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حاول أن تساهم بفكرة، بومضة من خواطر تفكيرك العلمي، بفائدة رأيتها في إحدى المواضيع العلمية، بجانب مضيء لمحته خلف ثنايا مفهوم هندسي ما. تأكد بأنك ستلتمس الفائدة في كل خطوة تخطوها، وترى غيرك يخطوها معك ...

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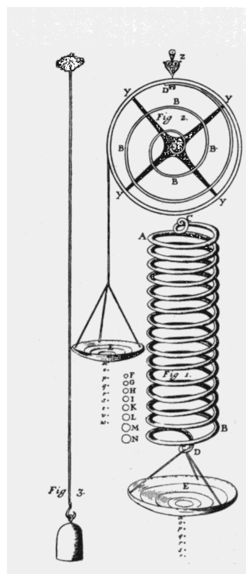
Lecture Notes

Introduction to

CONTINUUM MECHANICS

and Elements of

Elasticity/Structural Mechanics



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PREFACE

Une des questions fondamentales que l'ingénieur des Matériaux se pose est de connaître le comportement d'un materiel sous l'effet de contraintes et la cause de sa rupture. En définitive, c'est précisément la réponse à c/mat es deux questions qui vont guider le développement de nouveaux matériaux, et déterminer leur survie sous différentes conditions physiques et environnementales.

L'ingénieur en Matériaux devra donc posséder une connaissance fondamentale de la Mécanique sur le plan qualitatif, et être capable d'effectuer des simulations numériques (le plus souvent avec les Eléments Finis) et d'en extraire les résultats quantitatifs pour un problème bien posé.

Selon l'humble opinion de l'auteur, ces nobles buts sont idéalement atteints en trois étapes. Pour commencer, l'élève devra être confronté aux principes de base de la Mécanique des Milieux Continus. Une présentation détaillée des contraintes, déformations, et principes fondamentaux est essentiel. Par la suite une briefe introduction à l'Elasticité (ainsi qu'à la théorie des poutres) convaincra l'élève qu'un problème général bien posé peut avoir une solution analytique. Par contre, ceci n'est vrai (à quelques exceptions prêts) que pour des cas avec de nombreuses hypothèses qui simplifient le problème (élasticité linéaire, petites déformations, contraintes/déformations planes, ou axisymmetrie). Ainsi, la troisième et dernière étape consiste en une briefe introduction à la Mécanique des Solides, et plus précisément au Calcul Variationel. A travers la méthode des Puissances Virtuelles, et celle de Rayleigh-Ritz, l'élève sera enfin prêt à un autre cours d'éléments finis. Enfin, un sujet d'intérêt particulier aux étudiants en Matériaux a été ajouté, à savoir la Résistance Théorique des Matériaux cristallins. Ce sujet est capital pour une bonne compréhension de la rupture et servira de lien à un éventuel cours sur la Mécanique de la Rupture.

Ce polycopié a été entièrement préparé par l'auteur durant son année sabbatique à l'Ecole Polytechnique Fédérale de Lausanne, Département des Matériaux. Le cours était donné aux étudiants en deuxième année en Français.

Ce polycopié a été écrit avec les objectifs suivants. Avant tout il doit être complet et rigoureux. A tout moment, l'élève doit être à même de retrouver toutes les étapes suivies dans la dérivation d'une équation. Ensuite, en allant à travers toutes les dérivations, l'élève sera à même de bien connaître les limitations et hypothèses derrière chaque model. Enfin, la rigueur scientifique adoptée, pourra servir d'exemple à la solution d'autres problèmes scientifiques que l'étudiant pourrait être emmené à résoudre dans le futur. Ce dernier point est souvent négligé.

Le polycopié est subdivisé de façon très hiérarchique. Chaque concept est développé dans un paragraphe séparé. Ceci devrait faciliter non seulement la compréhension, mais aussi le dialogue entres élevés eux-mêmes ainsi qu'avec le Professeur.

Quand il a été jugé nécessaire, un bref rappel mathématique est introduit. De nombreux exemples sont présentés, et enfin des exercices solutionnés avec Mathematica sont présentés dans l'annexe.

L'auteur ne se fait point d'illusions quand au complet et à l'exactitude de tout le polycopié. Il a été entièrement développé durant une seule année académique, et pourrait donc bénéficier d'une révision extensive. A ce titre, corrections et critiques seront les bienvenues.

Enfin, l'auteur voudrait remercier ses élevés qui ont diligemment suivis son cours sur la Mécanique de Milieux Continus durant l'année académique 1997-1998, ainsi que le Professeur Huet qui a été son hôte au Laboratoire des Matériaux de Construction de l'EPFL durant son séjour à Lausanne.

Victor Saouma
Ecublens, Juin 1998

PREFACE

One of the most fundamental question that a Material Scientist has to ask him/herself is how a material behaves under stress, and when does it break. Ultimately, it its the answer to those two questions which would steer the development of new materials, and determine their survival in various environmental and physical conditions.

The Material Scientist should then have a thorough understanding of the fundamentals of Mechanics on the qualitative level, and be able to perform numerical simulation (most often by Finite Element Method) and extract quantitative information for a specific problem.

In the humble opinion of the author, this is best achieved in three stages. First, the student should be exposed to the basic principles of Continuum Mechanics. Detailed coverage of Stress, Strain, General Principles, and Constitutive Relations is essential. Then, a brief exposure to Elasticity (along with Beam Theory) would convince the student that a well posed problem can indeed have an analytical solution. However, this is only true for problems problems with numerous simplifying assumptions (such as linear elasticity, small deformation, plane stress/strain or axisymmetry, and resultants of stresses). Hence, the last stage consists in a brief exposure to solid mechanics, and more precisely to Variational Methods. Through an exposure to the Principle of Virtual Work, and the Rayleigh-Ritz Method the student will then be ready for Finite Elements. Finally, one topic of special interest to Material Science students was added, and that is the Theoretical Strength of Solids. This is essential to properly understand the failure of solids, and would later on lead to a Fracture Mechanics course.

These lecture notes were prepared by the author during his sabbatical year at the Swiss Federal Institute of Technology (Lausanne) in the Material Science Department. The course was offered to second year undergraduate students in French, whereas the lecture notes are in English. The notes were developed with the following objectives in mind. First they must be complete and rigorous. At any time, a student should be able to trace back the development of an equation. Furthermore, by going through all the derivations, the student would understand the limitations and assumptions behind every model. Finally, the rigor adopted in the coverage of the subject should serve as an example to the students of the rigor expected from them in solving other scientific or engineering problems. This last aspect is often forgotten.

The notes are broken down into a very hierarchical format. Each concept is broken down into a small section (a byte). This should not only facilitate comprehension, but also dialogue among the students or with the instructor.

Whenever necessary, Mathematical preliminaries are introduced to make sure that the student is equipped with the appropriate tools. Illustrative problems are introduced whenever possible, and last but not least problem set using *Mathematica* is given in the Appendix.

The author has no illusion as to the completeness or exactness of all these set of notes. They were entirely developed during a single academic year, and hence could greatly benefit from a thorough review. As such, corrections, criticisms and comments are welcome.

Finally, the author would like to thank his students who bravely put up with him and Continuum Mechanics in the AY 1997-1998, and Prof. Huet who was his host at the EPFL.

