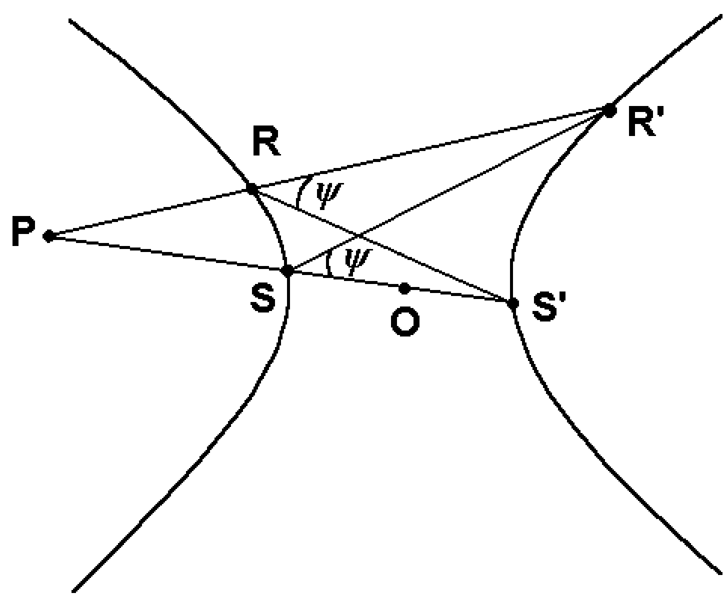
TREATISE OF PLANE GEOMETRY THROUGH GEOMETRIC ALGEBRA



Ramon González Calvet

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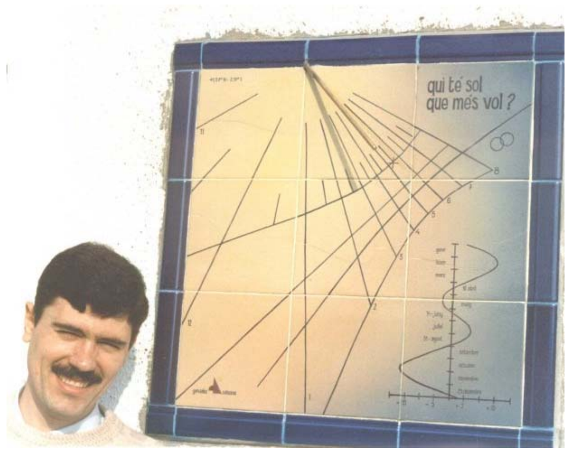

The geometric algebra, initially discovered by Hermann Grassmann (1809-1877) was reformulated by William Kingdon Clifford (1845-1879) through the synthesis of the Grassmann's extension theory and the quaternions of Sir William Rowan Hamilton (1805-1865). In this way the bases of the geometric algebra were established in the XIX century. Notwithstanding, due to the premature death of Clifford, the vector analysis –a remake of the quaternions by Josiah Willard Gibbs (1839-1903) and Oliver Heaviside (1850-1925)– became, after a long controversy, the geometric language of the XX century; the same vector analysis whose beauty attracted the attention of the author in a course on electromagnetism and led him -being still undergraduate- to read the Hamilton's *Elements of Quaternions*. Maxwell himself already applied the quaternions to the electromagnetic field. However the equations are not written so nicely as with vector analysis. In 1986 Ramon contacted Josep Manel Parra i Serra, teacher of theoretical physics at the Universitat de Barcelona, who acquainted him with the Clifford algebra. In the framework of the summer courses on geometric algebra which they have taught for graduates and teachers since 1994, the plan of writing some books on this subject appeared in a very natural manner, the first sample being the *Tractat de geometria plana mitjançant l'àlgebra geomètrica* (1996) now out of print. The good reception of the readers has encouraged the author to write the *Treatise of plane geometry through geometric algebra* (a very enlarged translation of the *Tractat*) and publish it at the Internet site http://campus.uab.es/~PC00018, writing it not only for mathematics students but also for any person interested in geometry. The plane geometry is a basic and easy step to enter into the Clifford-Grassmann geometric algebra, which will become the geometric language of the XXI century.

Dr. Ramon González Calvet (1964) is high school teacher of mathematics since 1987, fellow of the Societat Catalana de Matemàtiques (http://www-ma2.upc.es/~sxd/scma.htm) and also of the Societat Catalana de Gnomònica (http://www.gnomonica.org).

TREATISE OF PLANE GEOMETRY

THROUGH GEOMETRIC ALGEBRA

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To my son Pere, born with the book.

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PROLOGUE

The book I am so pleased to present represents a true innovation in the field of the mathematical didactics and, specifically, in the field of geometry. Based on the long neglected discoveries made by Grassmann, Hamilton and Clifford in the nineteenth century, it presents the geometry -the elementary geometry of the plane, the space, the spacetime- using the best algebraic tools designed specifically for this task, thus making the subject democratically available outside the narrow circle of individuals with the high visual imagination capabilities and the true mathematical insight which were required in the abandoned classical Euclidean tradition. The material exposed in the book offers a wide repertory of geometrical contents on which to base powerful, reasonable and up-to-date reintroductions of geometry to present-day high school students. This longed-for reintroductions may (or better should) take advantage of a combined use of symbolic computer programs and the cross disciplinary relationships with the physical sciences.

The proposed introduction of the geometric Clifford-Grassmann algebra in high school (or even before) follows rightly from a pedagogical principle exposed by William Kingdon Clifford (1845-1879) in his project of teaching geometry, in the University College of London, as a practical and empirical science as opposed to Cambridge Euclidean axiomatics: " ... for geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child". Fellow of the Royal Society at the age of 29, Clifford also gave a set of *Lectures on Geometry* to a Class of Ladies at South Kengsinton and was deeply concerned in developing with MacMillan Company a series of **inexpensive** "very good elementary schoolbook of arithmetic, geometry, animals, plants, physics ...". Not foreign to this proposal are Felix Klein lectures to teachers collected in his book *Elementary mathematics from an advanced standpoint*[1] and the advice of Alfred North Whitehead saying that "the hardest task in mathematics is the study of the elements of algebra, and yet this stage must precede the comparative simplicity of the differential calculus" and that "the postponement of difficulty mis no safe clue for the maze of educational practice"[2].

Clearly enough, when the fate of pseudo-democratic educational reforms, disguised as a back to basic leitmotifs, has been answered by such an acute analysis by R. Noss and P. Dowling under the title *Mathematics in the National Curriculum: The Empty Set?*[3], the time may be ripen for a reappraisal of true pedagogical reforms based on a real knowledge, of substantive contents, relevant for each individual worldview construction. We believe that the introduction of the *vital or experiential* plane, space and space-time geometries along with its proper algebraic structures will be a substantial part of a successful (high) school scientific curricula. Knowing, telling, learning why the sign rule, or the complex numbers, or matrices are mathematical structures correlated to the human representation of the real world are worthy objectives in mass education projects. And this is possible today if we learn to stand upon the shoulders of giants such as Leibniz, Hamilton, Grassmann, Clifford, Einstein, Minkowski, etc. To this aim this book, offered and opened to suggestions to the whole world of concerned people, may be a modest but most valuable step towards these very good schoolbooks that constituted one of the cheerful Clifford's aims.

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[1] Felix Klein, *Elementary mathematics from an advanced standpoint*. Dover (N. Y., 1924).
[2] A.N. Whitehead, *The aims of education*. MacMillan Company (1929), Mentor Books (N.Y., 1949).
[3] P. Dowling, R. Noss, eds., *Mathematics versus the National Curriculum: The Empty Set?*. The Falmer Press (London, 1990).

Finally, some words borrowed from Whitehead and Russell, that I am sure convey some of the deepest feelings, thoughts and critical concerns that Dr. Ramon González has had in mind while writing the book, and that fully justify a work that appears to be quite removed from today high school teaching, at least in Catalunya, our country.

> "Where attainable knowledge could have changed the issue, ignorance has the guilt of vice"[2].
> "The uncritical application of the principle of necessary antecedence of some subjects to others has, in the hands of dull people with a turn for organisation, produced in education the dryness of the Sahara"[2].
> "When one considers in its length and in its breadth the importance of this question of the education of a nation's young, the broken lives, the defeated hopes, the national failures, which result from the frivolous inertia with which it is treated, it is difficult to restrain within oneself a savage rage"[2].
> "A taste for mathematics, like a taste for music, can be generated in some people, but not in others. ... But I think that these could be much fewer than bad instruction makes them seem. Pupils who have not an unusually strong natural bent towards mathematics are led to hate the subject by two shortcomings on the part of their teachers. The first is that mathematics is not exhibited as the basis of all our scientific knowledge, both theoretical and practical: the pupil is convincingly shown that what we can understand of the world, and what we can do with machines, we can understand and do in virtue of mathematics. The second defect is that the difficulties are not approached gradually, as they should be, and are not minimised by being connected with easily apprehended central principles, so that the edifice of mathematics is made to look like a collection of detached hovels rather than a single temple embodying a unitary plan. It is especially in regard to this second defect that Clifford's book (*Common Sense of the Exact Sciences*) is valuable.(Russell)" [4].

An appreciation that Clifford himself had formulated, in his fundamental paper upon which the present book relies, relative to the *Ausdehnungslehre* of Grassmann, expressing "my conviction that its principles will exercise a vast influence upon the future of mathematical science".

Josep Manel Parra i Serra, June 2001

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[4] W. K. Clifford, *Common Sense of the Exact Sciences*. Alfred A. Knopf (1946), Dover (N.Y., 1955).

« On demande en second lieu, laquelle des deux qualités doit être préférée dans des *élémens*, de la facilité, ou de la rigour exacte. Je réponds que cette question suppose una chose fausse; elle suppose que la rigour exacte puisse exister sans la facilité & c'est le contraire; plus une déduction est rigoureause, plus elle est facile à entendre: car la rigueur consiste à reduire tout aux principes les plus simples. D'où il s'ensuit encore que la rigueur proprement dit entraîne nécessairement la méthode la plus naturelle & la plus directe. Plus les principles seront disposés dans l'ordre convenable, plus la déduction sera rigourease; ce n'est pas qu'absolument elle ne pût l'être si on suivonit une méthode plus composée, com a fait Euclide dans ses *élémens*: mais alors l'embarras de la marche feroit aisément sentir que cette rigueur précaire & forcée ne seroit qu'improprement telle. »[5]

["Secondly, one requests which of the two following qualities must be preferred within the *elements*, whether the easiness or the exact rigour. I answer that this question implies a falsehood; it implies that the exact rigour can exist without the easiness and it is the other way around; the more rigorous a deduction will be, the more easily it will be understood: because the rigour consists of reducing everything to the simplest principles. Whence follows that the properly called rigour implies necessarily the most natural and direct method. The more the principles will be arranged in the convenient order, the more rigorous the deduction will be; it does not mean that it cannot be rigorous at all if one follows a more composite method as Euclid made in his *elements*: but then the difficulty of the march will make us to feel that this precarious and forced rigour will only be an improper one."]

Jean le Rond D'Alembert (1717-1783)

[5] «Elémens des sciences» in *Encyclopédie, ou dictionaire raisonné des sciences, des arts et des métiers* (París, 1755).

PREFACE TO THE FIRST ENGLISH EDITION

The first edition of the *Treatise of Plane Geometry through Geometric Algebra* is a very enlarged translation of the first Catalan edition published in 1996. The good reception of the book (now out of print) encouraged me to translate it to the English language rewriting some chapters in order to make easier the reading, enlarging the others and adding those devoted to the non-Euclidean geometry.

The *geometric algebra* is the tool which allows to study and solve geometric problems through a simpler and more direct way than a purely geometric reasoning, that is, by means of the algebra of geometric quantities instead of *synthetic* geometry. In fact, the geometric algebra is the Clifford algebra generated by the Grassmann's outer product in a vector space, although for me, the geometric algebra is also the art of stating and solving geometric equations, which correspond to geometric problems, by isolating the unknown geometric quantity using the algebraic rules of the vectors operations (such as the associative, distributive and permutative properties). Following Peano[6]:

> "The geometric Calculus differs from the Cartesian Geometry in that whereas the latter operates analytically with coordinates, the former operates directly on the geometric entities".

Initially proposed by Leibniz[7] (*characteristica geometrica*) with the aim of finding an intrinsic language of the geometry, the geometric algebra was discovered and developed by Grassmann[8], Hamilton and Clifford during the XIX century. However, it did not become usual in the XX century ought to many circumstances but the *vector analysis* -a recasting of the Hamilton quaternions by Gibbs and Heaviside- was gradually accepted in physics. On the other hand, the geometry followed its own way aside from the vector analysis as Gibbs[9] pointed out:

> "And the growth in this century of the so-called synthetic as opposed to analytical geometry seems due to the fact that by the ordinary analysis geometers could not easily express, except in a cumbersome and unnatural manner, the sort of relations in which they were particularly interested"

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[6] Giuseppe Peano, «Saggio di Calcolo geometrico». Translated in *Selected works of Giuseppe Peano*, 169 (see the bibliography).

[7] C. I. Gerhardt, *G. W. Leibniz. Mathematical Schriften* **V**, 141 and *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematiker*, 570.

[8] In 1844 a prize (45 gold ducats for 1846) was offered by the Fürstlich Jablonowski'schen Gessellschaft in Leipzig to whom was capable to develop the *characteristica geometrica* of Leibniz. Grassmann won this prize with the memoir *Geometric Analysis*, published by this society in 1847 with a foreword by August Ferdinand Möbius. Its contents are essentially those of *Die Ausdehnungslehre* (1844).

[9] Josiah Willard Gibbs, «On Multiple Algebra», reproduced in *Scientific papers of J.W. Gibbs*, **II**, 98.

The work of revision of the history and the sources (see J. M. Parra[10]) has allowed us to synthesise the contributions of the different authors and completely rebuild the evolution of the geometric algebra, removing the conceptual mistakes which led to the vector analysis. This preface has not enough extension to explain all the history[11], but one must remember something usually forgotten: during the XIX century several points of view over what should become the geometric algebra came into competition. The Gibbs' vector analysis was one of these being not the better. In fact, the geometric algebra is a field of knowledge where different formulations are possible as Peano showed:

> "Indeed these various methods of geometric calculus do not at all contradict one another. They are various parts of the same science, or rather various ways of presenting the same subject by several authors, each studying it independently of the others.
>
> It follows that geometric calculus, like any other method, is not a system of conventions but a system of truth. In the same way, the methods of indivisibles (Cavalieri), of infinitesimals (Leibniz) and of fluxions (Newton) are the same science, more or less perfected, explained under different forms."[12]

The geometric algebra owns some fundamental geometric facts which cannot be ignored at all and will be recognised to it, as Grassmann hoped:

> "For I remain completely confident that the labour which I have expanded on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remained unused for another seventeen years or even longer, without entering into the actual development of science, still the time will come when it will be brought forth from the dust of oblivion, and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me in a position (which I have up to now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich further these ideas, nevertheless there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into living communication with contemporary developments. For truth is eternal and divine, and no phase in the development of truth, however small may be the region encompassed, can pass on without leaving

[10] Josep Manel Parra i Serra, «Geometric algebra versus numerical Cartesianism. The historical trend behind Clifford's algebra», in Brackx *et al*. ed., *Clifford Algebras and their Applications in Mathematical Physics*, 307-316, .
[11] A very complete reference is Michael J. Crowe, *A History of Vector Analysis. The Evolution of the Idea of a Vectorial System*.
[12] Giuseppe Peano, *op. cit.,* 168.

> a trace; truth remains, even though the garment in which poor mortals clothe it may fall to dust."[13]

As any other aspect of the human life, the history of the geometric algebra was conditioned by many fortuitous events. While Grassmann deduced the *extension theory* from philosophic concepts unintelligible for authors such as Möbius and Gibbs, Hamilton identified vectors and bivectors -the starting point of the great tangle of vector analysis- using a heavy notation[14]. Clifford had found the correct algebraic structure[15] which integrated the systems of Hamilton and Grassmann. However due to the premature death of Clifford in 1879, his opinion was not taken into account[16] and a long epistolary war was carried out by the quaternionists (specially Tait) against the defenders of the vector analysis, created by Gibbs[17], who did not recognise to be influenced by Grassmann and Hamilton:

> "At all events, I saw that the methods which I was using, while nearly those of Hamilton, were almost exactly those of Grassmann. I procured the two Ed. of the Ausdehnungslehre but I cannot say that I found them easy reading. In fact I have never had the perseverance to get through with either of them, and have perhaps got more ideas from his miscellaneous memoirs than from those works.
>
> I am not however conscious that Grassmann's writings exerted any particular influence on my Vector Analysis, although I was glad enough in the introductory paragraph to shelter myself behind one or two distinguished names [Grassmann and Clifford] in making changes of notation which I felt would be distasteful to quaternionists. In fact if you read that pamphlet carefully you will see that it all follows with the inexorable logic of algebra from the problem which I had set myself long before my acquaintance with Grassmann.
>
> I have no doubt that you consider, as I do, the methods of Grassmann to be superior to those of Hamilton. It thus seemed to me that it might [be] interesting to you to know how commencing with some knowledge of Hamilton's method and influenced simply by a desire to obtain the simplest algebra for the expression of the relations of Geom. Phys. etc. I was led essentially to Grassmann's algebra of vectors, independently of any influence from him or any one else."[18]

[13] Hermann Gunther Grassmann. Preface to the second edition of *Die Ausdehnungslehre* (1861). The first edition was published on 1844, hence the "seventeen years". Translated in Crowe, *op. cit.* p. 89.

[14] The *Lectures on Quaternions* was published in 1853, and the *Elements of Quaternions* posthumously in 1866.

[15] William Kingdon Clifford left us his synthesis in «Applications of Grassmann's Extensive Algebra».

[16] See «On the Classification of Geometric Algebras», unfinished paper whose abstract was communicated to the London Mathematical Society on March 10th, 1876.

[17] The first *Vector Analysis* was a private edition of 1881.

[18] Draft of a letter sent by Josiah Willard Gibbs to Victor Schlegel (1888). Reproduced by Crowe, *op. cit*. p. 153.

Before its beginning the controversy was already superfluous[19]. Notwithstanding the epistolary war continued for twelve years.

The vector analysis is a provisional solution[20] (which spent all the XX century!) adopted by everybody ought to its easiness and practical notation but having many troubles when being applied to three-dimensional geometry and unable to be generalised to the Minkowski's four-dimensional space. On the other hand, the geometric algebra is, by its own nature, valid in any dimension and it offers the necessary resources for the study and research in geometry as I show in this book.

The reader will see that the theoretical explanations have been completed with problems in each chapter, although this splitting is somewhat fictitious because the problems are demonstrations of geometric facts, being one of the most interesting aspects of the geometric algebra and a proof of its power. The usual numeric problems, which our pupils like, can be easily outlined by the teacher, because the geometric algebra always yields an immediate expression with coordinates.

I'm indebted to professor Josep Manel Parra for encouraging me to write this book, for the dialectic interchange of ideas and for the bibliographic support. In the framework of the summer courses on geometric algebra for teachers that we taught during the years 1994-1997 in the *Escola d'estiu de secundària* organised by the Col·legi Oficial de Doctors i Llicenciats en Filosofia i Lletres i en Ciències de Catalunya, the project of some books on this subject appeared in a natural manner. The first book devoted to two dimensions already lies on your hands and will probably be followed by other books on the algebra and geometry of the three and four dimensions.

Finally I also acknowledge the suggestions received from some readers.

Ramon González Calvet

Cerdanyola del Vallès, June 2001

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19 See Alfred M. Bork «"Vectors versus quaternions"—The letters in *Nature*».

20 The vector analysis bases on the duality of the geometric algebra of the three-dimensional space: the fact that the orientation of lines and planes is determined by three numeric components in both cases. However in the four-dimensional time-space the same orientations are respectively determined by four and six numbers.

CONTENTS

FIRST PART: THE VECTOR PLANE AND THE COMPLEX NUMBERS

Points and vectors are the main elements of the plane geometry. A *point* is conceived (but not defined) as a geometric element without extension, infinitely small, that has position and is located at a certain place on the plane. A *vector* is defined as an oriented segment, that is, a piece of a straight line having length and direction. A vector has no position and can be translated anywhere. Usually it is called a *free* vector. If we place the end of a vector at a point, then its head determines another point, so that a vector represents the translation from the first point to the second one.

Taking into account the distinction between points and vectors, the part of the book devoted to the Euclidean geometry has been divided in two parts. In the first one the vectors and their algebraic properties are studied, which is enough for many scientific and engineering branches. In the second part the points are introduced and then the affine geometry is studied.

All the elements of the geometric algebra (scalars, vectors, bivectors, complex numbers) are denoted with lowercase Latin characters and the angles with Greek characters. The capital Latin characters will denote points on the plane. As you will see, the geometric product is not commutative, so that fractions can only be written for real and complex numbers. Since the geometric product is associative, the inverse of a certain element at the left and at the right is the same, that is, there is a unique inverse for each element of the algebra, which is indicated by the superscript $^{-1}$. Also due to the associative property, all the factors in a product are written without parenthesis. In order to make easy the reading I have not numerated theorems, corollaries nor equations. When a definition is introduced, the definite element is marked with italic characters, which allows to direct attention and helps to find again the definition.

1. THE VECTORS AND THEIR OPERATIONS

A vector is an oriented segment, having length and direction but no position, that is, it can be placed anywhere without changing its orientation. The vectors can represent many physical magnitudes such as a force, a celerity, and also geometric magnitudes such as a translation.

Two algebraic operations for vectors are defined, the addition and the product, that generalise the addition and product of the real numbers.

Vector addition

The *addition* of two vectors $u + v$ is defined as the vector going from the end of the vector u to the head of v when the head of u contacts the end of v (upper triangle in the figure 1.1). Making the construction for $v + u$, that is, placing the end of u at the head of v (lower triangle in the figure 1.1) we see that the addition vector is the same. Therefore, the vector addition has the commutative property:

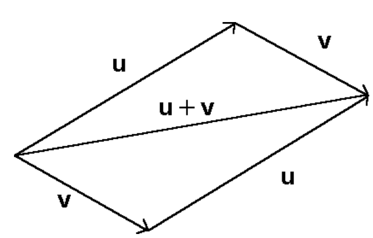


Figure 1.1

$$u + v = v + u$$

and the parallelogram rule follows: the addition of two vectors is the diagonal of the parallelogram formed by both vectors. The associative property follows from this definition because $(u+v)+w$ or $u+(v+w)$ is the vector closing the polygon formed by the three vectors as shown in the figure 1.2.

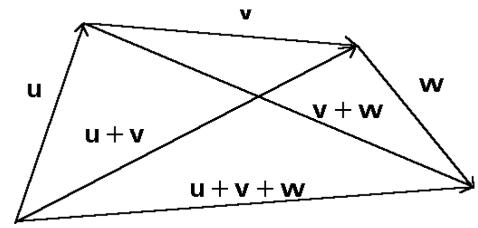


Figure 1.2

The neutral element of the vector addition is the *null vector*, which has zero length. Hence the *opposite vector* of u is defined as the vector $-u$ with the same orientation but opposite direction, which added to the initial vector gives the null vector:

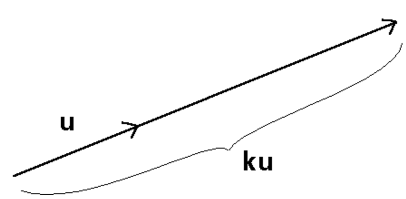
$$u + (-u) = 0$$

Product of a vector and a real number

One defines the product of a vector and a real number (or *scalar*) k, as a vector with the same direction but with a length increased k times (figure 1.3). If the real number is negative, then the direction is the opposite. The geometric definition implies the commutative property:

$$k\, u = u\, k$$

Figure 1.3



Two vectors u, v with the same direction are *proportional* because there is always a real number k such that $v = k\, u$, that is, k is the quotient of both vectors:

$$k = u^{-1}\, v = v\, u^{-1}$$

Two vectors with different directions are said to be *linearly independent*.

Product of two vectors

The product of two vectors will be called the *geometric product* in order to be distinguished from other vector products currently used. Nevertheless I hope that these other products will play a secondary role when the geometric product becomes the most used, a near event which this book will forward. At that time, the adjective «geometric» will not be necessary.

The following properties are demanded to the geometric product of two vectors:

1) To be distributive with regard to the vector addition:

$$u\,(\,v + w\,) = u\,v + u\,w$$

2) The square of a vector must be equal to the square of its length. By definition, the length (or *modulus*) of a vector is a positive number and it is noted by $|\,u\,|$:

$$u^2 = |\,u\,|^2$$

3) The mixed associative property must exist between the product of vectors and the product of a vector and a real number.

$$k\,(\,u\,v\,) = (\,k\,u\,)\,v = k\,u\,v$$

$$k\,(\,l\,u\,) = (\,k\,l\,)\,u = k\,l\,u$$

where k, l are real numbers and u, v vectors. Therefore, parenthesis are not needed.

These properties allows us to deduce the product. Let us suppose that c is the addition of two vectors a, b and calculate its square applying the distributive property:

$$c = a + b$$

$$c^2 = (\,a + b\,)^2 = (\,a + b\,)\,(\,a + b\,) = a^2 + a\,b + b\,a + b^2$$

We have to preserve the order of the factors because we do not know whether the product is commutative or not.

If a and b are orthogonal vectors, the Pythagorean theorem applies and then:

$$a \perp b \quad \Rightarrow \quad c^2 = a^2 + b^2 \quad \Rightarrow \quad a\,b + b\,a = 0 \quad \Rightarrow \quad a\,b = -\,b\,a$$

That is, <u>the product of two perpendicular vectors is anticommutative</u>.

If a and b are proportional vectors then:

$$a \parallel b \quad \Rightarrow \quad b = k\,a,\ \ k \text{ real} \quad \Rightarrow \quad a\,b = a\,k\,a = k\,a\,a = b\,a$$

because of the commutative and mixed associative properties of the product of a vector and a real number. Therefore <u>the product of two proportional vectors is commutative</u>. If c is the addition of two vectors a, b with the same direction, we have:

$$|\,c\,| = |\,a\,| + |\,b\,|$$

$$c^2 = a^2 + b^2 + 2\,|\,a\,|\,|\,b\,|$$

$$a\,b = |\,a\,|\,|\,b\,| \qquad \text{angle}(a,\,b) = 0$$

But if the vectors have opposite directions:

$$|\,c\,| = |\,a\,| - |\,b\,|$$

$$c^2 = a^2 + b^2 - 2\,|\,a\,|\,|\,b\,|$$

$$a\,b = -\,|\,a\,|\,|\,b\,| \qquad \text{angle}(a,\, b) = \pi$$

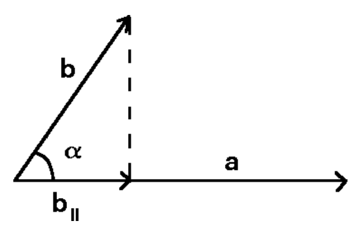
How is the product of two vectors with any directions? Due to the distributive property the product is resolved into one product by the proportional component $b_{||}$ and another by the orthogonal component $b_{\perp}$:

$$a\,b = a\,(\,b_{||} + b_{\perp}\,) = a\,b_{||} + a\,b_{\perp}$$

The product of one vector by the proportional component of the other is called the *inner product* (also *scalar product*) and noted by a point · (figure 1.4). Taking into account that the projection of b onto a is proportional to the cosine of the angle between both vectors, one finds:

$$a \cdot b = a\,b_{||} = |\,a\,|\,|\,b\,|\cos\alpha$$

Figure 1.4



The inner product is always a real number. For example, the work made by a force acting on a body is the inner product of the force and the walked space. Since the commutative property has been deduced for the product of vectors with the same direction, it follows also for the inner product:

$$a \cdot b = b \cdot a$$

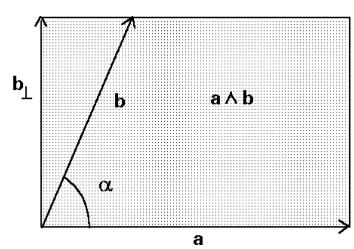
The product of one vector by the orthogonal component of the other is called the *outer product* (also *exterior product*) and it is noted with the symbol $\wedge$:

$$a \wedge b = a\,b_{\perp}$$

The outer product represents the area of the parallelogram formed by both vectors (figure 1.5):

$$|\,a \wedge b\,| = |\,a\,b_{\perp}\,| = |a|\,|b|\,|\sin\alpha|$$

Figure 1.5



Since the outer product is a product of orthogonal vectors, it is anticommutative:

$$a \wedge b = -\,b \wedge a$$

Some example of physical magnitudes which are outer products are the angular momentum, the torque, etc.

When two vectors are permuted, the sign of the oriented angle is changed. Then the cosine remains equal while the sine changes the sign. Because of this, the inner product is commutative while the outer product is anticommutative. Now, we can rewrite the geometric product as the sum of both products:

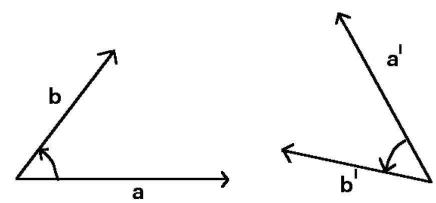
$$a\,b = a\cdot b + a\wedge b$$

From here, the inner and outer products can be written using the geometric product:

$$a\cdot b = \frac{a\,b + b\,a}{2}$$

$$a\wedge b = \frac{a\,b - b\,a}{2}$$

Figure 1.6



In conclusion, the geometric product of two proportional vectors is commutative whereas that of two orthogonal vectors is anticommutative, just for the pure cases of outer and inner products. The outer, inner and geometric products of two vectors only depend upon the moduli of the vectors and the angle between them. When both vectors are rotated preserving the angle that they form, the products are also preserved (figure 1.6).

How is the absolute value of the product of two vectors? Since the inner and outer product are linearly independent and orthogonal magnitudes, the modulus of the geometric product must be calculated through a generalisation of the Pythagorean theorem:

$$a\,b = a\cdot b + a\wedge b \quad\Rightarrow\quad |\,a\,b\,|^2 = |\,a\cdot b\,|^2 + |\,a\wedge b\,|^2$$

$$|\,a\,b\,|^2 = |\,a\,|^2\,|\,b\,|^2\,(\cos^2\alpha + \sin^2\alpha) = |\,a\,|^2\,|\,b\,|^2$$

That is, the modulus of the geometric product is the product of the modulus of each vector:

$$|\,a\,b\,| = |\,a\,|\,|\,b\,|$$

Product of three vectors: associative property

It is demanded as the fourth property that the product of three vectors be associative:

4) $u\,(\,v\,w\,) = (\,u\,v\,)\,w = u\,v\,w$

Hence we can remove parenthesis in multiple products and with the foregoing properties we can deduce how the product operates upon vectors.

We wish to multiply a vector *a* by a product of two vectors *b, c*. We ignore the result of the product of three vectors with different orientations except when two adjacent factors are proportional. We have seen that the product of two vectors depends only on the angle between them. Therefore the parallelogram formed by *b* and *c* can be rotated until b has, in the new orientation, the same direction as a. If b' and c' are the vectors b and c with the new orientation (figure 1.7) then:

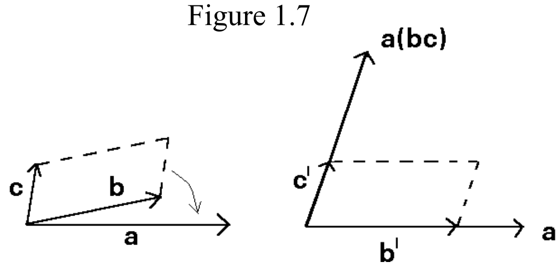
$$b\,c = b'\,c'$$

$$a\,(\,b\,c\,) = a\,(\,b'\ c'\,)$$

and by the associative property:

$$a\,(\,b\,c\,) = (\,a\,b'\,)\,c'$$

Figure 1.7



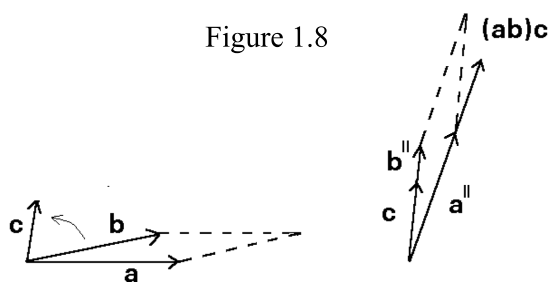
Since a and b' have the same direction, $a\ b' = |a|\,|b|$ is a real number and the triple product is a vector with the direction of c' whose length is increased by this amount:

$$a\,(\,b\,c\,) = |a|\,|b|\ c'$$

It follows that the modulus of the product of three vectors is the product of their moduli:

$$|a\,b\,c| = |a|\ |b|\ |c|$$

Figure 1.8



On the other hand, a can be firstly multiplied by b, and after this we can rotate the parallelogram formed by both vectors until b has, in the new orientation, the same direction as c (figure 1.8). Then:

$$(\,a\,b\,)\,c\ =\ a''(\ b''\,c\,) = a''\ |b|\ |c|$$

Although the geometric construction differs from the foregoing one, the figures clearly show that the triple product yields the same vector, as expected from the associative property.

$$(\,a\,b\,)\,c\ = a''\ |b|\ |c|\ =\ |c|\ |b|\ a'' = c\,b''\,a'' = c\,(\,b\,a\,)$$

That is, the triple product has the property:

$$a\,b\,c = c\,b\,a$$

which I call the *permutative* property: <u>every vector can be permuted with a vector located two positions farther in a product, although it does not commute with the neighbouring vectors</u>. The permutative property implies that any pair of vectors in a product separated by an odd number of vectors can be permuted. For example:

$$a\,b\,c\,d = a\,d\,c\,b = c\,d\,a\,b = c\,b\,a\,d$$

The permutative property is characteristic of the plane and it is also valid for the space whenever the three vectors are coplanar. This property is related with the fact that the product of complex numbers is commutative.

Product of four vectors

The product of four vectors can be deduced from the former reasoning. In order to multiply two pair of vectors, rotate the parallelogram formed by a and b until b' has the direction of c. Then the product is the parallelogram formed by a' and d but increased by the modulus of b and c:

$$a\,b\,c\,d = a'\,b'\,c\,d = a'\,|\,b\,|\,|\,c\,|\,d = |\,b\,|\,|\,c\,|\,a'\,d$$

Now let us see the special case when $a = c$ and $b = d$. If both vectors a, b have the same direction, the square of their product is a positive real number:

$$a \parallel b \qquad (a\,b)^2 = a^2\,b^2 > 0$$

If both vectors are perpendicular, we must rotate the parallelogram through $\pi/2$ until b' has the same direction as a (figure 1.9). Then a' and b are proportional but having opposite signs. Therefore, the square of a product of two orthogonal vectors is always negative:

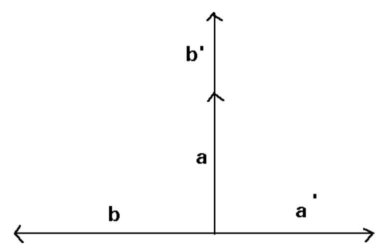


Figure 1.9

$$a \perp b \qquad (a\,b)^2 = a'\,b'\,a\,b = a'\,|\,b\,|\,|\,a\,|\,b = -a^2\,b^2 < 0$$

Likewise, the square of an outer product of any two vectors is also negative.

Inverse and quotient of two vectors

The *inverse* of a vector a is that vector whose multiplication by a gives the unity. Only the vectors which are proportional have a real product. Hence the inverse vector has the same direction and inverse modulus:

$$a^{-1} = a\,|\,a\,|^{-2} \qquad \Rightarrow \qquad a^{-1}\,a = a\,a^{-1} = 1$$

The quotient of vectors is a product for an inverse vector, which depends on the order of the factors because the product is not commutative:

$$a^{-1}\,b \neq b\,a^{-1}$$

Obviously the quotient of proportional vectors with the same direction and sense is equal to the quotient of their moduli. When the vectors have different directions, their quotient can be represented by a parallelogram, which allows to extend the concept of vector proportionality. We say that a is proportional to c as b is to d when their moduli are proportional and the angle between a and c is equal to the angle between b and d[1]:

$$a\,c^{-1} = b\,d^{-1} \quad \Leftrightarrow \quad |a|\,|c|^{-1} = |b|\,|d|^{-1} \qquad \text{and} \qquad \alpha(a, c) = \alpha(b, d)$$

Then the parallelogram formed by a and b is similar to that formed by c and d, being $\alpha(a, c)$ the angle of rotation from the first to the second one.

The inverse of a product of several vectors is the product of the inverses with the exchanged order, as can be easily seen from the associative property:

$$(\,a\,b\,c\,)^{-1} = c^{-1}\,b^{-1}\,a^{-1}$$

Hierarchy of algebraic operations

Like the algebra of real numbers, and in order to simplify the algebraic notation, I shall use the following hierarchy for the vector operations explained above:

1) The parenthesis, whose content will be firstly operated.
2) The power with any exponent (square, inverse, etc.).
3) The outer and inner product, which have the same hierarchy level but must be operated before the geometric product.
4) The geometric product.
5) The addition.

As an example, some algebraic expressions are given with the simplified expression at the left hand and its meaning using parenthesis at the right hand:

$$a \wedge b\,c \wedge d = (\,a \wedge b\,)\,(\,c \wedge d\,)$$

$$a^2\,b \wedge c + 3 = (\,(\,a^2\,)\,(\,b \wedge c\,)\,) + 3$$

$$a + b \cdot c\,d\,e = a + (\,(\,b \cdot c\,)\,d\,e\,)$$

Geometric algebra of the vectorial plane

The set of all the vectors on the plane together with the operations of vector addition and product of vectors by real numbers is a two-dimensional space usually called the *vector plane* V_2. The geometric product generates new elements (the complex numbers) not included in the vector plane. So, the geometric (or Clifford) algebra of a vectorial space is defined as the set of all the elements generated by products of vectors, for which the geometric product is an inner operation. The geometric algebra of the Euclidean vector plane is usually noted as $Cl_{2,0}(\mathbf{R})$ or simply as Cl_2. Making a parallelism with probability, the sample space is the set of elemental results of a certain

[1] William Rowan Hamilton defined the quaternions as quotients of two vectors in the way that similar parallelograms located at the same plane in the space represent the same quaternion (*Elements of Quaternions*, posthumously edited in 1866, Chelsea Publishers 1969, vol I, see p. 113 and fig. 34). In the vectorial plane a quaternion is reduced to a complex number. The quaternions were discovered by Hamilton (October 16th, 1843) before the geometric product by Clifford (1878).

random experiment. From the sample space Ω, the union $\cup$ and intersection $\cap$ generate the Boole algebra $A(\Omega)$, which includes all the possible events. In the same manner, the addition and geometric product generate the geometric algebra of the vectorial space. Then both sample and vectorial space play similar roles as generators of the Boole and geometric (Clifford) algebras respectively.

Exercises

1.1 Prove that the sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the four sides. Think about the sides as vectors.

1.2 Prove the following identity:

$$(a \cdot b)^2 - (a \wedge b)^2 = a^2\, b^2$$

1.3 Prove that: $a \wedge b\; c \wedge d = a \cdot (b\; c \wedge d) = (a \wedge b\; c) \cdot d$

1.4 Prove that: $a \wedge b\; c \wedge d + a \wedge c\; d \wedge b + a \wedge d\; b \wedge c = 0$

1.5 Prove the permutative property resolving b and c into the components which are proportional and orthogonal to the vector a.

1.6 Prove the Heron's formula for the area of the triangle:

$$A = \sqrt{s\,(s - |\,a\,|)\,(s - |\,b\,|)\,(s - |\,c\,|)}$$

where a, b and c are the sides and s the semiperimeter:

$$s = \frac{|\,a\,| + |\,b\,| + |\,c\,|}{2}$$

2. A BASE OF VECTORS FOR THE PLANE

Linear combination of two vectors

Every vector w on the plane is always a linear combination of two independent vectors u and v:

$$w = a\,u + b\,v \qquad a,\ b \text{ real}$$

Because of this, the plane has dimension equal to 2. In order to calculate the coefficients of linear combination a and b, we multiply w by u and v at both sides and subtract the results:

$$u\,w = a\,u^2 + b\,u\,v \quad \text{and} \quad w\,u = a\,u^2 + b\,v\,u \quad \Rightarrow \quad u\,w - w\,u = b\,(\,u\,v - v\,u\,)$$

$$v\,w = a\,v\,u + b\,v^2 \quad \text{and} \quad w\,v = a\,u\,v + b\,v^2 \quad \Rightarrow \quad w\,v - v\,w = a\,(\,u\,v - v\,u\,)$$

to obtain:

$$a = \frac{w \wedge v}{u \wedge v} \qquad\qquad b = \frac{u \wedge w}{u \wedge v}$$

The resolution of a vector as a linear combination of two independent vectors is a very frequent operation and also the foundation of the coordinates method.

Base and components

Any set of two independent vectors $\{e_1\ ,\ e_2\}$ can be taken as a *base* of the vector plane. Every vector u can be written as linear combination of the base vectors:

$$u = u_1\,e_1 + u_2\,e_2$$

The coefficients of this linear combination u_1, u_2 are the *components* of the vector in this base. Then a vector will be represented as a pair of components:

$$u = (\,u_1,\,u_2\,)$$

The components depend on the base, so that a change of base leads to a change of the components of the given vector.

We must only add components to add vectors:

$$v = (\,v_1,\,v_2\,)$$

$$u + v = (\,u_1 + v_1\,,\,u_2 + v_2)$$

The expression of the geometric product with components is obtained by means of the distributive property:

$$u\,v = (u_1\,e_1 + u_2\,e_2)\,(v_1\,e_1 + v_2\,e_2) = u_1\,v_1\,e_1^2 + u_2\,v_2\,e_2^2 + u_1\,v_2\,e_1\,e_2 + u_2\,v_1\,e_2\,e_1$$

$$u\,v = u_1\,v_1\,|e_1|^2 + u_2\,v_2\,|e_2|^2 + (u_1\,v_2 + u_2\,v_1)\,e_1 \cdot e_2 + (u_1\,v_2 - u_2\,v_1)\,e_1 \wedge e_2$$

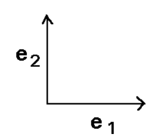
Hence the expression of the square of the vector modulus written in components is:

$$|u|^2 = u^2 = u_1^2\,|e_1|^2 + u_2^2\,|e_2|^2 + 2\,u_1\,u_2\,e_1 \cdot e_2$$

Orthonormal bases

Any base is valid to describe vectors using components, although the orthonormal bases, for which both e_1 and e_2 are unitary and perpendicular (such as the canonical base shown in the figure 2.1), are the more convenient and suitable:

Figure 2.1



$$e_1 \perp e_2 \qquad |e_1| = |e_2| = 1$$

For every orthonormal base :

$$e_1^2 = e_2^2 = 1 \qquad e_1\,e_2 = -\,e_2\,e_1$$

The product $e_1\,e_2$ represents a square of unity area. The square power of this product is equal to -1:

$$(e_1\,e_2)^2 = e_1\,e_2\,e_1\,e_2 = -\,e_1\,e_1\,e_2\,e_2 = -1$$

For an orthonormal base, the geometric product of two vectors becomes:

$$u\,v = u_1\,v_1 + u_2\,v_2 + (u_1\,v_2 - u_2\,v_1)\,e_1\,e_2$$

Note that the first and second summands are real while the third is an area. Therefore it follows that they are respectively the inner and outer products:

$$u \cdot v = u_1\,v_1 + u_2\,v_2 \qquad u \wedge v = (u_1\,v_2 - u_2\,v_1)\,e_1\,e_2$$

Also, the modulus of a vector is calculated from the self inner product:

$$|u|^2 = u_1^2 + u_2^2$$

Applications of the formulae for the products

The first application is the calculation of the angle between two vectors:

$$\cos\alpha = \frac{u_1 v_1 + u_2 v_2}{|u|\,|v|} \qquad \sin\alpha = \frac{u_1 v_2 - u_2 v_1}{|u|\,|v|}$$

The values of sine and cosine determine a unique angle α in the range $0<\alpha<2\pi$. The angle between two vectors is a sensed magnitude having positive sign if it is counterclockwise and negative sign if it is clockwise. Thus this angle depends on the order of the vectors in the outer (and geometric) product. For example, let us consider the vectors (figure 2.2) u and v:

$$u = (-2, 2) \qquad |u| = 2\sqrt{2} \qquad\qquad v = (4, 3) \qquad |v| = 5$$

$$\cos\alpha(u,v) = \cos\alpha(v,u) = -\frac{1}{5\sqrt{2}} \qquad\qquad \sin\alpha(u,v) = -\sin\alpha(v,u) = -\frac{7}{5\sqrt{2}}$$

$$\alpha(u, v) = 4.5705... \qquad\qquad \alpha(v, u) = 1.7127...$$

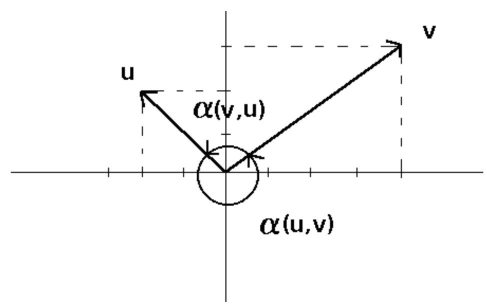
Also we may take the angle $\alpha(u, v) = 4.5705...-2\pi = -1.7127...$ The angle so obtained is always that going from the first to the second vector, being unique within a period.

Other application of the outer product is the calculus of areas. Using the expression with components, the area (considered as a positive real number) of the parallelogram formed by u and v is:

$$A = |u \wedge v| = |u_1 v_2 - u_2 v_1| = 14$$

When calculating the area of any triangle we must only divide the outer product of any two sides by 2.

Figura 2.2



Exercises

2.1 Let (u_1, u_2) and (v_1, v_2) be the components of the vectors u and v in the canonical base. Prove geometrically that the area of the parallelogram formed by both vectors is the modulus of the outer product $|u \wedge v| = u_1 v_2 - u_2 v_1$.

2.2 Calculate the area of the triangle whose sides are the vectors (3, 5), (–2, –3) and their addition (1, 2).

2.3 Prove the permutative property using components: $a\,b\,c = c\,b\,a$.

2.4 Calculate the angle between the vectors $u = 2\,e_1 + 3\,e_2$ and $v = -3\,e_1 + 4\,e_2$ in the canonical base.

2.5 Consider a base where e_1 has modulus 1, e_2 has modulus 2 and the angle between both vectors is $\pi/3$. Calculate the angle between $u = 2\,e_1 + 3\,e_2$ and $v = -3\,e_1 + 4\,e_2$.

2.6 In the canonical base $v = (3, -5)$. Calculate the components of this vector in a new base $\{u_1, u_2\}$ if $u_1 = (2, -1)$ and $u_2 = (5, -3)$.

3. THE COMPLEX NUMBERS

Subalgebra of the complex numbers

If $\{e_1, e_2\}$ is the canonical base of the vector plane V_2, its *geometric algebra* is defined as the vector space generated by the elements $\{1, e_1, e_2, e_1e_2\}$ together with the geometric product, so that the geometric algebra Cl_2 has dimension four. The unitary area e_1e_2 is usually noted as e_{12}. Due to the associative character of the geometric product, the geometric algebra is an associative algebra with identity. The complete table for the geometric product is the following:

	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

Note that the subset of elements containing only real numbers and areas is closed for the product:

	1	e_{12}
1	1	e_{12}
e_{12}	e_{12}	-1

This is the *subalgebra of complex numbers*. e_{12} , the *imaginary unity,* is usually noted as i. They are called *complex numbers* because their product is commutative like for the real numbers.

Binomial, polar and trigonometric form of a complex number

Every complex number z written in the *binomial* form is:

$$z = a + b\, e_{12} \qquad a, b \ \text{ real}$$

where a and b are the real and imaginary components respectively. The modulus of a complex number is calculated in the same way as the modulus of any element of the geometric algebra by means of the Pythagorean theorem:

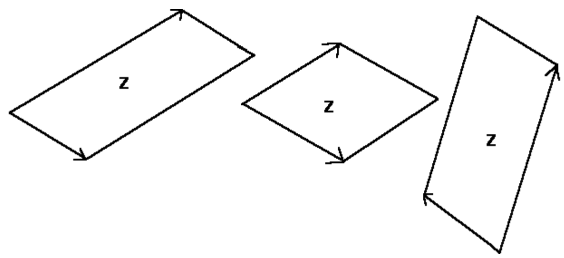
$$|z|^2 = |\, a + b\, e_{12}|^2 = a^2 + b^2$$

Since every complex number can be written as a product of two vectors u and v forming a certain angle α:

$z = u\,v = |u|\ |v|\ (\cos\alpha + e_{12}\sin\alpha)$

we may represent a complex number as a parallelogram with sides being the vectors u and v. But there are infinite pairs of vectors u' and v' whose product is the complex z provided that:

Figure 3.1



$|u|\,|v| = |u'|\,|v'|$ and $\alpha = \alpha'$

All the parallelograms having the same area and obliquity represent a given complex (they are equivalent) independently of the length and orientation of one side.(figure 3.1). The *trigonometric* and *polar* forms of a complex number z specifies its modulus $|z|$ and argument α:

$$z = |z|\ (\cos\alpha + e_{12}\sin\alpha) = |z|_{\alpha}$$

A complex number can be written using the exponential function, but firstly we must prove the Euler's identity:

$$\exp(\alpha\, e_{12}) = \cos\alpha + e_{12}\sin\alpha \qquad \alpha \text{ real}$$

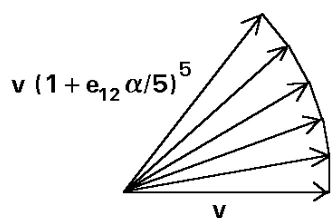
The exponential of an imaginary number is defined in the same manner as for a real number:

$$\exp(\alpha\, e_{12}) = \lim_{n\to\infty}\left(1 + \frac{\alpha\, e_{12}}{n}\right)^n$$

As shown in the figure 3.2 (for $n = 5$), the limit is a power of n rotations with angle α/n or equivalently a rotation of angle α.

Now a complex number written in *exponential* form is:

Figure 3.2



$$z = |z|\ \exp(\alpha\, e_{12})$$

Algebraic operations with complex numbers

Each algebraic operation is more easily calculated in a form than in another according to the following scheme:

addition / subtraction	$\leftrightarrow$	binomial form
product / quotient	$\leftrightarrow$	binomial or polar form
powers / roots	$\leftrightarrow$	polar form

The binomial form is suitable for the addition because both real components must be added and also the imaginary ones, e.g.:

$$z = 3 + 4\, e_{12} \qquad\qquad t = 2 - 5\, e_{12}$$

$$z + t = 3 + 4\, e_{12} + 2 - 5\, e_{12} = 5 - e_{12}$$

If the complex numbers are written in another form, they must be converted to the binomial form, e.g.

$$z = 2_{3\pi/4} \qquad t = 4_{\pi/6}$$

$$z + t = 2_{3\pi/4} + 4_{\pi/6} = 2\left(\cos\frac{3\pi}{4} + e_{12}\,\sin\frac{3\pi}{4}\right) + 4\left(\cos\frac{\pi}{6} + e_{12}\,\sin\frac{\pi}{6}\right)$$

$$= 2\left(-\frac{\sqrt{2}}{2} + e_{12}\,\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{3}}{2} + e_{12}\,\frac{1}{2}\right) = -\sqrt{2} + 2\sqrt{3} + e_{12}\left(\sqrt{2} + 2\right)$$

In order to write the result in polar form, the modulus must be calculated:

$$|\, z + t\,|^2 = (-\sqrt{2} + 2\sqrt{3}\,)^2 + (\sqrt{2} + 2\,)^2 = 20 + 4\,(\sqrt{2} - \sqrt{6}\,)$$

$$|\, z + t\,| = 2\sqrt{5 + \sqrt{2} - \sqrt{6}} = 3.9823...$$

and also the argument from the cosine and sine obtained as quotient of the real and imaginary components respectively divided by the modulus:

$$\cos\alpha = \frac{-\sqrt{2} + 2\sqrt{3}}{2\sqrt{5 + \sqrt{2} - \sqrt{6}}} \qquad \sin\alpha = \frac{\sqrt{2} + 2}{2\sqrt{5 + \sqrt{2} - \sqrt{6}}} \qquad \alpha = 1.0301...$$

$$z + t = 3.9823_{1.0301}$$

When multiplying complex numbers in binomial form, we apply the distributive property taken into account that the square of the imaginary unity is -1, e.g.:

$$(-2 + 5\, e_{12}\,)\,(\,3 - 4\, e_{12}\,) = -6 + 8\, e_{12} + 15\, e_{12} - 20\,(\,e_{12}\,)^2 = -6 + 20 + 23\, e_{12} = 14 + 23\, e_{12}$$

The exponential of an addition of arguments (real or complex) is equal to the product of exponential functions of each argument. Applying this characteristic property to the product of complex numbers in exponential form, we have:

$$z\,t = |z|\ \exp(\alpha\, e_{12}\,)\ |t|\ \exp(\,\beta\, e_{12}\,) = |z|\ |t|\ \exp[\,(\alpha + \beta\,)\, e_{12}\,]$$

from where the product of complex numbers in polar form is obtained by multiplying both moduli and adding both arguments.:

$$|z|_\alpha \; |t|_\beta = |z| \; |t| \; _{\alpha+\beta}$$

One may subtract 2π to the resulting argument in order to keep it between 0 and 2π. The product of two complex numbers z and t is the geometric operation consisting in the rotation of the parallelogram representing the first complex number until it touches the parallelogram representing the second complex number. When they contact in a unitary vector v (figure 3.3), the parallelogram formed by the other two vectors is the product of both complex numbers:

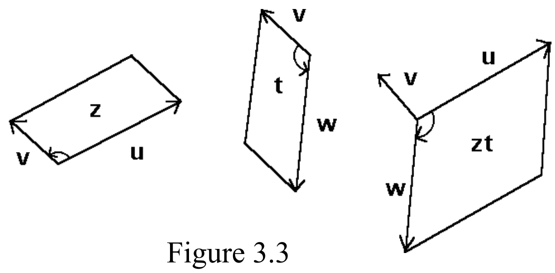


Figure 3.3

$$z = u\, v \qquad t = v\, w \qquad v^2 = 1$$

$$z\, t = u\, v\, v\, w = u\, w$$

This geometric construction is always possible because a parallelogram can be lengthened or widened maintaining the area so that one side has unity length.

The *conjugate* of a complex number (symbolised with an asterisk) is that number whose imaginary part has opposite sign:

$$z = a + b\, e_{12} \qquad z^* = a - b\, e_{12}$$

The geometric meaning of the conjugation is a permutation of the vectors whose product is the complex number (figure 3.4). In this case, the inner product is preserved while the outer product changes the sign. The product of a complex number and its conjugate is the square of the modulus:

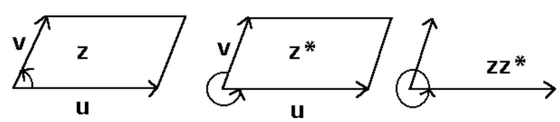


Figure 3.4

$$z\, z^* = u\, v\, v\, u = u^2 = |z|^2$$

The quotient of complex numbers is defined as the product by the inverse. The inverse of a complex number is equal to the conjugate divided by the square of the modulus:

$$z^{-1} = \frac{z^*}{|z|^2} = \frac{a - b\, e_{12}}{a^2 + b^2}$$

Then, let us see an example of quotient of complex numbers:

$$\frac{-2 + 5\, e_{12}}{3 - 4\, e_{12}} = \frac{(-2 + 5\, e_{12})(3 + 4\, e_{12})}{3^2 + (-4)^2} = \frac{-26 + 7\, e_{12}}{25}$$

With the polar form, the quotient is obtained by dividing moduli and subtracting arguments:

$$\frac{|\,z\,|_\alpha}{|\,t\,|_\beta} = \left(\frac{|\,z\,|}{|\,t\,|}\right)_{\alpha-\beta}$$

The best way to calculate the power of a complex number with natural exponent is through the polar form, although for low exponents the binomial form and the Newton formula is often used, e.g.:

$$(\,2 - 3\,e_{12}\,)^3 = 2^3 - 3 \cdot 2^2\,e_{12} + 3 \cdot 2\,e_{12}{}^2 - e_{12}{}^3 = 8 - 12\,e_{12} - 6 + e_{12} = 2 - 11\,e_{12}$$

From the characteristic property of the exponential function it follows that the modulus of a power of a complex number is the power of its modulus, and the argument of this power is the argument of the complex number multiplied by the exponent:

$$\left(|\,z\,|_\alpha\right)^n = |\,z\,|^n{}_{n\alpha}$$

This is a very useful rule for large exponents, e.g.:

$$(\,2 + 2\,e_{12}\,)^{1000} = (\,2\,\sqrt{2}\,_{\pi/4}\,)^{1000} = [(\,2\,\sqrt{2}\,\,)^{1000}]_{250\,\pi} = 2^{1500}{}_0 = 2^{1500}$$

When the argument exceeds 2π, divide by this value and take the remainder, in order to have the argument within the period $0<\alpha<2\pi$.

Since a root is the inverse operation of a power, its value is obtained by extracting the root of the modulus and dividing the argument by the index n. But a complex number of argument α may be also represented by the arguments $\alpha + 2\,\pi\,k$. Their division by the index n yields n different arguments within a period, corresponding to n different roots:

$$\sqrt[n]{z_\alpha} = \sqrt[n]{|\,z\,|}\;_{(\alpha+\,2\pi\,k)/n} \qquad\qquad k = 0, \ldots\, n-1$$

For example, the cubic roots of 8 are $\sqrt[3]{8_0} = \{2_0, 2_{2\pi/3}, 2_{4\pi/3}\}$. In the complex plane, the n-th roots of every complex number are located at the n vertices of a regular polygon.

Permutation of complex numbers and vectors

The permutative property of the vectors is intimately related with the commutative property of the product of complex numbers. Let z and t be complex numbers and a, b, c and d vectors fulfilling:

$$z = a\,b \qquad\qquad t = c\,d$$

Then the following equalities are equivalent:

$$z\,t = t\,z \qquad \Leftrightarrow \qquad a\,b\,c\,d = c\,d\,a\,b$$

A complex number z and a vector c do not commute, but they can be permuted by conjugating the complex number:

$$z\, c = a\, b\, c = c\, b\, a = c\, z^*$$

Every real number commute with any vector. However every imaginary number anticommute with any vector, because the imaginary unity e_{12} anticommute with e_1 as well as with e_2:

$$z\, c = -\, c\, z \qquad z \text{ imaginary}$$

The complex plane

In the *complex plane*, the complex numbers are represented taking the real component as the abscissa and the imaginary component as the ordinate. The vectorial plane differs from the complex plane in the fact that the vectorial plane is a plane of absolute directions whereas the complex plane is a plane of relative directions with respect to the real axis, to which we may assign any direction. As explained in more detail in the following chapter, the unitary complex numbers are rotation operators applied to vectors. The following equality shows the ambivalence of the Cartesian coordinates in the Euclidean plane:

$$e_1\left(x + y\, e_{12}\right) = x\, e_1 + y\, e_2$$

Due to a careless use, often the complex numbers have been improperly thought as vectors on the plane, furnishing the confusion between the complex and vector planes to our pupils. It will be argued that this has been very fruitful, but this argument cannot satisfy geometers, who search the fundamentals of the geometry. On the other hand, some physical magnitudes of a clearly vectorial kind have been taken improperly as complex numbers, specially in quantum mechanics. Because of this, I'm astonished when seeing how the inner and outer products transform in a special commutative and anti-commutative products of complex numbers. The relation between vectors and complex numbers is stated in the following way: If u is a fixed unitary vector, then every vector a is mapped to a unique complex z fulfilling:

$$a = u\, z \qquad \text{with} \quad u^2 = 1$$

Also other vector b is mapped to a complex number t:

$$b = u\, t$$

The outer and inner products of the vectors a and b can be written now using the complex numbers z and t:

$$a \wedge b = \frac{1}{2}(\, a\, b - b\, a\,) = \frac{1}{2}\;(\, u\, z\, u\, t - u\, t\, u\, z\,) = \frac{1}{2} u^2\,(\, z^*\, t - t^*\, z\,) =$$

$$= \frac{1}{2} \; (z^* \, t - t^* \, z) = (z_R \, t_I - z_I \, t_R) \, e_{12}$$

$$a \cdot b = \frac{1}{2} (a \, b + b \, a) = \frac{1}{2} (u \, z \, u \, t + u \, z \, u \, t) = \frac{1}{2} \; (z^* \, t + t^* \, z) = z_R \, t_R + z_I \, t_I$$

where z_R, t_R, z_I, t_I are the real and imaginary components of z and t. These products have been called improperly scalar and exterior products of complexes. So, I repeat again that complex quantities must be distinguished from vectorial quantities, and relative directions (complex numbers) from absolute directions (vectors). A guide for doing this is the reversion, under which the vectors are reversed while the complex number are not[1].

Complex analytic functions

The complex numbers are a commutative algebra where we can study functions as for the real numbers. A function $f(x)$ is said to be analytical if its complex derivative exists:

$$f(\mathrm{z}) \text{ analytical at } z_0 \quad \Leftrightarrow \quad \exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

This means that the derivative measured in any direction must give the same result. If $f(x) = a + b \, e_{12}$ and $z = x + y \, e_{12}$, the derivatives following the abscissa and ordinate directions must be equal:

$$f'(z) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial x} e_{12} = -\frac{\partial a}{\partial y} e_{12} + \frac{\partial b}{\partial y}$$

whence the Cauchy-Riemann conditions are obtained:

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad \text{and} \qquad \frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$$

Due to its linearity, now the derivative in any direction are also equal. A consequence of these conditions is the fact that the sum of both second derivatives (the Laplacian) vanishes, that is, both components are *harmonic* functions:

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} = 0$$

---

[1] A physical example is the alternating current. The voltage V and intensity I in an electric circuit are continuously rotating vectors. The energy E dissipated by the circuit is the inner product of both vectors, $E = V \cdot I$. The impedance Z of the circuit is of course a complex number (it is invariant under a reversion). The intensity vector can be calculated as the geometric product of the voltage vector multiplied by the inverse of the impedance $I = V \, Z^{-1}$. If we take as reference a continuously rotating direction, then V and I are replaced by pseudo complex numbers, but properly they are vectors.

The values of a harmonic function (therefore the value of *f*(z)) within a region are determined by those values at the boundary of this region. We will return to this matter later. The typical example of analytic function is the complex exponential:

$$\exp(x + y\,e_{12}) = \exp(x)(\cos y + e_{12}\,\sin y)$$

which is analytic in all the plane. The logarithm function is defined as the inverse function of the exponential. Since $z = |z|\exp(e_{12}\,\varphi)$ where φ is the argument of the complex, the principal branch of the logarithm is defined as:

$$\log z = \log|z| + e_{12}\varphi \qquad 0 \le \varphi < 2\pi$$

Also $\varphi + 2\pi k$ (k integer) are valid arguments for z yielding another branches of the logarithm[2]. In Cartesian coordinates:

$$\log(x + y\,e_{12}) = \log\sqrt{x^2 + y^2} + e_{12}\,\arccos\frac{x}{\sqrt{x^2 + y^2}} \qquad 0 \le \varphi < \pi$$

$$\log(x + y\,e_{12}) = \log\sqrt{x^2 + y^2} + e_{12}\left(\arccos\frac{x}{\sqrt{x^2 + y^2}} + \pi\right) \qquad \pi \le \varphi < 2\pi$$

At the positive real half axis, this logarithm is not analytic because it is not continuous.

Now let us see the Cauchy's theorem: if a function is analytic in a simply connected domain on the complex plane, then its integral following a closed way C within this domain is zero. If the analytic function is $f(z) = a + b\,e_{12}$ then the integral is:

$$\oint_C f(z)dz = \oint_C (a + b\,e_{12})(dx + dy\,e_{12}) = \oint_C (a\,dx - b\,dy) + e_{12}\oint_C (a\,dy + b\,dx)$$

Since C is a closed way, we may apply the Green theorem to write:

$$= -\iint_D \left(\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y}\right) dx\,dy + e_{12}\iint_D \left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) dx\,dy = 0$$

where D is the region bounded by the closed way C. Since *f*(z) fulfils the analyticity conditions everywhere within D, the integral vanishes.

From here the following theorem is deduced: if *f*(z) is an analytic function in a simply connected domain D and z_1 and z_2 are two points of D, then the definite integral between these points has a unique value independently of the integration trajectory, which is equal to the difference of the values of the primitive $F(z)$ at both points:

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) \qquad \text{if} \qquad f(z) = \frac{dF(z)}{dz}$$

---

[2] The logarithm is said to be *multi-valued*. This is also the case of the roots $\sqrt[n]{z}$.

If $f(z)$ is an analytic function (with a unique value) inside the region D bounded by the closed path C, then the *Cauchy integral formula* is fulfilled for a counterclockwise path orientation:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0)$$

Obviously, the integral does not vanish because the integrand is not analytic at z_0. However, it is analytic at the other points of the region D, so that the integration path from its beginning to its end passes always through an analyticity region, and by the former theorem the definite integral must have a constant value, independently of the fact that both extremes coincide. Now we integrate following a circular path $z = z_0 + r\exp(e_{12}\varphi)$, where the radius r is a real constant and the angle φ is a real variable. The evaluation of the integral gives $f(z_0)$:

$$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\exp(e_{12}\varphi))\, d\varphi = f(z_0)$$

because we can take any radius and also the limit $r \to 0$. The consequence of this theorem is immediate: the values of $f(z)$ at a closed path C determine its value at any z_0 inside the region bounded by C. This is a characteristic property of the harmonic functions, already commented above.

Let us rewrite the Cauchy integral formula in a more suitable form:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t - z}\, dt = f(z)$$

The first and successive derivative with respect to z are:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{(t - z)^2}\, dt = f'(z) \qquad\qquad \frac{n!}{2\pi\, e_{12}} \oint_C \frac{f(t)}{(t - z)^{n+1}}\, dt = f^{n}(z)$$

For $z = 0$ we have:

$$\frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t}\, dt = f(0) \qquad\qquad \frac{n!}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t^{n+1}}\, dt = f^{n}(0)$$

Now we see that these integrals always exist if $f(z)$ is analytic, that is, all the derivatives exist at the points where the function is analytic. In other words, the existence of the first derivative (analyticity) implies the existence of those with higher order.

The Cauchy integral formula may be converted into a power series of z:

$$f(z) = \frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t - z}\, dt = \frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)\, dt}{t\,(1 - z/t)} = \frac{1}{2\pi\, e_{12}} \oint_C \frac{f(t)}{t} \sum_{k=0}^{\infty} \left(\frac{z}{t}\right)^k dt$$

Rewriting this expression we find the Taylor series:

$$f(z)=\frac{1}{2\pi\, e_{12}}\sum_{k=0}^{\infty} z^k \oint_C \frac{f(t)}{t^{k+1}}\,dt=\sum_{k=0}^{\infty}\frac{f^k(0)}{k\,!}\,z^k$$

The only assumption made in the deduction is the analyticity of $f(z)$. So this series is convergent within the largest circle centred at the origin where the function is analytic (that is, the convergence circle touches the closest singular point). The Taylor series is unique for any analytic function. On the other hand, every analytic function has a Taylor series.

Instead of the origin we can take a series centred at another point z_0. In this case, following the same way as above, one arrives to the MacLaurin series:

$$f(z)=\sum_{k=0}^{\infty}\frac{f^k(z_0)}{k\,!}(z-z_0)^k$$

For instance, the Taylor series ($z_0 = 0$) of the exponential, taking into account the fact that all the derivatives are equal to the exponential and exp(0) = 1, is:

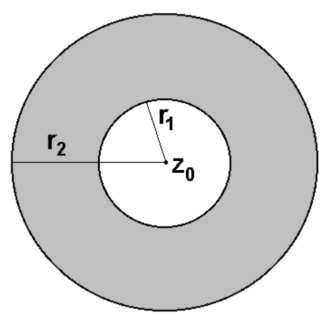
$$\exp(z)=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$$

The exponential has not any singular point. Then the radius of convergence is infinite.

In order to find a convergent series for a function which is analytic in an annulus although not at its centre (for $r_1 < |z - z_0| < r_2$ as shown in figure 3.5), we must add powers with negative exponents, obtaining the Lauren series:

$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k$$

Figure 3.5



The Lauren series is unique, and coincident with the McLaurin series if the function has not any singularity at the central region. The coefficients are obtained in the same way as above:

$$a_k=\frac{1}{2\pi\, e_{12}}\oint_C\frac{f(t)\,dt}{t^{k+1}}$$

where the path C encloses the central circle. When $f(x)$ is not analytic at some point of this central circle (e.g. z_0), the powers of negative exponent appear in the Laurent series. The coefficient a_{-1} is called the *residue* of f. The Lauren series is the addition of a series of powers with negative exponent and the McLaurin series:

$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k=\sum_{k=-\infty}^{-1}a_k(z-z_0)^k+\sum_{k=0}^{\infty}a_k(z-z_0)^k$$

and is convergent only when both series are convergent, so that the annulus goes from the radius of convergence of the first series (r_1) to the radius of convergence of the McLaurin series (r_2).

Let us review singular points, the points where an analytic function is not defined. An isolated singular point may be a *removable singularity*, an *essential singularity* or a *pole*. A point z_0 is a removable singularity if the limit of the function at this point exists and, therefore, we may remove the singularity taking the limit as the value of the function at z_0. A point z_0 is a pole of a function $f(z)$ if it is a zero of the function $1/f(z)$. Finally, z_0 is an essential singularity if both limits of $f(z)$ and $1/f(z)$ at z_0 do not exist.

The Lauren series centred at a removable singularity has not powers with negative exponent. The series centred at a pole has a finite number of powers with negative exponent, and that centred at an essential singularity has an infinite number of powers with negative exponent. Let us see some examples. The function $\sin z / z$ has a removable singularity at $z = 0$:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$

So the series has only positive exponents.

The function $\exp(1/z)$ has an essential singularity at $z = 0$. From the series of $\exp(z)$, we obtain a Lauren series with an infinite number of powers with negative exponent by changing z for $1/z$. Also the radius of convergence is infinite:

$$\exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots$$

Finally, the function $1/z^2 (z-1)$ has a pole at $z = 0$. Its Lauren series:

$$\frac{1}{z^2(z-1)} = -\frac{1}{z^2}\left(1 + z + z^2 + z^3 + z^4 \dots\right) = -\frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2 \dots$$

is convergent for $0<|z|<1$ since the function has another pole at $z = 1$.

To see the importance of the residue, let us calculate the integral of a function through an annular way from its Lauren series:

$$\oint_C f(z)\,dz = \sum_{k=-\infty}^{\infty} a_k \oint_C (z - z_0)^k\,dz$$

For $k\geq 0$ the integral is zero because $(z - z_0)^k$ is analytic in the whole domain enclosed by C. For $k<-2$ the integral is also zero because the path is inside a region where the powers are analytic:

$$\oint_C (z - z_0)^k\,dz = \lim_{z_1 \to z_2} \left[\frac{(z - z_0)^{k-1}}{k-1}\right]_{z_1}^{z_2} = 0$$

However $k = -1$ is a special case. Taking the circular path $z = r\exp(e_{12}\varphi)$ ($r_1 < r < r_2$) we have:

$$\oint_C \frac{dz}{z - z_0} = e_{12} \int_0^{2\pi} d\varphi = 2\pi \; e_{12}$$

So that the *residue theorem* is obtained:

$$\oint_C f(z)dz = 2\pi \; e_{12} \; a_{-1}$$

where a_{-1} is the coefficient of the Lauren series centred at the pole. If the path C encloses some poles, then the integral is proportional to the sum of the residues:

$$\oint_C f(z)dz = 2\pi \; e_{12} \sum \text{residues}$$

Let us see the case of the last example. If C is a path enclosing $z = 0$ and $z = 1$ then the residue for the first pole is 1 and that for the second pole –1 (for a counterclockwise path) so that the integral vanishes:

$$\oint_C \frac{dz}{z^2(z-1)} = \oint_C \left(\frac{1}{z-1} - \frac{1}{z} - \frac{1}{z^2} \right) dz = \oint_C \frac{dz}{z-1} - \oint_C \frac{dz}{z} = 0$$

The fundamental theorem of algebra

Firstly let us prove the *Liouville's theorem*: if $f(x)$ is analytic and bounded in the whole complex plane then it is a constant. If $f(x)$ is bounded we have:

$$|f(x)| < M$$

The derivative of $f(x)$ is always given by:

$$f'(z) = \frac{1}{2\pi \; e_{12}} \oint_C \frac{f(t)}{(t-z)^2} \, dt$$

Following the circular path $t - z = r\exp(e_{12}\varphi)$ we have:

$$f'(z) = \frac{1}{2\pi \; r} \int_0^{2\pi} f(r\exp(e_{12}\varphi)) \; \exp(-e_{12}\varphi) d\varphi$$

Using the inequality $\left| \int f(z) \, dz \right| \leq \int |f(z)| \, dz$, we find:

$$|f'(z)| \leq \frac{1}{2\pi \; r} \int_0^{2\pi} |f(r\exp(e_{12}\varphi))| \; d\varphi \leq \frac{1}{2\pi \; r} \int_0^{2\pi} M \; d\varphi = \frac{M}{2\pi \; r}$$

Since the function is analytic in the entire plane, we may take the radius r as large as we wish. In consequence, the derivative must be null and the function constant, which is the proof of the theorem.

A main consequence of the Liouville's theorem is the *fundamental theorem of algebra*: any polynomial of degree n has always n zeros (not necessarily different):

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = 0 \quad \Rightarrow \quad \exists \; z_i \quad i \in \{1, .., n\} \qquad p(z_i) = 0$$

where a_0, a_1, a_2, ... a_n are complex coefficients. For real coefficients, the zeros are whether real or pairs of conjugate complex numbers. The proof is by supposing that $p(z)$ has not any zero. In this case $f(z) = 1/p(z)$ is analytic and bounded (because $p(z)\rightarrow 0$ for $|z| \rightarrow \infty$) in the whole plane. From the Liouville's theorem $f(z)$ and $p(z)$ should be constant becoming in contradiction with the fact that $p(z)$ is a polynomial. In conclusion $p(z)$ has at least one zero.

According to the division algorithm, the division of the polynomial $p(z)$ by $z - b$ decreases the degree of the quotient $q(z)$ by a unity, and yields a complex number r as remainder:

$$p(z) = (z - b)\, q(z) + r$$

The substitution of z by b gives:

$$p(b) = r$$

That is, the remainder of the division of a polynomial by $z - b$ is equal to its numerical value for $z = b$. On the other hand, if b is a zero z_i, the remainder vanishes and we have an exact division:

$$p(z) = (z - z_i)\, q(z)$$

Again $q(z)$ has at least one zero. In each division, we find a new zero and a new factor, so that the polynomial completely factorises with as many zeros and factors as the degree of the polynomial, which ends the proof:

$$p(z) = a_n \prod_{i=1}^{n} (z - z_i) \qquad\qquad n = \text{degree of } p(z)$$

For example, let us calculate the zeros of the polynomial $z^3 - 5\, z^2 + 8\, z - 6$. By the Ruffini method we find the zero $z = 3$:

$$\begin{array}{r|rrr|r} & 1 & -5 & 8 & -6 \\ 3 & & 3 & -6 & 6 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

The zeros of the quotient polynomial $z^2 - 2z - 2$ are obtained through the formula of the equation of second degree:

$$z = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = 1 \pm \sqrt{-1} = 1 \pm e_{12}$$

yielding a pair of conjugate zeros. Then the factorisation of the polynomial is:

$$z^3 - 5\,z^2 + 8\,z - 6 = (\,z - 3\,)\,(\,z - [1 + e_{12}\,]\,)\,(\,z - [1 - e_{12}\,]\,)$$

Exercises

3.1 Multiply the complex numbers $z = 1 + 3\,e_{12}$ and $t = -2 + 2\,e_{12}$. Draw the geometric figure of their product and check the result found.

3.2 Prove that the modulus of a complex number is the square root of the product of this number by its conjugate.

3.3 Solve the equation $x^4 - 1 = 0$. Being of fourth degree, you must obtain four complex solutions.

3.4 For which natural values of n is fulfilled the following equation?

$$(\,1 + e_{12}\,)^n + (\,1 - e_{12}\,)^n = 0$$

3.5 Find the cubic roots of $-3 + 3\,e_{12}$.

3.6 Solve the equation:

$$z^2 + (\,-3 + 2\,e_{12}\,)\,z + 5 - e_{12} = 0$$

3.7 Find the analytical extension of the real functions $\sin x$ and $\cos x$.

3.8 Find the Taylor series of $\log\left(1 + \exp\left(-z\right)\right)$.

3.9 Calculate the Lauren series of $\dfrac{1}{z^2 + 2z - 8}$ in the annulus $1 < |\,z - 2\,| < 4$.

3.10 Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n \left(z + 1\right)^n}$ and its analytic function.

3.11 Calculate the Lauren series of $\dfrac{\sin z}{z^2}$ and the annulus of convergence.

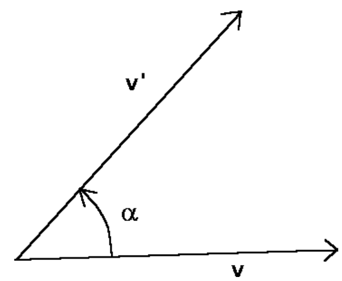
3.12 Prove that if $f(z)$ is analytic and does not vanish then it is a conformal mapping.

4. TRANSFORMATIONS OF VECTORS

The transformations of vectors are mappings from the vector plane to itself. Those transformations preserving the modulus of vectors, such as rotations and reflections, are called *isometries* and those which preserve angles between vectors are said to be *conformal*. Besides rotations and reflections, the inversions and dilatations are also conformal transformations.

Rotations

Figure 4.1



A *rotation* through an angle α is the geometric operation consisting in turning a vector until it forms an angle α with the previous orientation. The positive direction of angles is counterclockwise (figure 4.1). Under rotations the modulus of any vector is preserved. According to the definition of geometric product, the multiplication of a vector v by a unitary complex number with argument α produces a vector v' rotated through an angle α with regard to v.

$$v' = v\, 1_\alpha = v\, (\cos\alpha + e_{12}\sin\alpha)$$

This algebraic expression for rotations, when applied to a real or complex number instead of vector, modifies its value. However, real numbers are invariant under rotations and the parallelograms can be turned without changing the complex number which they represent. Therefore this expression for rotations, although being useful for vectors, is not valid for complex numbers. In order to remodel it, we factorise the unitary complex number into a product of two complex numbers with half argument. According to the permutative property, we can permute the vector and the first complex number whenever writing the conjugate:

$$v' = v\, 1_\alpha = v\, 1_{\alpha/2}\, 1_{\alpha/2} = 1_{-\alpha/2}\; v\, 1_{\alpha/2} = (\cos\frac{\alpha}{2} - e_{12}\sin\frac{\alpha}{2})\, v\, (\cos\frac{\alpha}{2} + e_{12}\sin\frac{\alpha}{2})$$

The algebraic expression for rotations now found preserves complex numbers:

$$z' = 1_{-\alpha/2}\; z\, 1_{\alpha/2} = z$$

Let us calculate the rotation of the vector $4\, e_1$ through $2\pi / 3$ by multiplying it by the unitary complex with this argument:

$$v' = 4\, e_1 \left(\cos\frac{2\pi}{3} + e_{12}\sin\frac{2\pi}{3} \right) = 4\, e_1 \left(-\frac{1}{2} + e_{12}\frac{\sqrt{3}}{2} \right) = -2\, e_1 + 2\sqrt{3}\, e_2$$

On the other hand, using the half angle $\pi/3$ we have:

$$v' = \left(\cos\frac{\pi}{3} - e_{12}\sin\frac{\pi}{3}\right) 4\, e_1 \left(\cos\frac{\pi}{3} + e_{12}\sin\frac{\pi}{3}\right)$$

$$= \left(\frac{1}{2} - e_{12}\,\frac{\sqrt{3}}{2}\right) 4\, e_1 \left(\frac{1}{2} + e_{12}\,\frac{\sqrt{3}}{2}\right) = -2\, e_1 + 2\sqrt{3}\, e_2$$

With the expression of half angle, it is not necessary that the complex number has unity modulus because:

$$z = |\,z\,|\; 1_{\alpha/2} \qquad\qquad z^{-1} = 1_{-\alpha/2}\; |\,z\,|^{-1}$$

Then, the rotation through an angle α can be written as:

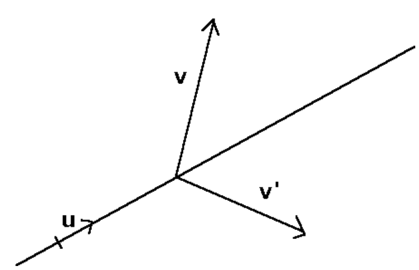
$$v' = z^{-1}\, v\, z$$

The composition of two successive rotations implies the product of both complex numbers, whose argument is the addition of the angles of both rotations.

$$v'' = 1_{-\beta/2}\, v'\, 1_{\beta/2} = 1_{-\beta/2}\, 1_{-\alpha/2}\, v\, 1_{\alpha/2}\, 1_{\beta/2} = 1_{-(\alpha+\beta)/2}\, v\, 1_{(\alpha+\beta)/2}$$

Reflections

Figure 4.2



A *reflection* of a vector in a direction is the geometric transformation which preserves the component having this direction and changes the sign of the perpendicular component (figure 4.2). The product of proportional vectors is commutative and that of orthogonal vectors is anti-commutative. Because of this, the reflected vector v' may be obtained as the multiplication of the vector v by the unitary vector u of the reflection axis at the left and right hand sides:

$$v' = u\, v\, u = u\,(\,v_{\|} + v_{\perp}\,)\,u = u\, v_{\|}\, u + u\, v_{\perp}\, u = v_{\|} - v_{\perp} \qquad \text{with} \quad u^2 = 1$$

where $v_{\|}$ and $v_{\perp}$ are the components of v proportional and perpendicular to u respectively.

Instead of the unitary vector u, any vector d having the axis direction can be introduced in the expression for reflections whenever we write its inverse at the left side of the vector:

$$v' = \frac{d\, v\, d}{|\,d\,|^2} = d^{-1} v\, d$$

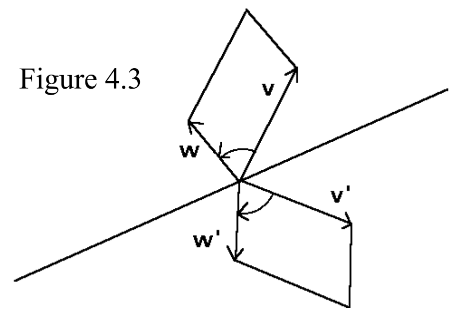
Although the reflection does not change the absolute value of the angle between vectors, it changes its sign. Under reflections, real numbers remain invariant but the complex numbers become conjugate because a reflection generates a symmetric parallelogram (figure 4.3) and changes the sign of the imaginary part:

$$z = a + b\, e_{12} \qquad a, b \text{ real}$$

$$z' = \frac{d\,\left(a + b\, e_{12}\right) d}{|\, d\,|^2}$$

$$= \frac{d^2 a - d^2 b\, e_{12}}{|\, d\,|^2} = a - b\, e_{12} = z^*$$

Figure 4.3



For example, let us calculate the reflection of the vector $3\, e_1 + 2\, e_2$ with regard to the direction $e_1 - e_2$. The resulting vector will be:

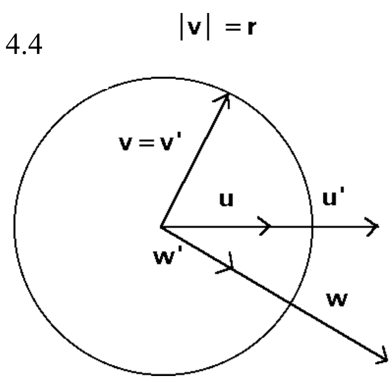
$$v' = \frac{1}{2} (\, e_1 - e_2\,)\,(\,3\, e_1 + 2\, e_2\,)\,(\, e_1 - e_2\,) = \frac{1}{2}\,(\, e_1 - e_2\,)\,(\,1 - 5\, e_{12}\,) = -\,2\, e_1 - 3\, e_2$$

Inversions

The *inversion* of radius r is the geometric transformation which maps every vector v on $r^2\, v^{-1}$, that is, on a vector having the same direction but with a modulus equal to $r^2 / |\, v\,|$:

$$v' = r^2\, v^{-1} \qquad r \text{ real}$$

Figure 4.4



This operation is a generalisation of the inverse of a vector in the geometric algebra (radius $r = 1$). It is called inversion of radius r, because all the vectors with modulus r, whose heads lie on a circumference with this radius, remain invariant (figure 4.4). The vectors whose heads are placed inside the circle of radius r transform into vectors having the head outside and reciprocally.

