\backslash K$, where $B(R)$ is the ball of radius R. Take $R \to \infty$; the integral over $S(R)$ tends to zero, hence it can be omitted in this formula. Passing to the limit as $\varepsilon \to 0$, we get (9). It is easy to see that right side of (9) satisfies the radiation condition. Indeed, we have for any $y \in \partial K$ the kernel in the second term (simple layer potential) fulfils this condition, since

$$\begin{aligned}\partial_r \frac{\exp(\imath k|x-y|)}{|x-y|} &= \left(\frac{x}{|x|}, \nabla\right)\frac{\exp(\imath k|x-y|)}{|x-y|} \\ &= \frac{(x, x-y)}{|x|\,|x-y|}\frac{\imath k\exp(\imath k|x-y|)}{|x-y|} + O\left(|x|^{-2}\right) \\ &= \frac{\imath k\exp(\imath k|x-y|)}{|x-y|} + O\left(|x|^{-2}\right)\end{aligned}$$

The kernel in the first term (double-layer) equals

$$\partial_\nu \frac{\exp((\imath k-\varepsilon)|x-y|)}{|x-y|} = \frac{(\nu, x-y)}{|x-y|}\frac{\exp(\imath k|x-y|)}{|x-y|} + O\left(|x|^{-2}\right)$$

and fulfils this condition too. □

The equation (9) is called the *Kirchhof* representation. The functions f, g are not arbitrary, in fact, $g = \Lambda f$, where Λ is a first order pseudodifferential operator on the boundary.

Exercise. Check the formula $\Delta_S = (\sin\theta)^{-1}\partial_\theta \sin\theta\partial_\theta + \sin^2\theta\partial_\phi^2$.

Problem. Show that the operator Δ_S is self-adjoint non-positive and the resolvent is compact.

Remark. The radiation condition is a method to single out a unique solution of the exterior problem. The real part of this solution is physically relevant, in particular,

$$\Re F_3 = \frac{\cos(k|x|)}{4\pi|x|}$$

is also a source function. Therefore we can replace $\imath$ to $-\imath$ simultaneously in (4), (6) and (9).

5.4 Scattering on an obstacle

The plane wave $u_i(x) = \exp(\imath k(\theta, x))$ is a solution of the Helmholtz equation in the free space X for arbitrary (incident) unit vector θ. Let K be a compact set in X, called *obstacle.* It impose a boundary value condition to any solution. There are several types of such conditions. We suppose that the obstacle is impenetrable and the field u satisfies the Dirichlet condition $u|\partial K = 0$. In this case the boundary ∂K is called also *soft* or pressure release surface in the context of the acoustic wave theory. In the case of Neumann condition $\partial_\nu u|\partial K = 0$ it is called *hard* surface, the third condition appears for *impedance* surface. The total field $u = u_i + u_s$ is the sum of the incident plane wave and the *scattered* field $u_s(\theta; x)$ in $X\backslash K$ such that u satisfies the Dirichlet condition

$$u_s|\partial K = -\exp(\imath k(\theta, x))\,|\partial K$$

and u_s fulfils the radiation condition. According to the above theorem the scattered field exists and unique. Moreover, by (9) it can be represented in the form

$$u_s(\theta; x) = \frac{\exp(\imath k|x|)}{4\pi|x|} A\left(\theta; \frac{x}{|x|}\right) + O\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to \infty \tag{5.10}$$

for a function A defined on the product $S^2 \times S^2$. This function is called the *scattering amplitude.*

The inverse obstacle problem: to determine the obstacle K from knowledge of the scattering amplitude (or from a partial knowledge).

Application: *radar imaging*.

Another kind of obstacle without sharp boundary surface is a non-homogeneity in the medium, i.e. a variable wave velocity and hence variable refraction coefficient $\mathbf{n} = \mathbf{n}(x)$. Suppose that the function $\mathbf{n}$ is smooth and is equal to a constant $\mathbf{n}_0$ in $X\backslash K$. Then again, for arbitrary unit vector θ there exists a field $u = u_i + u_s$ satisfying the Helmholtz equation

$$\left(\Delta + \omega^2\mathbf{n}^2(x)\right)u = 0$$

where $k = \omega\mathbf{n}_0$ and the scattered field u_s is of the form (9)-(10).

The inverse acoustic problem: to determine the function $\mathbf{n}$ from knowledge of a.

Application: *ultrasound tomography*.

Uniqueness theorems are proved. There is no analytic solution. For the inverse obstacle problem there are various reconstruction algorithms.

5.5 Interferation and diffraction

Take Helmholtz-Kirchhof formula

$$u(x) = \int_{\partial K}\left[f(y)\,\partial_\nu\frac{\exp(\imath k|x-y|)}{|x-y|} + g(y)\,\frac{\exp(\imath k|x-y|)}{|x-y|}\right]dS_y = v+w,\ f = u,\ g = -\partial_\nu u$$

Suppose that ∂K is the half-plane $\{y_1 \geq 0,\ y_2 \in \mathbb{R}\}$ and study the behaviour of the wave field near the light-shadow plane $L \doteq \{x_1 = 0\}$. Consider the second integral

$$w(x) = \int_{-\infty}^{\infty} \int_0^{\infty} g(y) \frac{\exp(\imath k |x-y|)}{|x-y|} dy_1 dy_2$$

We observe the amplitude $|w|^2$ of the wave field w on the screen $S = \{x_3 = r\}$. Suppose that g is a C^1-function decreasing at infinity. We can write $g(y) = g_0(y) + y_1 h(y)$ for a continuous functions g_0 and h such that g_0 does not depend on y_1 for $o \leq y_1 \leq 1$

$$w(x) = \int \int_0^{\infty} g_0 \frac{\exp(\imath k |x-y|)}{|x-y|} dy_1 dy_2 + W(x)$$

where W has a smoother singularity at L. We have $|x-y| = 1 + 1/2\left((x_1-y_1)^2 + (x_2-y_2)^2\right) + O\left((x-y)^4\right)$. The above integral can be aproximated by the product

$$\int_{-\infty}^{\infty} \exp\left(\imath k/2\,(y_2-x_2)^2\right) g(0,y_2)\, dy_2 \int_0^{\infty} \exp\left(\imath k/2\,(y_1-x_1)^2\right) dy_1$$

where $\int_0^{\infty} \exp\left(\imath k/2\,(y_1-x_1)^2\right) dy_1$ is called the *Fresnel* integral. The first factor is a smooth function of x_2 according to the stationary phase formula:

$$\int_{-\infty}^{\infty} \exp\left(\imath k/2\,(y_2-x_2)^2\right) g_0(y_2)\, dy_2 = \sqrt{\frac{2\pi\imath}{k}} g_0(x_2) + O\left(\frac{1}{k\sqrt{k}}\right)$$

A similar representation is valid for the first term v, hence the amplitude $|u| = |v+w|$ oscilates near the light-shadow border: the Fresnel diffraction:

Huygens-Fresnel Principle: the wave field generated by a hole in a screen can be obtained by superposition of elementary fields with the source points in the hole

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Chapter 6

Geometry of waves

6.1 Wave fronts

The wave equation in a non-homogeneous non-isotropical time independent medium in $V = X \times \mathbb{R}$ is

$$a(x, D_x, D_t) u = f, \text{ where} \tag{6.1}$$
$$a(x, D_x, D_t) \doteq \partial_t^2 - \sum_{ij} g^{ij}(x) \partial_i \partial_j + \sum b^i(x) \partial_i + c(x)$$

where $\partial_i = \partial / \partial x^i, i = 1, 2, 3,$ the functions g^{ij}, b^i, c are smooth in a domain $D \subset X$ and fulfil the condition

$$g^{ij}(x) \xi_i \xi_j \geq v_0^2(x) |\xi|^2, \ |\xi|^2 \doteq \sum \xi_i^2$$

for a positive function v_0. The medium is called *isotropic* if this is an equation for a function v_0 which is called the local velocity of the wave. The principal symbol of the equation is

$$\sigma_2(x; \xi, \tau) = -\tau^2 + g^{ij}(x) \xi_i \xi_j$$

A *wave front* is a hypersurface $W \subset D$ that is equal to the singularity set of a solution u of (1) for some $f \in C^\infty(D)$, i.e. W is the smallest closed subset of D such that $u \in C^\infty(D \backslash W)$. Take in account the following statement (Ch.4):

Theorem 1 *Suppose that the operator a with smooth coefficients is non-characteristic in y. Any generalized function u that satisfies the equation $a(x, D) u = 0$ is weakly smooth in y variable. The same is true for any solution of the equation $a(x, D) u = f$, where f is an arbitrary weakly y-smooth.*

It follows that any wave front W is characteristic at each point, i.e. satisfies the condition $\sigma_2(x; \xi, \tau) = 0$ for any point $(x, t) \in W$ and conormal vector (ξ, τ) to W at this point. If W is locally given by the equation $\phi(x, t) = 0$, then the covector $(\nabla \phi, \partial_t \phi)$ is conormal and the function ϕ has to fulfil the nonlinear equation in W :

$$\sigma_2(x; \nabla \phi, \partial_t \phi) = g^{ij}(x) \partial_i \phi \partial_j \phi - (\partial_t \phi)^2 = 0$$

This is called the *eikonal* equation, any function satisfying this equation such that $\partial_t \phi \neq 0$ is called an *eikonal* function. In particular, if $\phi(x, t) = t + \varphi(x)$, the eikonal equation is $g^{ij}(x) \partial_i \varphi \partial_j \varphi = 1$. For isotropical case $|\nabla \varphi| = \mathbf{n}(x)$.

6.2 Hamilton-Jacobi theory

Consider the first order equation in space-time $V = X \times \mathbb{R}$ of dimension n

$$h(x, t; \nabla\phi) = 0 \tag{6.2}$$

where $h(x, t; \eta)$ is a function that is homogeneous in $\eta = (\xi, \tau)$. Write the initial condition as follows $\phi(x, t_0) = \phi_0(x)$.

To solve this equation we consider the system of equations in the phase space $V \times V^*$:

$$h(x, t; \eta)\,|\Lambda = 0,\ \alpha|\Lambda = 0 \tag{6.3}$$

where $\alpha = \xi dx + \tau dt$ is the contact 1-form, Λ is unknown n-dimensional conical submanifold in the phase space. (A submanifold in $V \times V^*$ is called *conical,* if it is invariant under the mapping $(x, \eta) \mapsto (x, \lambda\eta)$ for any $\lambda > 0$. The unknown manifold Λ is Lagrangian, since of the second equation. Suppose that the form dt does not vanishes in Λ and the following initial condition is satisfied:

$$\Lambda|_{t=t_0} = \Lambda_0,\ \text{where } \Lambda_0 \subset X \times V^*$$

is a submanifold of dimension $n-1$ such that $h|\Lambda_0 = 0,\ \alpha|\Lambda_0 = 0$.

Proposition 2 *1 Let Λ be a solution of (3) such that the projection $p : \Lambda \to X$ is of rank $n-1$ at a point (x_0, t_0, η_0). Then there exists a solution of (2) in a neighborhood of (x_0, t_0) such that $d\phi(x_0, t_0) = \eta_0$ and $\phi(x, t) = 0$ in W.*

Proof. We can assume that $\tau \neq 0$ in Λ. The intersection $\Lambda_1 = \Lambda \cap \{\tau = 1\}$ is a manifold of dimension $n-1$ and the projection $p : \Lambda_1 \to X$ is a diffeomorphism in a neighborhood Λ^0 of the point $(x_0, t_0, \eta_0/\tau_0) \in \Lambda_1$. Therefore we have $\xi_j = \xi_j(x)$ in Λ^0 for some smooth functions $\xi_j, j = 1, ..., n$. We have $\xi dx + dt = 0$ in Λ_1, hence $d\xi \wedge dx = 0$. It follows that there exists a function $\varphi = \varphi(x)$ in a neighborhood of the point $(x_0, t_0) \in p(\Lambda^0)$ such that $d\varphi = \xi_j dx^j|\Lambda^0$:

$$\varphi(x) = -t_0 + \int_\zeta \xi_j(x)\,dx^j$$

where ζ is an arbitrary 1-chain that joins x_0 with x (i.e. $\partial\zeta = [x] - [x_0]$). Therefore $\alpha = d(\varphi(x) + t)$. This form vanishes in Λ^0, hence $t + \varphi(x) = \text{const}$ in Λ and in the image of Λ in V. We have $t_0 + \varphi(x_0) = 0$, hence $t + \varphi(x) = 0$ in Λ^0. Then the first equation (3) implies (2). Now we solve (3)

$$h(x, t; \eta)\,|\Lambda = 0,\ \alpha|\Lambda = 0 \tag{6.4}$$

Reminder: The *contraction* of a 2-form β by means of a field w is the 1-form $w \vee \beta$ such that $w \vee \beta\ (v) = \beta(w, v)$ for arbitrary v.
The *Hamiltonian* tangent field v is uniquely defined by the equation

$$v \vee d\alpha = -dh$$

Since $d\alpha = d\xi \wedge dx$, this is equivalent to

$$v = (h_\xi, h_\tau, -h_x, -h_t)$$

which is the standard form of the Hamiltonian field. We have $v(h) = 0$. Assume that $h_\tau \neq 0$ and consider the union $\Lambda \subset V \times V^*$ of all trajectories of the Hamiltonian field that start in Λ_0 this is a manifold and $dt(v) = h_\tau \neq 0$. We have $dh = 0$ in Λ, since $v(h) = 0$ and by the assumption $h|\Lambda_0 = 0$. Show that the Lie derivative of $\alpha = \xi dx + \tau dt$ along v vanishes: $L_v(\alpha) = 0$. Really, we have

$$L_v(\alpha) = d(v \vee \alpha) + v \vee d\alpha = d(\xi h_\xi + \tau h_\tau) - d(dh) = d(\xi h_\xi + \tau h_\tau) = mdh = 0$$

We have $\xi h_\xi + \tau h_\tau = mh$, where m is the degree of homogeneity of h. Therefore the right side equals mdh and vanishes too. We have $\alpha|\Lambda_0 = 0$ by the assumption, hence $\alpha|\Lambda = 0$. $\square$

Write the Hamiltonian system in coordinates

$$\frac{d}{ds}(x,t;\xi,\tau) = v(x,t;\xi,\tau), \text{ i.e. } \frac{dx}{ds} = h_\xi, \frac{dt}{ds} = h_\tau; \frac{d\xi}{ds} = -h_x, \frac{d\tau}{ds} = -h_t \tag{6.5}$$

where $h_\xi = \nabla_\xi h$ and so on. A trajectory of this system, for which $h = 0$ is called also the (zero) *bicharacteristic strip;* the projection of the strip to V is called a *ray.*

The covector $\eta = (\xi, \tau)$ is always orthogonal to the tangent dx/ds of the ray, since

$$\xi\frac{dx}{ds} + \tau\frac{dt}{ds} = \xi h_\xi(x;\xi,\tau) + \tau h_\tau(x;\xi,\tau) = \eta h_\eta(x;\eta) = mh(x;\eta) = 0$$

If the Hamiltonian function does not depend on time, we have $d\tau/ds = 0$.