Victor E. Saouma
Ecublens, June 1998

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C.1 Variational and Differential Operators C–2

NOTATION

Symbol	Definition	Dimension	SI Unit
	SCALARS		
A	Area	L^2	m^2
c	Specific heat		
e	Volumetric strain	N.D.	-
E	Elastic Modulus	$L^{-1}MT^{-2}$	Pa
g	Specicif free enthalpy	L^2T^{-2}	JKg^{-1}
h	Film coefficient for convection heat transfer		
h	Specific enthalpy	L^2T^{-2}	JKg^{-1}
I	Moment of inertia	L^4	m^4
J	Jacobian		
K	Bulk modulus	$L^{-1}MT^{-2}$	Pa
K	Kinetic Energy	L^2MT^{-2}	J
L	Length	L	m
p	Pressure	$L^{-1}MT^{-2}$	Pa
Q	Rate of internal heat generation	L^2MT^{-3}	W
r	Radiant heat constant per unit mass per unit time	$MT^{-3}L^{-4}$	Wm^{-6}
s	Specific entropy	$L^2T^{-2}\Theta^{-1}$	$JKg^{-1}K^{-1}$
S	Entropy	$ML^2T^{-2}\Theta^{-1}$	JK^{-1}
t	Time	T	s
T	Absolute temperature	Θ	K
u	Specific internal energy	L^2T^{-2}	JKg^{-1}
U	Energy	L^2MT^{-2}	J
U^*	Complementary strain energy	L^2MT^{-2}	J
W	Work	L^2MT^{-2}	J
$\mathcal{W}$	Potential of External Work	L^2MT^{-2}	J
Π	Potential energy	L^2MT^{-2}	J
α	Coefficient of thermal expansion	Θ^{-1}	T^{-1}
μ	Shear modulus	$L^{-1}MT^{-2}$	Pa
ν	Poisson's ratio	N.D.	-
ρ	mass density	ML^{-3}	Kgm^{-3}
γ_{ij}	Shear strains	N.D.	-
$\frac{1}{2}\gamma_{ij}$	Engineering shear strain	N.D.	-
λ	Lame's coefficient	$L^{-1}MT^{-2}$	Pa
Λ	Stretch ratio	N.D.	-
μ G	Lame's coefficient	$L^{-1}MT^{-2}$	Pa
λ	Lame's coefficient	$L^{-1}MT^{-2}$	Pa
Φ	Airy Stress Function		
Ψ	(Helmholtz) Free energy	L^2MT^{-2}	J
I_σ, I_E	First stress and strain invariants		
II_σ, II_E	Second stress and strain invariants		
III_σ, III_E	Third stress and strain invariants		
Θ	Temperature	Θ	K
	TENSORS order 1		
$\mathbf{b}$	Body force per unit massLT^{-2}	NKg^{-1}	
$\mathbf{b}$	Base transformation		
$\mathbf{q}$	Heat flux per unit area	MT^{-3}	Wm^{-2}
$\mathbf{t}$	Traction vector, Stress vector	$L^{-1}MT^{-2}$	Pa
$\widehat{\mathbf{t}}$	Specified tractions along Γ_t	$L^{-1}MT^{-2}$	Pa
$\mathbf{u}$	Displacement vector	L	m

$\hat{\mathbf{u}}(x)$	Specified displacements along Γ_u	L	m
$\mathbf{u}$	Displacement vector	L	m
$\mathbf{x}$	Spatial coordinates	L	m
$\mathbf{X}$	Material coordinates	L	m
$\boldsymbol{\sigma}_0$	Initial stress vector	$L^{-1}MT^{-2}$	Pa
$\sigma_{(i)}$	Principal stresses	$L^{-1}MT^{-2}$	Pa
	TENSORS order 2		
$\mathbf{B}^{-1}$	Cauchy's deformation tensor	N.D.	-
$\mathbf{C}$	Green's deformation tensor; metric tensor, right Cauchy-Green deformation tensor	N.D.	-
$\mathbf{D}$	Rate of deformation tensor; Stretching tensor	N.D.	-
$\mathbf{E}$	Lagrangian (or Green's) finite strain tensor	N.D.	-
$\mathbf{E}^*$	Eulerian (or Almansi) finite strain tensor	N.D.	-
$\mathbf{E}'$	Strain deviator	N.D.	-
$\mathbf{F}$	Material deformation gradient	N.D.	-
$\mathbf{H}$	Spatial deformation gradient	N.D.	-
$\mathbf{I}$	Idendity matrix	N.D.	-
$\mathbf{J}$	Material displacement gradient	N.D.	-
$\mathbf{k}$	Thermal conductivity	$LMT^{-3}\Theta^{-1}$	$Wm^{-1}K^{-1}$
$\mathbf{K}$	Spatial displacement gradient	N.D.	-
$\mathbf{L}$	Spatial gradient of the velocity		
$\mathbf{R}$	Orthogonal rotation tensor		
$\mathbf{T}_0$	First Piola-Kirchoff stress tensor, Lagrangian Stress Tensor	$L^{-1}MT^{-2}$	Pa
$\tilde{\mathbf{T}}$	Second Piola-Kirchoff stress tensor	$L^{-1}MT^{-2}$	Pa
$\mathbf{U}$	Right stretch tensor		
$\mathbf{V}$	Left stretch tensor		
$\mathbf{W}$	Spin tensor, vorticity tensor. Linear lagrangian rotation tensor		
$\boldsymbol{\varepsilon}_0$	Initial strain vector		
$\mathbf{k}$	Conductivity		
$\boldsymbol{\kappa}$	Curvature		
$\boldsymbol{\sigma}, \mathbf{T}$	Cauchy stress tensor	$L^{-1}MT^{-2}$	Pa
$\mathbf{T}'$	Deviatoric stress tensor	$L^{-1}MT^{-2}$	Pa
$\boldsymbol{\Omega}$	Linear Eulerian rotation tensor		
$\boldsymbol{\omega}$	Linear Eulerian rotation vector		
	TENSORS order 4		
$\mathbf{D}$	Constitutive matrix	$L^{-1}MT^{-2}$	Pa
	CONTOURS, SURFACES, VOLUMES		
$\mathcal{C}$	Contour line		
S	Surface of a body	L^2	m^2
Γ	Surface	L^2	m^2
Γ_t	Boundary along which surface tractions, $\mathbf{t}$ are specified	L^2	m^2
Γ_u	Boundary along which displacements, $\mathbf{u}$ are specified	L^2	m^2
Γ_T	Boundary along which temperatures, T are specified	L^2	m^2
Γ_c	Boundary along which convection flux, q_c are specified	L^2	m^2
Γ_q	Boundary along which flux, q_n are specified	L^2	m^2
Ω, V	Volume of body	L^3	m^3
	FUNCTIONS, OPERATORS		

$\tilde{u}$	Neighbour function to $u(x)$
δ	Variational operator
$\mathbf{L}$	Linear differential operator relating displacement to strains
$\nabla\phi$	Divergence, (gradient operator) on scalar $\lfloor \frac{\partial\phi}{\partial x} \quad \frac{\partial\phi}{\partial y} \quad \frac{\partial\phi}{\partial z} \rfloor^T$
$\boldsymbol{\nabla}\cdot\mathbf{u}$	Divergence, (gradient operator) on vector (div . $\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$
$\boldsymbol{\nabla}^2$	Laplacian Operator

Part I

CONTINUUM MECHANICS

Chapter 1

MATHEMATICAL PRELIMINARIES; Part I Vectors and Tensors

1 Physical laws should be independent of the position and orientation of the observer. For this reason, physical laws are **vector equations** or **tensor equations**, since both vectors and tensors transform from one coordinate system to another in such a way that if the law holds in one coordinate system, it holds in any other coordinate system.

