AN INVESTIGATION INTO THE RELATIONSHIPS BETWEEN LOGICAL OPERATIONS

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ABSTRACT

Boolean algebra (logical operations) is the backbone of computer software and hardware systems. Investigating new relationships between logical operations may help designing new computer algorithms. In this paper we propose to investigate the inverse relationships between the Exclusive-OR and Equivalence functions for three, four and five- variable functions. In addition, we propose to investigate the inverse relationships between the Inhibition and Implication functions for three and four-variable functions.

KEYWORDS

Exclusive -OR, Equivalence, Implication, Inhibition, Inverse.

1. INTRODUCTION

Boolean algebra is the backbone of computer software and hardware systems. Such a phenomenon may be exploited for the purpose of developing new computer applications.

In this paper several inverse relationships between logic operations will be investigated, namely, the inverse relationships between the exclusive-OR and the equivalence, and between the inhibition and the implication.

The previous work investigated the exclusive-OR for two, three and four variables, and the relationships between the exclusive-OR and the equivalence only for two variables [1-20]. In this paper, we propose to investigate the relationships between the exclusive-OR and the equivalence for three, four and five variables. In addition, the previous work investigated the inverse between the inhibition and the implication only for two variables [4, 5, 8, 10, 16]. In this paper, we propose to investigate the inverse between the inhibition and the implication for three and four variables. Also, in this paper, we conclude some useful valid implications.

2. RELATED WORK

In this section, we describe the previous work related to our paper.

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2.1. The Exclusive-OR and Equivalence Functions

2.1.1. The Exclusive-OR and Equivalence Functions for Two Variables

The Exclusive-OR and the equivalence functions are both commutative and associative and are the complements of each other for two-variable functions [1-6, 15-17] as follows:

The exclusive-OR (EXOR): $x \oplus y = x^{/} y + x y^{/}$

The complement:
$$\begin{aligned}(x \oplus y)^{/} &= \left(x^{/} y + x y^{/}\right)^{/} \\ &= \left(x + y^{/}\right)\left(x^{/} + y\right) \\ &= x^{/} y^{/} + xy\end{aligned}$$

The equivalence: $x \equiv y = x^{/} y^{/} + xy$

Therefore, the complement of EXOR is equal to the equivalence for two-variable functions. Furthermore, the EXOR is an odd function (is equal to one when the total number of 1's in the input variables is odd) [4, 5], and the equivalence is an even functions for two variables (is equal to one when the total number of 1's in the input variables is even), as described in Table 1.

Table 1. EXOR and Equivalence for two variables

x	**y**	**x ⊕ y**	**x ≡ y**
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

2.1.2. The Exclusive-OR Function for Three Variables

F1 =

$$\begin{aligned}(x \oplus y) \oplus z &= \left(x^{/} y + x y^{/}\right) \oplus z = \left(x^{/} y + x y^{/}\right)^{/} z + \left(x^{/} y + x y^{/}\right) z^{/} \\ &= \left(x^{/} y^{/} + xy\right) z + x^{/} y z^{/} + x y^{/} z^{/} \\ &= x^{/} y^{/} z + xyz + x^{/} y z^{/} + x y^{/} z^{/} \\ &= m_1 + m_2 + m_4 + m_7 \\ &= \sum (1,2,4,7) \quad \text{in sum of min. terms form [4, 5, 17].}\end{aligned}$$

2.1.3. The exclusive-OR Function for Four Variables

$$\begin{aligned}F_1 = (w \oplus x) \oplus (y \oplus z) &= \left(w^{/} x + w x^{/}\right) \oplus \left(y^{/} z + y z^{/}\right) \\ &= \left(w^{/} x + w x^{/}\right)^{/} \left(y^{/} z + y z^{/}\right) + \left(w^{/} x + w x^{/}\right)\left(y^{/} z + y z^{/}\right)^{/} \\ &= \left(w^{/} x^{/} + wx\right)\left(y^{/} z + y z^{/}\right) + \left(w^{/} x + w x^{/}\right)\left(y^{/} z^{/} + yz\right) \\ &= w^{/} x^{/} y^{/} z + w^{/} x^{/} y z^{/} + w x y^{/} z + w x y z^{/} + w^{/} x y^{/} z^{/} + w^{/} x y z + w x^{/} y^{/} z^{/} + w x^{/} y z \\ &= m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14} \\ &= \sum (1,2,4,7,8,11,13,14) \qquad [4,5]\end{aligned}$$