The inversion transforms complex numbers into proportional complex numbers with the same argument (figure 4.5):

$$v' = r^2\, v^{-1} \qquad w' = r^2\, w^{-1} \qquad z = v\, w$$

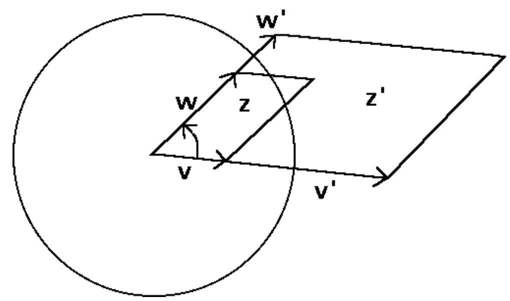
$$z' = v'\, w' = r^4\, v^{-1}\, w^{-1} = r^4\, v\, w\, v^{-2}\, w^{-2} = \frac{r^4\, z}{|\, z\,|^2}$$

Dilatations

The *dilatation* is that geometric transformation which enlarges or shortens a vector, that is, it increases (or decreases) k times the modulus of any vector preserving its orientation. The dilatation is simply the product by a real number k

$$v' = k\,v \qquad k \text{ real}$$

Figure 4.5



The most transformations of vectors that will be used in this book are combinations of these four elementary transformations. Many physical laws are invariant under some of these transformations. In geometry, from the vector transformations we define the transformations of points on the plane, indispensable for solving geometric problems.

Exercises

4.1 Calculate with geometric algebra what is the composition of a reflection with a rotation.

4.2 Prove that the composite of two reflections in different directions is a rotation.

4.3 Consider the transformation in which every vector v multiplied by its transformed v' is equal to a constant complex z^2. Resolve it into elementary transformations.

4.4 Apply a rotation of $2\pi/3$ to the vector $-3\,e_1 + 2\,e_2$ and find the resulting vector.

4.5 Find the reflection of the former vector in the direction $e_1 + e_2$.

SECOND PART: THE GEOMETRY OF THE EUCLIDEAN PLANE

A complete description of the Euclidean plane needs not only directions (given by vectors) but also positions. Points represent locations on the plane. Then, the plane R^2 is the set of all the points we may draw on a fictitious sheet of infinite extension.

Recall that points are noted with capital Latin letters, vectors and complex numbers with lowercase Latin letters, and angles with Greek letters. A vector going from the point P to the point Q will be symbolised by PQ and the line passing through both points will be distinguished as $\overline{PQ}$.

Translations

A *translation* v is the geometric operation which moves a point P in the direction and length of the vector v to give the point Q:

$$+ : R^2 \times V_2 \to R^2$$

$$(P, v) \to Q = P + v$$

Then the vector v is the oriented segment going from P to Q. The set of points together with vectors corresponding to translations is called *the affine plane*[1] $(R^2, V_2, +)$. From this equality a vector is defined as a subtraction of two points, usually noted as PQ:

$$v = Q - P = PQ$$

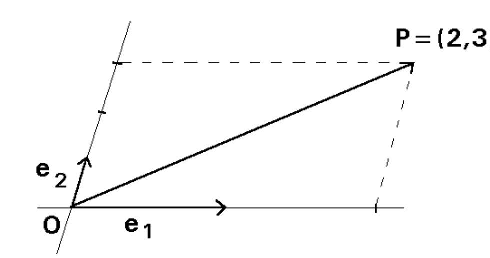
The sign of addition is suitable for translations, because the composition of translations results in an addition of their vectors:

$$Q = P + v \qquad R = Q + w \qquad R = (P + v) + w = P + (v + w)$$

5. POINTS AND STRAIGHT LINES

Coordinate systems

Figure 5.1



Given an origin of coordinates O, every point P on the plane is described by the *position vector* OP traced from the origin. The *coordinates* (c_1, c_2) of a point are the components of the postion vector in the chosen vector base $\{e_1, e_2\}$:

$$OP = c_1 e_1 + c_2 e_2$$

Then every point is given by means of a pair of

[1] The affine plane does not imply a priori an Euclidean or pseudo-Euclidean character.

coordinates:

$$P = (c_1, c_2)$$

For example, the position vector shown in the figure 5.1 is:

$$OP = 2\ e_1 + 3\ e_2$$

and hence the coordinates of P are (2, 3).

A *coordinate system* is the set $\{O;\ e_1,\ e_2\}$, that is, the origin of coordinates O together with the base of vectors. The coordinates of a point depend on the coordinate system to which they belong. If the origin or any base vector is changed, the coordinates of a point are also modified. For example, let us calculate the coordinates of the points P and Q in the figure 5.2. Both coordinate systems have the same vector base but the origin is different:

$$S = \{O;\ e_1,\ e_2\} \qquad S' = \{O';\ e_1,\ e_2\}$$

Since $OP = 2\ e_1 + 3\ e_2$ and $O'P = 2\ e_2$, the coordinates of the point P in the coordinate systems S and S' are respectively:

$$P = (2, 3)_S = (0, 2)_{S'}$$

Analogously:

$$OQ = -\,4\ e_1 + e_2 \quad \text{and} \quad O'Q = -\,6\ e_1$$

$$Q = (-4, 1)_S = (-6, 0)_{S'}$$

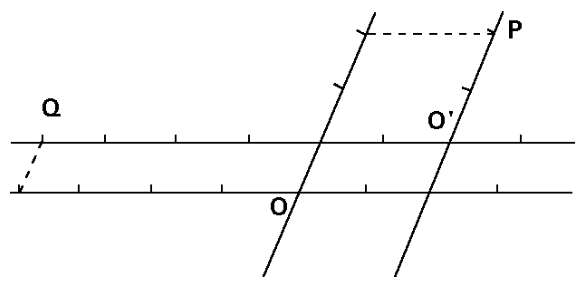


Figure 5.2

The *coordinate axes* are the straight lines passing through the origin and having the direction of the base vectors. All the points lying on a coordinate axis have the other coordinate equal to zero.

If a point Q is obtained from a point P by the translation v then:

$$Q = P + v \qquad \Leftrightarrow \qquad OQ = OP + v$$

that is, we must add the components of the translation vector v to the coordinates of the point P in order to obtain the coordinates of the point Q:

$$(q_1, q_2) = (p_1, p_2) + (v_1, v_2)$$

For example, let us apply the translation $v = 3\ e_1 - 5\ e_2$ to the point $P = (-6, 7)$:

$$Q = P + v = (-6, 7) + (3, -5) = (-3, 2)$$

On the other hand, given two points, the translation vector having these points as extremes is found by subtracting their coordinates. For instance, the vector from $P = (2, 5)$

to $Q = (-3, 4)$ is:

$$PQ = Q - P = (-3, 4) - (2, 5) = (-5, -1) = -5\, e_1 - e_2$$

Remember the general rule for operations between points and vectors:

coordinates = coordinates + components

components = coordinates – coordinates

Barycentric coordinates

Why is the plane described by complicated concepts such as translations and the coordinate system instead of using points as fundamental elements? This question was studied and answered by Möbius in *Der Barycentrische Calculus* (1827) and Grassmann in *Die Ausdehnungslehre* (1844). From the figure 5.3 it follows that:

$$OR = c_1\, e_1 + c_2\, e_2 = c_1\, OP + c_2\, OQ$$

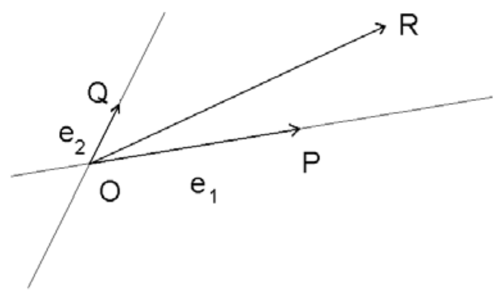


Figure 5.3

When writing all vectors as difference of points and isolating the generic point R, we obtain:

$$R = (1 - c_1 - c_2)\, O + c_1\, P + c_2\, Q$$

That is, a coordinate system is a set of three non-aligned points $\{O, P, Q\}$ so that every point on the plane is a linear combination of these three points with coefficients whose addition is equal to the unity. The coefficients $(1 - c_1 - c_2,\ c_1,\ c_2)$ are usually called *barycentric* coordinates, although they only differ from the usual coordinates in a third dependent coordinate.

Distance between two points and area

The *distance* between two points is the modulus of the vector going from one point to the other:

$$d(P, Q) = |PQ| = |Q - P|$$

The distance has the following properties:

1) The distance from a point to itself is zero: $d(P, P) = |PP| = 0$ and the reverse assertion: if the distance between two points is null then both points are coincident:

$$d(P, Q) = 0 \quad \Rightarrow \quad |PQ| = 0 \quad \Rightarrow \quad Q - P = 0 \quad \Rightarrow \quad P = Q$$

2) The distance has the symmetrical property:

$$d(P, Q) = |PQ| = |QP| = d(Q, P)$$

3) The distance fulfils the *triangular inequality*: the addition of the lengths of any two sides of a triangle is always higher than or equal to the length of the third side (figure 5.4).

$$d(P, Q) + d(Q, R) \geq d(P, R)$$

that is:

$$|PQ| + |QR| \geq |PR| = |PQ + QR|$$

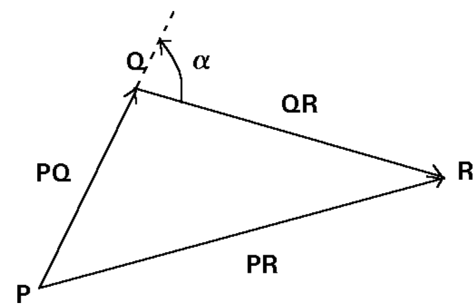


Figure 5.4

The proof of the triangular inequality is based on the fact that the product of the modulus of two vectors is always higher than or equal to the inner product:

$$|PQ|\ |QR| \geq PQ \cdot QR = |PQ|\ |QR| \cos \alpha$$

Adding the square of both vectors to the left and right hand sides, one has:

$$PQ^2 + QR^2 + 2\ |PQ|\ |QR| \geq PQ^2 + QR^2 + 2\ PQ \cdot QR$$

$$(|PQ| + |QR|)^2 \geq (PQ + QR)^2$$

$$|PQ| + |QR| \geq |PQ + QR|$$

For any kind of coordinate system the distance is calculated through the inner product:

$$d^2(P, Q) = PQ^2 = PQ \cdot PQ$$

The oriented area of a parallelogram is the outer product of both non parallel sides. If P, Q and R are three consecutive vertices of the parallelogram, the area is:

$$A = PQ \wedge QR$$

Then, the area of the triangle with these vertices is the half of the parallelogram area.

$$A = \frac{1}{2} PQ \wedge QR$$

If the coordinates are given in the system formed by three non-linear points $\{O, X, Y\}$:

$$P = (1 - x_P - y_P)\ O + x_P\ X + y_P\ Y$$

$$Q = (1 - x_Q - y_Q)\ O + x_Q\ X + y_Q\ Y$$

$$R = (1 - x_R - y_R)\, O + x_R\, X + y_R\, Y$$

it is easy to prove that:

$$PQ \wedge QR = \begin{vmatrix} 1 - x_P - y_P & x_P & y_P \\ 1 - x_Q - y_Q & x_Q & y_Q \\ 1 - x_R - y_R & x_R & y_R \end{vmatrix} OX \wedge OY$$

Then the absolute value of the area of the triangle *PQR* is equal to the product of the absolute value of the determinant of the coordinates and the area of the triangle formed by the three base points:

$$|A| = \left| \begin{vmatrix} 1 - x_P - y_P & x_P & y_P \\ 1 - x_Q - y_Q & x_Q & y_Q \\ 1 - x_R - y_R & x_R & y_R \end{vmatrix} \right| \frac{|OX \wedge OY|}{2}$$

Condition of alignment of three points

Three points *P*, *Q*, *R* are said to be *aligned*, that is, they lie on a line, if the vectors *PQ* and *PR* are proportional, and therefore commute.

$$P, Q, R \text{ aligned} \quad \Leftrightarrow \quad PR = k\, PQ \quad \Leftrightarrow \quad PQ \wedge PR = 0 \quad \Leftrightarrow$$

$$PQ\, PR - PR\, PQ = 0 \quad \Leftrightarrow \quad PQ\, PR = PR\, PQ \quad \Leftrightarrow \quad PR = PQ^{-1}\, PR\, PQ$$

The first equation yields a proportionality between components of vectors:

$$\frac{y_Q - y_P}{x_Q - x_P} = \frac{y_R - y_P}{x_R - x_P}$$

The second equation means that the area of the triangle *PQR* must be zero:

$$\begin{vmatrix} 1 - x_P - y_P & x_P & y_P \\ 1 - x_Q - y_Q & x_Q & y_Q \\ 1 - x_R - y_R & x_R & y_R \end{vmatrix} = 0$$

that is, the barycentric coordinates of the three points are linearly dependent. According to the previous chapter, the last equality means that the vector *PR* remains unchanged after a reflection in the direction *PQ*. Obviously, this is only feasible when *PR* is proportional to *PQ*, that is, when the three points are aligned.

Cartesian coordinates

The coordinates (x, y) of any system having an orthonormal base of vectors are called *Cartesian coordinates* from Descartes[2]. The horizontal coordinate x is called *abscissa* and the vertical coordinate y *ordinate*. In this kind of bases, the square of a vector is equal to the sum of the squares of both components, leading to the following formula for the distance between two points:

$$P = (x_P, y_P) \qquad\qquad Q = (x_Q, y_Q)$$

$$d(P,Q) = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$$

For example, the triangle with vertices $P = (2, 5)$, $Q = (3, 2)$ and $R = (-4, 1)$ have the following sides:

$$PQ = Q - P = (3, 2) - (2, 5) = (1, -3) = e_1 - 3\, e_2 \qquad |PQ| = \sqrt{10}$$

$$QR = R - Q = (-4, 1) - (3, 2) = (-7, -1) = -7\, e_1 - e_2 \qquad |QR| = \sqrt{50}$$

$$RP = P - R = (2, 5) - (-4, 1) = (6, 4) = 6\, e_1 + 4\, e_2 \qquad |PR| = \sqrt{40}$$

The area of the triangle is the half of the outer product of any pair of sides:

$$A = \frac{1}{2} PQ \wedge QR = \frac{1}{2}\,(e_1 - 3\, e_2) \wedge (-7\, e_1 - e_2) = -11\, e_{12} \qquad |A| = 11$$

Vectorial and parametric equations of a line

The condition of alignment of three points is the starting point to deduce the equations of a straight line. A line is determined whether by two points or by a point and a direction defined by its *direction vector*, which is not unique because any other proportional vector will also be a direction vector for this line. Known the direction vector v and a point P on the line, the *vectorial equation* gives the generic point R on the line:

$$r\!: \{v, P\} \qquad P = (x_p, y_P) \qquad R = (x, y)$$

$$PR = k\, v$$

Separating coordinates, the *parametric equation* of the line is obtained:

$$(x, y) = (x_p, y_p) + k\,(v_1, v_2)$$

Here the position of a point on the line depends on the real parameter k, usually identified

[2] If the x-axis is horizontal and the y-axis vertical, an orthonormal base of vectors is called *canonical*.

with the time in physics.

Knowing two points P, Q on the line, a direction vector v can be obtained by subtraction of both points:

$$r: \{P, Q\} \qquad v = Q - P$$

In this case, the vectorial and parametric equations become:

$$PR = k\,PQ \quad \Leftrightarrow \quad R = P + k\,(Q - P) = P\,(1 - k) + k\,Q$$

The parameter k indicates in which proportion the point R is closer to P than to Q. For example, the midpoint of the segment PQ can be calculated with $k = 1/2$. The figure 5.5 shows where the point R is located on the line PQ as a function of the parameter k. In this way, all the points of the line are mapped to the real numbers incorporating the order relation and the topologic properties of this set to the line[3].

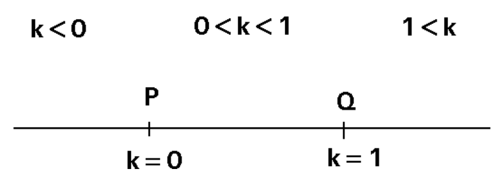


Figure 5.5

Algebraic equation and distance from a point to a line

If the point R belongs to the line, the vector PR and the direction vector v are proportional and commute:

$$PR\,v = v\,PR \quad \Leftrightarrow \quad PR\,v - v\,PR = 0 \quad \Leftrightarrow$$

$$PR \wedge v = 0 \quad \Leftrightarrow \quad PR = v^{-1}\,PR\,v$$

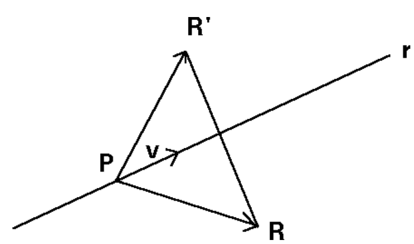


Figure 5.6

The last equation is the *algebraic equation* and shows that the vector PR remains invariant under a reflection in the direction of the line. That is, the point R only belongs to the line when it coincides with the point reflected in this line. Separating components, the *two-point equation* of the line arises:

$$\frac{x - x_P}{v_1} = \frac{y - y_P}{v_2} \quad \Leftrightarrow \quad \frac{x - x_P}{x_Q - x_P} = \frac{y - y_P}{y_Q - y_P}$$

[3] In the book *Geometria Axiomàtica* (Institut d'Estudis Catalans, Barcelona, 1993), Agustí Reventós defines a bijection between the points of a line and the real numbers preserving the order relation as a simplifying axiom of the foundations of geometry. I use implicitly this axiom in this book.

where $v = (v_1, v_2)$.

If the point R does not belong to the line, the reflected point R' differs from R. This allows to calculate the distance from R to the line as the half of the distance between R and R' (figure 5.6). The vector going from R to R' is equal to $PR' - PR$.

$$d(R, r) = \frac{1}{2} \mid R' - R \mid = \frac{1}{2} \mid PR' - PR \mid = \frac{1}{2} \mid v^{-1} PR \, v - PR \mid$$

We will obtain an easier expression for the distance by extracting the direction vector as common factor:

$$d(R, r) = \frac{1}{2} \mid v^{-1} (PR \, v - v \, PR) \mid = \frac{1}{2} \mid v^{-1} \mid \; \mid PR \, v - v \, PR \mid = \frac{\mid PR \wedge v \mid}{\mid v \mid}$$

In this formula, the distance from R to the line r is the height of the parallelogram formed by the direction vector and (figure 5.7).

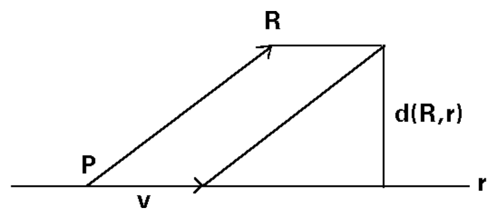


Figure 5.7

Similar line equations can also be deduced using the normal (perpendicular) vector n. Like for the direction vector, the normal vector is not unique because any other proportional vector may also be a normal vector. The normal vector anticommutes with the direction vector of the line, and therefore with the vector PR, P being a given point and R a general point on the line:

$$n \, PR = -PR \, n \quad \Leftrightarrow \quad n \, PR + PR \, n = 0 \quad \Leftrightarrow \quad n \cdot PR = 0 \quad \Leftrightarrow \quad PR = - n^{-1} \, PR \, n$$

The last equation is also called the *algebraic equation* and shows that the vector PR is reversed under a reflection in the perpendicular of the line. Taking components, the *general equation* of the line is obtained:

$$n_1 (x - x_P) + n_2 (y - y_P) = 0 \qquad \Leftrightarrow \qquad n_1 x + n_2 y + c = 0$$

where $n = (n_1, n_2)$ and c is a real constant.

When the point R does not lie on the line, the point R' reflected of R in the perpendicular does not belong to the line. Let PR'' be the reversed of PR', or equivalently R'' be the symmetric point of R with regard to P (figure 5.8). Then, the distance from R to the line r is the half of the distance from R to R'':

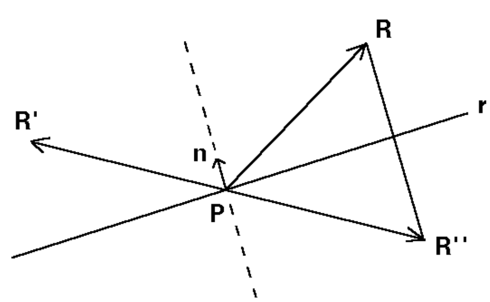


Figure 5.8

$$d(R, r) = \frac{1}{2} \mid PR'' - PR \mid$$

$$=\frac{1}{2}\left|-n^{-1}\,PR\,n-PR\,\right|=\frac{1}{2}\left|-n^{-1}\,(\,PR\,n+n\,PR\,)\,\right|=\frac{|\,PR\cdot n\,|}{|\,n\,|}$$

This expression is the modulus of the projection of the vector PR upon the perpendicular direction. Both formulas of the distance are equivalent because the normal and direction vectors anticommute:

$$n\,v=-\,v\,n \qquad \Leftrightarrow \qquad n\,v+v\,n=0 \qquad \Leftrightarrow \qquad n\cdot v=0$$

Written with components:

$$n_1\,v_1+n_2\,v_2=0$$

From each vector the other one is easily obtained by exchanging components and altering a sign:

$$n=(\,n_1\,,\,n_2\,)=(\,v_2\,,\,-v_1\,)$$

As an application, let us calculate the equation of the line r passing through the points (2, 3) and (7, 6). A direction vector is the segment having as extremes both points:

$$v=(7,6)-(2,3)=(5,3)$$

and an equation for this line is:

$$\frac{x-2}{5}=\frac{y-3}{3}$$

From the direction vector we calculate the perpendicular vector n and the general equation of the line:

$$n=(3,-5) \qquad\qquad 3\,(x-2)-5\,(y-3)=0 \qquad \Leftrightarrow \qquad 3\,x-5\,y+9=0$$

Now, let us calculate the distance from the point $R=(-2,1)$ to the line r. We choose the point (2, 3) as the point P on the line:

$$PR=R-P=(-2,1)-(2,3)=(-4,-2)$$

$$d(\,(-2,1),r\,)=\frac{|\,PR\wedge v\,|}{|\,v\,|}=\frac{|\,PR\cdot n\,|}{|\,n\,|}=\frac{|-4\cdot 3+(-2)\cdot(-5)|}{\sqrt{34}}=\sqrt{\frac{2}{17}}$$

For another point $R=(-3,0)$, we will find a null distance indicating that this point lies on r.

Slope and intercept equations of a line

If the ordinate is written as a function of the abscissa, the equation so obtained is the *slope-intercept equation*:

$$y = m\, x + b$$

Note that this expression cannot describe the vertical lines whose equation is x = constant. The coefficient m is called the *slope* because it is a measure of the inclination of the line. For any two points P and Q on the line we have:

$$y_P = m\, x_P + b$$

$$y_Q = m\, x_Q + b$$

Subtracting both equations:

$$y_Q - y_P = m\, (x_Q - x_P)$$

we see that the slope is the quotient of ordinate increment divided by the abscissa increment (figure 5.9):

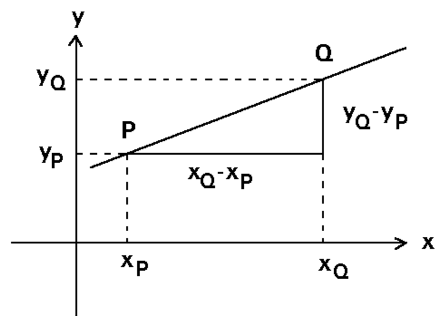


Figure 5.9

$$m = \frac{y_Q - y_P}{x_Q - x_P}$$

This quotient is also the trigonometric tangent of the oriented angle between the line and the positive abscissa semiaxis. For angles larger than $\pi/2$ the slope becomes negative.

b is the ordinate intercepted at the origin, the *y-intercept*:

$$x = 0 \quad \Rightarrow \quad y = b$$

If we know the slope and a point on a line, the *point-slope equation* may be used:

$$y - y_P = m\, (x - x_P)$$

Also, one may write for the *intercept equation* of a line:

$$\frac{x}{a} + \frac{y}{b} = 1$$

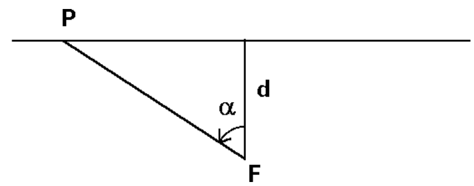
where a and b are the *x-intercept* and *y-intercept* of the line with each coordinate axis.

Polar equation of a line

In this equation, the distance from a fixed point F to the generic point P on the line is a function of the angle α with regard to the perpendicular direction (figure 5.10). If d is the distance from F to the line, then:

$$|FP| = \frac{d}{\cos\alpha}$$

Figure 5.10



This equation allows to relate easily the straight line with the circle and other conic sections.

Intersection of two lines and pencil of lines

The calculation of the intersection of two lines is a very usual problem. Let us suppose that the first line is given by the point P and the direction vector u and the second one by the point Q and the direction vector v. Denoting by R the intersection point, which belongs to both lines, we have:

$$R = P + k\,u = Q + l\,v \qquad \Leftrightarrow \qquad k\,u - l\,v = Q - P = PQ$$

In order to find the coefficients k and l, the vector PQ must be resolved into a linear combination of u and v, what results in:

$$k = PQ \wedge v\,(u \wedge v)^{-1} \qquad l = -\,u \wedge PQ\,(u \wedge v)^{-1}$$

The intersection point is also obtained by directly solving the system of the general equations of both lines:

$$\begin{cases} n_1\,x + n_2\,y + c = 0 \\ n_1'\,x + n_2'\,y + c' = 0 \end{cases}$$

The *pencil of lines* passing through this point is the set of lines whose equations are linear combinations of the equations of both given lines:

$$(1 - p)\,[\,n_1\,x + n_2\,y + c\,] + p\,[\,n'_1\,x + n'_2\,y + c'\,] = 0$$

where p is a real parameter and $-\infty \le p \le \infty$. Each line of the pencil determines a unique value of p, independently of the fact that the general equation for a line is not unique

If the equation system has a unique point as solution, all the lines of the pencil are concurrent at this point. But in the case of an incompatible system, all the lines of the pencil are parallel, that is, they

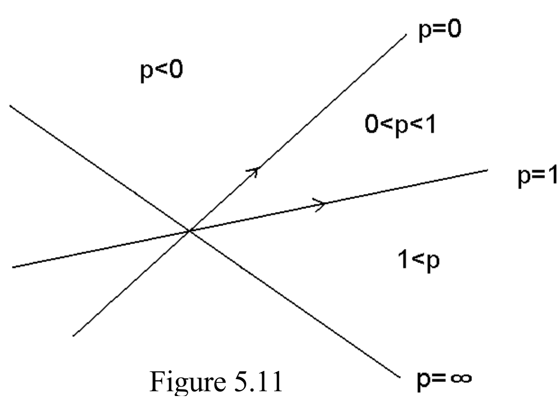


Figure 5.11

intersect in a point at infinity, which gives generality to the concept of pencil of lines.

If the common point R is known or given, then the equation of the pencil is written as:

$$[\,(1-p)\,n_1 + p\,n'_1\,]\,(x - x_R) + [\,(1-p)\,n_2 + p\,n'_2\,]\,(y - y_R) = 0$$

For $0<p<1$, the normal and direction vectors of the p-line are respectively comprised between the normal and direction vectors of both given lines[4]. In the other case the vectors are out of this region (figure 5.11)

For example, calculate the equation of the line passing through the point (3, 4) and the intersection of the lines $3\,x + 2\,y + 4 = 0$ and $2\,x - y + 3 = 0$. The equation of the pencil of lines is:

$$(1-p)\,(3\,x + 2\,y + 4) + p\,(2\,x - y + 3) = 0$$

The line of this pencil to which the point (3, 4) belongs must fulfil:

$$(1-p)\,(3\cdot 3 + 2\cdot 4 + 4) + p\,(2\cdot 3 - 4 + 3) = 0$$

yielding $p = 21/16$ and the line:

$$27\,x - 31\,y + 43 = 0$$

Which is the meaning of the coefficient of linear combination p? Let us write the pencil of lines P determined by the lines R and S as:

$$P = (1-p)\,R + p\,S$$

which implies the same relation for the normal vectors:

$$n_P = (1-p)\,n_R + p\,n_S$$

Taking outer products we obtain:

$$\frac{n_P \wedge n_R}{n_P \wedge n_S} = \frac{p}{p-1} \qquad \Rightarrow \qquad p = \frac{n_P \wedge n_R}{n_P \wedge (n_R - n_S)}$$

These equalities are also valid for direction vectors whenever they have the same modulus than the normal vectors. When the modulus of the normal vectors of the lines R and S are equal we can simplify:

$$\frac{p}{1-p} = \frac{\sin(RP)}{\sin(PS)}$$

In this case, the value $p = 1/2$ corresponds to the bisector line of R and S. We see from this expression and the foregoing ones that each value of p corresponds to a unique line.

[4] This statement requires that the angle from the direction vector to the normal vector be a positive right angle.

Dual coordinates

The *duality principle* states that any theorem relating incidence of lines and points implies a dual counterpart where points and lines have been exchanged. The phrase «two points determine a unique line passing through these points» has the dual statement «two lines determine a unique point, intersection of these lines». All the geometric facts have an algebraic counterpart, and the duality is not an exception. With the barycentric coordinates every point R on the plane can be written as linear combination of three non aligned points O, P and Q:

$$R = (1 - p - q)\, O + p\, P + q\, Q = (p, q)$$

where p and q are the coordinates and O the origin.

These three points determine three non parallel lines A, B, and C in the following way:

$$A = \overline{PQ} \qquad B = \overline{QO} \qquad C = \overline{OP}$$

Then the direction vectors $v_A = PQ$, $v_B = OP$, $v_C = QO$ are related by:

$$v_A + v_B + v_C = 0$$

Note that v_B and $-v_C$ are the base of the vectorial plane.

Any line D on the plane can be written as linear combination of these lines:

$$D = (1 - b - c)\, A + b\, B + c\, C$$

This means that the general equation (with point coordinates) of D is a linear combination of the general equations of the lines A, B and C. I call b and c the *dual coordinates* of the line D. In order to distinguish them from point coordinates, I shall write $D = [b, c]$. The choice of the equation for each line must be unambiguous and therefore the normal vector in the implicit equations for A, B and C will be obtained by turning the direction vector over $\pi/2$ counterclockwise.

Let us see some special cases. If the dual coordinate b is zero we have:

$$D = (1 - c)\, A + c\, C$$

which is the equation of the pencil of lines passing through the intersection of the lines A and C, that is, the point P. Then $P = [0, c]$ (for every c). Analogously, $c = 0$ determines the pencil of lines passing through the intersection of the lines A and B, which is the point Q, and then $Q = [b, 0]$ (for every b). The origin of coordinates is the intersection of the lines B and C and then $O = [b, 1 - b]$ (for every b). Compare the dual coordinates of these points:

$$O = [b, 1-b] \qquad P = [0, c] \qquad Q = [b, 0] \qquad \forall\, b, c$$

with the point coordinates of the lines A, B and C:

$$A=(p,\,1-p) \qquad B=(0,\,q) \qquad C=(p,\,0) \qquad \forall\, p,q$$

Let us see an example: calculate the dual Cartesian coordinates of the line $2x+3y+4=0$. The points of the Cartesian base are $O=(0,0)$, $P=(1,0)$ and $Q=(0,1)$. Then the lines of the Cartesian base are A: $-x-y+1=0$, B: $x=0$, C: $y=0$. We must solve the identity:

$$2x+3y+4 \equiv a'(-x-y+1)+b'x+c'y \qquad \forall\, x,y$$

$$x\,(2+a'-b')+y\,(3+a'-c')+4-a' \equiv 0$$

whose solution is:

$$a'=4 \qquad b'=6 \qquad c'=7$$

Dividing by the sum of the coefficients we obtain:

$$\frac{2x+3y+4}{17} \equiv \frac{4}{17}(-x-y+1)+\frac{6}{17}x+\frac{7}{17}y$$

from where the dual coordinates of this line are obtained as $[b,\,c]=[6/17,\,7/17]$. Let us see their meaning. The linear combination of both coordinates axes is a line of the pencil of lines passing through the origin:

$$\frac{6}{13}x+\frac{7}{13}y=0 \qquad \text{or} \qquad 6x+7y=0$$

This line intersects the third base line $-x-y+1=0$ at the point $(7,\,-6)$, whose pencil of lines is described by:

$$a(-x-y+1)+(1-a)\left(\frac{6}{13}x+\frac{7}{13}y\right)=0$$

Then $2x+3y+4=0$ is the line of this pencil determined by $a=4/17$.

On the other hand, how may we know whether three lines are concurrent and belong to the same pencil or not? The answer is that the determinant of the dual coordinates must be zero:

$$D=(1-b-c)\,A+b\,B+c\,C$$

$$E=(1-b'-c')\,A+b'\,B+c'\,C$$

$$F=(1-b''-c'')\,A+b''\,B+c''\,C$$

$$D,\,E \text{ and } F \text{ concurrent} \qquad \Leftrightarrow \qquad \begin{vmatrix} 1-b-c & b & c \\ 1-b'-c' & b' & c' \\ 1-b''-c'' & b'' & c'' \end{vmatrix}=0$$

When it happens, we will say that the lines are linearly dependent, and we can write anyone of them as linear combination of the others:

$$F = (1 - k)\, D + k\, E$$

We can express a point with an equation for dual coordinates in the same manner as we express a line with an equation for point coordinates. For example, the point (5, 3) is the intersection of the lines $x = 5$, $y = 3$ whose dual coordinates we calculate now:

$$x - 5 \equiv a'\left(1 - x - y\right) + b'\,x + c'\,y \qquad \Rightarrow \qquad a' = -5 \quad b' = -4 \quad c' = -5$$

From where a = 5/14, b = 4/14 and c = 5/14. The dual coordinates of x = 5 are [4/14, 5/14]. Analogously:

$$y - 3 \equiv a'\left(1 - x - y\right) + b'\,x + c'\,y \qquad \Rightarrow \qquad a' = -3 \quad b' = -3 \quad c' = -2$$

From where $a = b$ = 3/8, c = 2/8. The dual coordinates of y = 3 are [3/8, 2/8]. The linear combinations of both lines are the pencil of the point (5, 3), which is described by the parametric equation:

$$[b,\, c] = \left(1 - k\right)\left[\frac{4}{14},\, \frac{5}{14}\right] + k\left[\frac{3}{8}, \frac{2}{8}\right]$$

By removing the parameter k, the general equation is obtained:

$$\frac{b - 4/14}{5} = \frac{c - 5/14}{-6} \qquad \Leftrightarrow \qquad 12\,b + 10\,c - 7 = 0$$

That is, the *dual direction vector* of the point (5, 3) is [5, –6] and the *dual normal vector* is [6, 5]. In the dual plane, an algebra of dual vectors can be defined. A dual direction vector for a point may be obtained as the difference between the dual coordinates of two lines whose intersection be the point. Then, there exist dual translations of lines:

$$\text{dual} + :\; L \times W \to L \qquad\qquad L = \{\text{plane lines}\} \qquad W = \{\text{dual vectors}\}$$

$$(A,\, w) \to B = A + w$$

That is, we add a fixed dual vector w to the line A in order to obtain another line B.

Let us prove the following theorem: all the points whose dual direction vectors are proportional are aligned with the centroid of the coordinate system .

The proof begins from the dual continuous equation for a point P:

$$\frac{b - b_0}{v_1} = \frac{c - c_0}{v_2}$$

which can be written in a parametric form:

$$P: \quad [b,c] = (1-k)[b_0, c_0] + k\,[b_0 + `v_1,\; c_0 + v_2]$$

This equation means that P is the intersection point of the lines with dual coordinates $[b_0, c_0]$ and $[b_0 + v_1, c_0 + v_2]$. Then P must be obtained by solving the system of equations of each line for the point coordinates p, q (x, y if Cartesian):

$$\begin{cases} (1 - b_0 - c_0)A + b_0 B + c_0 C = 0 \\ (1 - b_0 - c_0 - v_1 - v_2)A + (b_0 + v_1)B + (c_0 + v_2)C = 0 \end{cases}$$

By subtraction of both equations, we find an equivalent system:

$$\begin{cases} (1 - b_0 - c_0)A + b_0 B + c_0 C = 0 \\ -(v_1 + v_2)A + v_1\, B + v_2\, C = 0 \end{cases}$$

Now, if we consider a set of points with the same (or proportional) dual direction vector, the first equation changes but the second equation remains constant (or proportional). That is, the first line changes but the second line remains constant. Therefore all points will be aligned and lying on the second line:

$$-(v_1 + v_2)A + v_1 B + v_2 C = 0$$

However in which matter do two points differ whether having or not a proportional dual direction vector? We cannot say that two points are aligned, because we need three points at least. Let us search the third point X, rewriting the foregoing equality:

$$\frac{v_1}{v_1 + v_2}(B - A) + \frac{v_2}{v_1 + v_2}(C - A) = 0$$

Now the variation of the components of the dual vector generates the pencil of the lines $B - A$ and $C - A$. That is, the intersection of both lines is the point X searched:

$$X: \quad \begin{cases} B - A = 0 \\ C - A = 0 \end{cases}$$

Then, two points have a proportional dual direction vector if they are aligned with the point X, intersection of the lines $B - A$ and $C - A$. However, note that the addition of coefficients of each line is zero instead of one, so that one dual coordinate of each line is infinite:

$$B - A = [\infty, c] \qquad\qquad C - A = [b, \infty]$$

I call X the *point at the dual infinity* or simply the *dual infinity point*. The dual infinity point has finite coordinates and a very well defined position. Then the points with proportional dual vector are always aligned with the dual infinity point, and I shall say that they are *parallel points* in a dual sense, of course. In order to precise which point is X let us take in mind that the lines $B - A$ and $C - A$ are the medians of the triangle OPQ (the point base). Hence the dual infinity point is the centroid of the base triangle *OPQ*. With Cartesian coordinates $O = (0, 0)$, $P = (1, 0)$, $Q = (0, 1)$ and $X = (1/3, 1/3)$. This ends the proof.

Summarising, we can say that two parallel points are aligned with the dual infinity point, which is the dual statement of the fact that two parallel lines meet at the infinity, that is, there is a *line at the infinity*, in the usual sense. This is the novelty of the projective geometry in comparison with the Euclidean geometry. In fact, the problem arises because the point coordinates take infinite values, but the line L at the infinity is a well defined line with finite dual coordinates [1/3, 1/3]:

$$L = \left[\frac{1}{3}, \frac{1}{3}\right] = \frac{A + B + C}{3}$$

This means that the line L at the infinity belongs to the pencil of the lines $(A + B)/2$ and C . But both lines are parallel because their direction vectors are proportional:

$$v_A + v_B = -v_C$$

That is, the lines $(A + B)/2$ and C meet at a point located at an infinite distance from the origin of coordinates. Since any other parallel line meets them also at the infinity, this argument does not suffice. However we can understand that the line L also belongs to the pencil of the lines A and $(B + C)/2$, which are also parallel with another direction, that is, they meet at another point of the infinity. Any other pencil we take has always its point of intersection located at the infinity. Then L only has points located at the infinity and because of this it is called the line at the infinity. Summarising we can say that the line at the infinity is the centroid of the three base lines in spite of the incompatibility of its equation:

$$L = \frac{A + B + C}{3} \qquad \Rightarrow \qquad \frac{(-x - y + 1) + x + y}{3} = 0 \qquad \Rightarrow \qquad \frac{1}{3} = 0$$

Moreover, we may interpret the parallel lines as those lines aligned (in a dual sense) with the line at the infinity:

$$E = [b_E, c_E] \qquad\qquad F = [b_F, c_F]$$

$$E \parallel F \qquad \Leftrightarrow \qquad \frac{b_F - b_E}{c_F - c_E} = \frac{b_F - 1/3}{c_F - 1/3}$$

This is a useful equality because it allows us to know whether two lines are parallel from their dual coordinates.

Many of the incidence theorems (and also their dual theorems) can be solved by means of line equations or dual coordinates. A proper example is the proof of the Desargues theorem.

The Desargues theorem

Given two triangles ABC and $A'B'C'$, let P be the intersection of the prolongation of the side AB with the side $A'B'$, Q the intersection of BC with $B'C'$, and R the intersection of CA with $C'A'$. The points P, Q and R are aligned if and only if the lines AA', BB' and CC' meet at the same point.

Proof $\Rightarrow$ The hypothesis states that the lines AA', BB' and CC' intersect at the same point O (figure 5.12):

$$O = a\,A + (\,1 - a\,)\,A'$$

$$O = b\,B + (\,1 - b\,)\,B'$$

$$O = c\,C + (\,1 - c\,)\,C'$$

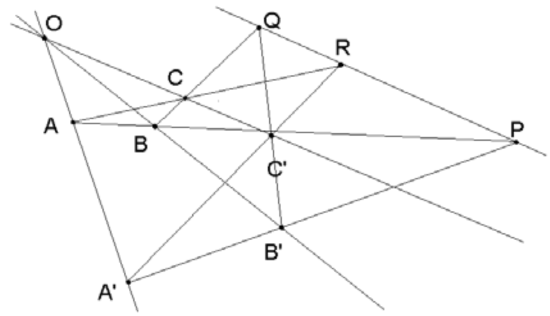


Figure 5.12

with a, b and c being real. Equating the first and second equations we obtain:

$$a\,A + (\,1 - a\,)\,A' = b\,B + (\,1 - b\,)\,B'$$

which can be rearranged as:

$$a\,A - b\,B = -\,(\,1 - a\,)\,A' + (\,1 - b\,)\,B'$$

Dividing by $a - b$ the sum of the coefficients becomes the unity, and then the equation represents the intersection of the lines AB and $A'B'$, which is the point P:

$$P = \frac{a}{a-b}A - \frac{b}{a-b}B = \frac{a-1}{a-b}A' - \frac{b-1}{a-b}B'$$

By equating the second and third equations for the point O, and the third and first ones, analogous equations for Q and R are obtained:

$$Q = \frac{b}{b-c}B - \frac{c}{b-c}C = \frac{b-1}{b-c}B' - \frac{c-1}{b-c}C'$$

$$R = \frac{c}{c-a}C - \frac{a}{c-a}A = \frac{c-1}{c-a}C' - \frac{a-1}{c-a}A'$$

Now we must prove that P, Q and R are aligned, that is, fulfil the equation:

$$R = d\,P + (\,1 - d\,)\,Q \qquad \text{with } d \text{ real}$$

With the substitution of the former equations into the last one we have:

$$\frac{c}{c-a}C - \frac{a}{c-a}A = d\left[\frac{a}{a-b}A - \frac{b}{b-a}B\right] + \left(1-d\right)\left[\frac{b}{b-c}B - \frac{c}{b-c}C\right]$$

Arranging all the terms at the left hand side, the expression obtained must be identical to zero because A, B and C are non aligned points:

$$\left[-\frac{d\,a}{a-b}-\frac{a}{c-a}\right]A+\left[\frac{d\,b}{a-b}-\frac{b(1-d)}{b-c}\right]B+\left[\frac{c}{c-a}+\frac{c(1-d)}{b-c}\right]C\equiv 0$$

This implies that the three coefficients must be null simultaneously yielding a unique value for d,

$$d=\frac{a-b}{a-c}$$

fact which proves the alignment of P, Q and R, and gives the relation for the distances between points:

$$QR\;QP^{-1}=(\,A'O\;\;A'A^{-1}-B'O\;\;B'B^{-1}\,)\,(\,A'O\;\;A'A^{-1}-C'O\;\;C'C^{-1}\,)^{-1}$$

Proof $\Leftarrow$ We will prove the Desargues theorem in the other direction following the same algebraic way but applying the duality, that is, I shall only change the words. *O, A, B, C, P, Q* and *R* will be now lines (figure 5.13).

The hypothesis states that the points AA', BB' and CC' belong to the same line O, that is, O belongs to the pencil of the lines A and A', but also to the pencil of the lines BB' and CC':

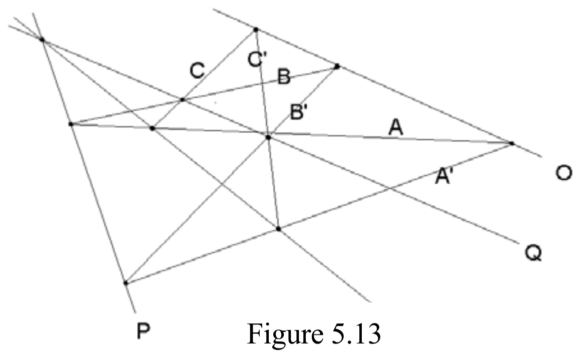


Figure 5.13

$$O=a\,A+(\,1-a\,)\,A'$$

$$O=b\,B+(\,1-b\,)\,B'$$

$$O=c\,C+(\,1-c\,)\,C'$$

with a, b and c being real coefficients. Equating the first and second equations we obtain:

$$a\,A+(\,1-a\,)\,A'=b\,B+(\,1-b\,)\,B'$$

which can be rearranged as:

$$a\,A-b\,B=-(\,1-a\,)\,A'+(\,1-b\,)\,B'$$

Dividing by $a-b$ the sum of the coefficients becomes the unity, and then the equation represents the line passing through the points AB and $A'B'$ (belonging to both pencils), which is the line P:

$$P=\frac{a}{a-b}A-\frac{b}{a-b}=\frac{a-1}{a-b}A'-\frac{b-1}{a-b}B'$$

By equating the second and third equations for the line O, and the third and first ones, analogous equations for Q and R are obtained:

$$Q = \frac{b}{b-c} B - \frac{c}{b-c} C = \frac{b-1}{b-c} B' - \frac{c-1}{b-c} C'$$

$$R = \frac{c}{c-a} C - \frac{a}{c-a} A = \frac{c-1}{c-a} C' - \frac{a-1}{c-a} A'$$

Now we must prove that the lines P, Q and R belong to the same pencil, that is, fulfil the equation:

$$R = d\,P + (\,1 - d\,)\,Q \qquad \text{with } d \text{ real}$$

With the substitution of the former equations into the last one we have:

$$\frac{c}{c-a} C - \frac{a}{c-a} A = d\left[\frac{a}{a-b} A - \frac{b}{b-a} B\right] + \left(1-d\right)\left[\frac{b}{b-c} B - \frac{c}{b-c} C\right]$$

Arranging all the terms at the left hand side, the expression obtained must be identical to zero because A, B and C are independent lines (not belonging to the same pencil):

$$\left[-\frac{d\,a}{a-b} - \frac{a}{c-a}\right] A + \left[\frac{d\,b}{a-b} - \frac{b(1-d)}{b-c}\right] B + \left[\frac{c}{c-a} + \frac{c(1-d)}{b-c}\right] C \equiv 0$$

This implies that the three coefficients must be null simultaneously yielding a unique value for d,

$$d = \frac{a-b}{a-c}$$

fact which proves that P, Q and R are lines of the same pencil.

Exercises

5.1 Let $A = (2, 4)$, $B = (4, -3)$ and $C = (2, -5)$ be three consecutive vertices of a parallelogram. Calculate the fourth vertex D and the area of the parallelogram.

5.2 Prove the Euler's theorem: for any four points A, B, C and D the product $AD\,BC + BD\,CA + CD\,AB$ vanishes if and only if A, B and C are aligned.

5.3 Consider a coordinate system with vectors $\{ e_1, e_2 \}$ where $|e_1| = 1$, $|e_2| = 1$ and the angle formed by both vectors is $\pi/3$.
a) Calculate the area of the triangle ABC being $A = (2, 2)$, $B = (4, 4)$, $C = (4, 2)$.
b) Calculate the distance between A and B, B and C, C and A.

5.4 Construct a trapezoid whose sides $|AB|$, $|BC|$, $|CD|$ and $|DA|$ are known and

AB is parallel to *CD*. Study for which values of the sides does the trapezoid exist.

5.5 Given any coordinate system with points $\{O, P, Q\}$ and any point R with coordinates (p, q) in this system, show that:

$$1-p-q=\frac{Area\ RPQ}{Area\ OPQ} \qquad p=\frac{Area\ ORQ}{Area\ OPQ} \qquad q=\frac{Area\ OPR}{Area\ OPQ}$$

5.6 Let the points $A = (2, 3)$, $B = (5, 4)$, $C = (1, 6)$ be given. Calculate the distance from C to the line AB and the angle between the lines AB and AC.

5.7 Given any barycentric coordinate system defined by the points $\{O, P, Q\}$ and the non aligned transformed points $\{O', P', Q'\}$, an *affinity* (or an *affine transformation*) is defined as that geometric transformation which maps each point D to D' in the following way:

$$D=(1-x-y)\,O+x\,P+y\,Q \rightarrow \qquad D'=(1-x-y)\,O'+x\,P'+y\,Q'$$

Prove that:
a) An affinity maps lines onto lines.
b) An affinity is equivalent to a linear mapping of the coordinates, that is, the coordinates of any transformed point are linear functions of the coordinates of the original point.
c) An affinity preserves the coordinates expressed in any other set of independent points different of the given base:

$$D=(1-b-c)\,A+b\,B+c\,C \rightarrow \qquad D'=(1-b-c)\,A'+b\,B'+c\,C'$$

d) An affinity maps a parallelogram to another parallelogram, and hence, parallel lines to parallel lines.
e) An affinity preserves the ratio $DE\,DF^{-1}$ for any three aligned points D, E and F.

5.8 If $\{A, B, C\}$ is a base of lines and $\{A', B', C'\}$ their transformed lines -the lines of each set being independent-, consider the geometric transformation which maps each line D to D' in the following way:

$$D=(1-b-c)\,A+b\,B+c\,C \rightarrow \qquad D'=(1-b-c)\,A'+b\,B'+c\,C'$$

a) Prove that every pencil of lines is mapped to another pencil of lines.
b) The dual coordinates of D' are linear functions of the dual coordinates of D.
c) This transformation preserves the coefficients which express a line as a linear combination of any three non concurrent lines, that is, in the foregoing mapping $\{A, B, C\}$ have not to be necessarily the dual coordinate base and can be any other set of independent lines.
d) Parallel points are mapped to parallel points.
e) For any three concurrent lines P, Q and R, the single ratio of the dual vectors $PQ\,PR^{-1}$ is preserved.
f) Using the formula of the *cross ratio* of a pencil of any four lines (*ABCD*) as a

function of their direction vectors[5]:

$$(ABCD) = \frac{v_A \wedge v_C \; v_B \wedge v_D}{v_A \wedge v_D \; v_B \wedge v_C}$$

show that it is preserved.

5.9 Calculate the dual coordinates of the lines $x - y + 1 = 0$ and $x - y + 3 = 0$. See that they are aligned in the dual plane with the line at the infinity, whose dual coordinates are [1/3, 1/3], and therefore are parallel.

5.10 Calculate the dual equations of the points (2, 1) and (–3, –1) and their direction vectors. See that they are parallel points. Hence prove that they are aligned with the dual infinity point, the centroid of the coordinate system (1/3, 1/3).

[5] This formula is deduced in the chapter devoted to the cross ratio.

6. ANGLES AND ELEMENTAL TRIGONOMETRY

Here, the basic identities of the elemental trigonometry are deduced in close connection with basic geometric facts, a very useful point of view for our pupils.

Sum of the angles of a polygon

Firstly let us see the special case of a triangle. For any triangle ABC the following identity holds:

$$CA\,AB^{-1}\,AB\,BC^{-1}\,BC\,CA^{-1} = 1$$

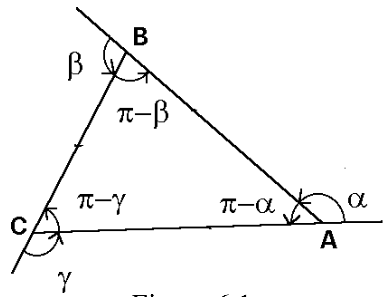


Figure 6.1

Let α, β and γ be the exterior angles between the sides CA and AB, AB and BC, BC and CA respectively (figure 6.1). Applying the definition of geometric quotient, the modulus of all the sides are simplified and only the exponentials of the arguments remain:

$$\exp(\alpha\, e_{12})\exp(\beta\, e_{12})\exp(\gamma\, e_{12}) = \exp[(\alpha+\beta+\gamma)e_{12}] = 1$$

The three angles have the same orientation, which we suppose positive, and are lesser than π. Hence, since the exponential is equal to the unity, the addition of the three angles must be equal to 2π:

$$\alpha+\beta+\gamma = 2\pi$$

The interior angles, those formed by AB and AC, BC and CA, CA and CB are supplementary of α, β, γ (figure 6.1). Therefore the sum of the angles of a triangle is equal to π:

$$(\pi-\alpha)+(\pi-\beta)+(\pi-\gamma) = \pi$$

This result is generalised to any polygon from the following identity:

$$AB\,BC^{-1}\,BC\;\ldots\;YZ^{-1}\,YZ\,ZA^{-1}\,ZA\,AB^{-1} = 1$$

Let α, β, $\gamma \ldots \omega$ be the exterior angles formed by the sides ZA and AB, AB and BC, BC and CD, ..., YZ and ZA, respectively. After the simplification of the modulus of all vectors, we have:

$$\exp[(\alpha+\beta+\gamma+\ldots+\omega)e_{12}] = 1$$

Let us suppose that the orientation defined by the vertices A, B, C ... Z is counterclockwise, although the exterior angles be not necessarily all positive[1]. Translating

[1] That is, it is not needed that the polygon be convex.

them to a common vertex, each angle is placed close each other following the order of the perimeter and summing one turn:

$$\alpha + \beta + \gamma + ... + \omega = 2\pi$$

The interior angles formed by the sides *AB* and *ZA*, *BC* and *BA*, *CD* and *CB*,... *ZA* and *ZY* are supplementary of α, β, γ, ... ω. Therefore the sum of the angles of a polygon is:

$$(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) + ... + (\pi - \omega) = n\,\pi - 2\pi = (n-2)\pi$$

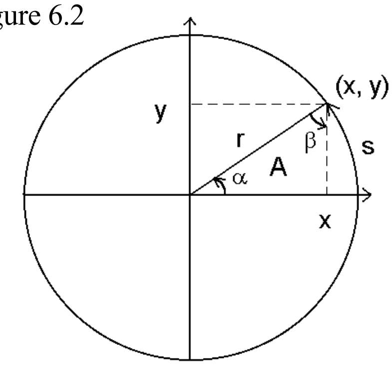
n being the number of sides of the polygon. The deduction for the clockwise orientation of the polygon is analogous with the only difference that the result is negative.

Definition of trigonometric functions and fundamental identities

Let us consider a circle with radius *r* (figure 6.2). The extreme of the radius is a point on the circumference with coordinates (*x*, *y*). The arc between the positive *X* semiaxis and this point (*x*, *y*) has an oriented length *s*, positive if counterclockwise and otherwise negative. Also, the *X*-axis, the arc of circumference and the radius delimit a sector with an oriented area *A*. An *oriented angle* α is defined as the quotient of the arc length divided by the radius[2]:

$$\alpha = \frac{s}{r}$$

Figure 6.2



Since the area of the sector is proportional to the arc length and the area of the circle is $2\pi r$, it follows that:

$$A = \frac{\alpha\, r^2}{2} \quad \Leftrightarrow \quad \alpha = \frac{2\,A}{r^2}$$

The *trigonometric functions*[3], *sine*, *cosine* and *tangent* of the angle α, are respectively defined as the ratios:

$$\sin\alpha = \frac{y}{r} \qquad \cos\alpha = \frac{x}{r} \qquad \mathrm{tg}\,\alpha = \frac{y}{x}$$

and the *cosecant*, *secant* and *cotangent* as their inverse fractions:

---

[2] With this definition it is said that the angle is given in *radians*, although an angle is a quotient of lengths and therefore a number without dimensions.

[3] Also called *circular functions* due to obvious reasons, to be distinguished from the *hyperbolic functions*.

$$\csc\alpha = \frac{r}{y} \equiv \frac{1}{\sin\alpha} \qquad \sec\alpha = \frac{r}{x} \equiv \frac{1}{\cos\alpha} \qquad \cot\alpha = \frac{x}{y} \equiv \frac{1}{\operatorname{tg}\alpha}$$

being r related with x and y by the Pythagorean theorem:

$$r^2 = x^2 + y^2$$

Then the radius vector v is:

$$v = x\, e_1 + y\, e_2 = r\,(\cos\alpha\; e_1 + \sin\alpha\; e_2)$$

From these definitions the fundamental identities follow:

$$\operatorname{tg}\alpha \equiv \frac{\sin\alpha}{\cos\alpha} \qquad \sin^2\alpha + \cos^2\alpha \equiv 1 \qquad 1 + \operatorname{tg}^2\alpha \equiv \frac{1}{\cos^2\alpha} \equiv \sec^2\alpha$$

If we take the opposite angle $-\alpha$ instead of α, the sign of y is changed while x and r are preserved, so we obtain the parity relations:

$$\sin(-\alpha) \equiv -\sin\alpha \qquad \cos(-\alpha) \equiv \cos\alpha \qquad \operatorname{tg}(-\alpha) \equiv -\operatorname{tg}\alpha$$

The sine and tangent are odd functions while the cosine is an even function.

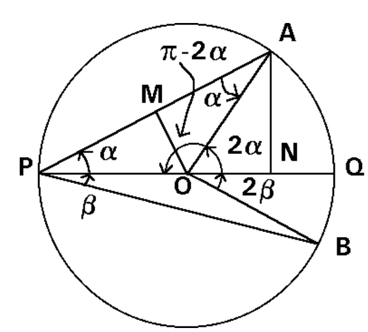
Look at the figure 6.2: $\beta = \pi/2 - \alpha$ is the *complementary* angle of α. If α is higher than $\pi/2$, the angle β becomes negative. On the other hand, if the angle α is negative then β is higher than $\pi/2$. The trigonometric ratios for β give the complementary angle identities:

$$\sin\beta = \frac{x}{r} \equiv \cos\alpha \qquad \cos\beta = \frac{y}{r} \equiv \sin\alpha \qquad \operatorname{tg}\beta = \frac{x}{y} \equiv \frac{1}{\operatorname{tg}\alpha}$$

Angle inscribed in a circle and double angle identities

Let us draw any diameter PQ and any radius OA (figure 6.3) in a circle with centre O. Since OP is also a radius, the triangle POA is isosceles and the angles OPA and PAO, which will be denoted as α, are equal. Because the addition of the three angles is equal to π, the angle AOP is $\pi - 2\alpha$. The angle QOA is supplementary of AOP, and therefore is equal to 2α, the double of the angle APQ. Let us draw from A a segment perpendicular to the diameter and touching it at the point N. By the definition of sine we have:

Figure 6.3



$$\sin 2\alpha = \frac{|NA|}{|OA|} = \frac{|NA|}{|PA|}\,\frac{|PA|}{|OA|}$$

The first quotient is $\sin\alpha$ for the triangle *PNA*. If *M* is the midpoint of the segment *PA*, then $|PA| = 2\ |MA|$ and the second quotient is equal to 2 $\cos\alpha$ for the triangle *MOA*:

$$\sin 2\alpha = 2\sin\alpha \frac{|MA|}{|OA|} \equiv 2\sin\alpha\ \cos\alpha$$

Through an analogous way we obtain $\cos 2\alpha$:

$$\cos 2\alpha = \frac{|ON|}{|OA|} = \frac{\left(|PN| - |PO|\right)}{|PA|}\frac{|PA|}{|OA|} \equiv 2\cos^2\alpha - 1 \equiv \cos^2\alpha - \sin^2\alpha$$

Also this result may be obtained from the second fundamental identity. In order to obtain the tangent of the double angle we make use of the first fundamental identity:

$$\mathrm{tg}\, 2\alpha \equiv \frac{\sin 2\alpha}{\cos 2\alpha} \equiv \frac{2\sin\alpha\ \cos\alpha}{\cos^2\alpha - \sin^2\alpha} \equiv \frac{2\,\mathrm{tg}\,\alpha}{1 - \mathrm{tg}^2\,\alpha}$$

Finally, let us draw any other segment *PB* (figure 6.3). The angle *BPQ* will be denoted as β. By the same arguments as above the angle *BOQ* is 2β. While the angle *BPA* is $\alpha + \beta$, the angle *BOA* is $2\alpha + 2\beta$: an angle inscribed in a circumference is equal to the half of the central angle (angle whose vertex is the centre of the circumference) which intercepts the same arc of circle. Consequently, all the angles inscribed in the same circle and intercepting the same arc are equal independently of the position of the vertex on the circle.

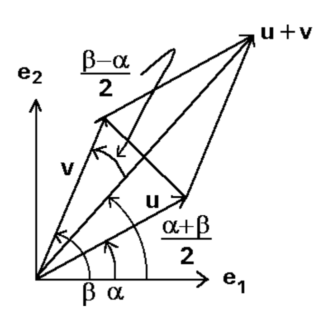
Addition of vectors and sum of trigonometric functions

Let us consider two unitary vectors forming the angles α and β with the e_1 direction (figure 6.4):

$$u = e_1 \cos\alpha + e_2 \sin\alpha \qquad\qquad v = e_1 \cos\beta + e_2 \sin\beta$$

$$u + v = e_1 \left(\cos\alpha + \cos\beta\right) + e_2 \left(\sin\alpha + \sin\beta\right)$$

Figure 6.4



The addition of both vectors, $u + v$, is the diagonal of the rhombus which they form, whence it follows that the long diagonal is the bisector of the angle $\alpha - \beta$ between both vectors. The short diagonal cuts the long diagonal perpendicularly forming four right triangles. Then the modulus of $u + v$ is equal to the double of the cosine of the half of this angle:

$$|u+v| = 2\cos\frac{\alpha-\beta}{2}$$

Moreover, the addition vector forms an angle $(\alpha+\beta)/2$ with the e_1 direction:

$$u+v = 2\cos\frac{\alpha-\beta}{2}\left(e_1\cos\frac{\alpha+\beta}{2} + e_2\sin\frac{\alpha+\beta}{2}\right)$$

By identifying this expression for $u + v$ with that obtained above, we arrive at two identities, one for each component:

$$\cos\alpha + \cos\beta \equiv 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha + \sin\beta \equiv 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

In a similar manner, but using a subtraction of unitary vectors, the other pair of identities are obtained:

$$\cos\alpha - \cos\beta \equiv -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta \equiv 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

The addition and subtraction of tangents are obtained through the common denominator:

$$\operatorname{tg}\alpha \pm \operatorname{tg}\beta \equiv \frac{\sin\alpha}{\cos\alpha} \pm \frac{\sin\beta}{\cos\beta} \equiv \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta} \equiv \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$$

Product of vectors and addition identities

Let us see at the figure 6.4, but now we calculate the product of both vectors:

$$v\,u = (e_1\cos\beta + e_2\sin\beta)(e_1\cos\alpha + e_2\sin\alpha) =$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta + e_{12}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

Since u and v are unitary vectors, their product is a complex number with unitary modulus and argument equal to the oriented angle between them:

$$v\,u = \cos(\alpha-\beta) + e_{12}\sin(\alpha-\beta)$$

The identification of both equations gives the trigonometric functions of the angles difference:

$$\sin(\alpha - \beta) \equiv \sin\alpha \; \cos\beta - \cos\alpha \, \sin\beta$$

$$\cos(\alpha - \beta) \equiv \cos\alpha \cos\beta + \sin\alpha \, \sin\beta$$

Taking into account that the sine and the cosine are odd and even functions respectively, one obtains the identities for the angles addition:

$$\sin(\alpha + \beta) \equiv \sin\alpha \; \cos\beta + \cos\alpha \, \sin\beta$$

$$\cos(\alpha + \beta) \equiv \cos\alpha \cos\beta - \sin\alpha \, \sin\beta$$

Rotations and De Moivre's identity

If v' is the result of turning the vector v over an angle α then:

$$v' = v\left(\cos\alpha + e_{12} \sin\alpha\right)$$

To repeat a rotation of angle α by n times is the same thing as to turn over an angle $n\alpha$:

$$v'' = v\left(\cos\alpha + e_{12} \sin\alpha\right)^n = v\left(\cos n\alpha + e_{12} \sin n\alpha\right)$$

$$\Rightarrow \qquad \cos n\alpha + e_{12} \sin n\alpha \equiv \left(\cos\alpha + e_{12} \sin\alpha\right)^n$$

This is the De Moivre's identity[4], which allows to calculate the trigonometric functions of multiplies of a certain angle through the binomial theorem. For example, for $n = 3$ we have:

$$\cos 3\alpha + e_{12} \sin 3\alpha \equiv \left(\cos\alpha + e_{12} \sin\alpha\right)^3$$

$$\equiv \cos^3\alpha - 3\cos\alpha \, \sin^2\alpha + e_{12}\left(3\cos^2\alpha \sin\alpha - \sin^3\alpha\right)$$

Splitting the real and imaginary parts one obtains:

$$\cos 3\alpha \equiv \cos^3\alpha - 3\cos\alpha \, \sin^2\alpha \qquad\qquad \sin 3\alpha \equiv 3\cos^2\alpha \, \sin\alpha - \sin^3\alpha$$

And dividing both identities one arrives at:

$$\mathrm{tg}\, 3\alpha \equiv \frac{3\,\mathrm{tg}\,\alpha - \mathrm{tg}^3\alpha}{1 - 3\,\mathrm{tg}^2\alpha}$$

---

[4] With the Euler's identity, the De Moivre's identity is:

$$\exp(n\alpha \, e_{12}) \equiv \left[\exp(\alpha \, e_{12})\right]^n$$

Inverse trigonometric functions

The arcsine, arccosine and arctangent are defined as the inverse functions of the sine, cosine and tangent. They are multivalued functions and the principal values are taken in the following intervals:

$$y = \arcsin x \quad \Leftrightarrow \quad x = \sin y \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$y = \arccos x \quad \Leftrightarrow \quad x = \cos y \quad \text{and} \quad 0 \leq y \leq \pi$$

$$y = \operatorname{arctg} x \quad \Leftrightarrow \quad x = \operatorname{tg} y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = \operatorname{arccot} x \quad \Leftrightarrow \quad x = \cot y \quad \text{and} \quad 0 < y < \pi$$

From the definitions the parity of the inverse functions follow immediately:

$$\arcsin x \equiv -\arcsin(-x) \qquad \arccos x \equiv \arccos(-x)$$

$$\operatorname{arctg} x \equiv -\operatorname{arctg}(-x)$$

Through the fundamental identities for the circular functions one obtains the following identities[5] for the inverse functions:

$$\arcsin x \equiv \left[\arccos\sqrt{1-x^2}\right] \equiv \operatorname{arctg}\frac{x}{\sqrt{1-x^2}}$$

$$\arccos x \equiv \left[\arcsin\sqrt{1-x^2}\right] \equiv \left[\operatorname{arctg}\frac{\sqrt{1-x^2}}{x}\right]$$

$$\operatorname{arctg} x \equiv \arcsin\frac{x}{\sqrt{1+x^2}} \equiv \left[\arccos\frac{1}{\sqrt{1+x^2}}\right]$$

where the brackets indicate that the identity only holds for positive values.

From the complementary angles identities we have:

$$\arcsin x \equiv \frac{\pi}{2} - \arccos x \qquad \operatorname{arctg} x \equiv \frac{\pi}{2} - \operatorname{arc\,cot} x$$

---

[5] The only inverse circular function predefined in the language Basic is the arctangent ATN(X), so these identities allows us to program the arcsine and arccosine.

Exercises

6.1 Prove the law of sines, cosines and tangents: if a, b and c are the sides of a triangle respectively opposite to the angles α, β and γ, then:

$$\frac{|a|}{\sin\alpha} = \frac{|b|}{\sin\beta} = \frac{|c|}{\sin\gamma} \qquad \text{(law of sines)}$$

$$c^2 = a^2 + b^2 - 2\,|a|\,|b|\,\cos\gamma \qquad \text{(law of cosines)}$$

$$\frac{|a|+|b|}{|a|-|b|} = \frac{\operatorname{tg}\dfrac{\alpha+\beta}{2}}{\operatorname{tg}\dfrac{\alpha-\beta}{2}} \qquad \text{(law of tangents)}$$

Hint: use the inner and outer products of sides.

6.2 Prove the following trigonometric identities:

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-a) \equiv 4\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}-1$$

$$\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha) \equiv -4\sin\frac{\alpha-\beta}{2}\,\sin\frac{\beta-\gamma}{2}\,\sin\frac{\gamma-\alpha}{2}$$

6.3 Express the trigonometric functions of 4α as a polynomial of the trigonometric functions of α.

6.4 Let P be a point on a circle arc whose extremes are the points A and B. Prove that the sum of the chords AP and PB is maximal when P is the midpoint of the arc AB.

6.5 Prove the Mollweide's formulas for a triangle:

$$\frac{|a|+|b|}{|c|} = \frac{\cos\dfrac{\alpha-\beta}{2}}{\sin\dfrac{\gamma}{2}} \qquad\qquad \frac{|a|-|b|}{|c|} = \frac{\sin\dfrac{\alpha-\beta}{2}}{\cos\dfrac{\gamma}{2}}$$

6.6 Deduce also the projection formulas:

$$|a| = |b|\cos\gamma + |c|\cos\beta \qquad |b| = |c|\cos\alpha + |a|\cos\gamma \qquad |c| = |a|\cos\beta + |b|\cos\alpha$$

6.7 Prove the half angle identities:

$$\sin\frac{\alpha}{2} \equiv \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} \equiv \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \operatorname{tg}\frac{\alpha}{2} \equiv \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \equiv \frac{1-\cos\alpha}{\sin\alpha} \equiv \frac{\sin\alpha}{1+\cos\alpha}$$

7. SIMILARITIES AND SINGLE RATIO

Two geometric figures are *similar* if they have the same shape. If the orientation of both figures is the same, they are said to be *directly* similar. For example, the dials of clocks are directly similar. On the other hand, two figures can have the same shape but different orientation. Then they are said to be *oppositely* similar. For example our hands are oppositely similar. However these intuitive concepts are insufficient and the similarity must be defined with more precision.

Direct similarity (similitude)

If two vectors u and v on the plane form the same angle as the angle between the vectors w and t, and they have proportional lengths (figure 7.1), then they are geometrically proportional:

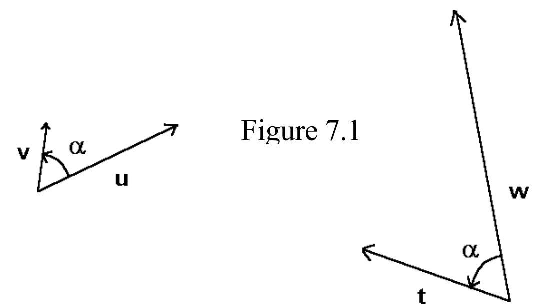


Figure 7.1

$$\left.\begin{array}{c}\alpha(u,v)=\alpha(w,t)\\ \dfrac{|u|}{|v|}=\dfrac{|w|}{|t|}\end{array}\right\} \Rightarrow \quad u\,v^{-1}=w\,t^{-1}$$

that is, the geometric quotient of u and v is equal to the quotient of w and t, which is a complex number. This definition of a geometric quotient is also valid for vectors in the space provided that the four vectors lie on the same plane. In this case, the geometric quotient is a quaternion, as Hamilton showed.

The geometric proportionality for vectors allows to define the similarity of triangles. Two triangles ABC and $A'B'C'$ are said to be *directly similar* and their vertices and sides denoted with the same letters are *homologous* if:

$$AB\,BC^{-1}=A'B'\,B'C'^{\,-1}$$

that is, if the geometric quotient of two sides of the first triangle is equal to the quotient of the homologous sides of the second triangle. Arranging the vectors of this equation one obtains the equality of the quotients of the homologous sides:

$$AB^{-1}\,A'B'=BC^{-1}\,B'C'$$

One can prove easily that the third quotient of homologous sides also coincides with the other quotients:

$$AB^{-1}\,A'B'=BC^{-1}\,B'C'=CA^{-1}\,C'A'=r$$

The *similarity ratio* r is defined as the quotient of every pair of homologous sides, which is a complex constant. The modulus of the similarity ratio is the size ratio and the argument is the angle of rotation of the triangle $A'B'C'$ with respect to the triangle ABC.

$$r = \frac{|A'B'|}{|AB|} \exp\left[\alpha(AB, A'B')e_{12}\right]$$

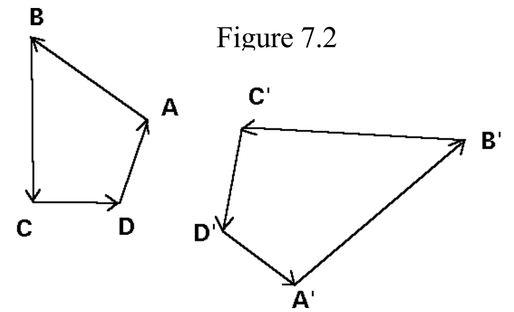


Figure 7.2

The definition of similarity is generalised to any pair of polygons in the following way. Let the polygons *ABC...Z* and *A'B'C'...Z'* be. They are said to be directly similar with similarity ratio *r* and the sides denoted with the same letters to be homologous if:

$$r = AB^{-1}\, A'B' = BC^{-1}\, B'C' = CD^{-1}\, C'D' = \ldots = YZ^{-1}\, Y'Z' = ZA^{-1}\, Z'A'$$

One of these equalities depends on the others and we do not need to know whether it is fulfilled. Here also, the modulus of *r* is the size ratio of both polygons and the argument is the angle of rotation. The fact that the homologous exterior and interior angles are equal for directly similar polygons (figure 7.2) is trivial because:

$$B'A'\, B'C'^{\,-1} = BA\, BC^{-1} \qquad \Rightarrow \qquad \text{angle } A'B'C' = \text{angle } ABC$$

$$C'B'\, C'D'^{\,-1} = CB\, CD^{-1} \qquad \Rightarrow \qquad \text{angle } B'C'D' = \text{angle } BCD \qquad \text{etc.}$$

The direct similarity is an equivalence relation since it has the reflexive, symmetric and transitive properties. This means that there are classes of equivalence with directly similar figures.

A similitude with $|r|=1$ is called a *displacement*, since both polygons have the same size and orientation.

Opposite similarity

Two triangles *ABC* and *A'B'C'* are *oppositely similar* and the sides denoted with the same letters are homologous if:

$$AB\, BC^{-1} = (\, A'B'\, B'C'^{\,-1}\,)^* = B'C'^{\,-1}\, A'B'$$

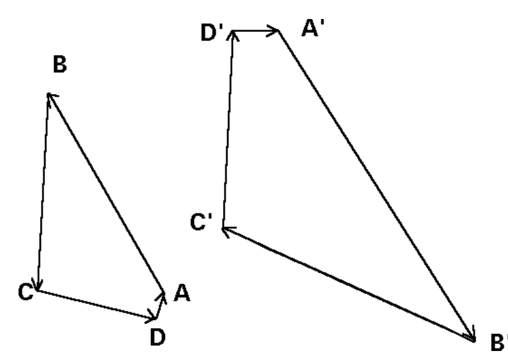


Figure 7.3

The former equality cannot be arranged in a quotient of a pair of homologous sides as we have made before. Because of this, the similarity ratio cannot be defined for the opposite similarity but only the size ratio, which is the quotient of the lengths of any two homologous sides. An opposite similarity is always the composition of a reflection in any line and a direct similarity.

$$AB\,BC^{-1} = v^{-1}\,A'B'\,B'C'^{-1}\,v \quad\Leftrightarrow\quad BC^{-1}\,v^{-1}\,B'C' = AB^{-1}\,v^{-1}\,A'B' \quad\Leftrightarrow$$

$$BC^{-1}\,(\,v^{-1}\,B'C'\,v\,) = AB^{-1}\,(\,v^{-1}\,A'B'^{-1}\,v\,) = r$$

where r is the ratio of a direct similarity whose argument is not defined but depends on the direction vector v of the reflection axis. Notwithstanding, this expression allows to define the opposite similarity of two polygons. So two polygons $ABC...Z$ and $A'B'C'...Z'$ are oppositely similar and the sides denoted with the same letters are homologous if for any vector v the following equalities are fulfilled:

$$AB^{-1}\,(\,v^{-1}\,A'B'^{-1}\,v\,) = BC^{-1}\,(\,v^{-1}\,B'C'\,v\,) = ... = ZA^{-1}\,(\,v^{-1}\,Z'A'\,v\,)$$

that is, if after a reflection one polygon is directly similar to the other. The opposite similarity is not reflexive nor transitive: if a figure is oppositely similar to another, and this is oppositely similar to a third figure, then the first and third figures are directly similar. Then there are not classes of oppositely similar figures.

An opposite similarity with $|r|=1$ is called a *reversal*, since both polygons have the same size both opposite orientations.

The theorem of Menelaus

For every triangle ABC (figure 7.4), three points D, E and F lying respectively on the sides BC, CA and AB or their prolongations are aligned if and only if:

$$AF\,FB^{-1}\,BD\,DC^{-1}\,CE\,EA^{-1} = -1$$

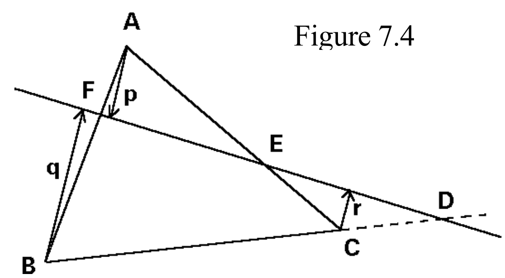


Figure 7.4

Proof $\Rightarrow$ Let us suppose that D, E and F are aligned on a crossing straight line. Let us denote by p, q and r the vectors with origin at the vertices A, B and C and going perpendicularly to the crossing line. Then every pair of right angle triangles having the hypotenuses on a common side of the triangle ABC are similar so that we have:

$$BF^{-1}\,AF = q^{-1}\,p \qquad\qquad CD^{-1}\,BD = r^{-1}\,q \qquad\qquad AE^{-1}\,CE = p^{-1}\,r$$

Multiplying the three equalities one obtains:

$$AE^{-1}\,CE\,CD^{-1}\,BD\,BF^{-1}\,AF = 1$$

Taking the complex conjugate expression and changing the sign of BF, CD and AE, the theorem is proved in this direction:

$$AF\,FB^{-1}\,BD\,DC^{-1}\,CE\,EA^{-1} = -1$$

Proof $\Leftarrow$ Being F a point on the line AB, D a point on the line BC and E a point on the line CA we have:

$$F = a\,A + (1-a)\,B \qquad D = b\,B + (1-b)\,C \qquad E = c\,C + (1-c)\,A$$

with a, b and c real. Then:

$$AF = (1-a)\,AB \qquad FB = a\,AB \qquad BD = (1-b)\,BC$$

$$DC = b\,BC \qquad CE = (1-c)\,CA \qquad EA = c\,CA$$

That the product of these segments is equal to -1 implies that:

$$\frac{(1-a)(1-b)(1-c)}{a\,b\,c} = -1 \qquad \Rightarrow \qquad c = \frac{(1-a)(1-b)}{1-a-b}$$

The substitution of c in the expression for E gives:

$$E = \frac{(1-a)(1-b)}{1-a-b}C - \frac{a\,b}{1-a-b}A$$

$$= \frac{1-a}{1-a-b}[(1-b)C + b\,B] - \frac{b}{1-a-b}[a\,A + (1-a)B]$$

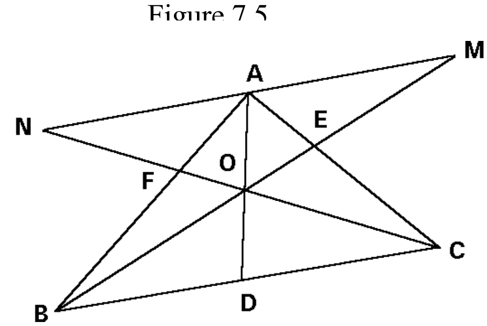
$$= \frac{1-a}{1-a-b}D - \frac{b}{1-a-b}F$$

That is, the point E belongs to the line FD, in other words, the point D, E and F are aligned, which is the prove:

$$E = d\,D + (\,1-d\,)\,F \qquad \text{with } d = \frac{1-a}{1-a-b}$$

The theorem of Ceva

Figure 7.5



Given a generic triangle ABC, we draw three segments AD, BE and CF from each vertex to a point of the opposite side (figure 7.5). The three segments meet in a unique point if and only if:

$$BD\,DC^{-1}\,CE\,EA^{-1}\,AF\,FB^{-1} = 1$$

In order to prove the theorem, we

enlarge the segments *BE* and *CF* to touch the line which is parallel to the side *BC* and passes through *A*. Let us denote the intersection points by *M* and *N* respectively. Then the triangle *BDO* is directly similar to the triangle *MAO*, and the triangle *CDO* is also directly similar to the triangle *NAO*. Therefore:

$$BD\,DO^{-1} = MA\,AO^{-1}$$

$$DO\,DC^{-1} = AO\,AN^{-1}$$

The multiplication of both equalities gives:

$$BD\,DC^{-1} = MA\,AN^{-1}$$

Analogously, the triangles *MEA* and *CEB* are similar as soon as *NFA* and *BFC*. Hence:

$$CE\,EA^{-1} = BC\,AM^{-1} \qquad AF\,FB^{-1} = AN\,BC^{-1}$$

The product of the three equalities yields:

$$BD\,DC^{-1}\,CE\,EA^{-1}\,AF\,FB^{-1} = MA\,AN^{-1}\,BC\,AM^{-1}\,AN\,BC^{-1} = 1$$

The sufficiency of this condition is proved in the following way: let *O* be the point of intersection of *BE* and *CF*, and *D'* the point of intersection of the line *AO* with the side *BC*. Let us suppose that the former equality is fulfilled. Then:

$$BD'\,D'C^{-1} = BD\,DC^{-1}$$

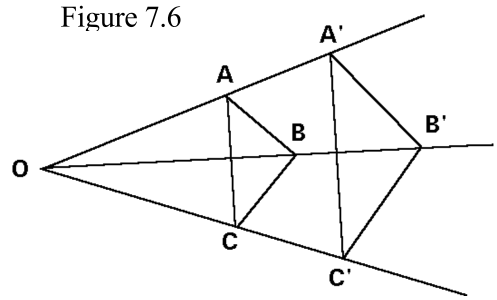
Since both *D* and *D'* lie on the line *BC*, it follows that $D = D'$.

Homothety and single ratio

A *homothety with centre O and ratio k* is the geometric transformation which dilates the distance from *O* to any point *A* in a factor *k*:

$$OA' = OA\,k$$

Figure 7.6



O is the unique invariant point of the homothety. If *k* is real number, the homothety is said to be *simple* (figure 7.6), otherwise is called *composite*. If *k* is a complex number, it may always be factorised in a product of the modulus and a unitary complex number:

$$k = |k|\,z \qquad \text{with } |z| = 1$$

The modulus of *k* is the ratio of a simple homothety with centre *O*, and *z* indicates an additional rotation with the same centre. Then, a composite homothety is equivalent to a simple homothety followed by a rotation. Direct similarities and homotheties are different names for the same transformations (exercise 7.5). Also homotheties with $|k|=1$ and displacements are equivalent.