Construction of wave fronts. Take a wave front W_0 at the time $t = t_0$ and consider all the trajectories of the Hamiltonian system that start at a point $(x, t_0; \xi, 1)$, where $x \in W_0, \xi$ is a covector that vanishes in $T_x(W_0)$ and fulfils the eikonal equation, i.e. $h(x, t_0; \xi, 1) = 0$. The union of these trajectories is just the front W in the domain $\{t > 0\}$.

The condition $p : \Lambda \to X$ is of maximal rank may be violated somewhere. Then the wave front get singularity and the corresponding solution has a *caustic.*

Proposition 3 *2 If a characteristic surface W is tangent to another characteristic surface W' at a point p, then they are tangent along a ray $\gamma \subset W \cap W'$ that contains p.*

Proof. Let $\eta = (\xi, 1)$ the normal covector at p to both surfaces. According to the above construction the front W is the projection of a conic Lagrange manifold Λ, which is a union of trajectories of (5) and W is the union of rays. The point (p, η) belongs to Λ, since the form $\xi dx + dt$ vanishes in $T(W)$. Let γ be the ray through p and Γ be the corresponding bicharacteristic strip through (p, η). It is contained in Λ since any solution of (5) is defined uniquely by initial data. Therefore $\gamma \subset W$ and similarly $\gamma \subset W'$. $\square$

6.3 Geometry of rays

If h does not depend on x and t, the trajectories are straight lines. One more case when the rays can be explicitly written is the following

Proposition 4 *If the velocity v is a linear function in X, the rays in the half-space $\{v > 0\}$ are circles with centers in the plane $\{v = 0\}$.*

Problem. To check this fact.

6.4 Legendre transformation and geometric duality

Definition. Let $f : X \to \mathbb{R}$ be a continuous function; the function g defined in X^* by

$$g(\xi) \doteq \sup_x \xi x - f(x)$$

is called Legendre transformation of f. If f is convex, g is defined in a convex subset of the dual space and is also convex. If g is defined everywhere in X^*, the Legendre transformation of g coincides with f, provided f is convex. This means that the graph of f is the envelope of hyperplanes $t = \xi x - g(\xi)\,,\ \xi \in X^*$.

If $f \in C^1(X)$ an arbitrary function, the Legendre transformation is defined as follows

$$g(\xi) = \xi x - f(x) \text{ as } \xi = \nabla f(x)$$

If $f \in C^2(X)$ and $\det \nabla^2 f \neq 0$ the equation $\nabla f(x) = \xi$ can be solved, at least, locally and the Legendre transform is defined as a multivalued function.

Example. For a non-singular quadratic form $q(x) = q_{ij}x^i x^j/2$ the Legendre transform is again a quadratic form, namely, $\tilde{q}(\xi) = q^{ij}\xi_i\xi_j/2$, where $\{q_{ij}\}$ is the inverse matrix to $\{q^{ij}\}$. Indeed, the system $\xi_i = \partial_i q(x) = q_{ij}x^j$ is solved by $x^i = q^{ij}\xi_j$. Then the Legendre transform equals

$$\xi_i q^{ij}\xi_j - q_{ij}q^{ik}q^{jl}\xi_k\xi_l/2 = \xi_i q^{ij}\xi_j/2 = \tilde{q}(\xi)$$

Definition. Let K be a compact set in X. The function

$$p_K^*(\xi) = \max_K \xi x$$

is called *Minkowski* functional of K. If K is convex and simmetric with respect to the origin, the functional p_K^* is a norm in X^* and the Minkowski functional of the unit ball $\{p_K^*(\xi) \leq 1\}$ is equal to the norm p_K in X generated by K.

Problem. Show that the Legendre transform of the function $(p_K)^2/2$ is equal to $(p_K^*)^2/2$, provided K is convex.

Definition. Let Y be a smooth conic hypersurface in X (i.e. Y is smooth in $X\setminus\{0\}$). The set Y^* of conormal vectors to $Y\setminus\{0\}$ is a cone in X^*. It is called the dual conic surface. If Y is strictly convex, i.e. the intersection $H \cap Y$ is strictly convex for any affine hypersurface in $X\setminus\{0\}$, then Y^* is smooth strictly convex hypersurface too.

Exercise. To check that, if Γ is the interior of the convex hypersurface Y, then the dual cone Γ^* as in Chapter 3 (MP3) is the interior of the Y^*.

Problem. Let f be a smooth homogeneous function in V of degree $d > 1$ such that ∇f does not vanish in $Y \doteq \{f = 0\}$. Show that the Legendre transform g is a homogeneous function of degree $d/(d-1)$ that vanishes in the dual cone Y^*.

6.5 Fermát principle

We have $\sigma_2(x;\xi,\tau) = q(x;\xi) - \tau^2$, where q is positive quadratic form of ξ. The Legendre transform of $q/2$ form with respect to ξ is the quadratic form $\tilde{q}(x;y)/2$, where

$$\tilde{q}(x;y) = g_{ij}(x)\,y^i y^j$$

Let γ be a smooth curve in the Euclidean space X given by the equation $x = x(r), a \leq r \leq b$; the integral

$$T(\gamma) = \int_a^b \sqrt{\tilde{q}(x(r), x'(r))} dr$$

is called the optical length of the curve γ (or the *action*). It is equal to the time of a motion along γ with the velocity $\sqrt{\tilde{q}(x(r), x'(r))}$ ($v = \mathbf{n}^{-1}$ in the isotropical case).

Proposition 5 *3 Each ray of the system (5) for the Hamiltonian function $h = \sigma_2/2$ is an extremal of the optical length integral $T(\gamma)$.*

PROOF. We compare the Euler-Lagrange equation

$$\frac{d}{dr}\frac{\partial F}{\partial x'} - \frac{\partial F}{\partial x} = 0 \tag{6.6}$$

for $F = \sqrt{\tilde{q}(x, x')}$ with the system (5). Suppose for simplicity that the medium is isotropic, i.e. $\sqrt{\tilde{q}(x, x')} = \mathbf{n}(x)|x'|$, $h(x; \xi, \tau) = \left(v^2(x)|\xi|^2 - \tau^2\right)/2$. Set

$$\xi = \frac{\partial F}{\partial x'} = \mathbf{n}(x)\frac{x'}{|x'|}, \ \frac{d}{ds} = \frac{1}{\mathbf{n}|x'|}\frac{d}{dr}$$

The Euler-Lagrange equation turns to

$$\frac{d\xi}{ds} = \frac{1}{\mathbf{n}|x'|}\frac{d}{dr}\frac{\partial F}{\partial x'} = \frac{1}{\mathbf{n}|x'|}\frac{\partial F}{dx} = \frac{\nabla \mathbf{n}}{\mathbf{n}|x'|} = -\nabla v^2 |\xi|^2/2 = -h'_x$$

since $|\xi| = \mathbf{n}$, whereas

$$\frac{dx}{ds} = \frac{x'}{\mathbf{n}|x'|} = v^2\xi = h'_\xi$$

These equations together with $\tau = 1$, $dt/ds = 1$ give (5). $\square$

Exercise. To generalize the proof for the case of anisotropic medium.

Corollary 6 ***Snell's law of refraction:*** $\mathbf{n}_1 \sin\varphi_1 = \mathbf{n}_2 \sin\varphi_2$.

Problem. To verify the Snell's law by means of the Fermát principle.

Corollary 7 *Rays of the equation (1) are geodesics of the metric $g = g_{ij}dx^i dx^j$ and vice versa.*

6.6 The major Huygens principle

The function

$$\sigma_2(x; \eta) = g^{ij}(x)\xi_i\xi_j - \tau^2, \ \eta = (\xi, \tau)$$

is the principal symbol of the equation (1). Fix x and consider the cone $K_x^* \doteq \{\sigma_2(x; \eta) = 0\}$ in V^*. It is called the *cone of normals* at x. The dual cone K_x in V; it is called the *cone of velocities* at x. It is given by the equation $\tilde{h}(x, y, y^0) = 0$, where

$$\tilde{h}(x; y, y_0) = \left(g_{ij}(x)y^i y^j - y^{02}\right)/2$$

is the Legendre transform of $h \doteq \sigma_2/2$ and (y, y^0) stands for a tangent vector to V at (x, t).

The major Huygens principle. *Let W_0 be a smooth wave front at a moment $t = t_0$. For a small time interval Δt and an arbitrary point $x \in W_0$ take the ellipsoid*

$$S_x \doteq \left\{ \tilde{h}\,(x; \Delta x, \Delta t) = 0 \right\} \tag{6.7}$$

Let $\tilde{W}$ be the envelope of these ellipsoids. The claim: the wave front $W_{\Delta t}$ at the moment $t = t_0 + \Delta t$ coincides with a component of $\tilde{W}$ up to $O\,(\Delta t^2)$.

This means, in fact, that the distance between the hypersurfaces is $O\,(\Delta t^2)$ in the standard C^1-metric.

Proof. An arbitrary point $x \in W_0$ is the end of a ray γ given by an equation $x = x\,(t)$, $0 \le t \le t_0$. According to (5), the extension of this ray for the time interval $[t_0, t_0 + \Delta t]$ is approximated by the line interval $[x, \tilde{x}]$, where $\tilde{x} = x + \Delta t\ x'$, $x' = h_\xi\,(x; \xi, 1)$ up to a term $O\,(\Delta t^2)$ and the point $(x; \xi, 1)$ belongs to the bicharacrestic strip that projects to γ. Check that the point $\tilde{x}$ belongs to S_x; we have $\Delta t^{-2}\tilde{h}\,(x, \Delta t\ x', \Delta t) = \tilde{h}\,(x, x', 1)$, since $\tilde{h}$ is a homogeneous quadratic function. By the involutivity of the Legendre transform, $\tilde{h}$ is the Legendre transform of h, i.e.

$$\tilde{h}\,(x, x', 1) = \tilde{\xi} x' + \tilde{\tau} - h\left(x; \tilde{\xi}, \tilde{\tau}\right)$$

where the point $\left(\tilde{\xi}, \tilde{\tau}\right)$ satisfies $h_\xi\left(x; \tilde{\xi}, \tilde{\tau}\right) = x'$, $h_\tau\,(x; \xi, \tau) = 1$. We find $\tilde{\tau} = 1$ form the second equation and $\tilde{\xi} = \xi$ from the first equation. Therefore

$$\tilde{h}\,(x, x', 1) = \xi x' + 1 - h\,(x; \xi, 1) = \xi h_\xi\,(x; \xi, 1) + h_\tau\,(x; \xi, 1) - h\,(x; \xi, 1) = h\,(x; \xi, 1) = 0$$

since h is homogeneous of degree 2. Therefore the point $\tilde{x} \in S_x$ is close to the front $W_{\Delta t}$.

Take another point $y = x + \Delta x \in S_x$; consider the piecewise curve $\gamma_y = \gamma \cup l_y$ where l_y denotes the interval $[x, x + \Delta x]$. The optical length of γ_y is equal the sum of optical lengths of the pieces, i.e. $T\,(\gamma_y) = t_0 + \Delta t = t$. It is the same as for the front $W_{\Delta t}$. The point $x + \Delta x$ belongs to a ray γ' that is close to γ. The time coordinate of this point in γ' is less that $t + \Delta t$ since of the Fermát principle. Therefore this point is *behind* the front $W_{\Delta t}$. This completes the proof. $\square$

6.7 Geometrical optics

This is the *ray method* (Debay's method) and similar methods for construction of high frequency approximations to solutions of the wave equation:

$$au_\omega = O\left(\omega^{-q}\right)$$

where a is a wave operator (1) or a similar operator. One looks for an approximate solution of the form (WKB-form)

$$u_\omega(x, t) = \exp(\imath\omega(\varphi(x) + t))(a_0(x) + \omega^{-1}a_1(x) + \ldots + \omega^{-k}a_k(x)) = \exp\,(\imath\omega t)\,U\,(x, \omega)$$

where the time frequency ω is a big parameter. Then the function

$$U(x,\omega) = \exp(\imath\omega(\varphi(x)))(a_0(x) + (\imath\omega)^{-1} a_1(x) + ... + (\imath\omega)^{-k} a_k(x)) \tag{6.8}$$

is an approximate solution of the Helmholtz equation

$$\left(\omega^2 + g^{ij}\partial_i\partial_j + b^j\partial_j + c\right) U(x,\omega) = O\left(\omega^{-k}\right)$$

The *phase* function φ satisfies the eikonal equation $1 - g^{ij}\partial_i\varphi\partial_j\varphi = 0$ and the *amplitude* functions $a_0, a_1, ..., a_k$ fulfil the recurrent differential equations, called *transport* equations

$$\begin{aligned} 2g^{ij}\partial_i\varphi\partial_j a_0 + \left(\partial_i g^{ij}\partial_j(\varphi) + b^j\partial_j\varphi\right) a_0 &= 0 : Ta_0 = 0 \\ 2g^{ij}\partial_i\varphi\partial_j a_1 + \left(\partial_i g^{ij}\partial_j\varphi + b^j\partial_j\varphi\right) a_1 &= -\left(\partial_i g^{ij}\partial_j + b^j\partial_j + c\right) a_0 \\ &\cdots \\ Ta_k &= L_k(a_0, ..., a_{k-1}) \end{aligned} \tag{6.9}$$

where the operator $T = 2g^{ij}\partial_i\varphi\partial_j + (\partial_i g^{ij}\partial_j(\varphi) + b^j\partial_j\varphi)$ acts along geodesic curves of the metric g. The principal term of (8) is called the approximation of geometrical optics. A *caustic* is an obstruction of the ray method.