2.2. The Inhibition and Implication Functions for Two Variables

The inhibition and implication functions are neither commutative nor associative and are the complements of each other for two-variable functions [4, 5, 8, 10, 16] as described below:

The inhibition: $x / y = x.y^{/}$

The complement of inhibition: $(x / y)^{/} = (x.y^{/})^{/} = x^{/} + y$

The implication: $x \supset y = x^{/} + y$

Therefore, the complement of inhibition is equal to the implication for two-variable functions. Table 2, describes the Inhibition and Implication for two variables. In this paper, we propose to investigate the relationships between the complement of inhibition and the implication for three and four variable functions.

Table 2. Inhibition and Implication for two variables

x	**y**	**x / y**	x ⊃ y
0	0	0	1
0	1	0	1
1	0	1	0
1	1	0	1

3. Our Work

3.1 The Exclusive-OR and Equivalence Functions

3.1.1 The Exclusive-OR and Equivalence Functions for Three Variables

F1 = (x ⊕ y) ⊕ z = Σ (1, 2, 4, 7), as described in section 2.1.2
F2 =

$$
\begin{aligned}
(x \equiv y) \equiv z &= (x^{/} y^{/} + xy) \equiv z = (x^{/} y^{/} + xy)^{/} z^{/} + (x^{/} y^{/} + xy) \; z \\
&= (x^{/} y + xy^{/}) \; z^{/} + x^{/} y^{/} z + xyz \\
&= x^{/} yz^{/} + xy^{/} z^{/} + x^{/} y^{/} z + xyz \\
&= m_1 + m_2 + m_4 + m_7 = \sum (1,2,4,7)
\end{aligned}
$$

The truth table for these functions is described in Table 3.

Table 3. EXOR and equivalence for three variables

x	**y**	**z**	$x \oplus y \oplus z$	$x \equiv y \equiv z$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

From Table 3, we conclude that the EXOR and equivalence are equal for three-variable functions, and they are odd functions.

3.1.2 The Exclusive-OR and Equivalence Functions for Four Variables

F1 = (w ⊕ x) ⊕ (y ⊕ z) = Σ (1, 2, 4, 7, 8, 11, 13, 14), as described in section 2.1.3

$$
\begin{aligned}
F_2 = (w \equiv x) \equiv (y \equiv z) &= \left(w^{/}x^{/} + wx\right) \equiv \left(y^{/}z^{/} + yz\right) \\
&= \left(w^{/}x^{/} + wx\right)^{/}\left(y^{/}z^{/} + yz\right)^{/} + \left(w^{/}x^{/} + wx\right)\left(y^{/}z^{/} + yz\right) \\
&= \left(w^{/}x + wx^{/}\right)\left(y^{/}z + yz^{/}\right) + \left(w^{/}x^{/} + wx\right)\left(y^{/}z^{/} + yz\right) \\
&= w^{/}xy^{/}z + w^{/}xyz^{/} + wx^{/}y^{/}z + wx^{/}yz^{/} + w^{/}x^{/}y^{/}z^{/} + w^{/}x^{/}yz + wxy^{/}z^{/} + wxyz \\
&= m_0 + m_3 + m_5 + m_6 + m_9 + m_{10} + m_{12} + m_{15} \\
&= \sum(0,3,5,6,9,10,12,15) \\
F_2^{/} &= \sum(1,2,4,7,8,11,13,14)
\end{aligned}
$$

Therefore, the complement of the equivalence is equal to the EXOR for 4-variable functions as described in Table 4, and the EXOR (F_1) is an odd function, but equivalence (F_2) is an even function.