1.1 Vectors

2 A vector is a directed line segment which can denote a variety of quantities, such as position of point with respect to another (**position vector**), a force, or a traction.

3 A vector may be defined with respect to a particular coordinate system by specifying the **components** of the vector in that system. The choice of the coordinate system is arbitrary, but some are more suitable than others (axes corresponding to the major direction of the object being analyzed).

4 The **rectangular Cartesian coordinate system** is the most often used one (others are the cylindrical, spherical or curvilinear systems). The rectangular system is often represented by three mutually perpendicular axes $Oxyz$, with corresponding **unit vector triad** $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (or $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) such that:

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}; \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}; \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}; \tag{1.1-a}$$

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \tag{1.1-b}$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \tag{1.1-c}$$

Such a set of base vectors constitutes an **orthonormal basis**.

5 An arbitrary vector $\mathbf{v}$ may be expressed by

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \tag{1.2}$$

where

$$v_x = \mathbf{v}\cdot\mathbf{i} = v\cos\alpha \tag{1.3-a}$$

$$v_y = \mathbf{v}\cdot\mathbf{j} = v\cos\beta \tag{1.3-b}$$

$$v_z = \mathbf{v}\cdot\mathbf{k} = v\cos\gamma \tag{1.3-c}$$

are the projections of $\mathbf{v}$ onto the coordinate axes, Fig. 1.1.

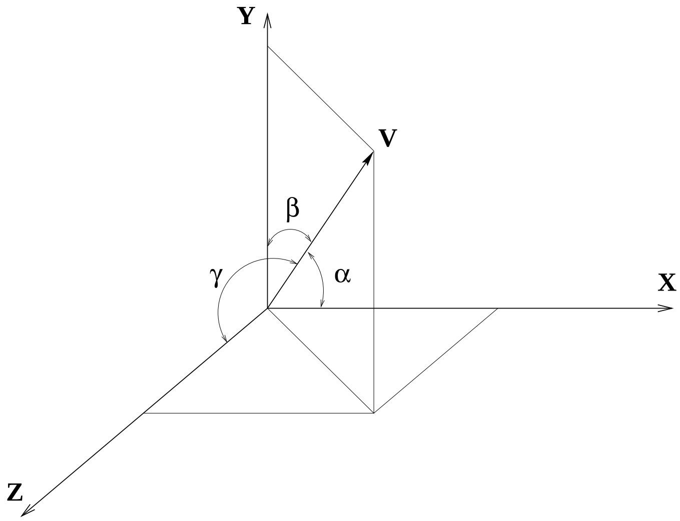


Figure 1.1: Direction Cosines (to be corrected)

6 The unit vector in the direction of **v** is given by

$$\mathbf{e}_v = \frac{\mathbf{v}}{v} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k} \tag{1.4}$$

Since **v** is arbitrary, it follows that any unit vector will have **direction cosines** of that vector as its **Cartesian components**.

7 The length or more precisely the magnitude of the vector is denoted by $\parallel \mathbf{v} \parallel = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

8 We will denote the **contravariant components** of a vector by superscripts v^k, and its **covariant components** by subscripts v_k (the significance of those terms will be clarified in Sect. 1.1.2.1.

1.1.1 Operations

Addition: of two vectors $\mathbf{a}+\mathbf{b}$ is geometrically achieved by connecting the tail of the vector **b** with the head of **a**, Fig. 1.2. Analytically the sum vector will have components $\lfloor\ a_1 + b_1 \quad a_2 + b_2 \quad a_3 + b_3\ \rfloor$.

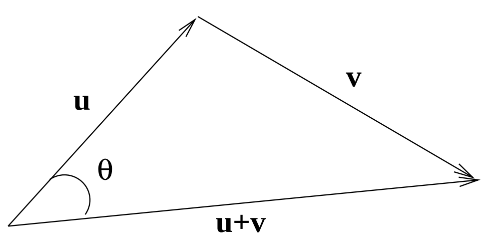


Figure 1.2: Vector Addition

Scalar multiplication: $\alpha\mathbf{a}$ will scale the vector into a new one with components $\lfloor\ \alpha a_1 \quad \alpha a_2 \quad \alpha a_3\ \rfloor$.

Vector Multiplications of **a** and **b** comes in three varieties:

Dot Product (or scalar product) is a scalar quantity which relates not only to the lengths of the vector, but also to the angle between them.

$$\boxed{\mathbf{a}\cdot\mathbf{b} \equiv \| \mathbf{a} \| \| \mathbf{b} \| \cos\theta(\mathbf{a},\mathbf{b}) = \sum_{i=1}^{3} a_i b_i} \quad (1.5)$$

where $\cos\theta(\mathbf{a},\mathbf{b})$ is the cosine of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. The dot product measures the relative orientation between two vectors.

The dot product is both *commutative*

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \quad (1.6)$$

and *distributive*

$$\alpha\mathbf{a}\cdot(\beta\mathbf{b} + \gamma\mathbf{c}) = \alpha\beta(\mathbf{a}\cdot\mathbf{b}) + \alpha\gamma(\mathbf{a}\cdot\mathbf{c}) \quad (1.7)$$

The dot product of $\mathbf{a}$ with a unit vector $\mathbf{n}$ gives the projection of $\mathbf{a}$ in the direction of $\mathbf{n}$.

The dot product of base vectors gives rise to the definition of the **Kronecker delta** defined as

$$\boxed{\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}} \quad (1.8)$$

where

$$\boxed{\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}} \quad (1.9)$$

Cross Product (or vector product) $\mathbf{c}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the vector

$$\boxed{\mathbf{c} = \mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3} \quad (1.10)$$

which can be remembered from the determinant expansion of

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (1.11)$$

and is equal to the area of the parallelogram described by $\mathbf{a}$ and $\mathbf{b}$, Fig. 1.3.

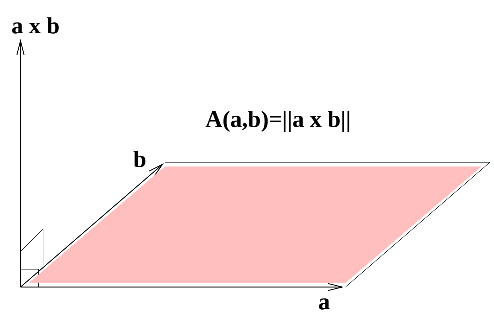


Figure 1.3: Cross Product of Two Vectors

$$\boxed{A(\mathbf{a},\mathbf{b}) = \| \mathbf{a}\times\mathbf{b} \|} \quad (1.12)$$

The cross product is not commutative, but satisfies the condition of **skew symmetry**

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \tag{1.13}$$

The cross product is distributive

$$\alpha\mathbf{a}\times(\beta\mathbf{b}+\gamma\mathbf{c}) = \alpha\beta(\mathbf{a}\times\mathbf{b}) + \alpha\gamma(\mathbf{a}\times\mathbf{c}) \tag{1.14}$$

Triple Scalar Product: of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is desgnated by $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and it corresponds to the (scalar) volume defined by the three vectors, Fig. 1.4.