6.8 Caustics

Take the manifold Λ in the phase space that is solution of the system (3)

$$h(x,t;\eta)\,|\Lambda = 0,\ \alpha|\Lambda = 0$$

If $\dim\Lambda = n = \dim V$, it is called *Lagrange* manifold. Consider the projection $p : \Lambda \to V$; the image $W = p(\Lambda)$ is called *wave front.* A point $(x,t) \in W$ is *regular*, if $(x,t) = p(\lambda),\ \lambda \in \Lambda$ and rank of p in λ is equal $n-1$ for any λ. The set of singular points is closed; its projection to X called the *caustic* of Λ.

6.9 Geometrical conservation law

We have found the conservation law for the global energy of a field u satisfying the selfadjoint wave equation (MP2) by means of

$$-\frac{\partial}{\partial t}\left\langle\frac{\partial^2 u}{\partial t^2}, u\right\rangle = -\frac{\partial}{\partial t}\sum\left\langle\frac{\partial}{\partial x_i}\left(v^2\frac{\partial u}{\partial x_i}\right), u\right\rangle$$

This identity can be written in the form

$$\operatorname{div} I_{x,t} = 0,\ \text{where } I_{x,t} \doteq \left(v^2\left(\nabla_x u\overline{u}_t - u_t\nabla\overline{u}\right), u_t\overline{u}_t\right)$$

The space-time field $I_{x,t}$ is interpreted as the *energy current.* For a time-harmonic solution $u(x,t) = \exp(\imath\omega t)\,U(x,\omega)$ the last component drops out and $u_t = \imath\omega u$. Therefore the energy current is represented by the field $I_x = v^2(\nabla_x u\overline{u}_t - u_t\nabla\overline{u})$. For the arbitrary selfadjoint wave operator

$$\left(\partial_t^2 - \partial_i g^{ij}\partial_j - c\right) u = 0$$

the energy current is

$$I^i = \frac{\omega}{2\sqrt{-1}} g^{ij} \left(\partial_j U \overline{U} - U \partial_j \overline{U} \right), \; i = 1, 2, 3$$

Substitute the WKB-development for (8) and take in account that the phase and amplitude functions are real:

$$I^i = \omega^2 g^{ij} \partial_j \varphi a_0 + O(\omega)$$

The vector $g^{ij} \partial_j \varphi = h_{\xi_i}(x, \nabla \varphi) = h_{\xi_i}(x, \xi) = dx^i/ds$ is equal to the tangent to the ray through a point $x \in X$. It follows

Corollary 8 *The energy flows along the rays in the approximation of geometrical optics.*

This fact can be explained in a different way. Consider the transport equation for the main term of the amplitude

$$T a_0 \doteq 2 g^{ij} \partial_i \varphi \partial_j a_0 + \partial_i g^{ij} \partial_j (\varphi) a_0 = 0$$

and write it in the form

$$2 d a_0 / ds + \partial_i \left(v^i \right) a_0 = 0 \tag{6.10}$$

where $\partial_i v^i = \operatorname{div} v(s)$, and $v^i \doteq h_{\xi_i} = g^{ij} \partial_j \varphi$ is the X-component of Hamiltonian field that generates the geodesic flow (5). We have $\operatorname{div} v(s) = (V(s))^{-1} dV(s)/ds$where $V(s)$ is the image of the volume element dx in X. Indeed, we have

$$L_v(dx) = d(v \vee dx) = \operatorname{div}(v) dx$$

Therefore (10) is equavalent to

$$\frac{d}{ds} \left(a_0 \sqrt{dx} \right) = 0$$

i.e. the *halfdensity* $a_0 \sqrt{dx}$ is preserved by the geodesic flow. The square of this haldehsity is the *energy* density $\left| a_0 \sqrt{dx} \right|^2 = |a_0|^2 dx$ of the wave field.

Corollary 9 *The energy density is preserved by the geodesic flow.*

Another conclusion is: a solution of the Helmholtz equation can be considered a halfdensity, whose square is the energy density. Also the halfdensity $a_0 \sqrt{dx \wedge dt}$ is preserved by this flow since dt is constant, since $v_t = h_t = 0$. Therefore a solution of the wave equation is a halfdensity in space-time.

Chapter 7

The method of Fourier integrals

7.1 Elements of simplectic geometry

Cotangent bundle. Let M be a manifold. Consider the set $T^*(M) = \cup_M T^*_x(M)$ together with the mapping $p : T^*(M) \to M$ that maps the fibre $T^*_x(M)$ to the point x. It maps an arbitrary element $\omega \in T^*_x(M)$ to the point x. The pair $T^*(M), p)$ is called *cotangent bundle* of the manifold M. The bundle possesses a smooth atlas: for an arbitrary chart (U, φ) in M one takes the set $T^*(U)$ as the domain of a chart in $T^*(M)$. Each element $\omega \in T^*_x(M)$ can be written in the form $\omega = \sum_1^m \xi_j dx_j$, where the coefficients $\xi_i \in$ are uniquely defined. The mapping

$$\varphi \circ p \times \xi : p^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m, \quad \xi(\omega) = (\xi_1, ..., \xi_m) \tag{7.1}$$

is a chart in $T^*(M)$. For another chart (U', φ') of this kind holds the relation $\psi\varphi' = \varphi$, where the transition mapping ψ is of the form $\psi((\xi_1, ..., \xi_n), x) = (\xi'_1, ..., \xi'_n), x)$. Here ξ'_i are coefficients of cotangent vectors in the second chart: $\omega = \sum \xi'_i dx'_i$. They are related to the coefficients in the first charts:

$$\xi_j(\omega) = \sum \frac{\partial x'_i}{\partial x_j} \xi'_i(\omega)$$

Here $J = \{\partial x'_i / \partial x_j\}$ is again the Jacobi matrix of the transition mapping ψ. Consequently the relation between the coefficients is linear and smooth with respect to the coordinates in U (as well as in U'). Therefore the transition mapping belongs to the class C^∞ and $T^*(M)$ has a smooth structure. The natural projection $p : T^*(M) \to M$ is a mapping of smooth manifolds. Each fibre $p^{-1}(x) = T^*_x(M)$ is a vector space (hence $T^*(M)$ is a vector bundle).

Remark. The union of sets $T_x(M)$, $x \in M$ has a structure of vector bundle too. It is called *tangent* bundle.

Canonical forms. The 1-form $\alpha_U = \sum \xi_i dx_i$ is defined for each chart (1). For another chart in $p^{-1}(U')$ the forms α_U and $\alpha_{U'}$ coincide in the intersection $p^{-1}(U) \cap p^{-1}(U')$. This follows from (2). Therefore there is well-defined a 1-form α in $T^*(M)$ such that $\alpha = \alpha_U$ for each chart U. It is called *canonical* 1-form in $T^*(M)$.

The form $\sigma = d\alpha$ is called *canonical* 2-form in the simplectic manifold $T^*(M)$. It is closed: $d\sigma = 0$. In local coordinates $\sigma = \sum_1^m d\xi_i \wedge dx_i$.

Definition. Let M be a manifold of dimension m. A submanifold $\Lambda \subset T^*(M)$ is called *Lagrange* manifold, if it satisfies the conditions $\dim \Lambda = m$ and $\sigma | N = 0$.

Proposition 1 ***1*** *Let Λ be a Lagrange manifold in $T^*(M)$ and $\lambda \in \Lambda$ be a point that is not a critical point of the projection $p : \Lambda \to M$. There exists a neighborhood U of $y = \pi(\lambda)$ and a real smooth function f in U such that the set of solutions of the system*

$$\xi_i = \frac{\partial f}{\partial x_i},\ i = 1, ..., m \tag{7.2}$$

coincides with Λ in a neighborhood of λ.

Proof. Let U be a simply connected neighborhood U of y such that the projection p is a diffeomorphism $p_U : \Lambda(U) \to U$, where we denote $\Lambda(U) = \Lambda \cap \pi^{-1}(U)$. Take a point $\xi \in \Lambda(U)$ and join the point $x = p(\xi)$ with the point y by a curve $\gamma_x \subset U$. We lift this curve by means of the mapping $(p_U)^{-1}$ and get a curve $\Gamma \subset \Lambda(U)$, which join the point ξ with λ. The integral $f(\lambda) = \int_\Gamma \alpha$ does not depend on the choice of the curve γ_x, because of the set $\Lambda(U) \sim U$ is simply connected and the form $\sigma = d\alpha$ vanishes in Λ. The function f is a primitive of the form α in U, i.e. $df = (p_U)^{-1*}\alpha$. This is equivalent to (2). $\square$

Definition. We call the image of projection $p : \Lambda \to M$ of a Lagrange manifold Λ the *locus* (or front) of this manifold. In the case of previous Proposition the locus is an open set in M. In the general case it is subset with singularities.

Definition. Denote by $T_0^*(M)$ the open subset in $T^*(M)$ of pairs (x, ξ), $\xi \neq 0$ .The multiplicative group of positive numbers $^+ = \{t > 0\}$ acts in $T_0^*(M)$ as follows $t : (x, \xi) \mapsto (x, t\xi)$. A trajectory of the group $^+$ is called ray. A subset $K \subset T_0^*(M)$ is called *conic*, if K is invariant with respect to the group, i.e. is a union of rays. We note that no conic Lagrange manifold can satisfy the conditions of Proposition 1. We generalize this proposition in the next section.

Proposition 2 ***2*** *A conic submanifold Λ of dimension* $\dim M$ *is a Lagrange manifold if and only if the canonic 1-form α vanishes in Λ.*

Proof. The part "if" is obvious: $\sigma|K = d\alpha|K = d(\alpha|K) = 0$. We need to check that the equation $\sigma|\Lambda = 0$ implies $\alpha|\Lambda = 0$. Consider the field $e = \sum \xi_i \partial/\partial\xi_i$ (Euler field) in the cotangent bundle. It satisfies the equation $e \vee \sigma = \alpha$. The Euler field is tangent to rays and hence to any conic submanifold. Therefore for any field v in $T^*(M)$ that is tangent to Λ we have

$$\alpha(v) = v \vee \alpha = v \vee (e \vee \sigma) = \sigma(e, v) = 0$$

Example. Let P be a submanifold of manifold M. Consider the set $N_P^*(M) \subset T^*(M)$ of points (x, ξ), $x \in P$ such that the form $\sum \xi_i dx_i$ vanishes in $T_x(P)$. It is called *conormal* bundle to P. This is obviously a conic Lagrange manifold.

7.2 Generating functions

We state a generalization of Proposition 1 for the case of critical point of the projection $p : \Lambda \to M$. First we state

Proposition 3 *3 Let Λ be Lagrange manifold in $T^*(M)$, λ a point in the manifold and r is the rank of the mapping $Dp : T_\lambda(\Lambda) \to T_y(M)$ Suppose that the forms $p^*(dx_1), ..., p^*(dx_r)$ are independent in $T_\lambda(\Lambda)$. The projection*

$$\rho = (x_1, ..., x_r; \xi_{r+1}, ..., \xi_m) : \Lambda \to \mathbb{R}^r \times \mathbb{R}^{m-r} \tag{7.3}$$

is a diffeomorphism of a neighborhood Λ' of the point λ.

Proof. The statement follows from the implicit function theorem, if we show that the point λ is not critical for the mapping ρ. Suppose the opposite. Then there exists a tangent vector $t \in T_\lambda(\Lambda)$, $t \neq 0$ such that $D\rho(t) = 0$. We write

$$t = \sum_{r+1}^{m} a_i \frac{\partial}{\partial x_i} + \sum_{1}^{r} b_j \frac{\partial}{\partial \xi_j}$$

Show that the coefficients a_i are equal zero. In virtue of the assumption for each $i = r+1, ..., n$ the restriction of the form dx_i to the space $T_\lambda(\Lambda)$ depends on the forms $dx_1, ..., dx_r$, i.e. we have $\tau | T_\lambda(\Lambda) = 0$, where $\tau \doteq dx_i - \sum_1^r c_j dx_j$. Therefore we have $0 = \tau(t) = a_i$.
The form $t \vee \sigma = \sum b_j dx_j$ is vanishes in $T_\lambda(\Lambda)$ too, since Λ is a Lagrange manifold. Therefore $t = 0$, which contradicts to the assumption. □

Theorem 4 *4 Let (3) be a coordinate system in a Lagrange manifold in a point λ_0. There exist smooth function $f = f(x_1, ..., x_r, \xi_{r+1}, ..., \xi_m)$ in a neighborhood of $\rho(\lambda_0)$ such that the set Λ coincides with the manifold*

$$\xi_1 = \frac{\partial f}{\partial x_1} \quad , ..., \quad \xi_r = \frac{\partial f}{\partial x_r} \tag{7.4}$$
$$x_{r+1} = -\frac{\partial f}{\partial \xi_{r+1}}, ..., x_m = -\frac{\partial f}{\partial \xi_m}$$

in a neighborhood of the point λ_0.