Table 4. EXOR and equivalence for four variables

w	**x**	**y**	**z**	w ⊕ x ⊕ y ⊕ z	w ≡ x ≡ y ≡ z
0	0	0	0	0	1
0	0	0	1	1	0
0	0	1	0	1	0
0	0	1	1	0	1
0	1	0	0	1	0
0	1	0	1	0	1
0	1	1	0	0	1
0	1	1	1	1	0
1	0	0	0	1	0
1	0	0	1	0	1
1	0	1	0	0	1
1	0	1	1	1	0
1	1	0	0	0	1
1	1	0	1	1	0
1	1	1	0	1	0
1	1	1	1	0	1

3.1.3 The Exclusive-OR and Equivalence Functions for Five Variables

$F_1 = v \oplus w \oplus x \oplus y \oplus z$

Where, $w \oplus x \oplus y \oplus z = \sum(1,2,4,7,8,11,13,14)$

Therefore,

$$F_1 = v \oplus (m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14})$$
$$= v(m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14})^{/} + v^{/}(m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14})$$
$$= v(m_0 + m_3 + m_5 + m_6 + m_9 + m_{10} + m_{12} + m_{15}) + v^{/}(m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14})$$
$$= (m_{16} + m_{19} + m_{21} + m_{22} + m_{25} + m_{26} + m_{28} + m_{31}) + (m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14})$$
$$= \sum(1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31)$$

Where,

$$vm_0 = vw^{/}x^{/}y^{/}z^{/} = m_{16}$$

And

$$v^{/}m_1 = v^{/}w^{/}x^{/}y^{/}z = m_1$$

$F_2 = v \equiv w \equiv x \equiv y \equiv z$

Where,

$$w \equiv x \equiv y \equiv z = \sum(0,3,5,6,9,10,12,15)$$

Therefore,

$$F_2 = v \equiv (m_0 + m_3 + m_5 + m_6 + m_9 + m_{10} + m_{12} + m_{15})$$
$$= v^{/}(m_0 + m_3 + m_5 + m_6 + m_9 + m_{10} + m_{12} + m_{15})^{/} + v\,(m_0 + m_3 + m_5 + m_6 + m_9 + m_{10} + m_{12} + m_{15})$$
$$= v^{/}(m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14}) + v\,(m_0 + m_3 + m_5 + m_6 + m_9 + m_{10} + m_{12} + m_{15})$$
$$= (m_1 + m_2 + m_4 + m_7 + m_8 + m_{11} + m_{13} + m_{14}) + (m_{16} + m_{19} + m_{21} + m_{22} + m_{25} + m_{26} + m_{28} + m_{31})$$
$$= \sum(1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31)$$

Since F1 equals F2, we conclude that the EXOR and equivalence are equal for five-variable functions, and they are odd functions, as described in Table 5.

Table 5. EXOR and equivalence for five variables

v	**w**	**x**	**y**	**z**	v ⊕ w ⊕ x ⊕ y ⊕ z	v ≡ w ≡ x ≡ y ≡ z
0	0	0	0	0	0	0
0	0	0	0	1	1	1
0	0	0	1	0	1	1
0	0	0	1	1	0	0
0	0	1	0	0	1	1
0	0	1	0	1	0	0
0	0	1	1	0	0	0
0	0	1	1	1	1	1
0	1	0	0	0	1	1
0	1	0	0	1	0	0
0	1	0	1	0	0	0
0	1	0	1	1	1	1
0	1	1	0	0	0	0
0	1	1	0	1	1	1
0	1	1	1	0	1	1
0	1	1	1	1	0	0
1	0	0	0	0	1	1
1	0	0	0	1	0	0
1	0	0	1	0	0	0

1	0	0	1	1	1	1
1	0	1	0	0	0	0
1	0	1	0	1	1	1
1	0	1	1	0	1	1
1	0	1	1	1	0	0
1	1	0	0	0	0	0
1	1	0	0	1	1	1
1	1	0	1	0	1	1
1	1	0	1	1	0	0
1	1	1	0	0	1	1
1	1	1	0	1	0	0
1	1	1	1	0	0	0
1	1	1	1	1	1	1

3.2 The Inhibition and Implication Functions

3.2.1 The Inhibition and Implication Functions for Three Variables

The inhibition:

$$(x/y)/z = (xy')/z = xy'z'$$

The complement of inhibition:

$$(xy'z')' = x' + y + z$$

The implication:

$$(x \supset y) \supset z = (x' + y) \supset z = (x' + y)' + z = xy' + z$$

Therefore, the complement of inhibition and the implication are not equal for three variables.