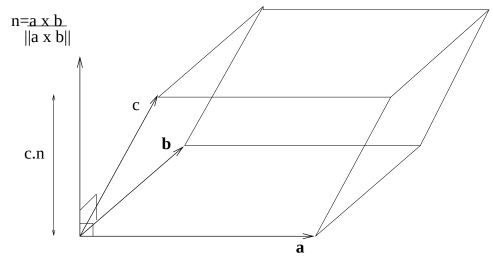


Figure 1.4: Cross Product of Two Vectors

$$V(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \tag{1.15}$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.16}$$

The triple scalar product of base vectors represents a fundamental operation

$$(\mathbf{e}_i\times\mathbf{e}_j)\cdot\mathbf{e}_k = \varepsilon_{ijk} \equiv \begin{cases} 1 & \text{if } (i,j,k) \text{ are in cyclic order} \\ 0 & \text{if any of } (i,j,k) \text{ are equal} \\ -1 & \text{if } (i,j,k) \text{ are in acyclic order} \end{cases} \tag{1.17}$$

The scalars ε_{ijk} is the **permutation tensor**. A cyclic permutation of 1,2,3 is $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$, an acyclic one would be $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$. Using this notation, we can rewrite

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} \Rightarrow c_i = \varepsilon_{ijk}a_jb_k \tag{1.18}$$

Vector Triple Product is a cross product of two vectors, one of which is itself a cross product.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \mathbf{d} \tag{1.19}$$

and the product vector $\mathbf{d}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

1.1.2 Coordinate Transformation

1.1.2.1 †General Tensors

9 Let us consider two bases $\mathbf{b}_j(x_1, x_2, x_3)$ and $\overline{\mathbf{b}}_j(\overline{x}_1, \overline{x}_2\overline{x}_3)$, Fig. 1.5. Each unit vector in one basis must be a linear combination of the vectors of the other basis

$$\overline{\mathbf{b}}_j = a^p_j\mathbf{b}_p \;\; \text{and} \;\; \mathbf{b}_k = b^k_q\overline{\mathbf{b}}_q \tag{1.20}$$

(summed on p and q respectively) where a_j^p (subscript new, superscript old) and b_q^k are the coefficients for the forward and backward changes respectively from $\overline{\mathbf{b}}$ to $\mathbf{b}$ respectively. Explicitly

$$\left\{\begin{array}{c}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{array}\right\}=\left[\begin{array}{ccc}b_1^1 & b_2^1 & b_3^1\\ b_1^2 & b_2^2 & b_3^2\\ b_1^3 & b_2^3 & b_3^3\end{array}\right]\left\{\begin{array}{c}\overline{\mathbf{e}}_1\\\overline{\mathbf{e}}_2\\\overline{\mathbf{e}}_3\end{array}\right\} \text{ and } \left\{\begin{array}{c}\overline{\mathbf{e}}_1\\\overline{\mathbf{e}}_2\\\overline{\mathbf{e}}_3\end{array}\right\}=\left[\begin{array}{ccc}a_1^1 & a_1^2 & a_1^3\\ a_2^1 & a_2^2 & a_2^3\\ a_3^1 & a_3^2 & a_3^3\end{array}\right]\left\{\begin{array}{c}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{array}\right\} \tag{1.21}$$

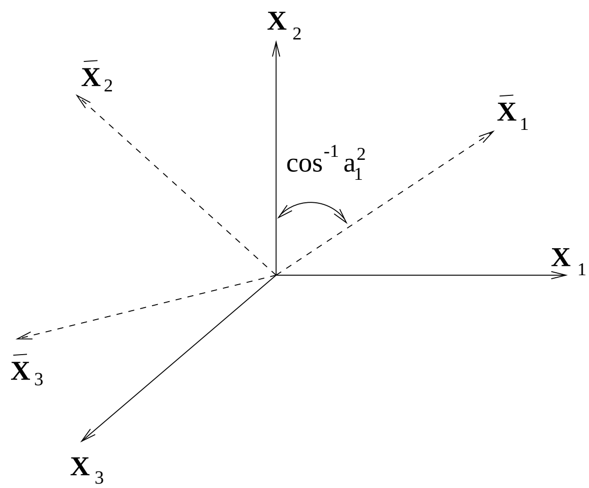


Figure 1.5: Coordinate Transformation

10 The transformation must have the determinant of its **Jacobian**

$$\boxed{J=\left|\begin{array}{ccc}\frac{\partial\overline{x}^1}{\partial x^1} & \frac{\partial\overline{x}^1}{\partial x^2} & \frac{\partial\overline{x}^1}{\partial x^3}\\ \frac{\partial\overline{x}^2}{\partial x^1} & \frac{\partial\overline{x}^2}{\partial x^2} & \frac{\partial\overline{x}^2}{\partial x^3}\\ \frac{\partial\overline{x}^3}{\partial x^1} & \frac{\partial\overline{x}^3}{\partial x^2} & \frac{\partial\overline{x}^3}{\partial x^3}\end{array}\right|\neq 0} \tag{1.22}$$

different from zero (the superscript is a label and not an exponent).

11 It is important to note that so far, the coordinate systems are completely general and may be Cartesian, curvilinear, spherical or cylindrical.

1.1.2.1.1 †Contravariant Transformation

12 The vector representation in both systems must be the same

$$\mathbf{v}=\overline{v}^q\overline{\mathbf{b}}_q=v^k\mathbf{b}_k=v^k(b_k^q\overline{\mathbf{b}}_q)\Rightarrow(\overline{v}^q-v^kb_k^q)\overline{\mathbf{b}}_q=\mathbf{0} \tag{1.23}$$

since the base vectors $\overline{\mathbf{b}}_q$ are linearly independent, the coefficients of $\overline{\mathbf{b}}_q$ must all be zero hence

$$\boxed{\overline{v}^q=b_k^qv^k \text{ and inversely } v^p=a_j^p\overline{v}^j} \tag{1.24}$$

showing that the forward change from components v^k to $\overline{v}^q$ used the coefficients b_k^q of the backward change from base $\overline{\mathbf{b}}_q$ to the original $\mathbf{b}_k$. This is why these components are called **contravariant**.

13 Generalizing, a **Contravariant Tensor of order one** (recognized by the use of the superscript) transforms a set of quantities r^k associated with point P in x^k through a coordinate transformation into

a new set $\overline{r}^q$ associated with $\overline{x}^q$

$$\boxed{\overline{r}^q = \underbrace{\frac{\partial \overline{x}^q}{\partial x^k}}_{b^q_k} r^k} \tag{1.25}$$

14 By extension, the **Contravariant tensors of order two** requires the tensor components to obey the following transformation law

$$\boxed{\overline{r}^{ij} = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s} r^{rs}} \tag{1.26}$$

1.1.2.1.2 Covariant Transformation

15 Similarly to Eq. 1.24, a **covariant component transformation** (recognized by subscript) will be defined as

$$\boxed{\overline{v}_j = a^p_j v_p \quad \text{and inversely} \quad v_k = b^k_q \overline{v}_q} \tag{1.27}$$

We note that contrarily to the contravariant transformation, the covariant transformation uses the same transformation coefficients as the ones for the base vectors.