Proof. We have $\sigma = d\alpha'$, where $\alpha' = \sum_1^r \xi_i dx_i - \sum_{r+1}^m x_j d\xi_j$. Choose a simply connected open set $W \subset \mathbb{R}^r \times \mathbb{R}^{m-r}$ such that the projection $\rho : \rho^{-1}(W) \to W$ is a diffeomorphism and set

$$f(x', \xi'') = \int_\gamma \alpha', \; x' = (x_1, ..., x_r), \; \xi'' = (\xi_{r+1}, ..., \xi_n)$$

where γ is an arbitrary curve in $\rho^{-1}(W) \subset \Lambda$ that connects λ_0 and $\lambda \doteq \rho^{-1}(x', \xi'')$. The integral does not depend on the curve γ in virtue of the equation $d\alpha' | \Lambda = \sigma | \Lambda = 0$. We have $df = \alpha'$, which implies the equations (4). □
We call f generating function for Λ in the point λ_0. It is unique up to an additive constant term.
Remark. The inverse statement is also true, since the set given by the equations (4) is a Lagrange manifold for arbitrary smooth f.

Proposition 5 *5 The Lagrange manifold generated by a function f is conic if f is homogeneous function of coordinates $\xi_{r+1}, ..., \xi_m$ of degree 1. Inversely any conic Lagrange manifold is generated by a homogeneous function of degree 1 in a conic neighborhood of a given point λ_0.*

Proof. Let f be homogeneous function of degree 1. Each derivative $\partial f / \partial x_j$ is homogeneous of degree 1 too and any derivative $\partial f / \partial \xi_i$ is homogeneous of degree 0. Therefore for arbitrary solution (x, ξ) any point $(x, t\xi)$, $t > 0$ satisfies the system (4).
Inversely, if Λ is conic, we can take a conic neighborhood W of the projection of the point $\lambda_0 = (x_0, \xi_0)$. Define a generating function f by means of the integral as above taken over a curve γ from the point $(x_0, 0)$ to λ. This is a generating function too. Check that it is homogeneous. Really for any $\lambda = (x, \xi)$ we can take the curve $\gamma = g \cup r$, where g joins the points x_0 and x and r is the ray $\{(x, t\xi), 0 \leq t \leq 1\}$. Therefore

$$f(\rho(\lambda)) = \int_\gamma \alpha' = \int_r \alpha' = \sum_{r+1}^{m} x_j \xi_j,$$

where $x_{r+1}, ..., x_m$ are functions of $x_1, ..., x_r$.

7.3 Fourier integrals

Fourier integral in an open set $X \subset \mathbb{R}^n$ is a functional of the form

$$I(\phi, a)\{\psi\} = \int_X \int_\Theta \exp(2\pi \imath \phi(x, \theta)) a(x, \theta) d\theta \psi(x) dx, \quad \psi \in \mathcal{D}(X) \tag{7.5}$$

The function ϕ is called phase and a amplitude. They are defined in $X \times \Theta$, where X is an open set in $\mathbb{R}^n$ and $\Theta = \mathbb{R}^N \setminus \{0\}$ is named *ancillary* space. The group $\mathbb{R}_+$ of positive numbers acts in the space $X \times \Theta$ by $(x, \theta) \mapsto (x, t\theta)$. Any set $\{(x, t\theta), t > 0\}$ is called *ray*; a *conic* set is a union of rays. A function f defined in $X \times \Theta$ is termed homogeneous of degree d, if $f(x, t\theta) = t^d f(x, \theta)$ for $t > 0$. The phase function is supposed to be real and homogeneous of degree 1.
We suppose that the amplitude satisfies the estimate $a(x, \theta) = O(|\theta|^\mu)$ for some μ that is locally uniform with respect to $x \in X$. If $\mu + N < 0$, the integral over the ancillary space converges and

$$|I(\phi, a)(\psi)| \leq \int C(x) |\psi(x)| dx \tag{7.6}$$

for some positive continuous function C. If the integral Fourier does not converges absolutely we apply a regularization procedure to turn it to a continuous functional in the space $\mathcal{D}(X)$. For this we suppose that the amplitude satisfies some special conditions.
Definition. Let $\mu, \rho \in \mathbb{R}$, $0 < \rho \leq 1$. The class $S^\mu_\rho = S^\mu_\rho(X \times \Theta)$ is the set of functions a in $X \times \Theta$ that satisfy for arbitrary $i \in \mathbb{Z}^n_+$, $j \in \mathbb{Z}^N_+$

$$\left| D^i_x D^j_\theta a(x, \theta) \right| \leq C_{ij}(x) (|\theta| + 1)^{\mu - \rho |j|}, \tag{7.7}$$

with a continuous function C_{ij} that does not depend on θ. We call the number $\nu \doteq \mu + N/2$ *order* of the Fourier integral $I(\phi, a)$.

Example. An arbitrary smooth homogeneous amplitude a of degree μ belongs to S_1^μ.
Definition. We say that an amplitude a is asymptotical homogeneous of order μ, if it has for any q the following development:

$$a = a_\mu + a_{\mu-1} + ... + a_{\mu-q} + r_q,$$

where each term a_ν is a smooth homogeneous amplitude of degree ν and $r_q \in S_1^{\mu-q-1}$.

We regularize the Fourier integral in following way:

$$I(\phi, a)(\psi) = \lim_{\varepsilon\searrow 0} \int_X \int_\Theta \exp(2\pi\imath\phi(x,\theta) - \varepsilon|\theta|)a(x,\theta)\psi(x)d\theta dx \tag{7.8}$$

The integral in righthand side obviously converges to any $\varepsilon > 0$.
Let $q \geq 0$ be an integer; we say that a distribution $u \in \mathcal{D}'(X)$ is of singular order $\leq q$, if

$$|u(\psi dx)| \leq \sum_{|i|\leq q} \int_X C(x) \left|D^i\psi(x)\right| dx$$

for a continuous function C. In particular, (6) implies that the distribution $I(\phi, a)$ is of singular order ≤ 0.

Theorem 6 *Let ϕ be arbitrary smooth real homogeneous of degree 1 function in $X \times \Theta$ without critical points and $a \in S_\rho^\mu$. The limit (8) exists for any test function ψ. The functional $I(\phi, a)$ is a distribution of singular order $\leq q$, if $\mu + N < q\rho$.*

Remark. At this stage we can consider the Fourier integral as a functional in the space $\mathcal{D}(X)$ of test densities $\rho = \psi dx$ as well. From this point of view the Fourier integral is a generalized function. A more natural approach is to handle it as a generalized halfdensity.

Proof. We call a differential operator A in $X \times \Theta$ *homogeneous* of degree α, if the function Af is homogeneous of degree $d + \alpha$ for an arbitrary homogeneous function f of arbitrary degree d. In particular, the fields

$$b(x,\theta)\frac{\partial}{\partial x_i},\ i = 1, ..., n,\ c(x,\theta)\frac{\partial}{\partial \theta_j},\ j = 1, ..., N$$

are homogeneous operators of degree -1, if the functions $b(x,\theta)$ and $c(x,\theta)$ are homogeneous of degree -1 and 0 respectively. The condition (7) implies that for arbitrary homogeneous operator A of degree -1 and amplitude $a \in S_\rho^\mu$ we have $Aa \in S_\rho^{\mu-\rho}$.

Pick out a function $\chi \in \mathcal{D}(\mathbb{R}^N)$ such that $\chi(\theta) = 1$ for $|\theta| \leq 1$. Write (5) in the form of sum of two integrals with the extra factors χ and $1 - \chi$. The first one is a proper integral which converges to $[I_0 dx](\psi)$ as $\varepsilon \to 0$, where

$$I_0(x) = \int \exp(2\pi\imath\phi(x,\theta))\chi(\theta)a(x,\theta)d\theta$$

is a continuous function. Set $\phi_\varepsilon = 2\pi\phi + \imath\varepsilon|\theta|$.

Lemma 7 *There exists a smooth family of tangent fields v_ε, $\varepsilon \geq 0$ of degree -1 in $X \times \Theta$ such that $v_\varepsilon(\phi_\varepsilon) = -\imath$.*

We postpone a proof of this Lemma. Write the second integral as follows

$$\int\int \exp(\imath\phi_\varepsilon)(1-\chi)a\psi dxd\theta = \int\int v_\varepsilon(\exp(\imath\phi_\varepsilon))(1-\chi)a\psi dxd\theta$$

Integrating by parts the right side, we get the integral

$$\int\int \exp(\imath\phi)v^*((1-\chi)a\psi)dxd\theta,$$

where v^* denotes the conjugated differential operator. This is an operator of degree -1 and we have

$$v_\varepsilon^*((1-\chi)a\psi) = [v_\varepsilon(\chi) - \operatorname{div}(v_\varepsilon)(1-\chi)]a\psi - (1-\chi)(v_\varepsilon(a)\psi + av_\varepsilon(\psi)),$$

whence

$$\int\int \exp(\imath\phi_\varepsilon)(1-\chi)a\psi dxd\theta = \int\int \exp(\imath\phi_\varepsilon)\left[a_0\psi + \sum_j a_j\frac{\partial\psi}{\partial x_j}\right]dxd\theta, \tag{7.9}$$

and

$$a_0 = [v_\varepsilon(\chi) - \operatorname{div}(v_\varepsilon)(1-\chi)]a + v_\varepsilon(a), \quad a_j = -av_\varepsilon(x_j),\ j = 1, ..., n$$

The amplitude a_j, $j = 1, ..., n$ belong to the class $S_\rho^{\mu-1}$ and satisfy (7) with constants C_{ij} that do not depend on ε, since the function $v_\varepsilon(x_j)$ is smooth and homogeneous of degree -1. The same is true for the function $[v_\varepsilon(\chi) - \operatorname{div}(v_\varepsilon)(1-\chi)]a$ since $\operatorname{div}(v_\varepsilon)$ is homogeneous of degree -1 and $v_\varepsilon(\chi)$ has compact support. The function $v_\varepsilon(a)$ belongs to the space $S_\rho^{\mu-\rho}$ and satisfy the corresponding inequality (7) with some constants that do not depend on ε. Taking in account the inequality $\rho \leq 1$, we conclude that the functions $a_0, ..., a_n$ satisfy (7) with some uniform constants and with the exponent $\mu - \rho$ instead of μ. If $\mu + N < \rho$, the integrals

$$\int \exp(\imath\phi_\varepsilon)|a_j|d\theta,\ j = 0, 1, ..., n$$

converges uniformly with respect to ε and we can pass on to the limit in (9). Thus we get the inequality

$$|\lim_{\varepsilon\searrow 0}\int_\Theta\int_X \exp(\imath\phi_\varepsilon)a(x,\theta)\psi(x)dxd\theta \leq C\left[\int |\psi|dx + \sum_1^n \int\left|\frac{\partial\psi}{\partial x_j}\right|dx\right]$$

where the constant C does not depend on ε. It follows that $I(\phi, a)$ is a distribution of order ≤ 1.

If the opposite inequality $\mu + N \geq \rho$ holds, we apply the same method to each term of (9) and get

$$I(\phi_\varepsilon, a)(\psi) = \int \exp(\imath\phi_\varepsilon)\sum_{ij}\left[a_{00}\psi + \sum a_{0j}\frac{\partial\psi}{\partial x_j} + \sum a_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}\right]dxd\theta,$$

where the amplitudes a_{ij} belong to $S_\rho^{\mu-2\rho}$ and satisfy (7) uniformly with respect to ε. We repeat these arguments q times until we reach the inequality $\mu + N < q\rho$. $\boxdot$

Proof of Lemma 1. We set

$$v = \sum b_j \frac{\partial}{\partial x_j} + \sum c_i \frac{\partial}{\partial \theta_i},$$

where

$$b_j = -\frac{\imath}{\sigma}\frac{\partial \phi}{\partial x_j},\ c_i = -\frac{\imath|\theta|^2}{\sigma}\frac{\partial \phi}{\partial \theta_i}, \quad \sigma = \sum \left|\frac{\partial \phi}{\partial x_j}\right|^2 + |\theta|^2 \sum \left|\frac{\partial \phi}{\partial \theta_i}\right|^2$$

The dominator σ does not vanish. We have $v(\phi_\varepsilon) = -\imath - \varepsilon v(|\theta|)$ where the function $v(|\theta|)$ is homogeneous of degree 0. We set $v_\varepsilon = (1 + \varepsilon \imath v(|\theta|))^{-1} v$. $\boxdot$

Lemma 8 *Let u be an arbitrary homogeneous tangent field in $X \times \Theta$ of degree -1 and Ω a smooth differential form of the highest degree in $X \times \Theta$ that vanishes in the complement of $K \times \Theta$ for a compact set $K \subset X$ such that the forms Ω and $L_u\Omega$ are integrable. We have*

$$\int L_u(\Omega) = 0 \tag{7.10}$$

Proof. Suppose that the form Ω has compact support and state the equation

$$\int \Phi_t^*(\omega) = \int \omega \tag{7.11}$$

for small t, where Φ_t means the flow generated by the field u. The integral of a form of the highest degree is invariant with respect to any isomorphism of the manifold. Take the t-derivative of (11) and get (10). For the given form Ω we consider the product $\Omega_k = h_k\Omega$ where $h_k(\theta) = h(k^{-1}\theta)$. The integral $\int \Omega_k$ converges to $\int \Omega$ as $k \to \infty$. We have

$$0 = \int L_u(\Omega_k) = \int u(h_k)\Omega + \int h_k L_u(\Omega)$$

We have $\int h_k L_u(\Omega) \to \int L_u(\Omega)$ as $k \to \infty$. At the other hand $u(h_k) = \sum c_j \partial h_k / d\theta_j = O(k^{-1})$ uniformly in $X \times \Theta$, since the functions $c_j = u(\theta_j)$, $i = 1, ..., N$ are homogeneous of degree 0. Therefore $\int u(h_k)\Omega \to 0$. $\square$