3.2.2 The Inhibition and Implication Functions for Four Variables

$$\begin{aligned}(w/x)/(y/z) &= (wx')/(yz') = (wx')\ (yz')' \\ &= (wx')\ (y' + z) = wx'y' + wx'z\end{aligned}$$

The complement of inhibition:

$$\begin{aligned}(wx'y' + wx'z)' &= (w' + x + y)\ (w' + x + z') \\ &= w' + w'x + w'z' + w'x + x + xz' + w'y + xy + yz' \\ &= w'(1 + x) + w'z' + x(1 + z') + w'y + xy + yz' \\ &= w' + w'z' + x + w'y + xy + yz' \\ &= w'(1 + z') + x(1 + y) + w'y + yz' \\ &= w' + x + w'y + yz' = w'(1 + y) + x + yz' \\ &= w' + x + yz'\end{aligned}$$

The implication:

$$\begin{aligned}(w \supset x) \supset (y \supset z) &= (w' + x) \supset (y' + z) \\ &= (w' + x)' + (y' + z) \\ &= wx' + y' + z\end{aligned}$$

Therefore, the complement of inhibition and the implication are not equal for four variables, and thus we conclude that the inhibition and implication functions are the complements of each other only for two variables.

3.3 Some Valid Implications

The EXOR, equivalence, and inhibition operations have the following relationships:

3.3.1 EXOR

Table 6. Valid implications for EXOR

x	**y**	**$x \oplus y$**	**$x + y$**	**$x \uparrow y$**
0	0	0	0	1
0	1	1	1	1
1	0	1	1	1
1	1	0	1	0

From Table 6, we conclude that $(x \oplus y) \supset (x + y)$, and $(x \oplus y) \supset (x \uparrow y)$ are valid.

3.3.2 Equivalence

Table 7. Valid implications for Equivalence

x	**y**	**$x \equiv y$**	**$x \supset y$**	**$y \supset x$**
0	0	1	1	1
0	1	0	1	0
1	0	0	0	1
1	1	1	1	1

From Table 7, we conclude that $(x \equiv y) \supset (x \supset y)$ and $(x \equiv y) \supset (y \supset x)$ are valid.

3.3.3 Inhibition (x / y)

Table 8. Valid implications for Inhibition (x / y)

x	**y**	**x / y**	**x**	**$x \oplus y$**	**$x + y$**	**y'**	**$y \supset x$**	**$x \uparrow y$**
0	0	0	0	0	0	1	1	1
0	1	0	0	1	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0

From Table 8, we conclude that the following are valid:

$(x / y) \supset x$
$(x / y) \supset (x \oplus y)$
$(x / y) \supset (x + y)$
$(x / y) \supset y'$
$(x / y) \supset (y \supset x)$
$(x / y) \supset (x \uparrow y)$

3.3.4 Inhibition (y / x)

Table 9. Valid implications for Inhibition (y / x)

x	y	y / x	y	$x \oplus y$	x + y	x'	$x \supset y$	$x \uparrow y$
0	0	0	0	0	0	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	1	0	0	1
1	1	0	1	0	1	0	1	0

From Table 9, we conclude that the following are valid:

$(y / x) \supset y$
$(y / x) \supset (x \oplus y)$
$(y / x) \supset (x + y)$
$(y / x) \supset x'$
$(y / x) \supset (x \supset y)$
$(y / x) \supset (x \uparrow y)$

4. CONCLUSIONS AND FUTURE WORK

In this paper we investigated the inverse relationships between the Exclusive-OR and Equivalence functions for three, four and five-variable functions. In addition, we investigated the inverse relationships between the Inhibition and Implication functions for three and four-variable functions. The results from our work showed that the EXOR and equivalence functions are the complements of each other when the number of variables in the function is even (two and four in this paper). The EXOR and equivalence functions are equals when the number of variables in the function is odd (three, and five in this paper). The EXOR is an odd function for five variables, and the equivalence is an even function for four variables and odd for three and five variables. In addition, the results from our work showed that the inhibition and implication functions are not the complements of each other for three and four-variable functions. In this paper, some useful valid implications are concluded. For EXOR, We concluded that $(x \oplus y) \supset (x + y)$, and $(x \oplus y) \supset (x \uparrow y)$ are valid. For equivalence we concluded that $(x \equiv y) \supset (x \supset y)$ and $(x \equiv y) \supset (y \supset x)$ are valid. For Inhibition (x / y), we concluded that the following relations are valid :