The *single ratio of three points A*, *B*, and *C* is defined as:

$$(A\;\;B\;\;C) = AB\,AC^{-1}$$

The single ratio is a real number when the three points are aligned, and a complex number in the other case. Under a homothety, the single ratio of any three points remains invariant. Let us prove this. Because any vector is dilated and rotated by a factor k, we have:

$$A'B' = OB' - OA' = OB\,k - OA\,k = AB\,k$$

$$A'C' = AC\;\;k \qquad \Rightarrow \qquad A'C'^{\,-1} = k^{-1}\,AC^{-1}$$

$$(A'\;\;B'\;\;C') = AB\,k\,k^{-1}\,AC^{-1} = AB\,AC^{-1} = (A\;\;B\;\;C)$$

If the single ratio is invariant for a geometric transformation, then it transforms triangles into directly similar triangles and hence also polygons into similar polygons:

$$(A\;\;B\;\;C) = (A'\;\;B'\;\;C') \qquad \Rightarrow \quad AB\,AC^{-1} = A'B'\,A'C'^{\,-1} \quad \Rightarrow AC^{-1}\,A'C' = AB^{-1}\,A'B'$$

Therefore, the homothety always transforms triangles into directly similar triangles as shown in the figure 7.6. It follows immediately that the homothety preserves the angles between lines. It is the simplest case of *conformal* transformations, the geometric transformations which preserve the angles between lines. Since a line may also be transformed into a curve, and for the general case a curve into a curve, the conformal transformations are those which preserve the angle between any pair of curves, that is, the angle between the tangent lines to both curves at the intersection point. A transformation is *directly* or *oppositely conformal* whether it preserves or changes the sense of the angle between two curves (figure 7.7). This condition is equivalent to the conservation of the single ratio of any three points at the limit of accumulation:

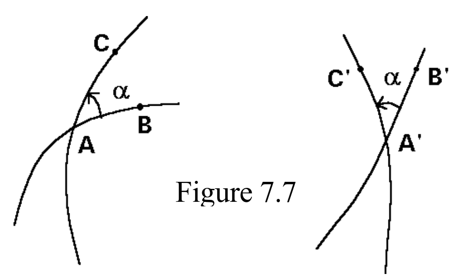


Figure 7.7

$$\lim_{B,C\to A} (A\,B\,C) = \lim_{B',C'\to A'} (A'\,B'\,C') \qquad \text{directly conformal}$$

$$\lim_{B,C\to A} (A\,B\,C) = \lim_{B',C'\to A'} (A'\,B'\,C')^{*} \qquad \text{oppositely conformal}$$

Exercises

7.1 Let T be the triangle with vertices (0, 0), (2, 0), (0, 1) and T' that with vertices (2,0), (5, 1) and (4, 2). Find which vertices are homologous and calculate the similarity ratio. Which is the size ratio? Which is the angle of rotation of the triangle T' with respect to T?

7.2 The altitude perpendicular to the hypotenuse divides a right triangle in two smaller right triangles. Show that they are similar and deduce the Pythagorean theorem.

7.3 Every triangle ABC with not vanishing area has a circumscribed circle. The line tangent to this circle at the point B cuts the line AC at the point M. Prove that:

$$MA\ MC^{-1} = AB^2\ BC^{-2}$$

7.4 Let ABC be an equilateral triangle inscribed inside a circle. If P is any point on the arc BC, show that $|PA| = |PB| + |PC|$.

7.5 Given two directly similar triangles ABC and $A'B'C'$, show that the centre O of the homothety that transform one triangle into another is equal to:

$$O = A - AA'\,(1 - AB^{-1}\,A'B')^{-1}$$

7.6 Draw any line passing through a fixed point P which cuts a given circle. Let Q and Q' be the intersection points of the line and the circle. Show that the product $PQ\ PQ'$ is constant for any line belonging to the pencil of lines of P.

7.7 Let a triangle have sides a, b and c. The bisector of the angle formed by the sides a and b divides the side c in two parts m and n. If m is adjacent to a and n to b respectively, prove that the following proportion is fulfilled:

$$\frac{|m|}{|a|} = \frac{|n|}{|b|}$$

8. PROPERTIES OF TRIANGLES

Area of a triangle

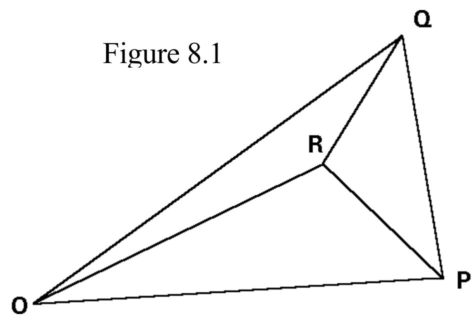
Since the area of a parallelogram is obtained as the outer product of two consecutive sides (taken as vectors, of course), the area of a triangle is the half of the outer product of any two of its three sides:

$$a_{PQR} = \frac{1}{2} PQ \wedge PR = \frac{1}{2} QR \wedge QP = \frac{1}{2} RP \wedge RQ$$

If the vertices P, Q and R are counterclockwise oriented, the area is a positive imaginary number. Otherwise, the area is a negative imaginary number. Note that the fundamental concept in geometry is the *oriented area*[1]. The modulus of the area may be useful in the current life but is insufficient for geometry. From now on I shall only regard oriented areas.

Writing the segments of the former equation as differences of points we arrive to:

Figure 8.1



$$a_{PQR} = \frac{1}{2} (P \wedge Q + Q \wedge R + R \wedge P)$$

which is a symmetric expression under cyclic permutation of the vertices. The position vector P goes from an arbitrary origin of coordinates to the point P. Then, $P \wedge Q$ is the double of the area of the triangle OPQ. Analogously $Q \wedge R$ is the double of the area of the triangle OQR and $R \wedge P$ is the double of the area of the triangle ORP. Therefore the former expression is equal to (figure 8.1):

$$a_{PQR} = a_{OPQ} + a_{OQR} + a_{ORP}$$

For the arrangement of points shown in the figure 8.1 the area of OPQ is positive, and the areas of OQR and ORP are negative, that is, the areas of ORQ and OPR are positive. Therefore, one would intuitively write, taking all the areas positive, that:

$$a_{PQR} = a_{OPQ} - a_{ORQ} - a_{OPR}$$

When one considers oriented areas, the equalities are wholly general and independent of the arrangement of the points[2].

---

[1] The integral of a function is also the oriented area of the region enclosed by the curve and the X-axis.

[2] About this topic, see A. M. Lopshitz, *Cálculo de las áreas de figuras orientadas*, Rubiños-1860 (1994).

Medians and centroid

The *medians* are the segments going from each vertex to the midpoint of the opposite side. Let us prove that the three medians meet in a unique point G called the *centroid* (figure 8.2). Since G is a point on the median passing through P and $(Q + R)/2$:

$$G = k\,P + (1-k)\frac{Q+R}{2} \qquad\qquad k \text{ real}$$

G lies also on the median passing through Q and $(P + R) / 2$:

$$G = m\,Q + (1-m)\frac{P+R}{2} \qquad\qquad m \text{ real}$$

Equating both expressions we find:

$$k\,P + (1-k)\frac{Q+R}{2} = m\,Q + (1-m)\frac{P+R}{2}$$

$$P\left(k - \frac{1}{2} + \frac{m}{2}\right) + Q\left(\frac{1}{2} - \frac{k}{2} - m\right) + R\left(-\frac{k}{2} + \frac{m}{2}\right) = 0$$

A linear combination of independent points can vanish only if every coefficients are null, a condition which leads to the following system of equations:

$$\begin{cases} k - \dfrac{1}{2} + \dfrac{m}{2} = 0 \\ \dfrac{1}{2} - \dfrac{k}{2} - m = 0 \\ -\dfrac{k}{2} + \dfrac{m}{2} = 0 \end{cases}$$

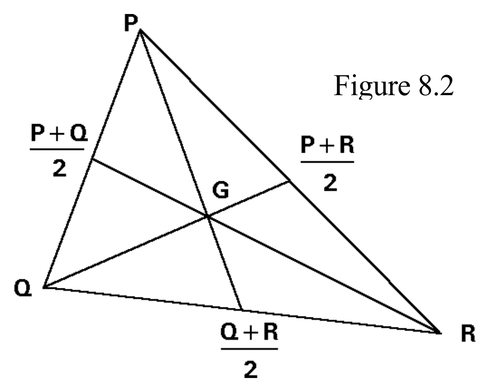


Figure 8.2

The solution of this system of equations is k =1/3 and m =1/3, indicating that the intersection of both medians are located at 1/3 distance from the midpoints. The substitution into the expression of G gives:

$$G = \frac{P+Q+R}{3}$$

This expression for the centroid is symmetric under permutation of the vertices. Therefore the three medians meet at the same point G, the centroid[3].

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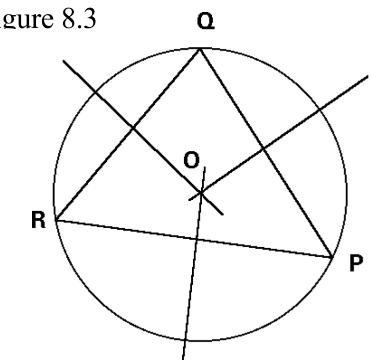
[3] The medians are a special case of *cevian lines* (lines passing through a vertex and not coinciding with the sides) and this statement is also proved by means of Ceva's theorem.

Perpendicular bisectors and circumcentre

The three perpendicular bisectors of the sides of a triangle meet in a unique point called the *circumcentre,* the centre of the circumscribed circle. Every point on the perpendicular bisector of PQ is equidistant from P and Q. Analogously every point on the perpendicular bisector of PR is equidistant from P and R. The intersection O of both perpendicular bisectors is simultaneously equidistant from P, Q and R. Therefore O also belongs to the perpendicular bisector of QR and the three bisectors meet at a unique point. Since O is equally distant from the three vertices, it is the centre of the circumscribed circle. Let us use this condition in order to calculate the equation of the circumcentre:

Figure 8.3



$$OP^2 = OQ^2 = OR^2 = d^2$$

where d is the radius of the circumscribed circle. Using the position vectors of each point we have:

$$(P - O)^2 = (Q - O)^2 = (R - O)^2$$

The first equality yields:

$$P^2 - 2\, P \cdot O + O^2 = Q^2 - 2\, Q \cdot O + O^2$$

By simplifying and arranging the terms containing O at the left hand side, we have:

$$2\,(Q - P) \cdot O = Q^2 - P^2$$

$$2\, PQ \cdot O = Q^2 - P^2$$

From the second equality we find an analogous result:

$$2\, QR \cdot O = R^2 - Q^2$$

Now we introduce the geometric product instead of the inner product in these equations:

$$PQ\;\, O + O\, PQ = Q^2 - P^2$$

$$QR\, O + O\, QR = R^2 - Q^2$$

By subtraction of the second equation multiplied on the right by PQ minus the first equation multiplied on the left by QR, we obtain:

$$PQ\;\, QR\;\, O - O\, PQ\;\, QR = PQ\;\, R^2 - PQ\;\, Q^2 - Q^2\;\, QR + P^2\;\, QR$$

By using the permutative property on the left hand side and simplifying the right hand side, we have:

$$PQ\, QR\, O - QR\, PQ\, O = P^2\, QR + Q^2\, RP + R^2\, PQ$$

$$2\,(\, PQ \wedge QR\,)\, O = P^2\, QR + Q^2\, RP + R^2\, PQ$$

Finally, the multiplication by the inverse of the outer product on the left gives:

$$O = (\, 2\, PQ \wedge QR\,)^{-1}\,(\, P^2\, QR + Q^2\, RP + R^2\, PQ\,)$$

$$= -\,(\, P^2\, QR + Q^2\, RP + R^2\, PQ\,)\,(\, 2\, PQ \wedge QR\,)^{-1}$$

a formula able to calculate the coordinates of the circumcentre. For example, let us calculate the centre of the circle passing through the points:

$$P = (\,2, 2\,) \qquad Q = (\,3, 1\,) \qquad R = (\,4, -2\,)$$

$$P^2 = 8 \qquad Q^2 = 10 \qquad R^2 = 20$$

$$QR = R - Q = e_1 - 3\, e_2 \qquad RP = P - R = -\,2\, e_1 + 4\, e_2 \qquad PQ = Q - P = e_1 - e_2$$

$$2\, PQ \wedge QR = -\,4\, e_{12}$$

$$O = -\,(\, 8\,(\, e_1 - 3\, e_2\,) + 10\,(\, -2\, e_1 + 4\, e_2\,) + 20\,(\, e_1 - e_2\,)\,)\,\frac{e_{12}}{4}$$

$$= -\, e_1 - 2\, e_2 = (\, -1, -2\,)$$

In order to deduce the radius of the circle, we take the vector OP:

$$OP = P - O = P + (\, P^2\, QR + Q^2\, RP + R^2\, PQ\,)\,(\, 2\, PQ \wedge QR\,)^{-1}$$

and extract the inverse of the area as a common factor:

$$OP = (\, 2\, P\;\; PQ \wedge QR + P^2\, QR + Q^2\, RP + R^2\, PQ\,)\,(\, 2\, PQ \wedge QR\,)^{-1} =$$

$$= [\, 2\, P\,(\, P \wedge Q + Q \wedge R + R \wedge P\,) + P^2\, QR + Q^2\, RP + R^2\, PQ\,]\,(\, 2\, PQ \wedge QR\,)^{-1} =$$

$$= [\, P\,(\, P\, Q - Q\, P + Q\, R - R\, Q + R\, P - P\, R\,) + P^2\,(\, R - Q\,) + Q^2\,(\, P - R\,) +$$

$$+ R^2\,(\, Q - P\,)\,]\;\;(\, 2\, PQ \wedge QR\,)^{-1}$$

The simplification gives:

$$OP = (\, P\, Q\, R - P\, R\, Q + P\, R\, P - P\, Q\, P + Q^2\, P - Q^2\, R + R^2\, Q - R^2\, P\,)\,(\, 2\, PQ \wedge QR\,)^{-1}$$

$$= -\,(Q - P)\,(R - Q)\,(P - R)\,(\, 2\, PQ \wedge QR\,)^{-1} = -\, PQ\;\; QR\;\; RP\,(\, 2\, PQ \wedge QR\,)^{-1}$$

Analogously:

$$OQ = -\,QR\;\; RP\;\; PQ\,(\,2\,PQ \wedge QR\,)^{-1} \qquad OR = -\,RP\;\; PQ\;\; QR\,(\,2\,PQ \wedge QR\,)^{-1}$$

The radius of the circumscribed circle is the length of any of these vectors:

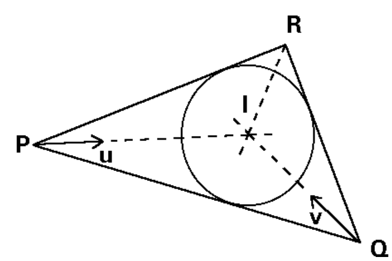
$$|\,OP\,| = \frac{|\,PQ\,|\,|\,QR\,|\,|\,RP\,|}{2\,|\,PQ \wedge QR\,|} = \frac{|\,PQ\,|}{2\sin QRP} = \frac{|\,QR\,|}{2\sin RPQ} = \frac{|\,RP\,|}{2\sin PQR}$$

where we find the law of sines.

Angle bisectors and incentre

The three bisector lines of the angles of a triangle meet in a unique point called the *incentre*. Every point on the bisector of the angle with vertex P is equidistant from the sides PQ and PR (figure 8.4). Also every point on the angle bisector of Q is equidistant from the sides QR and QP. Hence its intersection I is simultaneously equidistant from the three sides, that is, I is unique and is the centre of the circle inscribed into the triangle.

Figure 8.4



In order to calculate the equation of the angle bisector passing through P, we take the sum of the unitary vectors of both adjacent sides:

$$u = \frac{PQ}{|\,PQ\,|} + \frac{PR}{|\,PR\,|} \qquad\qquad v = \frac{QP}{|\,QP\,|} + \frac{QR}{|\,QR\,|}$$

The incentre I is the intersection of the angle bisector passing through P, whose direction vector is u, and that passing through Q, with direction vector v:

$$I = P + k\,u = Q + m\,v \qquad\qquad k,\ m \text{ real}$$

Arranging terms we find PQ as a linear combination of u and v:

$$k\,u - m\,v = Q - P = PQ$$

The coefficient k is:

$$k = \frac{PQ \wedge v}{u \wedge v} = \frac{|\,PQ\,|\,|\,RP\,|\,PQ \wedge QR}{PQ \wedge QR\,|\,RP\,| + QR \wedge RP\,|\,PQ\,| + RP \wedge PQ\,|\,QR\,|}$$

Since all outer products are equal because they are the double of triangle area, this expression is simplified:

$$k = \frac{|PQ|\,|RP|}{|RP| + |PQ| + |QR|}$$

Then, the centre of the circumscribed circle is:

$$I = P + k\,u = P + \frac{|PQ|\,|RP|}{|PQ| + |QR| + |RP|}\left(\frac{PQ}{|PQ|} + \frac{PR}{|PR|}\right)$$

By taking common denominator and simplifying, we arrive at:

$$I = \frac{P|QR| + Q|RP| + R|PQ|}{|QR| + |RP| + |PQ|}$$

For example, let us calculate the centre of the circle inscribed inside the triangle with vertices:

$P = (0, 0)$ $\quad Q = (0, 3)$ $\quad R = (4, 0)$

$|PQ| = 3$ $\quad |QR| = 5$ $\quad |RP| = 4$

$$I = \frac{5(0,0) + 4(0,3) + 3(4,0)}{5 + 4 + 3} = \frac{(12,12)}{12} = (1,1)$$

In order to find the radius, firstly we must obtain the segment *IP*:

$$IP = \frac{QP|RP| + RP|PQ|}{|QR| + |RP| + |PQ|}$$

The radius of the inscribed circle is the distance from *I* to the side *PQ*:

$$d(I, PQ) = \frac{|IP \wedge PQ|}{|PQ|} = \frac{|RP \wedge PQ|}{|PQ| + |QR| + |RP|}$$

whence the ratio of radius follows:

$$\frac{\text{radius of circumscribed circle}}{\text{radius of inscribed circle}} = \frac{1}{2}\,\frac{|PQ|\,|QR|\,|RP|}{|PQ| + |QR| + |RP|}$$

Altitudes and orthocentre

The *altitude* of a side is the segment perpendicular to this side (also called *base*) which passes through the opposite vertex. The three altitudes of a triangle intersect on a unique point called the *orthocentre*. Let us prove this statement calculating the intersection H of two altitudes. Since H belongs to the altitude passing through the vertex P and perpendicular to the base QR (figure 8.5), its equation is:

$$H = P + z\, QR \qquad z \text{ imaginary}$$

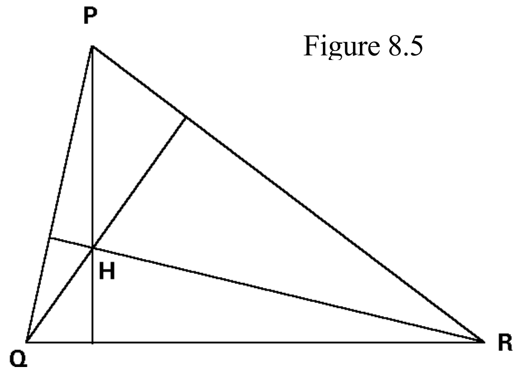


Figure 8.5

because the product by an imaginary number turns the vector QR over $\pi/2$, that is, $z\, PQ$ has the direction of the altitude. H also belongs to the altitude passing through Q and perpendicular to the base RP. Then its equation is:

$$H = Q + t\, RP \qquad t \text{ imaginary}$$

By equating both expressions:

$$P + z\, QR = Q + t\, RP$$

we arrive at a vector written as a linear combination of two vectors but with imaginary coefficients:

$$z\, QR - t\, RP = PQ$$

In this equation we must resolve PQ into components with directions perpendicular to the vectors QR and RP. The algebraic resolution follows the same way as for the case of real linear combination. Let us multiply on the right by RP:

$$z\, QR\, RP - t\, RP^2 = PQ\, RP$$

and on the left:

$$RP\, z\, QR - RP\, t\, RP = RP\, PQ$$

The imaginary numbers anticommute with vectors:

$$-z\, RP\, QR + t\, RP^2 = RP\, PQ$$

By adding both equalities we arrive at:

$$z\, (\, QR\, RP - RP\, QR\,) = PQ\, RP + RP\, PQ$$

$$z = \frac{PQ\, RP + RP\, PQ}{QR\, RP - RP\, QR} = \frac{PQ \cdot RP}{QR \wedge RP}$$

Analogously one finds t:

$$t = \frac{PQ \cdot QR}{QR \wedge RP}$$

Both expressions differ from the coefficients of real linear combination in the fact that the numerator is an inner product. The substitution in any of the first equations gives:

$$H = P + z\, QR = P + (\, PQ\, RP + RP\, PQ\,)\,(\, QR\, RP - RP\, QR\,)^{-1}\, QR =$$

The vector QR anticommutes with the outer product, an imaginary number:

$$H = P - (\, PQ\, RP + RP\, PQ\,)\, QR\,(\, QR\, RP - RP\, QR\,)^{-1} =$$

Extracting the area as common factor, we obtain:

$$H = [\, P\,(\, QR\, RP - RP\, QR\,) - (\, PQ\, RP + RP\, PQ\,)\, QR\,]\,(\, QR\, RP - RP\, QR\,)^{-1}$$

By using the fact that $PQ = Q - P$, etc., we arrive at:

$$H = (\, P\, QR\, RP - Q\, RP\, QR - P\, PQ\, QR + R\, PQ\, QR\,)\,(\, QR\, RP - RP\, QR\,)^{-1}$$

$$= (\, P^2\, QR + Q^2\, RP + R^2\, PQ + P\, QR\, P + Q\, RP\, Q + R\, PQ\, R\,)\,(\, QR\, RP - RP\, QR\,)^{-1}$$

$$= (\, P\;\; P \cdot QR + Q\;\; Q \cdot RP + R\;\; R \cdot PQ\,)\,(\, QR \wedge RP\,)^{-1}$$

This formula is invariant under cyclic permutation of the vertices. Therefore all the altitudes intersect on a unique point, the orthocentre.

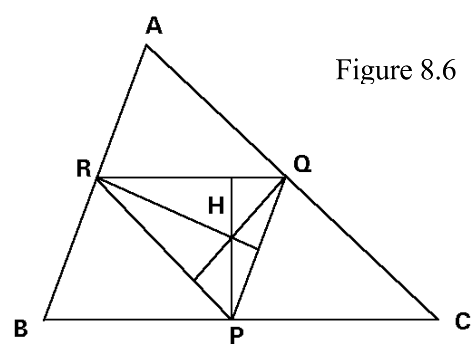


Figure 8.6

The equation of the orthocentre resembles that of the circumcentre. In order to see the relationship between both, let us draw a line passing through P and parallel to the opposite side QR, another one passing through Q and parallel to RP, and a third line passing through R and parallel to PQ (figure 8.6). Let A be the intersection of the line passing through Q and that passing through R, B be the intersection of the lines passing through R and P respectively, and C be the intersection of the lines passing through P and Q respectively. The triangle ABC is directly similar to the triangle PQR with ratio -2:

$$PQ = -\frac{1}{2}AB \qquad QR = -\frac{1}{2}BC \qquad RP = -\frac{1}{2}CA$$

and P, Q and R are the midpoints of the sides of the triangle ABC:

$$P = \frac{B + C}{2} \qquad Q = \frac{C + A}{2} \qquad R = \frac{A + B}{2}$$

Therefore the altitudes of the triangle PQR are the perpendicular bisectors of the sides of the triangle $ABC,$ and the orthocentre of the triangle PQR is the circumcentre of

ABC. In order to prove with algebra this obvious geometric fact, we must only deduce from the former relations the following equalities and substitute them into the orthocentre equation:

$$P\ P\cdot QR = \frac{B^3 - B\ C^2 + C\ B^2 - C^3}{8}$$

$$Q\ Q\cdot RP = \frac{C^3 - C\ A^2 + A\ C^2 - A^3}{8}$$

$$R\ R\cdot PQ = \frac{A^3 - A\ B^2 + B\ A^2 - B^3}{8}$$

By adding the three terms, the cubic powers vanish. On the other hand, the area of the triangle *PQR* is four times smaller than the area of the triangle *ABC*:

$$QR \wedge RP = \frac{BC \wedge CA}{4}$$

$$H = (\,-B\,C^2 + C\,B^2 - C\,A^2 + A\,C^2 - A\,B^2 + B\,A^2\,)\,(\,2\,BC \wedge CA\,)^{-1} =$$

$$= -(\,A^2\,BC + B^2\,CA + C^2\,AB\,)\,(\,2\,BC \wedge CA\,)^{-1}$$

This is just the equation of the circumcentre of the triangle *ABC*.

Euler's line

The centroid *G*, the circumcentre *O* and the orthocentre *H* of a triangle are always aligned on the *Euler's line*. To prove this fact, observe that the circumcentre and orthocentre equations have the triangle area as a "denominator", while the centroid equation does not, but we can introduce it:

$$G = \frac{P+Q+R}{3} = (\,P+Q+R\,)\,QR \wedge RP\,(\,3\,QR \wedge RP\,)^{-1}$$

Introducing the equality:

$$QR \wedge RP = P \wedge Q + Q \wedge R + R \wedge P = \frac{P\ Q - Q\ P + Q\ R - R\ Q + R\ P - P\ R}{2}$$

the centroid becomes:

$$G = (\,P+Q+R\,)\,(\,P\,Q - Q\,P + Q\,R - R\,Q + R\,P - P\,R\,)\,(\,6\,QR \wedge RP\,)^{-1}$$

$$= (\,P\,P\,Q - P\,Q\,P + P\,R\,P - P\,P\,R + Q\,P\,Q - Q\,Q\,P + Q\,Q\,R - Q\,R\,Q +$$

$$R\,Q\,R - R\,R\,Q + R\,R\,P - R\,P\,R\,)\,(\,6\,QR \wedge RP\,)^{-1}$$

$$= (\,-P^2\,QR + P\,QR\,P - Q^2\,RP + Q\,RP\,Q - R^2\,PQ + R\,PQ\,R\,)\,(\,6\,QR \wedge RP\,)^{-1}$$

from where the relation between the three points G, H and O follows immediately:

$$G = \frac{H + 2\ O}{3}$$

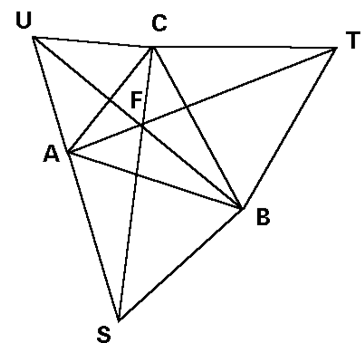
Hence the centroid is located between the orthocentre and the circumcentre, and its distance from the orthocentre is double of its distance from the circumcentre.

The Fermat's theorem

The geometric algebra allows to prove through a very easy and intuitive way the Fermat's theorem.

Over every side of a triangle ABC we draw an equilateral triangle (figure 8.7). Let T, U and S be the vertices of the equilateral triangles respectively opposite to A, B and C. Then the segments AT, BU and CS have the same length, form angles of $2\pi/3$ and intersect on a unique point F, called the *Fermat's point*. Moreover, the addition of the three distances from any point P to each vertex is minimal when P is the Fermat's point, provided that any of the interior angles of the triangle ABC be higher than $2\pi/3$.

Figure 8.7



Firstly, we must demonstrate that BU is obtained from AT by means of a rotation of $2\pi/3$, which will be represented by the complex number t:

$$t = \cos\frac{2\pi}{3} + e_{12}\sin\frac{2\pi}{3}$$

$$AT\,t = (AC + CT)\,t = AC\,t + CT\,t$$

By construction, the vector AC turned over $2\pi/3$ is the vector CU, and CT turned over $2\pi/3$ is BC, so:

$$AT\,t = CU + BC = BU$$

Analogously, one finds $CB = BU\,t$ and $AT = CS\,t$. That is, the vectors CS, BU and AT have the same length and each one is obtained from each other by successive rotations of $2\pi/3$.

Let us see that the sum of the distances from P to the three vertices A, B and C is minimal when P is the Fermat's point. We must prove firstly that the vectorial sum of PA turned $4\pi/3$, PB turned $2\pi/3$ and PC is constant independently of the point P. (figure 8.8). That is, for any two points P and P' it is always true that:

$$PA\,t^2 + PB\,t + PC = P'A\,t^2 + P'B\,t + P'C$$

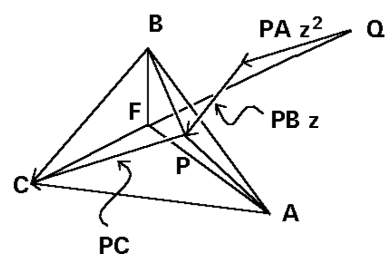
fact which is proved by arranging all the terms in one side of the equation:

$$PP'\,(\,t^2 + t + 1\,) = 0$$

This product is always zero since $t^2 + t + 1 = 0$. Hence, there is a unique point Q such that:

$$PA\,t^2 + PB\,t + PC = QC$$

Figure 8.8



For any point P, the three segments form a broken line as that shown in figure 8.8. Therefore, by the triangular inequality we have:

$$|PA| + |PB| + |PC| \geq |QC|$$

When P is the Fermat's point F, these segments form a straight line. Then, the addition of the distances from F to the three vertices is minimal provided that no angle of the triangle be higher than $2\pi/3$:

$$|FA| + |FB| + |FC| = |QC| \leq |PA| + |PB| + |PC|$$

In the other case, someone of the vectors $FA\,t^2$, $FB\,t$ or FC has a sense opposite to the others, so that its length is subtracted from the others and their sum is not minimal.

Exercises

8.1 The Napoleon's theorem. Over each side of a generic triangle draw an equilateral triangle. Prove that the centres of the three equilateral triangles also form an equilateral triangle.

8.2 Prove the Leibniz's theorem. Let P be any point on the plane and G the centroid of a triangle ABC. Then the following equality holds:

$$3\,PG^2 = PA^2 + PB^2 + PC^2 - \frac{1}{3}(\,AB^2 + BC^2 + CA^2\,)$$

8.3 Let A, B and C be three given points on the plane. Then any point G on the plane can be expressed as a linear combination of these three points (G is also considered as the centre of masses located at A, B and C with weights a, b and c [4]).

[4] See August Ferdinand Möbius, *Der Barycentrische Calcul*, Leipzig, 1827, p. 17 (facsimile edition of Georg Olms Verlag, Hildesheim,1976).

$G = a\,A + b\,B + c\,C \qquad\qquad \text{with} \quad a + b + c = 1$

Prove that:

a) a, b, c are the fractions of the area of the triangle ABC occupied by the triangles GBC, GCA and GAB respectively.

b) the geometric locus of the points P on the plane such that:

$$a\,PA^2 + b\,PB^2 + c\,PC^2 = k$$

is a circle with centre G (Apollonius' lost theorem).

8.4 Over every side of a convex quadrilateral $ABCD$, we draw the equilateral triangles ABP, BCQ, CDR and DAS. Prove that:

a) If $|AC| = |BD|$ then PR is perpendicular to QS.

b) If $|PR| = |QS|$ then AC is perpendicular to BD.

8.5 Show that the length of a median of a triangle ABC passing through A can be calculated from the sides as:

$$m_A^2 = \frac{AB^2 + AC^2}{2} - \frac{BC^2}{4}$$

8.6 In a triangle ABC we draw the bisectors of the angles A and B. Through the vertex C we draw the lines parallel to each angle bisector. Let us denote the intersections of each parallel line with the other bisector by D and E. Prove that if the line DE is parallel to the side AB then the triangle ABC is isosceles.

8.7 The bisection of a triangle (proposed by Cristóbal Sánchez Rubio). Let P and Q be two points on different sides of a triangle such that the segment PQ divides the triangle in two parts with the same area. Calculate the segment PQ in the following cases:

a) PQ is perpendicular to a given direction.

b) PQ has minimum length.

c) PQ passes through a given point inside the triangle.

9. CIRCLES

Algebraic and Cartesian equations

A circle with radius r and centre F is the locus of the points located at a distance r from F, so its equation is:

$$d(F, P) = |FP| = r \qquad \Leftrightarrow \qquad FP^2 = r^2$$

which can be written as:

$$(P - F)^2 = r^2 \quad \Rightarrow \qquad P^2 - 2\, P \cdot F + F^2 - r^2 = 0$$

By introducing the coordinates of $P = (x, y)$ and $F = (a, b)$ one obtains the analytic equation of the circle:

$$x^2 + y^2 - 2\, a\, x - 2\, b\, y + a^2 + b^2 - r^2 = 0$$

For example, the circle whose equation is:

$$x^2 + y^2 + 4\, x - 6\, y + 9 = 0$$

has the centre $F = (-2, 3)$ and radius 2.

There always exists a unique circle passing through any three non aligned points. In order to obtain this circle, one may substitute the coordinates of the points on the Cartesian equation of the circle, arriving at a system of three linear equations of three unknown quantities a, b and c:

$$\begin{cases} x_1^{\ 2} + y_1^{\ 2} + a\, x_1 + b\, y_1 + c = 0 \\ x_2^{\ 2} + y_2^{\ 2} + a\, x_2 + b\, y_2 + c = 0 \\ x_3^{\ 2} + y_3^{\ 2} + a\, x_3 + b\, y_3 + c = 0 \end{cases}$$

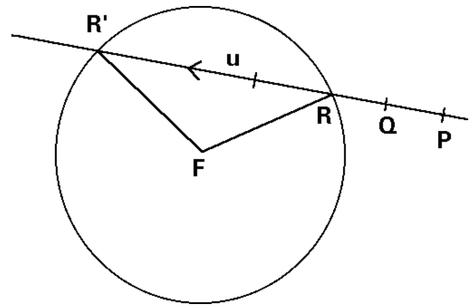
This is the classic way to find the equation of the circle, and from here the centre of the circle.

However, a more direct way is the calculation of the circumcentre of the triangle whose vertices are the three given points (see the previous chapter). Then the radius is the distance from the centre to any vertex.

Intersections of a line with a circle

The coordinates of the intersection of a line with a circle are usually calculated by substitution of the line equation into the circle equation, which leads to a second degree equation. Let us now follow an algebraic way. Let F be the centre of the circle with radius r, and R and R' the intersections of the line passing through two given points P and Q (figure 9.1):

Figure 9.1



$$\begin{cases} FR^2 = r^2 \\ R = P + k\, PQ \end{cases}$$

From the first equality we obtain:

$$(R - F)^2 = R^2 - 2\, R \cdot F + F^2 = r^2$$

The substitution of R given by the second equality yields:

$$(P + k\, PQ)^2 - 2\,(P + k\, PQ) \cdot F + F^2 = r^2$$

$$P^2 + 2\, k\, P \cdot PQ + k^2\, PQ^2 - 2\, P \cdot F - 2\, k\, PQ \cdot F + F^2 - r^2 = 0$$

By arranging the powers of k one obtains:

$$k^2\, PQ^2 + 2\, k\,(P \cdot PQ - F \cdot PQ) + P^2 - 2\, F \cdot P + F^2 - r^2 = 0$$

Taking into account that $PF = F - P$ one finds:

$$k^2\, PQ^2 - 2\, k\, PF \cdot PQ + PF^2 - r^2 = 0$$

This equation has solution whenever the discriminant be positive:

$$4\,(PF \cdot PQ)^2 - 4\, PQ^2\,(PF^2 - r^2) \geq 0$$

Introducing the identity $(PF \cdot PQ)^2 - (PF \wedge PQ)^2 = PQ^2\, PF^2$ the discriminant becomes:

$$4\,(PF \wedge PQ)^2 + 4\, PQ^2\, r^2 \geq 0$$

The solution of the second degree equation for k is:

$$k = PF \cdot PQ^{-1} \pm \sqrt{r^2\, PQ^{-2} + \left(PF \wedge PQ^{-1}\right)^2}$$

When $r = |PF \wedge PQ| / |PQ|$, the line is tangent to the circle. In this case the height of the parallelogram formed by the vectors PF and PQ is equal to the radius of the circle.

If u denote the unitary direction vector of the line PQ (figure 9.1):

$$u = \frac{PQ}{|PQ|}$$

then both intersection points are written as:

$$R = P + kPQ = P + u\left(PF \cdot u - \sqrt{r^2 + (PF \wedge u)^2} \right)$$

$$R' = P + k'PQ = P + u\left(PF \cdot u + \sqrt{r^2 + (PF \wedge u)^2} \right)$$

Power of a point with respect to a circle

Both intersection points R and R' have the following property: the product of the vectors going from a given point P to the intersections R and R' of any line passing through P with a given circle is constant:

$$PR\, PR' = (PF \cdot u)^2 - r^2 - (PF \wedge u)^2$$

$$= PF^2 - r^2 = PT^2$$

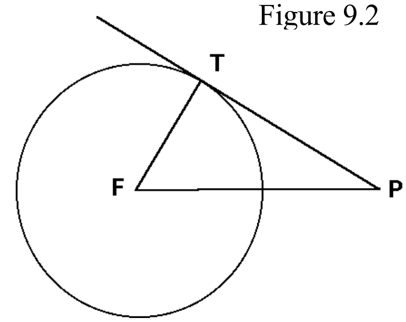


Figure 9.2

where T is the contact point of the tangent line passing through P (figure 9.2). The product $PR\, PR'$, which is independent of the direction u of the line and only depends on the point P, is called *the power of the point P with respect to the given circle*. The power of a point may be calculated through the substitution of its coordinates into the circle equation. If its centre is $F = (a, b)$ and $P = (x, y)$, then:

$$FP^2 - r^2 = (x - a)^2 + (y - b)^2 - r^2 = x^2 + y^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2 - r^2$$

The power is equal to zero when P belongs to the circle, positive when P lies outside the circle, and negative when P lies inside the circle.

Polar equation

We wish to describe the distance from a point P to the points R and R' on the circle as a function of the angle α between the line PR and the diameter (figure 9.3). The vectors PR and PR' are:

$$PR = u\left(PF \cdot u - \sqrt{r^2 - |\, PF \wedge u\,|^2} \right)$$

$$PR' = u\left(PF \cdot u + \sqrt{r^2 + |\, PF \wedge u\,|^2} \right)$$

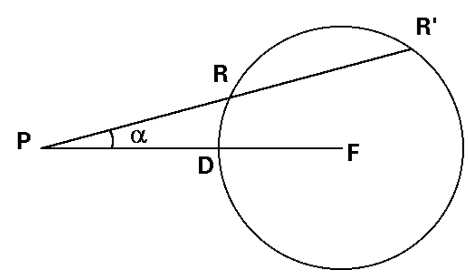


Figure 9.3

We arrive at the *polar equation* by taking the modulus of PR and PR', which is the distance to the circle:

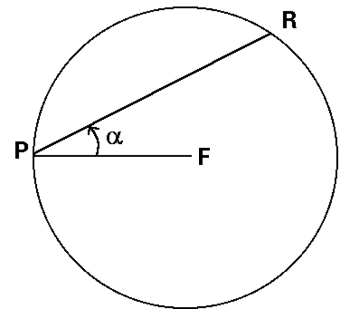
$$|PR| = |PF| \cos\alpha - \sqrt{r^2 - PF^2 \sin^2\alpha}$$

$$|PR'| = |PF| \cos\alpha + \sqrt{r^2 - PF^2 \sin^2\alpha}$$

If P is a point on the circle then $|PF| = r$ (figure 9.4) and the equation is simplified:

$$|PR| = 2\,r \cos\alpha$$

Figure 9.4



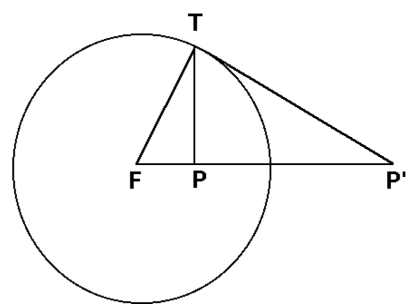
Inversion with respect to a circle

The point P' is the *inverse point* of P with respect to a circle of centre F and radius r if the vector FP' is the inverse vector of FP with radius k:

$$FP' = k^2\, FP^{-1} \qquad \Leftrightarrow \qquad FP\ \ FP' = k^2$$

Obviously, the product of the vectors FP and FP' is only real and positive when the points P and P' are aligned with the centre F and are located at the same side of F. The circle of centre F and radius k is called the *inversion circle* because its points remain invariant under this inversion. The inversion transforms points located inside the inversion circle into outside points and reciprocally.

Figure 9.5



To obtain geometrically the inverse of an inside point P, draw firstly (figure 9.5) the diameter passing through P; after this draw the perpendicular to this diameter from P which will cut the circle in the point T; finally draw the tangent with contact point T. The intersection P' of this tangent with the prolongation of the diameter is the inverse point of P. Note that the right triangles FPT and FTP' are oppositely similar:

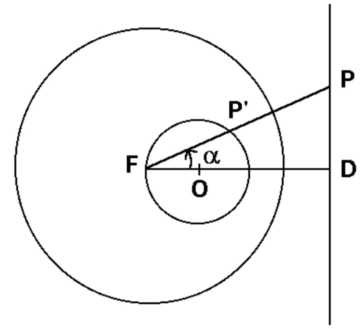
$$FP\,FT^{-1} = FP'^{-1}\,FT \;\Rightarrow\; FP'\,FP = FT^2 = k^2$$

To obtain the inverse of an outside point, make the same construction but in the opposite sense. Draw the tangent to the circle passing through the point (P' in the figure 9.5) and then draw the perpendicular to the diameter FP' passing through the contact point T of the tangent. The intersection P of the perpendicular with the diameter is the searched inverse point of P'.

Every line not containing the centre of inversion is transformed into a circle passing through the centre (figure 9.6) and reciprocally. To prove this statement let us take the polar equation of a straight line:

$$|FP| = \frac{d}{\cos\alpha}$$

Figure 9.6



Then the inverse of P with centre F has the equation:

$$|FP'| = \frac{k^2}{|FP|} = \frac{k^2\cos\alpha}{d}$$

which is the polar equation of a circle passing through F and with diameter k^2 / d . α is the angle between FP and FD, the direction perpendicular to the line; therefore, the centre O of the circle is located on FD.

Let us prove that an inversion transforms a circle not passing through its centre into another circle also not passing through its centre. Consider a circle of centre F and radius r which will be transformed under an inversion of centre P and radius k. Then any point R on the circle is mapped into another point S according to:

$$|PS| = \frac{k^2}{|PR|} = \frac{k^2}{|PF|\cos\alpha \pm \sqrt{r^2 - PF^2\sin^2\alpha}}$$

where the polar equation of the circle is used. The multiplication by the conjugate of the denominator gives:

$$|PS| = \frac{k^2\left(|PF|\cos\alpha \mp \sqrt{r^2 - PF^2\sin^2\alpha}\right)}{PF^2 - r^2}$$

which is the polar equation of a circle with centre G and radius s:

$$|PS| = |PG|\cos\alpha \mp \sqrt{s^2 - PG^2\sin^2\alpha}$$

where $\quad |PG| = \dfrac{k^2\,|PF|}{PF^2 - r^2} = \dfrac{k^2\,|PF|}{PT^2} \qquad s = \dfrac{k^2\, r}{PF^2 - r^2} = \dfrac{k^2\, r}{PT^2}$

$PF^2 - r^2 = PT^2$ is the power of the point P with respect to the circle of centre F. Then we see that the distance from the centre of inversion to the centre of the circle and the radius of this circle changes in the ratio of the square of the radius of inversion divided by the power of the centre of inversion with respect to the given circle.

What is the power of the centre of inversion P with respect to the new circle?:

$$PU^2 = PG^2 - s^2 = \frac{k^4\left(PF^2 - r^2\right)}{PT^4} = \frac{k^4}{PT^2}$$

That is, the product of the powers of the centre of inversion with respect to any circle and its transformed is constant and equal to the fourth power of the radius of inversion:

$$PT^2\, PU^2 = k^4$$

In the equation of the transformed circle the symbol $\mp$ appears instead of $\pm$. It means that the sense of the arc from R to R' is opposite to the sense of the arc from S to S'.

The nine-point circle

The Euler's theorem states that the midpoints of the sides of any triangle, the feet of the altitudes and the midpoints from each vertex to the orthocentre lie on a circle called the *nine-point circle*.

Let any triangle PQR whose orthocentre is H be. Now we evaluate the expression:

$$(P - H + Q - R)^2 - (P - H - Q + R)^2 = 4(P - H)\cdot(Q - R) = 0$$

which vanishes because the segment PH lying on the altitude passing through P is perpendicular to the base QR. Analogously:

$$(P - H + Q - R)^2 - (P + H - Q - R)^2 = 4(Q - H)\cdot(P - R) = 0$$

and

$$(P + H - Q - R)^2 - (P - H - Q + R)^2 = 4(R - H)\cdot(Q - P) = 0$$

Hence: $$(P + Q - R - H)^2 = (P - Q + R - H)^2 = (P - Q - R + H)^2$$

Since the sign may be opposite inside the square we have:

$$(P + Q - R - H)^2 = (P - Q + R - H)^2 = (P - Q - R + H)^2 =$$

$$= (-P - Q + R + H)^2 = (-P + Q - R + H)^2 = (-P + Q + R - H)^2$$

Now we introduce the point $N = (P + Q + R + H)/4$ to obtain:

$$\left(N - \frac{R+H}{2}\right)^2 = \left(N - \frac{Q+H}{2}\right)^2 = \left(N - \frac{Q+R}{2}\right)^2 = \left(N - \frac{P+Q}{2}\right)^2 = \left(N - \frac{P+R}{2}\right)^2 = \left(N - \frac{P+H}{2}\right)^2$$

That is, N is the centre of a circle (figure 9.7) passing through the three midpoints of the sides $(P + Q)/2$, $(Q + R)/2$ and $(R + P)/2$ and the three midpoints of the vertices and the orthocentre $(P + H)/2$, $(Q + H)/2$ and $(R + H)/2$.

Since we can write:

$$N = \frac{1}{2}\left(\frac{P+H}{2}\right) + \frac{1}{2}\left(\frac{Q+R}{2}\right)$$

it follows that $(P+H)/2$ and $(Q+R)/2$ are opposite extremes of the same diameter. Then the vectors $J-(P+H)/2$ and $J-(Q+R)/2$, which are orthogonal:

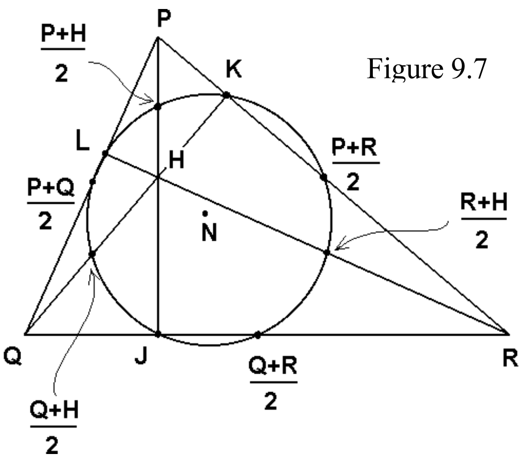


Figure 9.7

$$\left(J - \frac{P+H}{2}\right)\cdot\left(J - \frac{Q+R}{2}\right) = 0$$

are the sides of a right angle which intercepts a half circumference of the nine point circle. Therefore, this angle is inscribed and its vertex J also lies on the nine-point circle. Also, we may go through the algebraic way to arrive at the same conclusion. Developing the former inner product:

$$J^2 - J\cdot\frac{Q+R}{2} - \frac{P+H}{2}\cdot J + \frac{P+H}{2}\cdot\frac{Q+R}{2} = 0$$

Introducing the centre N of the nine-point circle, we have:

$$J^2 - 2\,J\cdot N + 2\,N\cdot\frac{Q+R}{2} - \left(\frac{Q+R}{2}\right)^2 = 0$$

and after adding and subtracting N^2 we arrive at:

$$(J-N)^2 - \left(\frac{Q+R}{2} - N\right)^2 = 0$$

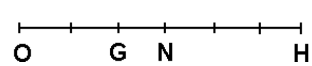
which shows that the length of JN is the radius of the nine-point circle.

From the formulas for the centroid and orthocentre one obtains the centre N of the nine-point circle:

$$N = \frac{3G+H}{4} = \left(P\;QR\;P + Q\;RP\;Q + R\;PQ\;R\right)\left(4PQ\wedge QR\right)^{-1}$$

that is, the point N also lies on the Euler's line. The relative distances between the circumcentre O, the centroid G, the nine-point circle centre N and the orthocentre H are shown in the figure 9.8.

Figure 9.8

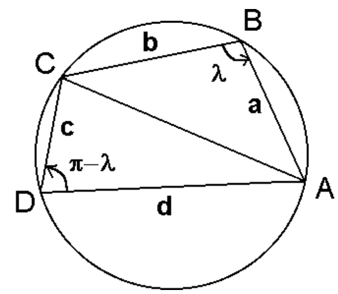


Cyclic and circumscribed quadrilaterals

First, let us see a statement valid for every quadrilateral: the area of a quadrilateral *ABCD* is the outer product of the diagonals *AC* and *BD*. The prove is written in one line. Since we can divide the quadrilateral in two triangles we have:

$$\begin{aligned}\text{Area} &= \frac{1}{2} AB \wedge BC + \frac{1}{2} AC \wedge CD \\ &= \frac{1}{2}(AC \wedge BC + AC \wedge CD) \\ &= \frac{1}{2} AC \wedge BD \\ &= \frac{1}{2}\,|AC|\,|BD|\sin(AC, BD)\, e_{12}\end{aligned}$$

Figure 9.9



The quadrilaterals inscribed in a circle are called *cyclic quadrilaterals*. They fulfil the Ptolemy's theorem: the product of the lengths of both diagonals is equal to the addition of the products of the lengths of opposite sides (figure 9.9):

$$|AC||BD| = |AB||CD| + |BC||DA|$$

A beautiful demonstration makes use of the cross ratio and it is proposed in the first exercise of the next chapter.

For the cyclic quadrilaterals, the sum of opposite angles is equal to π, because they are inscribed in the circle and intercept opposite arcs. Another interesting property is the Brahmagupta formula for the area. If the lengths of the sides are denoted by a, b, c and d and the semiperimeter by s then:

$$|\text{Area}| = \sqrt{(s-a)\,(s-b)\,(s-c)\,(s-d)} \qquad s = \frac{a+b+c+d}{2}$$

With this notation, the proof is written more briefly. The law of cosines applied to each triangle of quadrilateral gives:

$$\begin{cases} AC^2 = a^2 + b^2 - 2\,a\,b\cos\lambda \\ AC^2 = c^2 + d^2 - 2\,c\,d\cos(\pi - \lambda) \end{cases}$$

Equating both equations taking into account that the cosines of supplementary angles have opposite sign, we have:

$$a^2 + b^2 - 2\,a\,b\cos\lambda = c^2 + d^2 + 2\,c\,d\cos\lambda$$

$$a^2 + b^2 - c^2 - d^2 = 2\,(\,a\,b + c\,d\,)\cos\lambda$$

$$\frac{a^2+b^2-c^2-d^2}{2(a\,b+c\,d)}=\cos\lambda$$

The area of the quadrilateral is the sum of the areas of the triangles *ABC* and *CDA*:

$$|\,\text{Area}\,|=\frac{a\,b}{2}\sin\lambda+\frac{c\,d}{2}\sin(\pi-\lambda)$$

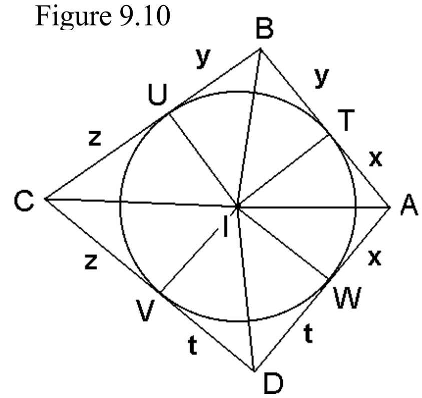
Both sines are equal; then we have:

$$\text{Area}^2=\frac{(a\,b+c\,d)^2}{4}\sin^2\lambda=\frac{(a\,b+c\,d)^2}{4}\left(1-\frac{(a^2+b^2-c^2-d^2)^2}{4(a\,b+c\,d)^2}\right)$$

$$=\frac{4(a\,b+c\,d)^2-(a^2+b^2-c^2-d^2)^2}{16}=(s-a)(s-b)(s-c)(s-d)$$

Another type of quadrilaterals are those where a circle can be inscribed, also called *circumscribed quadrilaterals*. For these quadrilaterals the bisectors of all the angles meet in the centre *I* of the inscribed circle, because it is equidistant from all the sides. When tracing the radii going to the tangency points, the quadrilateral is divided into pairs of opposite triangles (figure 9.10), and hence one deduces that the sums of the lengths of opposite sides are equal:

Figure 9.10



$$|AB|+|CD|=x+y+z+t=y+z+t+x=|BC|+|DA|$$

If a quadrilateral has inscribed and circumscribed circle, then we may substitute this condition ($a+c=b+d$) in the Brahmagupta formula to find:

$$|\,\text{Area}\,|=\sqrt{a\,b\,c\,d}=\sqrt{|AB|\,|BC|\,|CD|\,|DA|}$$

Since the sum of the angles of any quadrilateral is 2π and either of the four small quadrilaterals of a circumscribed quadrilateral has two right angles (figure 9.10), it follows that the central angles are supplementary of the vertices:

$$CBA\equiv UBT=\pi-TIU \qquad\qquad ADC\equiv WDV=\pi-VIW$$

If the quadrilateral is also cyclic then the angles *CBA* and *ADC* are inscribed and intercept opposite arcs of the outer circle, so that the segments joining opposite tangency points are orthogonal:

$$CBA + ADC = \pi \quad \Rightarrow \quad TIU + VIW = \pi \quad \Rightarrow \quad TV \perp UW$$

Angle between circles

Let us calculate the angle between a circle centred at O with radius r and another one centred at O' with radius r'. If $|O' - O| < r + r'$ then they intersect. The points of intersection P must fulfil the equations for both circles:

$$\begin{cases} (P-O)^2 = r^2 \\ (P-O')^2 = r'^2 \end{cases} \quad \Rightarrow \quad \begin{cases} P^2 - 2\,P\cdot O + O^2 = r^2 \\ P^2 - 2\,P\cdot O' + O'^2 = r'^2 \end{cases}$$

Adding both equations we obtain:

$$2\,P^2 - 2\,P\cdot(O+O') + O^2 + O'^2 = r^2 + r'^2$$

It is trivial that the angle of intersection is the supplementary of the angle between both radius, its cosines being equal:

$$\cos\alpha = \frac{OP\cdot O'P}{|OP|\,|OP'|} = \frac{P^2 - P\cdot(O+O') + O\cdot O'}{r\,r'}$$

The substitution of the first equality into the second gives:

$$\cos\alpha = \frac{r^2 + r'^2 - (O-O')^2}{2\,r\,r'}$$

The cosine of a right angle is zero; therefore two circles are orthogonal if and only if:

$$r^2 + r'^2 = (O-O')^2$$

which is the Pythagorean theorem.

Radical axis of two circles

The *radical axis* of two circles is the locus of the points which have the same power with respect to both circles:

$$OP^2 - r^2 = O'P^2 - r'^2$$

Let X be the intersection of the radical axis with the line joining both centres of the circles. Then:

$$OX^2 - r^2 = O'X^2 - r'^2 \qquad \Rightarrow \qquad OX^2 - O'X^2 = r^2 - r'^2$$

$$(OX + O'X) \cdot (OX - O'X) = r^2 - r'^2$$

$$2\left(X - \frac{O+O'}{2} \right) \cdot OO' = r^2 - r'^2$$

Since OX and $O'X$ are proportional to OO', the inner and geometric products are equivalent and we may isolate X:

$$X = \frac{O + O' + \left(r^2 - r'^2\right) OO'^{-1}}{2}$$

Now let us search which kind of geometric figure is the radical axis. Arranging as before the terms, we have:

$$2\left(P - \frac{O+O'}{2} \right) \cdot OO' = r^2 - r'^2$$

Introducing the point X we arrive at:

$$2\,(P - X) \cdot OO' = 0$$

That is, the radical axis is a line passing through X and perpendicular to the line joining both centres. If the circles intersect, then the radical axis is the straight line passing through the intersection points.

The *radical centre* of three circles is the intersection of their radical axis. Let O, O' and O'' be the centre of three circles with radii r, r' and r''. Let us denote with X (as above) the intersections of the radical axis of the first and second circles with the line OO', and with Y the intersection of the radical axis of the second and third circles with the line joining its centres $O'O''$. The radical centre P must fulfil the former equation for both radical axis:

$$\begin{cases} (P - X) \cdot OO' = 0 \\ (P - Y) \cdot O'O'' = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} P \cdot OO' = X \cdot OO' = \dfrac{O'^2 - O^2 + r^2 - r'^2}{2} \\ P \cdot O'O'' = Y \cdot O'O'' = \dfrac{O''^2 - O'^2 + r'^2 - r''^2}{2} \end{cases}$$

Now we have encountered the coefficients of linear combination of an imaginary decomposition. This process is explained in page 74 for the calculation of the orthocentre. However there is a difference: we wish not to resolve the vector into orthogonal components but from the known coefficients we wish to reconstruct the vector P. Then:

$$P = \frac{P \cdot O'O''}{OO' \wedge O'O''} OO' - \frac{P \cdot OO'}{OO' \wedge O'O''} O'O''$$

The substitution of the known values of the coefficients results in the following formula for the radical centre which is invariant under cyclic permutation of the circles:

$$P = (2\, OO' \wedge O'O'')^{-1}\left(\left(O''^2 - r''^2\right)OO' + \left(O^2 - r^2\right)O'O'' + \left(O'^2 - r'^2\right)O''O\right)$$

Exercises

9.1 Let A, B, C be the vertices of any triangle. The point M moves according to the equation:

$$MA^2 + MB^2 + MC^2 = k$$

Which geometric locus does the point M describe as a function of the values of the real parameter k?

9.2 Prove that the geometric locus of the points P such that the ratio of distances from P to two distinct points A and B is a constant k is a circle. Calculate the centre and the radius of this circle.

9.3 Prove the Simson's theorem: the feet of the perpendiculars from a point D upon the sides of a triangle ABC are aligned if and only if D lies on the circumscribed circle.

9.4 The Bretschneider's theorem: let a, b, c and d be the successive sides of a quadrilateral, m and n its diagonals and α and γ two opposite angles. Show that the following *law of cosines for a quadrilateral* is fulfilled:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - 2|\,a\,||\,b\,||\,c\,||\,d\,|\cos(\alpha + \gamma)$$

9.5 Draw three circles passing through each vertex of a triangle and the midpoints of the concurrent sides. Then join the centre of each circle and the midpoint of the opposite side. Show that the three segments obtained in this way intersect in a unique point.

9.6 Prove that the inversion is an opposite conformal transformation, that is, it preserves the angles between curves, but changes their sign.

10. CROSS RATIOS AND RELATED TRANSFORMATIONS

Complex cross ratio

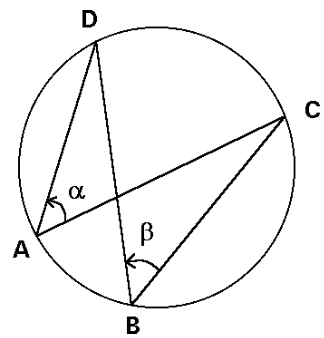
The *complex cross ratio* of any four points *A*, *B*, *C* and *D* on the plane is defined as the quotient of the following two single ratios:

$$(A\, B\, C\, D) = (A\, C\, D)\, (B\, C\, D)^{-1} = AC\, AD^{-1}\, BD\, BC^{-1}$$

If the four points are aligned, then the cross ratio is a real number, otherwise it is a complex number. Denoting the angles *CAD* and *CBD* as α and β respectively, the cross ratio is written as:

$$(A\, B\, C\, D) = \frac{AC\ AD\ BD\ BC}{|\, AD\,|^2\ |\, BC\,|^2}$$

Figure 10.1



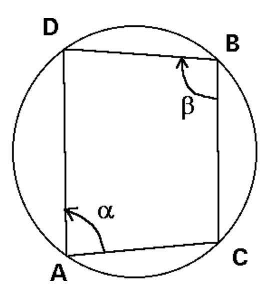
$$= \frac{|\, AC\,||\, BD\,|}{|\, AD\,||\, BC\,|} \exp[(\alpha - \beta)e_{12}]$$

If the four points are located in this order on a circle (figure 10.1), the cross ratio is a positive real number, because the inscribed angles α and β intercept the same arc *CD* and hence are equal, so:

$$(A\, B\, C\, D) = \frac{|\, AC\,||\, BD\,|}{|\, AD\,||\, BC\,|}$$

But if one of the points *C* or *D* is located between *A* and *B* (figure 10.2), then α and β have distinct arcs *CD* with opposite orientation. Since α and β are the half of these arcs, it follows that $\alpha - \beta = \pi$ and the cross ratio is a negative real number:

Figure 10.2



$$(A\, B\, C\, D) = - \frac{|\, AC\,||\, BD\,|}{|\, AD\,||\, BC\,|}$$

Analogously, if the four points are aligned and *C* or *D* is located between *A* and *B*, the cross ratio is negative[1].

Other definitions of the cross ratio differ from this one given here only in the order in which the vectors are taken. This question is equivalent to study how the value of the cross ratio changes under permutations of the points. For four points, there are 24 possible permutations, but only six different values for the cross ratio. The calculations are somewhat laborious and I shall only show one case.

[1] A straight line is a circle with infinite radius.

The reader may prove any other case as an exercise. I also indicate whether the permutations are even or odd, that is, whether they are obtained by an even or an odd number of exchanges of any two points from the initial definition $(A\,B\,C\,D)$:

(1) $(A\,B\,C\,D) = (B\,A\,D\,C) = (C\,D\,A\,B) = (D\,C\,B\,A) = r$ even

(2) $(B\,A\,C\,D) = (A\,B\,D\,C) = (D\,C\,A\,B) = (C\,D\,B\,A) = \dfrac{1}{r}$ odd

(3) $(A\,C\,B\,D) = (B\,D\,A\,C) = (C\,A\,D\,B) = (D\,B\,C\,A) = 1 - r$ odd

(4) $(C\,A\,B\,D) = (D\,B\,A\,C) = (A\,C\,D\,B) = (B\,D\,C\,A) = \dfrac{1}{1-r}$ even

(5) $(B\,C\,A\,D) = (A\,D\,B\,C) = (D\,A\,C\,B) = (C\,B\,D\,A) = \dfrac{r-1}{r}$ even

(6) $(C\,B\,A\,D) = (D\,A\,B\,C) = (A\,D\,C\,B) = (B\,C\,D\,A) = \dfrac{r}{r-1}$ odd

Let us prove, for example the equalities (3):

(3) $(A\,C\,B\,D) = AB\,AD^{-1}\,CD\,CB^{-1} = (AC + CB)\,AD^{-1}\,(CB + BD)\,CB^{-1} =$

$$= AC\,AD^{-1}\,CB\,CB^{-1} + AC\,AD^{-1}\,BD\,CB^{-1} + CB\,AD^{-1}\,CB\,CB^{-1} + CB\,AD^{-1}\,BD\,CB^{-1}$$

In the last term, under the reflection in the direction *CB,* the complex number $AD^{-1}\,BD$ becomes conjugate, that is, these vectors are exchanged:

$$= AC\,AD^{-1} + AC\,AD^{-1}\,BD\,CB^{-1} + CB\,AD^{-1} + BD\,AD^{-1}$$

$$= (AC + CB + BD)\,AD^{-1} - AC\,AD^{-1}\,BD\,BC^{-1} = AD\,AD^{-1} - r = 1 - r$$

Let us see that this value is the same for all the cross ratios of the case (3):

$$(B\,D\,A\,C) = BA\,BC^{-1}\,DC\,DA^{-1} = AB\,CB^{-1}\,CD\,AD^{-1} = AB\,AD^{-1}\,CD\,CB^{-1}$$

$$= (A\,C\,B\,D)$$

where in the third step the permutative property has been applied. Also by using this property one obtains:

$$(C\,A\,D\,B) = CD\,CB^{-1}\,AB\,AD^{-1} = CD\,AD^{-1}\,AB\,CB^{-1} = AB\,AD^{-1}\,CD\,CB^{-1}$$

$$= (A\,C\,B\,D)$$

$$(D\,B\,C\,A) = DC\,DA^{-1}\,BA\,BC^{-1} = BA\,DA^{-1}\,DC\,BC^{-1} = AB\,AD^{-1}\,CD\,CB^{-1}$$

$$= (A\,C\,B\,D)$$

Harmonic characteristic and ranges

From the cross ratio, one can find a quantity independent of the symbols of the points and only dependent on their locations. I call this quantity the *harmonic characteristic* -because of the obvious reasons that follow now- denoting it as [*A B C D*]. The simplest way to calculate it (but not the unique) is through the alternated addition of all the permutations of the cross ratio, condensed in the alternated sum of the six different values:

$$(A\,B\,C\,D) - (B\,A\,C\,D) - (A\,C\,B\,D) + (C\,A\,B\,D) + (B\,C\,A\,D) - (C\,B\,A\,D) =$$

$$= r - \frac{1}{r} - (1-r) + \frac{1}{1-r} + \frac{r-1}{r} - \frac{r}{r-1}$$

$$= \frac{(2r-1)(r+1)(r-2)}{r(1-r)}$$

When two points are permuted, this sum still changes the sign. The square of this sum is invariant under every permutation of the points, and is the suitable definition of the harmonic characteristic:

$$[A\,B\,C\,D] = \left[(A\,B\,C\,D) - (B\,A\,C\,D) - (A\,C\,B\,D) + (C\,A\,B\,D) + (B\,C\,A\,D) - (A\,B\,C\,D)\right]^2$$

$$[A\,B\,C\,D] = \frac{(2r-1)^2\,(r+1)^2\,(r-2)^2}{r^2\,(1-r)^2}$$

The first notable property of the harmonic characteristic is the fact that it vanishes for a harmonic range of points, that is, when the cross ratio (*A B C D*) takes the values 1/2, –1 or –2 dependent on the denominations of the points. The second notable property is the fact that the harmonic characteristic of a degenerate range of points (two or more coincident points) is infinite.

Perhaps, the reader believes that other harmonic characteristics may be obtained in many other ways, that is, by using other combinations of the permutations of the cross ratio. However, other attempts lead whether to the same function of r (or a linear dependence) or to constant values. For example, we can take the sum of all the squares, which is also invariant under any permutation of the points to find:

$$(A\,B\,C\,D)^2 + (B\,A\,C\,D)^2 + (A\,C\,B\,D)^2 + (C\,A\,B\,D)^2 + (B\,C\,A\,D)^2 + (C\,B\,A\,D)^2$$

$$= r^2 + \frac{1}{r^2} + (1-r)^2 + \frac{1}{(1-r)^2} + \frac{(r-1)^2}{r^2} + \frac{r^2}{(r-1)^2} = \frac{[A\,B\,C\,D] + 21}{2}$$

On the other hand, M. Berger (vol. I, p. 127) has used a function that is also linearly dependent on the harmonic characteristic:

$$\frac{4\left(r^2-r+1\right)^3}{27r^2\left(1-r\right)^2}=1+\frac{[A\,B\,C\,D]}{27}$$

Other useless possibilities are the sum of all the values of the permutations of the cross ratio, which is identical to zero, or their product giving always a unity result.

Four points A, B, C and D so ordered on a line are said to be a *harmonic range* if the segments AB, AC and AD are in harmonic progression, that is, the inverse vectors AB^{-1}, AC^{-1} and AD^{-1} are in arithmetic progression:

$$AB^{-1}-AC^{-1}=AC^{-1}-AD^{-1} \qquad \Leftrightarrow \qquad AC\,AD^{-1}\,BD\,BC^{-1}=2$$

The cross ratio of a harmonic range is 2, but if this order is altered then it can be -1 or $1/2$. Note that this definition also includes harmonic ranges of points on a circle. On the other hand, all the points of a harmonic range always lie on a line or a circle.

In order to construct the fourth point D forming a harmonic range with any three given points A, B and C (ordered in this way on a line) one must follow these steps (figure 10.3):

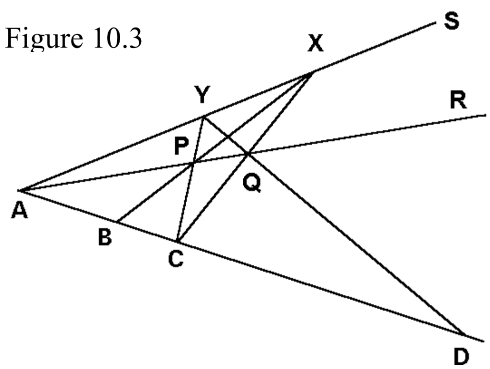
1) Draw the line passing through A, B and C. Trace two any distinct lines R and S also different from ABC and passing through A.

2) Draw any line passing through B. This line will cut the line R in the point P and the line S in the point X.

3) Draw the line passing through C and X. The intersection of this line with R will be denoted by Q. Draw also the line passing through C and P. This line intersects the line S in the point Y.

4) Now trace the line passing through Y and Q, which will cut the line ABC in the searched point D, which completes the harmonic range.

Figure 10.3



For a clearer displaying I have drawn some segments instead of lines, although the construction is completely general only for lines.

If A, B and C are not aligned, the point D which completes the harmonic range lies on a circle passing through the three given points (figure 10.4). In order to determine D one must make an inversion (which preserves the cross ratio if this is real, as shown below) with centre on the circle ABC, which transforms this circle into a line. In this line we determine the point D' forming a harmonic range with A', B', C' and then, by means of the same inversion, the point D is obtained. In the drawing method the radius and the exact position of the centre of inversion do not play any role whenever it lie on the circle ABC. Then we shall draw any line outside the circle, project A, B and C into this line, make the former construction and obtain D by projecting D' upon the circle.

The cross ratio becomes conjugated under circular inversion. In order to prove this statement, let us see firstly how the single ratio is transformed. If A', B', C' and D' are the transformed points of A, B, C and D under an inversion with centre O and any radius, we have:

$$A'C'\,A'D'^{-1} = (OC' - OA')\,(OD' - OA')^{-1} = (OC^{-1} - OA^{-1})\,(OD^{-1} - OA^{-1})^{-1}$$

$$= OC^{-1}\,(\,OA - OC\,)\,OA^{-1}\,[\,OD^{-1}\,(\,OA - OD\,)\,OA^{-1}\,]^{-1}$$

$$= OC^{-1}\,CA\,OA^{-1}\,[\,OD^{-1}\,DA\,OA^{-1}\,]^{-1} = OC^{-1}\,CA\,OA^{-1}\,OA\,DA^{-1}\,OD$$

$$= OC^{-1}\,AC\,AD^{-1}\,OD$$

Analogously we have:

$$B'D'\,B'C'^{-1} = OD^{-1}\,BD\,BC^{-1}\,OC$$

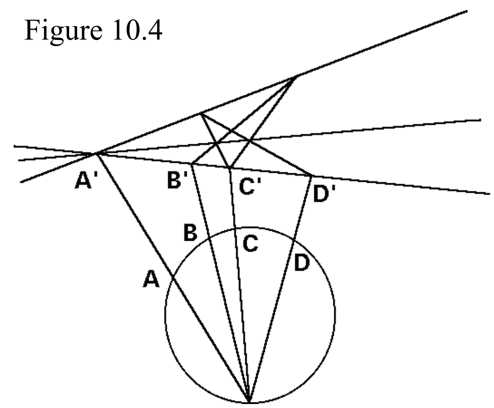
The product of both expressions is the cross ratio:

$$(\,A'\,B'\,C'\,D'\,) = A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1}$$

$$= OC^{-1}\,AC\,AD^{-1}\,BD\,BC^{-1}\,OC$$

$$= OC^{-1}\,(\,A\,B\,C\,D\,)\,OC = (\,A\,B\,C\,D\,)^*$$

Figure 10.4



So, the cross ratio of four points on a line or a circle, which is real, is preserved, otherwise it becomes conjugated.

Finally let us see that the single ratio is the limit of the cross ratio for one point at infinity:

$$(\,\infty\,B\,C\,D\,) = \lim_{A \to \infty} AC\,AD^{-1}\,BD\,BC^{-1} = BD\,BC^{-1} = (\,B\,D\,C\,)$$

Analogously $(\,A\,\infty\,C\,D\,) = (\,A\,C\,D\,)$, etc.

The homography (Möbius transformation)

The *homography* is defined as the geometric transformation that preserves the complex cross ratio of four points on the plane. In order to specify a homography one must gives only the images A', B' and C' of three points A, B, and C because the fourth image D' is determined by the conservation of their cross ratio:

$$(\,A'\,B'\,C'\,D'\,) = (\,A\,B\,C\,D\,)$$

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1} = AC\,AD^{-1}\,BD\,BC^{-1}$$

Arranging vectors we write:

$$A'D'^{-1}\,B'D' = A'C'^{-1}\,AC\,AD^{-1}\,BD\,BC^{-1}\,B'C' = A'C'^{-1}\,AC\;\,BC^{-1}\,B'C'\,AD^{-1}\,BD$$

After conjugation we find:

$$B'D'\,A'D'^{-1} = BD\,AD^{-1}\,B'C'\,BC^{-1}\,AC\,A'C'^{-1}$$

$$B'D'\,A'D'^{-1} = BD\,AD^{-1}\,z \qquad \text{where} \quad z = BC^{-1}\,B'C'\,A'C'^{-1}\,AC$$

$$D'B'\,D'A'^{-1} = DB\,DA^{-1}\,z \qquad \Rightarrow \qquad (\,D'\,B'\,A'\,) = (\,D\;B\;A\,)\,z$$

Hence the homography multiplies the single ratio by a complex number z. The modulus of the single ratio increases proportionally to the modulus of z and the angle BDA is increased in the argument of z:

$$z = |\,z\,|\;\exp[\alpha\, e_{12}\,]$$

$$\frac{|\,D'B'\,|}{|\,D'A'\,|} = |\,z\,|\frac{|\,DB\,|}{|\,DA\,|}$$

$$\text{angle } B'D'A' = \text{angle } BDA + \alpha$$

Then a homography can also be determined by specifying the images A' and B' of two points A and B, and a complex number z. The image D' of D is calculated by isolation in the former equation:

$$(\,D'A' + A'B'\,)\,D'A'^{-1} = DB\,DA^{-1}\,z$$

$$A'B'\,D'A'^{-1} = DB\,DA^{-1}\,z - 1$$

$$D'A' = (\,z\,DB\,DA^{-1} - 1\,)^{-1}\,A'B'$$

$$D' = A' + (\,1 - z\,DB\,DA^{-1}\,)^{-1}\,A'B'$$

The homography preserves the complex cross ratio of any four points. Analogously to the former equality, we find for any three points P, Q, R :

$$P' = A' + (\,1 - z\,PB\,PA^{-1}\,)^{-1}\,A'B'$$

$$Q' = A' + (\,1 - z\,QB\,QA^{-1}\,)^{-1}\,A'B'$$

$$R' = A' + (\,1 - z\,RB\,RA^{-1}\,)^{-1}\,A'B'$$

$$D'Q' = [\,(\,1 - z\,QB\,QA^{-1}\,)^{-1} - (\,1 - z\,DB\,DA^{-1}\,)^{-1}\,]\,A'B' =$$

$$= z\,(\,-DB\,DA^{-1} + QB\,QA^{-1}\,)\,(\,1 - z\,DB\,DA^{-1}\,)^{-1}\,(\,1 - z\,QB\,QA^{-1}\,)^{-1}\,A'B'$$

$$D'R' = z\,(\,-DB\,DA^{-1} + RB\,RA^{-1}\,)\,(\,1 - z\,DB\,DA^{-1}\,)^{-1}\,(\,1 - z\,RB\,RA^{-1}\,)^{-1}\,A'B'$$

$$P'Q' = z\,(\,-PB\,PA^{-1} + QB\,QA^{-1}\,)\,(\,1 - z\,PB\,PA^{-1}\,)^{-1}\,(\,1 - z\,QB\,QA^{-1}\,)^{-1}\,A'B'$$

$$P'R' = z\,(\,-PB\;PA^{-1} + RB\;RA^{-1}\,)\,(\,1 - z\;PB\;PA^{-1}\,)^{-1}\,(\,1 - z\;RB\;RA^{-1}\,)^{-1}\,A'B'$$

In the cross ratio, many factors are simplified:

$$D'Q'\;D'R'^{-1}\;P'R'\;P'Q'^{-1} = (\,-DB\;DA^{-1} + QB\;QA^{-1}\,)\,(\,-DB\;DA^{-1} + RB\;RA^{-1}\,)^{-1}$$

$$(\,-PB\;PA^{-1} + RB\;RA^{-1}\,)\,(\,-PB\;PA^{-1} + QB\;QA^{-1}\,)^{-1}$$

Writing the first factor as product of factors we obtain:

$$-DB\;DA^{-1} + QB\;QA^{-1} = -\,(\,DA + AB\,)\,DA^{-1} + (\,QA + AB\,)\,QA^{-1} =$$

$$= -AB\;DA^{-1} + AB\;QA^{-1} = AB\;DA^{-2}\,(\,-DA\;QA^{2} + DA^{2}\;QA\,)\,QA^{-2} =$$

$$= AB\;DA^{-2}\;DA\,(\,-QA + DA\,)\,QA\;QA^{-2}$$

$$= AB\;DA^{-1}\;QD\;QA^{-1}$$

In the same way each of the other three factors can be written as product of factors:

$$-DB\;DA^{-1} + RB\;RA^{-1} = AB\;DA^{-1}\;RD\;RA^{-1}$$

$$-PB\;PA^{-1} + RB\;RA^{-1} = AB\;PA^{-1}\;RP\;RA^{-1}$$

$$-PB\;PA^{-1} + QB\;QA^{-1} = AB\;PA^{-1}\;QP\;QA^{-1}$$

Then the cross ratio will be:

$$D'Q'\;D'R'^{-1}\;P'R'\;P'Q'^{-1} = (\,AB\;DA^{-1}\;QD\;QA^{-1}\,)\,(\,RA\;RA^{-1}\;DA\;AB^{-1}\,)$$

$$(\,AB\;PA^{-1}\;RP\;RA^{-1}\,)\,(\,QA\;QP^{-1}\;PA\;AB^{-1}\,)$$

where the brackets are not needed but only show the factors they come from. Applying the permutative property and simplifying, we arrive at:

$$D'Q'\;D'R'^{-1}\;P'R'\;P'Q'^{-1} = DQ\;DR^{-1}\;PR\;PQ^{-1}$$

This proves that the cross ratio of every set of four point is preserved under a homography. Four points lying on a line or a circle have a real cross ratio. Then the homography is a circular transformation since it transforms circles into circles in a general sense (taking the lines as circles with infinite radius).

The product of two circular inversions is always a homography because the cross ratio becomes conjugate under inversion and thus it is preserved under an even number of inversions. Let us calculate the homography resulting from a composition of two inversions. Let A be any point, A' be the transformed point under a first inversion with centre O and radius r, and A'' be the transformed point of A' under a second inversion with centre P and radius s, which is consequently the transformed point of A under the homography:

$$OA' = r^2\, OA^{-1} \quad PA'' = s^2\, PA'^{-1}$$

From where it follows that:

$$s^2\, PA''^{-1} = PA' = OA' - OP = r^2\, OA^{-1} - OP$$

$$PA''^{-1} = s^{-2}\,(\, r^2\, OA^{-1} - OP\,)$$

$$PA'' = s^2\,(r^2\, OA^{-1} - OP\,)^{-1} = s^2\,[(\, r^2 - OP\; OA\,)\; OA^{-1}]^{-1} = s^2\, OA\,(r^2 - OP\; OA\,)^{-1}$$

This equation allows to determine the image A'' of any point A having the centre O and P and the radius r and s as data. For instance, let us calculate the homography corresponding to:

$O = (\,0, 0\,) \qquad r = 2 \qquad P = (\,3, 0\,) \qquad s = 3$

$OP = 3\, e_1$

In order to specify the homography we determine the images of three points:

$A = (\,0, 2\,) \qquad B = (\,1, 1\,) \qquad C = (\,1, 0\,)$

$OA = 2\, e_2 \qquad OB = e_1 + e_2 \qquad OC = e_1$

$OP\; OA = 6\, e_{12} \qquad OP\; OB = 3 + 3\, e_{12} \qquad OP\; OC = 3$

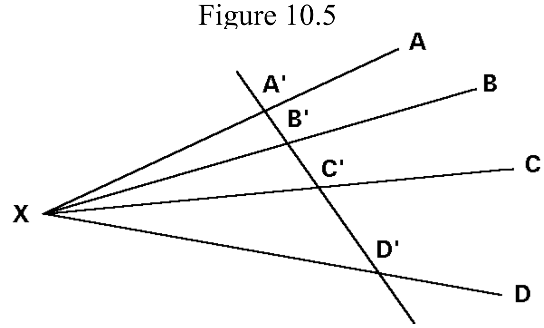
$$PA'' = \frac{-27\,e_1 + 18\,e_2}{13} \qquad PB'' = \frac{-9\,e_1 + 18\,e_2}{5} \qquad PC'' = 9\, e_1$$

$$A'' = \left(\frac{12}{13}, \frac{18}{13}\right) \qquad B'' = \left(\frac{6}{5}, \frac{18}{5}\right) \qquad C'' = (12, 0)$$

Projective cross ratio

The *projective cross ratio* of four points A, B, C and D with respect to a point X is defined as the cross ratio of the pencil of lines XA, XB, XC and XD, which is the cross ratio of the intersection points of these lines with any crossing line (figure 10.5). As it will be shown, the projective cross ratio is independent of the crossing line The intersection points A', B', C' and D' are

Figure 10.5



the projections with centre X of the points A, B, C and D upon the crossing line. Due to the alignment of these projections, the projective cross ratio is always a real number:

$$\{ X, A\,B\,C\,D \} = (\, A'\,B'\,C'\,D' \,) = A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1}$$

The projected points A', B', C' and D' are dependent on the crossing line and this expression is not suitable for calculations. Note that the segment $A'C'$ is the base of the triangle $XA'C'$, $B'D'$ is the base of the triangle $XB'D'$, etc. Since the altitudes of all these triangles are the distance from X to the crossing line, their bases are proportional to their areas, so we can write:

$$\{X, A\,B\,C\,D\} = \frac{XA' \wedge A'C' \;\; XB' \wedge B'D'}{XA' \wedge A'D' \;\; XB' \wedge B'C'}$$

The area of a triangle is the half of the outer product of any two sides. For example for the triangle $XA'C'$ we have:

$$XA' \wedge A'C' = XA' \wedge (\, A'X + XC' \,) = XA' \wedge XC'$$

Then, the cross ratio becomes:

$$\{X, A\,B\,C\,D\} = \frac{XA' \wedge XC' \;\; XB' \wedge XD'}{XA' \wedge XD' \;\; XB' \wedge XC'}$$

The simplification of the modulus gives:

$$\{X, A\,B\,C\,D\} = \frac{\sin A'XC' \,\sin B'XD'}{\sin A'XD' \,\sin B'XC'} = \frac{\sin AXC \,\sin BXD}{\sin AXD \,\sin BXC}$$

because the angles $A'XC'$ and AXC are identical, $B'XD'$ and BXD are also identical, etc. Now it is obvious that the projective cross ratio does not depend on the crossing line, but only on the angles between the lines of the pencil. If we introduce the modulus of XA, XB, etc., we find the outer products of these vectors:

$$\{X, A\,B\,C\,D\} = \frac{XA \wedge XC \;\; XB \wedge XD}{XA \wedge XD \;\; XB \wedge XC}$$

Now, the arbitrary crossing line of the pencil has disappeared in the formula, which has become suitable for calculations with Cartesian coordinates[2]. When the points A, B, C and D are aligned, the complex and projective cross ratios have the same value independently of the point X, because the segments AC, BD, AD and BC have the same direction, which can be taken as the crossing line:

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[2] The reader will find an application of this formula for the projective cross ratio to the error calculus in the electronic article "Estimation of the error committed in determining the place from where a photograph was taken", Ramon González, Josep Homs, Jordi Solsona (1999) http://www.terra.es/personal/rgonzal1 (translation of the paper published in the *Butlletí de la Societat Catalana de Matemàtiques* **12** [1997] 51-71).