Example. Let X be an open set in $\mathbb{R}^n$, $x_1, ..., x_n$ are coordinate functions. Take $\Omega = fg dx \wedge d\theta$ in (10) and get

$$\int u(f) g dx d\theta = \int f u^*(g) dx d\theta, \tag{7.12}$$

where the sum

$$u^* = -u - \operatorname{div} u,\ \operatorname{div} u \equiv \frac{L_u(dx \wedge d\theta)}{dx \wedge d\theta} = \sum \frac{\partial b_j}{\partial x_j} + \sum \frac{\partial c_i}{\partial \theta_i}$$

is the *conjugated* differential operator to the field u. We check the last equation by means of (4.4.8):

$$L_u(fgdx \wedge d\theta) = u(f)gdx \wedge d\theta + fu(g)dx \wedge d\theta + fgL_u(dx \wedge d\theta)$$
$$L_u(dx \wedge d\theta) = d(u \vee (dx \wedge d\theta)) = \sum_j (-1)^{j-1} d(b_j dx_1 \wedge ...\hat{\jmath}... \wedge dx_n \wedge d\theta)$$
$$+ \sum_i (-1)^{n+i-1} d(c_i dx \wedge d\theta_1 \wedge ...\hat{\imath}... \wedge d\theta_N) = \operatorname{div}(u)\, dx \wedge d\theta$$

Remark. We can use instead of $E_\varepsilon = \exp(-\varepsilon|\theta|)$ another sequence of decreasing functions, f.e. $E_\varepsilon = \exp(-\varepsilon|\theta|^2)$ in (8) . We get the same limit.
Non-degenerate phase. Let ϕ be a phase function in $X \times \Theta$. Consider the set

$$C(\phi) = \{(x,\theta) : d_\theta \phi = 0, \iff \phi'_{\theta_1} = ... = \phi'_{\theta_N} = 0\}$$

where $\phi'_{\theta_j} \doteq \partial\phi/\partial\theta_j$. This is the critical set of the projection $\{\phi = 0\} \to X$.
Definition. The phase function ϕ is called *non-degenerate*, if it has no critical points and the differential forms

$$d\left(\phi'_{\theta_1}\right), ..., d\left(\phi'_{\theta_N}\right)$$

are linearly independent in each point of the set $C(\phi)$. Suppose that ϕ is a non-degenerate phase. The critical set $C(\phi)$ is a conic subset of $X \times \Theta$ of dimension $n + N - N = n = \dim X$. This follows from the Implicit function theorem. Recall that $T^*_\circ(X)$ means the subset of $T^*(X)$ of nonzero cotangent vectors. Consider the mapping

$$\phi_* : C(\phi) \to T^*_\circ(X), \quad (x,\theta) \mapsto (x, d_x\phi(x,\theta))$$

It is well-defined since $d_x\phi$ does not vanish in the set, where $d_\theta\phi = 0$. This mapping is homogeneous of degree 1, since $d_x\phi(x, t\theta) = td_x\phi(x,\theta)$ for $t > 0$.

Proposition 9 *The differential $D\phi_* : T(C(\phi)) \to T(T^*_\circ(X))$ of the mapping ϕ_* is injective in each point of $C(\phi)$.*

Proof. The injectivity of $D\phi$ in a point $(x,\theta) \in C(\phi)$ is equivalent to the following implication:

$$v \in T_{(x,\theta)}(C(\phi)),\ D\phi_*(v) = 0 \Longrightarrow v = 0$$

Write $v = t + \tau$, $t \in T_x(X)$, $\tau \in T_\theta(\Theta)$ and calculate by means of local coordinates in X:

$$0 = D\phi_*(v) = \left(t; v(\phi'_{x_1}), ..., v(\phi'_{x_n})\right) \in T_\omega(T^*_\circ(X)) \tag{7.13}$$

We denote here $\omega = (x, d_x\phi(x,\theta))$ and use the natural isomorphism

$$T_\omega(T^*(X)) \cong T_x(X) \oplus \mathbb{R}^n$$

From (12) we conclude that $t = 0$ and $\tau(\phi'_{x_j}) = 0$, $j = 1, ..., n$ At the other hand the vector $\tau = v$ is tangent to $C(\phi)$, which means

$$\tau(\phi'_{\theta_i}) = 0,\ i = 1, ..., N$$

Extend the vector τ to the constant vector field $\tilde{\tau}$ in Θ. It commutes with the coordinate derivatives in $X \times \Theta$, consequently the last equations are equivalent to the following $d_\theta \tilde{\tau}\phi(\omega) = 0$. This is a linear relation between the forms $d\phi'_{\theta_1}, ..., d\phi'_{\theta_N}$. This relation is in fact trivial, since the phase ϕ is non-degenerate. □

Denote by $\Lambda(\phi)$ the image of the mapping $D\phi_*$. Take an arbitrary point $(x_0, \theta_0) \in X \times \Theta$. In virtue of Implicit function theorem there exists a neighborhood X_0 of x_0 and a neighborhood Θ_0 of θ_0 such that the restriction of $D\phi_*$ to $X_0 \times \Theta_0$ is a diffeomorphism to its image Λ_0. We can take for Θ_0 a conic neighborhood since the mapping $D\phi_*$ is homogeneous. The image Λ_0 is a conic submanifold of dimension $n = \dim X$; it is closed in a conic neighborhood of the point $\omega_0 = \phi_*(x_0, \theta_0)$. The variety $\Lambda(\phi)$ is a union of pieces Λ_0, hence it is a conic set too. If a neighborhood $X_1 \times \Theta_1$ overlaps with $X_0 \times \Theta_0$, then its image Λ_1 is a continuation of the manifold Λ_0. Taking a chain of continuations $\Lambda_0, \Lambda_1, ...$ we can reach a self-intersection point, if the mapping $D\phi_*$ is not an injection. In this case the set Ł(ϕ) may have singular points and we call it *variety*.

Proposition 10 *The set $\Lambda(\phi)$ is closed and locally equal a finite union of conic Lagrange manifolds.*

Proof. Show that the canonical 1-form α vanishes in any vector $w \in T_{(x,\xi)}(T^*(X))$, which belongs to the image of a tangent space $T_{(x,\theta)}(C(\phi))$. We have $\xi = d_x\phi(x, \theta)$ and $w = D\phi_*(v)$ for a tangent vector v to $C(\phi)$ at the point (x, θ). Therefore $v(f) = 0$ for arbitrary function f that vanishes in $C(\phi)$. Let t be the projection of w to X; it is equal the projection of v. We calculate

$$\alpha(w) = \xi dx(t) = d_x\phi(t) = t(\phi) = v(\phi),$$

where the righthand side is taken at the point $(x, \theta) \in C(\phi)$. It is equal zero, because of the function ϕ vanishes in $C(\phi)$. The last fact follows form the Euler identity $\phi = \sum \theta_i \phi'_{\theta_i}$. It follows that any piece Λ_0 of the set $\Lambda(\phi)$ is a Lagrange manifold.
Take an arbitrary point $\omega \in \Lambda(\phi)$, a neighborhood U of the point $x = p(\omega)$ such that its closure K is compact and check the set $\Lambda_K \doteq \Lambda(\phi) \cap p^{-1}(K)$ is closed. For this we take the unit sphere $S(\Theta)$ in the ancillary space and consider the mapping $\phi_* : C(\phi) \cap (K \times S(\Theta)) \to L_K)$. It is continuous and the first topological space is compact. Therefore the image is a closed subset of $\Lambda(\phi)$. The conic set $L_k(\phi)$ is generated by this subset and hence is also closed.
Show that Λ_K is covered by a finite number of Lagrange manifolds. The set $K \times \Theta$ can be covered by a finite number of conic neighborhoods $X_q \times \Theta_q$, $q = 1, ..., Q$ as above. The restriction of the mapping ϕ_* to each neighborhood of this form is a diffeomorphism to its image in virtue of the Implicit function theorem. The set Λ_K is contained in the union of Lagrange manifolds $\phi_*(X_q \times \Theta_q)$, which implies our assertion. □

Proposition 11 *Let Λ be a conic Lagrange manifold. For any point $\lambda \in \Lambda$ there exists a non-degenerate phase function ϕ such that $\lambda \in \Lambda(\phi) \subset \Lambda$.*

Proof. Take the generating function $f = \sum_{k+1}^m f_j \xi_j$ at λ constructed in Proposition 4.8.2 and consider $\theta = (\xi_{k+1}, ..., \xi_m)$ as ancillary variables. Here $f_j = f_j(x', \theta)$, $j = k+1, ..., n$ are smooth functions in W such that the equations

$$x_j = f_j(x', \theta),\ j = k+1, ..., m \tag{7.14}$$

are satisfied in Λ. We set

$$\phi(x,\theta) \doteq \sum x_j\xi_j - f(x',\theta) = \sum_{k+1}^{m}(x_j - f_j)\xi_j$$

and have $\partial\phi/\partial\xi_j = x_j - \partial f/\partial\xi_j = x_j - f_j(x',\theta)$, hence the critical set $C(\phi)$ coincides with (13) and ϕ is non-degenerate. Calculate the x-derivatives:

$$d_x\phi(x,\theta) = (-d_{x'}f,\theta) = (\xi'(x',\theta),\theta) = \xi|\Lambda$$

7.4 Lagrange distributions

Definition. Let X be an open set in $\mathbb{R}^n$ and Λ be a closed conic Lagrange submanifold in $T^*_\circ(X)$. We call an element $u \in \mathcal{D}'(X)$ Λ-distribution, (or Lagrange distribution), if it can written as a locally finite sum of Fourier integrals:

$$u = \sum I(\phi_j, a_j) + v, \quad v \in C^\infty,$$

where for each j the phase ϕ_j is non-degenerate in $X \times \Theta_j$ and

$$\Lambda(\phi_j) \subset \Lambda, \quad a_j \in S^{\mu_j}_\rho(X \times \Theta_j)$$

Definition. Suppose that all amplitudes are asymptotical homogeneous. We shall say that the Lagrange distribution u is of *order* $\leq \nu$, if u admits such a representation where all the Fourier integrals $I(\phi_j, a_j)$ are of the order $\leq \nu$.

Example 5.2. Let Y be a closed submanifold of X given by the equations $f_1(x) = ... = f_m(x) = 0$ such that the forms $df_1, ..., df_m$ are independent in each point of Y. Consider the functional

$$\delta_Y(\rho) = \int_Y \frac{\rho}{df_1 \wedge ... \wedge df_m}$$

on the space $\mathcal{D}(X)$ of test densities. The quotient is a density σ in Y such that $df_1 \wedge ... \wedge df_m \wedge \sigma = \rho$, hence the integral is well-defined. It is called the delta-function in Y.
Show that the delta-function is a Fourier integral with $N = m$ if X is an open set in R^n. Take the phase function $\phi(x,\theta) = \sum_1^m \theta_j f_j(x)$ and the amplitude $a = 1$. In the case $n = 1$ we have for any test density $\rho = \psi dx$

$$I(\rho) = \int\int \exp(2\pi\imath\theta f(x))d\theta\psi(x)dx = \int \exp(2\pi\imath\theta y)d\theta\,\frac{\psi}{f'}dy,$$

if we take $y = f(x)$ as an independent variable. The θ-integral is equal the delta-function, hence $I\{\rho\} = \rho/df|f = 0$, where ρ/df is a smooth function. In the case $m > 1$ we use this formula m times and get

$$I\{\rho\} = \int\int \exp(2\pi\imath\phi(x,\theta))d\theta\rho = \int_Y \frac{\rho}{df_1 \wedge ... \wedge df_m} = \delta_Y(\rho)$$

where δ_Y is the delta-function in the manifold Y. This is a Λ-distribution of order $(\dim X - \dim Y)/2$ for $\Lambda = N^*_Y$.

Properties. For a conic Lagrange manifold Λ we denote $\mathcal{D}'(\Lambda)$ the space of Λ-distributions.
I. We have $WF(u) \subset \Lambda$ for any $u \in \mathcal{D}'(\Lambda)$ according to Theorem 5.2.1.
Problem. Let Λ be an arbitrary closed conic Lagrange manifold and $\lambda \in \Lambda$ be an arbitrary point. To show that there is an element $u \in \mathcal{D}'_\Lambda$ such that $\lambda \in WF(u)$.
II. For any $u \in \mathcal{D}'(\Lambda)$ and any smooth differential operator a in X we have $au \in \mathcal{D}'(\Lambda)$. If u is of order $\leq \nu$, then Pu is of order $\nu + m$, where m is the order of a.
III. Restriction to a submanifold. Let Y be a closed submanifold in X such that $\Lambda \cap N^*(Y) = \emptyset$. Denote

$$\Lambda_Y = \{(y, \eta) : y \in Y, \eta = \xi | T_y(Y), (y, \xi) \in \Lambda\}$$

This is a conic Lagrange submanifold in $T^*(Y)$.

Proposition 12 *Any Λ-distribution u has a restriction u_Y that is a Λ_Y-distribution. If u is of order $\leq \nu$, the distribution u_Y is of order $\leq \nu$ too.*

IV. Product. If Λ' is another conic Lagrange manifold with no common points with $-\Lambda$, then for any Λ-distribution u and any Λ'-distribution u' the product uu' is well-defined as a distribution in X.