$(x / y) \supset x$
$(x / y) \supset (x \oplus y)$
$(x / y) \supset (x + y)$
$(x / y) \supset y'$
$(x / y) \supset (y \supset x)$
$(x / y) \supset (x \uparrow y)$,

And for Inhibition (y / x), we concluded the following relations are valid:

$(y / x) \supset y$
$(y / x) \supset (x \oplus y)$
$(y / x) \supset (x + y)$
$(y / x) \supset x'$
$(y / x) \supset (x \supset y)$
$(y / x) \supset (x \uparrow y)$

In future, we propose to investigate the relationships between the EXOR and the equivalence and the relationships between the Implication and Inhibition for more than five-variable functions.

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REFERENCES

[1] C.H. Roth, "Fundamentals of Logic Design", 4th Ed. West Publishing Company, 1992. pp. 51-53.
[2] C.H. Roth, "Fundamentals of Logic Design", 3rd Ed. West Publishing Company, 1985. pp. 48-50.
[3] F.W. John, "Digital Design: Principles and Practices". Prentice-Hall, 1994. pp. 336.
[4] M. M. Mano, "Digital Design", 3rd Ed. Prentice-Hall, 2002. pp. 51-57, 94-99.
[5] M. M. Mano, "Digital Design", 2nd Ed. Prentice-Hall, 1991. pp. 57-62, 142-148.
[6] B. N. Chatterji, "Digital Computer Technology" Khanna-Delhi, 1991. pp. 67-69.
[7] B. M. Alan, "Introduction to Logic Design". McGraw-Hill, 2002. pp. 65-70.
[8] M. M. Mano, "Computer System Architecture". Prentice-Hall, 1993. pp. 108-113.
[9] F.S. Donald and F.M. David, "Discrete Mathematics in Computer Science". Prentice-Hall, 1977. pp. 12-15.
[10] B.S. Angela, "Discrete Mathematics for Computer Science". West Publishing Company, 1987. pp. 42, 58, 59, 65.
[11] S. Anokh, "Introduction to Digital Computers". S. Chand & Company LTD, 1992. pp. 59-62.
[12] U. Kalay, M. A. Perkowski, and D. V. Hall, "A Minimal Universal Test Set for Self-Test of EXOR-Sum-of-Products Circutis". IEEE Trans. Computers, Vol. 49, no.3, pp. 267-276, Mar. 2000.
[13] R. Drechsler, H. Hengster, H. Schfer, J. Hartmann, and B. Becker, "Testability of 2-level AND/ EXOR Circuits". Proc. European Design and test Conf., Inst. of Comput. Sci. Albert Ludwigs-Univ. Freiburg, Mar. 1997.
[14] J. Saul, B. Eschermann, and J. Froessl, "Two-Level Logic Circuits Using EXOR Sums of Products". IEE Proc., Vol. 140, pp. 348-356, Nov. 1993.
[15] M.M. Mano, and R.K. Charles, "Logic and Computer Design Fundamentals", 4th Ed. Prentice-Hall, 2008. pp. 100-106.
[16] A. Saha, N. Manna, " Digital Principles & Logic Design". Jones and Bartlett Publishers, 2012. pp. 66-69.
[17] H.R. Charles, and L.L. Kinney, "Fundamentals of Logic Design", 6th Ed. Cengage Learning, 2010. pp. 64-66.
[18] B. Stephen, and V. Zvonko, "Digital Logic with VHDL Design", 3rd Ed. McGraw-Hill, 2009. pp. 139, 140.
[19] B.M. Alan, " Introduction to Logic Design", 3rd Ed. McGraw-Hill, 2010. pp. 63, 64.
[20] H.R. Kenneth, "Discrete Mathematics and Its Applications", 7th Ed. McGraw-Hill, 2011.

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