$\{X, A\,B\,C\,D\} = (\,A\,B\,C\,D\,)$

If A, B, C and D lie on a circle, the complex cross ratio is equal to the projective cross ratio taken from any point X on that circle. To prove this statement, make an inversion with centre X. Since the circle $ABCDX$ passes through the centre of inversion, it is transformed into a line, where the points A', B', C' and D' (transformed of A, B, C and D by the inversion) lie with the same cross ratio. Now observe that X, A and A' are aligned, also X, B and B', etc. So the projective cross ratio is equal to the complex cross ratio.

From the arguments given above, it follows that the cross ratio of a pencil of lines is the quotient of the outer products of their direction (or normal) vectors independently of their modulus:

$$\{X, A\,B\,C\,D\} = \frac{v_{XA} \wedge v_{XC}\;\; v_{XB} \wedge v_{XD}}{v_{XA} \wedge v_{XD}\;\; v_{XB} \wedge v_{XC}} = \frac{n_{XA} \wedge n_{XC}\;\; n_{XB} \wedge n_{XD}}{n_{XA} \wedge n_{XD}\;\; n_{XB} \wedge n_{XC}}$$

Instead of XA, XB, XC and XD, I shall denote by A, B, C and D the pencil of lines passing through X. Then the projective cross ratio $\{X, ABCD\}$ is written as $(ABCD)$. Let us suppose that these lines have the following dual coordinates expressed in the lines base $\{O, P, Q\}$:

$A = [x_A\,, y_A]$ $\qquad B = [x_B\,, y_B]$ $\qquad C = [x_C\,, y_C]$ $\qquad D = [x_D\,, y_D]$

Then the direction vector of the line A is:

$$v_A = (1 - x_A - y_A)v_O + x_A v_P + y_A v_Q$$

But the direction vectors of the base fulfil:

$$v_O + v_P + v_Q = 0$$

Hence:

$$v_A = (-1 + 2x_A + y_A)v_P + (-1 + x_A + 2y_A)v_Q$$

and analogously:

$$v_B = (-1 + 2x_B + y_B)v_P + (-1 + x_B + 2y_B)v_Q$$

$$v_C = (-1 + 2x_C + y_C)v_P + (-1 + x_C + 2y_C)v_Q$$

$$v_D = (-1 + 2x_D + y_D)v_P + (-1 + x_D + 2y_D)v_Q$$

The outer product of v_A and v_C is:

$$v_A \wedge v_C = \left(x_C - x_A - (y_C - y_A) + 3(x_A\, y_C - x_C\, y_A)\right) v_P \wedge v_Q$$

which leads to the fact that the cross ratio only depends on the dual coordinates:

$$(A\,B\,C\,D) = \frac{v_A \wedge v_C \;\; v_B \wedge v_D}{v_A \wedge v_D \;\; v_B \wedge v_C} =$$

$$= \frac{\left(x_C - x_A - (y_C - y_A) + 3(x_A\, y_C - x_C\, y_A)\right)\left(x_D - x_B - (y_D - y_B) + 3(x_B\, y_D - x_D\, y_B)\right)}{\left(x_D - x_A - (y_D - y_A) + 3(x_A\, y_D - x_D\, y_A)\right)\left(x_C - x_B - (y_C - y_B) + 3(x_B\, y_C - x_C\, y_B)\right)}$$

$$= \frac{(x_A - 1/3, y_A - 1/3) \wedge (x_C - 1/3, y_C - 1/3)\;\;(x_B - 1/3, y_B - 1/3) \wedge (x_D - 1/3, y_D - 1/3)}{(x_B - 1/3, y_B - 1/3) \wedge (x_C - 1/3, y_C - 1/3)\;\;(x_A - 1/3, y_A - 1/3) \wedge (x_D - 1/3, y_D - 1/3)}$$

$$= \frac{(x_A - x_C, y_A - y_C) \wedge (x_C - 1/3, y_C - 1/3)\;\;(x_B - x_D, y_B - y_D) \wedge (x_D - 1/3, y_D - 1/3)}{(x_B - x_C, y_B - y_C) \wedge (x_C - 1/3, y_C - 1/3)\;\;(x_A - x_D, y_A - y_D) \wedge (x_D - 1/3, y_D - 1/3)}$$

And taking into account the fact that the dual points corresponding to the lines of the pencil are aligned in the dual plane, it follows that:

$$(A\,B\,C\,D) = (x_A - x_C, y_A - y_C)(x_B - x_C, y_B - y_C)^{-1}(x_B - x_D, y_B - y_D)(x_A - x_D, y_A - y_D)^{-1}$$

$$= AC\; BC^{-1}\; BD\; AD^{-1} = AC\; AD^{-1}\; BD\; BC^{-1}$$

where the four dual vectors are proportional. That is: <u>the projective cross ratio of a pencil of lines is the cross ratio of the corresponding dual points</u>. Obviously, by means of the duality principle the dual proposition must also hold: <u>the cross ratio of four aligned points is equal to the projective cross ratio of the dual pencil of the corresponding lines on the dual plane</u>.

The points at the infinity and homogeneous coordinates

The points at the infinity play the same role as the points with finite coordinates in the projective geometry. Under a projectivity they can be mapped mutually without special distinction. In order to avoid the algebraic inconsistencies of the infinity, the *homogeneous coordinates* are defined as proportional to the barycentric coordinates with a no fixed homogeneous constant. The following expressions are equivalent and represent the same point:

$$(x,\ y) = (1 - x - y,\ x,\ y) = k\,(1 - x - y,\ x,\ y)$$

Then for instance:

$$(2, 3) = (-4, 2, 3) = (-20, 10, 15)$$

A point has finite coordinates if and only if the sum of the homogeneous coordinates are different from zero. In this case we can normalise the coordinates dividing by this sum in order to obtain the barycentric coordinates, and hence the Cartesian coordinates by omitting the first coordinate. On the other hand, the sum of the homogeneous coordinates sometimes vanishes and then they cannot be normalised (the normalisation implies a division by zero yielding infinite values). In this case, we are regarding a point at the infinity, whose algebraic operations should be performed with the finite values of the homogeneous coordinates[3]. For example:

$$(2, -1, -1) = (2\,\infty, -\infty, -\infty) = (\infty, \infty)$$

This is the point at the infinity in the direction of the bisector of the first quadrant. On the other hand, points at the infinity are equivalent to directions and hence to vectors. For example:

$$v = 2\,O - P - Q = -OP - OQ = -2\,e_1 - 2\,e_2$$

Every vector proportional to v also represents this direction and the point at the infinity given above.

Perspectivity and projectivity

A *perspectivity* is a central projection of a range of points on a line upon another line. Since the cross ratio of a pencil of lines is independent of the crossing line, it follows from the construction (figure 10.6) that a perspectivity preserves the projective cross ratio:

$$\{O, A\,B\,C\,D\} = \{O, A'\,B'\,C'\,D'\}$$

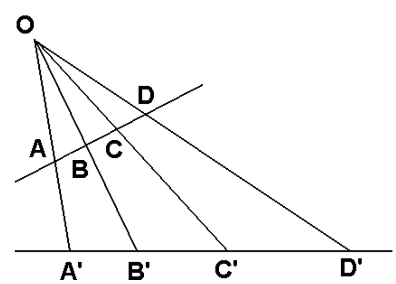


Figure 10.6

The composition of two perspectivities with different centres also maps ranges[4] of points to other ranges preserving their cross ratio. This is the most general transformation which preserves alignment and the projective cross ratio. A *projectivity* (also *projective transformation*) is defined as the geometric transformation which maps aligned points to aligned points preserving their cross ratio. A theorem of projective geometry states that any projectivity is expressible as the sequence of not more than three perspectivities. If the projectivity relates ranges on two distinct lines, two perspectivities suffice.

[3] In fact, the calculations are equally feasible with ∞ if we understand it strictly as a number. Then the equality $2\infty = \infty$ should be wrong. We could simplify by ∞, e.g.: $(2\,\infty, -\infty, 1) = (2, -1, 0)$ is a finite point and $(2\,\infty, -\infty, -\infty) = (2, -1, -1)$ a point at the infinity.

[4] A *range* of points is defined as a set of aligned points.

A projectivity is determined by giving three aligned points and their corresponding images, which must be also aligned. On the other hand, a projectivity is determined by giving four points no three of which are collinear (a quadrilateral) and their non-aligned images (another quadrilateral). Since the projectivity preserves the alignment and the incidence, we must take a linear transformation of the coordinates. However, the affinity is already a linear transformation of coordinates but only of the independent coordinates. The more general linear transformation must account for all the three barycentric coordinates. Thus the projectivity is given by a 3×3 matrix:

$$\begin{pmatrix} 1-x'-y' \\ x' \\ y' \end{pmatrix} = k \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} \qquad \text{with } \det h_{ij} \neq 0$$

The coefficients h_{ij} being given, the product of the matrices on the right hand side does not warrantee a set of coordinates with sum equal to 1. This means that the matrix h_{ij} of a perspectivity is defined except by the proportionality constant k (which depends on the coordinates), and the transformed point is obtained with homogeneous coordinates instead of barycentric ones. Nevertheless, this matrix maps collinear points to collinear points. This follows immediately from the matrices equalities:

$$\begin{pmatrix} k\,(1-x_A'-y_A') & l\,(1-x_B'-y_B') & m\,(1-x_C'-y_C') \\ k\,x_A' & l\,x_B' & m\,x_C' \\ k\,y_A' & l\,y_B' & m\,y_C' \end{pmatrix} = \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1-x_A-y_A & 1-x_B-y_B & 1-x_C-y_C \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix}$$

A set of points are collinear if the determinant of the barycentric coordinates vanishes. Since $\det h_{ij} \neq 0$, the implication of the alignment is bi-directional:

$$k\,l\,m \begin{vmatrix} 1-x_A'-y_A' & 1-x_B'-y_B' & 1-x_C'-y_C' \\ x_A' & x_B' & x_C' \\ y_A' & y_B' & y_C' \end{vmatrix} = \begin{vmatrix} 1-x_A-y_A & 1-x_B-y_B & 1-x_C-y_C \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{vmatrix} \det h_{ij}$$

As an example, let us transform the origin of coordinates under the following projectivity:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad (0,0) \to (1,2,1) = \left(\frac{1}{4}, \frac{2}{4}, \frac{1}{4} \right) = \left(\frac{1}{2}, \frac{1}{4} \right)$$

Analogously the transformed point of $(1, 0)$ is:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \quad \Rightarrow \quad (1, 0) \to (-1, 3, 2) = \left(\frac{3}{4}, \frac{1}{2} \right)$$

Also we may transform the point $(0, \infty)$ at the infinity, which has the homogeneous coordinates $(-1, 0, 1)$:

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad (0, \infty) \to (1, -1, -1) = (1, 1)$$

This case exemplifies how points at the infinity can be mapped under a projectivity to points with finite coordinates and vice versa. As a comparison, the affinity always maps points at the infinity to points at the infinity (although it does not leave them invariant). That is, the affinity always maps parallel lines to parallel lines, while under a projectivity parallel lines can be mapped to concurrent lines and vice versa[5]. Parallel lines are concurrent in a point at the infinity, which does not play any special role in the projectivity.

Let us see that four non collinear points (a quadrilateral) and their non collinear images (another quadrilateral) determine a unique projectivity. Let us suppose that the images of the points (0, 0), (0, 1), (1, 0) and (1, 1) are (a_1, a_2), (b_1, b_2), (c_1, c_2) and (d_1, d_2) where no three of which are collinear. The barycentric coordinates of these points are:

$$(0, 0) = (1, 0, 0) \qquad (1, 0) = (0, 1, 0) \qquad (0, 1) = (0, 0, 1)$$

$$(a_1, a_2) = (a_0, a_1, a_2) \quad \text{with} \quad a_0 = 1 - a_1 - a_2 \qquad \text{etc.}$$

The substitution into the matrix equality produces a system of three equations with three unknowns k, l and m:

$$\begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} k\,a_0 & l\,b_0 & m\,c_0 \\ k\,a_1 & l\,b_1 & m\,c_1 \\ k\,a_2 & l\,b_2 & m\,c_2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Since the points (0, 0), (1, 0) and (0, 1) are not collinear, the matrix is regular and the system has a unique solution, which proves that a projectivity is determined by the image of the base quadrilateral. If the vertices of the initial quadrilateral are not the base points, the matrix is calculated as the product of the inverse matrix of the projectivity which maps the base quadrilateral to the initial quadrilateral multiplied by the matrix of the projectivity which maps the point base to the final quadrilateral.

Let us calculate the projectivity which maps the quadrilateral with vertices (1, 1)-(2, 3)-(3, 1)-(1, 0) to the quadrilateral (4, 1)-(6, 0)-(4, 0)-(6, 1). The projectivity which

[5] Of course, the affinity is a special kind of projectivity.

maps the base quadrilateral (0, 0)-(1, 0)-(0, 1)-(1/3, 1/3) to the first quadrilateral is given by:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -k & -4\,l & -3\,m \\ k & 2\,l & 3\,m \\ k & 3\,l & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

whose solution is: $k = \dfrac{5}{4} \qquad l = -\dfrac{1}{2} \qquad m = \dfrac{1}{4}$

Taking the values $k = 5$, $l = -2$ $m = 1$, we find the matrix:

$$\begin{pmatrix} -5 & 8 & -3 \\ 5 & -4 & 3 \\ 5 & -6 & 1 \end{pmatrix} \text{ whose inverse matrix is } \frac{1}{20}\begin{pmatrix} 7 & 5 & 6 \\ 5 & 5 & 0 \\ -5 & 5 & -10 \end{pmatrix}$$

The projectivity which maps the base quadrilateral to the final quadrilateral is:

$$\begin{pmatrix} -6 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -4k & -5\,l & -3\,m \\ 4k & 6\,l & 4\,m \\ k & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

whose solution is: $k = 1$, $l = 1$, $m = -1$ and the square matrix:

$$\begin{pmatrix} -4 & -5 & 3 \\ 4 & 6 & -4 \\ 1 & 0 & 0 \end{pmatrix}$$

Then, the projectivity which maps the initial quadrilateral to the final one is the product:

$$\begin{pmatrix} -4 & -5 & 3 \\ 4 & 6 & -4 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 7 & 5 & 6 \\ 5 & 5 & 0 \\ -5 & 5 & -10 \end{pmatrix} = \begin{pmatrix} -68 & -30 & -54 \\ 78 & 30 & 64 \\ 7 & 5 & 6 \end{pmatrix}$$

For example, the point (2, 5/3) is transformed into:

$$\begin{pmatrix} -68 & -30 & -54 \\ 78 & 30 & 64 \\ 7 & 5 & 6 \end{pmatrix} \begin{pmatrix} -8 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 94 \\ -124 \\ 4 \end{pmatrix} = \left(\frac{62}{13}, -\frac{2}{13} \right)$$

A projectivity may be also defined in an alternative way: let *ABCD* be a quadrilateral and *A'B'C'D'* another quadrilateral. Then the mapping:

$$D = a\,A + b\,B + c\,C \qquad \rightarrow \qquad D' = a'\,A' + b'\,B' + c'\,C'$$

is a projectivity if the coefficients *a', b', c'* are homogeneous linear functions of *a*, *b*, *c*. For example in the foregoing example the quadrilateral (1, 1)-(2, 3)-(3, 1)-(1, 0) is mapped to the quadrilateral (4, 1)-(6, 0)-(4, 0)-(6, 1). Then we have:

$$(1,0) = (1-b-c)\,(1,1) + b\,(2,3) + c\,(3,1) \quad \Rightarrow \quad b = -\frac{1}{2} \quad c = \frac{1}{4} \quad a = 1-b-c = \frac{5}{4}$$

$$(6,1) = (1-b'-c')\,(4,1) + b'\,(6,0) + c'\,(4,0) \;\Rightarrow\; b' = 1 \quad c' = -1 \quad a' = 1-b'-c' = 1$$

Taking into account that for $a = 1$, $b = 0$, $c = 0 \rightarrow a' = 1$, $b' = 0$, $c' = 0$ and analogously for each of the first three points, the linear mapping of coefficients is a diagonal matrix which can be written simply:

$$a' = \frac{4}{5}a \qquad b' = -2\,b \qquad c' = -4\,c$$

If we wish to calculate the transformed point of (2, 5/3) we calculate firstly *a*, *b* and *c*:

$$P = \left(2, \frac{5}{3}\right) = (1-b-c)(1,1) + b\,(2,3) + c\,(3,1) \qquad \Rightarrow \qquad b = \frac{1}{3} \quad c = \frac{1}{3} \quad a = \frac{1}{3}$$

$$\Rightarrow \qquad a' = \frac{4}{15} \qquad b' = -\frac{2}{3} \qquad c' = -\frac{4}{3}$$

Normalising the coefficients, we find the transformed point *P'*:

$$P' = -\frac{4}{26}(4,1) + \frac{10}{26}(6,0) + \frac{20}{26}(4,0) = \left(\frac{62}{13}, -\frac{2}{13}\right)$$

This example shows the normalisation. In general, we say that the following transformation is a projectivity:

$$D = (1-x-y)\,O + x\,P + y\,Q \quad \rightarrow \quad D' = (1-x'-y')\,O' + x'\,P' + y'\,Q'$$

$$x' = \frac{l\,x}{k(1-x-y) + l\,x + m\,y} \qquad y' = \frac{m\,y}{k\,(1-x-y) + l\,x + m\,y}$$

Let us prove that a projectivity defined in this algebraic way leaves invariant the projective cross ratio (as evident by construction from perspectivities). Under a projectivity, four collinear points *A*, *B*, *C* and *D* are transformed into another four collinear points *A'*, *B'*, *C'* and *D'* . Their cross ratio is the quotient of differences of any coordinate (it is not needed that *x* and *y* are Cartesian coordinates) because these differences for *x* are proportional to the differences for *y* due to the alignment:

$$x'_C - x'_A = \frac{l\left[k\left(x_C - x_A\right) + \left(k-m\right)\left(x_A y_C - x_C y_A\right)\right]}{\left[k\left(1 - x_C - y_C\right) + l\,x_C + m\,y_C\right]\left[k\left(1 - x_A - y_A\right) + l\,x_A + m\,y_A\right]}$$

Let us suppose that the collinear points A, B, C and D lie on the line having the equation $y = s\,x + t$, where the constants s and t play the role of a "slope" and "ordinate intercept" although the x-axis and the y-axis be not orthogonal:

$$x'_C - x'_A = \frac{l\left(x_C - x_A\right)\left[k - \left(k-m\right)t\right]}{\left[k\left(1 - x_C - y_C\right) + l\,x_C + m\,y_C\right]\left[k\left(1 - x_A - y_A\right) + l\,x_A + m\,y_A\right]}$$

Now we see that a projectivity changes every single ratio due to the denominators, but the cross ratio is preserved because these denominators are cancelled in this case:

$$\left(A'\,B'\,C'\,D'\right) = \frac{\left(x_C' - x_A'\right)\,\left(x_D' - x_B'\right)}{\left(x_D' - x_A'\right)\,\left(x_C' - x_B'\right)} = \frac{\left(x_C - x_A\right)\,\left(x_D - x_B\right)}{\left(x_D - x_A\right)\,\left(x_C - x_B\right)} = \left(A\,B\,C\,D\right)$$

The projectivity as a tool for theorems demonstration

One of the most interesting applications of the projectivity is the demonstration of theorems of projective nature owing to the fact that the projectivity preserves alignment and incidence. For example, in the figure 10.3 I have displayed without proof the method of construction of the fourth point which completes a harmonic range from the other three. Now let us apply to this figure a projectivity which maps the point A to the infinity. In this case the lines $ABCD$, R and S become parallel (figure 10.7) and the cross ratio becomes a single ratio:

$$(A\,B\,C\,D) = (\infty\,B'\,C'\,D') = \frac{B'D'}{B'C'} = 2$$

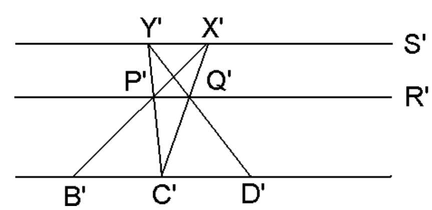
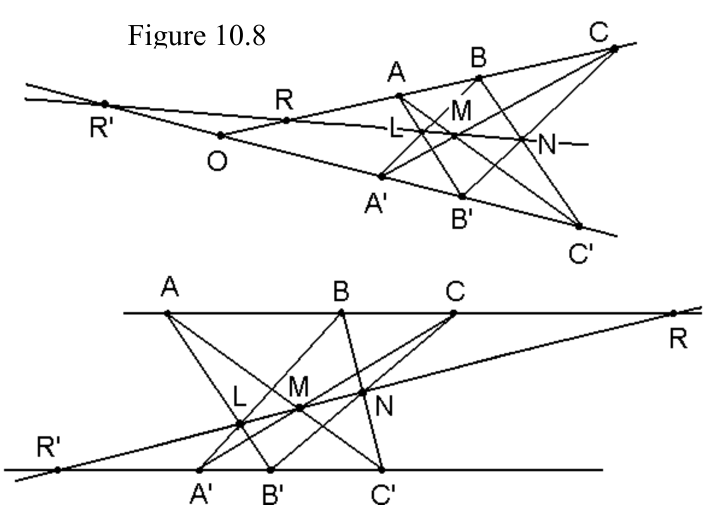


Figure 10.7

By similarity of triangles one sees that $B'D' = 2\,B'C'$, so that the cross ratio is 2 and proves that the points A, B, C and D in the figure 10.3 are a harmonic range.

Another example is the proof of the Pappus' theorem: the lines joining points with distinct letters belonging to two distinct ranges ABC and $A'B'C'$ intersect in three collinear points L, M and N (figure 10.8). Applying a perspectivity to make both ranges parallel, now we see that the triangles ABL and $A'B'L$ are

Figure 10.8



similar so that:

$$L = \frac{|A'B'|}{|AB| + |A'B'|} A + \frac{|AB|}{|AB| + |A'B'|} B' \quad \text{and also} \quad L = \frac{|A'B'|}{|AB| + |A'B'|} B + \frac{|AB|}{|AB| + |A'B'|} A'$$

We may gather both equations obtaining an expression more adequate to our purposes which is function of the midpoints:

$$L = \frac{|A'B'|}{|AB| + |A'B'|} \frac{A + B}{2} + \frac{|AB|}{|AB| + |A'B'|} \frac{A' + B'}{2}$$

Analogously for M and N we obtain:

$$M = \frac{|A'C'|}{|AC| + |A'C'|} \frac{A + C}{2} + \frac{|AC|}{|AC| + |A'C'|} \frac{A' + C'}{2}$$

$$N = \frac{|B'C'|}{|BC| + |B'C'|} \frac{B + C}{2} + \frac{|BC|}{|BC| + |B'C'|} \frac{B' + C'}{2}$$

These expressions fulfil the following equality what implies that the three points are aligned:

$$M = \frac{|AB| + |A'B'|}{|AC| + |A'C'|} L + \frac{|BC| + |B'C'|}{|AC| + |A'C'|} N$$

To prove this, begin from

$$\left(|AC| + |A'C'|\right) M = \left(|AB| + |A'B'|\right) L + \left(|BC| + |B'C'|\right) N \qquad \Rightarrow$$

$$|A'C'| \left(A + C\right) + |AC| \left(A' + C'\right) = |A'B'| \left(A + B\right) + |AB| \left(A' + B'\right) + |B'C'| \left(B + C\right) + |BC| \left(B' + C'\right)$$

Arranging A, B and C on the left hand side and A', B' and C' on the right hand side and taking into account that $|AC| = |AB| + |BC|$ and $|A'C'| = |A'B'| + |B'C'|$ we obtain the following vector equality:

$$-|B'C'| \, AB + |A'B'| \, BC = -|AB| \, B'C' + |BC| \, A'B'$$

which is an identity taking into account that the lines ABC and $A'B'C'$ are parallel and all vectors have the same direction and sense.

The linear relation between L, M and N is very useful to calculate the cross ratio ($LMNR$). In fact the coefficients of the midpoints are the relative altitudes of the point with respect to both parallel lines. Since this cross ratio is also the quotient of the relative altitudes, we have:

$$(LMNR)=\frac{\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'B'|}{|AB|+|A'B'|}\right)\dfrac{-|AC|}{|AC|+|A'C'|}}{\dfrac{-|AB|}{|AB|+|A'B'|}\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'C'|}{|AC|+|A'C'|}\right)}=\frac{|AC|}{|AB|}=(OABC)$$

The single ratio is the cross ratio corresponding to $O=\infty$. This identity of cross ratios is preserved under a projectivity and it is also valid for the upper scheme on the figure 10.8 when O is a point with finite coordinates. Analogously:

$$(LMNR')=\frac{\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'B'|}{|AB|+|A'B'|}\right)\dfrac{|A'C'|}{|AC|+|A'C'|}}{\dfrac{|A'B'|}{|AB|+|A'B'|}\left(\dfrac{|B'C'|}{|BC|+|B'C'|}-\dfrac{|A'C'|}{|AC|+|A'C'|}\right)}=\frac{|A'C'|}{|A'B'|}=(OA'B'C')$$

The homology

The points A', B' are *homologous* of the points A and B with respect to the *centre* O and the *axis of homology* given by the point F and the vector v (figure 10.9) if:

1) The centre O does not lie on the axis.

2) Every pair of homologous points and the homology centre are aligned, that is, O, A and A' are aligned and also O, B and B'.

3) Every pair of homologous lines AB and $A'B'$ intersects in a point on the homology axis.

A homology[6] is determined by an axis, a centre O and a pair of homologous points A and A'. To obtain the homologous of any point B, we draw the line AB, whose intersection with the homology axis will be denoted as Z. Then we draw the line $A'Z$ and the intersection with the line OB is the homologous point B'. By construction, the homology axis and the lines AB, $A'B'$ and OZ are concurrent in Z. The projective cross ratio of this pencil of lines is independent of the crossing line where it is measured. Let G and H be the intersections of the homology axis with the lines OA and OB respectively. Then we have:

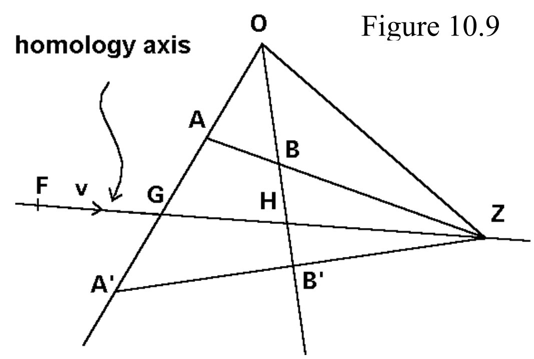


Figure 10.9

$$\{ Z, O\,A\,G\,A' \} = \{ Z, O\,B\,H\,B' \}$$

[6] The word *homology* is used by H. Eves (*A Survey of Geometry,* p. 105) with a different meaning. He names as *homology* a composite homothety, that is, a single homothety followed by a rotation.

Since the four points of each set are collinear, their projective cross ratio is identical to their complex cross ratio:

$$(\,O\,A\,G\,A'\,) = (\,O\,B\,H\,B'\,)$$

That is, the cross ratio of the four points O, B, H and B' is a real constant r for any point B and is called *homology ratio*. Hence we can write:

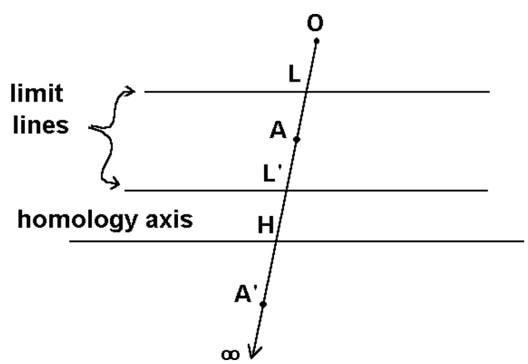
$$(\,O\,B\,H\,B'\,) = OH\,OB'^{-1}\,BB'\,BH^{-1} = r \neq 1,\,0$$

This equation leads to a simpler definition of the homology: B' is the homologous point of B with respect a given axis and centre O (outside the axis) with ratio r if O, B and B' are aligned and the cross ratio ($O\,B\,H\,B'$) is equal to r, being H the intersection point of the line OB with the homology axis. Hence, a homology is determined by giving a centre O, an axis and the homology ratio. If the homology ratio is the unity we have two degenerate cases: whether B lies on O and B' is any point, or B' lies on H (on the homology axis) and B is any point, that is, there is not one to one mapping, a not useful transformation. Then we must impose a ratio distinct of the unity. On the other hand, a null homology ratio implies that $BB' = 0$ and the homology becomes the identity.

In the special case $r = 2$ (when the points A and A' are harmonic conjugates with respect O and H) the transformation is called a *harmonic homology*.

The *first limit line* is defined as the set of points L on the plane having as homologous points those at infinity (figure 10.10):

Figure 10.10



$$(\,O\,L\,H\,\infty\,) = OH\,O\infty^{-1}\,L\infty\,LH^{-1} = r$$

The limit of the quotient of both distances tending to the infinity is the unity:

$$O\infty^{-1}\,L\infty = \lim_{X \to \infty} OX^{-1}\,LX = 1$$

Then: $\quad OH\,LH^{-1} = r \quad \Rightarrow \quad LH\,OH^{-1} = r^{-1}$

Because all the points H lie on the homology axis, all the points L form a line, the limit line.

The *second limit line* is defined as the points that are homologous of those at infinity:

$$(\,O\,\infty\,H\,L'\,) = OH\,OL'^{-1}\,\infty L'\,\infty H^{-1} = r$$

$$OH\,OL'^{-1} = r \qquad \Rightarrow \qquad OL'\,OH^{-1} = \mathrm{r}^{-1}$$

Therefore the distance from the second limit line to the homology axis is equal to the distance from the first limit line to the homology centre (figure 10.10), and both lines are located whether between or outside both elements according to the value of *r*:

$$OL' = LH = r^{-1}\,OH$$

Let us calculate the homologous point B' for every point B on the plane. Since H is a changing point on the homology axis, we write it as a function of a fixed point F also on the axis and its direction vector v (figure 10.9). At one time, the point H is the intersection of the line OB and the homology axis:

$$H = O + b\,OB = F + k\,v \qquad\qquad k,\,b \text{ real}$$

$$b\,OB - k\,v = OF \qquad \Rightarrow \qquad b = OF \wedge v\,(\,OB \wedge v\,)^{-1}$$

$$H = O + OF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB$$

Now we may calculate the segments OH and BH:

$$OH = OF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB$$

$$BH = BO + OF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB = [\,OF \wedge v\,(\,OB \wedge v\,)^{-1} - 1\,]\,OB =$$

$$= (\,OF \wedge v - OB \wedge v\,)\,(\,OB \wedge v\,)^{-1}\,OB = BF \wedge v\,(\,OB \wedge v\,)^{-1}\,OB$$

By substitution in the cross ratio we obtain:

$$OH\,OB'^{-1}\,BB'\,BH^{-1} = OH\,BH^{-1}\,BB'\,OB'^{-1} = OF \wedge v\,(\,BF \wedge v\,)^{-1}\,BB'\,OB'^{-1} = r$$

$$BB'\,OB'^{-1} = r\,v \wedge BF\,(\,v \wedge OF\,)^{-1}$$

and isolating OB' as a function of OB and other known data:

$$(\,-OB + OB'\,)\,OB'^{-1} = r\,v \wedge BF\,(\,v \wedge OF\,)^{-1} - OB\,OB'^{-1} = r\,v \wedge BF\,(\,v \wedge OF\,)^{-1} - 1$$

$$OB'^{-1} = OB^{-1}\,[\,1 - r\,v \wedge BF\,(\,v \wedge OF\,)^{-1}\,] = OB^{-1}\,(\,v \wedge OF - r\,v \wedge BF\,)\,(\,v \wedge OF\,)^{-1}$$

Inverting the vectors we find:

$$OB' = OB\,v \wedge OF\,(\,v \wedge OF - r\,v \wedge BF\,)^{-1} = OB\,(\,1 - r\,v \wedge BF\,(\,v \wedge OF\,)^{-1}\,)^{-1}$$

This equation allows to calculate the homologous point B' of any point B using the data of the homology: the centre O, the ratio r, and a direction vector v and a point F of the axis.

The homology preserves the projective cross ratio of any set of points. Let A', B', C', D' and E' be the transformed points of A, B, C, D and E under the homology. Then:

$$\{ E, A\, B\, C\, D \} = \{ E', A'\, B'\, C'\, D' \}$$

Let us prove this statement. Being A' and E' homologous points of A and E we have:

$$OA' = OA\, (\, 1 - r\, v \wedge AF\, (\, v \wedge OF\,)^{-1}\,)^{-1}$$

$$OE' = OE\, (\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,)^{-1}$$

Thus the segment $E'A'$ is:

$$E'A' = OA' - OE'$$

$$E'A' = OA\, (\, 1 - r\, v \wedge AF\, (\, v \wedge OF\,)^{-1}\,)^{-1} - OE\, (\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,)^{-1} =$$

$$= [\, OA\, (\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,) - OE\, (\, 1 - r\, v \wedge AF\, (\, v \wedge OF\,)^{-1}\,)\,]$$

$$[\, 1 - r\, v \wedge AF\, (\, v \wedge OF\,)^{-1}\,]^{-1}\, [\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,]^{-1} =$$

$$= [\, EA + r\, (\, -OA\, v \wedge EF + OE\, v \wedge AF\,)\, (\, v \wedge OF\,)^{-1}\,]$$

$$[\, 1 - r\, v \wedge AF\, (\, v \wedge OF\,)^{-1}\,]^{-1}\, [\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,]^{-1} =$$

$$= [\, EA\, (\, 1 - r\,) + r\, (\, OA\, v \wedge OE - OE\, v \wedge OA\,)\, (\, v \wedge OF\,)^{-1}\,]$$

$$[\, 1 - r\, v \wedge AF\, (\, v \wedge OF\,)^{-1}\,]^{-1}\, [\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,]^{-1} =$$

$$= [\, EA\, (\, 1 - r\,) + r\, v\, OA \wedge OE\, (\, v \wedge OF\,)^{-1}\,]$$

$$[\, 1 - r\, v \wedge AF\, (\, v \wedge OF\,)^{-1}\,]^{-1}\, [\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,]^{-1}$$

For the other vectors analogous expressions are obtained:

$$E'B' = [\, EB\, (\, 1 - r\,) + r\, v\, OB \wedge OE\, (\, v \wedge OF\,)^{-1}\,]$$

$$[\, 1 - r\, v \wedge BF\, (\, v \wedge OF\,)^{-1}\,]^{-1}\, [\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,]^{-1}$$

$$E'C' = [\, EC\, (\, 1 - r\,) + r\, v\, OC \wedge OE\, (\, v \wedge OF\,)^{-1}\,]$$

$$[\, 1 - r\, v \wedge CF\, (\, v \wedge OF\,)^{-1}\,]^{-1}\, [\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,]^{-1}$$

$$E'D' = [\, ED\, (\, 1 - r\,) + r\, v\, OD \wedge OE\, (\, v \wedge OF\,)^{-1}\,]$$

$$[\, 1 - r\, v \wedge DF\, (\, v \wedge OF\,)^{-1}\,]^{-1}\, [\, 1 - r\, v \wedge EF\, (\, v \wedge OF\,)^{-1}\,]^{-1}$$

Let us calculate the outer product $E'A' \wedge E'C'$:

$$E'A' \wedge E'C' = [\, EA \wedge EC\,(\,1 - \mathrm{r}\,)^2 + (\, EA \wedge v\, OC \wedge OE + v \wedge EC\, OA \wedge OE\,)\, r\,(\,1 - r\,)$$

$$(\, v \wedge OF\,)^{-1}\,]\,[\,1 - r\, v \wedge AF\,(\, v \wedge OF\,)^{-1}\,]^{-1}\,[\,1 - r\, v \wedge CF\,(\, v \wedge OF\,)^{-1}\,]^{-1}$$

$$[\,1 - r\, v \wedge EF\,(\, v \wedge OF\,)^{-1}\,]^{-2}$$

By substitution of the following identity, which may be proved by means of the geometric algebra (see exercise 1.4):

$$EA \wedge v\, OC \wedge OE + v \wedge EC\, OA \wedge OE = EA \wedge v\, EC \wedge OE + v \wedge EC\, EA \wedge OE =$$

$$EA \wedge EC \;\; v \wedge OE$$

$$E'A' \wedge E'C' = [\, EA \wedge EC\,(\,1 - r\,)^2 + EA \wedge EC\, v \wedge OE\, r\,(\,1 - r\,)\,(\, v \wedge OF\,)^{-1}\,]$$

$$[1 - r\, v \wedge AF\,(v \wedge OF\,)^{-1}\,]^{-1}\,[1 - r\, v \wedge CF\,(\, v \wedge OF\,)^{-1}\,]^{-1}\,[1 - r\, v \wedge EF\,(\, v \wedge OF\,)^{-1}\,]^{-2}$$

Extracting common factor:

$$E'A' \wedge E'C' = EA \wedge EC\,[\,(\,1 - r\,)^2 + v \wedge OE\, r\,(\,1 - r\,)\,(\, v \wedge OF\,)^{-1}\,]$$

$$[1 - r\, v \wedge AF\,(v \wedge OF\,)^{-1}\,]^{-1}\,[1 - r\, v \wedge CF\,(\, v \wedge OF\,)^{-1}\,]^{-1}\,[1 - r\, v \wedge EF\,(\, v \wedge OF\,)^{-1}\,]^{-2}$$

In the projective cross ratio, all the factors are simplified except the first outer product:

$$\frac{E'A' \wedge E'C'\; E'B' \wedge E'D'}{E'A' \wedge E'D'\; E'B' \wedge E'C'} = \frac{EA \wedge EC\; EB \wedge ED}{EA \wedge ED\; EB \wedge EC}$$

This proves that the projective cross ratio of any four points *A*, *B*, *C* and *D* with respect to a centre of projection *E* is equal to the projective cross ratio of the homologous points *A'*, *B'*, *C'* and *D'* with respect to the homologous centre of projection *E'*. It means that the homology is a special kind of projectivity where a line is preserved, the axis of homology. When the centre of projection is a point *H* on the homology axis, *H* and *H'* are coincident and we find in the former equality the initial condition of the homology again:

$$H = H' \qquad \{\, H, A'\, B'\, C'\, D'\,\} = \{\, H, A\; B\; C\; D\,\}$$

The following chapter is devoted to the conics and the Chasles' theorem, which states that the locus of the points from where the projective cross ratio of any four points is constant is a conic passing through these four points. Since the homology preserves the projective cross ratio, it transform conics into conics. Hence, a conic may be drawn as the homologous curve of a circle. Depending on the position of this circle there are three cases:

1) The circle does not cut the limit line *L* (whose homologous is the line at infinity). In this case the homologous curve is an ellipse, because very next points on the circle have also very next homologous points and therefore the homologous curve must be closed.

2) The circle touches the limit line *L*. Then its homologous curve is the parabola, because the homologous of the contact point is a point at infinity. The symmetry axis of

the parabola has the direction of the line passing through the centre of homology and the contact point.

3) The circle cuts the limit line L. Then the homologous curve is the hyperbola because the homologous points of both intersections are two points at infinity. The lines passing through the centre of homology and both intersection points have the direction of the asymptotes.

Exercises

10.1 Prove the Ptolemy's theorem: For a quadrilateral inscribed in a circle, the product of the lengths of both diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.

10.2 Find that the point D forming a harmonic range with A, B and C is given by the equation:

$$D = A + AB\,(\,1 - 2\,AC^{-1}\,BC\,)^{-1}$$

10.3 Show that the homography is a directly conformal transformation, that is, it preserves angles and their orientations.

10.4 Prove that if A', B', C' and D' are the homologous of A, B, C and D, and H is a point on the homology axis, then the equality $\{\,H, A'\,B'\,C'\,D'\,\} = \{\,H, A\;B\;C\;D\,\}$ holds. That is, show directly that:

$$\frac{HA' \wedge HC'\;HB' \wedge HD'}{HA' \wedge HD'\;HB' \wedge HC'} = \frac{HA \wedge HC\;HB \wedge HD}{HA \wedge HD\;HB \wedge HC}$$

10.5 A *special conformal transformation* of centre O and translation v is defined as the geometric operation which, given any point P, consists in the inversion of the vector OP, the addition of the translation v and the inversion of the resulting vector again.

$$OP'^{-1} = (\,OP^{-1} + v\,)^{-1}$$

Prove the following properties of this transformation:

a) It is an additive operation with respect to the translations: the result of applying firstly a transformation with centre O and translation v, and later a transformation with the same centre and translation w is identical to the result of applying a unique transformation with translation $v + w$.
b) It preserves the complex cross ratio and thus it is a special case of homography.

10.6 Prove that if a homography keeps invariant three or more points on the plane, then it is the identity.

10.7 An *antigraphy* is defined as that transformation which conjugates the complex cross ratio of any four points:

$$(A\ B\ C\ D) = (A'\ B'\ C'\ D')^*$$

Then an antigraphy is completely specified by giving the images A', B', C' of any three points A, B, C. Prove that:

a) The antigraphy is an opposite conformal transformation, that is, it changes the orientation of the angles but preserving their absolute values.
b) The composition of two antigraphies is a homography.
c) A product of three inversions is an antigraphy.
d) If an antigraphy has three invariant points, then it is a circular inversion.

10.8 Let A, B, C and A', B', C' be two sets of independent points. A projectivity has been defined as the mapping:

$$D = a\,A + b\,B + c\,C \qquad \rightarrow \qquad D' = a'\,A' + b'\,B' + c'\,C'$$

where the coefficients a', b', c' are homogeneous linear functions of a, b, c. Prove that collinear points are mapped to collinear points.

10.9 Prove the following theorem: let the sides of a hexagon *ABCDEF* pass alternatively through the points P and Q. Then the lines *AD*, *BE* and *CF* joining opposite vertices meet in a unique point. Hint: draw the hexagon in the dual plane.

11. CONICS

Conic sections

The *conic sections* (or *conics*) are the intersections of a conic surface with any transversal plane (figure 11.1). The *proper conics* are the curves obtained with a plane not passing through the vertex of the cone. If the plane contains the vertex we have improper conics, which reduce to a pair of straight lines or a point. In general, I shall regard only proper conics, taking into account that the other case can be usually obtained as a limit case.

Let us consider the two spheres inscribed in the cone which are tangent to the

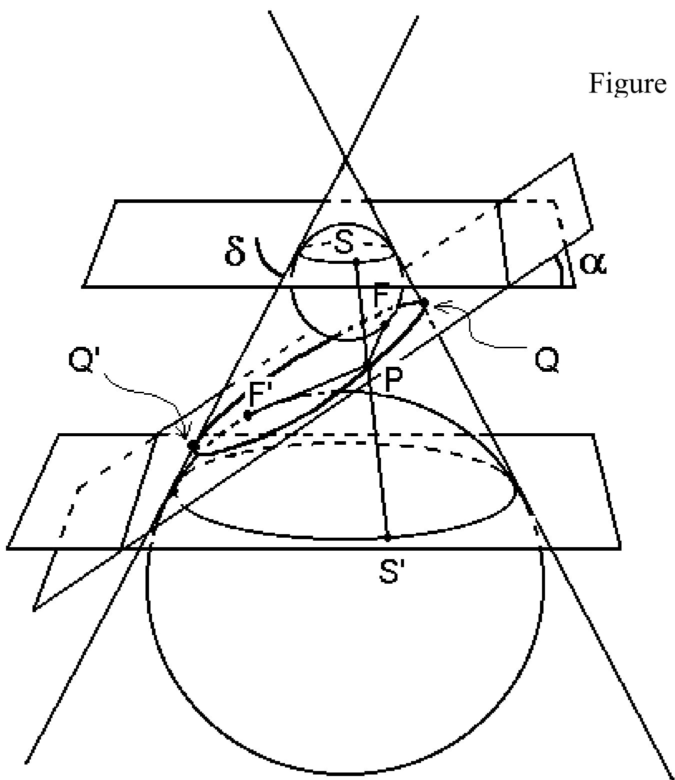


Figure 11.1

plane of the conic. The spheres touch this plane at the points F and F', the *foci* of the conic section. On the other hand, both spheres touch the cone surface at two tangency circles lying in planes perpendicular to the cone axis. The *directrices* of the conic are defined as the intersections of these planes with the plane of the conic.

Let P be a point on the conic. Since it lies on the plane of the conic, the segment going from P to the focus F is tangent to the sphere. Since it belongs to the cone surface, the generatrix passing through P is also tangent to the sphere. PF and PS are tangent to the same sphere so that their lengths are equal: $|PF|=|PS|$. Also PF' and PS' are tangent to the other sphere so that their lengths are also equal: $|PF'|=|PS'|$. On the other hand the segment SS' of any directrix has constant length, what implies that the addition of distances from any point P on the conic to both foci is constant:

$$|PF| + |PF'| = |PS| + |PS'| = |SS'| = \text{constant}$$

This discussion slightly changes for a plane which intercepts the upper and lower cones. In this case, the distances from P to the upper sphere and to the upper branch of the curve must be considered negative[1]. If the plane is parallel to a generatrix of the cone these distances become infinite. This unified point of view for all conics using negative or infinite distances when needed will be the guide of this chapter. I shall not separate equations but only specify cases for distinct conics.

A lateral view of figure 11.1 is given in the figure 11.2. In this figure the point P has been drawn in both extremes of the conic (and S and S' also twice), although we must consider any point on the conic section. As indicated previously $|PF| = |PS|$. But $|PS|$ is proportional to the distance from P to the upper plane containing S and this distance is also proportional to the distance from P to the directrix r (figure 11.2):

$$|PF| = |PS| = \frac{\sin\alpha}{\sin\delta}\,\mathrm{d}(P, r)$$

The quotient of the sines of both angles is called the *eccentricity* e of the conic:

$$\mathrm{e} = \frac{\sin\alpha}{\sin\delta} > 0$$

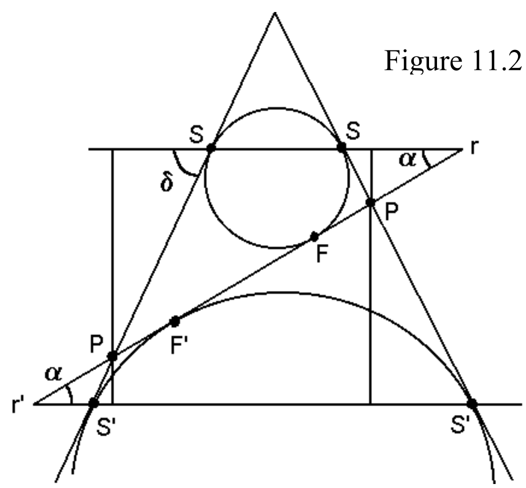


Figure 11.2

Then a conic can be also defined as the geometric locus of the points P whose distance from the focus F is proportional to the distance from the directrix r (figure 11.3).

$$|FP| = \mathrm{e}\,\mathrm{d}(P, r)$$

$|FP|$ is called the *focal radius*. Let T be the point of the directrix closest to the focus (figure 11.3). Then the *main axis of symmetry* of the conic is the line which is perpendicular to the directrix and passes through the focus. Under a reflection with respect the main axis of symmetry, the conic is preserved. The oriented distance from any point P to the directrix r is expressed, using the vector FT, as:

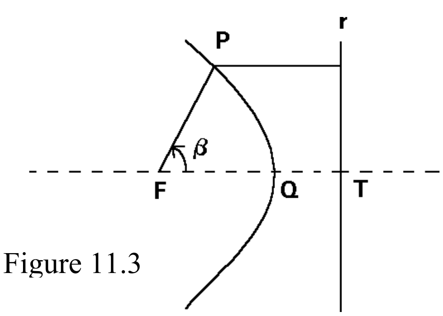


Figure 11.3

[1] To consider sensed distances is not a trouble but an advantage. For example, the distance from a point to a line can be also taken as an oriented distance, whose sign indicates the half-plane where the point lies.

$$\mathrm{d}(P,r)=\frac{FT\cdot PT}{|FT|}$$

Defined in this way, the distance from P to r is positive when P is placed on the half-plane that contains the focus of both in which the directrix divides the plane, and negative when P lies on the other half-plane. The equation of the conic becomes:

$$|FP|=\mathrm{e}\ \mathrm{d}(P,r)=\mathrm{e}\frac{FT\cdot PT}{|FT|}$$

$$|FP|\ |FT|=\mathrm{e}\ FT\cdot PT=\mathrm{e}\ FT\cdot(FT-FP)=\mathrm{e}\left(FT^2-FT\cdot FP\right)$$

Let us denote by β the angle between the main axis FT and the focal vector FP:

$$|FP|\ |FT|=\mathrm{e}\left(FT^2-|FT|\ |FP|\cos\beta\right)$$

$$|FP|\ |FT|\left(1+\mathrm{e}\ \cos\beta\right)=\mathrm{e}\ FT^2$$

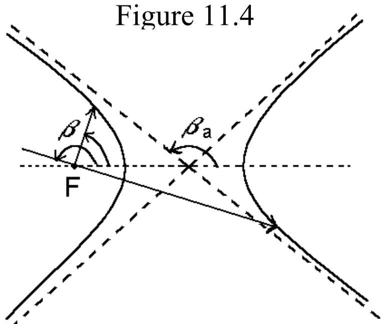
From here one obtains the *polar equation* of a conic:

$$|FP|=\frac{|FT|}{1+\mathrm{e}\ \cos\beta}$$

According to the value of the eccentricity there are three types of proper conics:

1) For 0 < e < 1, the inclination α of the plane of the conic is lower than the inclination δ of the cone generatrix and the denominator is always positive. Then the focal radius is always positive and the points P form a closed curve on the focal half-plane, an *ellipse*.
2) For e = 1, $\alpha = \delta$, the plane of the conic is parallel to a cone generatrix. Since the denominator vanishes for $\beta=\pi$, the points P form an open curve called *parabola*. Except for this value of β, the radius is always positive and the curve lies on the focal half-plane.
3) For e > 1, $\alpha>\delta$, the plane of the conic intercepts both upper and lower cones. The denominator vanishes twice so that it is an open curve with two branches called *hyperbola* (figure 11.4). The lowest positive angle that makes the denominator zero is the asymptote angle β_a :

Figure 11.4



$$\beta_a=\arccos\left(-\frac{1}{\mathrm{e}}\right)$$

The maximum eccentricity of the hyperbolas as cone sections is obtained with $\alpha=\pi/2$:

$$1 < \mathrm{e} \leq \frac{1}{\sin\delta}$$

The focal radius is positive (P lies on the focal branch) for the ranges of β:

$$0 < \beta < \beta_a \qquad \text{and} \qquad 2\pi - \beta_a < \beta < 2\pi$$

and negative (P lies on the non focal branch) for the range:

$$\beta_a < \beta < 2\pi - \beta_a$$

A negative focal radius means that we must take the sense opposite that determined by the angle (figure 11.4). For example, a radius –2 with an angle $7\pi/6$ is equivalent to a radius 2 with an angle of $\pi/6$. So the vertex T' of the non focal branch has an angle π (not zero as it would appear) and a negative focal radius.

Two foci and two directrices

Figure 11.5

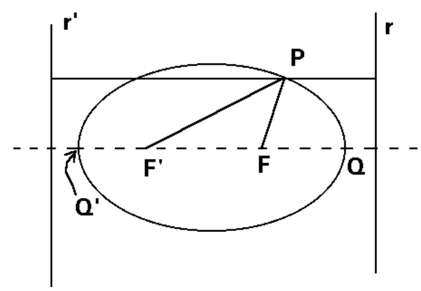
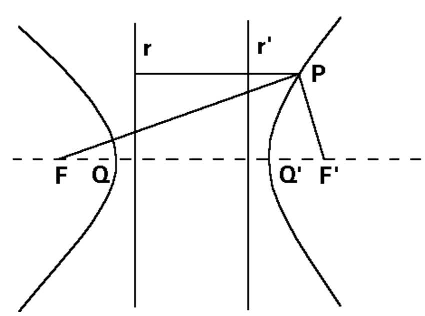


Figure 11.6



There are usually two spheres touching the cone surface and the conic plane, and hence there are also two foci. For the case of the ellipse, both tangent spheres are located in the same cone. For the case of the hyperbola, each sphere is placed in each cone so that each focus is placed next to each branch. In the case of the parabola, one sphere and the corresponding focus is placed at the infinity. For the improper conics, the tangent spheres have null radius and the foci are coincident with the vertex of the cone, which is the crossing point of both lines.

As the figure 11.1 shows for a proper conic, both directrices r and r' are parallel, and both foci F and F' are located on the axis of symmetry of the conic. Above we have already seen that the addition of oriented distances from both foci to any point P is constant. Let us see the relation with the distance between the directrices. If P is any point on the conic then:

$$|FP| + |F'P| = \mathrm{e}\,\mathrm{d}(P,r) + \mathrm{e}\,\mathrm{d}(P,r') = \mathrm{e}\,\mathrm{d}(r,r')$$

The sum of the oriented distances from any point P to any two parallel lines (the directrices in our case) is constant[2]. Then the addition of both focal radii of any point P

[2] Note that this statement is only right for oriented distances but not for positive distances.

on the conic is the product of the eccentricity multiplied by the oriented distance between both directrices. *P* can be any point, for example *Q*, the extreme of the conic:

$$|FP| + |F'P| = |FQ| + |F'Q| = |F'Q'| + |F'Q| = |QQ'|$$

whence it follows that the addition of both focal radii is $|QQ'|$, the *major diameter*. Specifying:

1) For the case of the ellipse (figure 11.5), both focal radii are positive and also the major diameter.
2) For the case of the parabola, one directrix is the line at the infinity and one focus is a point at the infinity on the main axis of symmetry. Hence the major diameter has an infinite value.
3) For the case of the hyperbola (figure 11.6), the focal radius of a point on the non focal branch is negative, its absolute value being higher than the other focal radius, which is positive, yielding a negative major diameter.

Vectorial equation

The polar equation of a conic may be written having as parameter the *focal distance* $|FQ|$ instead of the distance from the focus to the directrix $|FT|$:

$$|FP| = \frac{1+\mathrm{e}}{1+\mathrm{e}\ \cos\beta}|FQ|$$

Let Q' be the vertex of the conic closest to the focus F'. The distance from the focus F to Q' is found for $\beta = \pi$:

$$|FQ'| = \frac{1+\mathrm{e}}{1-\mathrm{e}}|FQ|$$

For the ellipse $|FQ'|$ is a positive distance; for the hyperbola it is a negative distance and for the parabola Q' is at infinity. Then the distance between both foci is:

$$|FF'| = |FQ'| - |F'Q'| = |FQ'| - |FQ| = \frac{2\ \mathrm{e}}{1-\mathrm{e}}|FQ|$$

On the other hand the major diameter is:

$$|QQ'| = |FQ| + |FQ'| = \frac{2|FQ|}{1-\mathrm{e}}$$

By dividing both equations, an alternative definition of the eccentricity is obtained:

$$\mathrm{e} = \frac{|FF'|}{|QQ'|}$$

The eccentricity is the ratio of the distance between both foci divided by the major diameter. For the case of ellipse, both distances are positive. For the case of the hyperbola, both distances are negative, so the eccentricity is always positive. When e = 0 both foci are coincident in the centre of a circle. Note that the circle is obtained when we cut the cone with a horizontal plane. In this case, the directrices are the line at the infinity.

The *vectorial equation* of a conic is obtained from the polar equation and contains the radius vector *FP*. Since *FP* forms with *FQ* an angle β (figure 11.3), *FP* is obtained from the unitary vector of *FQ* via multiplication by the exponential with argument β and by the modulus of *FP* yielding:

$$FP = \frac{1+\mathrm{e}}{1+\mathrm{e}\cos\beta} FQ\left(\cos\beta + e_{12}\sin\beta\right)$$

On the other hand, from the directrix property, one easily finds the following equation for a conic:

$$FP^2\, FT^2 = \mathrm{e}^2\,(\,FT^2 - FT\cdot FP\,)^2$$

F, *T* and e are parameters of the conic, and $P = (x, y)$ is the mobile point. Therefore from this equation we will also obtain a Cartesian equation of second degree. For example, let us calculate the Cartesian equation of an ellipse with eccentricity ½ and focus at the point (3, 4) and vertex at (4,5):

$$\mathrm{e} = 1/2 \qquad F = (\,3, 4\,) \qquad Q = (\,4, 5\,) \qquad P = (\,x, y\,)$$

$$FT = \frac{1+\mathrm{e}}{\mathrm{e}} FQ = 3e_1 + 3e_2 \qquad T = F + FT = (\,6, 7\,)$$

$$FP = (\,x-3\,)\,e_1 + (\,y-4\,)\,e_2$$

Then the equation of the conic is:

$$[\,(\,x-3\,)^2 + (\,y-4\,)^2\,]\;18 = \frac{1}{4}\,[\,18 - (\,3\,(x-3) + 3\,(\,y-4\,)\,)\,]^2$$

After simplification:

$$7\,x^2 - 2\,x\,y + 7\,y^2 - 22\,x - 38\,y + 31 = 0$$

The Chasles' theorem

According to this theorem[3], the projective cross ratio of any four given points *A*, *B*, *C* and *D* on a conic with respect to a point *X* also on this conic is constant independently of the choice of

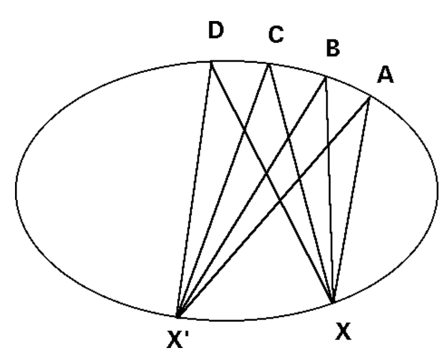


Figure 11.7

[3] Michel Chasles, *Traité des sections coniques*, Gauthier-Villars, Paris, 1865, p. 3.

the point X (figure 11.7):

$$\{ X, A\,B\,C\,D \} = \{ X', A\,B\,C\,D \}$$

To prove this theorem, let us take into account that the points A, B, C, D and X must fulfil the vectorial equation of the conic. Let us also suppose, without loss of generality, the main axis of symmetry having the direction e_1 (this supposition simplifies the calculations):

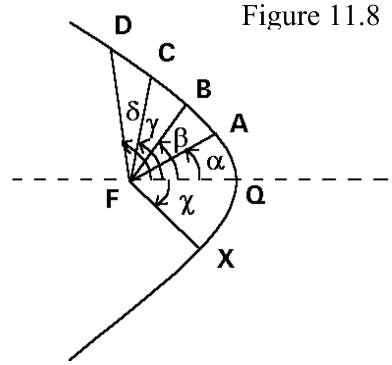


Figure 11.8

$$FQ = |FQ|\ e_1$$

Now on α, β, γ, δ and χ will be the angles which the focal radii FA, FB, FC, FD and FX form with the main axis with direction vector FQ (figure 11.8). Then:

$$XA = FA - FX = |FQ|\,(1+\mathrm{e})\left[\frac{e_1 \cos\alpha + e_2 \sin\alpha}{1+\mathrm{e}\cos\alpha} - \frac{e_1\cos\chi + e_2\sin\chi}{1+\mathrm{e}\cos\chi}\right]$$

Introducing a common denominator, we find:

$$XA = |FQ|\,\frac{(1+\mathrm{e})\left[e_1(\cos\alpha - \cos\chi) + e_2(\sin\alpha - \sin\chi + \mathrm{e}\sin\alpha\cos\chi - \mathrm{e}\cos\alpha\sin\chi)\right]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\chi)}$$

From XA and the analogous expression for XC, and after simplification we obtain:

$$XA \wedge XC = FQ^2\,\frac{e_{12}\,(1+\mathrm{e})^2(\sin\gamma\cos\alpha - \sin\alpha\cos\gamma + \sin\chi\cos\gamma - \sin\gamma\cos\chi + \sin\alpha\cos\chi - \sin\chi\cos\alpha)}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}$$

$$= FQ^2\,\frac{(1+\mathrm{e})^2\left[\sin(\gamma-\alpha) + \sin(\chi-\gamma) + \sin(\alpha - \chi)\right]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}\,e_{12}$$

Using the trigonometric identities of the half-angles, the sum is converted into a product of sines (exercise 6.2):

$$XA \wedge XC = -4\,FQ^2\,\frac{(1+\mathrm{e})^2\left[\sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\chi-\gamma}{2}\right)\sin\left(\frac{\alpha-\chi}{2}\right)\right]}{(1+\mathrm{e}\cos\alpha)(1+\mathrm{e}\cos\gamma)(1+\mathrm{e}\cos\chi)}\,e_{12}$$

In the same way the other outer products are obtained. The projective cross ratio is their quotient, where the factors containing the eccentricity or the angle χ are simplified:

$$\{X, A\,B\,C\,D\} = \frac{XA \wedge XC \;\; XB \wedge XD}{XA \wedge XD \;\; XB \wedge XC} = \frac{\sin\dfrac{\gamma-\alpha}{2}\,\sin\dfrac{\delta-\beta}{2}}{\sin\dfrac{\delta-\alpha}{2}\,\sin\dfrac{\gamma-\beta}{2}} = \frac{\sin\dfrac{AFC}{2}\,\sin\dfrac{BFD}{2}}{\sin\dfrac{AFD}{2}\,\sin\dfrac{BFC}{2}}$$

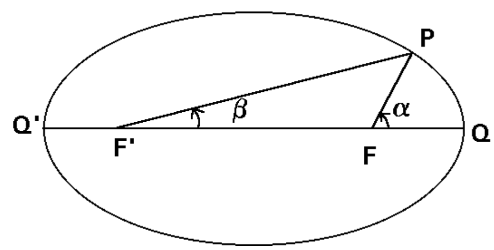
since $\gamma-\alpha$ is the angle *AFC*, etc. Therefore, the projective cross ratio of four points *A*, *B*, *C* and *D* on a conic is equal to the quotient of the sines of the focal half-angles, which do not depend on *X*, but only on the positions of *A*, *B*, *C* and *D*, fact which proves the Chasles' theorem. This statement is trivial for the case of the circumference, because the inscribed angles are the half of the central angles. However the inscribed angles on a conic vary with the position of the point *X* and they differ from the half focal angles. In spite of this, it is a notable result that the quotient of the sines of the inscribed angles (projective cross ratio) is equal to the quotient of the half focal angles. For the case of the hyperbola I remind you that the focal radius of a point on the non focal branch is oriented with the opposite sense and the focal angle is measured with respect this orientation.

Tangent and perpendicular to a conic

The vectorial equation of a conic with the major diameter oriented in the direction e_1 (figure 11.9) is:

$$FP = \frac{(1+\mathrm{e})\,|\,FQ\,|}{1+\mathrm{e}\cos\alpha}\left(e_1\cos\alpha + e_2\sin\alpha\right)$$

Figure 11.9



The derivation with respect the angle α gives:

$$\frac{d\,FP}{d\alpha} = \frac{(1+\mathrm{e})\,|\,FQ\,|}{(1+\mathrm{e}\;\cos\alpha)^2}\left[-\,e_1\sin\alpha + e_2\left(\mathrm{e}+\cos\alpha\right)\right]$$

This derivative has the direction of the line tangent to the conic at the point *P*, its unitary vector *t* being:

$$t = \frac{-\,e_1\sin\alpha + e_2\left(\mathrm{e}+\cos\alpha\right)}{\sqrt{1+\mathrm{e}^2+2\,\mathrm{e}\;\cos\alpha}}$$

The unitary normal vector *n* is orthogonal to the tangent vector:

$$n = \frac{e_1\left(\mathrm{e}+\cos\alpha\right) + e_2\sin\alpha}{\sqrt{1+\mathrm{e}^2+2\,\mathrm{e}\;\cos\alpha}}$$

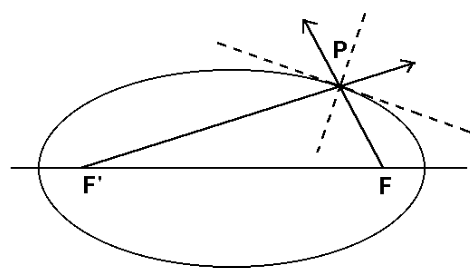
In the same way, *t* and *n* are obtained as functions of the angle β (figure 11.9) from the vectorial equation for the focus *F'*:

$$F'P = \frac{(1-e)\,|FQ|}{1 - e\cos\beta}\left(e_1 \cos\beta + e_2 \sin\beta\right)$$

$$t = \frac{-e_1 \sin\beta + e_2(-e + \cos\beta)}{\sqrt{1 + e^2 - 2e\cos\beta}}$$

$$n = \frac{e_1(-e + \cos\beta) + e_2 \sin\beta}{\sqrt{1 + e^2 - 2e\cos\beta}}$$

Figure 11.10

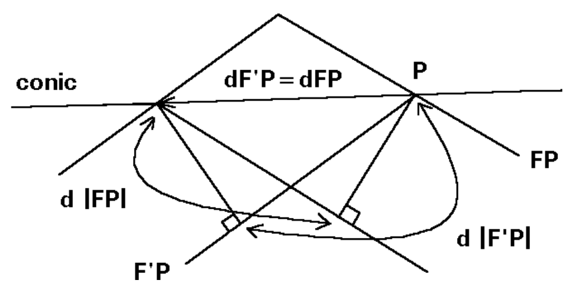


Let us see now that the normal vector has the direction of the bisector of the angle between both focal radii (figure 11.10). In order to prove this statement, let us consider an infinitesimal displacement of a point P on the conic (figure 11.11). The sum of the focal radii of any conic is constant, whence it follows that the addition of their differentials are null:

$$|FP| + |F'P| = \text{constant} \quad \Rightarrow \quad 0 = d|FP| + d|F'P|$$

The differential of the focal radius[4] is obtained by differentiating the square of the focal vector:

Figure 11.11



$$d\,FP^2 = 2\,FP \cdot d\,FP = 2\,|FP|\,d|FP|$$

$$d|FP| = -\frac{d\,FP \cdot FP}{|FP|}$$

Summing the differentials of both moduli we obtain:

$$0 = d|FP| + d|F'P| = -\frac{d\,FP \cdot FP}{|FP|} - \frac{d\,F'P \cdot F'P}{|F'P|} = -d\,FP \cdot \left(\frac{FP}{|FP|} + \frac{F'P}{|F'P|}\right)$$

since the differentials of the focal vectors are equal: $dP = d\,FP = d\,F'P$. In conclusion the differential vector (or the tangent vector) is orthogonal to the bisector of both focal vectors. The figure 11.11 clearly shows that both right triangles are opposite because they share the same hypotenuse $d\,FP$, and the legs $d|FP|$ and $d|F'P|$ have the same length. The reflection axis of both triangles, which is the shorter diagonal of the rhombus, is the bisector line of both focal vectors.

In the case of the parabola (figure 11.12), the focus F' is located at the infinity and $F'P$ is parallel to the main axis of symmetry. In the case of the hyperbola (figure 11.13) we must take into account that the radius of a point on the non focal branch is negative, so

[4] Note that it differs from the modulus of the differential of the focal vector: $d|FP| \neq |d\,FP|$.

it has the sense from P to the focus F'. The normal direction is the bisector of both oriented focal vectors.

These geometric features are the widely known optic reflection properties of the conics: parallel beams reflected by a parabola are concurrent at the focus. Also the beams emitted by a focus of an ellipse and reflected in its perimeter are concurrent in the other focus.

Figure 11.12

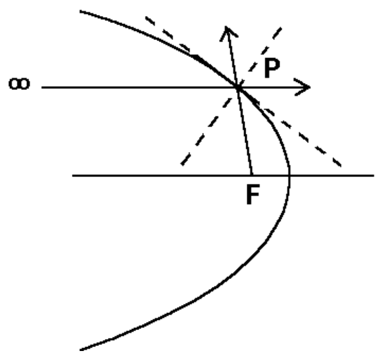
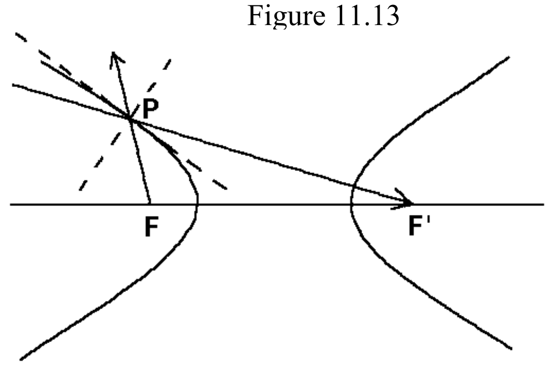


Figure 11.13



Central equations for the ellipse and hyperbola

We search a polar equation to describe a conic using its centre as the origin of the radius. Above we have seen that the eccentricity is the ratio of the distance between both foci and the major diameter:

$$\mathrm{e} = \frac{|FF'|}{|QQ'|}$$

If O is the centre of the conic then it is the midpoint of both foci:

$$O = \frac{F + F'}{2}$$

$$\mathrm{e} = \frac{|OF|}{|OQ|} \qquad 1 - \mathrm{e} = \frac{|FQ|}{|OQ|}$$

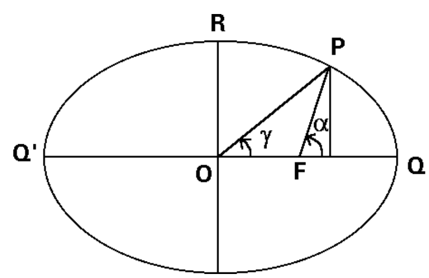


Figure 11.14

For the case of the ellipse all the quantities are positive. For the case of the hyperbola both $|OF|$ and $|OQ|$ are negative with $|FQ|$ positive. Let the angle QOP be γ (figure 11.14). Then:

$$|OP| \sin\gamma = |FP| \sin\alpha$$

$$|OP| \cos\gamma = |OF| + |FP| \cos\alpha$$

For the ellipse all the moduli are positive whereas for the hyperbola $|OP|$ and $|OF|$ are always negative (the centre is placed between both branches). For the parabola they become infinite. By substitution of the polar equation into these equalities we have:

$$|OP|\sin\gamma = |FQ|\frac{(1+\mathrm{e})\sin\alpha}{1+\mathrm{e}\cos\alpha} = |OQ|\frac{(1-\mathrm{e}^2)\sin\alpha}{1+\mathrm{e}\cos\alpha}$$

$$|OP|\cos\gamma = |OF| + \frac{|FQ|(1+\mathrm{e})\cos\alpha}{1+\mathrm{e}\cos\alpha} = |OQ|\left[\mathrm{e}+\frac{(1-\mathrm{e}^2)\cos\alpha}{1+\mathrm{e}\cos\alpha}\right] = |OQ|\frac{\mathrm{e}+\cos\alpha}{1+\mathrm{e}\cos\alpha}$$

Summing the squares of both former equalities we find:

$$OP^2\left(\frac{\sin^2\gamma}{1-\mathrm{e}^2}+\cos^2\gamma\right) = OQ^2 \qquad \Rightarrow \qquad |OP| = \frac{|OQ|}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2}+\cos^2\gamma}}$$

The direction of *OP* with respect to *OQ* is given by the angle γ and we may add it multiplying by the unitary complex with argument γ to obtain the central vectorial equation:

$$OP = \frac{OQ\,(\cos\gamma + e_{12}\sin\gamma)}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2}+\cos^2\gamma}}$$

Let *R* be the point *P* for $\gamma=\pi/2$. Then *OR* lies on the secondary axis of symmetry of the conic, which is perpendicular to *OQ*, the main axis of symmetry. $|OR|$ is the minor half-axis. The eccentricity relates both:

$$OR^2 = (1-\mathrm{e}^2)\,OQ^2$$

$$|OR| = \sqrt{1-\mathrm{e}^2}\,|OQ|$$

$$OR = \sqrt{1-\mathrm{e}^2}\,OQ\;e_{12}$$

Using the minor half-axis, the equation of the ellipse becomes:

$$OP = \frac{1}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2}+\cos^2\gamma}}\left(OQ\cos\gamma + OR\frac{\sin\gamma}{\sqrt{1-\mathrm{e}^2}}\right)$$

Hence it follows:

$$\frac{OP^2}{OR^2}\sin^2\gamma + \frac{OP^2}{OQ^2}\cos^2\gamma = 1$$

Taking OP and OQ as the coordinates axis and O the origin of coordinates, we have the canonical equation of a conic:

$$x = |OP| \cos \gamma \qquad\qquad y = |OP| \sin \gamma$$

$$\frac{x^2}{OQ^2} + \frac{y^2}{OR^2} = 1$$

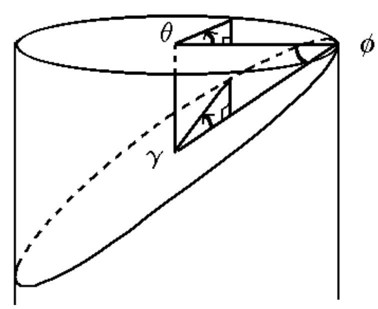
This equation of a conic is specific of this coordinate system and has a limited usefulness. If another system of coordinates is used (the general case) the equation is always of second degree but not so beautiful. For the ellipse both half-axis are positive real numbers. However for the hyperbola, the minor half-axis $|OQ|$ is imaginary and $OQ^2<0$ converting the sum of the squares in the former equation in a difference of real positive quantities.

Diameters and Apollonius' theorem

If e<1 (ellipse) we may introduce the angle θ in the following way:

$$\cos\theta = \frac{\cos\gamma}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2} + \cos^2\gamma}}$$

Figure 11.15



Then the central equation of an ellipse becomes[5]:

$$OP = OQ\cos\theta + OR\sin\theta$$

The angle θ has a direct geometric interpretation if we look at a section of a cylinder (figure 11.15):

$$\cos\theta = \frac{\cos\gamma\,\cos\phi}{\sqrt{\sin^2\gamma + \cos^2\gamma\cos^2\phi}}$$

From where it follows the relationship between the inclination of the cylindrical section and the eccentricity:

$$\cos\phi = \sqrt{1-\mathrm{e}^2}$$

---

[5] G. Peano (*Gli elementi di calcolo geometrico* [1891] in *Opere Scelte*, vol. III, Edizioni Cremone, [Roma, 1959], p.59) gives this central equation for the ellipse as function of the angle θ. He also used the parametric equations of the parabola, hyperbola, cycloid, epicycloid and the spiral of Archimedes.

Then θ is an angle between axial planes of the cylinder.

The equation of the ellipse may be written using another pair of axis turned by a cylindrical angle χ (figure 11.16):

$$OQ' = OQ \cos\chi + OR \sin\chi \qquad\qquad OR' = -OQ \sin\chi + OR \sin\chi$$

Each pair of OQ' and OR' obtained in this way are *conjugate central radii* and twice them are *conjugate diameters*. They are intersections of the plane of the ellipse with two axial planes of the cylinder that form a right angle. From the definition it is obvious that:

$$OQ'^2 + OR'^2 = OQ^2 + OR^2$$

The inverse relation between the conjugate radii are:

$$OQ = OQ' \cos\chi - OR' \sin\chi$$

$$OR = OQ' \sin\chi + OR' \cos\chi$$

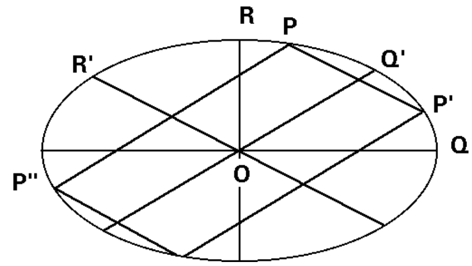


Figure 11.16

Then, the central radius of any point P is:

$$OP = (\, OQ' \cos\chi - OR' \sin\chi\,) \cos\theta + (\, OQ' \sin\chi + OR' \cos\chi\,) \sin\theta$$

$$= OQ' \cos(\theta - \chi) + OR' \sin(\theta - \chi)$$

Changing the sign of the cylindrical angle we obtain another point P' on the ellipse owing to the parity of trigonometric functions:

$$OP' = OQ' \cos(\theta - \chi) - OR' \sin(\theta - \chi) = OQ' \cos(\chi - \theta) + OR' \sin(\chi - \theta)$$

Then the chord PP' is parallel to the radius OR' (figure 11.16):

$$PP' = OP' - OP = -2\, OR' \sin(\theta - \chi)$$

Now, let us mention the Apollonius theorem: a diameter of a conic is formed by all the midpoints of the chords parallel to its conjugate diameter. To prove this statement, see that the central radius of the midpoint has the direction OQ':

$$\frac{OP + OP'}{2} = OQ' \cos(\theta - \chi)$$

The properties of the conjugate diameters are also applicable to the hyperbola, but taking into account that $OR^2 = (1 - e^2)\, OQ^2 < 0$, that is, the minor half-axis $|OR|$ is an imaginary number. In this case OR has the same direction than OQ:

$$OR = \sqrt{1-\mathrm{e}^2}\; OQ\; e_{12} = \sqrt{\mathrm{e}^2 - 1}\; OQ$$

And also the angle θ is an imaginary number, and the trigonometric functions are turned into hyperbolic functions of the real argument $\psi = \theta / e_{12}$:

$$\cos\theta = \frac{\cos\gamma}{\sqrt{\dfrac{\sin^2\gamma}{1-\mathrm{e}^2} + \cos^2\gamma}} = \cosh\psi \geq 1 \qquad \mathrm{e} > 1$$

Taking into account the relations between trigonometric and hyperbolic functions:

$$\cos\frac{\psi}{e_{12}} = \cosh\psi \qquad\qquad \sin\frac{\psi}{e_{12}} = \frac{\sinh\psi}{e_{12}}$$

the relation of the hyperbolic angle ψ with the Cartesian coordinates is:

$$\cosh\psi = \frac{x}{|\,OQ\,|} \qquad\qquad \sinh\psi = \frac{y}{|\,OR\,|}\, e_{12}$$

$$OP = \pm\left(OQ\cosh\psi + OR\; e_{12}\; \sinh\psi\right) = \pm\left(OQ\cosh\psi + \sqrt{e^2 - 1}\,\sinh\psi\; OQ\; e_{12}\right)$$

Let us define *OS*, which is taken usually as the minor half-axis of the hyperbola although this definition is not exact, as (figure 11.17):

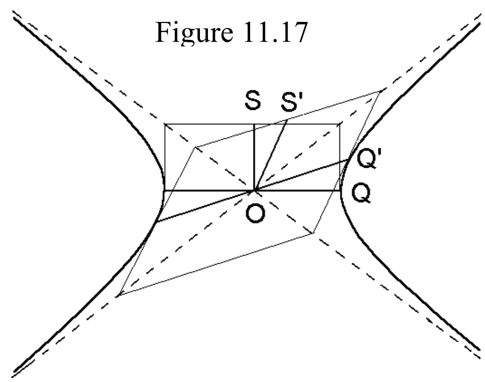
$$OS = \sqrt{\mathrm{e}^2 - 1}\; OQ\; e_{12}$$

Then the equation of the hyperbola is:

$$OP = OQ\cosh\psi + OS\,\sinh\psi$$

When the hyperbolic angle tends to infinity we find the asymptotes:

Figure 11.17



$$\psi \to \infty \qquad OP \to \pm\frac{OQ + OS}{2}\;\exp\psi$$

$$\psi \to -\infty \qquad OP \to \pm\frac{OQ - OS}{2}\;\exp(-\psi)$$

Two radii are conjugate if they are turned through the same hyperbolic angle φ (figure 11.17):

$$OQ' = OQ\cosh\varphi + OS\,\sinh\varphi$$

$$OS' = OQ \sinh\varphi + OS \cosh\varphi$$

Then the equality $OQ'^2 - OS'^2 = OQ^2 - OS^2$ holds and the central radius of any point of the hyperbola is:

$$OP = \pm\left(OQ' \cosh(\psi - \varphi) + OS' \sinh(\psi - \varphi)\right)$$

Also it is verified that the diameter is formed by the midpoints of the chords parallels to the conjugate diameter.

A complete understanding of the central equation of the hyperbola is found in the hyperbolic prism in a pseudo-Euclidean space. The equality $OQ'^2 - OS'^2 = OQ^2 - OS^2$ is the condition of hyperbolic prism of constant radius. Then the hyperbolas are planar section of this hyperbolic prism.