7.5 Hyperbolic Cauchy problem revisited

Consider a hyperbolic differential equation of order m in a space-time $X = X_0 \times \mathbb{R}$, where X_0 is an open set in $\mathbb{R}^n$

$$a(x, t; D_x, D_t)u = w \tag{7.15}$$

with smooth coefficients in X; $x = (x_1, ..., x_n)$ are spacial coordinates, t is the time variable. We denote by ξ, τ the corresponding coordinates for cotangent spacial and time vectors respectively. The principal symbol $\sigma_m = \sigma_m(x, t; \xi, \tau)$ of (14) is a polynomial in variables ξ, τ. We suppose that it has order m with respect to τ, which means that any hypersurface $t = \text{const}$ is non-characteristic for P. Consider the Cauchy problem in the domain $t > 0$ with the initial data

$$u(x, 0) = v_0(x), \frac{\partial u(x, 0)}{\partial t} = v_1(x), ..., \frac{\partial^{m-1} u(x, 0)}{\partial t^{m-1}} = v_{m-1}(x), \tag{7.16}$$

where $v_0, ..., v_{m-1}$ are some distributions.

Theorem 13 (Uniqueness) *Any strictly hyperbolic Cauchy problem (14),(15) has no more than one solution.*

Fix a point $y \in X_0$; let $E_y^j \in \mathcal{D}'(X \times \mathbb{R}_+)$, $j = 0, ..., m-1$ be the solution of the initial problem with $w = 0$, $v_i = \delta_j^i \delta_y$. The set of distributions $E_y^0, E_y^1, ..., E_y^{m-1}$ in $X \times_+ \times X$ is called *fundamental* solution of the Cauchy problem. Then one can solve the Cauchy problem with $w = 0$ and arbitrary distributions $u_0, ..., u_{m-1}$ by means of integration:

$$u = \sum_k \int_{X_0} E_y^k v_k(y) dy$$

This formula is valid, at least, for distributions v_k with compact support. In the global case we need an assumption on domain of dependence (see below). The general case is reduced to the case $w = 0$ by means of the Duhamel's method.

Remark. If the coefficients of the operator a do not depend on time, it is sufficient to construct the distribution E_y^{m-1} only, since we have $E_y^k = q_{m-1-k}(y, D)E_y^{m-1}$, $k < m-1$, where q_j is an appropriate differential operator of order j. Then the distribution $E_y = E_y^{m-1}$ is called the fundamental solution.

We describe now a more general construction. Therefore we can represent the symbol as the product of binomials:

$$\sigma_m(x, t; \xi, \tau) = q_0(x, t) \prod_1^m [\tau - \tau_j(x, t; \xi)],$$

where $\tau_1, ..., \tau_m$ are homogeneous functions of variables ξ of degree 1 and $q_0 \neq 0$. Let $\Lambda_0 \subset T_\circ^*(X_0)$ be an arbitrary Lagrange manifold. For any number $j = 1, ..., m$ we consider the Hamiltonian function $h_j(x, t; \tau, \xi) = \tau - \tau_j(x, t; \xi)$ in $T^*(X \times \mathbb{R}_+)$. We "lift" Λ_0 to the bundle $T^*(X \times \mathbb{R})$ taking the manifold

$$W_j = \{(x, 0; \xi, \tau_j(x, 0; \xi)),\ (x, \xi) \in \Lambda\}$$

which is contained in the hypersurface $h_j = 0$. The canonical form α vanishes in W_j. Now we take the Hamiltonian flow generated by h_j

$$\frac{dx}{dt} = \frac{\partial h_j}{\partial \xi}, \quad \frac{d\xi}{dt} = -\frac{\partial h_j}{\partial x} \quad \frac{d\tau}{dt} = -\frac{\partial h_j}{\partial t} \tag{7.17}$$

with initial data from W_j. Denote by Λ_j the union of trajectories of this flow. This is a Lagrange manifold Λ_j in $T^*(X\times)\mathbb{R}$ in virtue of Proposition ?4.7.1. The union $\Lambda = \cup_1^m \Lambda_j$ is also a Lagrange manifold possibly with self-intersection. Note that h_j vanishes in W_j and hence in Λ_j, since it is constant on any trajectory of (16).

Theorem 14 *There exists a neighborhood Y of the hyperplane X_0 in X such that for arbitrary Λ_0-distributions $v_0, ..., v_{m-1}$ the Cauchy problem (14),(15) has a solution u that is a Λ-distribution in Y. If v_k is a Λ_0-distribution of order $\leq \nu + k$ for some ν and $k = 0, 1..., m-1$, then the solution u is of order $\leq \nu$.*

Proof. We describe in short the construction of u. Take an arbitrary point $\lambda \in \Lambda_0$, a local coordinate system (x', θ) for Λ_0, where $x' = (x_1, ..., x_r)$ and $\theta = (\xi_{r+1}, ..., \xi_n)$, $N = n-r$. Let (x_0', θ_0) be the coordinates of λ and $x_0 \doteq p(\lambda) \in X_0$. Take a phase function $\phi_0 = \phi_0(x, \theta)$ in a conic neighborhood of (x_0, θ_0) that generates Λ_0 in a neighborhood of λ. We can write the initial data $v_0, ..., v_{m-1}$ as Fourier integrals with the phase function ϕ_0 and some asymptotical homogeneous amplitudes $b_0, ..., b_{m-1}$ in a neighborhood of (x_0, θ_0), where b_k is of order $\leq \nu - N/2 + k$ for $k = 0, ..., m-1$. The functions $(x', t; \theta)$ form a local coordinate system in Λ_j for any j and we can choose a generating phase function in the form ϕ_j such that $\phi_j(x, 0; \theta) = \phi_0(x, \theta)$. Set $u_{j,\lambda} \doteq I(\phi_j, a_j)$, where a_j are unknown homogeneous amplitudes of degree $\nu - N/2$ and substitute it in the equation. We get a Λ-distribution $w = au_{j,\lambda}$ with the symbol

$$\sigma(w) = \sum_j (-\imath L + s)\, \sigma(u_{j,\lambda}),$$

where $L = L_{p_m}$ is the Lie derivative. The term of degree $\nu + m$ vanishes according to Proposition 5.6.1 since the symbol $\sigma_m = \prod h_k$ vanishes in Λ_j. The next term is calculated by means of Theorem 6.1.1. where s is the subprincipal symbol of P. The degree of this term is equal $\nu + m - 1$. We choose the amplitudes a_j in such a way that the symbol of w vanishes. For this we solve first the equations $(-\imath L + s)\,\sigma(u_j) = 0$. According to (5.5.1) we have $\sigma(u) = a_j\psi_j$, where ψ_j is a non-vanishing halfdensity depending only on the phase function ϕ_j and a_j be the principal homogeneous term of A_j of degree $\nu - N/2$. Dividing the above equation by this halfdensity we get an equation

$$L(a_j) + g_j a_j = 0 \tag{7.18}$$

where g_j is a known function. This is an ordinary equation along the trajectories of the field (16). It has a unique solution for an arbitrary initial data $a_j(x, 0; \theta)$. We specify these data to satisfy the initial condition (15) for the Cauchy problem. This gives the equations

$$(2\pi\imath)^k \sum_j (\phi_j')^k a_{j,\lambda}(x, 0; \theta) = g_\lambda(x, \theta) b_k(x, \theta), \quad k = 0, ..., m-1, \tag{7.19}$$

where we denote $\phi' = \partial\phi/\partial t$ and introduce a factor g_λ that is a smooth homogeneous function of degree 0 supported by a compact conic neighborhood V of (x_0, θ_0) (i.e. the intersection $\operatorname{supp} g_\lambda \cap S^*(X_0)$ is compact). In the k-the equation both sides are homogeneous of the same degree $\nu - N/2 + k$. To solve this system we consider the matrix $W = \{(\phi_j')^k\}$, where $\phi' = d\phi/dt$. We have

$$\det W = \prod_{j<k} (\phi_j' - \phi_k'),$$

hence the matrix W is invertible, if $\phi_j' \neq \phi_k'$ for $j < k$. We have $\phi_j' = \tau_j(x, t; d_x\phi_j)$, since the function h_j vanishes in Λ_j. Therefore $d_x\phi_j(x, 0; \theta) = d_x\phi_0(x; \theta) \neq 0$ in $C(\phi_0)$. The functions $\tau_j(x, 0; d_x\phi_0)$, $j = 1, ..., m$ are different, because of the operator is strictly hyperbolic and $d_x\phi_0 \neq 0$. Therefore the matrix is invertible and the system (18) has a solution $a_{1,\lambda}(x, \theta), ..., a_{m,\lambda}(x, \theta)$, whose components are smooth and homogeneous of degree $\nu - N/2$ and compactly supported in V. Then we solve the transport equations (17) with the initial condition $a_{j,\lambda}(x, \theta)$ for the j-th equation. The solution $a_{j,\lambda}(x, t; \theta)$ exists and is uniformly bounded in a conic neighborhood of λ in Λ_j. The Fourier integral $u_{j,\lambda} \doteq I(\phi_j, a_{j,\lambda})$, $j = 1, ..., m$ satisfies the equation in the first and second highest orders, i.e. the amplitude of $Pu_{j,\lambda}$ is of of order $\leq \nu - N/2 + m - 2$. Set $u_\nu = \sum u_{j,\lambda}$ for an appropriate partition of unity $\{g_\lambda\}$ in a neighborhood of Λ_0. This distribution satisfies the equation $au_\nu = w_1$, where w_1 is a Λ-distribution of order $\leq \nu + m - 2$ and initial conditions $v_k - \partial_t^k u_\nu | t = v_k'$, where v_k' is a Λ_0-distribution of order $\leq \nu + k - 1$ for $k = 0, ..., m-1$.
For the next approximation we look for a new homogeneous amplitudes $a_{j,\lambda}'$ of degree $\nu - N/2 - 1$ and take $u_{j,\lambda}' = I(\phi_j, a_{j,\lambda}')$. Calculating the symbol, we find

$$\sigma(a(u_{j,\lambda} + u_{j,\lambda}')) = (-\imath L + s)\,\sigma(u_{j,\lambda}') + q_1,$$

where q_1 is a asymptotically homogeneous halfdensity of order $\leq \nu + m - 2$ that only depends on $u_{j,\lambda}$. We need to solve the equation

$$(-\imath L + s)\, \sigma(u'_{j,\lambda}) = -q_1$$

in Λ_j with the initial data taken from the system

$$(2\pi\imath)^k \sum_j (\phi'_j)^k a'_{j,\lambda}(x, 0; \theta) = g_\lambda(x, \theta) b'_k(x, \theta), \quad k = 0, ..., m - 1$$

Here b'_k are some homogeneous amplitudes of degree $\nu - N/2 + k - 1$, $k = 0, ...m - 1$. In fact we take for b'_k the principal homogeneous terms of amplitudes of Fourier integrals representing new initial data $v'_k = v_k - \partial_t^k u_\nu | t = 0$. Solving these systems we get amplitudes a'_j and set $u_{\nu-1} = \sum g_{\jmath,\lambda} u'_{j,\lambda}$. The sum $u_\nu + u_{\nu-1}$ is the second approximation. It satisfies the equation $P(u_\nu + u_{\nu-1}) = w_2$, where w_2 is a Λ-distribution of order $\leq \nu + m - 3$ and initial conditions $v'_k - \partial_t^k u_{\nu-1} | t = v''_k$, where v''_k is Λ_0-distribution of order $\leq \nu + k - 2$.
Iterating these arguments we construct an infinite series

$$u_\nu + u_{\nu-1} + u_{\nu-2} + ...$$

of Λ-distributions of orders $\nu, \nu - 1, ...$ We can modify the above construction in such a way that all the amplitudes in the the term $U_{\nu-k}$ vanish in the ball $|\theta| \leq k$. Then this series converges to a Λ-distribution u. It satisfies the conditions: Pu is smooth in Y and the initial conditions are satisfied up to smooth functions. Such distribution u is called *parametrix* of the problem. To get an exact solution from a parametrix one need to find a smooth solution u_∞ to the Cauchy problem with smooth righthand side and initial functions. This can be done by means of reduction to an integral equation. □
GLOBAL EXISTENCE. To prove the global existence in $Y = X_0 \times \mathbb{R}_+$ more conditions on behavior of bicharacteristics are necessary.
Definition. Let $(x, t) \in X \times_+$. The domain of dependence $D(x, t)$ is the union of trajectories of the systems (17), where $j = 1, ..., m$ going in the backward direction, i.e. for times in the interval $[0, t]$. For a set $K \subset X$ we call the union $\cup D(x, t)$, $(x, t) \in K$ domain of dependence of K.

Theorem 15 *Suppose that $X_0 = \mathbb{R}^n$ for an arbitrary compact set $K \subset X_0 \times [0, T)$ its domain of dependence is also a compact set. Then the statement of the above theorem holds for $Y = X_0 \times [0, T)$.*

Proof. The construction of the previous theorem gives a solution u that exists in a neighborhood Y of X_0. Choose a hypersurface $X_f = \{t = f(x)\}$ in Y such that f is a smooth positive function and P is strictly hyperbolic with respect to conormal bundle $N^*(X_f)$. This means that the polynomial $\sigma_m(x, f(x); \xi, \tau df(x))$ is of degree m with respect to τ and all his roots are real and different. If ∇f decrease sufficiently fast at infinity the bundle $N^*(X_f)$ has no common points with Λ. Therefore our solution u has restriction to X_f and this restriction is a Λ_f-distribution as well as restrictions of its conormal derivatives. We take the restriction of the derivatives as new initial conditions

in X_f and solve again the Cauchy problem in a neighborhood Y_f of X_f. This solution u_f agrees with u. They make together a solution of the Cauchy problem in $Y_0 \cup Y_f$. Then we choose a hypersurface $X_g = \{t = g(x)\}$ in Y_f such that $g > f$ and so on. From the condition of theorem follows that we can regulate this construction in such a way that the union of all neighborhoods $Y_0, Y_f, Y_g, ...$ coincides with $X_0 \times [0, T)$. $\boxdot$
Take an arbitrary point $y \in X_0$ and consider the Lagrange variety $T_y^*(X_0)$. Apply the construction of Theorem 6.2.1 taking for Λ_0 this manifold. Let Λ_y be the corresponding Lagrange manifold over X.