16 Finally transformation of tensors of order one and two is accomplished through

$$\overline{r}_q = \frac{\partial x^k}{\partial \overline{x}^q} r_k \tag{1.28}$$

$$\overline{r}_{ij} = \frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j} r_{rs} \tag{1.29}$$

1.1.2.2 Cartesian Coordinate System

17 If we consider two different sets of cartesian orthonormal coordinate systems $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \overline{\mathbf{e}}_3\}$, any vector $\mathbf{v}$ can be expressed in one system or the other

$$\mathbf{v} = v_j \mathbf{e}_j = \overline{v}_j \overline{\mathbf{e}}_j \tag{1.30}$$

18 To determine the relationship between the two sets of components, we consider the dot product of $\mathbf{v}$ with one (any) of the base vectors

$$\overline{\mathbf{e}}_i \cdot \mathbf{v} = \overline{v}_i = v_j(\overline{\mathbf{e}}_i \cdot \mathbf{e}_j) \tag{1.31}$$

(since $\overline{v}_j(\overline{\mathbf{e}}_j \cdot \overline{\mathbf{e}}_i) = \overline{v}_j \delta_{ij} = \overline{v}_i$)

19 We can thus define the nine scalar values

$$\boxed{a^j_i \equiv \overline{\mathbf{e}}_i \cdot \mathbf{e}_j = \cos(\overline{x}_i, x_j)} \tag{1.32}$$

which arise from the dot products of base vectors as the **direction cosines**. (Since we have an orthonormal system, those values are nothing else than the cosines of the angles between the nine pairing of base vectors.)

20 Thus, one set of vector components can be expressed in terms of the other through a **covariant transformation** similar to the one of Eq. 1.27.

$$\boxed{\begin{array}{rcll} \overline{v}_j &=& a_j^p v_p & (1.33) \\ v_k &=& b_q^k \overline{v}_q & (1.34) \end{array}}$$

we note that the free index in the first and second equations appear on the upper and lower index respectively.

21 Because of the orthogonality of the unit vector we have $a_p^s a_q^s = \delta_{pq}$ and $a_r^m a_r^n = \delta_{mn}$.

22 As a further illustration of the above derivation, let us consider the transformation of a vector $\mathbf{V}$ from $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ coordinate system to $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, Fig. 1.6:

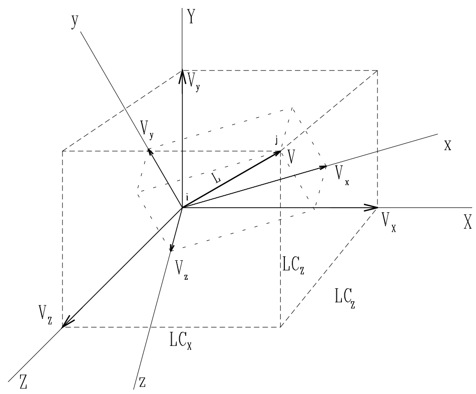


Figure 1.6: Arbitrary 3D Vector Transformation

23 Eq. 1.33 would then result in

$$V_x = a_x^X V_X + a_x^Y V_Y + a_x^Z V_Z \tag{1.35}$$

or

$$\left\{ \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right\} = \left[\begin{array}{ccc} a_x^X & a_x^Y & a_x^Z \\ a_y^X & a_y^Y & a_y^Z \\ a_z^X & a_z^Y & a_z^Z \end{array} \right] \left\{ \begin{array}{c} V_X \\ V_Y \\ V_Z \end{array} \right\} \tag{1.36}$$

and a_i^j is the direction cosine of axis i with respect to axis j

- $a_x^j = (a_x X, a_x^Y, a_x^Z)$ direction cosines of $\mathbf{x}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$
- $a_y^j = (a_y X, a_y^Y, a_y^Z)$ direction cosines of $\mathbf{y}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$
- $a_z^j = (a_z X, a_z^Y, a_z^Z)$ direction cosines of $\mathbf{z}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$

24 Finally, for the 2D case and from Fig. 1.7, the transformation matrix is written as

$$T = \left[\begin{array}{cc} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{array} \right] = \left[\begin{array}{cc} \cos\alpha & \cos\beta \\ \cos\gamma & \cos\alpha \end{array} \right] \tag{1.37}$$

but since $\gamma = \frac{\pi}{2} + \alpha$, and $\beta = \frac{\pi}{2} - \alpha$, then $\cos\gamma = -\sin\alpha$ and $\cos\beta = \sin\alpha$, thus the transformation matrix becomes

$$T = \left[\begin{array}{cc} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{array} \right] \tag{1.38}$$

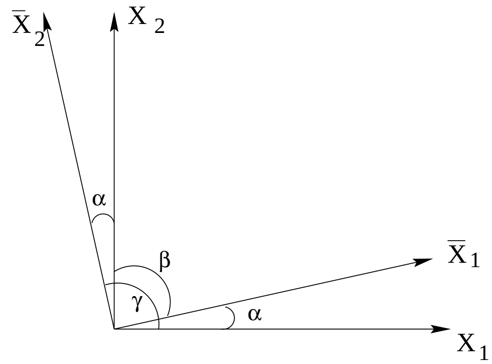


Figure 1.7: Rotation of Orthonormal Coordinate System

1.2 Tensors

25 We now seek to generalize the concept of a vector by introducing the **tensor** (**T**), which essentially *exists to operate on vectors* **v** *to produce other vectors* (or on tensors to produce other tensors!). We designate this operation by **T·v** or simply **Tv**.

26 We hereby adopt the **dyadic** notation for tensors as **linear vector operators**

$$\mathbf{u} = \mathbf{T}\cdot\mathbf{v} \quad \text{or} \quad u_i = T_{ij}v_j \tag{1.39-a}$$
$$\mathbf{u} = \mathbf{v}\cdot\mathbf{S} \quad \text{where} \quad \mathbf{S} = \mathbf{T}^T \tag{1.39-b}$$

27 † In general the vectors may be represented by either covariant or contravariant components v_j or v^j. Thus we can have different types of linear transformations

$$\begin{aligned} u_i &= T_{ij}v^j; \quad u^i = T^{ij}v_j \\ u_i &= T_i^{\cdot j}v_j; \quad u^i = T^i_{\cdot j}v^j \end{aligned} \tag{1.40}$$

involving the **covariant components** T_{ij}, the **contravariant components** T^{ij} and the **mixed components** $T^i_{\cdot j}$ or $T_i^{\cdot j}$.

28 Whereas a tensor is essentially an operator on vectors (or other tensors), it is also a physical quantity, independent of any particular coordinate system yet specified most conveniently by referring to an appropriate system of coordinates.

29 Tensors frequently arise as physical entities whose components are the coefficients of a linear relationship between vectors.

30 A tensor is classified by the rank or order. A Tensor of order zero is specified in any coordinate system by one coordinate and is a scalar. A tensor of order one has three coordinate components in space, hence it is a vector. In general 3-D space the number of components of a tensor is 3^n where n is the order of the tensor.

31 A force and a stress are tensors of order 1 and 2 respectively.

1.2.1 Indicial Notation

32 Whereas the Engineering notation may be the simplest and most intuitive one, it often leads to long and repetitive equations. Alternatively, the tensor and the dyadic form will lead to shorter and more compact forms.

33 While working on general relativity, Einstein got tired of writing the summation symbol with its range of summation below and above (such as $\sum_{i=1}^{n=3} a_{ij} b_i$) and noted that most of the time the upper range (n) was equal to the dimension of space (3 for us, 4 for him), and that when the summation involved a product of two terms, the summation was over a repeated index (i in our example). Hence, he decided that there is no need to include the summation sign $\sum$ if there was repeated indices (i), and thus any repeated index is a **dummy index** and is summed over the range 1 to 3. An index that is not repeated is called **free index** and assumed to take a value from 1 to 3.