Conic passing through five points

Five non collinear points determine a conic. Since every conic has a Cartesian second degree equation:

$$a\,x^2 + b\,y^2 + c\,x\,y + d\,x + e\,y + f = 0$$

where there are five independent parameters, the substitution of the coordinates of five points leads to a linear system with five equation, with a hard solving. A briefer way to obtain the Cartesian equation of the conic passing through these points is through the Chasles' theorem. Let us see an example:

$A = (1, 1)$ $\quad B = (2, 3)$ $\quad C = (1, -1)$ $\quad D = (0, 0)$ $\quad E = (-1, 3)$

$EA = (2, -2)$ $\quad EB = (3, 0)$ $\quad EC = (2, -4)$ $\quad ED = (1, -3)$

The projective cross ratio of the four points A, B, C and D on the conic is:

$$r = \{E, A\,B\,C\,D\} = \frac{EA \wedge EC \;\; EB \wedge ED}{EA \wedge ED \;\; EB \wedge EC} = \frac{-4(-9)}{-4(-12)} = \frac{3}{4}$$

If the point $X = (x, y)$ then:

$XA = (1-x, 1-y)$ $\quad XB = (2-x, 3-y)$ $\quad XC = (1-x, -1-y)$ $\quad XD = (-x, -y)$

According to the Chasles' theorem, the projective cross ratio is constant for any point X on the conic:

$$\frac{3}{4} = \frac{XA \wedge XC \;\; XB \wedge XD}{XA \wedge XD \;\; XB \wedge XC} = \frac{(2x-2)(3x-2y)}{(x-y)(4x-y-5)} \quad \Rightarrow \quad 0 = 12x^2 - 3\,y^2 - x\,y - 9\,x + y$$

Another way closest to geometric algebra is as follows. Let us take A, B and C as a point base of the plane and express D and X in this base

$$D = d_A A + d_B B + d_C C \qquad \text{with} \qquad d_A + d_B + d_C = 1$$

$$X = x_A A + x_B B + x_C C \qquad \text{with} \qquad x_A + x_B + x_C = 1$$

The projective cross ratio is the quotient of the areas of the triangles XAC, XBD, XAD and XBC:

$$r = \frac{XA \wedge XC \; XB \wedge XD}{XA \wedge XD \; XB \wedge XC} = \frac{\begin{vmatrix} x_A & x_B & x_C \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_A & x_B & x_C \\ 0 & 1 & 0 \\ d_A & d_B & d_C \end{vmatrix}}{\begin{vmatrix} x_A & x_B & x_C \\ 1 & 0 & 0 \\ d_A & d_B & d_C \end{vmatrix} \begin{vmatrix} x_A & x_B & x_C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{x_B (x_A d_C - x_C d_A)}{(x_B d_C - x_C d_B) x_A}$$

which yields the following equation:

$$0 = r\, x_C\, x_A\, d_B + (1 - r)\, x_A\, x_B\, d_C - x_B\, x_C\, d_A$$

The fifth point E also lying on the conic fulfils this equation:

$$0 = r\, e_C\, e_A\, d_B + (1 - r)\, e_A\, e_B\, d_C - e_B\, e_C\, d_A$$

which results in a simplified expression for the cross ratio:

$$r = \frac{\dfrac{d_A}{e_A} - \dfrac{d_C}{e_C}}{\dfrac{d_B}{e_B} - \dfrac{d_C}{e_C}}$$

The substitution of the cross ratio r in the equation of the conic gives:

$$d_A\, e_A\, x_B\, x_C\, (d_B\, e_C - d_C\, e_B) + d_B\, e_B\, x_C\, x_A\, (d_C\, e_A - d_A\, e_C) + d_C\, e_C\, x_A\, x_B\, (d_A\, e_B - d_B\, e_A) =$$

$$\begin{vmatrix} d_A\, e_A & d_B\, e_B & d_C\, e_C \\ d_A\, x_A & d_B\, x_B & d_C\, x_C \\ e_A\, x_A & e_B\, x_B & e_C\, x_C \end{vmatrix} = 0$$

Conic equations in barycentric coordinates and tangential conic

The Cartesian equation of a conic:

$$a\, x^2 + b\, y^2 + c\, x\, y + d\, x + e\, y + f = 0$$

is written in barycentric coordinates as a bilinear mapping:

$$\begin{pmatrix} 1-x-y & x & y \end{pmatrix} \begin{pmatrix} f & \dfrac{d}{2}+f & \dfrac{e}{2}+f \\ \dfrac{d}{2}+f & a+d+f & \dfrac{c+d+e}{2}+f \\ \dfrac{e}{2}+f & \dfrac{c+d+e}{2}+f & b+e+f \end{pmatrix} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0$$

Let us calculate now the dual conic of a given conic. This is defined as the locus of the points which are dual of the lines tangent to the conic. Then the conic is the envelope of the tangents. Let us differentiate the Cartesian equation:

$$\delta x\,(2\,a\,x + c\,y + d) + \delta y\,(2\,b\,y + c\,x + e) = 0$$

where δ indicates the ordinary differential in order to avoid confusion with the coefficient *d*. The equation of the line touching the conic at (x_0, y_0) is:

$$(2\,a\,x_0 + c\,y_0 + d)(x - x_0) + (2\,b\,y_0 + c\,x_0 + e)(y - y_0) = 0$$

$$(2\,a\,x_0 + c\,y_0 + d)\,x + (2\,b\,y_0 + c\,x_0 + e)\,y - 2\,a\,x_0^2 - 2\,b\,y_0^2 - 2\,c\,x_0\,y_0 - d\,x_0 - e\,y_0 = 0$$

$$(2\,a\,x_0 + c\,y_0 + d)\,x + (2\,b\,y_0 + c\,x_0 + e)\,y + d\,x_0 + e\,y_0 + 2f = 0$$

$$\begin{aligned}(d\,x_0 + e\,y_0 + 2f)(1-x-y) + ((2\,a+d)\,x_0 + (c+e)\,y_0 + d + 2\,f)\,x + \\ + ((2\,b+e)\,y_0 + (c+d)\,x_0 + e + 2f)\,y = 0\end{aligned}$$

If we denote by *t*, *u* and *v* the dual coordinates, the dual homogeneous coordinates *t'*, *u'* and *v'* are linear functions of the barycentric point coordinates:

$$\begin{bmatrix} t' \\ u' \\ v' \end{bmatrix} = \begin{pmatrix} 2\,f & d+2\,f & e+2\,f \\ d+2\,f & 2\,a+2\,d+2\,f & c+d+e+2\,f \\ e+2\,f & c+d+e+2\,f & 2\,b+2\,e+2\,f \end{pmatrix} \begin{pmatrix} 1-x_0-y_0 \\ x_0 \\ y_0 \end{pmatrix}$$

Now we see that this is twice the matrix of the conic equation (the matrix of a conic is defined except by a common factor). Let **M** be the matrix of the conic, **X** the matrix of the point coordinates, and **U** the matrix of the dual (homogeneous or normalised) coordinates.

$$\mathbf{U} = \mathbf{M}\,\mathbf{X}$$

$$\mathbf{M}^{-1}\,\mathbf{U} = \mathbf{X}$$

$$\mathbf{X}^{\mathrm{T}} = \mathbf{U}^{\mathrm{T}}\,\mathbf{M}^{-1}$$

because like **M**, the inverse matrix $\mathbf{M}^{-1}$ is also symmetric and does not change under transposition. The substitution in the equation of the conic $\mathbf{X}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{X} = 0$ gives:

$$\mathbf{U}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathbf{U} = 0$$

That is, <u>the matrix of the dual conic equation is the inverse of the matrix of the point conic</u>. Every tangent line of the conic is mapped onto a point of the dual conic and vice versa. Let us consider for instance the ellipse:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \qquad \Leftrightarrow \qquad (1-x-y \quad x \quad y)\begin{pmatrix} 16 & 16 & 16 \\ 16 & 15 & 16 \\ 16 & 16 & 12 \end{pmatrix}\begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0$$

The inverse matrix is:

$$\begin{pmatrix} 16 & 16 & 16 \\ 16 & 15 & 16 \\ 16 & 16 & 12 \end{pmatrix}^{-1} = \begin{pmatrix} -19/16 & 1 & 1/4 \\ 1 & -1 & 0 \\ 1/4 & 0 & -1/4 \end{pmatrix}$$

So the equation of the dual conic is:

$$[t \quad u \quad v]\begin{pmatrix} -19 & 16 & 4 \\ 16 & -16 & 0 \\ 4 & 0 & -4 \end{pmatrix}\begin{bmatrix} t \\ u \\ v \end{bmatrix} = 0$$

where we can substitute indistinctly homogeneous or normalised coordinates. If we take $t = 1 - u - v$ then we obtain the Cartesian equation:

$$-67\,u^2 - 31\,v^2 - 78\,u\,v + 70\,u + 46\,v - 19 = 0$$

A concrete case is the tangent line at (0, 2) with equation $y = 2$. The dual coordinates of this line are obtained as follows:

$$y - 2 \equiv t'\,(1-x-y) + u'\,x + v'\,y \qquad \Rightarrow \qquad [t' \quad u' \quad v'] = [-2, -2, -1] = \left[\frac{2}{5}, \frac{1}{5}\right]$$

Let us check that this dual point lies on the dual conic:

$$[2 \quad 2 \quad 1]\begin{pmatrix} -19 & 16 & 4 \\ 16 & -16 & 0 \\ 4 & 0 & -4 \end{pmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 0$$

Polarities

A *correlation* is defined as the geometric transformation which maps collinear points onto a pencil of lines, in other words, a linear mapping of points to lines (dual points), which may be represented with a matrix **M**:

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \mathbf{M} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix}$$

where a', b' and c' are homogeneous dual coordinates.

A *polarity* is a correlation whose square is the identity. This means that by applying it twice, a point is mapped onto itself and a line is mapped onto itself. Let us consider the line $[a, b, c]$:

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0 \qquad \Rightarrow \qquad \begin{bmatrix} a & b & c \end{bmatrix} \mathbf{M}^{-1} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = 0$$

On the other hand:

$$\begin{bmatrix} a' & b' & c' \end{bmatrix} \begin{pmatrix} 1-x'-y' \\ x' \\ y' \end{pmatrix} = 0 \qquad \Rightarrow \qquad \begin{bmatrix} a' & b' & c' \end{bmatrix} \mathbf{M}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

since $[\, a'' \;\; b'' \;\; c'' \,] = k\, [\, a \;\; b \;\; c \,]$ where k is a homogeneous constant. Whence it follows that the matrix **M** is symmetric for a polarity.

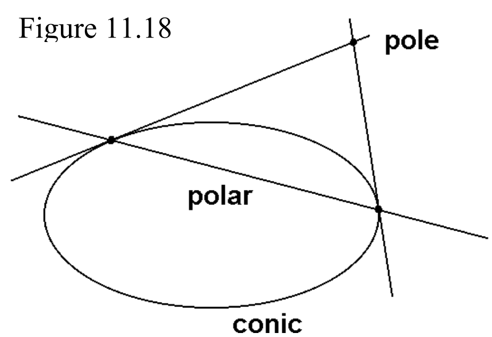
Under a polarity a point is mapped to its *polar line*. This point is also called the *pole* of the line. The polarity transforms the pole into the polar and the polar into the pole. In general the polar does not include the pole, except by a certain subset of points on the plane. The set of points belonging to their own polar line (*self-conjugate points)* fulfil the equation:

$$\begin{pmatrix} 1-x-y & x & y \end{pmatrix} \mathbf{M} \begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} = 0$$

This is just the equation of a point conic. The set of lines passing through their own poles (*self-conjugate lines*) fulfil the equation:

$$\begin{bmatrix} a & b & c \end{bmatrix} \mathbf{M}^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Figure 11.18



which is just the equation of its dual (*tangential*) conic. Then, a polarity has an associated point and tangential conics, and every conic defines a polarity. However depending on the eigenvalues of the matrix, there are polarities that do not have any associated conic.

The polar of a point outside its associated conic is the line passing through the points of contact of the tangents drawn from the point (figure 11.18). The reason is the following: the polarity maps the tangent lines of the associated conic to the tangency points, so its intersection (the pole) must be transformed into the line passing through the tangency points.

Reduction of the conic matrix to a diagonal form

Since the matrix of a polarity is symmetric, we can reduce it to a diagonal form. Let **D** be the diagonal matrix and **B** the exchange base matrix. Then:

$$\mathbf{M} = \mathbf{B}^{-1}\,\mathbf{D}\,\mathbf{B}$$

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

There exist three eigenvectors of the conic matrix. These eigenvectors are also characteristic vectors of the dual conic matrix with inverse eigenvalues:

$$\mathbf{M}^{-1} = \mathbf{B}^{-1}\,\mathbf{D}^{-1}\,\mathbf{B}$$

$$\mathbf{D}^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{pmatrix} \qquad\qquad d_i\,\mathbf{V}_i = \mathbf{M}\,\mathbf{V}_i \quad \Leftrightarrow \quad \mathbf{M}^{-1}\,\mathbf{V}_i = \frac{1}{d_i}\,\mathbf{V}_i$$

The eigenvectors are three points (equal for both punctual and dual planes) that I will call the *eigenpoints* of the polarity. Let **X** be a point on the conic given in Cartesian coordinates. Then:

$$\mathbf{X}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{X} = \mathbf{X}^{\mathrm{T}}\,\mathbf{B}^{-1}\,\mathbf{D}\,\mathbf{B}\,\mathbf{X} = \mathbf{V}^{\mathrm{T}}\,\mathbf{D}\,\mathbf{V}$$

where **V** = **B X** are the coordinates of the point in the base of eigenpoints. If we name with t, u, v the eigencoordinates, then the equation of the conic is simply:

$$d_1\,t^2 + d_2\,u^2 + d_3\,v^2 = 0$$

The eigenpoints of a polarity do not never belong to the associated conic since they have the eigencoordinates (1,0,0), (0,1,0) and (0,0,1) and $d_i \neq 0$. On the other hand, the conic only exist if the eigenvalues have different signs (one positive and two negative or one negative and two positive). This fact classifies the polarities in two classes: those having an associated conic, and those not having an associated conic. In fact, if we state a level, a certain value of the matrix product, there can be a conic or not. That is, there is a family of conics for different levels and there is a threshold value, from where the conic does not exist.

Using a base of points on the conic

If instead of Cartesian coordinates we use the coordinates of a base of three points A, B and C lying on the conic, the coefficients on the matrix diagonal vanish. Then the equation of the conic is:

$$\frac{1}{2}\begin{pmatrix} x_A & x_B & x_C \end{pmatrix}\begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{12} & 0 & c_{23} \\ c_{13} & c_{23} & 0 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = c_{23}\, x_B\, x_C + c_{31}\, x_C\, x_A + c_{12}\, x_A\, x_B = 0$$

With the same base, the equation of the conic passing through D and whose projective cross ratio $\{X, A\, B\, C\, D\}$ is r is:

$$0 = r\, x_C\, x_A\, d_B + (1-r)\, x_A\, x_B\, d_C - x_B\, x_C\, d_A$$

From where we have:

$$\begin{pmatrix} x_A & x_B & x_C \end{pmatrix}\begin{pmatrix} 0 & (1-r)\, d_C & r\, d_B \\ (1-r)\, d_C & 0 & -d_A \\ r\, d_B & -d_A & 0 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} = 0$$

This conic matrix is not constant, and has the determinant:

$$\det = -2\, r\,(1-r)\, d_A\, d_B\, d_C$$

so the matrix with constant coefficients is:

$$\mathbf{C} = \frac{1}{\sqrt[3]{-r\,(1-r)\, d_A\, d_B\, d_C}}\begin{pmatrix} 0 & (1-r)\, d_C & r\, d_B \\ (1-r)\, d_C & 0 & -d_A \\ r\, d_B & -d_A & 0 \end{pmatrix}$$

Exercises

11.1 Prove that the projective cross ratio of any four points on a conic is equal to the ratio of the sines of the cylindrical half-angles. That is, if:

$OA = OQ \cos\alpha + OR \sin\alpha$ $\qquad$ $OB = OQ \cos\beta + OR \sin\beta$, etc.

Then $$\{X, ABCD\} = \frac{\sin\dfrac{\gamma-\alpha}{2}\sin\dfrac{\delta-\beta}{2}}{\sin\dfrac{\delta-\alpha}{2}\sin\dfrac{\gamma-\beta}{2}}$$

11.2 Prove that an affinity transforms a circle into an ellipse, and using this fact prove the Newton's theorem: Given two directions, the ratio of the product of distances from any point on the plane to both intersections with an ellipse along the first direction and the same product of distances along the other direction is independent of the location of this point on the plane.

11.3 Prove that a line touching an ellipse is parallel to the conjugate diameter of the diameter whose end-point is the tangency point. Prove also that the parallelogram circumscribed around an ellipse has constant area (Apollonius' theorem).

11.4 Prove the former theorem applied to a hyperbola: the parallelogram formed by two conjugate diameters has constant area.

11.5 Given a pole, trace a line passing through the pole and cutting the conic. Prove that the pole, and the intersections of this line with the conic and with the polar form a harmonic range.

11.6 Calculate the equation of the point and tangential conic passing through the points (1, 1), (2, 3), (1, 4), (0, 2) and (2, 5).

11.7 The Pascal's theorem: Let *A*, *B*, *C*, *D*, *E* and *F* be six distinct points on a proper conic, Let *P* be the intersection of the line *AE* with the line *BF*, *Q* the intersection of the line *AD* with *CF*, and *R* the intersection of the line *BD* with *CE*. Show that *P*, *Q* and *R* are collinear.

11.8 The Brianchon's theorem. Prove that the diagonals joining opposite vertices of a hexagon circumscribed around a proper conic are concurrent. Hint: take the dual plane.

THIRD PART: PSEUDO-EUCLIDEAN GEOMETRY

This part is devoted to the hyperbolic plane, where vectors and numbers have a pseudo Euclidean modulus, that is, a modulus of a Minkowski space. The bidimensional geometric algebra already includes the Euclidean and pseudo Euclidean planes. In fact, the geometric algebra does not make any special distinction between both kinds of planes. On the other hand, the plane geometric algebra can be represented by real 2×2 matrices, which helps us to define some concepts with more precision.

12. MATRIX REPRESENTATION AND HYPERBOLIC NUMBERS

All the associative algebras with neutral elements for the addition and product can be represented with matrices. The representations can have different dimensions, but the most interesting is the minimal representation which is an isomorphism.

Rotations and the representation of complex numbers

We have seen that a rotation of a vector v over an angle α is written in geometric algebra as:

$$v' = v\ (\cos\alpha + e_{12}\sin\alpha)$$

Separating the components:

$$v_1' = v_1\cos\alpha - v_2\sin\alpha$$

$$v_2' = v_1\sin\alpha + v_2\cos\alpha$$

and writing them in matrix form, we have:

$$\begin{pmatrix} v_1' & v_2' \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

Identifying this equation with the first one:

$$\cos\alpha + e_{12}\sin\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

we obtain the matrix representation for the complex numbers:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad z = a + b\,e_{12} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Now we wish to obtain also the matrix representation for vectors. Note that the matrix form of the rotation of a vector gives us the first row of the vector representation, which may be completed with a suitable second row:

$$\begin{pmatrix} v_1' & v_2' \\ v_2' & -v_1' \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

from where we arrive to the matrix representation for vectors:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad v = v_1\, e_1 + v_2\, e_2 = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}$$

These four matrices are a basis of the real matrix space $\mathbf{M}_{2\times 2}(\mathbf{R})$ and therefore the geometric algebra of the vectorial plane V_2 is isomorphic to this matrix space:

$$Cl(V_2) \equiv Cl_{2,0} = \mathbf{M}_{2\times 2}(\mathbf{R})$$

The moduli of a complex number and a vector are related with the determinant of the matrix:

$$|\, z \,|^2 = a^2 + b^2 = \det\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad |\, v \,|^2 = v_1^{\;2} + v_2^{\;2} = -\det\begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}$$

Let us think about a heterogeneous element being a sum of a complex and a vector. Its geometric meaning is unknown for us but we can write it with a matrix:

$$z + v = a + b\, e_{12} + v_1\, e_1 + v_2\, e_2 = \begin{pmatrix} a + v_1 & b + v_2 \\ -b + v_2 & a - v_1 \end{pmatrix}$$

The determinant of this matrix

$$\det(z + v) = a^2 + b^2 - v_1^{\;2} - v_2^{\;2}$$

reminds us the Pythagorean theorem in a pseudo Euclidean plane.

The subalgebra of the hyperbolic numbers

In the matrix representation of the geometric algebra, we can see that the diagonal matrices are a subalgebra:

$$a + b\, e_1 = \begin{pmatrix} a + b & 0 \\ 0 & a - b \end{pmatrix}$$

The product of elements of this kind is commutative (and also associative). Because of this, we may talk about «numbers»:

$$(\, a + b\, e_1\,)\,(\, c + d\, e_1\,) = a\, c + b\, d + (\, a\, d + b\, c\,)\, e_1$$

Using the Hamilton's notation of pairs of real numbers, we write:

$$(a, b)(c, d) = (a\,c + b\,d\,,\, a\,d + b\,c)$$

From the determinant we may define their pseudo Euclidean *modulus* $|z|$:

$$\det(a + b\,e_1) = a^2 - b^2 = |a + b\,e_1|^2$$

The numbers having constant modulus lie on a hyperbola. Because of this, they are called *hyperbolic numbers*. Due to the fact that e_1 cannot be a privileged direction on the plane, any other set of elements having the form:

$$a + k\,u$$

with a and k real, and u being a fixed unitary vector, is also a set of hyperbolic numbers.

By defining the *conjugate* of a hyperbolic number as that number with the opposite vector component:

$$(a + b\,e_1)^* = a - b\,e_1$$

we see that the square of the modulus of a hyperbolic number is equal to the product of this number by its conjugate:

$$|z|^2 = z\,z^*$$

Hence the inverse of a hyperbolic number follows:

$$z^{-1} = \frac{z^*}{|z|^2}$$

Since the modulus is the square root of the determinant, and the determinant of a matrix product is equal to the product of determinants, it follows that the modulus of a product of hyperbolic numbers is equal to the product of moduli:

$$|z\,t| = |z|\,|t|$$

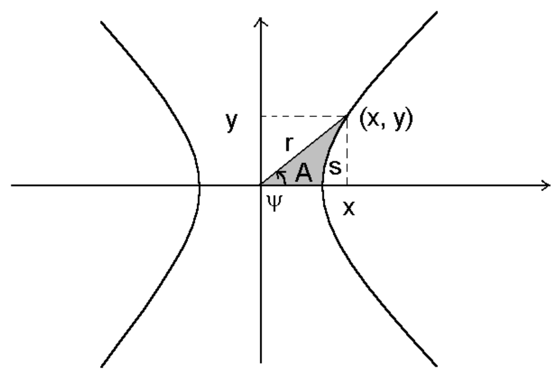
The elements $(1 + e_1)/2$ and $(1 - e_1)/2$, which are idempotents, and their multiples have a null modulus and no inverse. They form two ideals, that is, the product of any hyperbolic number by a multiple of an idempotent (or any other multiple) yields also a multiple of the idempotent.

Hyperbolic trigonometry

Let us consider the locus of the points (hyperbolic numbers) located at a constant distance r from the origin (hyperbolic numbers with constant determinant equal to r^2), which lie on the hyperbola $x^2 - y^2 = r^2$ (figure 12.1). I shall call r the *hyperbolic radius* following the analogy with the circle. The extreme of the radius is a point on the hyperbola with coordinates (x, y). The arc between the positive X half axis and this point (x, y) has an oriented length s. On the other hand, the radius, the hyperbola and the X-axis delimit a sector with an oriented area A. The hyperbolic angle (or argument) ψ is defined as the quotient of the arc length divided by the radius[1]:

$$\psi = \frac{s}{r}$$

Figure 12.1



It follows from this definition that the oriented area A is[2]:

$$A = \frac{\psi\, r^2}{2} \qquad \Rightarrow \qquad \psi = \frac{2A}{r^2}$$

The hyperbolic sine, cosine and tangent are defined as the following quotients:

$$\sinh\psi = \frac{y}{r}$$

$$\cosh\psi = \frac{x}{r} \qquad\qquad \mathrm{tgh}\,\psi = \frac{y}{x}$$

These definitions yield the three fundamental identities of the hyperbolic trigonometry:

$$\mathrm{tgh}\,\psi \equiv \frac{\sinh\psi}{\cosh\psi} \qquad\qquad \cosh^2\psi - \sinh^2\psi \equiv 1 \qquad\qquad 1 - \mathrm{tgh}^2\,\psi \equiv \frac{1}{\cosh^2\psi}$$

Now we search an explicit expression of the hyperbolic functions in terms of elemental functions such as polynomials, exponential, logarithm, etc. The differential of the arc length (being real) is related with the differentials of the coordinates by the pseudo Pythagorean theorem:

$$ds^2 = dy^2 - dx^2 \qquad \Rightarrow \qquad 1 = \left(\frac{d\left(\sinh\psi\right)}{d\psi}\right)^2 - \left(\frac{d\left(\cosh\psi\right)}{d\psi}\right)^2$$

That is, we have the following system with one differential equation:

---

[1] Note that $x^2 - y^2 = r^2 > 0$ while $s^2 < 0$. However I overcome this trouble taking in these definitions r and s and also the area A as real numbers but being oriented, that is, with sign.

[2] The formula of the sector area is obtained by an analogous argument from Archimedes: the total area is the addition of areas of the infinitesimal triangles with altitude equal to r and base equal to ds . r being constant, the area of the hyperbola sector is $A = r\, s\, /\, 2$. Obviously the radius is orthogonal to each infinitesimal piece of arc of the hyperbola. This question and the concept and calculus of areas are studied with more detail in the following chapter.

$$\begin{cases} 1 = \cosh^2\psi - \sinh^2\psi \\ 1 = (\sinh'\psi)^2 - (\cosh'\psi)^2 \end{cases}$$

whose solution, according to the initial conditions given by the geometric definition (sinh 0 = 0 and cosh 0 = 1), is:

$$\sinh\psi = \frac{\exp\psi - \exp(-\psi)}{2} \qquad\qquad \cosh\psi = \frac{\exp\psi + \exp(-\psi)}{2}$$

Hyperbolic exponential and logarithm

Within the hyperbolic numbers one can define and study functions and develop a *hyperbolic analysis*, with the aid of matrix functions. In the matrix algebra, the exponential function of a matrix A is defined as:

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

The matrices which represent the hyperbolic numbers (e_1 direction) are diagonal. Then their exponential matrix has the exponential of each element in the diagonal:

$$\exp(x + y\, e_1) = \exp\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix} = \begin{pmatrix} \exp(x+y) & 0 \\ 0 & \exp(x-y) \end{pmatrix}$$

By extracting the common factor and introducing the hyperbolic functions we arrive at:

$$\exp(x + y\, e_1) = \exp(x)\begin{pmatrix} \cosh y + \sinh y & 0 \\ 0 & \cosh y - \sinh y \end{pmatrix}$$

which is the analogous of the Euler's identity:

$$\exp(x + y\, e_1) = \exp(x)(\cosh y + e_1 \sinh y)$$

From this exponential identity, we can find the logarithm function:

$$\log(x + y\, e_1) = \frac{1}{2}\log\left(x^2 - y^2\right) + e_1 \arg\operatorname{tgh}\left(\frac{y}{x}\right) = \frac{1}{2}\log\left(x^2 - y^2\right) + \frac{e_1}{2}\log\left(\frac{x+y}{x-y}\right)$$

for $-x < y < x$. This condition is the set of hyperbolic numbers with positive determinant, which is called a *sector*[3]. The two ideals generated by the idempotents $(1+e_1)/\sqrt{2}$ and $(1-e_1)/\sqrt{2}$ separate two sectors of hyperbolic numbers, one with

[3] According to the relativity theory, the region of the space-time accessible to our knowledge must fulfil the inequality $c^2\, t^2 - x^2 \geq 0$, being x the space coordinate, t the time and c the light celerity.

positive determinant and real modulus and another with negative determinant and imaginary modulus.

The characteristic property of the exponential function is:

$$\exp(z+t)=\exp(z)\exp(t)$$

z, t being hyperbolic numbers. Taking exponentials with unity determinant:

$$\exp((\psi+\chi)e_1)\equiv\exp(\psi\, e_1)\exp(\chi\, e_1)$$

and splitting the real and vectorial parts, we obtain the addition identities:

$$\cosh(\psi+\chi)\equiv\cosh\psi\cosh\chi+\sinh\psi\sinh\chi$$

$$\sinh(\psi+\chi)\equiv\sinh\psi\cosh\chi+\cosh\psi\sinh\chi$$

Also through the equality:

$$\exp(n\psi\, e_1)\equiv(\exp(\psi\, e_1))^n$$

the analogous of Moivre's identity is found:

$$\cosh(n\psi)+e_1\sinh(n\psi)\equiv(\cosh\psi+e_1\sinh\psi)^n$$

For example, for $n = 3$ it becomes:

$$\cosh 3\psi=\cosh^3\psi+3\cosh\psi\sinh^2\psi$$

$$\sinh 3\psi=3\cosh^2\psi\sinh\psi+\sinh^3\psi$$

Polar form, powers and roots of hyperbolic numbers

The exponential allows us to write any hyperbolic number in polar form. For example:

$$z=13+5\,e_1 \qquad |\,z\,|=\sqrt{13^2-5^2}=12 \qquad \arg z=\arg\operatorname{tgh}\frac{13}{5}=\log\frac{3}{2}$$

$$z=12\exp\left(e_1\log\frac{3}{2}\right)=12\left(\cosh\left(\log\frac{3}{2}\right)+e_1\sinh\left(\log\frac{3}{2}\right)\right)=12_{\log\frac{3}{2}}$$

The product of hyperbolic numbers, like that of complex numbers, is found by the multiplication of the moduli and the addition of the arguments, while the division is obtained as the quotient of moduli and difference of arguments. The division is not possible when the denominator is a multiple of an idempotent (modulus zero).

The power of a hyperbolic number has modulus equal to the power of the modulus, and argument the addition of arguments. Here the periodicity is absent, and distinct arguments always correspond to distinct numbers. The roots must be viewed with more detail. For example, there are four square roots of $13 + 12\, e_1$:

$$13+12\,e_1 = (3+2\,e_1)^2 = (2+3\,e_1)^2 = (-3-2\,e_1)^2 = (-2-3\,e_1)^2$$

that is, the equation of second degree $z^2 - 13 - 12\, e_1 = 0$ has four solutions. Recently G. Casanova[4] has proved that n^2 is the maximum number of hyperbolic solutions (including real values) of an algebraic equation of n^{th} degree. In the case of the equation of second degree we can use the classical formula whenever we know to solve the square roots. For example, let us consider the equation:

$$z^2 - 5\,z + 5 + e_1 = 0$$

whose solutions, according to the second degree formula, are:

$$z = \frac{5 \pm \sqrt{25 - 4\cdot 1\cdot (5+e_1)}}{2\cdot 1} = \frac{5 \pm \sqrt{5 - 4\,e_1}}{2}$$

Now, we must calculate all the square roots of the number $5 - 4\, e_1$. In order to find them we obtain its modulus and argument:

$$|\,5-4\,e_1| = \sqrt{5^2 - (-4)^2} = 3 \qquad \arg(\,5-4\,e_1) = \arg\operatorname{tgh}\frac{-4}{5} = -\log 3$$

The square root has half argument and the square root of the modulus:

$$\sqrt{5-4\,e_1} = \sqrt{3}_{-\frac{\log 3}{2}} = \sqrt{3}\,\left[\cosh\!\left(-\log\sqrt{3}\right) + e_1\,\sinh\!\left(-\log\sqrt{3}\right)\right] = 2 - e_1$$

There are four square roots of this number:

$$2-e_1,\; -2+e_1,\; 1-2\,e_1,\; -1+2\,e_1$$

and so four solutions of the initial equation:

$$z_1 = \frac{5+2-e_1}{2} = \frac{7-e_1}{2} \qquad z_2 = \frac{5-2+e_1}{2} = \frac{3+e_1}{2}$$

$$z_3 = \frac{5+1-2\,e_1}{2} = 3-e_1 \qquad z_4 = \frac{5-1+2\,e_1}{2} = 2+e_1$$

Let us study the second degree equation with real coefficients using the matrix representation:

[4] Gaston Casanova, *Advances in Applied Clifford Algebras* **9**, 2 (1999) 215-219.

$$a\,z^2 + b\,z + c = 0 \qquad a,\ b \text{ and } c \text{ real} \qquad \text{and } z = x + e_1\,y$$

Dividing the second degree equation by a we obtain the equivalent equation:

$$z^2 + \frac{b}{a}\,z + \frac{c}{a} = 0$$

which is fulfilled by the number z, but also by its matrix representation. So, according to the Hamilton-Cayley theorem, it is the characteristic polynomial of the matrix of z, whose eigenvalues are the elements on the diagonal $x + y$, $x - y$ (whenever they are distinct, that is, z is not real). Since b is the opposite of the sum of eigenvalues, and c is their product, we have:

$$z^2 - 2x\,z + x^2 - y^2 = 0$$

which gives the equalities:

$$x = -\frac{b}{2a} \qquad \frac{c}{a} = x^2 - y^2 \qquad \Rightarrow \qquad y^2 = \frac{b^2 - 4\,a\,c}{4\,a^2}$$

and the solutions:

$$z = \frac{-\,b \pm e_1\sqrt{b^2 - 4\,a\,c}}{2\,a}$$

The two real solutions must be added to these values, obtaining the four expected. Anyway, the equation has only solutions if the discriminant is positive. Let us see an example:

$$z^2 + 3\,z + 2 = 0$$

$$z_1 = \frac{-3+1}{2} = -1 \qquad z_2 = \frac{-3-1}{2} = -2 \qquad z_3 = \frac{-3+e_1}{2} \qquad z_4 = \frac{-3-e_1}{2}$$

On the other hand, we may calculate for instance the cubic root of 14–13 e_1. The modulus and argument of this number are:

$$\left|\,14 - 13\,e_1\right| = \sqrt{14^2 - 13^2} = \sqrt{27} \qquad \arg\!\left(14 - 13\,e_1\right) = \arg\operatorname{tgh}\left(-\frac{13}{14}\right) = -\log\sqrt{27}$$

From where it follows that the unique cubic root is:

$$\sqrt[3]{14 - 13\,e_1} = \sqrt{3}_{\,-\frac{\log\sqrt{27}}{3}} = \sqrt{3}_{\,-\log\sqrt{3}} = 2 - e_1$$

In order to study with more generality the number of roots, let us consider again the matrix representation of a hyperbolic number:

$$a + b\, e_1 = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

Now, it is obvious that there is only a unique root with odd index, which always exists for every hyperbolic number:

$$\sqrt[n]{a + b\, e_1} = \begin{pmatrix} \sqrt[n]{a+b} & 0 \\ 0 & \sqrt[n]{a-b} \end{pmatrix} \qquad n \text{ odd}$$

On the other hand, if $a + b > 0$ and $a - b > 0$ (first half-sector) there are four roots with even index, one on each half-sector (I follow the anticlockwise order as usually):

n even

$$\begin{pmatrix} \sqrt[n]{a+b} & 0 \\ 0 & \sqrt[n]{a-b} \end{pmatrix} \in 1^{\text{st}} \text{ half-sector} \qquad \begin{pmatrix} \sqrt[n]{a+b} & 0 \\ 0 & -\sqrt[n]{a-b} \end{pmatrix} \in 2^{\text{nd}} \text{ half-sector}$$

$$\begin{pmatrix} -\sqrt[n]{a+b} & 0 \\ 0 & -\sqrt[n]{a-b} \end{pmatrix} \in 3^{\text{rd}} \text{ half-sector} \qquad \begin{pmatrix} -\sqrt[n]{a+b} & 0 \\ 0 & \sqrt[n]{a-b} \end{pmatrix} \in 4^{\text{th}} \text{ half-sector}$$

If the number belongs to a half-sector different from the first, some of the elements on the diagonal is negative and there is not any even root. This shows a panorama of the hyperbolic algebra far from that of the complex numbers.

Hyperbolic analytic functions

Which conditions must fulfil a hyperbolic function $f(z)$ of a hyperbolic variable z to be analytic? We wish that the derivative be well defined:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

that is, this limit must be independent of the direction of Δz. If $f(z) = a + b\, e_1$ and the variable $z = x + y\, e_1$, then the derivative calculated in the direction $\Delta z = \Delta x$ is:

$$f'(z) = \frac{\partial a}{\partial x} + e_1 \frac{\partial b}{\partial x}$$

while the derivative calculated in the direction $\Delta z = e_1\, \Delta y$ becomes:

$$f'(z) = e_1 \frac{\partial a}{\partial y} + \frac{\partial b}{\partial y}$$

Both expressions must be equal, which results in the *conditions of hyperbolic analyticity*:

$$\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y} \qquad \text{and} \qquad \frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$$

Note that the exponential and logarithm fulfil these conditions and therefore they are hyperbolic analytic functions. More exactly, the exponential is analytic in all the plane while the logarithm is analytic for the sector of hyperbolic numbers with positive determinant.

By derivation of both identities one finds that the analytic functions satisfy the hyperbolic partial differential equation (also called wave equation):

$$\frac{\partial^2 a}{\partial x^2} - \frac{\partial^2 a}{\partial y^2} = \frac{\partial^2 b}{\partial x^2} - \frac{\partial^2 b}{\partial y^2} = 0$$

Now, we must state the main integral theorem for hyperbolic analytic functions: <u>if a hyperbolic function is analytic in a certain domain on the hyperbolic plane, then its integral following a closed way *C* within this domain is zero</u>. If the hyperbolic function is $f(z) = a + b\,e_1$ then the integral is:

$$\oint_C f(z)\,dz = \oint_C (a + be_1)(dx + dy\,e_1) = \oint_C (a\,dx + b\,dy) + e_1 \oint_C (a\,dy + b\,dx)$$

Since *C* is a closed path, we may apply the Green theorem to write:

$$= \iint_D \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx \wedge dy + e_1 \iint_D \left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} \right) dx \wedge dy = 0$$

where *D* is the region bounded by the closed way *C*. Since *f*(*z*) fulfils the analyticity conditions everywhere within *D*, the integral vanishes.

From here other theorems follow like for the complex analysis, e. g.: <u>if *f*(*z*) is a hyperbolic analytic function in a domain *D* simply connected and z_1 and z_2 are two points on *D* then the definite integral</u>:

$$\int_{z_2}^{z_1} f(z)\,dz$$

<u>between these points has a unique value independently of the integration path</u>.

Let us see an example. Consider the function *f*(*z*) = 1 / (*z* − 1). The function is only defined if the inverse of *z* − 1 exists, which implies $|z-1| \neq 0$. Of course, this function is not analytic at *z* = 1, but neither for the points:

$$|z-1|^2 = 0 \quad \Leftrightarrow \quad (x-1)^2 - y^2 = 0 \quad \Leftrightarrow \quad (x+y-1)(x-y-1) = 0$$

The lines *x* + *y* = 1 and *x* − *y* = 1 break the analyticity domain into two sectors. Let us calculate the integral:

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1}$$

through two different trajectories within a sector. The first one is a straight path given by the parametric equation $z = 5 + t\, e_1$ (figure 12.2):

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} = \int_{-3}^{3} \frac{dt}{4+t\, e_1} e_1 = \int_{-3}^{3} \frac{(4-t\, e_1)\, dt}{16-t^2} e_1$$

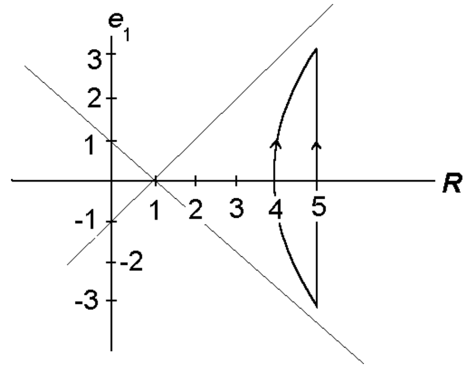


Figure 12.2

Ought to symmetry the integral of the odd function is zero:

$$= \int_{-3}^{3} \frac{4\, dt}{16-t^2} e_1 = \left[\frac{e_1}{2} \log\left| \frac{4+t}{4-t} \right| \right]_{-3}^{3} = e_1 \log 7$$

The second path (figure 12.2) is the hyperbola going from the point (5, –3) to (5, 3):

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} = \int_{(5,-3)}^{(5,3)} \frac{(z^*-1)\, dz}{(z-1)(z^*-1)} = \int_{(5,-3)}^{(5,3)} \frac{(z^*-1)\, dz}{\det(z) - 2\operatorname{Re}(z) + 1}$$

Introducing the parametric equation of this path, $z = 4\,(\cosh t + e_1 \sinh t)$ we have:

$$= \int_{-\log 2}^{\log 2} \frac{(4\cosh t - 4e_1 \sinh t - 1)(4\sinh t + 4\, e_1 \sinh t)}{16 - 8\cosh t + 1} dt = 4 \int_{-\log 2}^{\log 2} \frac{e_1\,(4-\cosh t) - 4\sinh t}{17 - 8\cosh t} dt$$

Due to symmetry, the integral of the hyperbolic sine (an odd function) divided by the denominator (an even function) is zero. Then we split the integral in two integrals and find its value:

$$= \frac{15}{2} e_1 \int_{-\log 2}^{\log 2} \frac{dt}{17 - 8\cosh t} + e_1 \int_{-\log 2}^{\log 2} \frac{dt}{2} = \frac{e_1}{2} \left[\log \frac{-8\exp(t) + 2}{-8\exp(t) + 32} \right]_{-\log 2}^{\log 2} + e_1 \log 2 = e_1 \log 7$$

Now we see that the integral following both paths gives the same result, as indicated by the theorem. In fact, the analytical functions can be integrated directly by using the indefinite integral:

$$\int_{(5,-3)}^{(5,3)} \frac{dz}{z-1} = \left[\log(z-1) \right]_{(5,-3)}^{(5,3)} = \left[\frac{1}{2} \log\left((x-1)^2 - y^2 \right) + e_1 \operatorname{arg\,tgh}\left(\frac{y}{x-1} \right) \right]_{(5,-3)}^{(5,3)}$$

$$= \left[\frac{e_1}{2} \log\left| \frac{x-1+y}{x-1-y} \right| \right]_{(5,-3)}^{(5,3)} = e_1 \log 7$$

Analyticity and square of convergence of the power series

A matrix function $f(A)$ can be developed as a Taylor series of powers of the matrix A:

$$f(A) = \sum_{n=0}^{\infty} a_n A^n \qquad\qquad A = x + y\, e_1 = \begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix}$$

The series is convergent when all the eigenvalues of the matrix A are located within the radius r of convergence of the complex series:

$$\left| \sum_{n=0}^{\infty} a_n z^n \right| < \infty \qquad\qquad \text{for} \qquad | z | < r$$

which leads us to the following conditions:

$$| x + y | < r \qquad \text{and} \qquad | x - y | < r$$

Therefore, the region of convergence of a Taylor series of hyperbolic variable is a square centred at the origin of coordinates with vertices (r, 0)-(0, r)-($-r$, 0)-(0, $-r$). Note that from both conditions we obtain:

$$\left| \det(x + y\, e_1) \right| = \left| x^2 - y^2 \right| < r^2 \, .$$

which is a condition similar to that for complex numbers, that is, the modulus must be lower than the radius of convergence. However this condition is not enough to ensure the convergence of the series. Let us see, for example, the function $f(z) = 1/(1 - z)$:

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

The radius of convergence of the complex series is r = 1, so that the square of the convergence of the hyperbolic series is (1, 0)-(0, 1)-(−1,0)-(0,−1). Otherwise, we have formerly seen that $1/(z - 1)$ (and hence $f(z)$) is not analytic at the lines $x + y - 1 = 0$ and $x - y - 1 = 0$ belonging to the boundary of the square of convergence. Now we find a phenomenon which also happens within the complex numbers: at some point on the boundary of the region of convergence of the Taylor series, the function is not analytic.

Another example is the Riemann's function:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z \log n}$$

Now, we take z instead of real being a hyperbolic number:

$$\zeta(x + y\, e_1) = \sum_{n=1}^{\infty} e^{-x \log n} \left(\cosh(y \log n) - e_1 \sinh(y \log n) \right)$$

$$= \sum_{n=1}^{\infty} n^{-x} \left(\frac{n^{y} + n^{-y}}{2} - \frac{n^{y} - n^{-y}}{2} e_1 \right) = \sum_{n=1}^{\infty} \left(n^{-x+y} \frac{1-e_1}{2} + n^{-x-y} \frac{1+e_1}{2} \right)$$

The series is convergent only if $-x + y < -1$ and $-x - y < -1$ or:

$$1 - x < y < x - 1$$

which is the positive half-sector beginning at $(x, y) = (1, 0)$. You can see that it is an analytic function within this domain. We can rewrite the Riemann's function in the form:

$$\zeta(x + y\, e_1) = \frac{1+e_1}{2} \zeta(x+y) + \frac{1-e_1}{2} \zeta(x-y)$$

and then define the Riemann's zeta function extended to the other sector taking into account the complex analytic continuation when needed for $\zeta(x + y)$ or $\zeta(x - y)$:

$$\zeta(1-z) = \frac{2}{(2\pi)^{z}} \zeta(z)\, \Gamma(z) \cos\frac{\pi\, z}{2} \quad \text{and} \quad \zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^{t} - 1} dt \qquad 0 < \mathrm{Re}(z) < 1$$

Now a very interesting theorem arises: <u>every analytic function $f(z)$ can be always written in the following form</u>:

$$f(x + y\, e_1) = \frac{1+e_1}{2} f(x+y) + \frac{1-e_1}{2} f(x-y)$$

Let us prove this statement calculating the partial derivatives of each part a and b of the analytic function:

$$\frac{\partial a}{\partial x} = \frac{1}{2}\left[\frac{df(x+y)}{d(x+y)} + \frac{df(x-y)}{d(x-y)} \right] \qquad \frac{\partial b}{\partial x} = \frac{1}{2}\left[\frac{df(x+y)}{d(x+y)} - \frac{df(x-y)}{d(x-y)} \right]$$

$$\frac{\partial a}{\partial y} = \frac{1}{2}\left[\frac{df(x+y)}{d(x+y)} - \frac{df(x-y)}{d(x-y)} \right] \qquad \frac{\partial b}{\partial y} = \frac{1}{2}\left[\frac{df(x+y)}{d(x+y)} + \frac{df(x-y)}{d(x-y)} \right]$$

These derivatives fulfil the analyticity condition provided that the derivatives of the real function at $x + y$ and $x - y$ exist. On the other hand, this expression is a method to get the analytical continuation of any real function. For example, let us construct the analytical continuation of $f(x) = \cos x$:

$$\cos(x + y\, e_1) = \frac{1+e_1}{2} \cos(x+y) + \frac{1-e_1}{2} \cos(x-y)$$

$$= \frac{1}{2}\left[\cos(x+y) + \cos(x-y) \right] + \frac{e_1}{2}\left[\cos(x+y) - \cos(x-y) \right]$$

$$= \cos x \cos y - e_1 \sin x \sin y$$

The other consequence of this way of analytic continuation is the fact that if $f(z)$ loses analyticity for the real value x_0 then it is neither analytic at the lines with slope 1 and −1 passing through $(x_0, 0)$. Since we have supposed that the function is analytic except for a certain real value, the former equality holds although the analyticity is lost at this point. But then the function cannot be analytic at $(x_0 + t, \; -t)$ nor $(x_0 - t, \; t)$ because:

$$f(x_0 + t - t\,e_1) = \frac{1+e_1}{2} f(x_0 + 2t) + \frac{1-e_1}{2} f(x_0)$$

$$f(x_0 - t + t\,e_1) = \frac{1+e_1}{2} f(x_0) + \frac{1-e_1}{2} f(x_0 - 2t)$$

For example, look at the function $f(z) = 1/z$, which is not analytic for the real value $x = 0$. Then it will not be analytic at the lines $y = x$ and $y = -x$. We can see it through the decomposition of the function:

$$\frac{1}{z} = \frac{1}{x + y\,e_1} = \frac{x - y\,e_1}{x^2 - y^2} = \frac{1+e_1}{2}\,\frac{1}{x+y} + \frac{1-e_1}{2}\,\frac{1}{x-y}$$

Now we may return to the question of the convergence of the Taylor series. A function can be only developed in a power series in the neighbourhood of a point where it is analytic. The series is convergent till where the function breaks the analyticity, so that the lines $y = x - x_0$ and $y = -x + x_0$ (being x_0 the real value for which $f(x)$ is not analytic) are boundaries of the convergence domain. On the other hand, due to the symmetry of the powers of z, the convergence domain forms a square around the centre of development. This implies that there are not multiply connected domains. The Gruyère cheese picture of a complex domain is not possible within the hyperbolic numbers, because if the function is not analytic at a certain point, then it is not analytic for all the points lying on a cross which passes through this point. The hyperbolic domains are multiply separated, in other words, formed by disjoined rectangular regions without holes.

About the isomorphism of Clifford algebras

Until now, I have only used the geometric algebra generated by the Euclidean plane vectors (usually noted as $Cl_{2,0}(\mathbf{R})$). This algebra already contains the hyperbolic numbers and hyperbolic vectors. However an isomorphic algebra $Cl_{1,1}(\mathbf{R})$ generated by hyperbolic vectors (time-space) is more used in relativity:

$$e_0^{\,2} = 1 \qquad e_1^{\,2} = -1 \qquad e_0\,e_1 = -e_1\,e_0 \qquad (e_0\,e_1)^2 = 1$$

The isomorphism is:

$$\begin{array}{ccc} Cl_{2,0}(\mathbf{R}) & \leftrightarrow & Cl_{1,1}(\mathbf{R}) \\ 1 & \leftrightarrow & 1 \\ e_1 & \leftrightarrow & e_{01} \\ e_2 & \leftrightarrow & e_0 \\ -e_{12} & \leftrightarrow & e_1 \end{array}$$

A hyperbolic number has the expression $z = x + y\, e_{01}$ and a hyperbolic vector $v = v_0\, e_0 + v_1\, e_1$.

This description perhaps could be satisfactory for a mathematician or a physicist, but not for me. I think that in fact both algebras only differ in the notation but not in their nature, so that they are the same algebra. Moreover, they are equal to the matrix algebra, which is the expected algebra for a space of multiple quantities (usually said *vectors*). The plane geometric algebra is the algebra of the 2×2 real matrices[5].

$$Cl_{2,0}(\mathbf{R}) = Cl_{1,1}(\mathbf{R}) = \mathbf{M}_{2\times 2}(\mathbf{R})$$

This distinct notation only expresses the fact that the plane geometric algebra is equally generated (in the Grassmann's sense) by Euclidean vectors or hyperbolic vectors. The vector plane (Euclidean or hyperbolic) is the quotient space of the geometric algebra divided by an even subalgebra (complex or hyperbolic numbers). Just this is the matter of the next chapter.

Exercises

12.1 Calculate the square roots of $5 + 4\, e_1$.

12.2 Solve the following equation: $2\, z^2 + 3\, z - 17 + 3\, e_1 = 0$

12.3 Solve directly the equation: $z^2 - 6\, z + 5 = 0$

12.4 Find the analytical continuation of the function $f(x) = \sin x$.

12.5 Find $\cosh 4\psi$ and $\sinh 4\psi$ as functions of $\cosh\psi$ and $\sinh\psi$.

12.6 Construct the analytical continuation of the real logarithm and see that it is identical to the logarithm found from the hyperbolic exponential function.

12.7 Calculate the integral $\int_{-e_1}^{e_1} z^2\, dz$ following a straight path $z = t\, e_1$ and a circular path $z = \cos t + e_1 \sin t$, and see that the result is identical to the integration via the primitive.

12.8 Prove that if $f(z)$ is a hyperbolic analytic function and does not vanish then it is a hyperbolic conformal mapping.

[5] In comparison, the 2×2 complex matrices (Pauli matrices) are a representation of the algebra of the tridimensional space.

13. THE HYPERBOLIC OR PSEUDO-EUCLIDEAN PLANE

Hyperbolic vectors

The elements not belonging to the subalgebra of the hyperbolic numbers form a quotient space[1], but not an algebra:

$$v = v_2 e_2 + v_{21}\, e_{21} = \begin{pmatrix} 0 & v_2 - v_{21} \\ v_2 + v_{21} & 0 \end{pmatrix}$$

In other words, their linear combinations are in the space, but the products are hyperbolic numbers. These elements play the same role as the Euclidean vectors with respect to the complex numbers. Because of this, they are called *hyperbolic vectors*. Like the hyperbolic numbers, the hyperbolic vectors have also a pseudo-Euclidean determinant:

$$\det(v_2\, e_2 + v_{21}\, e_{21}) = v_{21}{}^2 - v_2{}^2$$

Following the analogy with Euclidean vectors:

$$\det(w_1\, e_1 + w_2\, e_2) = -(w_1\, e_1 + w_2\, e_2)^2 = -|\, w\,|^2$$

the determinant of the hyperbolic vectors is also equal to its square with opposite sign:

$$\det(v_2\, e_2 + v_{21}\, e_{21}) = -(v_2\, e_2 + v_{21}\, e_{21})^2 = -|\, v\,|^2$$

whence the modulus of a hyperbolic vector can be defined:

$$|\, v\,| = \sqrt{v_2^2 - v_{21}^2}$$

Like for Euclidean vectors, the inverse of a hyperbolic vector is equal to this vector divided by its square (or square of the modulus):

$$(v_2\, e_2 + v_{21}\, e_{21})^{-1} = \frac{v_2\, e_2 + v_{21}\, e_{21}}{v_2{}^2 - v_{21}{}^2}$$

As before, there is not any privileged direction in the plane. Then every subspace whose elements have the form:

---

[1] Every Euclidean vector is obtained from e_1 through a rotation and dilation given by the complex number with the same components: $v_1\, e_1 + v_2\, e_2 = e_1\, (v_1 + v_2\, e_{12})$ so that the complex and vector planes are equivalent. This statement also holds in hyperbolic geometry: every hyperbolic vector is obtained from e_2 through a rotation and dilation given by the hyperbolic number with the same components: $v_2\, e_2 + v_{21}\, e_{21} = e_2\, (v_2 + v_{21}\, e_1)$ or in relativistic notation: $v_0\, e_0 + v_1\, e_1 = e_0\, (v_0 + v_1\, e_{01})$ so that both hyperbolic planes are equivalent. That reader accustomed to the relativistic notation should remember the isomorphism: $e_0 \leftrightarrow e_2$, $e_1 \leftrightarrow e_{21}$, $e_{01} \leftrightarrow e_1$.

$$v_w\, w + v_{21}\, e_{21}$$

where w is a unitary Euclidean vector perpendicular to the unitary Euclidean vector u, is also a subspace of hyperbolic vectors complementary of the hyperbolic numbers with the direction of u.

The hyperbolic vectors have the following properties, which you may prove:

1) The product of two hyperbolic vectors is always a hyperbolic number. This fact is shown by the following table, where the products of all the elements of the geometric algebra are summarised:

×	hyp. numbers	hyp. vectors
hyp. numbers	hyp. numbers	hyp. vectors
hyp. vectors	hyp. vectors	hyp. numbers

2) The conjugate of a product of two hyperbolic vectors x and y is equal to the product with these vectors exchanged:

$$(x\, y)^* = y\, x$$

3) The product of three hyperbolic vectors x, y and z fulfils the permutative property:

$$x\, y\, z = z\, y\, x$$

When a hyperbolic number n and a hyperbolic vector x are exchanged, the hyperbolic number becomes conjugated according to the permutative property:

$$n\, x = x\, n^*$$

4) The modulus of a product of hyperbolic vectors is equal to the product of moduli:

$$|v\, w| = |v|\, |w|$$

This property follows immediately from the fact that the determinant of a product of matrices is equal to the product of determinants of each matrix.

Inner and outer products of hyperbolic vectors

For any two hyperbolic vectors r and s such as:

$$r = r_2\, e_2 + r_{21}\, e_{21} \qquad s = s_2\, e_2 + s_{21}\, e_{21}$$

their inner and outer products are defined by means of the geometric (or matrix) product in the following way:

$$r \cdot s = \frac{r\, s + s\, r}{2} = r_2\, s_2 - r_{21}\, s_{21}$$

$$r \wedge s = \frac{r\,s - s\,r}{2} = (r_2\, s_{21} - r_{21}\, s_2)\, e_1$$

Then the product of two vectors can be written as the addition of both products:

$$r\,s = r \cdot s + r \wedge s$$

If the outer product of two hyperbolic vectors vanishes, they are proportional:

$$r \wedge s = 0 \qquad \Leftrightarrow \qquad \frac{r_2}{s_2} = \frac{r_{12}}{s_{12}} \qquad \Leftrightarrow \qquad r \parallel s$$

Two hyperbolic vectors are said to be *orthogonal* if their inner product vanishes:

$$r \perp s \quad \Leftrightarrow \quad r \cdot s = 0 \quad \Leftrightarrow \quad r_2\, s_2 - r_{21}\, s_{21} = 0 \quad \Leftrightarrow \quad \frac{r_2}{r_{21}} = \frac{s_{21}}{s_2}$$

When the last condition is fulfilled, we see both vectors being symmetric with respect to any quadrant bisector. When the ratio of components is equal to ±1 (directions of quadrant bisectors), the vectors have null modulus and are self-orthogonal. For ratios differing from ±1, one vector has negative determinant and the other positive determinant, so that they belong to different sectors. That is, there is not any pair of orthogonal vectors within the same sector.

Angles between hyperbolic vectors

The oriented angle α between two Euclidean vectors u and v (of the form $x\, e_1 + y\, e_2$) is obtained from the complex exponential function:

$$\exp(\alpha\, e_{12}) = \frac{u\,v}{|u|\,|v|} \quad \Leftrightarrow \quad \begin{cases} \cos\alpha = \dfrac{u \cdot v}{|u|\,|v|} \\ \sin\alpha = \dfrac{u \wedge v\, e_{21}}{|u|\,|v|} \end{cases} \quad \Leftrightarrow \quad \alpha = \operatorname{arctg} \frac{u \wedge v\, e_{21}}{u \cdot v} + (\pi\ e_{12})$$

Then, the oriented hyperbolic angle ψ between two hyperbolic vectors u and v (of the form $x\, e_2 + y\, e_{21}$) is defined as:

$$\exp(\psi\, e_1) = \frac{u\,v}{|u|\,|v|} \quad \Leftrightarrow \quad \begin{cases} \cosh\psi = \dfrac{u \cdot v}{|u|\,|v|} \\ \sinh\psi = \dfrac{u \wedge v\, e_1}{|u|\,|v|} \end{cases} \quad \Leftrightarrow$$

$$\psi = \arg\operatorname{tgh}\left(\frac{u \wedge v\, e_1}{u \cdot v}\right) + (\pi\ e_{12}) = \frac{1}{2} \log\left(\frac{u \cdot v + u \wedge v\, e_1}{u \cdot v - u \wedge v\, e_1}\right) + (\pi\ e_{12})$$

The parenthesis for $\pi\ e_{12}$ indicates that this angle is added to the arctangent or not depending on the quadrant (Euclidean plane) or half sector (hyperbolic plane).

Let us analyse whether these expressions are suitable. The hyperbolic cosine is always greater or equal to 1. It cannot describe the angle between two vectors belonging to opposite half sectors, which have a negative inner product. So we must keep the complex analytic continuation of the hyperbolic sine and cosine:

$$\sinh(x + e_{12}\ y) \equiv \sinh x \cos y + e_{12} \cosh x \sin y$$

$$\cosh(x + e_{12}\ y) \equiv \cosh x \cos y + e_{12} \sinh x \sin y$$

to get $\sinh(\psi + e_{12}\,\pi) \equiv -\sinh\psi$ and $\cosh(\psi + e_{12}\,\pi) \equiv -\cosh\psi$ with ψ real. Therefore opposite hyperbolic vectors form a circular angle of π radians[2].

Let us consider the angle between vectors lying on different sectors. The modulus of one vector is real while that of the other is imaginary what implies that the hyperbolic sine and cosine of the angle are imaginary. Using again the complex analytic continuation of the hyperbolic sine and cosine we find these imaginary values:

$$\sinh\left(\psi \pm e_{12}\,\frac{\pi}{2}\right) \equiv \pm e_{12} \cosh\psi \qquad \psi \text{ real}$$

$$\cosh\left(\psi \pm e_{12}\,\frac{\pi}{2}\right) \equiv \pm e_{12} \sinh\psi$$

Then, which is the angle between two orthogonal vectors? Since they have a null inner product while their outer product equals in real value the product of both moduli and the hyperbolic cosine is $\pm e_{12}$, they form an angle of $\pi\ e_{12}/2$ or $3\pi\ e_{12}/2$, that is, orthogonal hyperbolic vectors form a circular right angle:

$$\psi_\perp = \frac{1}{2}\log(-1) + (\pi\, e_{12}) = \frac{\pi}{2} e_{12} + (\pi\, e_{12})$$

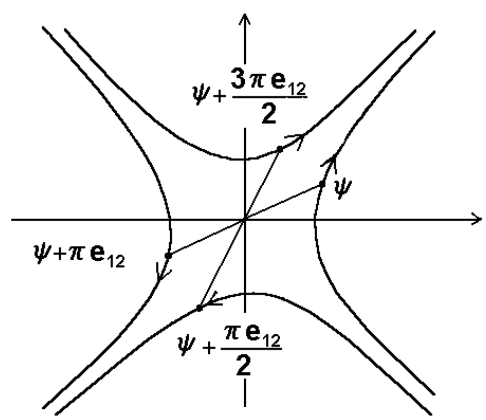


Figure 13.1

The analytic continuations of the hyperbolic trigonometric functions must be consistent with their definitions given in the page 142. For example, in the lowest half

[2] The so called "antimatter" is not really any special kind of particles, but matter with the energy-momentum vector (E, p) lying on the negative half sector instead of the positive one (usual matter). Also we may wonder whether matter having (E, p) on the imaginary sector exists since, from a geometric point of view, there is not any obstacle. It is known that the particle dynamics for $c<V<\infty$ is completely symmetric to the dynamics for $0\le V<c$, but with an imaginary mass. In this case, the head of the energy-momentum vector is always located on the hyperbola having the y-axis (p axis) as principal axis of symmetry. According to Einstein's formula, for $V = \infty$ we have $E = 0$ and $p = m\ c$, that is, the particle has a null energy! The technical question is how can a particle trespass the light barrier?

sector, the modulus of the hyperbolic radius is imaginary. The abscissa x and ordinate y divided by r, the modulus of the radius vector, is also imaginary so the angle is a real quantity plus a right angle and the hyperbolic sine and cosine are exchanged. The values of the angles consistent with the analytic continuation are displayed in the figure 13.1.

Congruence of segments and angles

Two segments (vectors) are *congruent* (have *equal length*) if their determinants are equal, in other words, when one segment can be obtained from the other through an isometry. Segments having equal length always lie on the same sector.

Two hyperbolic angles are said to be *congruent* (or *equal*) if they have the same moduli, that is, if they intercept arcs of hyperbola with the same hyperbolic length.

A triangle is said to be *isosceles* if it has two congruent sides. In colloquial language we talk about "sides with equal length", even "equal sides". Let us see the *isosceles triangle theorem*: the angles adjacent to the base of an isosceles triangle are equal. The proof uses the outer product. The triangle area is the half of the outer product of any two sides, as proved below in the section on the area. Suppose that the sides a and b have the same modulus. Then:

$$|a|\,|c|\,|\sinh\beta| = |b|\,|c|\,|\sinh\alpha| \qquad \Rightarrow \qquad |\beta| = |\alpha|$$

Isometries

In the Euclidean plane, an isometry is a geometric transformation that preserves the modulus of vectors and complex numbers. Now I give a more general definition: a geometric transformation is an *isometry* if it preserves the determinant of every element of the geometric algebra. In fact, the isometries are the inner automorphism of matrices:

$$\mathbf{A'} = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{B} \Rightarrow \qquad \det \mathbf{A'} = \det \mathbf{A}$$

When **B** represents a Euclidean vector, the isometry is a reflection in the direction of **B**. If a complex number of argument α is represented by **B**, the transformation is a rotation of angle 2α.

We wish now to obtain the isometries for hyperbolic vectors. The hyperbolic rotation of a hyperbolic vector (figure 13.2) is obtained through the product by an exponential having unity determinant:

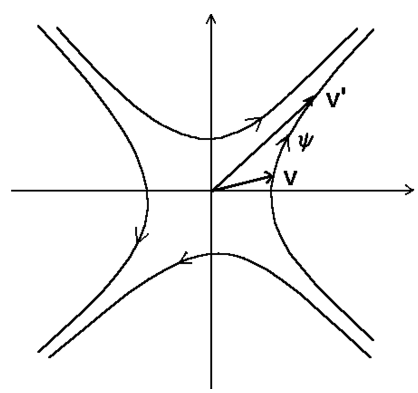


Figure 13.2

$$v'_2 e_2 + v'_{21}\, e_{21} = (v_2 e_2 + v_{21}\, e_{21})(\cosh\psi + \sinh\psi\, e_1)$$

Writing the components, we have:

$$\begin{cases} v'_2 = v_2 \cosh\psi + v_{21}\sinh\psi \\ v'_{21} = v_2 \sinh\psi + v_{21}\cosh\psi \end{cases}$$

which is the Lorentz transformation[3] of the relativity. When a vector is turned through a positive angle ψ, its extreme follows the hyperbola in the direction shown by the arrowheads in the figure 13.2, as is deduced from the components. So, they indicate the geometric positive sense of hyperbolic angles (a not trivial question). Since the points of intersections with the axis corresponds to $\psi = 0$ plus multiples of $\pi\, e_{12}/2$, the signs of the hyperbolic angles are determined: positive in the first and third quadrant, and negative in the second and fourth quadrant.

We can also write the hyperbolic rotation as an inner automorphism of matrices by using the half argument identity for hyperbolic trigonometric functions:

$$\cosh\psi + e_1 \sinh\psi \equiv \left(\cosh\frac{\psi}{2} + e_1 \sinh\frac{\psi}{2}\right)^2$$

and the permutative property:

$$v'_2 e_2 + v'_{21}\, e_{21} = \left(\cosh\frac{\psi}{2} - \sinh\frac{\psi}{2} e_1\right)(v_2 e_2 + v_{21}\, e_{21})\left(\cosh\frac{\psi}{2} + \sinh\frac{\psi}{2} e_1\right)$$

If $z = \exp(e_1 \psi/2)$, then the rotation is written as:

$$v' = z^{-1}\, v\, z$$

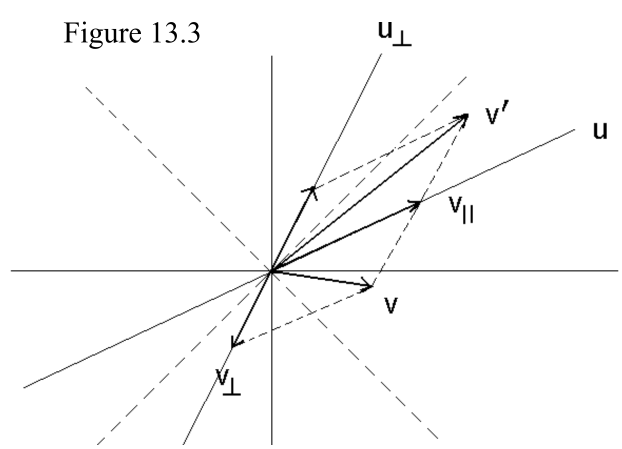
Now it is not needed that z be unitary and can have any modulus. Also, we see that the hyperbolic rotation leaves the hyperbolic numbers invariant.

Analogously to reflections in the Euclidean plane, the hyperbolic reflection of a hyperbolic vector v with respect to a direction given by the vector u is:

$$v' = u^{-1}\, v\, u = u^{-1}\, (\, v_{\|} + v_{\perp})\, u = v_{\|} - v_{\perp}$$

because the proportional vectors commute and those which are orthogonal anticommute (also in the hyperbolic case, because the inner product is zero). Every pair of orthogonal directions (for example $u_{\|}$ and $u_{\perp}$ in figure 13.3) are always seen by us as having symmetry with respect to the quadrant bisectors. Since the determinant is preserved, a vector and its reflection always belong to the same sector, the hyperbolic angle between each vector and the direction of reflection being equal with opposite sign.

Figure 13.3



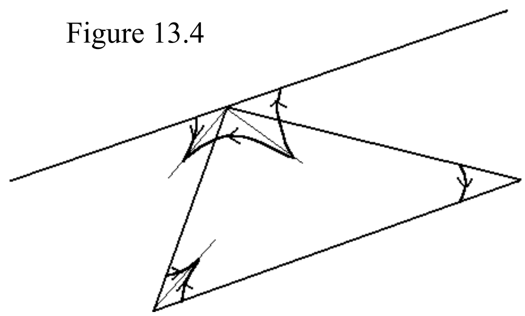
[3] The hyperbolic vector in the space-time is $ct\, e_0 + x\, e_1$ where x is the space coordinate, t the time and c the light celerity. The argument of the hyperbolic rotation ψ is related with the relative velocity V of both frames of reference through $\psi = \text{arg tgh } V/c$.

Finally, note that the extremes of a vector and its transformed vector under a hyperbolic reflection (or any isometry) lie on an equilateral hyperbola.

Theorems about angles

<u>The sum of the oriented angles of a hyperbolic triangle is minus two right circular angles</u>. To prove this theorem, draw a line parallel to a side and apply the Z-theorem (figure 13.4): they will form an angle between opposite directions, which has the value $-\pi\ e_{12}$. The minus sign is caused by the fact that, following the positive orientation of angles, a right angle is subtracted each time an asymptote is trespassed.

Figure 13.4



Look at the figure 13.5 where an angle inscribed in an equilateral hyperbola is drawn. A diameter divides the inscribed angle into the angles α and β. Then we draw two radius from the origin to both extremes of the angle. All the radius have the same length, so that the upper and lowest triangle are isosceles and have two equal angles. Since the sum of the angles of each hyperbolic triangle is $-\pi\ e_{12}$, the third angle is $-\pi\ e_{12}-2\alpha$ and $-\pi\ e_{12}-2\beta$ respectively. So the supplementary angles are 2α and 2β, and hence the theorem follows: <u>the inscribed angle on a hyperbola $x^2 - y^2 = r^2$ is the half of the central angle (intercepted arc) so that it is constant and independent of the location of its vertex</u>.

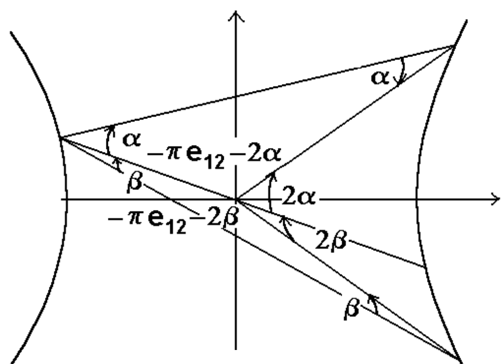


Figure 13.5

For example, take the points (5, 3) and (5, −3) as extremes of an inscribed angle with the vertex placed at the negative branch, for example at (−4, 0) or (−5, −3). Even you can take the vertex at the positive branch, e. g. at (4, 0). Anyway its value is log 2. Now calculate the central angle or intercepted arc (with vertex at the origin) and see that its value is log 4, twice the inscribed angle.

Distance between points

The *distance between two points* on the hyperbolic plane is defined as the modulus of the vector going from one point to the another:

$$d(P,Q) = |\,PQ\,| = \sqrt{(x_Q - x_P)^2 - (y_Q - y_P)^2}$$

In the hyperbolic plane, the distance is a real or imaginary positive number (depending on the sector), which has the following properties:

1) $d(P,Q) = d(Q,P)$
2) If the vectors *PQ*, *QR* and *PR* lie on the positive half sector then they fulfil the triangular inequality:

   $$d(P,R) \geq d(P,Q) + d(Q,R)$$

   The prove is obtained by means of the inner product:

$$PR^2 = (PQ + QR)^2 = PQ^2 + QR^2 + 2\, PQ \cdot QR$$

$$= PQ^2 + QR^2 + 2\,|\,PQ\,|\,|\,QR\,|\cosh\psi \geq PQ^2 + QR^2 + 2\,|\,PQ\,|\,|\,QR\,|$$

The extraction of the square root yields the triangular inequality:

$$|\,PR\,| \geq |\,PQ\,| + |\,QR\,|$$

So the sum of two sides of a triangle is smaller than the third side whenever they are taken in the positive half sector. For example, let us apply this statement to the triangle with vertices A = (5, 3), B = (1, 0) and C = (10, 1). Taking the modulus of the sides positive, we have:

$$|\,BC\,| = \sqrt{80} \qquad |\,BA\,| = \sqrt{7} \qquad |\,AC\,| = \sqrt{21}$$

$$|\,BC\,| \geq |\,BA\,| + |\,AC\,| \qquad \Rightarrow \qquad \sqrt{80} \geq \sqrt{7} + \sqrt{21}$$

This result must be commented with more detail. Firstly, we see that the straight path has the highest length among the possible paths between two given points. In the Minkowski's space-time this fact causes the *twin paradox*. Since the pseudo-Euclidean length of the path is the proper time for each person, the brother who followed the straight path -going in an inertial frame- has aged more than the brother who followed another path -subjected to accelerations-.

Area on the hyperbolic plane

Now I give a general geometric definition of area valid for both Euclidean and hyperbolic planes. If a parallelogram has orthogonal sides, then the modulus of its *area* is equal to the product of the lengths of the orthogonal sides. In Euclidean geometry a parallelogram with orthogonal sides is called a *rectangle*. In the hyperbolic plane, two sides are orthogonal if their directions are seen by us as symmetric with respect to the direction of the quadrant bisector. So, I have preferred to avoid the word *rectangle* in this case, while the term *parallelogram* is still valid within this context.

Which is the suitable algebraic expression to calculate the area of any parallelogram? Suppose that it is the outer product. If a and b are the sides of the parallelogram then:

$$|A| = |a \wedge b| = |a||b||\sinh \psi(a,b)|$$

Let us see the consistence of this expression. First at all, recall that the angle between orthogonal hyperbolic vectors is a right circular angle:

$$\psi_{\perp} = \pm\frac{\pi}{2} e_{12}$$

so that the area of a parallelogram with orthogonal sides is the product of their lengths as expected:

$$|A| = |a||b|\left|\sinh\left(\pm\frac{\pi}{2} e_{12}\right)\right| = |a||b|\sin\frac{\pi}{2} = |a||b|$$

For any parallelogram not having orthogonal sides, the area is the product of the *base* (one side) for the *altitude* (the projection of the other side onto the direction orthogonal to the base) and this product is only given by the outer product because its anticommutativity removes the proportional projection:

$$|A| = |a \wedge b| = |(a_{\parallel} + a_{\perp}) \wedge b| = |a_{\perp}||b|$$

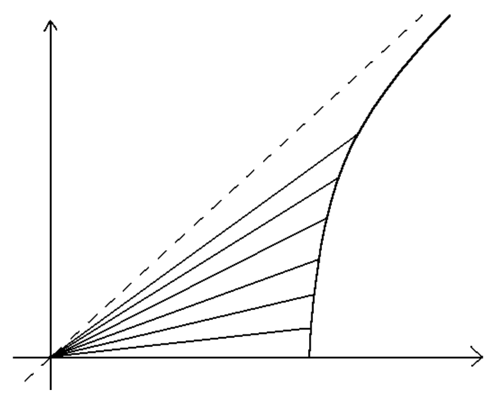


Figure 13.6

The expression for the area in Cartesian components is equal to that for the Euclidean plane. It means that we can calculate areas graphically in the usual way, fact that has allowed till now to define the hyperbolic trigonometric functions from the area scanned by the hyperbola radius in the Euclidean plane, in spite of not being their proper plane. However, as I have explained in the footnote at the page 142, we must perceive the pseudo-Euclidean nature of the area in the hyperbolic plane. The figure 13.6 displays how are the radii perpendicular to the hyperbola because the bisector of the radius and the tangent vector is parallel to the quadrant bisector. Also it may be proved analytically by means of differentiation of the equation $x^2 - y^2 = r^2$:

$$2\,x\,dx - 2\,y\,dy = 0 \quad \Leftrightarrow \quad r \cdot dr = 0$$

showing that the radius vector and its differential are orthogonal for the hyperbola with constant radius. On the other hand, observe that the triangles drawn in the figure 13.6 are isosceles and have two equal angles approaching the right angle value at the limit of null area.

Diameters of the hyperbola and Apollonius' theorem

At the page 129, I postponed an interpretation of the central equation for the hyperbola analogous to the cylindrical angles for the ellipse because it is properly of pseudo-Euclidean nature. Here I develop this interpretation.

If P is any point on a hyperbola with major and minor half-axis OQ and OS (figure 11.17 reproduced below) then its central equation is:

$$OP = \pm \frac{OQ \cosh \chi + OS \dfrac{\sinh \chi}{\sqrt{e^2 - 1}}}{\sqrt{\cosh^2 \chi - \dfrac{\sinh^2 \chi}{e^2 - 1}}}$$

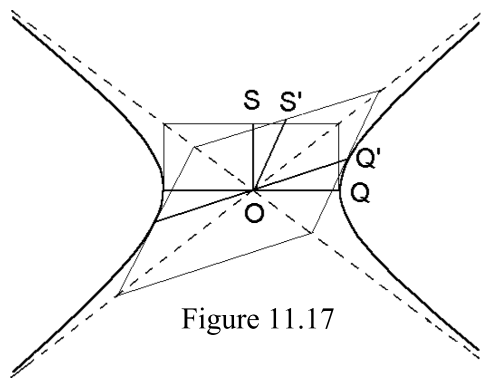


Figure 11.17

Since OQ and OS are orthogonal, the square of this equation fulfils:

$$\frac{OP^2}{OQ^2} \cosh^2 \chi + \frac{OP^2}{OS^2} \sinh^2 \chi = 1$$

Both half-axis are related with the eccentricity through:

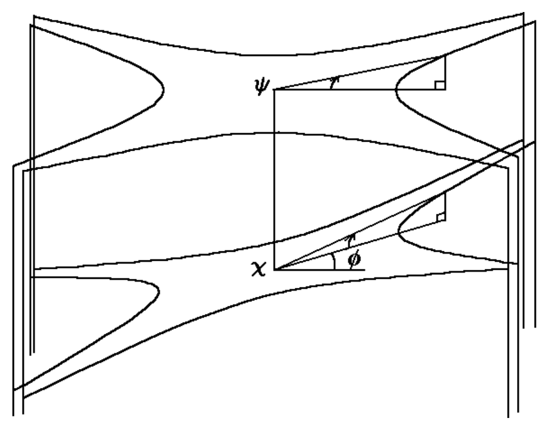
$$OS^2 = \left(1 - e^2\right) OQ^2$$

Note that $OS^2 < 0$ since it is the square of an ordinate on the hyperbolic plane.

Introducing the hyperbolic angle ψ in the following way:

$$\cosh \psi = \frac{\cosh \chi}{\sqrt{\cosh^2 \chi - \dfrac{\sinh^2 \chi}{e^2 - 1}}}$$

Figure 13.7



now the equation of the hyperbola becomes:

$$OP = \pm \left(OQ \cosh \psi + OS \sinh \psi \right)$$

Observe in the figure 13.7 that any hyperbola can be obtained as an intersection of a transverse plane with the equilateral hyperbolic prism. The plane of the acute hyperbola (with $1 < e < \sqrt{2}$) forms an angle ϕ with respect to the horizontal, and hence:

$$\cosh \psi = \frac{\cosh \chi \cos \phi}{\sqrt{\cosh^2 \chi \cos^2 \phi - \sinh^2 \chi}}$$

So the eccentricity e of the hyperbola is related with the obliquity ϕ of the transverse plane through the relationship:

$$\cos\phi = \sqrt{\mathrm{e}^2 - 1} \quad \text{with} \quad 1 < \mathrm{e} < \sqrt{2}$$

The Apollonius' conjugate diameters of any hyperbola are the intersections of the transverse plane with a pair of orthogonal axial planes; in other words, two radii are conjugate (figure 11.17) if their projections onto the horizontal plane are turned through the same hyperbolic angle φ:

$$OQ' = OQ \cosh\varphi + OS \sinh\varphi$$

$$OS' = OQ \sinh\varphi + OS \cosh\varphi$$

Our Euclidean eyes see the horizontal projections as symmetric lines with respect to the quadrant bisector. However, they are actually orthogonal because:

$$OQ'^2 - OS'^2 = OQ^2 - OS^2$$

and, therefore, can be taken as a new system of orthogonal coordinates. Even we can draw a new picture with the new diameters on the Cartesian axis.

The central equation of the hyperbola using the rotated axis is:

$$OP = \pm\left(OQ' \cosh(\psi - \varphi) + OS' \sinh(\psi - \varphi)\right)$$

which shows that a hyperbolic rotation of the coordinate axis has been made with respect to the principal diameter of the hyperbola.

The law of sines and cosines

Since the modulus of the area is identical on the Euclidean and hyperbolic planes, a parallelogram is divided by its diagonal in two triangles of equal area. This statement is somewhat subtle since the Euclidean congruence of triangles is not valid in the hyperbolic plane. I shall return to this question later. Now we only need to know that the area of a hyperbolic triangle is the half of the outer product of any two sides.

Following the perimeter of a triangle, let *a*, *b*, and *c* be its sides respectively opposite to the angles α, β and γ. Then the angles formed by the oriented sides are supplementary of the angles of the triangle and:

$$a \wedge b = b \wedge c = c \wedge a \quad \Rightarrow \quad -|a||b| \sinh\gamma = -|b||c| \sinh\alpha = -|c||a| \sinh\beta$$

$$\frac{|a|}{\sinh\alpha} = \frac{|b|}{\sinh\beta} = \frac{|c|}{\sinh\gamma}$$

which is the *law of sines*.

From $a + b + c = 0$, we have:

$$a^2 = (-b-c)^2 = b^2 + c^2 + 2\,b\cdot c \quad \Rightarrow \quad a^2 = b^2 + c^2 - 2\,|b|\,|c|\cosh\alpha$$

which is the *law of cosines*. And also:

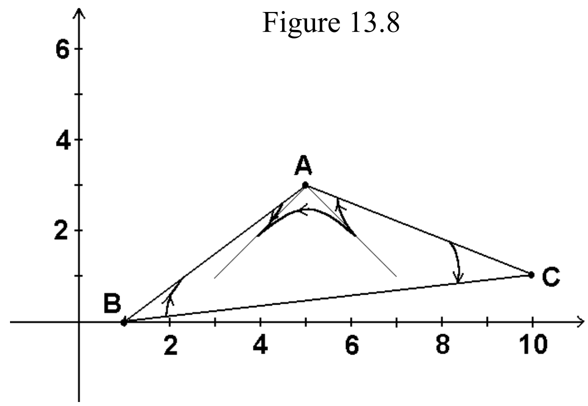
$$b^2 = a^2 + c^2 - 2\,|a|\,|c|\cosh\beta$$

$$c^2 = a^2 + b^2 - 2\,|a|\,|b|\cosh\gamma$$

When applying both theorems, we must take care with the sides having imaginary length and the signs of the angles and trigonometric functions.

As an application of the law of sines and cosines, consider the hyperbolic triangle with vertices A=(5, 3), B=(1, 0), C=(10, 1), whose sides belong to the real sector (figure 13.8):

Figure 13.8



$$AB = B - A = -4\,e_2 - 3\,e_{21} \qquad |AB| = \sqrt{(-4)^2 - (-3)^2} = \sqrt{7}$$

$$BC = C - B = 9\,e_2 + e_{21} \qquad |BC| = \sqrt{9^2 - 1^2} = \sqrt{80}$$

$$CA = A - C = -5\,e_2 + 2\,e_{21} \qquad |CA| = \sqrt{(-5)^2 - 2^2} = \sqrt{21}$$

$$BC^2 = CA^2 + AB^2 - 2\,|CA|\,|AB|\cosh\alpha \qquad \cosh\alpha = -\frac{26}{7\sqrt{3}}$$

$$CA^2 = AB^2 + BC^2 - 2\,|AB|\,|BC|\cosh\beta \qquad \cosh\beta = \frac{33}{4\sqrt{35}}$$

$$AB^2 = BC^2 + CA^2 - 2\,|BC|\,|CA|\cosh\gamma \qquad \cosh\gamma = \frac{47}{4\sqrt{105}}$$

From where it follows that:

$$\alpha = -1.3966... - \pi\, e_{12} \qquad \beta = 0.8614... \qquad \gamma = 0.5352...$$

I have obtained their signs from the definition of the angles $\alpha = BAC$, $\beta = CBA$, $\gamma = ACB$ and the geometric plot (figure 13.8). Note that $\alpha + \beta + \gamma = -\pi\, e_{12}$ and they fulfil the law of sines:

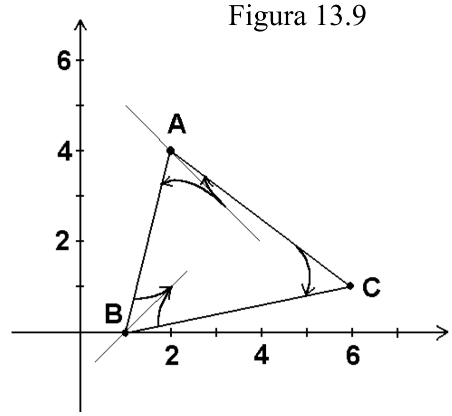
$$\frac{|BC|}{\sinh\alpha} = \frac{|CA|}{\sinh\beta} = \frac{|AB|}{\sinh\gamma}$$

because $\sinh\alpha = \sinh(-1.3966+ \pi e_{12}) = -\sinh(-1.3966) = \sinh 1.3966$.