Corollary 16 *Suppose that for any compact set $K \subset X$ its domain of dependence is again a compact set. Then for any $y \in X_0$ there exist fundamental system $E_y^0, E_y^1, ..., E_y^{m-1}$, where E_y^k is a Λ_y-distribution of order $\leq (n-1)/2 - k$.*

For each k, $0 \leq k < m$ we apply Theorem 6.2.1 to the initial data $v_k = \delta_y$, $v_j = 0$, $j \neq k$. The delta-distribution δ_y is a Λ_y-distribution of order $(n-1)/2$. Therefore the solution of the Cauchy problem is a Λ_y-distribution of order $(n-1)/2 - k$. $\boxdot$
Remark. We have $WF(E_y^k) \subset \Lambda_y$ according to Property I of Sec.5.3. Therefore supp E_y^k is contained in the locus $L_y = p(\Lambda_y)$. The locus is the union of all bicharacteristic curves γ starting at y. If the coefficients of the symbol σ_m are constant, these curves are straight lines and L_y is a cone with the vertex at y. In general case the locus L_y is called *ray conoid*.
Another geometrical construction of the conoid can be done in "dual" terms. Take coordinates $x_1, ..., x_n$ in X_0 that vanish at y and consider the phase function $\phi_0 = \xi x \doteq \xi_1 x_1 + ... + \xi_n x_n$. It generates the Lagrange manifold $T_y^*(X_0)$. Any phase function ϕ_j has the form $\phi_j(x, t; \xi) = \xi x + \tau_j(x, 0; \xi)t + O(t^2)$, since $h_j(x, t; \phi_j', \nabla\phi_j) = 0$. For any $\xi \neq 0$ the hypersurface $H_j(\xi) = \{\phi_j = 0\}$ is smooth and tangent to the hyperplane $\xi x + \tau(y, 0; \xi)t = 0$ at y. Consider the family of varieties $H_j(\omega)$ where ω ranges in the unit sphere in $T_y^*(X_0)$ and j runs from 1 to m.

Proposition 17 *The conoid L_y is contained in the envelope of the family $\{H_j(\omega)\}$.*

Proof. Apply Proposition 5.4.1 to the fundamental distributions:

$$E_y^k = \sum_j \int_{S(\Theta)} (\phi_j(x, \omega) + 0\imath)^{k+1-n} a_j(x, \omega) d\omega,$$

Here a_j are smooth functions in $U \times S(\Theta)$, where U is a neighborhood of y. This is true, if $n > k + 1$. We see that the kernel $(\phi_j + 0\imath)^{k+1-n}$ is singular only in $H_j(\omega)$, hence the integral is smooth in the compliment to the envelope of the family as above. If $n \leq k + 1$ a similar formula holds with the extra factor $\log |\phi_j|$ in the integrand. This implies the same conclusion.

References
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Chapter 8

Electromagnetic waves

8.1 Vector analysis

Vector operations: Let X be an oriented Euclidean 3-space X with a frame (e_1, e_2, e_3). For vectors $U = u_1 e_1 + u_2 e_2 + u_3 e_3, V = .., W = ... \in X$

$$\begin{aligned}
U \times V &= \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ e_1 & e_2 & e_3 \end{pmatrix} = -V \times U \\
(U \times V, W) &= \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \\
U \times (V \times W) &= -(U, V)\, W + (U, W)\, V \neq (U \times V) \times W
\end{aligned}$$

For a smooth vector field V and a function a

$$\begin{aligned}
\nabla \times V = \operatorname{rot} V = \operatorname{curl} V &= \det \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \\ e_1 & e_2 & e_3 \end{pmatrix} \\
&= (\partial_2 v_3 - \partial_3 v_2) e_1 + (\partial_3 v_1 - \partial_1 v_3) e_2 + (\partial_1 v_2 - \partial_2 v_1) e_3 \\
(\nabla, V) &= \operatorname{div} V = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 \\
(\nabla, \nabla \times V) &= 0 \\
\nabla \times (\nabla \times V) &= -\Delta V - \nabla (\nabla, V) \\
(\nabla, aV) &= (\nabla a, V) + a\, (\nabla, V) \\
\nabla \times aV &= \nabla a \times V + a \nabla \times V \\
(\nabla, V \times U) &= (\nabla \times V, U) - (V, \nabla \times U)
\end{aligned}$$

Orthogonal transformations. Let U, V be vectors, i.e. they transform as the frame vectors e_j by means of the group $O(X)$. Then $U \times V$ is a *pseudovector* (axial) vector, i.e. $A(U \times V) = \operatorname{sgn}(\det A)\ (AU \times AV)$, $A \in O(X)$. A pseudovector is covariant for the subgroup $SO(X)$ and does not change under the symmetry $x \mapsto -x$. If U is a vector, U is a pseudovector, then $U \times V$ is a vector.

8.2 Maxwell equations

The *electric field* E, the *magnetic field* H, the *electric induction* D and the *magnetic induction* B in the Euclidean space-time $X \times \mathbb{R}$ are related by the Maxwell system of equations

$$\nabla \times H = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial D}{\partial t} \quad \text{(Ampère, Biot-Savart-Laplace's law)} \tag{8.1}$$

$$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \quad \text{(Faraday's law)} \tag{8.2}$$

$$(\nabla, B) = 0 \quad \text{(Gauss's law)} \tag{8.3}$$

$$(\nabla, D) = 4\pi\rho \quad \text{(corollary of Coulomb's law)} \tag{8.4}$$

with the sources: the *charge density* ρ and the *current* j. The term $\partial D / \partial t$ is called the Maxwell displacement current. The Gauss' units system - *centimeter, gram, second* - is used; $c \approx 3 \cdot 10^{10}\, \mathrm{cm} / \sec$.

E, D are vector fields, i.e. they are covariant to the orthogonal group $O(X)$ and

H, B are pseudovector field (axial vectors), i.e. they are covariant to the special orthogonal group $SO(X)$ and do not change under the symmetry $x \mapsto -x$.

$\dim E = \dim D = \dim H = \dim B = L^{-1/2} M^{1/2} T^{-1}$.

Integral form of the Maxwell system in the oriented space-time

$$\int_{\partial S} (H, dl) = \frac{4\pi}{c} \int_S (j, ds) + \frac{1}{c} \frac{\partial}{\partial t} \int_S (D, ds)$$

$$\int_{\partial S} (E, dl) = -\frac{1}{c} \frac{\partial}{\partial t} \int_S (B, ds)$$

$$\int_{\partial U} (B, ds) = 0$$

$$\int_{\partial U} (D, ds) = 4\pi \int_U \rho dx$$

where

ds is the oriented surface element: $ds = t_1 \times t_2 |ds|$; (t_1, t_2) is an orthonormal basis of tangent fields in the surface S that define the orientation of S;

dx is the volume form (not a density!) in X.

Conservation law for charge. The charge and the current are not arbitrary: applying $\nabla \times$ to the first equation and ∂_t to the forth one, we get $(\nabla, j) + \partial_t \rho = 0$ and in the integral form

$$\int_{\partial U} (j, ds) + \frac{\partial}{\partial t} \int_U \rho dx = 0$$

This is a conservation law for charge: if there is now current through the boundary ∂U, then the charge $\int_U \rho dx$ is constant.

Symmetry. The system is invariant for the transformations:

$$E' = \cos\theta \cdot E + \sin\theta \cdot H, \; H' = \cos\theta \cdot H - \sin\theta \cdot E$$

i.e. with respect to the group $U(1)$. This is a very simple example of gauge invariant system. Another example: the Dirac-Maxwell system; the group is infinite.

Potentials. The equation (2) with constant coefficients can be solved:

$$B = \nabla \times A,\ E = -\nabla A^0 - \frac{1}{c}\frac{\partial A}{\partial t}$$

A, A^0 are the vector and the scalar potentials. Physical sense: Aharonov-Bohm' quantum effect.

Material equations. To complete the Maxwell system one use material equations $D = D(E, H)$, $B = B(E, H)$. In the simplest form:

$$D = \varepsilon E,\ B = \mu H$$

ε is the (scalar) electric *permittivity*, μ is the (scalar) magnetic *permeability.* They are dimensionless positive coefficients depending on the medium; $\varepsilon = \mu = 1$ for vacuum, otherwise $\varepsilon \geq 1, \mu \geq 1$. The velocity of electromagnetic waves is equal to $v = c/\sqrt{\varepsilon\mu}$.

The principal symbol of the Maxwell system is the 8×6-matrix

$$\sigma_1 = \begin{pmatrix} -\tilde{\varepsilon}\tau I_3 & \xi \times \cdot \\ \xi \times \cdot & \tilde{\mu}\tau I_3 \\ 0 & \mu(\xi, \cdot) \\ \varepsilon(\xi, \cdot) & 0 \end{pmatrix}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and I_3 stands for the unit 3×3 matrix and $\tilde{\varepsilon} = \varepsilon/c,\ \tilde{\mu} = \mu/c$. There are 28 6×6-minors. One of them is

$$\det \begin{pmatrix} \tilde{\varepsilon}\tau & 0 & 0 & 0 & -\gamma & \beta \\ 0 & \tilde{\varepsilon}\tau & 0 & \gamma & 0 & -\alpha \\ 0 & 0 & \tilde{\varepsilon}\tau & -\beta & \alpha & 0 \\ 0 & \gamma & -\beta & \tilde{\mu}\tau & 0 & 0 \\ -\gamma & 0 & \alpha & 0 & \tilde{\mu}\tau & 0 \\ \beta & -\alpha & 0 & 0 & 0 & \tilde{\mu}\tau \end{pmatrix} = \tilde{\tau}^2 \left(\tilde{\tau}^2 - \xi^2\right)^2,$$

where $\xi = (\alpha, \beta, \gamma)$, $\xi^2 = \alpha^2 + \beta^2 + \gamma^2$, $\tilde{\tau} = (\tilde{\varepsilon}\tilde{\mu})^{1/2}\tau$.

Let $A = \mathbb{C}[\alpha, \beta, \gamma, \tau]$ be the algebra of polynomials and J be the ideal generated by all 6×6-minors of σ_1. We have $J = \left(v^2(x)\xi^2 - \tau^2\right)^2 \cdot \mathfrak{m}^2$, where $v \doteq (\tilde{\varepsilon}\tilde{\mu})^{-1/2}$ is the velocity of electro-magnetic waves in the medium and $\mathfrak{m} \subset A$ is the maximal ideal of the point $(0, 0)$. Note that $h = v^2(x)\xi^2 - \tau^2$ is the Hamiltonian function of the wave equation with the velocity v. On the other hand, each component of the field (E, H) satisfies the wave equation with the principal symbol $h(x; \xi, \tau)$.

8.3 Harmonic analysis of solutions

Consider, first, the wave equation in $X \times \mathbb{R}$ with a constant velocity v

$$\left(\frac{\partial^2}{\partial t^2} - v^2\Delta\right) u = 0$$

The symbol is $\sigma_2 = h = v^2\xi^2 - \tau^2$. The characteristic variety is the cone $\{h(\xi, \tau) = 0\} \subset \mathbb{C}^4$. A general solution is equal to a superposition of exponential solutions $\exp(\imath((\xi, x) + \tau t))$; the algebraic condition is that $h(\xi, \tau) = 0$.

Theorem 1 *Let Ω be a convex open set in space-time. An arbitrary generalized solution of the wave equation in Ω can be written in the form*

$$u(x) = \int_{h=0} \exp(\imath((\xi, x) + \tau t))\, m, \tag{8.5}$$

where m is a complex-valued density supported by the variety $\{h = 0\}$ such that for an arbitrary compact $K \subset \Omega$ we have

$$\int \exp(p_K(\operatorname{Im}(\xi, \tau))) \left(|\xi|^2 + |\tau|^2 + 1\right)^{-q} |m| < \infty$$

for some $q = q(K)$. Vice versa, for any density that fulfils this condition the integral (5) is a generalized solution of the wave equation in Ω.

The function p_K is the Minkowski functional of K. The density m is not unique.

Maxwell system. Suppose that the coefficients ε and μ are constant and $j = 0, \rho = 0$. The plane waves

$$E = \exp(\imath((\xi, x) + \tau t))\, e,\ H = \exp(\imath((\xi, x) + \tau t))\, h \tag{8.6}$$

If the vectors e, h satisfying

$$\tilde{\varepsilon}\tau e + \xi \times h = 0,\ \xi \times e - \tilde{\mu}\tau h = 0,\ (\xi, h) = 0,\ (\xi, e) = 0$$

then the plane wave (5) satisfies the Maxwell system in the free medium. Moreover, an arbitrary solution is a superposition of the plane waves. Take the 6×6-matrix

$$\begin{pmatrix} \xi \times \xi \times \cdot & -\tilde{\varepsilon}\tau\xi \times \cdot \\ \tilde{\mu}\tau\xi \times \cdot & \xi \times \xi \times \cdot \end{pmatrix} \tag{8.7}$$

$$= \begin{pmatrix} -\beta^2 - \gamma^2 & \alpha\beta & \alpha\gamma & 0 & \tilde{\varepsilon}\tau\gamma & -\tilde{\varepsilon}\tau\beta \\ \alpha\beta & -\alpha^2 - \gamma^2 & \beta\gamma & -\tilde{\varepsilon}\tau\gamma & 0 & \tilde{\varepsilon}\tau\alpha \\ \alpha\gamma & \beta\gamma & -\alpha^2 - \beta^2 & \tilde{\varepsilon}\tau\beta & -\tilde{\varepsilon}\tau\alpha & 0 \\ 0 & -\tilde{\mu}\tau\gamma & \tilde{\mu}\tau\beta & -\beta^2 - \gamma^2 & \alpha\beta & \alpha\gamma \\ \tilde{\mu}\tau\gamma & 0 & -\tilde{\mu}\tau\alpha & \alpha\beta & -\alpha^2 - \gamma^2 & \beta\gamma \\ -\tilde{\mu}\tau\beta & \tilde{\mu}\tau\alpha & 0 & \alpha\gamma & \beta\gamma & -\alpha^2 - \beta^2 \end{pmatrix}$$

Each line of this matrix satisfies (6), since $\xi \times \xi \times V = -|\xi|^2 V + \xi(\xi, V)$.