34 Hence, this so called **indicial notation** is also referred to **Einstein's notation**.

35 The following rules define indicial notation:

1. If there is one letter index, that index goes from i to n (range of the tensor). For instance:

$$a_i = a^i = \lfloor\ a_1 \quad a_2 \quad a_3\ \rfloor = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\} \qquad i = 1, 3 \tag{1.41}$$

   assuming that $n = 3$.

2. A repeated index will take on all the values of its range, and the resulting tensors summed. For instance:

$$a_{1i} x_i = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \tag{1.42}$$

3. Tensor's order:

   - First order tensor (such as force) has only one free index:

$$a_i = a^i = \lfloor\ a_1 \quad a_2 \quad a_3\ \rfloor \tag{1.43}$$

     other first order tensors $a_{ij} b_j$, F_{ikk}, $\varepsilon_{ijk} u_j v_k$
   - Second order tensor (such as stress or strain) will have two free indeces.

$$D_{ij} = \left[\begin{array}{ccc} D_{11} & D_{22} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{array} \right] \tag{1.44}$$

     other examples A_{ijip}, $\delta_{ij} u_k v_k$.
   - A fourth order tensor (such as Elastic constants) will have four free indeces.

4. Derivatives of tensor with respect to x_i is written as $,i$. For example:

$$\frac{\partial \Phi}{\partial x_i} = \Phi_{,i} \quad \frac{\partial v_i}{\partial x_i} = v_{i,i} \quad \frac{\partial v_i}{\partial x_j} = v_{i,j} \quad \frac{\partial T_{i,j}}{\partial x_k} = T_{i,j,k} \tag{1.45}$$

36 Usefulness of the indicial notation is in presenting systems of equations in compact form. For instance:

$$x_i = c_{ij} z_j \tag{1.46}$$

this simple compacted equation, when expanded would yield:

$$\begin{array}{rcl} x_1 & = & c_{11} z_1 + c_{12} z_2 + c_{13} z_3 \\ x_2 & = & c_{21} z_1 + c_{22} z_2 + c_{23} z_3 \\ x_3 & = & c_{31} z_1 + c_{32} z_2 + c_{33} z_3 \end{array} \tag{1.47-a}$$

Similarly:

$$A_{ij} = B_{ip} C_{jq} D_{pq} \tag{1.48}$$

$$\begin{aligned}
A_{11} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\
A_{12} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\
A_{21} &= B_{21}C_{11}D_{11} + B_{21}C_{12}D_{12} + B_{22}C_{11}D_{21} + B_{22}C_{12}D_{22} \\
A_{22} &= B_{21}C_{21}D_{11} + B_{21}C_{22}D_{12} + B_{22}C_{21}D_{21} + B_{22}C_{22}D_{22} \qquad (1.49\text{-a})
\end{aligned}$$

37 Using indicial notation, we may rewrite the definition of the dot product

$$\boxed{\mathbf{a}\cdot\mathbf{b} = a_i b_i} \qquad (1.50)$$

and of the cross product

$$\boxed{\mathbf{a}\times\mathbf{b} = \varepsilon_{pqr} a_q b_r \mathbf{e}_p} \qquad (1.51)$$

we note that in the second equation, there is one free index p thus there are three equations, there are two repeated (dummy) indices q and r, thus each equation has nine terms.

1.2.2 Tensor Operations

1.2.2.1 Sum

38 The sum of two (second order) tensors is simply defined as:

$$\boxed{\mathbf{S}_{ij} = \mathbf{T}_{ij} + \mathbf{U}_{ij}} \qquad (1.52)$$

1.2.2.2 Multiplication by a Scalar

39 The multiplication of a (second order) tensor by a scalar is defined by:

$$\boxed{\mathbf{S}_{ij} = \lambda \mathbf{T}_{ij}} \qquad (1.53)$$

1.2.2.3 Contraction

40 In a contraction, we make two of the indeces equal (or in a mixed tensor, we make a ubscript equal to the superscript), thus producing a tensor of order two less than that to which it is applied. For example:

$$\begin{array}{rcllcl}
T_{ij} & \rightarrow & T_{ii}; & 2 & \rightarrow & 0 \\
u_i v_j & \rightarrow & u_i v_i; & 2 & \rightarrow & 0 \\
A^{mr}_{..sn} & \rightarrow & A^{mr}_{..sm} = B^{r}_{.s}; & 4 & \rightarrow & 2 \\
E_{ij} a_k & \rightarrow & E_{ij} a_i = c_j; & 3 & \rightarrow & 1 \\
A^{mpr}_{qs} & \rightarrow & A^{mpr}_{qr} = B^{mp}_{q}; & 5 & \rightarrow & 3
\end{array} \qquad (1.54)$$

1.2.2.4 Products

1.2.2.4.1 Outer Product

41 The outer product of two tensors (not necessarily of the same type or order) is a set of tensor components obtained simply by writing the components of the two tensors beside each other with no repeated indices (that is by multiplying each component of one of the tensors by every component of the other). For example

$$a_i b_j = T_{ij} \quad (1.55\text{-a})$$
$$A^i B_j^{.k} = C^{i.k}.j \quad (1.55\text{-b})$$
$$v_i T_{jk} = S_{ijk} \quad (1.55\text{-c})$$

1.2.2.4.2 Inner Product

42 The inner product is obtained from an outer product by contraction involving one index from each tensor. For example

$$a_i b_j \rightarrow a_i b_i \quad (1.56\text{-a})$$
$$a_i E_{jk} \rightarrow a_i E_{ik} = f_k \quad (1.56\text{-b})$$
$$E_{ij} F_{km} \rightarrow E_{ij} F_{jm} = G_{im} \quad (1.56\text{-c})$$
$$A^i B_i^{.k} \rightarrow A^i B_i^{.k} = D^k \quad (1.56\text{-d})$$

1.2.2.4.3 Scalar Product

43 The scalar product of two tensors is defined as

$$\boxed{\mathbf{T}:\mathbf{U} = T_{ij}U_{ij}} \quad (1.57)$$

in any rectangular system.

44 The following **inner-product** axioms are satisfied:

$$\mathbf{T}:\mathbf{U} = \mathbf{U}:\mathbf{T} \quad (1.58\text{-a})$$
$$\mathbf{T}:(\mathbf{U}+\mathbf{V}) = \mathbf{T}:\mathbf{U}+\mathbf{T}:\mathbf{V} \quad (1.58\text{-b})$$
$$\alpha(\mathbf{T}:\mathbf{U}) = (\alpha\mathbf{T}):\mathbf{U} = \mathbf{T}:(\alpha\mathbf{U}) \quad (1.58\text{-c})$$
$$\mathbf{T}:\mathbf{T} > 0 \text{ unless } \mathbf{T} = \mathbf{0} \quad (1.58\text{-d})$$

1.2.2.4.4 Tensor Product

45 Since a tensor primary objective is to operate on vectors, the tensor product of two vectors provides a fundamental building block of second-order tensors and will be examined next.