Consider now another triangle $A = (2, 4)$, $B = (1,0)$ and $C = (6, 1)$, having sides on real and imaginary sectors (figure 13.9):

$$AB = B - A = -e_2 - 4\, e_{21}$$

Figura 13.9



$$|\,AB\,| = \sqrt{(-1)^2 - (-4)^2} = \sqrt{15}\, e_{12}$$

$$BC = C - B = 5\, e_2 + e_{21}$$

$$|\,BC\,| = \sqrt{5^2 - 1^2} = \sqrt{24}$$

$$CA = A - C = -4\, e_2 + 3\, e_{21}$$

$$|\,CA\,| = \sqrt{(-4)^2 - 3^2} = \sqrt{7}$$

$$BC^2 = CA^2 + AB^2 - 2\,|\,CA\,|\,|\,AB\,| \cosh\alpha \qquad \cosh\alpha = \frac{16\, e_{12}}{\sqrt{105}}$$

$$CA^2 = AB^2 + BC^2 - 2\,|\,AB\,|\,|\,BC\,| \cosh\beta \qquad \cosh\beta = \frac{-\, e_{12}}{6\sqrt{10}}$$

$$AB^2 = BC^2 + CA^2 - 2\,|\,BC\,|\,|\,CA\,| \cosh\gamma \qquad \cosh\gamma = \frac{23}{2\sqrt{42}}$$

From the last hyperbolic cosine, which is real, we find the hyperbolic sine for γ, which is a positive angle as shown by the plot:

$$\sinh\gamma = \frac{19}{2\sqrt{42}}$$

which we may use in the law of sines:

$$\frac{\sqrt{24}}{\sinh\alpha} = \frac{\sqrt{7}}{\sinh\beta} = \frac{\sqrt{15}\, e_{12}}{19/2\sqrt{42}} \qquad \Rightarrow \qquad \sinh\alpha = -\frac{19\, e_{12}}{\sqrt{105}} \qquad \sinh\beta = -\frac{19\, e_{12}}{6\sqrt{10}}$$

Recalling that for ψ real:

$$\sinh\left(\psi \pm e_{12}\,\frac{\pi}{2}\right) \equiv \pm e_{12} \cosh\psi \qquad \cosh\left(\psi \pm e_{12}\,\frac{\pi}{2}\right) \equiv \pm e_{12} \sinh\psi$$

the angles α, β and γ follow:

$$\alpha = -1.2284... - \frac{\pi}{2} e_{12} \qquad \beta = 0.0527... - \frac{\pi}{2} e_{12} \qquad \gamma = 1.1757$$

Observe that the addition of the three angles is $-\pi\ e_{12}$, as expected. According to the definition $\alpha = BAC$, $\beta = CBA$, $\gamma = ACB$, let you see the consistence with the geometric plot. The angle γ has positive sign as shown by the figure 13.9. The bisector of α parallel to the quadrant bisector divides it in two angles, one real and the other complex (with imaginary part $\pi\ e_{12}\ /2$). The algebraic addition of both angles is α.. Taking into account that the second has opposite orientation and must be subtracted and predominates over the first, the negative value of α is explained.

Hyperbolic similarity

Two triangles ABC and $A'B'C'$ are said to be *directly similar*[4] and their vertices and sides denoted with the same letters are *homologous* if:

$$AB\ BC^{-1} = A'B'\ B'C'^{-1} \qquad \Rightarrow \qquad AB^{-1}\ A'B' = BC^{-1}\ B'C'$$

One can prove easily that the third quotient of homologous sides also coincides with the other quotients:

$$AB^{-1}\ A'B' = BC^{-1}\ B'C' = CA^{-1}\ C'A' = r$$

The *similarity ratio* r is defined as the quotient of every pair of homologous sides, which is a hyperbolic number. The modulus of the similarity ratio is the size ratio and the argument is the angle of rotation of the triangle $A'B'C'$ with respect to the triangle ABC.

$$r = \frac{|A'B'|}{|AB|} \exp\left[\alpha\left(AB, A'B'\right) e_1\right]$$

The definition of similarity is generalised to any pair of polygons in the following way. The polygons $ABC...Z$ and $A'B'C'...Z'$ are said to be directly similar with similarity ratio r and the sides denoted with the same letters to be homologous if:

$$r = AB^{-1}\ A'B' = BC^{-1}\ B'C' = CD^{-1}\ C'D' = ... = YZ^{-1}\ Y'Z' = ZA^{-1}\ Z'A'$$

Here also, the modulus of r is the size ratio of both polygons and the argument is the angle of rotation. The fact that the homologous exterior and interior angles are equal for directly similar polygons is trivial because:

$$B'A'\ B'C'^{-1} = BA\ BC^{-1} \qquad \Rightarrow \qquad \text{angle } A'B'C' = \text{angle } ABC$$

$$C'B'\ C'D'^{-1} = CB\ CD^{-1} \qquad \Rightarrow \qquad \text{angle } B'C'D' = \text{angle } BCD \qquad \text{etc.}$$

---

[4] A direct similarity is also called a *similitude* and an opposite similarity sometimes an *antisimilitude*.

The direct similarity is an equivalence relation since it has the reflexive, symmetric and transitive properties. This means that there are classes of equivalence with directly similar figures.

A similitude with $|r|=1$ is a *displacement*, since both polygons have the same size and orientation.

Two triangles ABC and $A'B'C'$ are *oppositely similar* and the sides denoted with the same letters are *homologous* if:

$$AB\,BC^{-1} = (\,A'B'\,B'C'^{-1}\,)^* = B'C'^{-1}\,A'B'$$

where the asterisk denotes the hyperbolic conjugate. The former equality cannot be arranged into quotients of pairs of homologous sides as done before. Because of this, the similarity ratio cannot be defined for the opposite similarity but only the size ratio, which is the quotient of the lengths of any two homologous sides. An opposite similarity is always the composition of a reflection in any line and a direct similarity.

$$AB\,BC^{-1} = v^{-1}\,A'B'\,B'C'^{-1}\,v \quad \Leftrightarrow \quad BC^{-1}\,v^{-1}\,B'C' = AB^{-1}\,v^{-1}\,A'B' \quad \Leftrightarrow$$

$$BC^{-1}\,(\,v^{-1}\,B'C'\,v\,) = AB^{-1}\,(\,v^{-1}\,A'B'^{-1}\,v\,) = r$$

where r is the ratio of a direct similarity whose argument is not defined but depends on the direction vector v of the reflection axis. Notwithstanding, this expression allows to define the opposite similarity of two polygons. So two polygons $ABC...Z$ and $A'B'C'...Z'$ are oppositely similar and the sides denoted with the same letters are homologous if for any hyperbolic vector v the following equalities are fulfilled:

$$AB^{-1}\,(\,v^{-1}\,A'B'^{-1}\,v\,) = BC^{-1}\,(\,v^{-1}\,B'C'\,v\,) = ... = ZA^{-1}\,(\,v^{-1}\,Z'A'\,v\,)$$

that is, if after a reflection one polygon is directly similar to the other. The opposite similarity is not reflexive nor transitive and there are not classes of oppositely similar figures.

An opposite similarity with $|r|=1$ is called a *reversal*, since both polygons have the same size but opposite orientations.

Obviously the plot of similar figures on the hyperbolic plane breaks our Euclidean intuition about figures with the same form. The algebraic definition is very rigorous and clear, but we must change our visual illusions. So I recommend the exercise 13.6.

This chapter is necessarily unfinished because the geometry of the hyperbolic plane can be developed and studied with the same extension and profundity as for the Euclidean plane. Then, it is obvious that many theorems should follow in the same way as Euclidean geometry. As a last example I will define the power of a point with respect to a hyperbola.

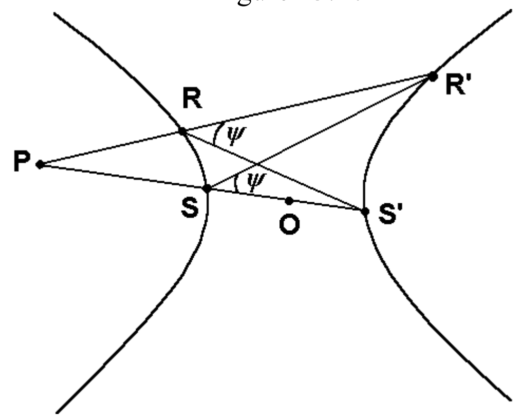
Power of a point with respect to a hyperbola with constant radius

The locus of the points placed at a fixed distance r from a given point O is an equilateral hyperbola centred at this point with equation:

$$|OP| = r \qquad\qquad (x - x_O)^2 - (y - y_O)^2 = r^2$$

The *power of a point* with respect to this hyperbola is the product of both oriented distances from P to the intersections R and R' of a line passing through P with the hyperbola. The power of a point is constant for every line of the pencil of lines of P. In the proof the inscribed angle theorem is used (figure 13.10): the angles $S'RR'$ and $S'SR'$ are equal so the triangles SPR' and RPS' are oppositely similar. Then:

Figure 13.10



$$PR'\, PS^{-1} = PR^{-1} PS' \;\Rightarrow\; PR\, PR' = PS'\, PS$$

Developing the product of distances on the line passing through the centre of the hyperbola we find:

$$PS\, PS' = (PO + OS)(PO + OS') = PO^2 + OS\, OS' = (x_P - x_O)^2 - (y_P - y_O)^2 - r^2$$

that is, the power of a point is obtained by substitution of the coordinates of P on the hyperbola equation.

Exercises

13.1 Let A = (2, 2), B = (1, 0) and C = (5, 3) be the vertices of a hyperbolic triangle. Calculate all the sides and angles and also the area.

13.2 Turn the vector 2 e_2 + e_{21} through an angle ψ = log 2. Do a reflection in the direction 3 $e_2 - e_{21}$. Make also an inversion with radius 3.

13.3 Find the direction and normal vectors of the line $y = 2x + 1$ and calculate the perpendicular line passing through the point (3, 1).

13.4 Calculate the power of the point P = (–7, 3) with respect to the hyperbola $x^2 - y^2 = 16$ using the intersections of the lines $y = 3$, $y = -3x - 18$ and that passing through the centre of the hyperbola $y = -3x / 7$. See that in all cases the power of P is equal to the value found by substitution in the Cartesian equation.

13.5 In this chapter I have deduced the law of sines and cosines. Therefore a law of tangents should be expected. Find and prove it.

13.6 Check that the triangle A = (0, 0), B = (5, 0) and C = (5, 3) is directly similar to the triangle A' = (0, 0), B' = (25, –15), C' = (16, 0). Find the similarity ratio and the rotation and dilation of the corresponding homothety. Draw the triangles and you will astonish.

FOURTH PART: PLANE PROJECTIONS OF TRIDIMENSIONAL SPACES

The complete study of the geometric algebra of the tridimensional spaces falls out the scope of this book. However, due to the importance of the Earth charts and of the Lobachevsky's geometry, the first one being more practical and the second one more theoretical, I have written this last section. In order to make the explanations clearer, the tridimensional geometric algebra has been reduced to the minimal concepts, enhancing the plane projections.

The geometric quality of being Euclidean or pseudo-Euclidean is not the signature + or − of a coordinate, but the fact that two coordinates have the same or different signature, in other words, it is a characteristic of a plane. For instance, a plane with signatures + + is equivalent, from a geometric point of view, to another with − − . Therefore, only two kinds of three-dimensional spaces exist: the room space where all the planes are Euclidean (signatures + + + or − − −), and the pseudo-Euclidean space, which has one Euclidean plane and two orthogonal pseudo-Euclidean planes (signatures + − − or + + −).

14. SPHERICAL GEOMETRY IN THE EUCLIDEAN SPACE

The geometric algebra of the Euclidean space

A *vector* of the Euclidean space is an oriented segment in this space with direction and sense, although it can represent other physical magnitudes such as forces, velocities, etc. The set of all the segments (geometric vectors) together with their addition (parallelogram rule) and the product by real numbers (dilation of vectors) has a structure of vector space, symbolised with V_3. Every vector in V_3 is of the form:

$$v = v_1\, e_1 + v_2\, e_2 + v_3\, e_3$$

where e_i are three unitary perpendicular vectors, which form the base of the space. If we define an associative product (*geometric* or *Clifford product*) being a generalisation of that defined for the plane in the first chapter of this book, we will arrive to:

$$e_i^2 = 1 \qquad \text{and} \qquad e_i\, e_j = -e_j\, e_i \ \text{ for } i \neq j$$

In general, the square of a vector is the square of its modulus and perpendicular vectors anticommute whereas proportional vectors commute.

From the base vectors one deduces that the geometric algebra generated by the space V_3 has eight components:

$$Cl(V_3) = Cl_{3,\,0} = \langle 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123} \rangle$$

Let us see with more detail the product of two vectors:

$$v\, w = (v_1\, e_1 + v_2\, e_2 + v_3\, e_3)(w_1\, e_1 + w_2\, e_2 + w_3\, e_3) = v_1\, w_1 + v_2\, w_2 + v_3\, w_3$$

$$+ (v_2\, w_3 - v_3\, w_2)\, e_{23} + (v_3\, w_1 - v_1\, w_3)\, e_{31} + (v_1\, w_2 - v_2\, w_1)\, e_{12}$$

The product (or quotient) of two vectors is said a *quaternion*[1]. The quaternions are the even subalgebra of $Cl_{3,0}$ that generalise the complex numbers to the space. Splitting a quaternion in the real and bivector parts, we obtain the *inner* (or *scalar*) product and the *outer* (or *exterior*) product respectively:

$$v \cdot w = v_1\, w_1 + v_2\, w_2 + v_3\, w_3$$

$$v \wedge w = (v_2\, w_3 - v_3\, w_2)\, e_{23} + (v_3\, w_1 - v_1\, w_3)\, e_{31} + (v_1\, w_2 - v_2\, w_1)\, e_{12}$$

The *bivectors* are oriented plane surfaces and indicate the direction of planes in the space. Who be acquainted with the vector analysis will say that both vectors and bivectors are the same thing. This confusion was originated by Hamilton[2] himself, and continued by the founders of vector analysis, Gibbs and Heaviside. However, vectors and bivectors are different things just as physicists have experienced and know long time ago. The proper vectors are called usually "polar vectors" while the pseudo-vectors that actually are bivectors are usually called "axial vectors". The following magnitudes are vectors: of course a geometric segment, but also a velocity, an electric field, the momentum, etc. On the other hand, the oriented area is, of course, a bivector, but also the angular momentum, the angular velocity and the magnetic field. As a criterion to distinguish both kind of magnitudes one uses the reversal of coordinates, which changes the sense of vectors while leaves bivectors invariant.

The product of two bivectors yields a real number plus a bivector. Both parts can be separated as the symmetric and antisymmetric product. The symmetric product is a real number and its negative value will be denoted here with the symbol $\bullet$ while the antisymmetric product is also a bivector and will be noted here with the symbol $\times$:

$$v \bullet w = -\frac{1}{2}(v\, w + w\, v) = v_{23}\, w_{23} + v_{31}\, w_{31} + v_{12}\, w_{12}$$

$$\begin{aligned} v \times w &= -\frac{1}{2}(v\, w - w\, v) \\ &= (v_{31}\, w_{12} - v_{12}\, w_{31})\, e_{23} + (v_{12}\, w_{23} - v_{23}\, w_{12})\, e_{31} + (v_{23}\, w_{31} - v_{31}\, w_{23})\, e_{12} \end{aligned}$$

$$v\, w = -v \bullet w - v \times w$$

Let us see what happens with the outer product of three vectors. According to the extension theory of Grassmann, the product $u \wedge v \wedge w$ is the oriented volume generated by the surface represented by the bivector $u \wedge v$ when it is translated parallelly along the segment w:

---

[1] Hamilton discovered the quaternions in October 16th 1843 and defined them as quotients of two vectors. From this definition he deduced the properties of the product of quaternions. I recommend you the reading of the initial chapters of the *Elements of Quaternions* because of its pedagogic importance.

[2] This confusion is due to the fact that vectors and bivectors are dual spaces of the algebra $Cl_{3,0}$. However, this duality does not exist at higher dimensions, although there is also duality among other spaces.

$$u \wedge v \wedge w = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} e_{123}$$

Finally, let us see how is the product of three vectors u, v and w. The vector v can be resolved into a component coplanar with u and v and another component perpendicular to the plane u-v:

$$u\, v\, w = u\, v_{\parallel}\, w + u\, v_{\perp}\, w$$

Now let us analyse the permutative property. In the plane we found $u\, v\, w - w\, v\, u = 0$. In the space the permutative property becomes[3]:

$$u\, v\, w - w\, v\, u = u\, v_{\perp}\, w - w\, v_{\perp}\, u = v_{\perp}\left(-u\, w + w\, u\right)$$

$$= -2\, v_{\perp}\, u \wedge w = -2\, v \wedge u \wedge w = 2\, u \wedge v \wedge w$$

I take the same algebraic hierarchies as in chapter 1: all the abridged products must be operated before the geometric product, convention which is coherent with the fact that in many algebraic situations, the abridged products must be developed in sums of geometric products.

Spherical trigonometry

In this section the relations for the sides and angles of the spherical triangles are deduced. I will take for convenience the sphere having unity radius, although the trigonometric identities are equally valid for a sphere of any radius.

Let us consider any three points A, B and C on the sphere with unity radius centred at the origin (figure 14.1). Then $|A| = |B| = |C| = 1$. The angles formed by each pair of sides will be denoted by α, β and γ, and the sides respectively opposite to these angles will be symbolised by a, b and c respectively. Then a is the arc of the great circle passing through the points B and C, that is, the angle between these vectors:

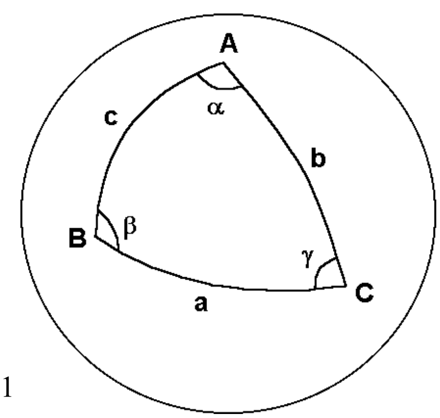


Figure 14.1

$$\sin a = |\, B \wedge C \,|$$

[3] From this result it follows that $u \wedge v \wedge w = (u\, v\, w - u\, w\, v + v\, w\, u - v\, u\, w + w\, u\, v - w\, v\, u) / 6$. In other words, the outer product is a fully antisymmetric product. However although being beautiful, it is not so useful for geometric algebra as the permutative property $u \wedge v \wedge w = (u\, v\, w - w\, v\, u)/2$.

Note that in this equality the sine is positive and therefore $a \leq \pi$. So, the spherical trigonometry is deduced for *strict triangles*, those having a, b, $c \leq \pi$.

Also, α is the angle between the sides b and c of the triangle, that is, the angle between the planes passing through the origin, A and B, and the origin, A and C. Since the direction of a plane is given by its bivector, which can be obtained through the outer product, we have:

$$\sin\alpha = \frac{|(A \wedge B) \times (A \wedge C)|}{|A \wedge B||A \wedge C|}$$

Now we write the products of the numerator using the geometric product:

$$\sin\alpha = \frac{|-(A\,B - B\,A)(A\,C - C\,A) + (A\,C - C\,A)(A\,B - B\,A)|}{8|A \wedge B||A \wedge C|}$$

$$\sin\alpha = \frac{|-A\,B\,A\,C + A\,B\,C\,A + B\,A^2\,C - B\,A\,C\,A + A\,C\,A\,B - A\,C\,B\,A - C\,A^2\,B + C\,A\,B\,A|}{8|A \wedge B||A \wedge C|}$$

We extract the vector A as common factor at the left, but without writing it because $|A| = 1$:

$$\sin\alpha = \frac{|-B\,A\,C + B\,C\,A + A\,B\,C - A^{-1}\,B\,A\,C\,A + C\,A\,B - C\,B\,A - A\,C\,B + A^{-1}\,C\,A\,B\,A|}{8|A \wedge B||A \wedge C|}$$

Applying the permutative property to the suitable pairs of products, we have:

$$\sin\alpha = \frac{|6\,A \wedge B \wedge C + 2\,A^{-1}\,A \wedge B \wedge C\,A|}{8|A \wedge B||A \wedge C|} = \frac{|A \wedge B \wedge C|}{|A \wedge B||A \wedge C|}$$

since the volume $A \wedge B \wedge C$ is a pseudoscalar, which commutes with all the elements of the algebra. Now the *law of sines for spherical triangles* follows:

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c} = \frac{|A \wedge B \wedge C|}{|A \wedge B||B \wedge C||C \wedge A|} \qquad \text{(I)}$$

Let us see the law of cosines. Since $e_{23}^2 = e_{31}^2 = e_{12}^2 = -1$ then:

$$\cos\alpha = \frac{(A \wedge B) \bullet (A \wedge C)}{|A \wedge B||A \wedge C|} = \frac{(A \wedge B) \bullet (A \wedge C)}{\sin c \sin b}$$

$$\sin b \sin c \cos\alpha = -\frac{1}{8}\left[(A\,B - B\,A)(A\,C - C\,A) + (A\,C - C\,A)(A\,B - B\,A)\right]$$

$$= -\frac{1}{8}\left(A\,B\,A\,C - A\,B\,C\,A - B\,A^2\,C + B\,A\,C\,A + A\,C\,A\,B - A\,C\,B\,A - C\,A^2\,B + C\,A\,B\,A\right)$$

Now taking into account that:

$$4\,A\cdot B\;A\cdot C = \left(A\,B + B\,A\right)\left(A\,C + C\,A\right) = A\,B\,A\,C + A\,B\,C\,A + B\,A^2\,C + B\,A\,C\,A$$

But also:

$$4\,A\cdot B\;A\cdot C = \left(A\,C + C\,A\right)\left(A\,B + B\,A\right) = A\,C\,A\,B + A\,C\,B\,A + C\,A^2\,B + C\,A\,B\,A$$

and adding the needed terms, we find:

$$\sin b \sin c \cos\alpha = -\frac{1}{8}\left(8\,A\cdot B\;A\cdot C - 2\,A\,B\,C\,A - 2\,B\,A^2\,C - 2\,A\,C\,B\,A - 2\,C\,A^2\,B\right)$$

Extracting common factors and using $A^2 = B^2 = 1$, we may write:

$$\sin b \sin c \cos\alpha = -\frac{1}{8}\left(8\,A\cdot B\;A\cdot C - 2\,A\left(B\,C + C\,B\right)A - 2\left(B\,C + C\,B\right)\right) = -A\cdot B\;A\cdot C + B\cdot C$$

$$\sin b \sin c \cos\alpha = -\cos c \cos b + \cos a$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha \qquad \text{(II)}$$

which is the *law of cosines for sides*. The substitution of cos c by means of the law of cosines gives:

$$\cos a = \cos b\left(\cos a \cos b + \sin a \sin b \cos\gamma\right) + \sin b \sin c \cos\alpha$$

$$\cos a\left(1 - \cos^2 b\right) = \cos b \sin a \sin b \cos\gamma + \sin b \sin c \cos\alpha$$

and the simplification of sin b:

$$\cos a \sin b = \cos b \sin a \cos\gamma + \sin c \cos\alpha$$

The substitution of $\sin c = \sin a \sin\gamma / \sin\alpha$ yields:

$$\cos a \sin b = \cos b \sin a \cos\gamma + \frac{\sin a \sin\gamma \cos\alpha}{\sin\alpha}$$

Dividing by sin a :

$$\cot a \sin b = \cos b \cos\gamma + \sin\gamma \cot\alpha \qquad \text{(III)}$$

The dual spherical triangle

The perpendicular to the plane containing each side of a spherical triangle cuts the spherical surface in a point. The three points A', B' and C' obtained in this geometric way form the *dual triangle*. The algebraic way to calculate them is the duality operation, which maps bivectors into the perpendicular vectors. The dual of any element is obtained as the product by the pseudoscalar unity $-e_{123}$, which commutes with all the elements of the algebra:

$$A' = -e_{123}\,\frac{B \wedge C}{|B \wedge C|} \qquad B' = -e_{123}\,\frac{C \wedge A}{|C \wedge A|} \qquad C' = -e_{123}\,\frac{A \wedge B}{|A \wedge B|}$$

The inner product of two vectors yields:

$$B' \cdot C' = \frac{(C \wedge A) \bullet (A \wedge B)}{|C \wedge A||A \wedge B|} = -\frac{(A \wedge B) \bullet (A \wedge C)}{|A \wedge B||A \wedge C|} \quad \Leftrightarrow \quad \cos a' = -\cos \alpha$$

which shows that the angle a' and α are supplementary, and so also the other angles:

$$a' = \pi - \alpha \qquad b' = \pi - \beta \qquad c' = \pi - \gamma$$

It is trivial that the dual of the dual triangle is the first triangle, and hence:

$$\alpha' = \pi - a \qquad \beta' = \pi - b \qquad \gamma' = \pi - c$$

We may apply the laws of sines and cosines to the dual triangle. The law of sines is self-dual and may be written in a more symmetric form:

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c} = \frac{|A \wedge B \wedge C|}{|A' \wedge B' \wedge C'|} \qquad \text{(I)}$$

On the other hand, the law of cosines yields a new result when applying duality:

$$-\cos \alpha' = \cos \gamma' \cos \beta' - \sin \beta' \sin \gamma' \cos a'$$

Removing the marks, since this law must be valid for any spherical triangle, we find the *law of cosines for angles*:

$$\cos \alpha = -\cos \gamma \cos \beta + \sin \beta \sin \gamma \cos a \qquad \text{(IV)}$$

Also, applying the equality (III) to the dual triangle we find:

$$\cot \alpha \sin \beta = -\cos \beta \cos c + \sin c \cot a \qquad \text{(V)}$$

The five Bessel's equalities (I to V) allow to solve every spherical triangle knowing any three of its six elements.

Right spherical triangles and Napier's rule

For the case of a right angle spherical triangle, the five Bessel's formulas are reduced to a simpler form, and then they may be remembered with the help of the Napier's pentagon rule (figure 14.2). Draw the angles and the sides of the triangle following the order of the perimeter removing the right angle and writing instead of the legs (the sides adjacent to the right angle) the complementary arcs. Then follow the Pentagon rules:

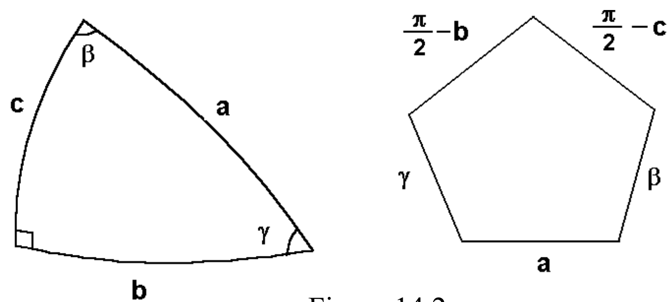


Figure 14.2

1) The cosine of every element is equal to the product of the cotangents of the adjacent elements.
2) The cosine of every element is equal to the product of the sines of the nonadjacent elements.

For example:

$$\cos a = \sin\left(\frac{\pi}{2} - b\right)\sin\left(\frac{\pi}{2} - c\right) = \cos b \cos c$$

also: $$\cos\left(\frac{\pi}{2} - b\right) = \cot\gamma \cot\left(\frac{\pi}{2} - c\right) \qquad \Rightarrow \qquad \sin b = \cot\gamma \operatorname{tg} c$$

This rule is applied to the right side triangles in the same way: remove the right side and write the complementary of the adjacent angles.

Area of a spherical triangle

A *lune* is a two-sided polygon on the sphere defined by two great circles. The area of a lune is proportional to the angle α between both great circles. For an angle $\pi/2$ its area is π, therefore the area of a lune with angle α is 2α. Now let us consider in the sphere shown by figure 14.3 the three lunes having the angles α, β and γ. Then:

$$\begin{cases} s + t = 2\,\alpha \\ s + u' = 2\,\beta \\ s + v = 2\,\gamma \end{cases}$$

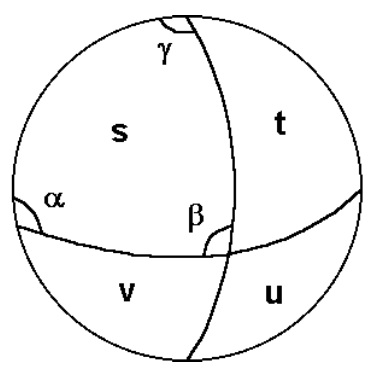


Figure 14.3

where u' is the area of the antipodal triangle of u. Both triangles u and u' have the same angles and area and the system can be rewritten:

$$\begin{cases} s+t=2\,\alpha \\ s+u=2\,\beta \\ s+v=2\,\gamma \end{cases}$$

Adding the three equations, we find:

$$3\,s+t+u+v=2\,(\alpha+\beta+\gamma)$$

The four triangles s, t, u and v fill an hemisphere:

$$s+t+u+v=2\,\pi$$

so the area of the triangle s is the *spherical excess*, that is, the addition of the three angles minus π:

$$s=\alpha+\beta+\gamma-\pi$$

Properties of the projections of the spherical surface

No chart of the spherical surface preserving the scale of distances everywhere exists, that is, we cannot depict any map with distances proportional to those measured on the sphere. The *distortion* is the variation of the scale and the angles. Usually there is a line of zero distortion, where the scale is constant.

Since the scale of distances is never preserved for all the points, projections with other interesting properties have been searched in cartography. A projection is said to be *conformal* if it preserves in the map the angles between great circles on the sphere. A projection is said to be *equivalent* if it preserves the area, that is if the figures on the sphere are projected into figures on the map having the same area. A projection is said to be *equidistant* if the scale of distances is preserved, not everywhere but on the line perpendicular to the line of zero distortion, or radially outwards from a point of zero distortion.

The more general concept of projection is any one-to-one mapping of any point (x, y, z) on the sphere into a point (u, v) on the plane, although the main types of projections are azimuthal, cylindrical and conic. An *azimuthal* projection is a standard projection into a plane, which may be considered touching the sphere. In the tangency point there is zero distortion and the bearings or azimuths from this point are correctly shown. A *cylindrical* projection is a projection into a cylindrical surface around the sphere that will be unrolled. A *conic* projection is a projection into a conical surface tangent to the sphere that will also be unrolled.

Now, I review the main and more used projections beginning with the azimuthal projections.

The central or gnomonic projection

Let us consider the sphere with unity radius (figure 14.4) centred at the origin. Any point on the sphere has the coordinates (x, y, z) fulfilling:

$$x^2 + y^2 + z^2 = 1$$

Every point on the upper hemisphere is projected into another point on the plane $z = 1$ using as centre of projection the centre of the sphere. Let u and v be the Cartesian coordinates on the projection plane. Taking similar triangles the following relations are found:

$$u = \frac{x}{z} \qquad v = \frac{y}{z}$$

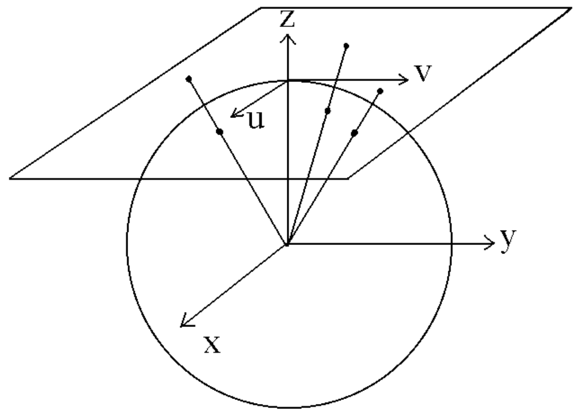


Figure 14.4

from where we obtain:

$$x = \frac{u}{\sqrt{u^2 + v^2 + 1}} \qquad y = \frac{v}{\sqrt{u^2 + v^2 + 1}} \qquad z = \frac{1}{\sqrt{u^2 + v^2 + 1}}$$

The differential of the arc length is obtained through the differentiation of the above relationships:

$$ds = dx\, e_1 + dy\, e_2 + dz\, e_3$$

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{\left(1 + v^2\right)du^2 - 2\,u\,v\,du\,dv + \left(1 + u^2\right)dv^2}{\left(1 + u^2 + v^2\right)^2}$$

The geodesics of the sphere are the great circles, which are the intersections with planes passing through its centre. These planes cut the projection plane in straight lines, which are the projections of the geodesics. In other words, any great circle is projected into a line on the projection plane. Taking as equation of the line:

$$v = k\,u + l$$

with k and l constant, the substitution in ds gives:

$$ds = \frac{\sqrt{1 + k^2 + l^2}}{u^2\left(1 + k^2\right) + 2\,k\,l\,u + 1 + l^2}\,du$$

By integration we arrive to the following primitive:

$$s = \operatorname{arctg} \frac{u\left(1 + k^2\right) + k\,l}{\sqrt{1 + k^2 + l^2}} + \text{const}$$

The arc length between two points on this great circle is the difference of this primitive between both points. However it is more advantageous to write it using cosines instead of tangents by means of the trigonometric identity:

$$\cos(s_B - s_A) \equiv \frac{1 + \operatorname{tg} s_A \operatorname{tg} s_B}{\sqrt{1 + \operatorname{tg}^2 s_A}\sqrt{1 + \operatorname{tg}^2 s_B}}$$

After removing *k* and *l* using the equation of the line, we arrive to:

$$\cos(s_B - s_A) = \frac{1 + u_A u_B + v_A v_B}{\sqrt{1 + u_A{}^2 + v_A{}^2}\sqrt{1 + u_B{}^2 + v_B{}^2}}$$

$$= \frac{(u_A e_1 + v_A e_2 + e_3) \cdot (u_B e_1 + v_B e_2 + e_3)}{|u_A e_1 + v_A e_2 + e_3| \; |u_B e_1 + v_B e_2 + e_3|}$$

a trivial result because the arc length is the angle between the position vectors of both points, and also of the proportional vectors going to the projection plane. However, the interest of this result is its analogy with the result found for the hyperboloidal surface.

From this value of the cosine, we may obtain the sine of the arc:

$$\sin(s_B - s_A) = \sqrt{1 - \cos^2(s_B - s_A)} = \frac{\sqrt{(u_A v_B - u_B v_A)^2 + (u_A - v_A)^2 + (u_B - v_B)^2}}{\sqrt{1 + u_A{}^2 + v_A{}^2}\sqrt{1 + u_B{}^2 + v_B{}^2}}$$

$$= \frac{|(u_A e_1 + v_A e_2 + e_3) \wedge (u_B e_1 + v_B e_2 + e_3)|}{|u_A e_1 + v_A e_2 + e_3| \; |u_B e_1 + v_B e_2 + e_3|}$$

which is also a trivial result, since the sine of the angle is proportional to the modulus of the outer product.

Let us see the area function. The differential of the area is easily obtained taking into account that it is a bivector and using the outer product of the differentials of the coordinates:

$$dA = \sqrt{(dx \wedge dy)^2 + (dy \wedge dz)^2 + (dz \wedge dx)^2} = \frac{du \wedge dv}{(1 + u^2 + v^2)^{3/2}}$$

This result shows that the central projection is not *equivalent* and the distortion increases with the distance to the origin.

Let us consider a plane passing through the centre of the sphere, which cuts its surface in the great circle determined by the equation system:

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ a\,x + b\,y + z = 0 \quad a, b \quad \text{real} \end{cases}$$

Then the angle between two great circles is the angle between both central planes:

$$\cos\alpha = \frac{(a\, e_{23} + b\, e_{31} + e_{12}) \bullet (a'\, e_{23} + b'\, e_{31} + e_{12})}{|\, a\, e_{23} + b\, e_{31} + e_{12} \,|\; |\, a'\, e_{23} + b'\, e_{31} + e_{12} \,|} = \frac{a\, a' + b\, b' + 1}{\sqrt{a^2 + b^2 + 1}\,\sqrt{a'^2 + b'^2 + 1}}$$

The sides of a spherical triangle are great circles; therefore the central projection of a spherical triangle is a triangle whose sides are straight lines.

Stereographic projection

In the stereographic projection the point of view is placed on the spherical surface. As before, every point (x, y, z) on the sphere with unity radius centred at the origin fulfils the equation:

$$x^2 + y^2 + z^2 = 1$$

We project the spherical surface into the plane z = 0 locating the centre of projection at the pole (0, 0, −1) (figure 14.5). The upper hemisphere is projected inside the circle of unity radius while the lowest hemisphere is projected outside. If u and v are the Cartesian coordinates on the projection plane, we have by similar triangles:

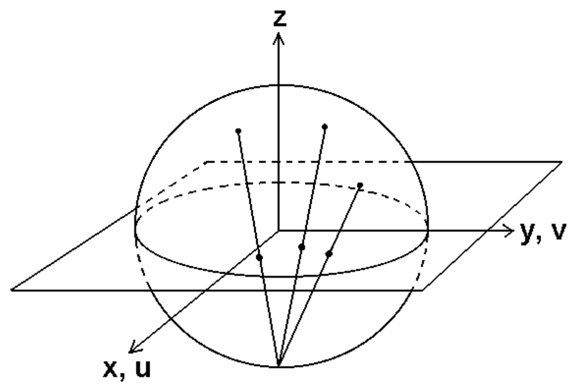


Figure 14.5

$$\frac{x}{u} = z + 1 \qquad \frac{y}{v} = z + 1$$

Using the equation of the sphere one arrives to:

$$x = \frac{2\,u}{1 + u^2 + v^2} \qquad y = \frac{2\,v}{1 + u^2 + v^2} \qquad z = \frac{2}{1 + u^2 + v^2} - 1$$

from where the differential of the arc length is obtained:

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{4\left(du^2 + dv^2\right)}{\left(1 + u^2 + v^2\right)^2}$$

Now we see that this projection is not equidistant and the distortion increases with the distance to the origin. The factor 4 indicates that the lengths at the origin of coordinates on the plane are the half of the lengths on the sphere. Taking instead of the plane z = 0, the plane z = 1 this factor becomes 1. Then, we can state correctly the scale of the chart (for example, a polar chart).

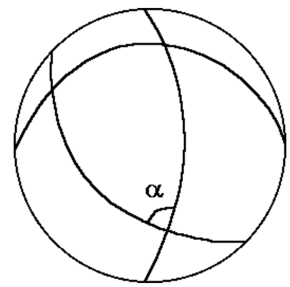
The geodesic lines (great circles) are intersection of the sphere surface with planes passing through the origin, which have the equation:

$$z = a\,x + b\,y$$

The substitution by the stereographic coordinates yields:

$$\left(u+a\right)^2+\left(v+b\right)^2=a^2+b^2+1$$

Figure 14.6



which is the equation of a circle centred at ($-a$, $-b$) with radius $r=\sqrt{a^2+b^2+1}$. Observe that this radius is the hypotenuse of a right angle triangle having as legs the distance to the origin and the unity. That is, the great circles on the sphere are shown in the stereographic projection as circles that intersect the circle $x^2+y^2=1$ in extremes of diameters (figure 14.6). Usually only the projection of the upper hemisphere is used, so that the great circles are represented as circle arcs inside the circle $x^2+y^2\leq 1$. The angle α between two of these circles, as explained in the page 89, is obtained from their radii r and r' and centres O and O' through:

$$\cos\alpha=\frac{r^2+r'^2-\left(O-O'\right)^2}{2\,r\,r'}=\frac{a\,a'+b\,b'+1}{\sqrt{a^2+b^2+1}\sqrt{a'^2+b'^2+1}}$$

Just this is the angle between the planes $a\,x+b\,y-z=0$ and $a'\,x+b'\,y-z=0$, that is, the angle between the two great circles represented by the projected circles. Therefore, the stereographic projection is a conformal projection of the spherical surface.

If we calculate the differential of area we find:

$$dA=\frac{4\,du\wedge dv}{\left(1+u^2+v^2\right)^2}$$

Now we see that this projection is not equivalent since distortion of areas increases with the distance from the origin (as commented above, the factor 4 becomes 1 projecting into the plane $z=1$).

Orthographic projection

The orthographic projection of the sphere is a parallel projection (the point of view is placed at the infinity). If we make a projection parallel to the z-axis upon the plane $z=0$, the Cartesian coordinates on the map are identical to x, y and we have:

$$z=\sqrt{1-x^2-y^2} \qquad dz=-\frac{x\,dx+y\,dy}{\sqrt{1-x^2-y^2}}$$

$$ds^2=dx^2+dy^2+dz^2=\frac{dx^2+dy^2+2\,x\,y\,dx\,dy}{1-x^2-y^2}$$

$$dA = \sqrt{\frac{1+x^2+y^2}{1-x^2-y^2}}\, dx \wedge dy$$

Introducing the distance r to the origin of coordinates and the angle φ with respect to the x-axis, we have:

$$x = r\cos\varphi \qquad\qquad y = r\sin\varphi$$

$$ds^2 = \frac{dr^2 + r^2 d\varphi^2 + 2\, r\, dr\, d\varphi}{1-r^2}$$

An example of orthographic projection is the Earth image that appears in many TV news.

Spherical coordinates and cylindrical equidistant (Plate Carrée) projection

For a sphere with unity radius, the spherical coordinates[4] are related with the Cartesian coordinates by means of:

$$x = \sin\theta\cos\varphi$$

$$y = \sin\theta\sin\varphi$$

$$z = \cos\theta$$

Then the differential of arc length is:

$$ds = (\cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi)\, e_1 + (\cos\theta\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\varphi)\, e_2 - \sin\theta\, d\theta\, e_3$$

Introducing the unitary vectors e_θ and e_φ as:

$$e_\theta = \cos\theta\cos\varphi\, e_1 + \cos\theta\sin\varphi\, e_2 - \sin\theta\, e_3$$

$$e_\varphi = -\sin\varphi\, e_1 + \cos\varphi\, e_2$$

the differential of arc length in spherical coordinates becomes:

$$ds = d\theta\, e_\theta + \sin\theta\, d\varphi\, e_\varphi$$

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

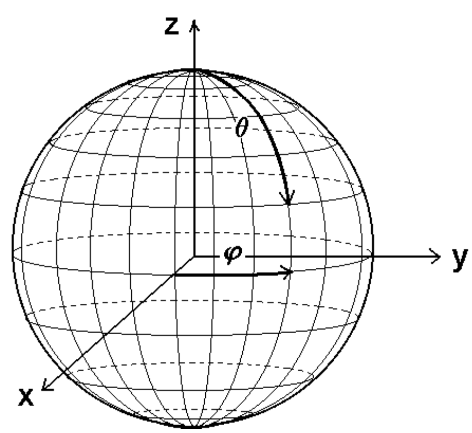


Figure 14.7

[4] For geographical coordinates, θ is the colatitude and φ the longitude.

Note that e_θ and e_φ are orthogonal vectors, since their inner product is zero. At $\theta = 0$, ds depends only on $d\theta$ and there is a pole. Then θ is the arc length from the pole to the given point (figure 14.7), while φ is the arc length over the equator ($\theta = \pi/2$, $\sin\theta = 1$). The meridians ($\varphi =$ constant) are geodesics, but the parallels ($\theta =$ constant) are not. Exceptionally, the equator is also a geodesic.

In the Plate Carrée projection $u = \varphi$ and $v = \pi/2 - \theta$ (latitude) so that the meridians and parallels are shown in a squared graticule. This projection is equidistant for any meridian whereas the distortion in the parallels increases, as well as for every cylindrical projection, as we separate from the equator. Let us see the area:

$$dA = \sin\theta \, d\theta \wedge d\varphi = \cos v \, du \wedge dv$$

The ratio of the real area with respect to the represented area on the chart is equal to the cosine of the latitude and therefore the projection is not equivalent.

Mercator's projection

The Mercator's projection[5] is defined as the cylindrical conformal projection. If we wish to preserve the angles between curves, we must enlarge the meridians by the same amount as the parallels are enlarged in a cylindrical projection, that is, by the factor $1/\sin\theta$ (the secant of the geographical latitude):

$$dv = \frac{-d\theta}{\sin\theta} \qquad \Rightarrow \qquad v = -\log \operatorname{tg}\frac{\theta}{2}$$

The differential of arc length is:

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 = \sin^2\theta \left(du^2 + dv^2\right) = \frac{4\exp(2v)}{\left(\exp(2v)+1\right)^2}\left(du^2 + dv^2\right)$$

$$ds = \frac{2\exp(v)}{\exp(2v)+1}\sqrt{du^2 + dv^2}$$

where the distances are increased in an amount independent of the direction and proportional to the inverse of the sine of θ. The differential of area is:

$$dA = \sin\theta \, d\theta \wedge d\varphi = \frac{4\exp(2v)}{\left(\exp(2v)+1\right)^2} du \wedge dv$$

[5] The difference between the U.T.M. (Universal Transverse Mercator) projection and the Mercator planisphere is not geometric but geographic: in the U.T.M. the cylinder of projection is tangent to a meridian instead of the equator. All the Earth has been divided in zones of 6º of longitude, where the cylinder of projection is tangent to the central meridian (3º, 9º, 15º ...).

Peters' projection

The Peters' projection is the cylindrical equivalent projection. If we wish to preserve area, we must shorten the meridians in the same amount as the parallels are enlarged in the cylindrical projection, what yields:

$$dv = -\sin\theta \, d\theta \qquad \Rightarrow \qquad v = \cos\theta$$

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 = \left(1 - v^2\right) du^2 + \frac{dv^2}{1 - v^2}$$

$$dA = \sin\theta \, d\theta \wedge d\varphi = du \wedge dv$$

which displays clearly the equivalence of the projection. Observe that $v = \cos\theta$ means the sphere is projected following planes perpendicular to the cylinder of projection (figure 14.8).

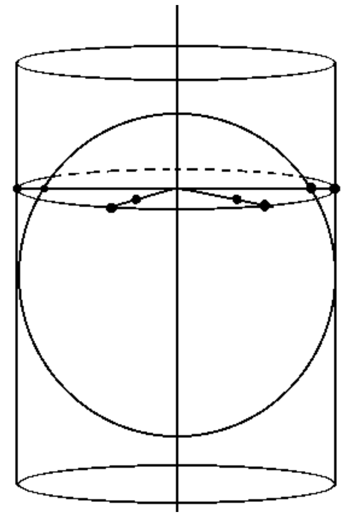


Figure 14.8

Conic projections

These projections are made into a cone surface tangent to the sphere (figure 14.9). Because the cone surface unrolled is a plane circular sector, they are often used to display middle latitudes, while the azimuthal projections are mainly used for poles. In a conic projection a small circle (a parallel) is shown as a circle with zero distortion. The characteristic parameter of a conic projection is the constant of the cone $n = \cos\,\theta_0$, being θ_0 the angle of inclination of the generatrix of the cone and also the angle from the axis of the cone to any point of tangency with the sphere. Since the graticule of the conic projections is radial, to use the radius r and the angle χ is more convenient:

$$dr = f(\theta)\, d\theta \qquad\qquad d\chi = n \, d\varphi$$

The differentials of arc length and area for a conic projection are:

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 = \frac{dr^2}{[f(\theta)]^2} + \frac{\sin^2\theta}{n^2} d\chi^2$$

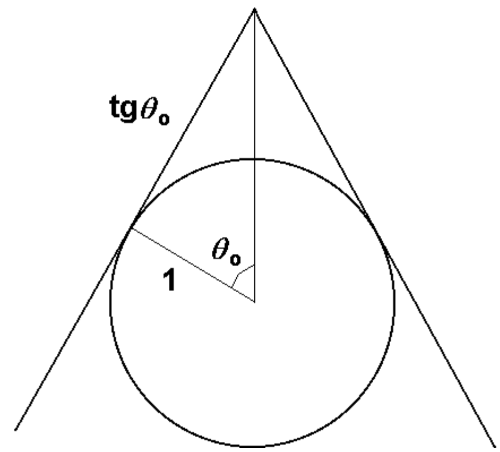
$$dA = \sin\theta \, d\theta \wedge d\varphi = \frac{\sin\theta}{n \, f(\theta)} dr \wedge d\chi$$

Let us see as before the three special cases: equidistant, conformal and equivalent projections. The differential of area for polar coordinates r, χ is $dA = r \, dr \wedge d\chi$. If the projection is equivalent, we must identify both dA to find :

$$\frac{d}{d\theta}\left[\frac{\sin\theta}{n\, f(\theta)}\right]=f(\theta) \qquad \Rightarrow \qquad f(\theta)=\frac{1}{\sqrt{n}}\cos\frac{\theta}{2} \quad \text{and} \quad r=\frac{2}{\sqrt{n}}\sin\frac{\theta}{2}$$

$$ds^2=\frac{n}{1-\frac{n\,r^2}{4}}dr^2+\frac{1-\frac{r^2\,n}{4}}{n}r^2 d\chi^2$$

Figure 14.9



If the projection is equidistant, the meridians have zero distortion so $d\theta = dr/n$ and:

$$ds^2=\frac{1}{n^2}\left(dr^2+\sin^2\left(\frac{r}{n}\right)d\chi^2\right)$$

If the projection is conformal then $ds^2 \propto dr^2 + r^2 d\chi^2$ so:

$$ds^2=\frac{\sin^2\theta}{n^2 r^2}\left(dr^2+r^2\, d\chi^2\right)$$

Solving the differential equation:

$$d\theta=\frac{\sin\theta}{n\,r}dr$$

with the boundary condition $\mathrm{tg}\,\theta_0 = r_0$ as shown by the figure 14.9 we find the Lambert's conformal projection:

$$r=\mathrm{tg}\,\theta_0\left(\frac{\mathrm{tg}\frac{\theta}{2}}{\mathrm{tg}\frac{\theta_0}{2}}\right)^n$$

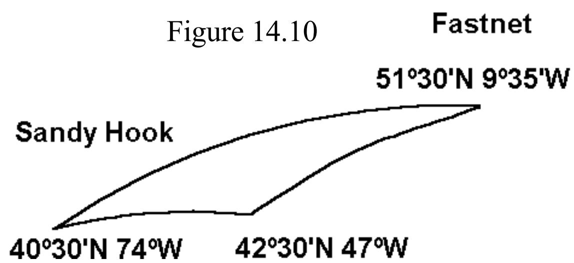
Exercises

14.1 We have seen the gnomonic projection of great circles being always straight lines. Then, why does the shadow of a gnomon of a sundial follow a hyperbola on a plane surface instead of a line during a day? Why does this hyperbola become a straight line the March 21 and September 23?

14.2 Three points on the sphere are projected by means of the stereographic projection into a circle of unity radius with the coordinates A = (–0.5, 0.5), B = (0, –2/3) and C = (2/3, 0). Calculate the sides, angles and area of the triangle which they form.

14.3 Built at Belfast, the Titanic begun its first and last travel in Southampton the April 10th 1912. After visiting Cherbourg the Titanic weighed anchor in the Cork harbour with bearing New York the 11th April. Calculate the length of the shortest trajectory from Fastnet (Ireland) at 51º30'N 9º35'W to Sandy Hook (New York) at 40º30'N 74ºW. From January 15th to July 15th the ships had to follow the orthodrome (great circle) from Fastnet to the point 42º30'N 47ºW and from this point to Sandy Hook passing at twenty miles from the floating lighthouse of Nantucket. Calculate the length of the route the Titanic should have followed. Take as an averaged radius of the Earth 6366 km. The Titanic sank the 15th April at the position 41º46'N 50º14'W. Is this point on the obliged track?

Figure 14.10



14.4 The figure 14.11 is a photograph of the Hale-Bopp comet. The orientation of the camera is unknown, but the W of Cassiopeia constellation appears in the photograph. The declination D of a star is the angular distance from the celestial equator to the star (measured on the great circle passing through the celestial pole and the star). The right ascension A is the arc of celestial equator measured eastward from the vernal equinox (one of the intersections of the celestial equator with the ecliptic, also called Aries point) to the foot of the great circle passing through the star and the pole. The right ascension and declination of the stars are constant while those of the comets, planets, the Sun and the Moon are variable. The data of Cassiopeia constellation are:

star	Magnitude	A	D
α Cassiopeia (Schedir)	2.2	0h 40min 6.4s = 10.0267º	56º 29' 57" N
β Cassiopeia (Caph)	2.3	0h 8min 48.1s = 2.2004º	59º 6' 40" N
γ Cassiopeia (Tsih)	2.4	0h 56min 16.9s = 14.0704º	60º 40' 44" N
δ Cassiopeia (Ruchbah)	2.7	1h 25min 21.2s = 21.3383º	60º 11' 57" N

In your system of coordinates, take the x-axis as the Aries point and the z-axis as the north pole. Then $D = \pi/2-\theta$ and $A=\varphi$.

a) Calculate the focal distance of the photograph, that is, the distance from the point of view to the plane of the photograph. Knowing that the negative was universal (24×36 mm), calculate the focal distance of the camera.
b) Calculate the orientation (right ascension and declination) of the photographic camera.
c) Calculate the coordinates of the Hale-Bopp comet.

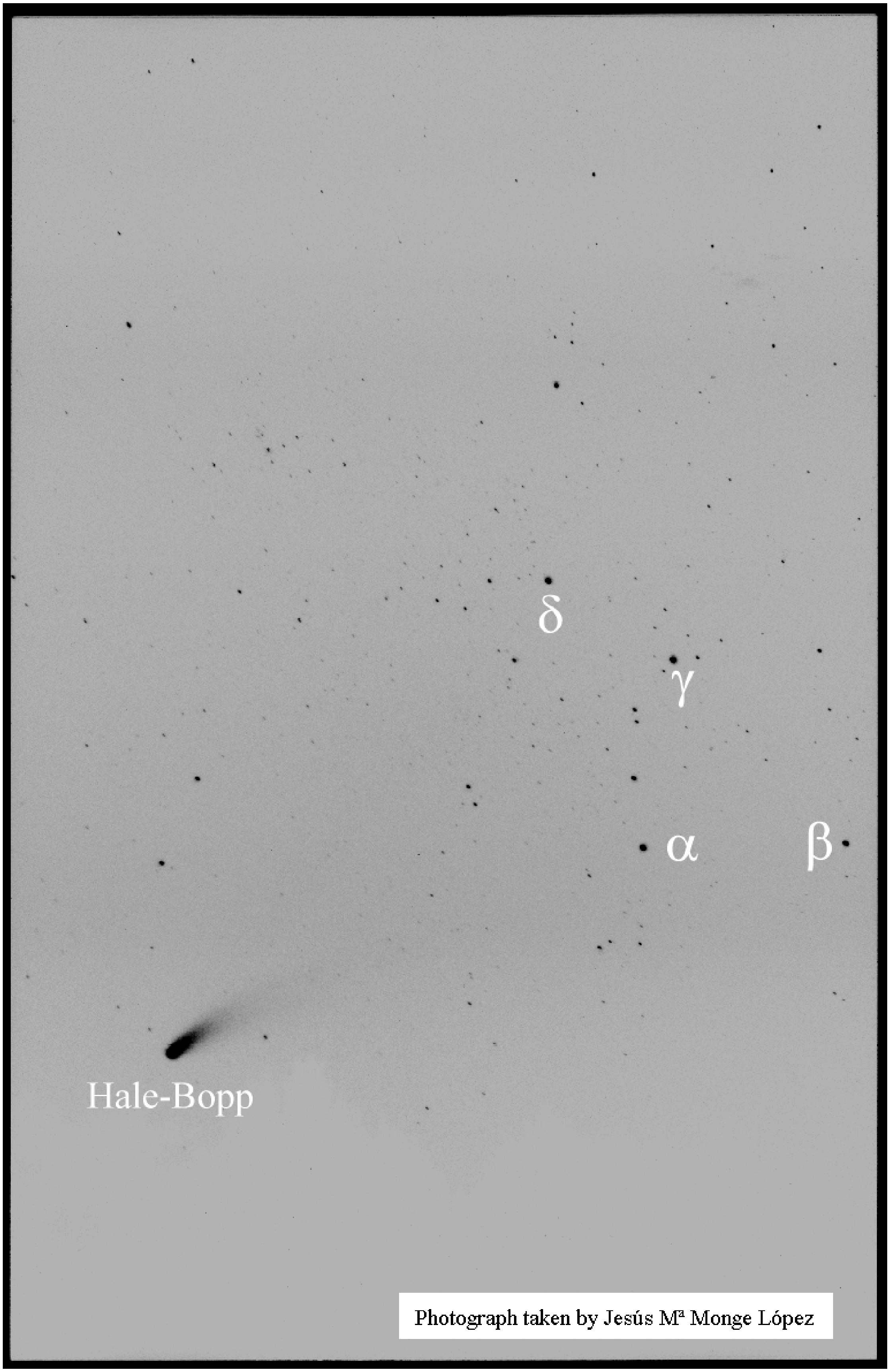


Figure 14.11

15. HYPERBOLOIDAL GEOMETRY IN THE PSEUDO-EUCLIDEAN SPACE (LOBACHEVSKY'S GEOMETRY)

The geometric algebra of the pseudo-Euclidean space

A *vector* of the pseudo-Euclidean space is an oriented segment in this space with direction and sense. The set of all segments (vectors) together with their addition (parallelogram rule) and the product by real numbers (dilation of vectors) has a structure of vector space, symbolised here with W_3. Every vector in W_3 is of the form:

$$v = v_1\, e_1 + v_2\, e_2 + v_3\, e_3$$

where e_i are three unitary perpendicular vectors, which constitute the base of the space. The modulus of a vector is:

$$|\, v\,|^2 = -v_1^2 - v_2^2 + v_3^2$$

It determines the geometric properties of the space, very different from the Euclidean space. Now we define an associative product (*geometric* or *Clifford product*) being a generalisation of those defined for the Euclidean and hyperbolic planes. Imposing the condition that the square of the modulus be equal to the square of the vector, we find:

$$|\, v\,|^2 = v^2$$

$$e_1^2 = -1 \qquad e_2^2 = -1 \qquad e_3^2 = 1 \qquad \text{and} \qquad e_i\, e_j = -e_j\, e_i \ \text{ for } i \neq j$$

From the base vectors one deduces that the geometric algebra generated by the space W_3 has eight components:

$$Cl(W_3) = Cl_{1,\,2} = \langle\, 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123}\, \rangle$$

Let us see with more detail the product of two vectors:

$$v\, w = (v_1\, e_1 + v_2\, e_2 + v_3\, e_3)(w_1\, e_1 + w_2\, e_2 + w_3\, e_3) = -v_1\, w_1 - v_2\, w_2 + v_3\, w_3$$
$$+ (v_2\, w_3 - v_3\, w_2)\, e_{23} + (v_3\, w_1 - v_1\, w_3)\, e_{31} + (v_1\, w_2 - v_2\, w_1)\, e_{12}$$

I shall call the product (or quotient) of two vectors a *tetranion*. The tetranions are the even subalgebra of $Cl_{1,\,2}$ that generalises the complex and hyperbolic numbers to the pseudo-Euclidean space. Splitting a tetranion in the real and bivector parts, we obtain the *inner* (or *scalar*) product and the *outer* (or *exterior*) product respectively:

$$v \cdot w = -v_1\, w_1 - v_2\, w_2 + v_3\, w_3$$

$$v \wedge w = (v_2\, w_3 - v_3\, w_2)\, e_{23} + (v_3\, w_1 - v_1\, w_3)\, e_{31} + (v_1\, w_2 - v_2\, w_1)\, e_{12}$$

Here also, the bivectors are oriented plane surfaces indicating the direction of planes in the pseudo-Euclidean space. As before, vectors and bivectors are different things. This fact has been experienced also by physicists: in the Minkowski's space, the electromagnetic field is a bivector while the tetrapotential is a vector. On the other hand, the oriented area is, of course, a bivector. As a criterion to distinguish both kind of magnitudes one also uses the reversal of coordinates, which changes the sense of vectors while leaves bivectors invariant.

Two vectors are said to be *orthogonal* if their inner product vanishes:

$$v \perp w \quad \Leftrightarrow \quad v \cdot w = 0$$

So the outer product is the product by the orthogonal component and the inner product is the product by the proportional component:

$$v \cdot w = v\, w_{\parallel} \qquad\qquad v \wedge w = v\, w_{\perp}$$

The product of two bivectors yields a tetranion. The real and bivector parts can be separated as the symmetric and antisymmetric product. Since:

$$e_{23}^2 = e_{31}^2 = 1 \qquad \text{and} \qquad e_{12}^2 = -1$$

the symmetric product is a real number whose negative value will be denoted here with the symbol •, whereas the antisymmetric product, denoted here with the symbol ×, is also a bivector:

$$v \bullet w = -\frac{1}{2}\left(v\, w + w\, v\right) = -v_{23}\, w_{23} - v_{31}\, w_{31} + v_{12}\, w_{12}$$

$$v \times w = -\frac{1}{2}\left(v\, w - w\, v\right)$$

$$= -\left(v_{31}\, w_{12} - v_{12}\, w_{31}\right) e_{23} - \left(v_{12}\, w_{23} - v_{23}\, w_{12}\right) e_{31} + \left(v_{23}\, w_{31} - v_{31}\, w_{23}\right) e_{12}$$

$$v\, w = -v \bullet w - v \times w$$

The outer product of three vectors has the same expression as for Euclidean geometry and this is a natural outcome of the extension theory: the product $u \wedge v \wedge w$ is the oriented volume generated by the surface represented by the bivector $u \wedge v$ when it is translated parallelly along the segment w:

$$u \wedge v \wedge w = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} e_{123}$$

Finally, let us see how the product of three vectors u, v and w is. The vector v can be resolved into a component coplanar with u and v and another component perpendicular to the plane u-v:

$$u\,v\,w = u\,v_{\parallel}\,w + u\,v_{\perp}\,w$$

Now let us analyse the permutative property. In both Euclidean and hyperbolic planes we found $u\ v\ w - w\ v\ u = 0$. In the pseudo-Euclidean space the permutative property becomes:

$$u\,v\,w - w\,v\,u = u\,v_{\perp}\,w - w\,v_{\perp}\,u = v_{\perp}\,(-u\,w + w\,u)$$

$$= -2\,v_{\perp}\,u \wedge w = -2\,v \wedge u \wedge w = 2\,u \wedge v \wedge w$$

I take the same algebraic hierarchies as in the former chapter: all the abridged products must be operated before the geometric product, convention adequate to the fact that in many algebraic situations, the abridged products must be developed in sums of geometric products.

The hyperboloid of two sheets

According to Hilbert (*Grundlagen der Geometrie*, Anhang V) the complete Lobachevsky's "plane" cannot be represented by a smooth surface with constant curvature as proposed by Beltrami. But this result only concerns surfaces in the Euclidean space. The surface whose points are placed at a fixed distance from the origin in a pseudo-Euclidean space (the two-sheeted hyperboloid) is the surface searched by Hilbert which realises the Lobachevsky's geometry[1]. It is known that it has a characteristic distance like the radius of the sphere. Since all the spheres are similar, we need only study the unitary sphere. Likewise, all the hyperboloidal surfaces $z^2-x^2-y^2 = r^2$ are similar and the hyperboloid with unity radius (figure 15.1):

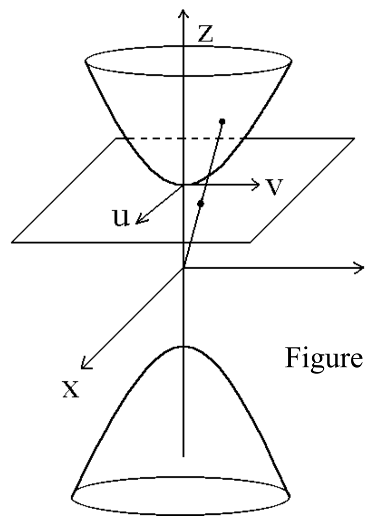


Figure 15.1

$$z^2 - x^2 - y^2 = 1$$

where (x, y, z) are Cartesian although not Euclidean[2] coordinates, suffices to study the whole Lobachesky's geometry.

[1] The reader will find a complete study of the hyperboloidal surface in Faber, *Foundations of Euclidean and Non-Euclidean Geometry*, chap. VII. "The Weierstrass model".

[2] For me, Cartesian coordinates are orthogonal coordinates in a flat space (with null curvature), independently of the Euclidean or pseudo-Euclidean nature of the space. Lines and planes (linear relations between Cartesian coordinates) can always be drawn in both spaces, although the pseudo-Euclidean orthogonality is not shown as a right angle, and the pseudo-Euclidean distance is not the viewed length. The incompleteness of Cartesian coordinates was already criticised by Leibniz in a letter to Huygens in 1679 proposing a new geometric calculus: «Car premierement je puis exprimer parfaitement par ce calcul toute la nature ou définition de la figure (ce que l'algèbre ne fait jamais, car disant que $x^2 + y^2$ *aeq.* a^2 est l'equation du cercle, il

The fact that the hyperboloidal surface owns the Lobachevsky's geometry will be evident through the different projections here reviewed. If we followed the same order as in the former chapter, we should firstly deal with the Lobachevskian trigonometry. However, the concept of arc length on the hyperboloid is much less intuitive than on the sphere. Therefore we must see the concept and determination of the arc length before studying the hyperboloidal trigonometry.

The central projection (Beltrami disk)

In this projection the centre of the hyperboloid is the centre of projection. Thus, we project every point on the upper sheet of the hyperboloid $z^2-x^2-y^2 = 1$ into another point on the plane $z = 1$ taking the origin $x = 0$, $y = 0$, $z = 0$ as centre of projection (figure 15.1). Let u and v be Cartesian coordinates on the plane $z = 1$, which touches the hyperboloid in the vertex. By similar triangles[3]:

$$u = \frac{x}{z} \qquad v = \frac{y}{z}$$

From where we obtain:

$$x = \frac{u}{\sqrt{1-u^2-v^2}} \qquad y = \frac{v}{\sqrt{1-u^2-v^2}} \qquad z = \frac{1}{\sqrt{1-u^2-v^2}}$$

The differential of the arc length upon the hyperboloid is:

$$ds = dx\, e_1 + dy\, e_2 + dz\, e_3$$

$$ds^2 = -dx^2 - dy^2 + dz^2 = -\frac{\left(1-v^2\right)du^2 + 2\,u\,v\,du\,dv + \left(1-u^2\right)dv^2}{\left(1-u^2-v^2\right)^2}$$

which is the negative value of the usual metric of the Lobachevsky's surface in the Beltrami disk. The negative sign indicates that the arc length is comparable with lengths in the x-y plane. The whole upper sheet of the hyperboloid is projected inside the Beltrami's circle $u^2+v^2 \le 1$ with unity radius. However, if we project the one-sheeted hyperboloid $z^2-x^2-y^2 = -1$ into the plane $z = 1$ we find the same metric but positive, what indicates the lengths being comparable to those measured on the z-axis. All the upper half of the one-sheeted hyperboloid is projected outside the circle of unity radius, giving rise to a new geometry not studied up till now.

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faut expliquer par la figure ce que c'est *x* et *y*.» (Josep Manel Parra, "Geometric Algebra versus numerical cartesianism" in F. Brackx, R. Delanghe, H. Serras, *Clifford Algebras and Their Applications in Mathematical Physics*).

[3] Now to consider that the legs of both right angle triangles are horizontal and vertical, and therefore perpendicular is enough for the deduction. Notwithstanding, in a pseudo-Euclidean space one must use the algebraic definition of similitude: a triangle with sides a and b is directly similar to a triangle with sides c and d if and only if $a\,c^{-1} = b\,d^{-1}$. This condition implies that both quotients of the modulus are equal and both arguments also.

Let us consider a plane passing through the origin of coordinates (central plane). Its intersection with the two-sheeted hyperboloid is the hyperbola determined by the equations system:

$$\begin{cases} z^2 - x^2 - y^2 = 1 \\ z = a\,x + b\,y \qquad a, b \ \ \text{real} \end{cases}$$

We search the Frenet's trihedron. By differentiation we find a linear differential equations system :

$$\begin{cases} z\,dz = x\,dx + y\,dy \\ dz = a\,dx + b\,dy \end{cases}$$

where *dx* and *dy* can be isolated:

$$dx = \frac{y - b\,z}{a\,y - b\,x}\,dz \qquad\qquad dy = \frac{a\,z - x}{a\,y - b\,x}\,dz$$

Then the differential of the arc length of this hyperbola is:

$$ds = dx\, e_1 + dy\, e_2 + dz\, e_3 = \left(\frac{y - b\,z}{a\,y - b\,x}\, e_1 + \frac{a\,z - x}{a\,y - b\,x}\, e_2 + e_3 \right) dz$$

and its square: $$ds^2 = -dx^2 - dy^2 + dz^2 = \frac{1 - a^2 - b^2}{(a\,y - b\,x)^2}\, dz^2$$

Now we obtain the unitary vector *t* tangent to the hyperbola:

$$t = \frac{dx}{ds}\, e_1 + \frac{dy}{ds}\, e_2 + \frac{dz}{ds}\, e_3 = \frac{(y - b\,z)\, e_1 + (a\,z - x)\, e_2 + (a\,y - b\,x)\, e_3}{\sqrt{1 - a^2 - b^2}}$$

Observe that *t* and *ds* have imaginary components since a plane only cuts the two-sheeted hyperboloid if $a^2 + b^2 > 1$. Anyway, the geometric vector can be taken with real components, although then its modulus is -1.

The derivative of the tangent vector with respect to the arc length is not only proportional but equal to the normal vector *n* of the hyperbola:

$$n = \frac{dt}{ds} = -\left(x\, e_1 + y\, e_2 + z\, e_3 \right) \qquad\qquad \Rightarrow \qquad \kappa = \left| \frac{dt}{ds} \right| = 1$$

because the position vector of every point on the two sheet hyperboloid has unitary modulus. Hence the curvature κ of the hyperbola is constant and equal to 1. All the hyperbolas being intersections of the hyperboloid with planes passing through the origin and a given point on the hyperboloid have the same radius of curvature *r*:

$$r = \left(\frac{dt}{ds} \right)^{-1} = -\left(x\, e_1 + y\, e_2 + z\, e_3 \right)$$

Two consequences are deduced from this fact: firstly their common radius of curvature is perpendicular to the surface, and secondly, these hyperbolas are geodesics of the hyperboloid. Because all the hyperbolas lie on planes passing through the origin, their central projections are straight lines. In other words, every geodesic hyperbola is projected into a segment on the Beltrami disk. Taking as equation of the line:

$$v = k\, u + l$$

with k and l constant, the substitution in ds gives[4]:

$$\frac{ds}{e_{123}} = \frac{\sqrt{1 + k^2 - l^2}}{-u^2\left(1 + k^2\right) - 2\, k\, l\, u + 1 - l^2}\, du$$

By integration we arrive at the following primitive:

$$\frac{s}{e_{123}} = \operatorname{arg\,tgh} \frac{u\left(1 + k^2\right) + k\, l}{\sqrt{1 + k^2 - l^2}} + \text{const} = \frac{1}{2} \log \frac{u + \dfrac{k\, l + \sqrt{1 + k^2 - l^2}}{1 + k^2}}{u + \dfrac{k\, l - \sqrt{1 + k^2 - l^2}}{1 + k^2}} + \text{const}$$

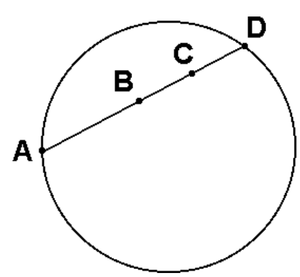
The line $v = k\, u + l$ cuts the Beltrami's circle $u^2 + v^2 = 1$ at the points A and D whose coordinates are the solution of the system of the equations:

$$\begin{cases} v = k\, u + l \\ u^2 + v^2 = 1 \end{cases} \Rightarrow \qquad u_A = \frac{-k\, l - \sqrt{1 + k^2 - l^2}}{k^2 + 1} \qquad u_D = \frac{-k\, l + \sqrt{1 + k^2 - l^2}}{k^2 + 1}$$

Thus the primitive may be rewritten:

$$\frac{s}{e_{123}} = \frac{1}{2} \log \frac{u - u_A}{u - u_D} + \text{const}$$

Figure 15.2



Then, the arc length between two points B and C on this geodesic hyperbola is the difference of this primitive between both points:

$$\frac{s_C - s_B}{e_{123}} = \frac{1}{2} \log \frac{\left(u_C - u_A\right)\left(u_D - u_B\right)}{\left(u_D - u_A\right)\left(u_C - u_B\right)} = \frac{1}{2} \log\left(A\, B\, C\, D\right)$$

---

[4] As said before, the arc length s on the two-sheeted hyperboloid has imaginary values. I'm sorry by the inconvenience of that reader accustomed to take real values. However the coherence of all the geometric algebra force to take account of the imaginary unity e_{123}. This also explains why Lobachevsky found the hyperboloidal trigonometry by choosing imaginary values for the sides in the spherical trigonometry.

that is, the half of the logarithm of the cross ratio of the four points (figure 15.2) on the Beltrami disk.

However it is more advantageous to write it using cosines instead of tangents by means of the trigonometric identity:

$$\cosh\frac{s_C - s_B}{e_{123}} \equiv \frac{1-\mathrm{tgh}\,\dfrac{s_B}{e_{123}}\,\mathrm{tgh}\,\dfrac{s_C}{e_{123}}}{\sqrt{1-\mathrm{tgh}^2\,\dfrac{s_B}{e_{123}}}\sqrt{1-\mathrm{tgh}^2\,\dfrac{s_C}{e_{123}}}}$$

After removing *k* and *l* using the equation of the line in the primitive, we arrive at:

$$\cosh\frac{s_C - s_B}{e_{123}} = \frac{1-u_B\,u_C - v_B\,v_C}{\sqrt{1-u_B{}^2 - v_B{}^2}\sqrt{1-u_C{}^2 - v_C{}^2}}$$

where the expression is real for all the points on the hyperboloid, whose projections lie on the Beltrami disk $u^2 + v^2 \leq 1$. This expression is equivalent to write:

$$\cosh\frac{s_C - s_B}{e_{123}} = \frac{(u_B\,e_1 + v_B\,e_2 + e_3)\cdot(u_C\,e_1 + v_C\,e_2 + e_3)}{|(u_B\,e_1 + v_B\,e_2 + e_3)|\;|(u_C\,e_1 + v_C\,e_2 + e_3)|}$$

Since the vector $u_A\,e_1 + v_A\,e_2 + e_3$ is proportional to the position vector, we have:

$$\cosh\frac{s_C - s_B}{e_{123}} = \cos(s_C - s_B) = \frac{(x_B\,e_1 + y_B\,e_2 + z_B\,e_3)\cdot(x_C\,e_1 + y_C\,e_2 + z_C\,e_3)}{|(x_B\,e_1 + y_B\,e_2 + z_B\,e_3)|\;|(x_C\,e_1 + y_C\,e_2 + z_C\,e_3)|}$$

which is the expected expression to calculate angles between vectors in the pseudo-Euclidean space. This result must be commented in more detail. Note that every pair of vectors going from the origin to the two-sheeted hyperboloid always lie on a central plane having hyperbolic nature, since it cuts the cone $z^2 - x^2 - y^2 = 0$ of vectors with zero length. Because of this, the angle measured on this plane is hyperbolic.

A hyperboloidal triangle is the region of the hyperboloid bounded by three geodesic hyperbolas. The length of a side of the triangle can be calculated by integration of the arc length of the hyperbola according to the last result. On the other hand, the angle of a vertex of a hyperboloidal triangle is the angle between the tangent vectors of the hyperbolas of each side at the vertex:

$$\cos\alpha = t\cdot t' = \frac{-(y-b\,z)(y-b'\,z)-(a\,z-x)(a'\,z-x)+(a\,y-b\,x)(a'\,y-b'\,x)}{\sqrt{1-a^2-b^2}\,\sqrt{1-a'^2-b'^2}}$$

$$= \frac{1-a\,a'-b\,b'}{\sqrt{1-a^2-b^2}\,\sqrt{1-a'^2-b'^2}}$$

Note that every pair of tangent vectors lie on a plane of Euclidean nature, a plane parallel to a central plane not cutting the cone of zero length. So the angles measured on this plane are circular. We arrive at the same conclusion by proving that the absolute value of the cosine (and also its square) is lesser than the unity:

$$\frac{(1-a\,a'-b\,b')^2}{(1-a^2-b^2)^2\,(1-a'^2-b'^2)^2}<1$$

The denominator is positive (although both squares are negative because $a^2+b^2>1$) and it can pass to the right hand side of the inequality without changing the sense. Removing the denominator and after simplification:

$$-2\,a\,a'-2\,b\,b'+2\,a\,a'\,b\,b'<-a^2-b^2-a'^2-b'^2+a^2\,b'^2+a'^2\,b^2$$

and arranging terms we find an equivalent inequality:

$$a^2+a'^2-2\,a\,a'+b^2+b'^2-2\,b\,b'<a^2\,b'^2+a'^2\,b^2-2\,a\,a'\,b\,b'$$

$$(a-a')^2+(b-b')^2<(a\,b'-b\,a')^2$$

Just this is the condition for the existence of the intersection point where two central planes and the hyperboloid meet as I shall show now. The planes with equations $a\,x+b\,y=z$ and $a'\,x+b'\,y=z$ are projected into the lines $a\,u+b\,v=1$ and $a'\,u+b'\,v=1$ on the Beltrami disk. Then the point of intersection of both planes and the hyperboloidal surface is projected into the point on the Beltrami disk given by the system of equations:

$$\begin{cases} a\,u+b\,v=1 \\ a'\,u+b'\,v=1 \end{cases}$$

whose solution is: $$u=\frac{b'-b}{a\,b'-a'\,b} \qquad v=\frac{-a'+a}{a\,b'-a'\,b}$$

This point lies inside the Beltrami circle (and the point of intersection of the hyperboloid and both planes exists) if and only if:

$$u^2+v^2=\frac{(a-a')^2+(b-b')^2}{(a\,b'-b\,a')^2}<1$$

which is the condition above obtained.

Note that the expression for the angle between geodesic hyperbolas coincides with the angle between the bivectors of the planes containing these hyperbolas:

$$\cos\alpha=\frac{(a\,e_{23}+b\,e_{31}+e_{12})\bullet(a'\,e_{23}+b'\,e_{31}+e_{12})}{|\,a\,e_{23}+b\,e_{31}+e_{12}\,|\;|\,a'\,e_{23}+b'\,e_{31}+e_{12}\,|}$$

It is not a surprising result, because it happens likewise for the sphere: the angle between two maximum circles is the angle between the central planes containing them.

The planes passing through the origin only cut the two-sheeted hyperboloid if $a^2 + b^2 > 1$. These planes also cut the cone of zero length $z^2 - x^2 - y^2 = 0$ so they are hyperbolic and the angles measured on them are hyperbolic arguments. On the other hand, the planes with $a^2 + b^2 < 1$ do not cut the hyperboloid neither the cone and have Euclidean nature. In the Lobachevskian trigonometry, the planes containing the sides of a hyperboloidal triangle are always hyperbolic, while the planes touching the hyperboloid and containing the angles between sides are always Euclidean, and hence the angles are circular. Resuming, real modulus of a bivector ($0 < -a^2 - b^2 + 1$) indicates an Euclidean plane and imaginary modulus ($0 > -a^2 - b^2 + 1$) a hyperbolic plane.

Now we already have all the formulas to deduce the hyperboloidal trigonometry.

Hyperboloidal (Lobachevskian) trigonometry

The Lobachevskian trigonometry is the study of the trigonometric relations for the sides and angles of geodesic triangles on two-sheeted hyperboloids in the pseudo-Euclidean space. We will take for convenience the hyperboloid with unity radius $z^2 - x^2 - y^2 = 1$, although the trigonometric identities are equally valid for a hyperboloid with any radius.

Consider three points A, B and C on the unitary hyperboloid (figure 15.3). Then $|A| = |B| = |C| = 1$. The angles formed by each pair of sides will be denoted by α, β and γ, and the real value of the sides respectively opposite to these angles will be symbolised by a, b and c respectively. Then a is the real value of the arc of the geodesic hyperbola passing through the points B and C, that is, the hyperbolic angle between these vectors:

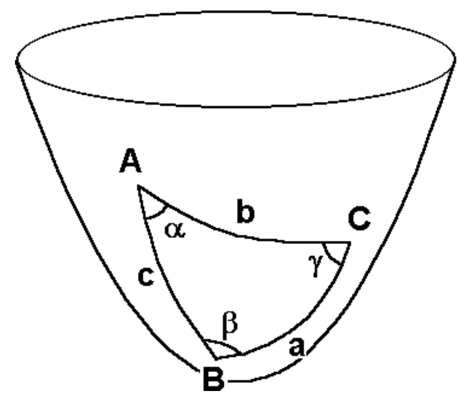


Figure 15.3

$$\sinh a = \frac{|B \wedge C|}{e_{123}}$$

As said above, the planes intersecting the hyperboloid have imaginary modulus. So we must divide by the pseudoscalar imaginary unity, which is also the volume element. In this equality the hyperbolic sine is taken positive and also a.