Theorem 2 *Let (e_j, h_j), $j = 1, 2$ be arbitrary lines of the matrix (7) and Ω be an arbitrary convex domain in the space-time $X \times \mathbb{R}$. An arbitrary generalized solution of the Maxwell system without sources in Ω can be written in the form*

$$E = \int_{h=0} \exp(\imath((\xi, x) + \tau t)) \left[e_1 m^1(\xi, \tau) + e_2 m^2(\xi, \tau)\right],$$
$$H = \int_{h=0} \exp(\imath((\xi, x) + \tau t)) \left[h_1 m^1(\xi, \tau) + h_2 m^2(\xi, \tau)\right],$$

where m^1, m^2 are some complex-valued densities supported in the variety $\{h = 0\}$ such that

$$\int \exp(p_K(\operatorname{Im}(\xi, \tau))) \left(|\xi|^2 + |\tau|^2 + 1\right)^{-q} \left(\left|m^1\right| + \left|m^2\right|\right) < \infty$$

for an arbitrary compact set $K \subset \Omega$ and some constant $q = q(K)$.

8.4 Cauchy problem

Write the Maxwell system with sources: $\tilde{j} = 4\pi c^{-1} j,\ \tilde{\rho} = 4\pi c^{-1}\rho$:

$$\begin{aligned} \nabla \times H - \partial_t (\tilde{\varepsilon} E) &= \tilde{j} \\ \nabla \times E + \partial_t (\tilde{\mu} H) &= 0 \\ (\nabla, \tilde{\mu} H) &= 0 \\ (\nabla, \tilde{\varepsilon} E) &= \tilde{\rho} \end{aligned} \tag{8.8}$$

and variable coefficients $\varepsilon = \varepsilon(x),\ \mu = \mu(x)$. This is a overdetermined system: the conservation law $(\nabla, j) + \partial_t \rho = 0$ is a necessary condition for existence of a solution. The system is hyperbolic in a sense; we can solve for the Cauchy problem for this system

$$E(x,0) = E_0(x),\ \partial_t E(x,0) = E_1(x),\ H(x,0) = H_0(x),\ \partial_t H(x,0) = H_1(x)$$

provided more necessary conditions are satisfied:

$$\begin{aligned} &\nabla \times H_0 - \tilde{\varepsilon} E_1 = j(x,0),\ \nabla \times E_0 + \tilde{\mu} H_1 = 0, \\ &\qquad \nabla(\tilde{\mu} H_0) = \nabla(\tilde{\mu} H_1) = 0,\ (\nabla, \varepsilon E_0) = \rho(x,0),\ (\nabla, \varepsilon E_1) = \partial_t \rho(x,0) \end{aligned}$$

These equations together with the conservation law are the *consistency* conditions.

Theorem 3 *Suppose that the coefficients ε, μ are smooth functions in X and the sources $j, \rho \in \mathcal{D}'(X \times \mathbb{R})$ and the functions $E_0, E_1, H_0, H_1 \in \mathcal{D}'(X)$ satisfy the consistency conditions. Then the Cauchy problem for the Maxwell system has unique solution in the space $\mathcal{D}'(X \times \mathbb{R})$.*

Proof. For unknown E, H we denote by $F_i, i = 1,2,3,4$ the left sides of the equations (8) respectively. We find

$$\begin{aligned} -\partial_t F_1 + \nabla \times \tilde{\mu}^{-1} F_2 &\equiv \tilde{\varepsilon} \partial_t^2 E + \nabla \times \left(\tilde{\mu}^{-1} \nabla \times E\right) = -\partial_t \tilde{j} \\ \partial_t F_2 + \nabla \times \tilde{\varepsilon}^{-1} F_1 &\equiv \tilde{\mu} \partial_t^2 H + \nabla \times \left(\tilde{\varepsilon}^{-1} \nabla \times H\right) = \nabla \times \tilde{\varepsilon}^{-1} \tilde{j} \end{aligned}$$

We have

$$\begin{aligned} \nabla \times \tilde{\mu}^{-1} \nabla \times E &\equiv \tilde{\mu}^{-1} (-\Delta E + \nabla (\nabla, E)) + \nabla \tilde{\mu}^{-1} \times (\nabla \times E) \\ &= \tilde{\mu}^{-1} \left(-\Delta E + \nabla \left(\tilde{\varepsilon}^{-1} F_4\right) - \nabla \left(\tilde{\varepsilon}^{-1} (\nabla \tilde{\varepsilon}, E)\right)\right) + \nabla \tilde{\mu}^{-1} \times (\nabla \times E) \end{aligned}$$

Therefore

$$\begin{aligned} &- \partial_t F_1 + \nabla \times \tilde{\mu}^{-1} F_2 + \tilde{\mu}^{-1} \nabla \left(\tilde{\varepsilon}^{-1} F_4\right) \\ &\equiv \tilde{\varepsilon} \partial_t^2 E - \tilde{\mu}^{-1} \Delta E - \tilde{\mu}^{-1} \nabla \left(\tilde{\varepsilon}^{-1} (\nabla \tilde{\varepsilon}, E)\right) + \nabla \tilde{\mu}^{-1} \times \nabla \times E \\ &= -\partial_t j - \tilde{\mu}^{-1} \nabla (\tilde{\varepsilon} \rho) \end{aligned}$$

which implies the equation for the electric field

$$\tilde{\varepsilon} \partial_t^2 E - \tilde{\mu}^{-1} \Delta E - \tilde{\mu}^{-1} \nabla \left(\tilde{\varepsilon}^{-1} (\nabla, \tilde{\varepsilon} E)\right) + \nabla \tilde{\mu}^{-1} \times \nabla \times E = S_E \doteq -\partial_t \tilde{j} - \tilde{\mu}^{-1} \nabla (\tilde{\varepsilon} \tilde{\rho})$$

The principal part is the wave operator with velocity since $v = (\tilde{\varepsilon}\tilde{\mu})^{-1/2}$. The Cauchy problem for this equation and initial data E_0, E_1 has unique generalized solution E in $X \times \mathbb{R}$. Apply the operator $(\nabla,)$ to this equation and get by the consistency of the source

$$\begin{aligned} -\partial_t (\nabla, F_1) - \left(\nabla, \tilde{\mu}^{-1}\nabla\left(\tilde{\varepsilon}^{-1}F_4\right)\right) &= \left(\nabla, \partial_t \tilde{j} - \tilde{\mu}^{-1}\nabla(\tilde{\varepsilon}\tilde{\rho})\right) \\ &= -W\tilde{\rho}, \end{aligned}$$

where

$$W\tilde{\rho} \doteq \partial_t^2 \tilde{\rho} - \left(\nabla, \tilde{\mu}^{-1}\nabla(\tilde{\varepsilon}\tilde{\rho})\right)$$

On the other hand

$$-\partial_t (\nabla, F_1) + \left(\nabla, \tilde{\mu}^{-1}\nabla\left(\tilde{\varepsilon}^{-1}F_4\right)\right) = \partial_t^2 (\nabla, \tilde{\varepsilon}E) + \left(\nabla, \tilde{\mu}^{-1}\nabla(\nabla, \tilde{\varepsilon}E)\right) = W(\nabla, \tilde{\varepsilon}E) = WF_4$$

hence $W(F_4 - \tilde{\rho}) = 0$. The function $F_4 - \tilde{\rho}$ vanishes for $t = 0$ together with the first time derivative in virtue of the consistency conditions.

Lemma 4 *The Cauchy problem for the operator W has no more than one solution*

From the Lemma we conclude that $F_4 = \tilde{\rho}$, which proves the forth equation.
Similarly we find

$$\begin{gathered} \partial_t F_2 + \nabla \times \tilde{\varepsilon}^{-1}F_1 - \tilde{\varepsilon}^{-1}\nabla\left(\tilde{\mu}^{-1}F_3\right) \equiv \\ \tilde{\mu}\partial_t^2 H - \tilde{\varepsilon}\Delta H - \tilde{\varepsilon}^{-1}\nabla\left(\tilde{\mu}^{-1}(\nabla\tilde{\mu}, H)\right) + \nabla\tilde{\varepsilon}^{-1} \times \nabla \times H = S_H \doteq \left(\nabla, \tilde{\varepsilon}^{-1}\tilde{j}\right) \end{gathered}$$

This equation has the same principal part up to a scalar factor and we can solve the Cauchy problem for initial data H_0, H_1. Arguing as above, we check that this solution fulfils the third equation. Then we have the system

$$\begin{aligned} -\partial_t F_1 + \nabla \times \tilde{\mu}^{-1}F_2 &= S_E \\ \partial_t F_2 + \nabla \times \tilde{\varepsilon}^{-1}F_1 &= S_H \end{aligned}$$

Apply the operator $-\partial_t$ to the first equation and the operator $\nabla \times \tilde{\mu}^{-1}$ to the second and take the sum

$$\begin{aligned} \partial_t^2 F_1 + \nabla \times \tilde{\mu}^{-1}\left(\nabla \times \tilde{\varepsilon}^{-1}F_1\right) &= \partial_t S_E + \nabla \times \tilde{\mu}^{-1}S_H \\ &= -\partial_t\left(\partial_t \tilde{j} + \tilde{\mu}^{-1}\nabla(\tilde{\varepsilon}\tilde{\rho})\right) + \nabla \times \tilde{\mu}^{-1}\left(\left(\nabla, \tilde{\varepsilon}^{-1}\tilde{j}\right)\right) \end{aligned} \tag{8.9}$$

We have

$$\begin{aligned} \nabla \times \tilde{\mu}^{-1}\left(\nabla \times \tilde{\varepsilon}^{-1}F_1\right) &= \nabla\left(\tilde{\mu}^{-1}\right) \times \left(\nabla \times \tilde{\varepsilon}^{-1}F_1\right) + \tilde{\mu}^{-1}\nabla \times \nabla \times \tilde{\varepsilon}^{-1}F_1 \\ &= ... - \tilde{\mu}^{-1}\Delta + \nabla\left(\nabla, \tilde{\varepsilon}^{-1}F_1\right) \\ &= ... - \tilde{\mu}^{-1}\Delta\tilde{\varepsilon}^{-1}F_1 + \nabla\left(\nabla\left(\tilde{\varepsilon}^{-1}\right), F_1\right) + \nabla\left(\tilde{\varepsilon}^{-1}(\nabla, F_1)\right) \end{aligned}$$

and the last term vanishes since $(\nabla, F_1) = 0$. Therefore the left side of (9) is equal to UF_1, where

$$U \doteq \partial_t^2 - \tilde{\mu}^{-1}\Delta\tilde{\varepsilon}^{-1} + \nabla\left(\tilde{\mu}^{-1}\right) \times \left(\nabla \times \tilde{\varepsilon}^{-1}\cdot\right) + \nabla\left(\nabla\left(\tilde{\varepsilon}^{-1}\right), \cdot\right)$$

The principal part is again the wave operator with the velocity v. The right side of (9) is equal to $U\tilde{\rho}$ in virtue of the conservation law. Thus we have $U(F_1 - \tilde{\rho}) = 0$. We argue as above and check the first equation. The second one can verified in the same way. □

PROOF OF LEMMA. We will to show that $Wu = 0$ and $u(x,0) = u_t(x,0) = 0$ implies $u = 0$. Suppose for simplicity that $u(\cdot, t) \in H_2^2(X)$ for any value of time. Then we can show the integral conservation law

$$\partial_t \int \left(\tilde{\varepsilon} u_t^2 + \tilde{\mu}^{-1} |\nabla(\tilde{\varepsilon} u)|^2\right) dx = 2 \int \tilde{\varepsilon} u_t \left(u_{tt} - \left(\nabla \tilde{\mu}^{-1}, \nabla\right)(\tilde{\varepsilon} u)\right) dx = 0$$

It follows that integral of $\left(\tilde{\varepsilon} u_t^2 + \tilde{\mu}^{-1} |\nabla(\tilde{\varepsilon} u)|^2\right) dx$ does not depend of time. It vanishes for $t = 0$, hence vanishes for all times. To remove the assumption we continue $u = 0$ for $t < 0$ and change the variables $t' = t + \delta |x - x_0|^2$, $x' = x$, where $\delta > 0$, x_0 is arbitrary. The function u has compact support in each hypersurface $t' = \tau$ for any τ. □

8.5 Local conservation laws

The quadratic forms

$$\varepsilon E^2 = \varepsilon(E, E), \quad \mu H^2 = \mu(H, H), \quad S \doteq \frac{v}{4\pi} E \times H$$

are called electric energy, magnetic energy and energy *flux* (Poynting vector), respectively. We have $\dim(\varepsilon E^2 dx) = \dim(\mu H^2 dx) = \dim S dx = M(L/T)^2$ which equals the dimension of energy.

Consider the Hamiltonian flow F generated by the function h. Its projection to $X \times \mathbb{R}$ is the geodesic flow of the metric $g = v^{-2} ds^2$.

Theorem 5 *The densities $\varepsilon E^2 dx, \mu H^2 dx$ are equal and is preserved by the flow F in the approximation of geometrical optics. The vector field E is orthogonal to H and both are orthogonal to any trajectory of F. Moreover the halfdensities*

$$\mu^{-1/2} E \sqrt{dx}, \ \varepsilon^{-1/2} H \sqrt{dx}$$

keep parallel along any trajectory of F.

References

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