46 The **Tensor Product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a second order tensor $\mathbf{u} \otimes \mathbf{v}$ which in turn operates on an arbitrary vector $\mathbf{w}$ as follows:

$$\boxed{[\mathbf{u} \otimes \mathbf{v}]\mathbf{w} \equiv (\mathbf{v}\cdot\mathbf{w})\mathbf{u}} \tag{1.59}$$

In other words when the tensor product $\mathbf{u} \otimes \mathbf{v}$ operates on $\mathbf{w}$ (left hand side), the result (right hand side) is a vector that points along the direction of $\mathbf{u}$, and has length equal to $(\mathbf{v}\cdot\mathbf{w})||\mathbf{u}||$, or the original length of $\mathbf{u}$ times the dot (scalar) product of $\mathbf{v}$ and $\mathbf{w}$.

47 Of particular interest is the tensor product of the base vectors $\mathbf{e}_i \otimes \mathbf{e}_j$. With three base vectors, we have a set of nine second order tensors which provide a suitable basis for expressing the components of a tensor. Again, we started with base vectors which themselves provide a basis for expressing any vector, and now the tensor product of base vectors in turn provides a formalism to express the components of a tensor.

48 The **second order tensor T** can be expressed in terms of its components T_{ij} relative to the base tensors $\mathbf{e}_i \otimes \mathbf{e}_j$ as follows:

$$\mathbf{T} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}[\mathbf{e}_i \otimes \mathbf{e}_j] \tag{1.60-a}$$

$$\mathbf{T}\mathbf{e}_k = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\,[\mathbf{e}_i \otimes \mathbf{e}_j]\,\mathbf{e}_k \tag{1.60-b}$$

$$[\mathbf{e}_i \otimes \mathbf{e}_j]\,\mathbf{e}_k = (\mathbf{e}_j\cdot\mathbf{e}_k)\mathbf{e}_i = \delta_{jk}\mathbf{e}_i \tag{1.60-c}$$

$$\mathbf{T}\mathbf{e}_k = \sum_{i=1}^{3} T_{ik}\mathbf{e}_i \tag{1.60-d}$$

Thus T_{ik} is the ith component of $\mathbf{T}\mathbf{e}_k$. We can thus define the tensor component as follows

$$\boxed{T_{ij} = \mathbf{e}_i\cdot\mathbf{T}\mathbf{e}_j} \tag{1.61}$$

49 Now we can see how the second order tensor $\mathbf{T}$ operates on any vector $\mathbf{v}$ by examining the components of the resulting vector $\mathbf{T}\mathbf{v}$:

$$\mathbf{T}\mathbf{v} = \left[\sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}[\mathbf{e}_i \otimes \mathbf{e}_j]\right]\left(\sum_{k=1}^{3} v_k\mathbf{e}_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} T_{ij}v_k[\mathbf{e}_i \otimes \mathbf{e}_j]\mathbf{e}_k \tag{1.62}$$

which when combined with Eq. 1.60-c yields

$$\mathbf{T}\mathbf{v} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}v_j\mathbf{e}_i \tag{1.63}$$

which is clearly a vector. The ith component of the vector $\mathbf{T}\mathbf{v}$ being

$$(\mathbf{T}\mathbf{v})_i = \sum_{i=1}^{3} T_{ij}v_j \tag{1.64}$$

50 The identity tensor $\mathbf{I}$ leaves the vector unchanged $\mathbf{I}\mathbf{v} = \mathbf{v}$ and is equal to

$$\boxed{\mathbf{I} \equiv \mathbf{e}_i \otimes \mathbf{e}_i} \tag{1.65}$$

51 A simple example of a tensor and its operation on vectors is the **projection** tensor $\mathbf{P}$ which generates the projection of a vector v on the plane characterized by a normal $\mathbf{n}$:

$$\mathbf{P} \equiv \mathbf{I} - \mathbf{n} \otimes \mathbf{n} \tag{1.66}$$

the action of $\mathbf{P}$ on $\mathbf{v}$ gives $\mathbf{Pv} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$. To convince ourselves that the vector $\mathbf{Pv}$ lies on the plane, its dot product with $\mathbf{n}$ must be zero, accordingly $\mathbf{Pv}\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} - (\mathbf{v}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{n}) = 0\surd$.

1.2.2.5 Product of Two Second-Order Tensors

52 The product of two tensors is defined as

$$\boxed{\mathbf{P} = \mathbf{T}\cdot\mathbf{U}; \qquad P_{ij} = T_{ik}U_{kj}} \tag{1.67}$$

in any rectangular system.

53 The following axioms hold

$$\begin{aligned}
(\mathbf{T}\cdot\mathbf{U})\cdot\mathbf{R} &= \mathbf{T}\cdot(\mathbf{U}\cdot\mathbf{R}) && \text{(1.68-a)}\\
\mathbf{T}\cdot(\mathbf{R}+\mathbf{U}) &= \mathbf{T}\cdot\mathbf{R}+\mathbf{t}\cdot\mathbf{U} && \text{(1.68-b)}\\
(\mathbf{R}+\mathbf{U})\cdot\mathbf{T} &= \mathbf{R}\cdot\mathbf{T}+\mathbf{U}\cdot\mathbf{T} && \text{(1.68-c)}\\
\alpha(\mathbf{T}\cdot\mathbf{U}) &= (\alpha\mathbf{T})\cdot\mathbf{U} = \mathbf{T}\cdot(\alpha\mathbf{U}) && \text{(1.68-d)}\\
\mathbf{1T} &= \mathbf{T}\cdot\mathbf{1} = \mathbf{T} && \text{(1.68-e)}
\end{aligned}$$

Note again that some authors omit the dot.

Finally, the operation is not commutative

1.2.3 Dyads

54 The *indeterminate vector product* of $\mathbf{a}$ and $\mathbf{b}$ defined by writing the two vectors in juxtaposition as $\mathbf{ab}$ is called a **dyad**. A **dyadic D** corresponds to a tensor of order two and is a linear combination of dyads:

$$\mathbf{D} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 \cdots \mathbf{a}_n\mathbf{b}_n \tag{1.69}$$

The **conjugate dyadic** of $\mathbf{D}$ is written as

$$\mathbf{D}_c = \mathbf{b}_1\mathbf{a}_1 + \mathbf{b}_2\mathbf{a}_2 \cdots \mathbf{b}_n\mathbf{a}_n \tag{1.70}$$

1.2.4 Rotation of Axes

55 The rule for changing second order tensor components under rotation of axes goes as follow:

$$\begin{aligned}
\overline{u}_i &= a_i^j u_j && \text{From Eq. 1.33}\\
&= a_i^j T_{jq} v_q && \text{From Eq. 1.39-a}\\
&= a_i^j T_{jq} a_p^q \overline{v}_p && \text{From Eq. 1.33}
\end{aligned} \tag{1.71}$$

But we also have $\overline{u}_i = \overline{T}_{ip}\overline{v}_p$ (again from Eq. 1.39-a) in the barred system, equating these two expressions we obtain

$$\overline{T}_{ip} - (a_i^j a_p^q T_{jq})\overline{v}_p = 0 \tag{1.72}$$

hence

$$\boxed{\begin{aligned}
\overline{T}_{ip} &= a_i^j a_p^q T_{jq} \text{ in Matrix Form } [\overline{T}] = [A]^T[T][A] && (1.73)\\
T_{jq} &= a_i^j a_p^q \overline{T}_{ip} \text{ in Matrix Form } [T] = [A][\overline{T}][A]^T && (1.74)
\end{aligned}}$$

By extension, higher order tensors can be similarly transformed from one coordinate system to another.