α is the circular angle between the sides b and c of the triangle, that is, the angle between the planes passing through the origin, A and B, and the origin, A and C respectively. Since the direction of a plane is given by its bivector, which can be obtained through the outer product, we have:

$$\sin\alpha = -\frac{|(A\wedge B)\times(A\wedge C)|}{|A\wedge B||A\wedge C|}$$

where the minus sign is due to the imaginary modulus of the outer products of the denominators. Now we write the products of the numerator using the geometric product:

$$\sin\alpha = -\frac{|-(A\,B - B\,A)(A\,C - C\,A) + (A\,C - C\,A)(A\,B - B\,A)|}{8|A\wedge B||A\wedge C|}$$

$$\sin\alpha = -\frac{|-A\,B\,A\,C + A\,B\,C\,A + B\,A^2\,C - B\,A\,C\,A + A\,C\,A\,B - A\,C\,B\,A - C\,A^2\,B + C\,A\,B\,A|}{8|A\wedge B||A\wedge C|}$$

We extract the vector A as common factor at the left, but without writing it because $|A| = 1$:

$$\sin\alpha = -\frac{|-B\,A\,C + B\,C\,A + A\,B\,C - A^{-1}\,B\,A\,C\,A + C\,A\,B - C\,B\,A - A\,C\,B + A^{-1}\,C\,A\,B\,A|}{8|A\wedge B||A\wedge C|}$$

Applying the permutative property to the suitable pairs of products, we have:

$$\sin\alpha = -\frac{|6\,A\wedge B\wedge C + 2\,A^{-1}\,A\wedge B\wedge C\,A|}{8|A\wedge B||A\wedge C|} = -\frac{|A\wedge B\wedge C|}{|A\wedge B||A\wedge C|}$$

since the volume $A \wedge B \wedge C$ is a pseudoscalar, which commutes with all the elements of the algebra. Now the *law of sines for hyperboloidal triangles* follows:

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c} = \frac{|A\wedge B\wedge C|}{|A\wedge B||B\wedge C||C\wedge A|\,e_{123}} \qquad \text{(I)}$$

Let us see the law of cosines. Since $e_{23}^2 = e_{31}^2 = -e_{12}^2 = 1$ then:

$$\cos\alpha = \frac{(A\wedge B)\bullet(A\wedge C)}{|A\wedge B||A\wedge C|} = -\frac{(A\wedge B)\bullet(A\wedge C)}{\sinh c\,\sinh b}$$

$$\sinh b\,\sinh c\,\cos\alpha = \frac{1}{8}\left((A\,B - B\,A)(A\,C - C\,A) + (A\,C - C\,A)(A\,B - B\,A)\right)$$

$$= \frac{1}{8}\left(A\,B\,A\,C - A\,B\,C\,A - B\,A^2\,C + B\,A\,C\,A + A\,C\,A\,B - A\,C\,B\,A - C\,A^2\,B + C\,A\,B\,A\right)$$

Now taking into account that:

$$4\, A\cdot B\; A\cdot C = (A\,B + B\,A)(A\,C + C\,A) = A\,B\,A\,C + A\,B\,C\,A + B\,A^2\,C + B\,A\,C\,A$$

But also:

$$4\, A\cdot B\; A\cdot C = (A\,C + C\,A)(A\,B + B\,A) = A\,C\,A\,B + A\,C\,B\,A + C\,A^2\,B + C\,A\,B\,A$$

and adding the needed terms, we find:

$$\sinh b \sinh c \cos\alpha = \frac{1}{8}\left(8\, A\cdot B\; A\cdot C - 2\,A\,B\,C\,A - 2\,B\,A^2\,C - 2\,A\,C\,B\,A - 2\,C\,A^2\,B\right)$$

Extracting common factors and using $A^2 = B^2 = 1$, we may write:

$$\sinh b \sinh c \cos\alpha = \frac{1}{8}\left(8\, A\cdot B\; A\cdot C - 2\,A\,(B\,C + C\,B)\,A - 2\,(B\,C + C\,B)\right) = A\cdot B\; A\cdot C - B\cdot C$$

$$\sinh b \sinh c \cos\alpha = \cosh c \cosh b - \cosh a$$

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha \qquad \text{(II)}$$

which is the *law of cosines for sides*. The substitution of cosh c by means of the law of cosines gives:

$$\cosh a = \cosh b\,(\cosh a \cosh b - \sinh a \sinh b \cos\gamma) - \sinh b \sinh c \cos\alpha$$

$$\cosh a\,(1 - \cosh^2 b) = -\cosh b \sinh a \sinh b \cos\gamma - \sinh b \sinh c \cos\alpha$$

and the simplification of $-\sinh b$:

$$\cosh a \sinh b = \cosh b \sinh a \cos\gamma + \sinh c \cos\alpha$$

The substitution of $\sinh c = \sinh a \sin\gamma / \sin\alpha$ yields:

$$\cosh a \sinh b = \cosh b \sinh a \cos\gamma + \frac{\sinh a \sin\gamma \cos\alpha}{\sin\alpha}$$

Dividing by sinh a :

$$\coth a \sinh b = \cosh b \cos\gamma + \sin\gamma \cot\alpha \qquad \text{(III)}$$

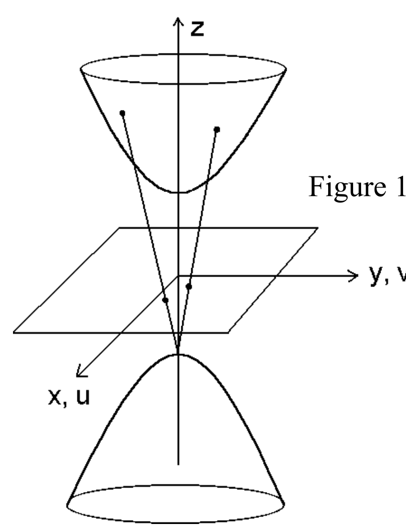


Figure 15.4

Stereographic projection (Poincaré disk)

We project the hyperboloidal surface into the plane $z = 0$ (figure 15.4) taking as centre of projection the lower pole (0, 0, −1). The upper sheet is projected inside the circle of unity radius while the lower sheet is projected outside. Let u and v be the

Cartesian coordinates of the projection plane. By similar triangles we have:

$$\frac{x}{u} = z + 1 \qquad \frac{y}{v} = z + 1$$

where (x, y, z) are the coordinates of a point on the hyperboloid. Using its equation $z^2 - x^2 - y^2 = 1$, one arrives at:

$$x = \frac{2\,u}{1 - u^2 - v^2} \qquad y = \frac{2\,v}{1 - u^2 - v^2} \qquad z = \frac{2}{1 - u^2 - v^2} - 1$$

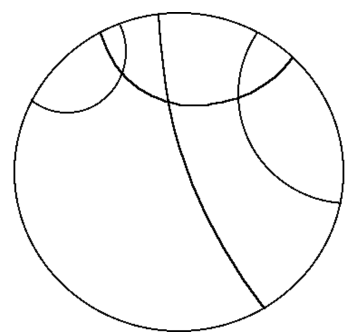
from where the differential of the arc length is obtained:

$$ds^2 = -dx^2 - dy^2 + dz^2 = -\frac{4\left(du^2 + dv^2\right)}{\left(1 - u^2 - v^2\right)^2}$$

The geodesic hyperbolas are intersection of the hyperboloid with central planes having the equation:

$$z = a\,x + b\,y$$

Figure 15.5



The substitution by the stereographic coordinates yields:

$$\left(u - a\right)^2 + \left(v - b\right)^2 = a^2 + b^2 - 1$$

which is the equation of a circle centred at (a, b) with radius $r = \sqrt{a^2 + b^2 - 1}$. That is, the geodesic hyperbolas on the hyperboloid are shown as circles in the stereographic projection (figure 15.5). The central planes cutting the two-sheeted hyperboloid fulfil $a^2 + b^2 > 1$, so that the radius is real and the centres of these circles are always placed outside the Poincaré disk.

The angle α between two circles is obtained through:

$$\cos\alpha = \frac{r^2 + r'^2 - \left(O - O'\right)^2}{2\,r\,r'} = \frac{a\,a' + b\,b' - 1}{\sqrt{a^2 + b^2 - 1}\sqrt{a'^2 + b'^2 - 1}}$$

Just this is the angle between two geodesic hyperbolas (between the tangent vectors t and t') found above. Therefore, the stereographic projection (Poincaré disk) is a conformal projection of the hyperboloidal surface.

On the other hand, the geodesic circles are always orthogonal to the Poincaré's circle $u^2 + v^2 = 1$ (r'=1, O'= (0, 0)) because:

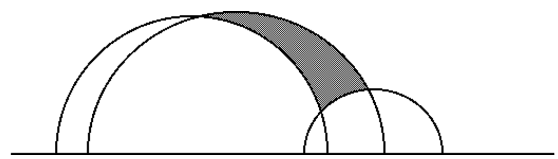
$$\cos\alpha = \frac{r^2 + 1 - O^2}{r} = 0$$

The differential of area is easily obtained through the modulus of the differential bivector:

$$dA = \sqrt{(dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2} = \frac{4\, du \wedge dv}{\left(1 - u^2 - v^2\right)^2}$$

When doing an inversion of the Poincaré disk centred at a point lying on the limit circle, another projection of the Lobachevsky's geometry is obtained: the Poincaré's half plane. Here the geodesics are semicircles orthogonal to the base line of the half plane. Since the inversion is a conformal transformation, the upper half plane is also a conformal projection of the hyperboloid. The figure 15.6 displays the projection of a hyperboloidal triangle and its sides in the Poincaré's upper half plane.

Figure 15.6



Azimuthal equivalent projection

This projection preserves the area and is similar to the azimuthal equivalent projection of the sphere usually used in the polar maps of the Earth.

The differential of the area in the pseudo-Euclidean space is a bivector whose square is:

$$dA^2 = (dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2$$

The substitution of the hyperboloid equation:

$$z^2 - x^2 - y^2 = 1 \qquad \Rightarrow \qquad dz = \frac{x}{z}\,dx + \frac{y}{z}\,dy$$

yields: $$dA^2 = \frac{1}{z^2}(dx \wedge dy)^2$$

Every azimuthal projection has plane coordinates u and v proportional to x and y, being the proportionality constant only function of z:

$$u = x\, f(z) \qquad v = y\, f(z)$$

Then by differentiation of these equalities and substitution of dz, which is a linear combination of dx and dy we arrive at:

$$du \wedge dv = \left(f^2 + \frac{z^2 - 1}{z}\, f\, f' \right) dx \wedge dy$$

Identifying both area differentials we find the following differential equation:

$$f^2 + \frac{z^2 - 1}{z} f\, f' = \frac{1}{z}$$

I introduce the auxiliary function $g = f^2$:

$$g + \frac{z^2 - 1}{2\,z} g' = \frac{1}{z}$$

Rewriting the equation with differentials, we arrive at an exact differential:

$$(z\,g - 1)\,dz + \frac{z^2 - 1}{2} dg = 0 \quad \Leftrightarrow \quad d\left(\frac{z^2 - 1}{2} g - z\right) = 0$$

$$g(z) = \frac{2\,(z + \text{const})}{\text{z}^2 - 1} \quad \Rightarrow \quad f(z) = \sqrt{\frac{2\,(z + \text{const})}{\text{z}^2 - 1}}$$

Next to the pole x and u are coincident and also y and v. Then $f(1)=1$, which implies the integration constant be -1 and then[5]:

$$f(z) = \sqrt{\frac{2\,(z - 1)}{\text{z}^2 - 1}} = \sqrt{\frac{2}{z + 1}} \quad \Rightarrow \quad u = x\sqrt{\frac{2}{z + 1}} \qquad v = y\sqrt{\frac{2}{z + 1}}$$

Weierstrass coordinates and cylindrical equidistant projection

These are a set of coordinates similar to the spherical coordinates, but on the hyperboloid surface in the pseudo-Euclidean space.

For a hyperboloid with unity radius, the Weierstrass coordinates are:

$$x = \sinh\psi \cos\varphi$$

$$y = \sinh\psi \sin\varphi$$

$$z = \cosh\psi$$

Then the differential of arc length is:

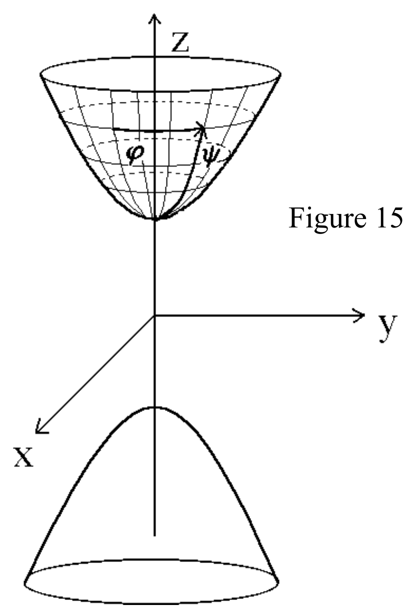


Figure 15.7

[5] As a comparison in the central projection $f(z) = 1/z$ while for the stereographic projection $f(z) = 1 / (z + 1)$.

$$ds = \left(\cosh\psi\,\cos\varphi\,d\psi - \sinh\psi\,\sin\varphi\,d\varphi\right)e_1 + \left(\cosh\psi\,\sin\varphi\,d\psi + \sinh\psi\,\cos\varphi\,d\varphi\right)e_2 + \sinh\psi\,d\psi\,e_3$$

Introducing the unitary vectors e_ψ and e_φ as:

$$e_\psi = \cosh\psi\,\cos\varphi\,e_1 + \cosh\psi\,\sin\varphi\,e_2 + \sinh\psi\,e_3$$

$$e_\varphi = -\sin\varphi\,e_1 + \cos\varphi\,e_2$$

the differential of arc length in hyperboloidal coordinates becomes:

$$ds = d\psi\,e_\psi + \sinh\psi\,d\varphi\,e_\varphi \qquad\qquad ds^2 = -\left(d\psi^2 + \sinh^2\psi\,d\varphi^2\right)$$

Note that e_ψ and e_φ are orthogonal vectors, since their inner product is zero. At $\psi = 0$, ds only depends on $d\psi$, so that there is a pole. Then ψ is the arc length from the pole to the given point on the hyperboloid surface (figure 15.7), while φ is the arc length over the equator, the parallel determined by $\sinh\psi = 1$ and $\psi = \log(1+\sqrt{2})$. The meridians (φ = constant) are geodesics, while the parallels (ψ = constant) including the equator are not geodesics because their planes do not pass through the origin of coordinates.

In the cylindrical projections, the hyperboloid is projected onto a cylinder passing through its equator, whose axis is the z axis. The chart is the Euclidean map obtained unrolling the cylinder[6]. Using φ and ψ as the coordinates u and v of the chart, the hyperboloid surface is projected into a rectangle with 2π width and infinite height. The area on this chart is:

$$dA = \sinh\psi\,d\varphi \wedge d\psi = \sinh v\,du \wedge dv$$

This projection is equidistant for the meridians (φ = constant) and for the equator ($\psi = \log(1+\sqrt{2})$).

Cylindrical conformal projection

If we wish to preserve the angles between curves, we must enlarge the meridians by the same amount as the parallels are enlarged in a cylindrical projection, that is by a factor $1/\sinh\psi$:

$$dv = \frac{d\psi}{\sinh\psi} \quad\Rightarrow\quad v = \log \operatorname{tgh}\frac{\psi}{2}$$

The differential of arc length and area are:

[6] Every plane tangent to the two-sheeted hyperboloid has Euclidean nature. So the Lobachevsky's surface can be only projected onto a Euclidean chart, and the cylinder of projection is not hyperbolic and does not belong properly to the pseudo-Euclidean space.

$$ds^2 = -\left(d\psi^2 + \sinh^2\psi \, d\varphi^2\right) = -\sinh^2\psi \left(du^2 + dv^2\right) = \frac{-4\exp(2v)}{(1-\exp(2v))^2}\left(du^2 + dv^2\right)$$

$$dA = \sinh\psi \, d\varphi \wedge d\psi = \frac{4\exp(2v)}{(1-\exp(2v))^2} du \wedge dv$$

Cylindrical equivalent projection

If we wish to preserve area, we must shorten the meridians in the same amount as the parallels are enlarged in the cylindrical projection, what yields:

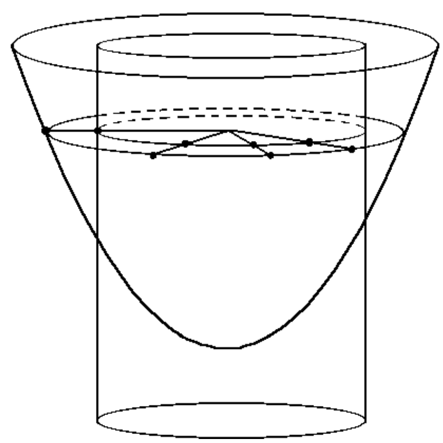
$$dv = \sinh\psi \, d\psi \qquad \Rightarrow \qquad v = \cosh\psi$$

$$ds^2 = -\left(d\psi^2 + \sinh^2\psi \, d\varphi^2\right) = -\left(v^2 - 1\right)du^2 - \frac{1}{v^2-1} dv^2$$

$$dA = \sinh\psi \, d\varphi \wedge d\psi = du \wedge dv$$

which clearly displays the equivalence of the projection. Observe that $v = \cosh\psi$ means the hyperboloid is projected following planes perpendicular to the cylinder of projection (figure 15.8).

Figure 15.8



Conic projections

These projections are made into a cone surface tangent to the hyperboloid. Here to take the cone $z^2 - x^2 - y^2 = 0$ is the most natural choice, although as before, the cone of projection has Euclidean nature and does not belong properly to the pseudo-Euclidean space, where it should be the cone of zero length. The cone surface unrolled is a circular sector. In a conic projection a parallel is shown as a circle with zero distortion. The characteristic parameter of a conic projection is the constant of the cone $n = \cos\theta_0$, being θ_0 the Euclidean angle of inclination of the generatrix of the cone. The relation with the hyperbolic angle ψ_0 of the parallel touching the cone (which is represented without distortion in the projection) and θ_0 is:

$$n = \cos\theta_0 = \frac{1}{\sqrt{1 + \mathrm{tgh}^2\,\psi_0}}$$

Since the graticule of the conic projections is radial, is more convenient to use the radius r and the angle χ :

$$dr = f(\psi)\, d\psi \qquad\qquad d\chi = n\, d\varphi$$

The differentials of arc length and area for a conic projection are:

$$ds^2 = -\left(d\psi^2 + \sinh^2\psi\, d\varphi^2\right) = -\left(\frac{dr^2}{[f(\psi)]^2} + \frac{\sinh^2\psi}{n^2}\, d\chi^2\right)$$

$$dA = \sinh\psi\, d\varphi \wedge d\psi = \frac{\sinh\psi}{n\, f(\psi)}\, dr \wedge d\chi$$

Let us see as before the three special cases: equidistant, conformal and equivalent projections. The differential of area for polar coordinates r, χ is $dA = r\, dr \wedge d\chi$. If the projection is equivalent, we must identify both dA to find:

$$\frac{d}{d\psi}\left[\frac{\sinh\psi}{n\, f(\psi)}\right] = f(\psi) \qquad \Rightarrow \qquad f(\psi) = \frac{1}{\sqrt{n}}\cosh\frac{\psi}{2} \quad \text{and} \quad r = \frac{2}{\sqrt{n}}\sinh\frac{\psi}{2}$$

$$ds^2 = -\left(\frac{n}{1+\dfrac{n\, r^2}{4}}\, dr^2 + \frac{1+\dfrac{n\, r^2}{4}}{n}\, r^2 d\chi^2\right)$$

If the projection is equidistant, the meridians have zero distortion so $d\psi = dr/n$ and:

$$ds^2 = -\frac{1}{n^2}\left(dr^2 + \sinh^2\left(\frac{r}{n}\right) d\chi^2\right)$$

If the projection is conformal then $ds^2 \propto dr^2 + r^2 d\chi^2$ so:

$$ds^2 = -\frac{\sinh^2\psi}{n^2 r^2}\left(dr^2 + r^2\, d\chi^2\right)$$

Then we solve the differential equation:

$$d\psi = \frac{\sinh\psi}{n\, r}\, dr$$

with the boundary condition $\operatorname{tgh}\psi_0 = r_0$ to find the conformal projection:

$$r = \mathrm{tgh}\,\psi_0 \left(\frac{\mathrm{tgh}\,\dfrac{\psi}{2}}{\mathrm{tgh}\,\dfrac{\psi_0}{2}} \right)^n$$

On the congruence of geodesic triangles

Two geodesic triangles on the hyperboloid having the same angles also have the same sides and are said to be *congruent*. This follows immediately from the rotations in the pseudo-Euclidean space, which are expressed by means of tetranions in the same way as quaternions are used in the rotations of Euclidean space. However, this falls out of the scope of this book and will not be treated. The reader must perceive, in spite of his Euclidean eyes[7], that all the points on the hyperboloid are equivalent because the curvature is always the unity and the surface is always perpendicular to the radius. So the pole (vertex of the hyperboloid) is not any special point, and any other point may be chosen as a new pole provided that the new axis are obtained from the old ones through a rotation.

Comment about the names of the non-Euclidean geometry

Finally it is apparent that the different kinds of non-Euclidean geometry have been often misnamed. The Lobachevskian surface was improperly called the "hyperbolic plane" in contradiction with the fact that it has constant curvature and therefore is not flat. Certainly, in the Lobachevsky's geometry the hyperbolic functions are widely used, but in the same way as the circular functions work in the spherical trigonometry, as Lobachevsky himself showed. In the pseudo-Euclidean plane the geometry of the hyperbola is properly realised, the hyperbolic functions being also defined there. Thus the pseudo-Euclidean plane should be named properly the *hyperbolic plane*, while the Lobachevsky's geometry might be alternatively called *hyperboloidal geometry*. The plane models of the Lobachevsky's geometry (Beltrami disk, Poincaré disk and half plane, etc.) must be understood as plane projections of the hyperboloid in the pseudo-Euclidean space, which is its proper space, and then the Lobachevskian trigonometry may be called *hyperboloidal trigonometry*.

On the other hand, the space-time was the first physical discovery of a pseudo-Euclidean plane, but one no needs to appeal to relativity because, as the lector has viewed in this book, both signatures of the metrics (Euclidean and pseudo-Euclidean) are included in the plane geometric algebra and also in the algebras of higher dimensions.

Exercises

15.1. According to the Lobachevsky's axiom of parallelism, at least two parallel lines pass through an exterior point of a given line. The two lines approaching a given line at

[7] In fact, eyes are not Euclidean but a very perfect camera where images are projected on a spherical surface. The principles of projection are likewise applicable to the pseudo-Euclidean space. Our mind accustomed to the ordinary space (we learn its Euclidean properties in the first years of our life) deceives us when we wish perceive the pseudo-Euclidean space.

the infinity are called «parallel lines» while those not cutting and not approaching it are called «ultraparallel» or «divergent lines». Lobachevsky defined the angle of parallelism $\Pi(s)$ as the angle which forms the line parallel to a given line with its perpendicular, which is a function of the distance s between the intersections on the perpendicular. Prove that $\Pi(s) = 2$ arctg(exp($-s$)) using the stereographic projection.

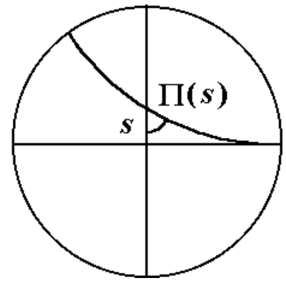


Figure 15.9

15.2. Find the equivalent of the Pythagorean theorem in the Lobachevskian geometry by taking a right angle triangle.

15.3. A *circle* is defined as the curve which is the intersection of the hyperboloid with a plane not passing through the origin with a slope less than the unity. Show that:

a) The curve obtained is an ellipse.

b) The distance from a point that we call the *centre* to every point of this ellipse is constant, and may be called the *radius of the circle*.

c) This ellipse is projected as a circle in the stereographic projection.

15.4. A *horocycle* is the intersection of the hyperboloid and a plane with unity slope ($a^2 + b^2 = 1$). Show that the horocycles are projected as circles touching the limit circle $x^2 + y^2 = 1$ in the stereographic projection. Find the centre of a horocycle.

15.5 a) Show that the differential of area in the Beltrami projection is:

$$dA = \frac{du \wedge dv}{\left(1 - u^2 - v^2\right)^{3/2}}$$

b) By integrating dA prove that the area between two lines forming an angle φ and the common "parallel" line (figure 15.10-a) is equal to $\pi - \varphi$.

c) Prove that the area of every triangle (figure 15.10-b) is equal to its angular defect $\pi - \alpha - \beta - \gamma$.

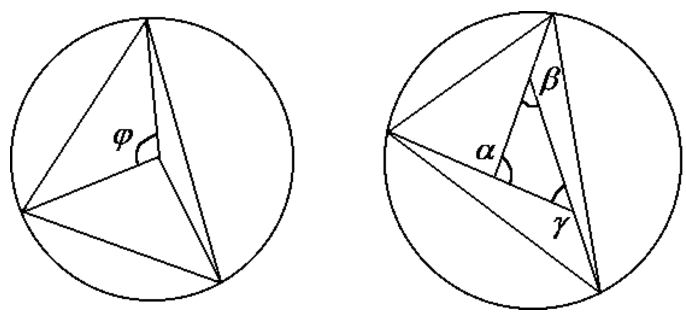


Figure 15.10

15.6 Use the azimuthal conformal projection to show that the area of a circle (hyperboloidal segment) with radius ψ is equal to 2π (cosh $\psi - 1$). This result is analogous to the area of a spherical segment $2\pi(1 - \cos\theta)$.

16. SOLUTIONS OF THE PROPOSED EXERCISES

1. The vectors and their operations

1.1 Let a, b be the different sides of the parallelogram and c, d both diagonals. Then:

$$c = a + b \qquad\qquad d = a - b$$

$$c^2 + d^2 = (a + b)^2 + (a - b)^2 = a^2 + 2\,a \cdot b + b^2 + a^2 - 2\,a \cdot b + b^2 =$$

$$= 2\,a^2 + 2\,b^2$$

which are the sum of the squares of the four sides of the parallelogram.

1.2 $$(a \cdot b)^2 - (a \wedge b)^2 = \frac{(a\,b + b\,a)^2}{4} + \frac{(a\,b - b\,a)^2}{4}$$

$$= \frac{a\,b\,a\,b + b\,a\,b\,a + a\,b^2\,a + b\,a^2\,b}{4} - \frac{a\,b\,a\,b + b\,a\,b\,a - a\,b^2\,a - b\,a^2\,b}{4}$$

$$= a^2\,b^2$$

1.3 Let us prove that $a \wedge b\ \ c \wedge d = (a \wedge b\ \ c) \cdot d$:

$$4\,a \wedge b\ \ c \wedge d = (a\,b - b\,a)\,(c\,d - d\,c) = a\,b\,c\,d - a\,b\,d\,c - b\,a\,c\,d + b\,a\,d\,c =$$

$$= a\,b\,c\,d - d\,b\,a\,c - b\,a\,c\,d + d\,a\,b\,c = (a\,b\,c - b\,a\,c)\,d + d\,(a\,b\,c - b\,a\,c) =$$

$$= 2\,(a\,b\,c - b\,a\,c) \cdot d = 4\,(a \wedge b\ \ c) \cdot d$$

In the same way the identity $a \wedge b\ \ c \wedge d = \mathrm{a} \cdot (b\ \ c \wedge d)$ is proved.

1.4 Using the identity of the previous exercise we have:

$$a \wedge b\ \ c \wedge d + a \wedge c\ \ d \wedge b + a \wedge d\ \ b \wedge c =$$

$$= a \cdot (b\ \ c \wedge d) + a \cdot (c\ \ d \wedge b) + a \cdot (d\ \ b \wedge c) =$$

$$= a \cdot (b\ \ c \wedge d + c\ \ d \wedge b + d\ \ b \wedge c) =$$

$$= a \cdot (b\,c\,d - b\,d\,c + c\,d\,b - c\,b\,d + d\,b\,c - d\,c\,b) = a \cdot 0 = 0$$

where the summands are simplified taking into account the permutative property.

1.5 Let us denote the components proportional and perpendicular to the vector a with $\|$ and $\perp$. Then using the fact that orthogonal vectors anticommute and those with the same direction commute, we have:

$$a\,b\,c = a\,(\,b_{\|} + b_{\perp}\,)\,(\,c_{\|} + c_{\perp}\,) = (\,b_{\|} - b_{\perp}\,)\,a\,(\,c_{\|} + c_{\perp}\,) = (\,b_{\|} - b_{\perp}\,)\,(\,c_{\|} - c_{\perp}\,)\,a =$$

$$= (\,b_{\|}\,c_{\|} - \,b_{\|}\,c_{\perp}\, - b_{\perp}\,c_{\|} + b_{\perp}\,c_{\perp}\,)\,a = (\,c_{\|}\,b_{\|} + c_{\perp}\,b_{\|} + c_{\|}\,b_{\perp} + c_{\perp}\,b_{\perp}\,)\,a =$$

$$= (\,c_{\|} + c_{\perp}\,)\,(\,b_{\|} + b_{\perp}\,)\,a = c\,b\,a$$

1.6 Let a, b, c be the sides of a triangle and h the altitude corresponding to the base b. Then the altitude divides the triangle in two right triangles where the Pythagorean theorem may be applied:

$$|\,b\,| = \sqrt{a^2 - h^2} + \sqrt{c^2 - h^2}$$

Isolating the altitude as function of the sides we obtain:

$$h^2 = a^2 - \frac{\left(a^2 + b^2 - c^2\right)^2}{4\,b^2}$$

Then the square of the area of the triangle is:

$$A^2 = \frac{b^2 h^2}{4} = \frac{4\,a^2 b^2 - \left(a^2 + b^2 - c^2\right)^2}{16} = \frac{-\,a^4 - b^4 - c^4 + 2\,a^2 b^2 + 2\,a^2 c^2 + 2\,b^2 c^2}{16}$$

Introducing the semiperimeter $s = \dfrac{1}{2}\,(\,|\,a\,| + |\,b\,| + |\,c\,|\,)$ we find:

$$A = \sqrt{s\,(s - |\,a\,|)\,(\,s - |\,b\,|)\,(\,s - |\,c\,|\,)}$$

2. A base of vectors for the plane

2.1 Draw, for example, two vectors of the first quadrant and mark u_1, u_2, v_1 and v_2 on the Cartesian axis. Then calculate the area of the parallelogram from the areas of the rectangles:

$$A = (\,u_1 + v_1\,)\,(\,u_2 + v_2\,) - u_1\,u_2 - v_1\,v_2 - 2\,v_1\,u_2 = u_1\,v_2 - u_2\,v_1$$

2.2 The area of a triangle is the half of the area of the parallelogram formed by any two sides. Therefore:

$$A = (\,3\,e_1 + 5\,e_2\,) \wedge (\,-2\,e_1 - 3\,e_2\,) = (\,-9 + 10\,)\,e_{12} = e_{12} \qquad \Rightarrow \qquad |\,A\,| = 1$$

2.3 $\quad a\,b\,c = (\,a_1\,e_1 + a_2\,e_2\,)\,(\,b_1\,e_1 + b_2\,e_2\,)\,(\,c_1\,e_1 + c_2\,e_2\,)$

$$= [\,a_1\,b_1 + a_2\,b_2 + e_{12}\,(\,a_1\,b_2 - a_2\,b_1\,)\,]\,(\,c_1\,e_1 + c_2\,e_2\,)$$

$$= e_1\,(a_1\,b_1\,c_1 + a_2\,b_2\,c_1 + a_1\,b_2\,c_2 - a_2\,b_1\,c_2) + e_2\,(a_1\,b_1\,c_2 + a_2\,b_2\,c_2 - a_1\,b_2\,c_1 + a_2\,b_1\,c_1)$$

$= (c_1 e_1 + c_2 e_2) [b_1 a_1 + b_2 a_2 + e_{12} (b_1 a_2 - b_2 a_1)] =$

$= (c_1 e_1 + c_2 e_2) (b_1 e_1 + b_2 e_2) (a_1 e_1 + a_2 e_2) = c\, b\, a$

2.4 $u\, v = (2 e_1 + 3 e_2) (-3 e_1 + 4 e_2) = 6 + 17 e_{12}$

$|u\, v| = \sqrt{325}$ $\qquad \alpha(u, v) = 1.2315 = 70º33'36''$

2.5 For this base we have:

$e_1 \cdot e_2 = | e_1 | | e_2 | \cos \pi/3 = 1$ $\qquad | e_1 \wedge e_2 | = | e_1 | | e_2 | | \sin \pi/3 | = \sqrt{3}$

Applying the original definitions of the modulus of a vector one finds:

$u^2 = (2 e_1 + 3 e_2)^2 = 4 e_1^2 + 9 e_2^2 + 12\, e_1 \cdot e_2 = 4 + 36 + 12 = 52 \quad \Rightarrow \quad |u| = \sqrt{52}$

$v^2 = (-3 e_1 + 4 e_2)^2 = 9 e_1^2 + 16 e_2^2 - 24 e_1 \cdot e_2 = 9 + 64 - 24 = 49 \quad \Rightarrow \quad |v| = \sqrt{49}$

$u\, v = (2 e_1 + 3 e_2) (-3 e_1 + 4 e_2) = -6 e_1^2 + 12 e_2^2 + 8 e_1 e_2 - 9 e_2 e_1 =$

$= 42 - e_1 \cdot e_2 + 17 e_1 \wedge e_2 = 41 + 17 e_1 \wedge e_2 \quad \Rightarrow \quad |u\, v| = \sqrt{2548}$

$$\cos\alpha = \frac{u \cdot v}{|u\, v|} = \frac{41}{\sqrt{2548}} \qquad \sin\alpha = \pm \frac{|u \wedge v|}{|u\, v|} = \frac{17\sqrt{3}}{\sqrt{2548}}$$

$\alpha(u, v) = 0.6228 = 35º\ 41'\ 5''$ oriented with the sense from e_1 to e_2 .

When e_1 and e_2 are not perpendicular, $e_1 e_2 \neq - e_2 e_1$ and e_{12} has not the meaning of a pure area but the outer product $e_1 \wedge e_2$ with an area of $\sqrt{3}$. Also observe that $|u\, v| = |u|\,|v|$.

2.6 In order to find the components of v for the new base $\{u_1, u_2\}$, we must resolve the vector v into a linear combination of u_1 and u_2 .

$v = c_1 u_1 + c_2 u_2$

$$c_1 = \frac{v \wedge u_2}{u_1 \wedge u_2} = \frac{3(-3) - (-5)5}{-1} = -16 \qquad c_2 = \frac{u_1 \wedge v}{u_1 \wedge u_2} = \frac{2(-5) - (-1)3}{-1} = 7$$

so in the new base $v = (-16, 7)$.

3. The complex numbers

3.1 $z\, t = (1 + 3 e_{12}) (-2 + 2 e_{12}) = -2 - 6 + (2 - 6) e_{12} = -8 - 4 e_{12}$

3.2 Every complex number z can be written as product of two vectors a and b, its modulus being the product of the moduli of both vectors:

$$z = a\,b \qquad \Rightarrow \qquad |z| = |a|\,|b| \qquad \Rightarrow \qquad |z|^2 = a^2\,b^2 = a\,b\,b\,a = z\,z^*$$

3.3 The equation $x^4 - 1 = 0$ is solved by extraction of the fourth roots:

$$x^4 = 1 \quad \begin{cases} x^2 = 1 & \Rightarrow \quad x_1 = 1,\ x_2 = -1 \\ x^2 = -1 & \Rightarrow \quad x_3 = e_{12}\,,\ x_4 = -e_{12} \end{cases}$$

3.4 Passing to the polar form we have:

$$\left(\sqrt{2}_{\pi/4}\right)^n = -\left(\sqrt{2}_{-\pi/4}\right)^n$$

$$\sqrt{2}^n{}_{n\pi/4} = -\sqrt{2}^n{}_{-n\pi/4} \qquad \Rightarrow \qquad \sqrt{2}^n{}_{n\pi/4} = \sqrt{2}^n{}_{-n\pi/4+\pi}$$

Therefore the arguments must be equal except for k times 2π:

$$\frac{n\pi}{4} = -\frac{n\pi}{4} + \pi + 2\,k\pi \qquad \Rightarrow \qquad \frac{n\pi}{2} = \pi + 2\,k\pi \qquad \Rightarrow \qquad n = 2 + 4\,k$$

3.5 The three cubic roots of $-3 + 3\,e_{12}$ are $\sqrt[3]{18}_{\pi/4}$, $\sqrt[3]{18}_{11\pi/12}$, $\sqrt[3]{18}_{19\pi/12}$.

3.6 Using the formula of the equation of second degree we find: $z_1 = 2-3\,e_{12}$, $z_2 = 1+e_{12}$.

3.7 We suppose that the complex analytic extension f has a real part a of the form:

$$a = \sin x\, K(y) \qquad \text{with } K(0) = 1$$

Applying the first Cauchy-Riemann condition, we find the imaginary part b of f:

$$\frac{\partial a}{\partial x} = \cos x\, K(y) = \frac{\partial b}{\partial y} \qquad \Rightarrow \qquad b = \cos x \int K(y)\, dy$$

Applying the second Cauchy-Riemann condition:

$$\frac{\partial a}{\partial y} = \sin x\, K'(y) = -\frac{\partial b}{\partial x} = \sin x \int K(y)\, dy$$

we arrive at a differential equation for $K(y)$ whose solution is the hyperbolic cosine:

$$\left.\begin{aligned} K'(y) &= \int K(y)\, dy \\ K(0) &= 1 \end{aligned}\right\} \qquad \Rightarrow \qquad K(y) = \cosh y$$

Hence the analytical extension of the sine is:

$$\sin(x + e_{12}\, y) = \sin x \cosh y + e_{12} \cos x \sinh y$$

The analytical extensions of the trigonometric functions may be also obtained from the exponential function. In the case of the cosine, we have:

$$\cos z = \frac{\exp(e_{12}\, z) + \exp(-e_{12}\, z)}{2} = \frac{\exp(-y)(\cos x + e_{12} \sin x) + \exp(y)(\cos x - e_{12} \sin x)}{2}$$

$$\cos z = \cos x \cosh y - e_{12} \sin x \sinh y$$

3.8 Let us calculate some derivatives at $z = 0$:

$$f(z) = \log(1 + \exp(-z)) \qquad f(0) = \log 2$$

$$f'(z) = \frac{-\exp(-z)}{1 + \exp(-z)} = \frac{-1}{\exp(z) + 1} \qquad f'(0) = -\frac{1}{2}$$

$$f''(z) = \frac{\exp(z)}{(\exp(z) + 1)^2} \qquad f''(0) = \frac{1}{4}$$

$$f'''(z) = \frac{\exp(z)}{(\exp(z) + 1)^2} - \frac{2 \exp(2z)}{(\exp(z) + 1)^3} \qquad f'''(0) = 0$$

$$f^{IV}(z) = \frac{\exp(z)}{(\exp(z) + 1)^2} - \frac{2 \exp(2z)}{(\exp(z) + 1)^3} - \frac{4 \exp(2z)}{(\exp(z) + 1)^3} + \frac{6 \exp(3z)}{(\exp(z) + 1)^4} \qquad f^{IV}(0) = -\frac{1}{8}$$

Hence the Taylor series is:

$$f(z) = \log 2 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^4}{8} + \dots$$

Observe that $f(e_{12}\, \pi) = \log(0)$ is divergent. This is the singularity nearest the origin. Therefore the radius of convergence of the series is π.

3.9 Separating fractions:

$$f(z) = \frac{1}{z^2 + 2z - 8} = \frac{1}{6(z - 2)} - \frac{1}{6(z + 4)}$$

and developing the second fraction:

$$\frac{1}{z+4}=\frac{1}{6+z-2}=\frac{\frac{1}{6}}{1+\frac{z-2}{6}}=\frac{1}{6}\left(1-\frac{z-2}{6}+\left(\frac{z-2}{6}\right)^2-\left(\frac{z-2}{6}\right)^3+...\right)$$

we find: $$f(z)=\frac{1}{6(z-2)}-\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{6^{n+2}}$$

This Lauren series is convergent in the annulus $0<|z-2|<6$, which contains the required annulus $1<|z-2|<4$.

3.10 Taking into account that $(1-x)^{-1}=1+x+x^2+...$, the function defined by the series is:

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{4^n(z+1)^n}=\frac{1}{1-\frac{1}{4(z+1)}}-1=\frac{1}{4z+3}$$

Now we see that at $z=-3/4$ the function has a pole. Therefore the radius of convergence of this series (centred at $z=-1$) is $|-3/4-(-1)|=1/4$.

3.11 From the successive derivatives of the sines calculated at $z=0$, one obtains the Taylor series for $\sin z$:

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-...=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

so the Lauren series for the given function is:

$$\frac{\sin z}{z^2}=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n-1}}{(2n+1)!}=\frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}-...$$

Since the pole $z=0$ is the unique singularity (see that the analytical extension of the real sine in the exercise 3.7 has no singularities), the annulus of convergence is $0<|z|<\infty$.

3.12 If $f(z)$ is analytic then the derivative at each point is unique and we can write for two different directions dz_1 and dz_2:

$$df_1=dz_1\,f'(z) \qquad\qquad df_2=dz_2\,f'(z)$$

According to the relationship between the complex and vectorial planes, we can multiply by e_1 at the left in order to turn the complex differentials into vectors:

$$df=da+db\,e_{12}\ \xrightarrow{e_1}\ df=da\,e_1+db\,e_2 \qquad\qquad dz=dx+dy\,e_{12}\ \xrightarrow{e_1}\ dz=dx\,e_1+dy\,e_2$$

Now let us calculate the geometric product of the vector differentials:

$$df_1\,df_2=dz_1\,f'(z)\,dz_2\,f'(z)=dz_1\,dz_2\,[f'(z)]^*\,f'(z)=|f'(z)|^2\,dz_1\,dz_2$$

Since $|f'(z)|^2$ is real, both geometric products have the same argument, $\alpha(df_1, df_2) = \alpha(dz_1, dz_2)$, and hence the transformation is conformal.

4. Transformations of vectors

4.1 If d is the direction vector of the reflection, the vector v' reflected of v with respect to this direction is given by:

$$v' = d^{-1}\, v\, d$$

If v'' is obtained from v' by a rotation through an angle α, and z is a unitary complex number with argument $\alpha/2$ then:

$$v'' = z^{-1}\, v'\, z \qquad\qquad z = \cos\frac{\alpha}{2} + e_{12}\, \sin\frac{\alpha}{2}$$

Joining both equations:

$$v'' = z^{-1}\, d^{-1}\, v\, d\, z = c^{-1}\, v\, c$$

That is, one obtains a reflection with respect another direction with vector $c = d\, z$, the product of the direction vector of the initial reflection and the complex number with half argument.

4.2 Let w'' be the transformed vector of w by two consecutive reflections with respect different directions u and v:

$$w'' = v^{-1}\, w'\, v = v^{-1}\, u^{-1}\, w\, u\, v$$

Then we can write:

$$w'' = z^{-1}\, w\, z \qquad\qquad z = u\, v \qquad\qquad z \text{ being a complex number}$$

so that it is equivalent to a rotation with an angle equal to the double of that formed by both direction vectors.

4.3 If the product of each transformed vector v' by the initial vector v is equal to a complex number z^2 (v' and v always form a constant angle) then:

$$v\, v' = z^2$$

$$v' = v^{-1}\, z^2 = z^*\, v^{-1}\, z = z^{-1}\, |\, z\,|^2\, v^{-1}\, z = |\, z\,|^2\, (\, z^{-1}\, v\, z\,)^{-1}$$

which represents an inversion with radius $|\, z\,|^2$ followed by a rotation with an angle equal to the double of the argument of z. As the algebra shows, both elemental transformations commute.

5. Points and straight lines

5.1 If A, B, C and D are located following this order on the perimeter of the parallelogram, then AB=DC:

$$AB = DC \quad \Rightarrow \quad B - A = C - D \quad \Rightarrow \quad D = C - B + A$$

$$D = (2, -5) - (4, -3) + (2, 4) = (0, 2)$$

The area is: $\left| AB \wedge BC \right| = \left| (2\, e_1 - 7\, e_2) \wedge (-2\, e_1 - 2\, e_2) \right| = \left| -18\, e_{12} \right| = 18$

5.2 The Euler's theorem:

$$AD\; BC + BD\; CA + CD\; AB = (D - A) (C - B) + (D - B) (A - C) + (D - C) (B - A)$$

$$= C\,A - A\,C + A\,B - B\,A + B\,C - C\,B = 2 (C \wedge A + A \wedge B + B \wedge C) =$$

$$= 2 (B - A) \wedge (C - B) = 2\, AB \wedge BC$$

This product only vanishes if A, B and C are collinear. On the other hand we see that the product is the oriented area of the triangle ABC.

5.3 a) The area of the triangle ABC is the outer product of two sides:

$AB = B - A = (4, 4) - (2, 2) = 2\, e_1 + 2\, e_2$ $\qquad$ $AC = C - A = (4, 2) - (2, 2) = 2\, e_1$

$$AB \wedge AC = (2\, e_1 + 2\, e_2) \wedge 2\, e_1 = 4\, e_2 \wedge e_1$$

$$\left| AB \wedge AC \right| = 4 \left| e_2 \wedge e_1 \right| = 4 \left| e_2 \right| \left| e_1 \right| \left| \sin 60^\circ \right| = 4\sqrt{3}$$

b) The distance from A to B is the length of the vector AB, etc:

$$AB^2 = (2\, e_1 + 2\, e_2)^2 = 4\, e_1^2 + 4\, e_2^2 + 8\, e_1 \cdot e_2 = 4 + 16 + 16 \cos \pi/3 = 28$$

$AC^2 = (2\, e_1)^2 = 4$ $\qquad$ $BC^2 = (-2\, e_2)^2 = 4\, e_2^2 = 16$

$d(A, B) = \left| AB \right| = \sqrt{28}$ $\qquad$ $d(B, C) = \left| BC \right| = 4$ $\qquad$ $d(C, A) = \left| CA \right| = 2$

5.4 A side of the trapezoid is a vectorial sum of the other three sides:

$$AD = AB + BC + CD$$

$$AD^2 = (AB + BC + CD)^2 = AB^2 + BC^2 + CD^2 + 2\, AB \cdot BC + 2\, AB \cdot CD + 2\, BC \cdot CD$$

$$= AB^2 + BC^2 + CD^2 + 2\, AB \cdot BC - 2 \left| AB \right| \left| CD \right| + 2\, BC \cdot CD$$

where the fact that AB and CD be vectors with the same direction and contrary sense has been taken into account. Arranging terms:

$$AD^2 - AB^2 - BC^2 - CD^2 + 2\ |AB|\ |CD|\ = 2\ AB \cdot BC + 2\ BC \cdot CD$$

$$AD^2 - AB^2 - BC^2 - CD^2 + 2\ |\mathrm{AB}|\ |\mathrm{CD}|\ = 2\ BC \cdot (\ AB + CD\ )$$

Without loss of generality we will suppose that $|AB| > |CD|$:

$$|AB + CD| = |AB| - |CD|$$

Since the angle ABC is the supplement of the angle formed by the vectors AB and BC, we have:

$$AD^2 - AB^2 - BC^2 - CD^2 + 2\ |AB|\ |CD|\ = -2\ |BC|\ (\ |AB| - |CD|\ ) \cos ABC$$

whence we obtain the angle ABC:

$$\cos ABC = \frac{-AD^2 + AB^2 + BC^2 + CD^2 - 2|AB||CD|}{2\ |BC|\big(|AB| - |CD|\big)}$$

The trapezoid can only exist for the range $-1 < \cos ABC < 1$, that is:

$$2\ |BC|\ (\ |AB| - |CD|\ ) > -AD^2 + AB^2 + BC^2 + CD^2 - 2\ |AB|\ |CD|\ >$$

$$> -2\ |BC|\ (|AB| - |CD|)$$

5.5 $R = (\ 1 - p - q\ )\ O + p\ P + q\ Q \quad \Rightarrow \quad RP = (\ 1 - p - q\ )\ OP + q\ QP$

$$\Rightarrow \quad RP \wedge PQ = (\ 1 - p - q\ )\ OP \wedge PQ$$

whence it follows that: Area $RPQ = (\ 1 - p - q\ )$ Area OPQ

5.6 The direction vector of the straight line r is AB:

$AB = B - A = (5, 4) - (2, 3) = 3\ e_1 + e_2 \qquad AC = C - A = (1, 6) - (2, 3) = -\ e_1 + 3\ e_2$

The distance from the point C to the line r is:

$$d(C, r) = \frac{|AC \wedge AB|}{|AB|} = \frac{|(-e_1 + 3e_2) \wedge (3e_1 + e_2)|}{\sqrt{10}} = \sqrt{10}$$

The angle between the vectors AB and AC is deduced by means of the sine and cosine:

$$\cos\alpha = \frac{AB \cdot AC}{|AB|\,|AC|} = 0 \qquad e_{12} \sin\alpha = \frac{AB \wedge AC}{|AB|\,|AC|} = e_{12}$$

Therefore, $\alpha = \pi/2$. The angle between two lines is always comprised from $-\pi/2$ to $\pi/2$

because a rotation of 2π around the intersection point does not alter the lines. When the angle exceeds these boundaries, you may add or subtract π.

5.7 a) Three points D, E, F are aligned if they are linearly dependent, that is, if the determinant of the coordinates vanishes.

$$D = \left(1 - x_D - y_D\right) O + x_D\, P + y_D\, Q$$

$$E = \left(1 - x_E - y_E\right) O + x_E\, P + y_E\, Q$$

$$F = \left(1 - x_F - y_F\right) O + x_F\, P + y_F\, Q$$

$$\begin{vmatrix} 1 - x_D - y_D & x_D & y_D \\ 1 - x_E - y_E & x_E & y_E \\ 1 - x_F - y_F & x_F & y_F \end{vmatrix} = 0$$

where the barycentric coordinate system is given by the origin O and points P, Q (for example the Cartesian system is determined by $O = (0, 0)$, $P = (1, 0)$ and $Q = (0, 1)$). The transformed points D', E' and F' have the same coordinates expressed for the base O', P' and Q'. Then the determinant is exactly the same, so that it vanishes and the transformed points are aligned. Therefore any straight line is transformed into another straight line.
b) Let O', P' and Q' be the transformed points of O, P and Q by the given affinity:

$$O' = \left(o_1, o_2\right) \qquad P' = \left(p_1, p_2\right) \qquad Q' = \left(q_1, q_2\right)$$

and consider any point R with coordinates (x, y):

$$R = (x, y) = (1 - x - y)\, O + x\, P + y\, Q$$

Then R', the transformed point of R, is:

$$R' = (1 - x - y)\, O' + x\, P' + y\, Q' = (1 - x - y)\,(o_1, o_2) + x\,(p_1, p_2) + y\,(q_1, q_2) =$$

$$= \left(x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1,\quad x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2\right) = (x', y')$$

where we see that the coordinates x' and y' of R' are linear functions of the coordinates of R:

$$x' = x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1$$

$$y' = x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2$$

In matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} p_1 - o_1 & q_1 - o_1 \\ p_2 - o_2 & q_2 - o_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} o_1 \\ o_2 \end{pmatrix}$$

Because every linear (and non degenerate) mapping of coordinates can be written in this regular matrix form, now we see that it is always an affinity.

c) Let us consider any three non aligned points A, B, C and their coordinates:

$$A = (1 - x_A - y_A)\, O + x_A P + y_A Q$$

$$B = (1 - x_B - y_B)\, O + x_B P + y_B Q$$

$$C = (1 - x_C - y_C)\, O + x_C P + y_C Q$$

In matrix form:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 - x_A - y_A & x_A & y_A \\ 1 - x_B - y_B & x_B & y_B \\ 1 - x_C - y_C & x_C & y_C \end{pmatrix} \begin{pmatrix} O \\ P \\ Q \end{pmatrix}$$

A certain point D is expressed with coordinates whether for $\{ O, P, Q \}$ or $\{ A, B, C \}$:

$$D = (1 - b - c) A + b\, B + c\, C = (1 - x_D - y_D) O + x_D\, P + y_D\, Q$$

In matrix form:

$$D = \begin{pmatrix} 1 - x_D - y_D & x_D & y_D \end{pmatrix} \begin{pmatrix} O \\ P \\ Q \end{pmatrix} = \begin{pmatrix} 1 - b - c & b & c \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\begin{pmatrix} 1 - x_D - y_D & x_D & y_D \end{pmatrix} = \begin{pmatrix} 1 - b - c & b & c \end{pmatrix} \begin{pmatrix} 1 - x_A - y_A & x_A & y_A \\ 1 - x_B - y_B & x_B & y_B \\ 1 - x_C - y_C & x_C & y_C \end{pmatrix}$$

which leads to the following system of equations:

$$\begin{cases} x_D = (1 - b - c)\, x_A + b\, x_B + c\, x_C \\ y_D = (1 - b - c)\, y_A + b\, y_B + c\, y_C \end{cases}$$

An affinity does not change the coordinates x, y of A, B, C and D, but only the point base -$\{O', P', Q'\}$ instead of $\{O, P, Q\}$-. Therefore the solution of the system of equations for b and c is the same. Then we can write:

$$D' = (1 - b - c) A' + b\, B' + c\, C'$$

d) If the points D, E, F and G are the consecutive vertices in a parallelogram then:

$$DE = GF \quad \Rightarrow \quad G = D - E + F$$

The affinity preserves the coordinates expressed in any base $\{D, E, F\}$. Then the transformed points form also a parallelogram:

$$G' = D' - E' + F' \;\Rightarrow\; D'E' = G'F'$$

e) For any three aligned points D, E, F the single ratio r is:

$$DE\,DF^{-1} = r \qquad \Rightarrow \qquad DE = r\,DF \qquad \Rightarrow \qquad E = (1 - r)\,D + r\,F$$

The ratio r is a coordinate within the straight line DF and it is not changed by the affinity:

$$E' = (1 - r)\,D' + r\,F' \qquad \Rightarrow \qquad D'E'\,D'F'^{\,-1} = r$$

5.8 This exercise is the dual of the problem 2. Then I have copied and pasted it changing the words for a correct understanding.
a) Three lines D, E, F are concurrent if they are linearly dependent, that is, if the determinant of the dual coordinates vanishes:

$$D = \left(1 - x_D - y_D\right) O + x_D\, P + y_D\, Q$$

$$E = \left(1 - x_E - y_E\right) O + x_E\, P + y_E\, Q$$

$$F = \left(1 - x_F - y_F\right) O + x_F\, P + y_F\, Q$$

$$\begin{vmatrix} 1 - x_D - y_D & x_D & y_D \\ 1 - x_E - y_E & x_E & y_E \\ 1 - x_F - y_F & x_F & y_F \end{vmatrix} = 0$$

where the dual coordinate system is given by the lines O, P and Q. For example, the Cartesian system is determined by $O = [0, 0]$ (line $-x - y + 1 = 0$), $P = [1, 0]$ (line $x = 0$) and $Q = [0, 1]$ (line $y = 0$). The transformed lines D', E' and F' have the same coordinates expressed for the base O', P' and Q'. Then the determinant also vanishes and the transformed lines are concurrent. Therefore any pencil of lines is transformed into another pencil of lines.
b) Let O', P' and Q' be the transformed lines of O, P and Q by the given transformation:

$$O' = [o_1, o_2] \qquad\qquad P' = [p_1, p_2] \qquad\qquad Q' = [q_1, q_2]$$

and consider any line R with dual coordinates $[x, y]$:

$$R = [x, y] = (1 - x - y)\,O + x\,P + y\,Q$$

Then R', the transformed line of R, is:

$$R' = (1 - x - y)\,O' + x\,P' + y\,Q' = (1 - x - y)\,[\,o_1, o_2\,] + x\,[\,p_1, p_2\,] + y\,[\,q_1, q_2\,] =$$

$$= [\,x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1,\;\; x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2\,] = [\,x', y'\,]$$

where we see that the coordinates x' and y' of R' are linear functions of the coordinates of R:

$$x' = x\,(p_1 - o_1) + y\,(q_1 - o_1) + o_1$$

$$y' = x\,(p_2 - o_2) + y\,(q_2 - o_2) + o_2$$

In matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} p_1 - o_1 & q_1 - o_1 \\ p_2 - o_2 & q_2 - o_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} o_1 \\ o_2 \end{bmatrix}$$

Because any linear mapping (non degenerate) of dual coordinates can be written in this regular matrix form, now we see that it is always an affinity.

c) Let us consider any three non concurrent lines A, B, C and their coordinates:

$$A = \left(1 - x_A - y_A\right) O + x_A P + y_A Q$$

$$B = \left(1 - x_B - y_B\right) O + x_B P + y_B Q$$

$$C = \left(1 - x_C - y_C\right) O + x_C P + y_C Q$$

In matrix form:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 - x_A - y_A & x_A & y_A \\ 1 - x_B - y_B & x_B & y_B \\ 1 - x_C - y_C & x_C & y_C \end{bmatrix} \begin{bmatrix} O \\ P \\ Q \end{bmatrix}$$

A certain line D is expressed with dual coordinates whether for $\{O, P, Q\}$ or $\{A, B, C\}$:

$$D = \left(1 - b - c\right) A + b\,B + c\,C = \left(1 - x_D - y_D\right) O + x_D\,P + y_D\,Q$$

In matrix form:

$$D = \begin{bmatrix} 1 - x_D - y_D & x_D & y_D \end{bmatrix} \begin{bmatrix} O \\ P \\ Q \end{bmatrix} = \begin{bmatrix} 1 - b - c & b & c \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{bmatrix} 1 - x_D - y_D & x_D & y_D \end{bmatrix} = \begin{bmatrix} 1 - b - c & b & c \end{bmatrix} \begin{bmatrix} 1 - x_A - y_A & x_A & y_A \\ 1 - x_B - y_B & x_B & y_B \\ 1 - x_C - y_C & x_C & y_C \end{bmatrix}$$

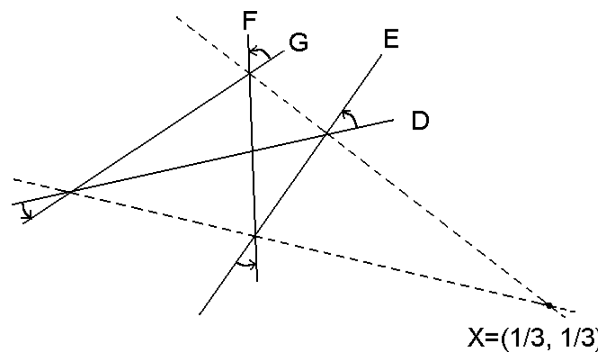
which leads to the following system of equations:

$$\begin{cases} x_D = (1-b-c)\,x_A + b\,x_B + c\,x_C \\ y_D = (1-b-c)\,y_A + b\,y_B + c\,y_C \end{cases}$$

An affinity does not change the coordinates x, y of A, B, C and D, but only the lines base -$\{O', P', Q'\}$ instead of $\{O, P, Q\}$-. Therefore the solution of the system of equations for b and c is the same. Then we can write:

$$D' = (1-b-c)\,A' + b\,B' + c\,C'$$

Figure 16.1



d) The affinity maps parallel points into parallel points (points aligned with the point (1/3, 1/3), point at the infinity in the dual plane). If the lines D, E, F and G are the consecutive vertices in a dual parallelogram (figure 16.1) then:

$$DE = GF \quad \Rightarrow \quad G = D - E + F$$

Where DE is the dual vector of the intersection point of the lines D and E, and GF the dual vector of the intersection point of G and F. Obviously, the points EF and DG are also parallel because from the former equality it follows:

$$EF = DG$$

The affinity preserves the coordinates expressed in any base $\{D, E, F\}$. Then the transformed lines form also a dual parallelogram:

$$G' = D' - E' + F' \quad \Rightarrow \quad D'E' = G'F'$$

e) For any three concurrent lines D, E, F the single dual ratio r is:

$$DE\,DF^{-1} = r \qquad \Rightarrow \qquad DE = r\,DF \qquad \Rightarrow \qquad E = (\,1 - r\,)\,D + r\,F$$

The ratio r is a coordinate within the pencil of lines DF and it is not changed by the affinity:

$$E' = (\,1 - r\,)\,D' + r\,F' \qquad \Rightarrow \qquad D'E'\,D'F'^{\,-1} = r$$

f) Let us consider four concurrent lines A, B, C and D with the following dual coordinates expressed in the lines base $\{O, P, Q\}$:

$$A = [x_A\,, y_A] \qquad B = [x_B\,, y_B] \qquad C = [x_C\,, y_C] \qquad D = [x_D\,, y_D]$$

Then the direction vector of the line *A* is:

$$v_A = \left(1 - x_A - y_A\right) v_O + x_A v_P + y_A v_Q$$

But the direction vectors of the base fulfils:

$$v_O + v_P + v_Q = 0$$

Then:

$$v_A = \left(-1 + 2x_A + y_A\right) v_P + \left(-1 + x_A + 2y_A\right) v_Q$$

And analogously:

$$v_B = \left(-1 + 2x_B + y_B\right) v_P + \left(-1 + x_B + 2y_B\right) v_Q$$

$$v_C = \left(-1 + 2x_C + y_C\right) v_P + \left(-1 + x_C + 2y_C\right) v_Q$$

$$v_D = \left(-1 + 2x_D + y_D\right) v_P + \left(-1 + x_D + 2y_D\right) v_Q$$

The outer product is:

$$v_A \wedge v_C = \left(x_C - x_A - \left(y_C - y_A\right) + 3\left(x_A\, y_C - x_C\, y_A\right)\right) v_P \wedge v_Q$$

Then the cross ratio only depends on the dual coordinates but not on the direction of the base vectors (see chapter 10):

$$\left(ABCD\right) = \frac{v_A \wedge v_C \;\; v_B \wedge v_D}{v_A \wedge v_D \;\; v_B \wedge v_C} =$$

$$= \frac{\left(x_C - x_A - \left(y_C - y_A\right) + 3\left(x_A\, y_C - x_C\, y_A\right)\right)\left(x_D - x_B - \left(y_D - y_B\right) + 3\left(x_B\, y_D - x_D\, y_B\right)\right)}{\left(x_D - x_A - \left(y_D - y_A\right) + 3\left(x_A\, y_D - x_D\, y_A\right)\right)\left(x_C - x_B - \left(y_C - y_B\right) + 3\left(x_B\, y_C - x_C\, y_B\right)\right)}$$

Hence it remains invariant under an affinity. Each outer product can be written as an outer product of dual vectors obtained by subtraction of the coordinates of each line and the infinite line:

$$v_A \wedge v_C = 3\left[x_A - \frac{1}{3}, y_A - \frac{1}{3}\right] \wedge \left[x_C - \frac{1}{3}, y_C - \frac{1}{3}\right] = 3\, LA \wedge LC$$

where $LA = A - L$ is the dual vector going from the line *L* at the infinity to the line *A*, etc. Then we can write this useful formula for the cross ratio of four lines:

$$(ABCD) = \frac{LA \wedge LC \;\; LB \wedge LD}{LA \wedge LD \; LB \wedge LC}$$

This exercise links with the section *Projective cross ratio* in the chapter 10.

5.9 To obtain the dual coordinates of the first line, solve the identity:

$$x - y - 1 \equiv a'(-x - y + 1) + b'x + c'y \qquad \Rightarrow \qquad a' = -1 \quad b' = 0 \quad c' = -2$$

$$\Rightarrow \qquad a = \frac{1}{3} \quad b = 0 \quad c = \frac{2}{3} \qquad \Rightarrow \qquad [0, 2/3]$$

For the second line:

$$x - y + 3 \equiv a'(-x - y + 1) + b'x + c'y \qquad \Rightarrow \qquad a' = 3 \quad b' = 4 \quad c' = 2$$

$$\Rightarrow \qquad a = \frac{3}{9} \quad b = \frac{4}{9} \quad c = \frac{2}{9} \qquad \Rightarrow \qquad [4/9, 2/9]$$

Both lines are aligned in the dual plane with the line at the infinity (whose dual coordinates are [1/3, 1/3]) since the determinant of the coordinates vanishes:

$$\begin{vmatrix} 1/3 & 0 & 2/3 \\ 2/9 & 4/9 & 2/9 \\ 1/3 & 1/3 & 1/3 \end{vmatrix} = 0$$

Therefore they are parallel (this is also trivial from the general equations).

5.10 The point (2, 1) is the intersection of the lines $x - 2 = 0$ and $y - 1 = 0$, whose dual coordinates are:

$$x - 2 = 0 \qquad \Rightarrow \qquad [1/5, 2/5]$$

$$y - 1 = 0 \qquad \Rightarrow \qquad [1/2, 0]$$

The difference of dual coordinates gives a dual direction vector for the point:

$$v = [1/2, 0] - [1/5, 2/5] = [3/10, -4/10]$$

and hence we obtain the dual continuous and general equations for the point:

$$\frac{b - 1/2}{3} = \frac{c}{-4} \qquad \Leftrightarrow \qquad 4b + 3c = 2$$

The point (–3, –1) is the intersection of the lines $x + 3 = 0$ and $y + 1 = 0$, whose dual coordinates are:

$x + 3 = 0 \qquad \Rightarrow \qquad [4/10, 3/10]$

$y + 1 = 0 \qquad \Rightarrow \qquad [1/4, 2/4]$

The difference of dual coordinates gives a dual direction vector for the point:

$$w = [1/4, 2/4] - [4/10, 3/10] = [-3/20, 4/20]$$

and hence we obtain the dual continuous and general equations for the point:

$$\frac{b - 1/4}{-3} = \frac{c - 2/4}{4} \qquad \Leftrightarrow \qquad 8b + 6c = 5$$

Note that both dual direction vectors v and w are proportional and the points are parallel in a dual sense. Hence the points are aligned with the centroid (1/3, 1/3) of the coordinate system.

6. Angles and elemental trigonometry

6.1 Let a, b and c be the sides of a triangle taken anticlockwise. Then:

$$a + b + c = 0 \quad \Rightarrow \quad c = -a - b \quad \Rightarrow \quad c^2 = (a + b)^2 = a^2 + b^2 + 2\, a \cdot b$$

$$c^2 = a^2 + b^2 + 2\, |a|\, |b| \cos(\pi - \gamma) = a^2 + b^2 - 2\, |\mathrm{a}|\, |\mathrm{b}| \cos \gamma \quad \text{(law of cosines)}$$

The area s of a triangle is the half of the outer product of any pair of sides:

$$2\, s = a \wedge b = b \wedge c = c \wedge a$$

$$|a|\, |b| \sin(\pi - \gamma)\, e_{12} = |b|\, |c| \sin(\pi - \alpha)\, e_{12} = |c|\, |a| \sin(\pi - \beta)\, e_{12}$$

$$\frac{\sin\gamma}{|c|} = \frac{\sin\alpha}{|a|} = \frac{\sin\beta}{|b|} \qquad \text{(law of sines)}$$

From here one quickly obtains:

$$\frac{|a| + |b|}{|a|} = \frac{\sin\alpha + \sin\beta}{\sin\alpha} \qquad\qquad \frac{|a| - |b|}{|b|} = \frac{\sin\alpha - \sin\beta}{\sin\beta}$$

By dividing both equations and introducing the identities for the addition and subtraction of sines we arrive at:

$$\frac{|a|+|b|}{|a|-|b|}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\frac{|a|+|b|}{|a|-|b|}=\frac{\operatorname{tg}\frac{\alpha+\beta}{2}}{\operatorname{tg}\frac{\alpha-\beta}{2}} \qquad \text{(law of tangents)}$$

6.2 Let us substitute the first cosine by the half angle identity and convert the addition of the two last cosines into a product:

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=2\cos^2\frac{\alpha-\beta}{2}-1+\cos\frac{\beta-\alpha}{2}\cos\frac{\beta-2\gamma+\alpha}{2}$$

Let us extract common factor and convert the addition of cosines into a product:

$$=2\cos\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha-2\gamma+\beta}{2}\right)-1$$

$$=2\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha-\gamma}{2}\cos\frac{\gamma-\beta}{2}-1$$

$$=2\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}-1$$

The identity for the sines is proved in a similar way.

6.3 Using the De Moivre's identity:

$$\cos 4\alpha+e_{12}\sin 4\alpha\equiv(\cos\alpha+e_{12}\sin\alpha)^4$$

After developing the right hand side we find:

$$\sin 4\alpha\equiv 4\cos^3\alpha\,\sin\alpha-4\cos\alpha\,\sin^3\alpha \qquad \cos 4\alpha\equiv\cos^4\alpha-6\cos^2\alpha\,\sin^2\alpha+\sin^4\alpha$$

And dividing both identities: $\operatorname{tg}4\alpha=\dfrac{4\operatorname{tg}\alpha-4\operatorname{tg}^3\alpha}{1-6\operatorname{tg}^2\alpha+\operatorname{tg}^4\alpha}$

6.4 Let α, β and γ be the angles of the triangle *PAB* with vertices *A*, *B* and *P* respectively. The angle γ embracing the arc *AB* is constant for any point *P* on the arc *AB*. By the law of sines we have:

$$\frac{|PA|}{\sin\alpha}=\frac{|PB|}{\sin\beta}=\frac{|PC|}{\sin\gamma}$$

The sum of both chords is:

$$|PA| + |PB| = |PC| \frac{\sin\alpha + \sin\beta}{\sin\gamma}$$

Converting the sum of sines into a product we find:

$$|PA| + |PB| = |PC| \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\sin\gamma} = |PC| \frac{2\sin\frac{\pi-\gamma}{2}\cos\frac{\alpha-\beta}{2}}{\sin\gamma}$$

Since γ is constant, the maximum is attained for $\cos(\alpha/2 - \beta/2)=1$, that is, when the triangle *PAB* is isosceles and *P* is the midpoint of the arc *AB*.

6.5 From the sine theorem we have:

$$\frac{|a| + |b|}{|c|} = \frac{\sin\alpha + \sin\beta}{\sin\gamma} = \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}} = \frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}$$

Following the same way, we also have:

$$\frac{|a| - |b|}{|c|} = \frac{\sin\alpha - \sin\beta}{\sin\gamma} = \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}} = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\gamma}{2}}$$

6.6 Take *a* as the base of the triangle and draw the altitude. It divides *a* in two segments which are the projections of the sides *b* and *c* on *a*:

$$|a| = |b|\cos\gamma + |c|\sin\beta \qquad \text{and so forth.}$$

6.7 From the double angle identities we have:

$$\cos\alpha \equiv \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} \equiv 1 - 2\sin^2\frac{\alpha}{2} \equiv 2\cos^2\frac{\alpha}{2} - 1$$

From the last two expressions the half-angle identities follow:

$$\sin\frac{\alpha}{2} \equiv \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} \equiv \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Making the quotient:

$$\mathrm{tg}\,\alpha \equiv \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \equiv \frac{1-\cos\alpha}{\sin\alpha} \equiv \frac{\sin\alpha}{1+\cos\alpha}$$

7. Similarities and single ratio

7.1 First at all draw the triangle and see that the homologous vertices are:

$$P = (\,0, 0\,) \qquad P' = (\,4, 2\,)$$
$$Q = (\,2, 0\,) \qquad Q' = (\,2, 0\,)$$
$$R = (\,0, 1\,) \qquad R' = (\,5, 1\,)$$

$$PQ = 2\,e_1 \qquad QR = -2\,e_1 + e_2 \qquad RP = -e_2$$

$$P'Q' = -2\,e_1 + 2\,e_2 \qquad Q'R' = 3\,e_1 + e_2 \qquad R'P' = -e_1 + e_2$$

$$r = P'Q'\,PQ^{-1} = Q'R'\,QR^{-1} = R'P'\,RP^{-1} = -1 - e_{12} = \sqrt{2}_{\,5\pi/4}$$

The size ratio is $|\,r\,| = \sqrt{2}$ and the angle between the directions of homologous sides is $5\pi/4$.

7.2 Let *ABC* be a right triangle being *C* the right angle. The altitude *CD* cutting the base *AB* in *D* splits *ABC* in two right angle triangles: *ADC* and *BDC*. In order to simplify I introduce the following notation:

$$AB = c \qquad BC = a \qquad CA = b \qquad AD = x \qquad DB = c - x$$

The triangles *CBA* and *DCA* are oppositely similar because the angle *CAD* is common and the other one is a right angle. Hence:

$$b\,x^{-1} = (\,c\,b^{-1}\,)^* = b^{-1}\,c \qquad \Rightarrow \qquad b^2 = c\,x$$

The triangles *DBC* and *CBA* are also oppositely similar, because the angle *DBC* is common and the other one is a right angle. Hence:

$$a\,(\,c - x\,)^{-1} = (\,c\,a^{-1}\,)^* = a^{-1}\,c \qquad \Rightarrow \qquad a^2 = c\,(\,c - x\,)$$

Summing both results the Pythagorean theorem is obtained:

$$a^2 + b^2 = c\,(\,c - x\,) + c\,x = c^2$$

7.3 First at all we must see that the triangles *ABM* and *BCM* are oppositely similar. Firstly, they share the angle *BMC*. Secondly, the angles *MBC* and *BAM* are equal because they embrace the same arc *BC*. In fact, the limiting case of the angle *BAC* when *A* moves to *B* is the angle *MBA*. Finally the angle *BCM* is equal to the angle *ABM* because the sum of the angles of a triangle is π. The opposite similarity implies:

$$MA\,AB^{-1} = BC^{-1}\,MB \qquad \Rightarrow \qquad MA = BC^{-1}\,MB\,AB$$

$$MB\,AB^{-1} = BC^{-1}\,MC \qquad \Rightarrow \qquad MC^{-1} = AB\,MB^{-1}\,BC^{-1}$$

Multiplying both expressions we obtain:

$$MA\,MC^{-1} = BC^{-1}\,MB\,AB^{2}\,MB^{-1}\,BC^{-1} = AB^{2}\,BC^{-2}$$

7.4 Let Q be the intersection point of the segments PA and BC. The triangle PQC is directly similar to the triangle PBA and oppositely similar to the triangle BQA:

$$BA\,PA^{-1} = QC\,PC^{-1} \;\Rightarrow\; QC = BA\,PA^{-1}\,PC \;\Rightarrow\; |QC| = |BA|\,|PA|^{-1}\,|PC|$$

$$BQ\,BA^{-1} = PA^{-1}\,PB \;\Rightarrow\; BQ = PA^{-1}\,PB\,BA \;\Rightarrow\; |BQ| = |PA|^{-1}\,|PB|\,|BA|$$

Summing both expressions: $|BC| = |BQ| + |QC| = |BA|\,(\,|PB| + |PC|\,)\,|PA|^{-1}$

The three sides of the triangle are equal; therefore: $|PA| = |PB| + |PC|$

7.5 Let us firstly calculate the homothety ratio k, which is the quotient of homologous sides and the similarity ratio of both triangles:

$$AB^{-1}\,A'B' = k$$

Secondly we calculate the centre of the homothety by isolation from:

$$OA' = OA\,k$$

$$OA' - OA = OA\,(\,1 - k\,)$$

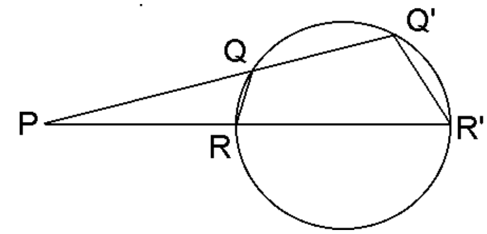
$$AA'\,(\,1 - k\,)^{-1} = OA$$

Since the segment AA' is known, this equation allows us to calculate the centre O :

$$O = A - AA'\,(\,1 - k\,)^{-1} = A - AA'\,(\,1 - AB^{-1}\,A'B'\,)^{-1}$$

7.6 Draw the line passing through P and the centre of the circle. This extended diameter cuts the circle in the points R and R'. See that the angles $R'RQ$ and $QQ'R'$ are supplementary because they intercept opposite arcs of the circle. Then the angles PRQ and $R'Q'P$ are equal. Therefore the triangle QRP and $R'Q'P$ are oppositely similar and we have:

Figure 16.2



$$PR^{-1}\,PQ = PR'\,PQ'^{-1} \quad\Rightarrow\quad PQ\,PQ' = PR\,PR'$$

Since PR and PR' are determined by P and the circle, the product $PQ\,PQ'$ (the *power* of P) is constant independently of the line PQQ'.

7.7 The bisector d of the angle ab divides the triangle abc in two triangles, which are obviously not similar! However we may apply the law of sines to both triangles to find:

$$\frac{|m|}{\sin ad} = \frac{|a|}{\sin dm} \qquad\qquad \frac{|n|}{\sin db} = \frac{|b|}{\sin nd}$$

The angles *nd* and *dm* are supplementary and sin *nd* = sin *dm*. On the other hand, the angles *ad* and *db* are equal because of the angle bisector. Therefore it follows that:

$$\frac{|m|}{|a|} = \frac{|n|}{|b|}$$

8. Properties of the triangles

8.1 Let A, B and C be the vertices of the given triangle with anticlockwise position. Let S, T and U be the vertices of the three equilateral triangles drawn over the sides AB, BC and CA respectively. Let P, Q and R be the centres of the triangles ABS, BCT and CAU respectively. The side CU is obtained from AC through a rotation of $2\pi/3$:

$$CU = AC\,t \qquad \text{with} \qquad t = 1_{2\pi/3} = \cos\frac{2\pi}{3} + e_{12}\sin\frac{2\pi}{3}$$

CR is 2/3 of the altitude of the equilateral triangles ACU ; therefore is 1/3 of the diagonal of the parallelogram formed by CA and CU:

$$CR = \frac{CA + CU}{3} = \frac{CA(1-t)}{3}$$

The same argument applies to the other equilateral triangles:

$$BQ = \frac{BC(1-t)}{3} \qquad\qquad AP = \frac{AB(1-t)}{3}$$

From where P, Q and R as functions of A, B and C are obtained:

$$P = A + \frac{AB(1-t)}{3} \qquad Q = B + \frac{BC(1-t)}{3} \qquad R = C + \frac{CA(1-t)}{3}$$

Let us calculate the vector PQ:

$$PQ = Q - P = B - A + \frac{(C - B - B + A)(1-t)}{3}$$

Introducing the centroid: $\quad G = \dfrac{A + B + C}{3}$

$$PQ = AB + BG\,(1-t) = AG - BG\,t$$

Analogously:

$$QR = BG - CG\,t \qquad\qquad RP = CG - AG\,z$$

Now we apply a rotation of $2\pi/3$ to PQ in order to obtain QR:

$$PQ\,t = (\,AG - BG\,t\,)\,t = AG\,t - BG\,t^2$$

Since t is a third root of the unity ($1 + t + t^2 = 0$) then:

$$PQ\,t = AG\,t + BG + BG\,t = AG\,t + BG\,t + CG\,t + BG - CG\,t$$

$$= 3\,GG + BG - CG\,t = BG - CG\,t = QR$$

Through the same way we find:

$$QR\,t = RP \qquad\qquad RP\,t = PQ$$

Therefore P, Q and R form an equilateral triangle with centre in G, the centroid of the triangle ABC:

$$\frac{P+Q+R}{3} = \frac{A+B+C}{3} + \frac{AB+BC+CA}{3}(1-t) = G$$

8.2 The substitution of the centroid G of the triangle ABC gives:

$$\begin{aligned}
3\,PG^2 = 3\left[P - \frac{A+B+C}{3}\right]^2 &= 3\,P^2 - 2\,P\cdot(A+B+C) + \frac{(A+B+C)^2}{3} \\
&= 3\,P^2 - 2\,P\cdot(A+B+C) + \frac{A^2 + B^2 + C^2 + 2\,A\cdot B + 2\,B\cdot C + 2\,C\cdot A}{3} \\
&= \left[(A-P)^2 + (B-P)^2 + (C-P)^2\right] + \frac{2\left(-A^2 - B^2 - C^2 + A\cdot B + B\cdot C + C\cdot A\right)}{3} \\
&= PA^2 + PB^2 + PC^2 - \frac{(B-A)^2 + (C-B)^2 + (A-C)^2}{3} \\
&= PA^2 + PB^2 + PC^2 - \frac{AB^2 + BC^2 + CA^2}{3}
\end{aligned}$$

8.3 a) Let us calculate the area of the triangle GBC:

$$GB \wedge BC = [\,-a\,A + (\,1 - b\,)\,B - c\,C\,] \wedge BC = [\,-a\,A + (\,a + c\,)\,B - c\,C\,] \wedge BC =$$

$$= (\,a\,AB + c\,CB\,) \wedge BC = a\,AB \wedge BC \qquad \Rightarrow \qquad a = \frac{GB \wedge BC}{AB \wedge BC}$$

The proof is analogous for the triangles GCA and GAB.

b) Let us develop PG^2 following the same way as in the exercise 8.2:

$$PG^2 = [P - (a\,A + b\,B + c\,C)\,]^2 = P^2 - 2\,P \cdot (a\,A + b\,B + c\,C) + (a\,A + b\,B + c\,C)^2$$

$$= (a + b + c)\,P^2 - 2\,a\,P \cdot A - 2\,b\,P \cdot B - 2\,c\,P \cdot C + a^2 A^2 + b^2 B^2 + c^2 C^2 +$$

$$+ 2\,a\,b\,A \cdot B + 2\,b\,c\,B \cdot C + 2\,c\,a\,C \cdot A$$

$$= a\,(A - P\,)^2 + b\,(B - P\,)^2 + c\,(C - P\,)^2 + a\,(a - 1\,)\,A^2 + b\,(b - 1)\,B^2$$

$$+ c\,(c - 1\,)\,C^2 + 2\,a\,b\,A \cdot B + 2\,b\,c\,B \cdot C + 2\,c\,a\,C \cdot A$$

$$= a\,PA^2 + b\,PB^2 + c\,PC^2 - a\,(b + c\,)\,A^2 - b\,(c + a\,)\,B^2 - c\,(a + b\,)\,C^2$$

$$+ 2\,a\,b\,A \cdot B + 2\,b\,c\,B \cdot C + 2\,c\,a\,C \cdot A$$

$$= a\,PA^2 + b\,PB^2 + c\,PC^2 - a\,b\,(B - A\,)^2 - b\,c\,(C - B\,)^2 - c\,a\,(A - C\,)^2$$

$$= a\,PA^2 + b\,PB^2 + c\,PC^2 - a\,b\,AB^2 - b\,c\,BC^2 - c\,a\,CA^2$$

The Leibniz's theorem is a particular case of the Apollonius' lost theorem for $a = b = c = 1/3$.

8.4 Let us consider the vertices A, B, C and D ordered clockwise on the perimeter. Since AP is AB turned $\pi/3$, BQ is BC turned $\pi/3$, etc, we have:

$$z = \cos \pi/3 + e_{12} \sin \pi/3$$

$$AP = AB\,z \qquad BQ = BC\,z \qquad CR = CD\,z \qquad DS = DA\,z$$

$$PR \cdot QS = (PA + AC + CR\,) \cdot (QB + BD + DS\,)$$

$$= [\,(CD - AB\,)\,z + AC\,] \cdot [\,(DA - BC\,)\,z + BD\,]$$

$$= [\,(-AC + BD\,)\,z + AC\,] \cdot [\,(-AC - BD\,)\,z + BD\,] =$$

$$= [\,(-AC + BD\,)\,z\,] \cdot [\,(-AC - BD\,)\,z\,] + [\,(-AC + BD\,)\,z\,] \cdot BD$$

$$+ AC \cdot [\,(-AC - BD\,)\,z + BD\,]$$

The inner product of two vectors turned the same angle is equal to that of these vectors before the rotation. We use this fact for the first product. Also we must develop the other products in geometric products and permute vectors and the complex number z:

$$PR \cdot QS = (-AC + BD\,) \cdot (-AC - BD\,) + (-AC\,BD - BD\,AC - AC^2 + BD^2\,)\,\frac{z + z^*}{2}$$

$$+ AC \cdot BD = \frac{AC^2 - BD^2}{2}$$

Therefore $|AC| = |BD| \Leftrightarrow PR \cdot QS = 0$. The statement b) is proved through an analogous way.

8.5 The median is the segment going from a vertex to the midpoint of the opposite side:

$$m_A^2 = \left(\frac{B+C}{2} - A\right)^2 = \left(\frac{AB}{2} + \frac{AC}{2}\right)^2 = \frac{AB^2}{4} + \frac{AC^2}{4} + \frac{AB \cdot AC}{2}$$

$$= \frac{AB^2}{2} + \frac{AC^2}{2} + \frac{2\, AB \cdot AC - AB^2 - AC^2}{4} = \frac{AB^2}{2} + \frac{AC^2}{2} - \frac{BC^2}{4}$$

8.6 If E is the intersection point of the bisector of B with the line parallel to the bisector of A, then the following equality holds:

$$E = B + b\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right) = C + a\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right)$$

where a, b are real. Arranging terms we obtain a vectorial equality:

$$-a\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right) + b\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right) = BC$$

The linear decomposition yields after simplification:

$$a = -\frac{|BC|\,|CA|}{|AB| + |BC| + |CA|}$$

In the same way, if D is the intersection of the bisector of A with the line parallel to the bisector of B we have:

$$D = A + c\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right) = C + d\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right)$$

where c and d are real. Arranging terms:

$$-c\left(\frac{AB}{|AB|} + \frac{AC}{|AC|}\right) + d\left(\frac{BA}{|BA|} + \frac{BC}{|BC|}\right) = CA$$

After simplification we obtain:

$$d = -\frac{|BC|\,|CA|}{|AB| + |BC| + |CA|}$$

Now we calculate the vector ED:

$$ED = D - E = \left(\frac{2\,AB}{|\,AB\,|} - \frac{BC}{|\,BC\,|} + \frac{AC}{|\,AC\,|} \right) \frac{|\,BC\,|\;|\,CA\,|}{|\,AB\,| + |\,BC\,| + |\,CA\,|}$$

Then the direction of the line ED is given by the vector v:

$$v = \frac{2\,AB}{|\,AB\,|} - \frac{BC}{|\,BC\,|} + \frac{AC}{|\,AC\,|} = AB \left(\frac{2}{|\,AB\,|} + \frac{1}{|\,AC\,|} \right) + BC \left(-\frac{1}{|\,BC\,|} + \frac{1}{|\,AC\,|} \right)$$

When the vector v has the direction AB, the second summand vanishes, $|AC| = |BC|$ and the triangle becomes isosceles.

8.7 Let us indicate the sides of the triangle ABC with a, b and c in the following form:

$$a = BC \qquad b = CA \qquad c = AB$$

Suppose without loss of generality that P lies on the side BC and Q on the side AC . Hence:

$$P = k\,B + (1 - k)\,C \qquad\qquad Q = l\,A + (1 - l)\,C$$

$$CP = k\,CB = -\,k\,a \qquad\qquad CQ = l\,CA = l\,b$$

where k and l are real and $0 < k, l < 1$. Now the segment PQ is obtained:

$$PQ = k\,a + l\,b$$

Since the area of the triangle CPQ must be the half of the area of the triangle ABC, it follows that:

$$CP \wedge CQ = \frac{1}{2} CB \wedge CA \qquad \Rightarrow \qquad (-\,k\,a) \wedge (l\,b) = -\frac{1}{2} a \wedge b \qquad \Rightarrow \qquad k\,l = \frac{1}{2}$$

The substitution into PQ gives:

$$PQ = k\,a + \frac{1}{2k} b$$

a) If u is the vector of the given direction, PQ is perpendicular when $PQ \cdot u = 0$ which results in:

$$\left(k\,a + \frac{1}{2k} b \right) \cdot u = 0$$

whence one obtains:

$$k = \sqrt{-\frac{b \cdot u}{2\,a \cdot u}} \qquad\qquad PQ = a\sqrt{-\frac{b \cdot u}{2\,a \cdot u}} + b\sqrt{-\frac{a \cdot u}{2\,b \cdot u}}$$

In this case the solutions only exist when $a \cdot u$ and $b \cdot u$ have different signs. However we can also choose the point P (or Q) lying on the another side c, what gives an analogous solution containing the inner product $c \cdot u$. Note that if $a \cdot u$ and $b \cdot u$ have the same sign, then $c \cdot u$ have the opposite sign because:

$$a + b + c = 0 \qquad \Rightarrow \qquad c \cdot u = -\, a \cdot u - b \cdot u$$

what warrants there is always a solution.

b) Let us calculate the square of PQ:

$$PQ^2 = k^2 a^2 + \frac{1}{4k^2} b^2 + a \cdot b$$

By equating the derivative to zero, one obtains the value of k for which PQ^2 is minimum:

$$k = \sqrt{\frac{|\,b\,|}{2|\,a\,|}}$$

Then we obtain the segment PQ and its length:

$$PQ = a\sqrt{\frac{|\,b\,|}{2|\,a\,|}} + b\sqrt{\frac{|\,a\,|}{2|\,b\,|}} \qquad\qquad |\,PQ\,| = \sqrt{|\,a\,||\,b\,| + a \cdot b}$$

c) Every point may be written as linear combination of the three vertices of the triangle the sum of the coefficients being equal to the unity:

$$R = x\,A + y\,B + (1 - x - y)C \qquad \Leftrightarrow \qquad CR = x\,CA + y\,CB$$

where all the coefficients are comprised between 0 and 1:

$$0 < x, y,\ 1 - x - y < 1 \qquad x = \frac{CR \wedge CB}{CA \wedge CB} \qquad y = \frac{CA \wedge CR}{CA \wedge CB}$$

Now the point R must lie on the segment PQ, that is, P, Q and R must be aligned. Then the determinant of their coordinates will vanish:

$$\det(P, Q, R) = \begin{vmatrix} 0 & k & 1-k \\ l & 0 & 1-l \\ x & y & 1-x-y \end{vmatrix} = \begin{vmatrix} 0 & k & 1-k \\ \dfrac{1}{2\,k} & 0 & 1-\dfrac{1}{2\,k} \\ x & y & 1-x-y \end{vmatrix} = 0$$

$$2\,x\,k^2 - k + y = 0 \qquad \Rightarrow \qquad k = \frac{1 \pm \sqrt{1 - 8\,x\,y}}{4\,x} \qquad \text{and} \qquad l = \frac{1}{2\,k} = \frac{1 \mp \sqrt{1 - 8\,x\,y}}{4\,y}$$

There is only solution for a positive discriminant:

$$1 > 8\,x\,y$$

The limiting curve is an equilateral hyperbola on the plane *x*-*y*. Since the triangle may be obtained through an affinity, the limiting curve for the triangle is also a hyperbola although not equilateral.

If the extremes P and Q could move along the prolongations of the sides of the triangle without limitations, this would be the unique condition. However in this problem the point P must lie between C and B, and Q must lie between C and A. It means the additional condition:

$$\frac{1}{2} < k < 1 \qquad \Leftrightarrow \qquad \frac{1}{2} < l < 1 \qquad \Leftrightarrow \qquad \frac{1}{2} < \frac{1 \pm \sqrt{1 - 8\,x\,y}}{4\,x} < 1$$

From the first solution (root with sign +) we have:

$$2\,x - 1 < \sqrt{1 - 8\,x\,y} < 4\,x - 1$$

only existing for $x > 1/4$. For $1/4 < x < 1/2$ the left hand side is negative so that the unique restriction is the right hand side. By squaring it we obtain:

$$1 - 8\,x\,y < (4\,x - 1)^2 \qquad \Rightarrow \qquad 0 < 2\,x + y - 1 \qquad \text{for} \qquad \frac{1}{4} < x < \frac{1}{2}$$

For $x > 1/2$ the right hand side is higher than 1 so that the unique restriction is the left hand side:

$$(2\,x - 1)^2 < 1 - 8\,x\,y \qquad \Rightarrow \qquad x + 2\,y - 1 < 0 \qquad \text{for} \qquad x > \frac{1}{2}$$

For the second solution (root with sign –) we have:

$$2\,x - 1 < -\sqrt{1 - 8\,x\,y} < 4\,x - 1$$

only existing for $x < 1/2$. For $1/4 < x < 1/2$ the right hand side is positive so that the unique restriction is the left hand side:

$$(2\,x - 1)^2 > 1 - 8\,x\,y \qquad \Rightarrow \qquad x + 2\,y - 1 > 0 \qquad \text{for} \qquad \frac{1}{4} < x < \frac{1}{2}$$

For $x < 1/4$ both members are negative and after squaring we have:

$$(2x-1)^2 > 1-8xy > (4x-1)^2 \qquad \Rightarrow \qquad x+2y-1>0>2x+y-1 \quad \text{for } x<\frac{1}{4}$$

All these conditions are plotted in the figure 16.3.

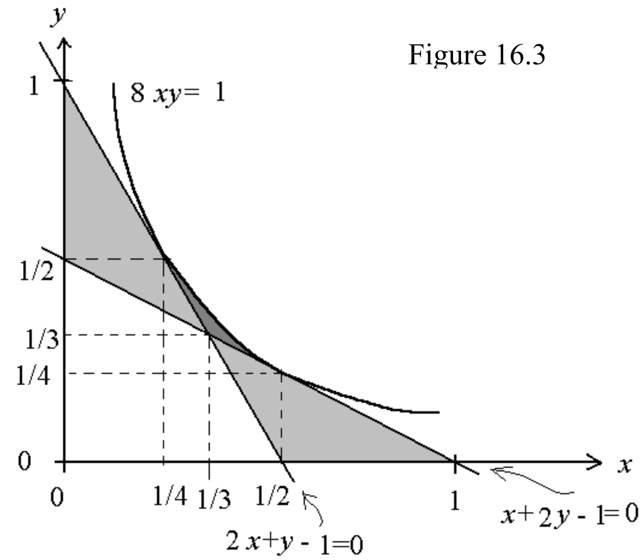


Figure 16.3

It is easily proved that the limiting lines touch the hyperbola at the points (1/4, 1/2) and (1/2, 1/4). For both triangles painted with light grey there is a unique solution. In the region painted with dark grey there are two solutions, that is, there are two segments passing through the point *R* and dividing the triangle in two parts with equal area. By means of an affinity the plot in the figure 16.3 is transformed into any triangle (figure 16.4). The affinity is a linear transformation of the coordinates, which converts the limiting lines into the medians of the triangle intersecting at the centroid and the hyperbola into another non equilateral hyperbola:

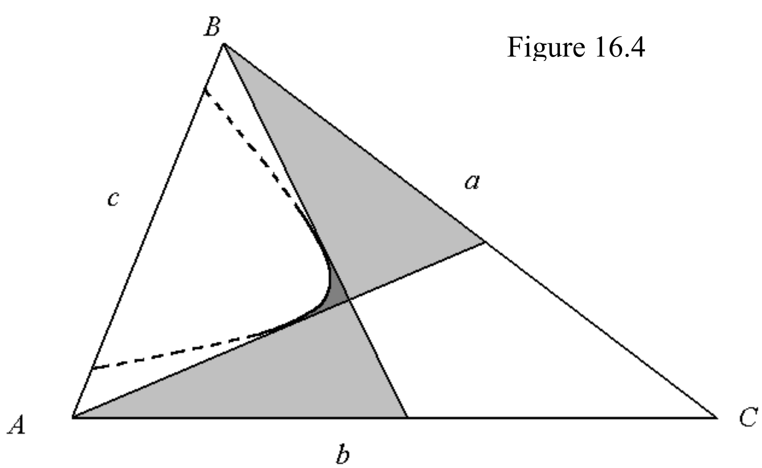


Figure 16.4

Observe that the medians divide the triangle *ABC* in six triangles. If a point *R* lies outside the shadowed triangles, we can always support the extremes of the segment *PQ* passing through R in another pair of sides. This means that the division of the triangle is always possible provided of the suitable choice of the sides. On the other hand, if we consider all the possibilities, the points neighbouring the centroid admit three segments.

9. Circles

9.1 $$(A-M)^2+(B-M)^2+(C-M)^2=k$$

$$3M^2-2M\cdot(A+B+C)+A^2+B^2+C^2=k$$

Introducing the centroid $G=(A+B+C)/3$:

$$M^2-2M\cdot G+G^2=\frac{k}{3}-\frac{2\left(A^2+B^2+C^2+A\cdot B+B\cdot C+C\cdot A\right)}{9}$$

$$(M-G)^2=\frac{k}{3}-\frac{(B-A)^2+(C-B)^2+(A-C)^2}{9}$$

$$GM^2=\frac{k}{3}-\frac{AB^2+BC^2+CA^2}{9}=r^2$$

M runs on a circumference with radius r centred at the centroid of the triangle when k is higher than the arithmetic mean of the squares of the three sides:

$$k\geq\frac{AB^2+BC^2+CA^2}{3}$$

9.2 Let A and B be the fixed points and P any point on the searched geometric locus. If the ratio of distances from P to A and B is constant it holds that:

$$|PA|=k\,|PB| \quad\Leftrightarrow\quad PA^2=k^2\,PB^2 \quad \text{with } k \text{ being a real positive number}$$

$$A^2-2A\cdot P+P^2=k^2(B^2-2B\cdot P+P^2)$$

$$(1-k^2)P^2-2(A-k^2B)\cdot P=k^2B^2-A^2$$

$$P^2-\frac{2P\cdot(A-k^2B)}{1-k^2}=\frac{k^2B^2-A^2}{1-k^2}$$

We indicate by O the expression: $O=\dfrac{A-k^2B}{1-k^2}$. Adding O^2 to both members we obtain:

$$OP^2=\frac{k^2B^2-A^2}{1-k^2}+\frac{\left(A-k^2B\right)^2}{\left(1-k^2\right)^2}$$

$$OP^2 = \frac{k^2\left(B^2 - 2\,A\cdot B + A^2\right)}{\left(1-k^2\right)^2} = \frac{k^2\;AB^2}{\left(1-k^2\right)^2}$$

Therefore, the points *P* form a circumference centred at *O*, which is a point of the line *AB*, with radius:

$$|\,OP\,| = \frac{k\,|\,AB\,|}{1-k^2}$$

9.3 Let us draw the circumference circumscribed to the triangle *ABC* (figure 16.5) and let *D* be any point on this circumference. Let *P*, *Q* and *R* be the orthogonal projections of *P* on the sides *AB*, *BC* and *CA* respectively. Note that the triangles *ADP* and *CDR* are similar because of $DAB = \pi - BCD = DCR$. The similarity of both triangles is written as:

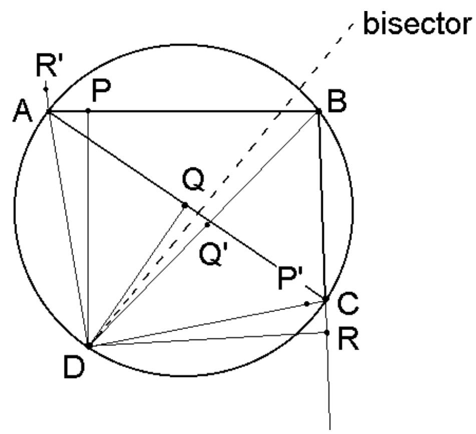


Figure 16.5

$$DC\;DA^{-1} = DR\;DP^{-1}$$

The triangles *ADQ* and *BDR* are also similar because of the equality of the inscribed angles $CAD = CBD$:

$$DB\;DA^{-1} = DR\;DQ^{-1}$$

Since the angles $RDC = PDA$ and $RDB = QDA$, the bisector of the angle *BDQ* is also bisector of the angle *CDP* and *RDA*, being its direction vector:

$$v = \frac{DA}{|\,DA\,|} + \frac{DR}{|\,DR\,|} = \frac{DB}{|\,DB\,|} + \frac{DQ}{|\,DQ\,|} = \frac{DC}{|\,DC\,|} + \frac{DP}{|\,DP\,|}$$

Let *P′*, *Q′* and *R′* be the reflected points of *P*, *Q* and *R* with respect to the bisector:

$$DR' = v^{-1}\;DR\;v$$

Multiplying by *DA* we have:

$$DR'\;DA = v^{-1}\;DR\;v\;DA = |\,DR\,|\;\;|\,DA\,|$$

which is an expected result since the reflection of *R* with respect the bisector of the angle *RDA* yields always a point *R′* aligned with *D* and *A*. Analogously for *B* and *C* we find that the points *D*, *B* and *Q′* are aligned and also the points *D*, *C* and *P′*:

$$DQ'\;DB = v^{-1}\;DQ\;v\;DR\;DQ^{-1}\;DA = v^{-1}\;DR\;v\;DA = |\,DR\,|\;\;|\,DA\,|$$

$$DP'\,DC = v^{-1}\,DP\,v\,DR\,DP^{-1}\,DA = v^{-1}\,DR\,v\,DA = |DR|\;|DA|$$

Now we see that the points R', Q' and P' are the transformed of A, B and C under an inversion with centre D:

$$DR'\,DA = DQ'\,DB = DP'\,DC$$

Since D lies on the circumference passing through A, B and C, the inversion transforms them into aligned points and the reflection preserves this alignment so that P, Q and R are aligned.

9.4 If $m = a + b$ and $n = b + c$ then:

$$m^2 n^2 = (a+b)^2 (b+c)^2 = \left(a^2 + a\,b + b\,a + b^2\right)\left(b^2 + b\,c + c\,b + c^2\right)$$

$$= a^2 b^2 + a^2 b\,c + a^2 c\,b + a^2 c^2 + a\,b^3 + a\,b^2 c + a\,b\,c\,b + a\,b\,c^2 +$$

$$+ b\,a\,b^2 + b\,a\,b\,c + b\,a\,c\,b + b\,a\,c^2 + b^4 + b^3 c + b^2 c\,b + b^2 c^2$$

Taking into account that $d = -(a + b + c)$ then:

$$d^2 = (a+b+c)^2 = a^2 + b^2 + c^2 + a\,b + b\,a + b\,c + c\,b + c\,a + a\,c$$

so we may rewrite the former equality:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 + a^2 b\,c + a^2 c\,b + a\,b\,c\,b + a\,b\,c^2 + b\,a\,b\,c + b\,a\,c^2$$

We apply the permutative property to some terms:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 + a^2 b\,c + a\,b\,c\,a + a\,b\,c\,b + a\,b\,c^2 + b\,a\,b\,c + c\,a\,b\,c$$

in order to arrive at a fully symmetric expression:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - d\,a\,b\,c - a\,b\,c\,d$$

Now we express the product of each pair of vectors using the exponential function of their angle:

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - |\,a\,||\,b\,||\,c\,||\,d\,|\left(\exp\left((2\pi - \gamma - \alpha)\,e_{12}\right) + \exp\left((2\pi - \beta - \delta)\,e_{12}\right)\right)$$

Adding the arguments of the exponential, simplifying and taking into account that $\alpha + \beta + \gamma + \delta = \pi$.

$$m^2 n^2 = a^2 c^2 + b^2 d^2 - 2\,|\,a\,||\,b\,||\,c\,||\,d\,|\cos(\alpha + \gamma)$$

9.5 Let A, B and C be the midpoints of the sides of any triangle PQR :

$$A=\frac{Q+R}{2} \qquad B=\frac{R+P}{2} \qquad C=\frac{P+Q}{2}$$

The centre of the circumferences RAB, PBC and QCA will be denoted as D, E and F respectively. The circumference PBC is obtained from the circumscribed circumference PQR by means of a homothety with centre P. Then if O is the circumcentre of the triangle PQR, the centre of the circumference PBC is located at half distance from R:

$$E=\frac{P+O}{2} \qquad F=\frac{Q+O}{2}$$

$$EA=\frac{PQ}{2}+\frac{OR}{2} \qquad FB=-\frac{PQ}{2}+\frac{OR}{2} \qquad EF=\frac{PQ}{2}$$

Let Z be the intersection of EA with FB. Then:

$$Z=a\ A+(1-a)E=b\ B+(1-b)F$$

Arranging terms we find:

$$a\ EA-b\ FB=EF$$

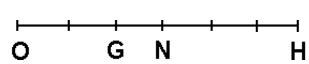
The linear decomposition gives:

$$a=\frac{EF\wedge FB}{EA\wedge FB} \qquad -b=\frac{EA\wedge EF}{EA\wedge FB}$$

$$Z=\frac{EF\wedge FB}{EA\wedge FB}A+\frac{FA\wedge FB}{EA\wedge FB}E=\frac{A}{2}+\frac{E}{2}=\frac{P+Q+R+O}{4}$$

which is invariant under cyclic permutation of the vertices P, Q and R. Then the lines EA, FB and DC intersect in a unique point Z. On the other hand, the point Z lies on the Euler's line (figure 9.8):

Figure 9.8



$$Z=\frac{3\,G+O}{4}$$

9.6 We must prove that the inversion changes the single ratio of three very close points by its conjugate value, because the orientation of the angles is changed. Let us consider an inversion with centre O and radius r and let the points A', B', C' be the transformed of A, B, C under this inversion. Then:

$$OA'=r^2\,OA^{-1} \qquad OB'=r^2\,OB^{-1} \qquad OC'=r^2\,OC^{-1}$$

$$A'B'=OB'-OA'=r^2\,(OB^{-1}-OA^{-1})=r^2\,OA^{-1}\,(OA-OB)\,OB^{-1}$$

$$= -r^2\, OA^{-1}\, AB\, OB^{-1}$$

In the same way: $A'C' = -r^2\, OA^{-1}\, AC\, OC^{-1}$

Let us calculate the single ratio of the transformed points:

$$(A', B', C') = A'B'\, A'C'^{-1} = OA^{-1}\, AB\, OB^{-1}\, OC\, AC^{-1}\, OA =$$

$$= OB^{-1}\, AB\, AC^{-1}\, OC = (OA + AB)^{-1}\, AB\, AC^{-1}\, (OA + AC)$$

When B and C come near A, AB and AC tend to zero and the single ratio for the inverse points becomes the conjugate value of that for the initial points:

$$\lim_{B, C \to A} (A', B', C') = OA^{-1}\, AB\ AC^{-1}\, OA = AC^{-1}\, AB = (A, B, C)^*$$

10. Cross ratios and related transformations

10.1 Let $ABCD$ be a quadrilateral inscribed in a circle. We must prove the following equality:

$$|BC|\ |AD| + |AB|\ |CD| = |BD|\ |AC|$$

which is equivalent to:

$$\frac{|BC||AD|}{|BD||AC|} + \frac{|AB||CD|}{|BD||AC|} = 1$$

Now we identify these quotients with cross ratios of four points A, B, C and D on a circle:

$$(B\ A\ C\ D) + (B\ C\ A\ D) = 1$$

equality that always holds because if $(A\ B\ C\ D) = r$ then:

$$\frac{1}{r} + \frac{r-1}{r} = 1$$

When the points do not lie on a circle, the cross ratio is not the quotient of moduli but a complex number:

$$\frac{|BC||AD|}{|BD||AC|} \exp[e_{12}(CBD - CAD)] + \frac{|AB||CD|}{|BD||AC|} \exp[e_{12}(ABD - ACD)] = 1$$

According to the triangular inequality, the modulus of the sum of two complex numbers is lower than or equal to the sum of both moduli:

$$\frac{|BC||AD|}{|BD||AC|}+\frac{|AB||CD|}{|BD||AC|}\geq 1 \quad \Rightarrow \quad |BC||AD|+|AB||CD|\geq |BD||AC|$$

10.2 If A, B, C and D form a harmonic range their cross ratio is 2:

$$AC\,AD^{-1}\,BD\,BC^{-1}=2$$

In order to isolate D we must write BD as addition of BA and AD:

$$AC\,AD^{-1}\,(\,BA+AD\,)\,BC^{-1}=2$$

$$AC\,AD^{-1}\,BA\,BC^{-1}=2-AC\,BC^{-1}$$

$$AD^{-1}=AC^{-1}\,(\,2-AC\,BC^{-1}\,)\,BC\,BA^{-1}=(\,2\,AC^{-1}\,BC-1\,)\,BA^{-1}$$

$$AD=AB\,(\,1-2\,AC^{-1}\,BC\,)^{-1}$$

10.3 The homography preserves the cross ratio:

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1}=AC\,AD^{-1}\,BD\,BC^{-1}$$

which may be rewritten as:

$$A'C'\,A'D'^{-1}\,(\,B'A'+A'D'\,)\,(\,B'A'+A'C'\,)^{-1}=AC\,AD^{-1}\,(\,BA+AD\,)\,(\,BA+AC\,)^{-1}$$

When C and D approach to A, the vectors AC, AD, $A'C'$ and $A'D'$ tend to zero. So the single ratio of three very close points remains constant:

$$\lim_{C,D\to A} A'C'\;A'D'^{-1}=\lim_{C,D\to A} AC\;AD^{-1}$$

and therefore the angle between tangent vectors of curves. So the homography is a directly conformal transformation.

10.4 If F is a point on the homology axis we must prove that:

$$\{\,F,A\,B\,C\,D\,\}=\{\,F,A'\,B'\,C'\,D'\,\}$$

As we have seen, a point A' homologous of A is obtained through the equation:

$$OA'=OA\,[\,1-r\,v\wedge FA\,(\,v\wedge FO\,)^{-1}\,]^{-1}$$

where O is the centre of homology, F a point on the axis, v the direction vector of the axis and r the homology ratio. Then the vector FA' is:

$$FA'=FO+OA'=FO+OA\,[\,1-r\,v\wedge FA\,(\,v\wedge FO\,)^{-1}\,]^{-1}$$

$$= [\, FO\,(\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,) + OA\,]\,[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

$$= [\, FA - r\, FO\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

Analogously:

$$FB' = [\, FB - r\, FO\, v \wedge FB\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FB\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

$$FC' = [\, FC - r\, FO\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

$$FD' = [\, FD - r\, FO\, v \wedge FD\,(\, v \wedge FO\,)^{-1}\,]\,[\, 1 - r\, v \wedge FD\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

The product of two outer products is a real number so that we must only pay attention to the order of the first loose vectors FA and FO, etc:

$$FA' \wedge FC' = [\, FA \wedge FC - r\,(\, FA \wedge FO\, v \wedge FC + FO \wedge FC\, v \wedge FA\,)\,(\, v \wedge FO\,)^{-1}\,]$$
$$[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}\;\; [\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

Using the identity of the exercise 1.4, we have:

$$FA' \wedge FC' = [\, FA \wedge FC - r\, FA \wedge FC\, v \wedge FO\,(\, v \wedge FO\,)^{-1}\,]$$
$$[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}\;\; [\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$
$$= FA \wedge FC\,(\, 1 - r\,)\,[\, 1 - r\, v \wedge FA\,(\, v \wedge FO\,)^{-1}\,]^{-1}\;\; [\, 1 - r\, v \wedge FC\,(\, v \wedge FO\,)^{-1}\,]^{-1}$$

In the projective cross ratio all the factors except the first outer product are simplified:

$$\frac{FA' \wedge FC'\; FB' \wedge FD'}{FA' \wedge FD'\; FB' \wedge FC'} = \frac{FA \wedge FC\; FB \wedge FD}{FA \wedge FD\; FB \wedge FC}$$

10.5 a) Let us prove that the special conformal transformation is additive:

$$OP' = (\, OP^{-1} + v\,)^{-1} \quad \Rightarrow \quad OP'^{-1} = OP^{-1} + v$$

$$OP'' = (\, OP'^{-1} + w\,)^{-1} = (\, OP^{-1} + v + w\,)^{-1}$$

b) Let us extract the factor OP from OP':

$$OP'^{-1} = (\, 1 + v\, OP\,)\, OP^{-1} \quad \Rightarrow \quad OP' = OP\,(\, 1 + v\, OP\,)^{-1}$$

Then we have:

$$OA' = OA\,(\, 1 + v\, OA\,)^{-1} \qquad OB' = OB\,(\, 1 + v\, OB\,)^{-1}$$

$$OC' = OC\,(\, 1 + v\, OC\,)^{-1} \qquad OD' = OD\,(\, 1 + v\, OD\,)^{-1}$$

$$A'C' = OC' - OA' = OC\,(1 + v\,OC)^{-1} - OA\,(1 + v\,OA)^{-1} =$$

$$= [\,OC\,(1 + v\,OA) - OA\,(1 + v\,OC)\,]\,(1 + v\,OC)^{-1}\,(1 + v\,OA)^{-1}$$

$$= AC\,(1 + v\,OC)^{-1}\,(1 + v\,OA)^{-1}$$

Analogously:

$$B'D' = BD\,(1 + v\,OB)^{-1}\,(1 + v\,OD)^{-1}$$

$$A'D' = AD\,(1 + v\,OA)^{-1}\,(1 + v\,OD)^{-1}$$

$$B'C' = BC\,(1 + v\,OB)^{-1}\,(1 + v\,OC)^{-1}$$

From where it follows that the complex cross ratio is preserved and it is a special case of homography:

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1} = AC\,(1 + v\,OC)^{-1}\,(1 + v\,OD)\,AD^{-1}$$

$$BD\,(1 + v\,OD)^{-1}\,(1 + v\,OC)^{-1}\,BC^{-1} = AC\,AD^{-1}\,BD\,BC^{-1}$$

10.6 If the points A, B and C are invariant under a certain homography, then for any other point D' the following equality is fulfilled:

$$(\,A\,B\,C\,D\,) = (\,A\,B\,C\,D'\,)$$

$$AC\,AD^{-1}\,BD\,BC^{-1} = AC\,AD'^{-1}\,BD'\,BC^{-1}$$

The simplification of factors yields:

$$AD^{-1}\,BD = AD'^{-1}\,BD' \quad \Rightarrow \quad AD^{-1}\,(\,BA + AD\,) = AD'^{-1}\,(\,BA + AD'\,) \quad \Rightarrow$$

$$\Rightarrow \quad AD^{-1}\,BA = AD'^{-1}\,BA \quad \Rightarrow \quad AD^{-1} = AD'^{-1} \quad \Rightarrow \quad D = D'$$

10.7. **a)** From the definition of antigraphy we have:

$$A'C'\,A'D'^{-1}\,B'D'\,B'C'^{-1} = (\,AC\,AD^{-1}\,BD\,BC^{-1}\,)^* = BC^{-1}\,BD\,AD^{-1}\,AC$$

If C and D approach A, then BC, $BD \rightarrow BA$:

$$\lim_{C', D' \to A} A'C'\,A'D'^{-1} = \lim_{C, D \to A} AD^{-1}\,AC$$

So the antigraphy is an opposite conformal transformation.
b) The composition of two antigraphies is always a homography because twice the conjugation of a complex number is the identity, so that the complex cross ratio is preserved.
c) The inversion conjugates the cross ratio. Consequently an odd number of inversions is

always an antigraphy.

d) If an antigraphy has three invariant points, then all the points lying on the circumference passing through these points are invariant. Let us prove this statement: if *A*, *B* and *C* are the invariant points and *D* belongs to the circumference passing through these points, then the cross ratio is real:

$$(A\ B\ C\ D\,) = (A\ B\ C\ D\,)^* = (A\ B\ C\ D'\,)$$

from where $D = D'$ as proved in the exercise 10.6. If *D* does not belong to this circumference, $D' \neq D$ and the cross ratio $(\,A\ B\ C\ D\,)$ becomes conjugate. A geometric transformation that preserves a circumference and conjugates the cross ratio can only be an inversion because there is a unique point D' for each point *D* fulfilling this condition. The centre and radius of inversion are those for the circumference *ABC*.

10.8 Consider the projectivity:

$$D = a\,A + b\,B + c\,C \qquad \rightarrow \qquad D' = a'\,A' + b'\,B' + c'\,C'$$

If A', B' and C' are the images of *A*, *B* and *C* then the coefficients a', b', c' must be proportional to *a*, *b*, *c*:

$$a' = \frac{k\,a}{k\,a + l\,b + m\,c} \qquad b' = \frac{l\,b}{k\,a + l\,b + m\,c} \qquad c' = \frac{m\,c}{k\,a + l\,b + m\,c}$$

where the values are already normalised. Three points are collinear if the determinant of the barycentric coordinates is zero, what only happens if the initial points are also collinear:

$$\begin{vmatrix} a'_D & b'_D & c'_D \\ a'_E & a'_E & a'_E \\ a'_F & a'_F & a'_F \end{vmatrix} = \frac{k\ l\ m \begin{vmatrix} a_D & b_D & c_D \\ a_E & a_E & a_E \\ a_F & a_F & a_F \end{vmatrix}}{\left(k\ a_D + l\ b_D + m\ c_D\right)\left(k\ a_E + l\ b_E + m\ c_E\right)\left(k\ a_F + l\ b_F + m\ c_F\right)} = 0$$

10.9 In the dual plane the six sides of the hexagon are six points. Since the sides *AB*, *CD* and *EF* (or their prolongations) pass through *P* and *BC*, *DE* and *FA* pass through *Q*, this means that the dual points are alternatively aligned in two dual lines *P* and *Q*. By the Pappus' theorem the dual points *AD*, *BE* and *CF* lie on a dual line *X*, that is, these lines joining opposite vertices of the hexagon are concurrent in the point *X*. Moreover, as proved for the Pappus' theorem the cross ratios fulfil the equalities in the left hand side:

$$(BE\ AD\ FC\ P) = (PQ\ AB\ FE\ CD) \qquad \Leftrightarrow \qquad \{X, B\ A\ F\ P\} = \{P, Q\ A\ F\ C\}$$

$$(BE\ AD\ FC\ Q) = (PQ\ DE\ CB\ FA) \qquad \Leftrightarrow \qquad \{X, B\ A\ F\ Q\} = \{Q, P\ D\ C\ F\}$$

Since the cross ratio of a pencil of lines is equal to the cross ratio of the dual points, the equalities in the right hand side follow immediately.

11. Conics

11.1 The starting point is the central equation of a conic:

$OA = OQ \cos\alpha + OR \sin\alpha \qquad OB = OQ \cos\beta + OR \sin\beta$

$OC = OQ \sin\gamma + OR \sin\gamma \qquad OD = OQ \cos\delta + OR \sin\delta$

$OX = OQ \cos\chi + OR \sin\chi$

$XA = OA - OX = OQ\,(\cos\alpha - \cos\chi) + OR\,(\sin\alpha - \sin\chi)$

$XC = OC - OA = OQ\,(\cos\gamma - \cos\chi) + OR\,(\sin\gamma - \sin\chi)$

$$XA \wedge XC = OQ \wedge OR\,[(\cos\gamma - \cos\chi)(\sin\alpha - \sin\chi) - (\cos\alpha - \cos\chi)(\sin\gamma - \sin\chi)]$$

$$= OQ \wedge OR\,[\cos\gamma \sin\alpha - \cos\gamma \sin\chi - \cos\chi \sin\alpha - \cos\alpha \sin\gamma$$

$$+ \cos\alpha \sin\chi + \cos\chi \sin\gamma]$$

$$= OQ \wedge OR\,[\sin(\alpha - \gamma) + \sin(\gamma - \chi) + \sin(\chi - \alpha)]$$

$$= -4\, OQ \wedge OR \sin\frac{\alpha-\gamma}{2} \sin\frac{\gamma-\chi}{2} \sin\frac{\chi-\alpha}{2}$$

From where it follows that:

$$\frac{XA \wedge XC \; XB \wedge XD}{XA \wedge XD \; XB \wedge XC} = \frac{\sin\frac{\alpha-\gamma}{2} \sin\frac{\beta-\delta}{2}}{\sin\frac{\alpha-\delta}{2} \sin\frac{\beta-\gamma}{2}}$$

11.2 The central equation of an ellipse of centre O with semiaxis OQ and OR is:

$$OP = OQ \cos\theta + OR \sin\theta \quad \Rightarrow \quad P = O\,(1 - \cos\theta - \sin\theta) + Q \cos\theta + R \sin\theta$$

There is always an affinity transforming an ellipse into a circle:

$O' = (0, 0) \qquad Q' = (1, 0) \qquad R' = (0, 1)$

$$P' = O'\,(1 - \cos\theta - \sin\theta) + Q' \cos\theta + R' \sin\theta = (\cos\theta, \sin\theta)$$

Let us now consider any point E and a circle. A line passing through E cuts the circle in the points A and B; another line passing also through E but with different direction cuts the ellipse in the points C and D. Since the power of a point E with respect to the circle is constant we have:

$$EA\ EB = EC\ ED \qquad\qquad EA\ EB^{-1}\ |EB|^2 = EC\ ED^{-1}\ |ED|^2$$

The affinity preserves the single ratio of three aligned points:

$$EA\ EB^{-1} = E'A'\ E'B'^{-1} \qquad\qquad EC\ ED^{-1} = E'C'\ ED'^{-1}$$

Then:

$$E'A'\ E'B'^{-1}\ |EB|^2 = E'C'\ E'D'^{-1}\ |ED|^2$$

$$\frac{E'A'\ E'B'}{E'C'\ E'D'} = \frac{|E'B'|^2\ |ED|^2}{|EB|^2\ |E'D'|^2}$$

and this quotient is constant for two given directions because the preservation of the single ratio implies that the distances in each direction are enlarged by a constant ratio:

$$\frac{|EA|}{|E'A'|} = \frac{|EB|}{|E'B'|}$$

11.3 If a diameter is formed by the midpoints of the chords parallel to the conjugate diameter, then it is obvious that the tangent to the point where the length of the chord vanishes is also parallel to the conjugate diameter. This evidence may be proved by derivation of the central equation:

$$OP = OQ \cos\theta + OR \sin\theta$$

$$\frac{d\,OP}{d\theta} = -OQ \sin\theta + OR \cos\theta$$

That is, OP and $dOP / d\theta$ (having the direction of the tangent to P) are conjugate radius. The area a of the parallelogram circumscribed to the ellipse is the outer product of both conjugate diameters, which is independent of θ:

$$a = 4\ (OQ \cos\theta + OR \sin\theta) \wedge (-OQ \sin\theta + OR \cos\theta) = 4\ OQ \wedge OR$$

11.4 The area of a parallelogram formed by two conjugate diameters of the hyperbola is also independent of ψ:

$$a = 4\ (OQ \cosh\psi + OR \sinh\psi) \wedge (OQ \sinh\psi + OR \cosh\psi) = 4\ OQ \wedge OR$$

11.5 Let us prove the statement for a circle. The intersections R and R' of a circumference with a line passing through P are given by (fig. 9.3):

$$|PR| = |PF| \cos\alpha - \sqrt{r^2 - PF^2 \sin^2\alpha}$$

$$|PR'| = |PF|\cos\alpha + \sqrt{r^2 - PF^2\sin^2\alpha}$$

The point M located between R and R' forming harmonic range with P, R and R' fulfils the condition:

$$(P\,R\,M\,R') = \frac{|PM||RR'|}{|PR'||RM|} = 2$$

Then: $$|PM| = \frac{2|PR||PR'|}{|PR| + |PR'|}$$

The numerator is the power of P (constant for any line passing through P) and after operations we obtain the polar equation of the geometric locus of the points M:

$$|PM| = \frac{PT^2}{|PF|\cos\alpha}$$

which is the polar line. When α is the angle TPF (figure 9.2) we have $|PM| = |PT|$, that is, the polar passes through the points T of tangency to the circumference.

Now, by means of any projectivity (e.g. a homology) the circle is transformed into any conic. Since it preserves the projective cross ratio, the points M also form an aligned harmonic range in the conic, so the polar also passes through the touching points of the tangents drawn from P.

11.6 Let the points be denoted by:

$$A = (1,1) \qquad B = (2,3) \qquad C = (1,4) \qquad D = (0,2) \qquad E = (2,5)$$

Then: $EA \wedge EC = -3 \qquad EB \wedge ED = -4 \qquad EA \wedge ED = -5 \qquad EB \wedge EC = -2$

The cross ratio is:

$$\{E, A\,B\,C\,D\} = \frac{EA \wedge EC\;EB \wedge ED}{EA \wedge ED\;EB \wedge EC} = \frac{6}{5}$$

Now a generic point of the conic will be $X = (x, y)$:

$$XA \wedge XC = -3\,x + 3 \qquad XB \wedge XD = x - 2\,y + 4$$

$$XA \wedge XD = -x - y + 2 \qquad XB \wedge XC = -x - y + 5$$

Applying the Chasles' theorem we find the equation of the conic:

$$\frac{XA\wedge XC\ XB\ XD}{XA\wedge XD\ XB\wedge XC}=\frac{(-3\,x+3)(x-2\,y+4)}{(-x-y+2)(-x-y+5)}=\frac{6}{5}$$

After simplifying we arrive at the point equation:

$$7\,x^2+2\,y^2-6\,x\,y+x-4\,y=0$$

which may be written in the form:

$$\begin{pmatrix}1-x-y & x & y\end{pmatrix}\begin{pmatrix}0 & 1/2 & -2\\ 1/2 & 8 & -9/2\\ -2 & -9/2 & -2\end{pmatrix}\begin{pmatrix}1-x-y\\ x\\ y\end{pmatrix}=0$$

The inverse matrix of the conic is:

$$\begin{pmatrix}0 & 1/2 & -2\\ 1/2 & 8 & -9/2\\ -2 & -9/2 & -2\end{pmatrix}^{-1}=\frac{1}{90}\begin{pmatrix}145 & -40 & -55\\ -40 & 16 & 4\\ -55 & 4 & 1\end{pmatrix}$$

So the equation of the tangential conic is:

$$\begin{bmatrix}1-u-v & u & v\end{bmatrix}\begin{pmatrix}145 & -40 & -55\\ -40 & 16 & 4\\ -55 & 4 & 1\end{pmatrix}\begin{bmatrix}1-u-v\\ u\\ v\end{bmatrix}=0$$

$$241\,u^2+256\,v^2+488\,u\,v-370\,u-400\,v+145=0$$

11.7 If we take *A*, *B* and *C* as the base of the points, the coordinates in this base of every point *X* will be expressed as:

$$X=(x_A, x_B, x_C) \qquad\qquad x_A+x_B+x_C=1$$

If *P* is the intersection point of the lines *AE* and *BF*, then *A*, *P* and *E* are aligned and also *B*, *P* and *F*, conditions which we may express in coordinates:

$$\begin{vmatrix}1 & 0 & 0\\ p_A & p_B & p_C\\ e_A & e_B & e_C\end{vmatrix}=0 \quad \begin{vmatrix}0 & 1 & 0\\ p_A & p_B & p_C\\ f_A & f_B & f_C\end{vmatrix}=0 \;\Rightarrow\; (p_A, p_B, p_C)=\frac{(e_C\ f_A, e_B\ f_C, e_C\ f_C)}{e_C\ f_A+e_B\ f_C+e_C\ f_C}$$

If *Q* is the intersection point of the lines *AD* and *CF*, then *A*, *Q* and *D* are aligned and also *C*, *Q* and *F*, conditions which we may express in coordinates:

$$\begin{vmatrix} 1 & 0 & 0 \\ q_A & q_B & q_C \\ d_A & d_B & d_C \end{vmatrix} = 0 \quad \begin{vmatrix} 0 & 0 & 1 \\ q_A & q_B & q_C \\ f_A & f_B & f_C \end{vmatrix} = 0 \quad \Rightarrow \quad (q_A, q_B, q_C) = \frac{(d_B\, f_A, d_B\, f_B, d_C\, f_B)}{d_B\, f_A + d_B\, f_B + d_C\, f_B}$$

If R is the intersection point of the lines BD and CE, then B, R and D are aligned and also C, R and E, conditions which we may express in coordinates:

$$\begin{vmatrix} 0 & 1 & 0 \\ r_A & r_B & r_C \\ d_A & d_B & d_C \end{vmatrix} = 0 \quad \begin{vmatrix} 0 & 0 & 1 \\ r_A & r_B & r_C \\ e_A & e_B & e_C \end{vmatrix} = 0 \quad \Rightarrow \quad (r_A, r_B, r_C) = \frac{(d_A\, e_A, d_A\, e_B, d_C\, e_A)}{d_A\, e_A + d_A\, e_B + d_C\, e_A}$$

The points P, Q and R will be aligned if and only if the determinant of their coordinates vanishes. Setting aside the denominators, we have the first step:

$$\begin{vmatrix} p_A & p_B & p_C \\ q_A & q_B & q_C \\ r_A & r_B & r_C \end{vmatrix} \propto \begin{vmatrix} e_C f_A & e_B f_C & e_C f_C \\ d_B f_A & d_B f_B & d_C f_B \\ d_A e_A & d_A e_B & d_C e_A \end{vmatrix} \propto \begin{vmatrix} \dfrac{f_A}{f_C} & \dfrac{e_B}{e_C} & 1 \\ \dfrac{f_A}{f_B} & 1 & \dfrac{d_C}{d_B} \\ 1 & \dfrac{e_B}{e_A} & \dfrac{d_C}{d_A} \end{vmatrix} \propto \begin{vmatrix} \dfrac{1}{f_C} & \dfrac{1}{e_C} & \dfrac{1}{d_C} \\ \dfrac{1}{f_B} & \dfrac{1}{e_B} & \dfrac{1}{d_B} \\ \dfrac{1}{f_A} & \dfrac{1}{e_A} & \dfrac{1}{d_A} \end{vmatrix}$$

In the second step each row has been divided by a product of two coordinates. In the third step each column has been divided by a coordinate. And finally, after transposition and exchange of the first and third columns (turning the elements 90º in the matrix) and multiplying each column by a product of three coordinates we obtain:

$$\begin{vmatrix} p_A & p_B & p_C \\ q_A & q_B & q_C \\ r_A & r_B & r_C \end{vmatrix} \propto \begin{vmatrix} \dfrac{1}{f_A} & \dfrac{1}{f_B} & \dfrac{1}{f_C} \\ \dfrac{1}{e_A} & \dfrac{1}{e_B} & \dfrac{1}{e_C} \\ \dfrac{1}{d_A} & \dfrac{1}{d_B} & \dfrac{1}{d_C} \end{vmatrix} \propto \begin{vmatrix} d_A e_A & d_B e_B & d_C e_C \\ d_A f_A & d_B f_B & d_C f_C \\ e_A f_A & e_B f_B & e_C f_C \end{vmatrix} = 0$$

The last determinant is zero if and only if the point F lies on the conic passing through A, B, C, D and E because then it fulfils the conic equation:

$$\begin{vmatrix} d_A e_A & d_B e_B & d_C e_C \\ d_A x_A & d_B x_B & d_C x_C \\ e_A x_A & e_B x_B & e_C x_C \end{vmatrix} = 0$$

11.8 The Brianchon's theorem is the dual of the Pascal's theorem. In the dual plane, the point conic is represented by the tangential conic, the lines tangent to the point conic are represented by points lying on the tangential conic and the diagonals of the circumscribed hexagon are represented by the points of intersection P, Q and R of the former exercise. So the algebraic deduction is the same but in the dual plane.

12. Matrix representation and hyperbolic numbers

12.1 Let us calculate the modulus and argument:

$$|\,5+4\,e_1| = \sqrt{5^2-4^2} = 3 \qquad \arg(5+4\,e_1) = \frac{1}{2}\arg\operatorname{tgh}\frac{4}{5} = \log 3$$

Then:

$$\sqrt{5+4\,e_1} = \sqrt{3}_{\frac{\log 3}{2}} = \sqrt{3}\left(\cosh\left(\log\sqrt{3}\right) + e_1\,\sinh\left(\log\sqrt{3}\right)\right) = 2+e_1$$

The four square roots are:

$$2+e_1 \qquad 1+2\,e_1 \qquad -2-e_1 \qquad -1-2e_1$$

12.2 $\quad 2\,z^2+3\,z-17+3\,e_1=0 \quad \Rightarrow \quad z=\dfrac{-3\pm\sqrt{9-4\cdot 2\cdot(-17+3\,e_1)}}{2\cdot 2}$

We must calculate the square root of the discriminant:

$$\sqrt{145-24\,e_1} = \sqrt{143}_{\frac{\log(11/13)}{2}} = 12-e_1$$

So the four solutions are:

$$z_1=\frac{9-e_1}{4} \qquad z_2=\frac{-15+e_1}{4} \qquad z_3=\frac{-1-6\,e_1}{2} \qquad z_4=-1+3\,e_1$$

12.3 Let us apply the new formula for the second degree equation:

$$z^2-6\,z+5=0 \qquad z=\frac{6\pm\sqrt{36-4\cdot 1\cdot 5}}{2\cdot 1}=\frac{6\pm 4}{2} \qquad \text{and}$$

$$z=\frac{6\pm e_1\sqrt{36-4\cdot 1\cdot 5}}{2\cdot 1}=\frac{6\pm 4\,e_1}{2}$$

12.4 $\sin(x+y\,e_1) = \dfrac{1+e_1}{2}\sin(x+y) + \dfrac{1-e_1}{2}\sin(x-y)$

$$= \sin x\cos y + e_1\,\cos x\sin y$$

12.5 From the analogous of Moivre's identity we have:

$$(\cosh\psi + e_1 \sinh\psi)^4 \equiv \cosh 4\psi + e_1 \sinh 4\psi$$

$$\cosh 4\psi \equiv \cosh^4\psi + 6\cosh^2\psi \sinh^2\psi + \sinh^4\psi$$

$$\sinh 4\psi \equiv 4\cosh^3\psi \sinh\psi + 4\cosh\psi \sinh^3\psi$$

12.6 The analytical continuation of the real logarithm is:

$$\log(x + y\, e_1) = \frac{1+e_1}{2}\log(x+y) + \frac{1-e_1}{2}\log(x-y)$$

$$= \frac{1}{2}\log(x^2 - y^2) + \frac{e_1}{2}\log\frac{x+y}{x-y}$$

It may be rewritten in the form:

$$= \log\sqrt{x^2 - y^2} + e_1 \operatorname{arg\,tgh}\frac{y}{x}$$

12.7 For the straight path $z = t\, e_1$ we have:

$$\int_{-e_1}^{e_1} z^2 dz = e_1 \int_{-1}^{1} t^2 dt = e_1 \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{2\, e_1}{3}$$

For the circular path $z = \cos t + e_1 \sin t$ we have:

$$\int_{-e_1}^{e_1} z^2 dz = \int_{-\pi/2}^{\pi/2} (\cos t + e_1 \sin t)^2 (-\sin t + e_1 \cos t)\, dt$$

$$= \left[\cos t + \frac{2}{3}\cos^3 t + e_1\left(\sin t - \frac{2}{3}\sin^3 t\right)\right]_{-\pi/2}^{\pi/2} = \frac{2\, e_1}{3}$$

Using the indefinite integral, we find also the same result:

$$\int_{-e_1}^{e_1} z^2 dz = \left[\frac{z^3}{3}\right]_{-e_1}^{e_1} = \frac{2\, e_1}{3}$$

12.8 The proof is analogous to the exercise 3.12: turn the hyperbolic numbers df, dz into hyperbolic vectors by multiplying them at the left by e_2.

13. The hyperbolic or pseudo-Euclidean plane

13.1 If the vertices of the triangle are $A = (2, 2)$, $B = (1, 0)$ and $C = (5, 3)$ then:

$$AB = (-1, -2) \qquad BC = (4, 3) \qquad CA = (-3, -1)$$

$$|\,AB\,| = \sqrt{3}\, e_{12} \qquad |\,BC\,| = \sqrt{7} \qquad |\,CA\,| = \sqrt{8}$$

From the cosine theorem we have:

$$AB^2 = BC^2 + CA^2 - 2\,|\,BC\,|\,|\,CA\,| \cosh\gamma \quad \Rightarrow \quad \cosh\gamma = \frac{9}{2\sqrt{14}} \quad \Rightarrow \quad \gamma \cong 0.6264$$

taking into account that $\gamma = ACB$ is a positive angle. From the sine theorem we have:

$$\frac{|\,BC\,|}{\sinh\alpha} = \frac{|\,CA\,|}{\sinh\beta} = \frac{|\,AB\,|}{\sinh\gamma} \quad \Rightarrow \quad \frac{\sqrt{7}}{\sinh\alpha} = \frac{\sqrt{8}}{\sinh\beta} = \frac{\sqrt{3}\, e_{12}}{\dfrac{5}{2\sqrt{14}}}$$

whence the angles α and β follow:

$$\sinh\alpha = -\frac{5\, e_{12}}{2\sqrt{6}} \quad \Rightarrow \quad \alpha \cong -0.2027 - \frac{\pi}{2} e_{12}$$

$$\sinh\beta = -\frac{5\, e_{12}}{\sqrt{21}} \quad \Rightarrow \quad \beta \cong -0.4236 - \frac{\pi}{2} e_{12}$$

The plot helps to choose the right sign of the angles. Anyway a wrong choice would be revealed by the cosine theorem:

$$BC^2 = CA^2 + AB^2 - 2\,|\,CA\,|\,|\,AB\,| \cosh\alpha$$

$$CA^2 = AB^2 + BC^2 - 2\,|\,AB\,|\,|\,BC\,| \cosh\beta$$

13.2 The rotation of the vector is obtained with the multiplication by a hyperbolic number:

$$v' = v\, z_\alpha = (2\, e_2 + e_{21})(\cosh\log 2 + e_1 \sinh\log 2) = (2\, e_2 + e_{21})\left(\frac{5}{4} + \frac{3}{4} e_1\right)$$
$$= \frac{13}{4} e_2 + \frac{11}{4} e_{21}$$

The reflection is obtained by multiplying on the right and on the left by the direction vector of the reflection and its inverse respectively:

$$v' = d^{-1} v\, d = (3\, e_2 - e_{21})^{-1}\, (2\, e_2 + e_{21})(3\, e_2 - e_{21}) = \frac{1}{8}(3\, e_2 - e_{21})(7 - 5\, e_1) = \frac{13\, e_2 - 11\, e_{21}}{4}$$

The inversion with radius $r = 3$ is:

$$v' = r^2 v^{-1} = 9\left(2\,e_2 + e_{21}\right)^{-1} = 3\left(2\,e_2 + e_{21}\right) = 6\,e_2 + 3\,e_{21}$$

13.3 The two-point equation of the line $y = 2\,x + 1$ is:

$$\frac{x+1}{1} = \frac{y}{2}$$

which gives us the direction vector:

$$v = e_1 + 2\,e_2$$

The direction vector is the normal vector of the perpendicular line. The general equation of a line in the hyperbolic plane is:

$$n \cdot PR = 0 \qquad \Rightarrow \qquad n_x\left(x - x_P\right) - n_y\left(y - y_P\right) = 0$$

Note the minus sign since this line lies on the hyperbolic plane. The substitution of the components of the direction vector of the first line and the coordinates of the point R = (3, 1) through which the line passes results in the equation:

$$1 \cdot (x-3) - 2 \cdot (y-1) = 0 \qquad \Rightarrow \qquad x - 2\,y - 1 = 0$$

The fact that both lines and the first quadrant bisector intersect in a unique point is circumstantial. Only it is required that the bisectors of both lines be parallel to the quadrant bisectors.

13.4 The line y =3 intersects the hyperbola $x^2 - y^2 = 16$ in the points R = (–5, 3) and R' = (5, 3). Then the power of P = (–7, 3) is:

$$PR\ PR' = 2\,e_2 \cdot 12\,e_2 = 24$$

The line $y = -\,3\,x - 18$ cuts the hyperbola in the points T = (–5, –3) and T' = (–17/2, 15/2). Then the power of P is:

$$PT\ PT' = \left(2\,e_2 - 6\,e_{21}\right) \cdot \left(-\frac{3\,e_2}{2} + \frac{9\,e_{21}}{2}\right) = -3 + 27 = 24$$

The line $y = -3\,x / 7$ cuts the hyperbola in the points:

$$S = \left(-\frac{7\sqrt{10}}{5}, \frac{3\sqrt{10}}{5}\right) \qquad \text{and} \qquad S' = \left(\frac{7\sqrt{10}}{5}, -\frac{3\sqrt{10}}{5}\right)$$

The power is:

$$PS\ PS' = \left(\left(-\frac{7\sqrt{10}}{5} + 7\right)e_2 + \left(\frac{3\sqrt{10}}{5} - 3\right)e_{21}\right) \cdot \left(\left(\frac{7\sqrt{10}}{5} + 7\right)e_2 + \left(-\frac{3\sqrt{10}}{5} - 3\right)e_{21}\right)$$

$$= \frac{147}{5} - \frac{27}{5} = 24$$

Also the substitution of the coordinates gives 24:

$$x_P^2 - y_P^2 - 16 = 49 - 9 - 16 = 24$$

13.5 From the law of sines we find:

$$\frac{|a|}{|b|} = \frac{\sinh\alpha}{\sinh\beta} \qquad \Rightarrow \qquad \frac{|a|+|b|}{|a|-|b|} = \frac{\sinh\alpha + \sinh\beta}{\sinh\alpha - \sinh\beta}$$

Introducing the identity for the addition and subtraction of sines we arrive at the law of tangents:

$$\frac{|a|+|b|}{|a|-|b|} = \frac{\operatorname{tgh}\dfrac{\alpha+\beta}{2}}{\operatorname{tgh}\dfrac{\alpha-\beta}{2}}$$

13.6 The first triangle has the vertices $A = (0, 0)$, $B = (5, 0)$, $C = (5, 3)$ and the sides:

$$AB = 5\, e_2 \qquad BC = 3\, e_{21} \qquad CA = -5\, e_2 - 3\, e_{21}$$

The second triangle has the vertices $A' = (0, 0)$, $B' = (25, -15)$, $C' = (16, 0)$ and the sides:

$$A'B' = 25\, e_2 - 15\, e_{21} \qquad B'C' = -9\, e_2 + 15\, e_{21} \qquad C'A' = -16\, e_2$$

Both triangles are directly similar since:

$$AB^{-1}\, A'B' = BC^{-1}\, B'C' = CA^{-1}\, C'A' = 5 - 3\, e_1 = r$$

which is the similarity ratio r. The size ratio is the modulus of the similarity ratio:

$$|r| = \sqrt{5^2 - 3^2} = 4$$

that is, the second triangle is four times larger than the first. The angle of rotation between both figures is the argument of the similarity ratio:

$$\psi = \arg(5 - 3\, e_1) = \arg\operatorname{tgh}\left(-\frac{3}{5}\right) = -\log 2 = -0.6931...$$

Now plot the vertices of each triangle in the hyperbolic plane and break your Euclidean illusions about figures with the same shape.

14. Spherical geometry in the Euclidean space

14.1 The shadow of a gnomon follows a hyperbola on a plane (a quadrant) because the Sun describes approximately a parallel around the North pole during a day. The altitude of this parallel from the celestial equator is the declination, which changes slowly from one day to another. The March 21 and September 23 are the equinoxes when the Sun follows the equator. Since it is a great circle, its central projection is a straight line named the *equinoctial line*.

14.2 From the stereographic coordinates we find the Cartesian coordinates of the points:

$$A = -\frac{2}{3}e_1 + \frac{2}{3}e_2 + \frac{1}{3}e_3 \qquad B = -\frac{12\, e_2}{13} + \frac{5\, e_3}{13} \qquad C = \frac{12}{13}e_1 + \frac{5}{13}e_3$$

Their outer products are equal to the cosines of the sides:

$$\cos a = B \cdot C = \frac{25}{169} \qquad a = 1.4223$$

$$\cos b = C \cdot A = -\frac{19}{39} \qquad b = 2.0797$$

$$\cos c = A \cdot B = -\frac{19}{39} \qquad c = 2.0797$$

Now we calculate the bivectors of the planes containing the sides:

$$A \wedge B = \frac{22\, e_{23} + 10\, e_{31} + 24\, e_{12}}{39} \qquad |\, A \wedge B\,| = \frac{\sqrt{1160}}{39}$$

$$B \wedge C = \frac{-60\, e_{23} + 60\, e_{31} + 144\, e_{12}}{169} \qquad |\, B \wedge C\,| = \frac{\sqrt{27936}}{169}$$

$$C \wedge A = \frac{-10\, e_{23} - 22\, e_{31} + 24\, e_{12}}{39} \qquad |\, C \wedge A\,| = \frac{\sqrt{1160}}{39}$$

The cosines of the angles between planes are obtained through the analogous of scalar product for bivectors:

$$\cos\alpha = -\frac{(C \wedge A)\cdot(A \wedge B)}{|\, C \wedge A\,||\, A \wedge B\,|} = -\frac{136}{1160} \qquad \alpha = 1.6882$$

$$\cos\beta = -\frac{(A \wedge B)\cdot(B \wedge C)}{|\, A \wedge B\,||\, B \wedge C\,|} = -\frac{2736}{\sqrt{1160 \cdot 27936}} \qquad \beta = 2.0721$$

$$\cos\gamma = -\frac{(B\wedge C)\cdot(C\wedge A)}{|B\wedge C||C\wedge A|} = -\frac{2736}{\sqrt{27936\cdot 1160}} \qquad \gamma = 2.0721$$

The area of the triangle is the spherical excess:

$$area = \alpha + \beta + \gamma - \pi = 2.6908$$

14.3 Let us calculate the Cartesian coordinates of Fastnet (point *F*), the point in the middle Atlantic (point *A*) and Sandy Hook (point *S*). Identifying the geographical longitude with the angle φ and the colatitude with the angle θ we have:

$$x = \sin\theta\cos\varphi \quad y = \sin\theta\sin\varphi \qquad z = \cos\theta$$

$$F = (\theta = 38^{o}30', \varphi = -9^{o}35') = 0.6138\, e_1 - 0.1036\, e_2 + 0.7826\, e_3$$

$$A = (\theta = 47^{o}30', \varphi = -47^{o}) = 0.5028\, e_1 - 0.5392\, e_2 + 0.6756\, e_3$$

$$S = (\theta = 49^{o}30', \varphi = -74^{o}) = 0.2096\, e_1 - 0.7309\, e_2 + 0.6494\, e_3$$

The length of the arcs is obtained from the inner products of the position vectors:

$$\cos FS = F\cdot S = 0.7126 \qquad FS = 44^{o}33' = 4950 \text{ km}$$

$$\cos FA = F\cdot A = 0.8932 \qquad FA = 26^{o}43' = 2969 \text{ km}$$

$$\cos AS = A\cdot S = 0.9382 \qquad AS = 20^{o}14' = 2250 \text{ km}$$

Now we see that the track is 269 km longer than the shortest path between Fastnet and Sandy Hook.

Let us investigate whether the Titanic followed the obliged track, that is whether the point *T* where the tragedy happened lies on the line *AS*, by calculating the determinant of the three points:

$$T = (\theta = 48^{o}14', \varphi = -50^{o}14') = 0.4771\, e_1 - 0.5733\, e_2 + 0.6661\, e_3$$

$$A\wedge T\wedge S = \begin{vmatrix} 0.5028 & 0.4771 & 0.2096 \\ -0.5392 & -0.5733 & -0.7309 \\ 0.6756 & 0.6661 & 0.6494 \end{vmatrix} e_{123} = -0.0050\, e_{123}$$

The negative sign indicates that the orientation of the three points is clockwise (figure 16.6). In other words, the Titanic was shipping at the South of the obliged track. We can calculate the distance from *T* to the track *AS* taking the right angle triangle. By the Napier's rule we have:

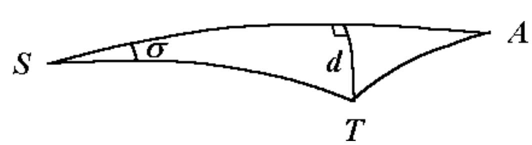


Figure 16.6

$$\sin d = \sin\sigma \sin ST = \frac{|A \wedge T \wedge S|}{|A \wedge S|} = \frac{0.0050}{0.3461} = 0.0144 \quad d = 49' = 92 \text{ km}$$

14.4 After removing the margins and subtracting the coordinates of the centre of the photograph in the bitmap file, I have obtained a pair of coordinates u' and v' in pixels, which are proportional to u and v:

star	u'	v'
α Cassiopeia	173	–139
β Cassiopeia	349	–135
γ Cassiopeia	199	24
δ Cassiopeia	89	93
Hale-Bopp comet	–240	–320

On the other hand, the right ascension A and declination D of these stars are known data. From the spherical coordinates and taking $\theta = 90º - D$ and $\varphi = A$ one obtains their Cartesian equatorial coordinates in the following way:

$$x = \cos D \cos A \qquad y = \cos D \sin A \qquad z = \sin D$$

The Aries point has the equatorial coordinates (1,0,0). Using these formulas I have found the coordinates:

star	x	y	z
α Cassiopeia	0.543519	0.096098	0.833878
β Cassiopeia	0.512996	0.019711	0.858216
γ Cassiopeia	0.475011	0.119054	0.871889
δ Cassiopeia	0.462917	0.180840	0.867759

A photograph is a central projection. So the arch s between two stars A and B is related with their coordinates (u, v) on the projection plane by:

$$\cos s_{AB} = \frac{1 + u_A u_B + v_A v_B}{\sqrt{1 + u_A^2 + v_A^2}\sqrt{1 + u_B^2 + v_B^2}}$$

The focal distance f is the distance between the projection plane (the photograph) and the centre of projection. When the focal distance is unknown we only can measure proportional coordinates u' and v' instead of u and v, and the foregoing formula becomes:

$$\cos s_{AB} = \frac{f^2 + u'_A u'_B + v'_A v'_B}{\sqrt{f^2 + u'^2_A + v'^2_A}\sqrt{f^2 + u'^2_B + v'^2_B}}$$

which is a second degree equation on f^2:

$$\left(f^2 + u'^2_A + v'^2_A\right)\left(f^2 + u'^2_B + v'^2_B\right)\cos^2 s_{AB} = \left(f^2 + u'_A u'_B + v'_A v'_B\right)^2$$

$$0 = f^4\left(1 - \cos^2 s_{AB}\right) + f^2\left[2\left(u'_A u'_B + v'_A v'_B\right) - \left(u'^2_A + v'^2_A + u'^2_B + v'^2_B\right)\cos^2 s_{AB}\right]$$
$$+ \left(u'_A u'_B + v'_A v'_B\right)^2 - \left(u'^2_A + v'^2_A\right)\left(u'^2_B + v'^2_B\right)\cos^2 s_{AB}$$

All the coefficients of the equation are known, because $\cos s_{AB} = A \cdot B$ is calculated from the Cartesian equatorial coordinates obtained from the right ascension and declination of both stars. The focal distances so obtained are:

star	f
α-β Cassiopeia	2012.7
α-γ Cassiopeia	2010.6
α-δ Cassiopeia	2017.2
β-γ Cassiopeia	2003.0
β-δ Cassiopeia	2019.1
γ-δ Cassiopeia	2049.1
mean value	2018.6

The width of the photograph is 750 pixels and the height 1166. If 750 pixels corresponds to 24 mm and 1166 to 36 mm, we find a focal distance of the camera equal to 64.6 mm and 62.3 mm respectively. However the author of the photograph indicated me that his camera has a focal distance of 58 mm, what implies that the original image was cut a 7 % in the photographic laboratory. This is a customary usage in photography, so we do not know the enlargement proportion of a paper copy and we must calculate the focal distance directly from the photograph, which is always somewhat higher than that calculated from the focal distance of the camera. On the other hand, the calculus of the focal distance is very sensitive to the errors and truncation of decimals. The best performance is to write a program to calculate the mean focal distance from the original data or to make us the paper copies without cutting the image. Known the focal distance, we are able to calculate the normalised coordinates u and v:

star	u	v
α Cassiopeia	0.085703	–0.068860

β Cassiopeia	0.172892	–0.066878
γ Cassiopeia	0.098583	0.011889
δ Cassiopeia	0.044090	0.046072
Hale-Bopp comet	–0.118894	–0.158526

and from here the inner product between stars and the comet:

$$\cos(A,H)=\frac{1+u_A\,u_H+v_A\,v_H}{\sqrt{1+u_A^2+v_A^2}\sqrt{1+u_H^2+v_H^2}}$$

$$\cos(B,H)=\frac{1+u_B\,u_H+v_B\,v_H}{\sqrt{1+u_B^2+v_B^2}\sqrt{1+u_H^2+v_H^2}}$$

The unknown equatorial coordinates of the comet H are calculated from a system of three equations:

$$\begin{cases} A\cdot H=\cos(A,H)\\ B\cdot H=\cos(B,H)\\ \quad H^2=1\end{cases}$$

Let us introduce the Clifford product instead of the inner product and multiply the first equation by B and the second one by A on the right:

$$\begin{cases} \dfrac{A\,H+H\,A}{2}=\cos(A,H) \quad\Rightarrow\quad \dfrac{A\,H\,B+H\,A\,B}{2}=\cos(A,H)\,B\\ \dfrac{B\,H+H\,B}{2}=\cos(B,H) \quad\Rightarrow\quad \dfrac{B\,H\,A+H\,B\,A}{2}=\cos(B,H)\,A\\ \qquad H^2=1\end{cases}$$

Using the permutative property (in the space $u\;v\;w-w\;v\;u=2\;u\wedge v\wedge w$), the difference of both equations is equal to:

$$A\wedge H\wedge B+H\;A\wedge B=\cos(A,H)\,B-\cos(B,H)\,A$$

or equivalently: $-H\wedge A\wedge B+H\;A\wedge B=\cos(A,H)\,B-\cos(B,H)\,A$

The outer product by H is the geometric product by its perpendicular component $H_\perp$, so we have:

$$H_\parallel\;A\wedge B=\cos(A,H)\,B-\cos(B,H)\,A$$

where we can isolate the coplanar component:

$$H_{\parallel} = \left(\cos(A,H)\, B - \cos(B,H)\, A\right)\left(A \wedge B\right)^{-1}$$

Observe that this result can be only obtained and expressed through the geometric product, and not through the more usual inner and outer products. The component perpendicular to the plane *AB* is proportional to the dual vector of the outer product of this vectors. Known the coplanar component we may now fix the modulus of the perpendicular component because $H^2=1$:

$$H_{\perp} = \pm \frac{e_{123}\ A \wedge B}{|\, A \wedge B\,|} \sqrt{1 - H_{\parallel}^2}$$

There are two solutions, but according to the statement of the problem only one of them is valid. Since each pair of stars gives two values of *H* (the position vector of the comet), we may distinguish the true solution because it has proper values for different pairs of stars. The mean value with the standard deviation so obtained is:

$$H = (0.661606 \pm 0.000741)\, e_1 + (0.236258 \pm 0.001298)\, e_2 + (0.711659 \pm 0.000302)\, e_3$$

from where the following equatorial coordinates of the Hale-Bopp comet are obtained:

$$A_H = 1\text{h } 18\text{min } 36\text{s} \qquad D_H = 45^{\circ}22'12''$$

In the ephemeris (http://www.xtec.es/recursos/astronom/hb/ephjan97.txt) for the comet we find:

Date	Right ascension (*A*)	Declination (*D*)
1997 March 28	01h 09min 18.03s	+45º 36’ 30.6”
1997 March 29	01h 19min 06.07s	+45º 25’ 15.6”
1997 March 30	01h 28min 44.92s	+45º 10’ 51.9”

So I conclude that this photograph was taken the March 29 of 1997.

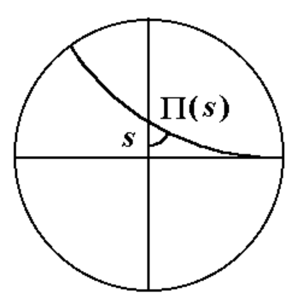
On the other hand, I have also calculated the orientation *O* of the camera (of the line passing through the centre of the photograph):

$$O = (0.513505 \pm 0.000673)\, e_1 + (0.201570 \pm 0.000471)\, e_2 \pm (0.834075 \pm 0.000360)\, e_3$$

$$A_O = 1\text{h } 25\text{min } 44\text{s} \qquad D_O = 56^{0}31'11''$$

15. Hyperboloidal geometry in the pseudo-Euclidean space (Lobachevsky’s geometry)

15.1 The circle being projection of a parallel line (geodesic hyperbola) shown in the figure 15.9 passes through the point (1, 0) which implies the equation:



$$(u-1)^2+(v-b)^2=b^2$$

According to this equation, the intersection of this line with the v-axis has the coordinates $\left(0, b-\sqrt{b^2-1}\right)$ (the other intersection point falls outside the Poincaré disk). The inverse of the slope at this point with opposite sign is the tangent of the angle of parallelism:

Figure 15.9

$$\operatorname{tg}\Pi(s)=-\left(\frac{du}{dv}\right)_{u=0}=\sqrt{b^2-1} \qquad \Rightarrow \qquad \cos\Pi(s)=\frac{1}{b}$$

Now let us calculate the distance s from the origin to this point. Its hyperbolic cosine is the inner product of the unitary position vectors of the origin and the point of intersection:

$$\cosh s=e_3\cdot\left(\left(\frac{b-\sqrt{b^2-1}}{-b^2+1+b\sqrt{b^2-1}}\right)e_2+\frac{b}{\sqrt{b^2-1}}e_3\right)=\frac{b}{\sqrt{b^2-1}}$$

$$s=\log\sqrt{\frac{b+1}{b-1}} \quad \exp(-s)=\sqrt{\frac{b-1}{b+1}}=\sqrt{\frac{1-\cos\Pi(s)}{1+\cos\Pi(s)}}=\operatorname{tg}\frac{\Pi(s)}{2}$$

so we find the Lobachevsky's formula for the angle of parallelism:

$$\Pi(s)=2\operatorname{arctg}(\exp(-s))$$

15.2 The law of cosines in hyperboloidal trigonometry is:

$$\cosh a=\cosh b\cosh c-\sinh b\sinh c\cos a$$

For a right angle triangle $\alpha=\pi/2$:

$$\cosh a=\cosh b\cosh c$$

In the limit of small arcs it becomes the Pythagorean theorem:

$$1+\frac{a^2}{2}+O(a^4)=\left(1+\frac{b^2}{2}+O(b^4)\right)\left(1+\frac{c^2}{2}+O(c^4)\right)$$

$$a^2+O(a^4)=b^2+c^2+O(b^2c^2,b^4,c^4)$$

15.3 A «circle» is given by the equation system:

$$\begin{cases} z^2 - x^2 - y^2 = 1 \\ a\,x + b\,y + c = z \end{cases}$$

a) The intersection of a quadric (hyperboloid) with a plane is always a conic. Since the curve is closed, it must be an ellipse (for horizontal planes it is a circle).
b) We hope that the centre of the ellipse be the intersection of the plane with its perpendicular line passing through the origin (this occurs also in spherical geometry). The plane has a bivector:

$$a\,e_{23} + b\,e_{31} - e_{12}$$

The vector perpendicular, the dual vector is:

$$a\,e_1 + b\,e_2 + e_3$$

because the product of the vector and bivector is equal to a pure volume element:

$$(a\,e_{23} + b\,e_{31} - e_{12})(a\,e_1 + b\,e_2 + e_3) = \left(a^2 + b^2 - 1\right)e_{123}$$

The *axis* of a circle is the line passing through the origin and perpendicular to the plane containing this circle. The centre of the circle is the intersection point of the plane and axis of this circle, given by the equation system:

$$\begin{cases} a\,x + b\,y + c = z \\ \dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{1} \end{cases} \Rightarrow \quad x = \frac{a\,c}{1 - a^2 - b^2} \qquad y = \frac{b\,c}{1 - a^2 - b^2} \qquad z = \frac{c}{1 - a^2 - b^2}$$

The distance from any point of the circle to its centre is constant:

$$\left(z - \frac{c}{1 - a^2 - b^2}\right)^2 - \left(x - \frac{a\,c}{1 - a^2 - b^2}\right)^2 - \left(y - \frac{b\,c}{1 - a^2 - b^2}\right) =$$

$$1 - \frac{2\,c^2}{1 - a^2 - b^2} + \frac{c^2\left(1 - a^2 - b^2\right)}{\left(1 - a^2 - b^2\right)^2} = 1 - \frac{c^2}{1 - a^2 - b^2}$$

Since $c > 1$ and $a^2 + b^2 < 1$ this value is always negative, what means that the radius is a distance comparable with distances on the Euclidean plane *x*-*y*. Taking the real value, the radius of a circle is:

$$r = \sqrt{\frac{c^2}{1 - a^2 - b^2} - 1}$$

However this is not the radius measured on the hyperboloid and obtained from its projections. The intersection of the axis with the hyperboloid gives its hyperboloidal centre:

$$\begin{cases} z^2 - x^2 - y^2 = 1 \\ \dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{1} \end{cases} \quad \Rightarrow \quad x = \frac{a}{1 - a^2 - b^2} \qquad y = \frac{b}{1 - a^2 - b^2} \qquad z = \frac{1}{1 - a^2 - b^2}$$

The hyperboloidal radius is the arc length on the hyperboloid from any point of the circle to its hyperboloidal centre, which should be constant. Making the inner product of the position vectors of a point on the circle and its centre, we find:

$$\cosh \psi = (x\, e_1 + y\, e_2 + z\, e_3) \cdot \left(\frac{a}{1 - a^2 - b^2} e_1 + \frac{b}{1 - a^2 - b^2} e_2 + \frac{z}{1 - a^2 - b^2} e_3 \right) =$$
$$\frac{-a\,x - b\,y + z}{1 - a^2 - b^2} = \frac{c}{1 - a^2 - b^2}$$

So the hyperboloidal radius is:

$$\psi = \operatorname{arg\,cosh} \frac{c}{1 - a^2 - b^2}$$

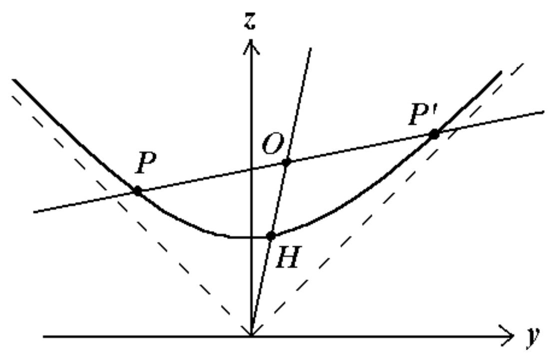


Figure 16.7

The figure 16.7 shows a lateral view of a plane with $a = 0$ and equation $b\, x + c = z$. The points P and P' lie on the circle. O is the centre of the circle on its plane and H is the centre on the hyperboloid. Then OH is the axis of the circle, which is perpendicular (in a pseudo-Euclidean way) to the circle plane, that is, their bisector has a unity slope. Note that according to a known property of the hyperbola, O must be the midpoint of the chord PP'. The plane radius is the distance PO or $P'O$, while the hyperboloidal radius is the arc length PH or $P'H$.

c) The coordinates of the stereographic projection are:

$$x = \frac{2\,u}{1 - u^2 - v^2} \qquad y = \frac{2\,v}{1 - u^2 - v^2} \qquad z = \frac{1 + u^2 + v^2}{1 - u^2 - v^2}$$

Let us make the substitution of these coordinates in the equation of the circle plane:

$$\frac{2\,a\,u + 2\,b\,v}{1 - u^2 - v^2} + c = \frac{1 + u^2 + v^2}{1 - u^2 - v^2}$$

After simplification we arrive to an equation of a circle:

$$\left(u - \frac{a}{c+1} \right)^2 + \left(v - \frac{b}{c+1} \right)^2 = \frac{a^2 + b^2 + c^2 - 1}{(c+1)^2}$$

15.4 A horocycle is a circle having $a^2 + b^2 = 1$. Therefore its equation in the stereographic projection is:

$$\frac{c^2}{(c+1)^2} = \left(u - \frac{a}{c+1}\right)^2 + \left(v - \frac{b}{c+1}\right)^2$$

Let us search the intersection points (if they exist) with the limit circle $u^2 + v^2 = 1$. Then we solve the system of both equations and find that it has a unique solution:

$$u = a \qquad\qquad v = b$$

Since they meet in a unique point and $a^2 + b^2 = 1$, the horocycle projection is tangent to the limit circle. The centre of the horocycle is the intersection of the hyperboloid with its axis. However this axis has unity slope, so the intersection lies at the infinity.

15.5 By differentiation of the coordinates in the Beltrami projection we find:

$$dx = \frac{(1-v^2)\,du + u\,v\,dv}{(1-u^2-v^2)^{3/2}} \qquad dy = \frac{(1-u^2)\,dv + u\,v\,du}{(1-u^2-v^2)^{3/2}}$$

$$dz = \frac{u\,du + v\,dv}{(1-u^2-v^2)^{3/2}}$$

The differential of area is a bivector and we search its modulus:

$$dA = \sqrt{(dx \wedge dy)^2 - (dy \wedge dz)^2 - (dz \wedge dx)^2} = \frac{du \wedge dv}{(1-u^2-v^2)^{3/2}}$$

Note the outer products of the differentials of coordinates so $dx \wedge dy = -dy \wedge dx$. Now we must integrate the differential of area of a doubly asymptotic triangle with angle φ (figure 15.10a). We change to polar coordinates r and θ in the Beltrami disk:

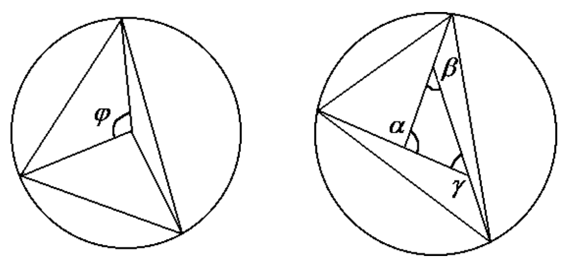


Figure 15.10

$$r = \sqrt{u^2 + v^2} \qquad\qquad \operatorname{tg}\theta = \frac{v}{u}$$

Since the equation of the line is $r = \dfrac{\cos(\varphi/2)}{\cos\theta}$ the integral becomes:

$$A = \int_0^{\frac{\cos(\varphi/2)}{\cos\theta}} \int_{-\varphi/2}^{\varphi/2} \frac{r\,dr \wedge d\theta}{(1-r^2)^{3/2}} = \int_{-\varphi/2}^{\varphi/2} \left[\frac{1}{\sqrt{1-r^2}}\right]_0^{\frac{\cos(\varphi/2)}{\cos\theta}} d\theta = \int_{-\varphi/2}^{\varphi/2} \frac{\cos\theta}{\sqrt{1-\cos^2\dfrac{\varphi}{2} - \sin^2\theta}}\,d\theta - \int_{-\varphi/2}^{\varphi/2} d\theta =$$

$$= \left[\arcsin \frac{\sin\theta}{\sin\dfrac{\varphi}{2}} - \theta \right]_{-\varphi/2}^{\varphi/2} = \pi - \varphi$$

This expression is applied to each asymptotic triangle surrounding the central triangle (figure 15.10b). Since the area of a triangle having the three vertices in the limit circle is $3\pi-2\pi = \pi$, the area of the central triangle follows:

$$A = \pi - \alpha - \beta - \gamma$$

15.6 In the azimuthal projection, the radius r of a circle centred at the origin is:

$$u = x\sqrt{\frac{2}{z+1}} \qquad v = y\sqrt{\frac{2}{z+1}} \qquad r^2 = u^2 + v^2 = 2\left(z-1\right)$$

Since the projection is equivalent, we can evaluate the area directly in the plane:

$$A = \pi\, r^2 = 2\,\pi\left(\,z-1\right) = 2\,\pi\left(\cosh\psi - 1\right)$$

where ψ is the Weierstrass coordinate, that is, the radius of the circle on the hyperboloid.

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CLICAL (Clifford Calculator) http://www.helsinki.fi/~lounesto/CLICAL.htm
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Subject pages

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The geometry of the sphere http://math.rice.edu/~pcmi/sphere/
Geometric Calculus: Research and Development http://modelingnts.la.asu.edu/
The Geometric Algebra Research Group (Cambridge) http://www.mrao.cam.ac.uk/~clifford/
Clifford Research Group (Gent) http://cage.rug.ac.be/~fb/crg/

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CHRONOLOGY OF THE GEOMETRIC ALGEBRA

1679 Letters of Leibniz to Huygens on the *characteristica geometrica*.
1799 Publication of *Om Directionens analytiske Betegning* by Caspar Wessel with scarce diffusion.
1805 Birth of William Rowan Hamilton at Dublin.
1806 Publication of *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* by Jean Robert Argand.
1809 Birth of Hermann Günther Grassmann at Stettin.
1818 Death of Wessel
1822 Death of Argand.
1827 Publication of *Der barycentrische Calcul* by Möbius at Leipzig.
1831 Birth of James Clerk Maxwell at Edinburgh.
1831 Birth of Peter Guthrie Tait.
1839 Birth of Josiah Willard Gibbs at New Haven.
1843 Discovery of the quaternions by Hamilton.
1844 Publication of *Die Lineale Ausdehnungslehre* (first edition), where Grassmann presents the anticommutative product of geometric unities (outer product).
1845 Birth of William Kingdon Clifford at Exeter.
1847 Publication of *Geometric Analysis* with a foreword by Möbius, memoir with which Grassmann won the prize to whom developed the Leibniz's *characteristica geometrica*.
1850 Birth of Oliver Heaviside at London.
1853 Publication of *Lectures on Quaternions* where Hamilton introduces the nabla operator (gradient).
1862 Publication of the second edition of *Die Ausdehnungslehre.*
1864 Publication of *A dynamical theory of the electromagnetic field* by Maxwell, where he defines the divergence and the curl.
1865 Death of Hamilton.
1866 Posthumous publication of Hamilton's *Elements of Quaternions*.
1867 Publication of *Elementary Treatise on Quaternions* by Tait.
1873 Publication of *Introduction to Quaternions* by Kelland and Tait.
1873 Maxwell publishes the *Treatise on Electricity and Magnetism* where he writes the equations of electromagnetism with quaternions.
1877 Publication of the Grassmann's paper «Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre».
1877 Death of Grassmann.
1878 Publication of the paper «Applications of Grassmann's Extensive Algebra» by Clifford where he makes the synthesis of the systems of Grassmann and Hamilton.
1879 Death of Maxwell.
1879 Death of Clifford.
1880 Publication of Lipschitz' *Principes d'un calcul algebraic*.
1881 Private printing of *Elements of Vector Analysis* by Gibbs.
1886 Publication of Lipschitz' *Untersuchungen uber die Summen von Quadraten*.
1886 Publication of the Gibbs' paper «On multiple algebra».
1888 Publication of Peano's *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazione della logica deduttiva*.
1891 Oliver Heaviside publishes «The elements of vectorial algebra and analysis» in *The Electrician Series*.
1895 Publication of Peano's «Saggio di Calcolo Geometrico».
1901 Death of Tait.
1901 Wilson publishes the Gibbs' lessons in *Vector Analysis*.
1903 Death of Gibbs.

1925 Death of Heaviside.
1926 Wolfgang Pauli introduces his matrices to explain the electronic spin.
1928 Publication of the paper «The Quantum Theory of Electron», where Paul A. M. Dirac defines a set of 4×4 anticommutative matrices built from the Pauli's matrices.

This comparative diagram of the life and works of the authors of (or related with) the geometric algebra visualises and summarises the chronology. The XIX century may be properly called *the century of the geometric algebra*. Note the premature death of Clifford, which caused the delay in the development of the geometric algebra along the XX century.

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                 1800      1820      1840      1860      1880      1900
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Wessel**************
Argand*************************
Hamilton              *******************************
Grassmann               ***********************************
Maxwell                            *************************
Tait                               ************************************
Gibbs                                  *********************************
Clifford                                  ******************
Heaviside                                   **************************************


Essai sur ...         |
Der barycentrische Calcul        |
Die Ausdehnungslehre                     |
Lectures on Quaternions                       |
Elements on Quaternions                             |
Elementary Treatise on Quaternions                   |
Treatise on Electricity and Magnetism                   |
Applications of Grassmann's Extensive Algebra             |
Elements of Vector Analysis                                 |
The Electrician Series                                           |
Vector Analysis                                                       |
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