56 If we consider the 2D case, From Eq. 1.38

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.75\text{-a})$$

$$T = \begin{bmatrix} T_{xx} & T_{xy} & 0 \\ T_{xy} & T_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.75\text{-b})$$

$$\overline{T} = A^T T A = \begin{bmatrix} \overline{T}_{xx} & \overline{T}_{xy} & 0 \\ \overline{T}_{xy} & \overline{T}_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.75\text{-c})$$

$$= \begin{bmatrix} \cos^2\alpha T_{xx} + \sin^2\alpha T_{yy} + \sin 2\alpha T_{xy} & \frac{1}{2}(-\sin 2\alpha T_{xx} + \sin 2\alpha T_{yy} + 2\cos 2\alpha T_{xy} & 0 \\ \frac{1}{2}(-\sin 2\alpha T_{xx} + \sin 2\alpha T_{yy} + 2\cos 2\alpha T_{xy} & \sin^2\alpha T_{xx} + \cos\alpha(\cos\alpha T_{yy} - 2\sin\alpha T_{xy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1.75\text{-d})$$

alternatively, using $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, this last equation can be rewritten as

$$\boxed{\begin{Bmatrix} \overline{T}_{xx} \\ \overline{T}_{yy} \\ \overline{T}_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} T_{xx} \\ T_{yy} \\ T_{xy} \end{Bmatrix}} \quad (1.76)$$

1.2.5 Trace

57 The **trace** of a second-order tensor, denoted tr $\mathbf{T}$ is a scalar invariant function of the tensor and is defined as

$$\boxed{\text{tr } \mathbf{T} \equiv T_{ii}} \quad (1.77)$$

Thus it is equal to the sum of the diagonal elements in a matrix.

1.2.6 Inverse Tensor

58 An inverse tensor is simply defined as follows

$$\boxed{\mathbf{T}^{-1}(\mathbf{T}\mathbf{v}) = \mathbf{v} \text{ and } \mathbf{T}(\mathbf{T}^{-1}\mathbf{v}) = \mathbf{v}} \quad (1.78)$$

alternatively $\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$, or $T^{-1}_{ik}T_{kj} = \delta_{ij}$ and $T_{ik}T^{-1}_{kj} = \delta_{ij}$

1.2.7 Principal Values and Directions of Symmetric Second Order Tensors

59 Since the two fundamental tensors in continuum mechanics are of the second order and symmetric (stress and strain), we examine some important properties of these tensors.

60 For every symmetric tensor T_{ij} defined at some point in space, there is associated with each direction (specified by unit normal n_j) at that point, a vector given by the inner product

$$v_i = T_{ij}n_j \quad (1.79)$$

If the direction is one for which v_i is **parallel** to n_i, the inner product may be expressed as

$$T_{ij} n_j = \lambda n_i \tag{1.80}$$

and the direction n_i is called **principal direction** of T_{ij}. Since $n_i = \delta_{ij} n_j$, this can be rewritten as

$$(T_{ij} - \lambda \delta_{ij}) n_j = 0 \tag{1.81}$$

which represents a system of three equations for the four unknowns n_i and λ.

$$\begin{aligned} (T_{11} - \lambda) n_1 + T_{12} n_2 + T_{13} n_3 &= 0 \\ T_{21} n_1 + (T_{22} - \lambda) n_2 + T_{23} n_3 &= 0 \\ T_{31} n_1 + T_{32} n_2 + (T_{33} - \lambda) n_3 &= 0 \end{aligned} \tag{1.82-a}$$

To have a non-trivial slution ($n_i = 0$) the determinant of the coefficients must be zero,

$$\boxed{|T_{ij} - \lambda \delta_{ij}| = 0} \tag{1.83}$$

61 Expansion of this determinant leads to the following **characteristic equation**

$$\boxed{\lambda^3 - I_T \lambda^2 + II_T \lambda - III_T = 0} \tag{1.84}$$

the roots are called the **principal values** of T_{ij} and

$$I_T = T_{ij} = \text{tr } T_{ij} \tag{1.85}$$

$$II_T = \frac{1}{2}(T_{ii} T_{jj} - T_{ij} T_{ij}) \tag{1.86}$$

$$III_T = |T_{ij}| = \det T_{ij} \tag{1.87}$$

are called the first, second and third **invariants** respectively of T_{ij}.

62 It is customary to order those roots as $\lambda_1 > \lambda_2 > \lambda_3$

63 For a symmetric tensor with real components, the principal values are also real. If those values are distinct, the three principal directions are mutually orthogonal.

1.2.8 Powers of Second Order Tensors; Hamilton-Cayley Equations

64 When expressed in term of the principal axes, the tensor array can be written in matrix form as

$$\mathcal{T} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 \\ 0 & \lambda_{(2)} & 0 \\ 0 & 0 & \lambda_{(3)} \end{bmatrix} \tag{1.88}$$

65 By direct matrix multiplication, the quare of the tensor T_{ij} is given by the inner product $T_{ik} T_{kj}$, the cube as $T_{ik} T_{km} T_{mn}$. Therefore the nth power of T_{ij} can be written as

$$\mathcal{T}^n = \begin{bmatrix} \lambda^n_{(1)} & 0 & 0 \\ 0 & \lambda^n_{(2)} & 0 \\ 0 & 0 & \lambda^n_{(3)} \end{bmatrix} \tag{1.89}$$

Since each of the principal values satisfies Eq. 1.84 and because the diagonal matrix form of $\mathcal{T}$ given above, then the tensor itself will satisfy Eq. 1.84.

$$\boxed{\mathcal{T}^3 - I_T\mathcal{T}^2 + II_T\mathcal{T} - III_T\mathcal{I} = 0} \tag{1.90}$$

where $\mathcal{I}$ is the identity matrix. This equation is called the **Hamilton-Cayley equation**.

Chapter 2

KINETICS

Or How Forces are Transmitted

2.1 Force, Traction and Stress Vectors

1 There are two kinds of **forces** in continuum mechanics

body forces: act on the elements of volume or mass inside the body, e.g. gravity, electromagnetic fields. $d\mathbf{F} = \rho \mathbf{b} dVol$.

surface forces: are contact forces acting on the free body at its bounding surface. Those will be defined in terms of **force per unit area.**

2 The surface force per unit area acting on an element dS is called **traction** or more accurately **stress vector**.

$$\int_S \mathbf{t} dS = \mathbf{i} \int_S t_x dS + \mathbf{j} \int_S t_y dS + \mathbf{k} \int_S t_z dS \tag{2.1}$$

Most authors limit the term traction to an actual bounding surface of a body, and use the term **stress vector** for an imaginary interior surface (even though the state of stress is a tensor and not a vector).

3 The traction vectors on planes perpendicular to the coordinate axes are particularly useful. When the vectors acting at a point on three such mutually perpendicular planes is given, the **stress vector** at that point on any other arbitrarily inclined plane can be expressed in terms of the first set of tractions.

4 A **stress**, Fig 2.1 is a second order cartesian tensor, σ_{ij} where the 1st subscript (i) refers to the direction of outward facing normal, and the second one (j) to the direction of component force.

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \left\{ \begin{array}{c} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{array} \right\} \tag{2.2}$$

5 In fact the nine rectangular components σ_{ij} of $\boldsymbol{\sigma}$ turn out to be the three sets of three vector components $(\sigma_{11}, \sigma_{12}, \sigma_{13})$, $(\sigma_{21}, \sigma_{22}, \sigma_{23})$, $(\sigma_{31}, \sigma_{32}, \sigma_{33})$ which correspond to

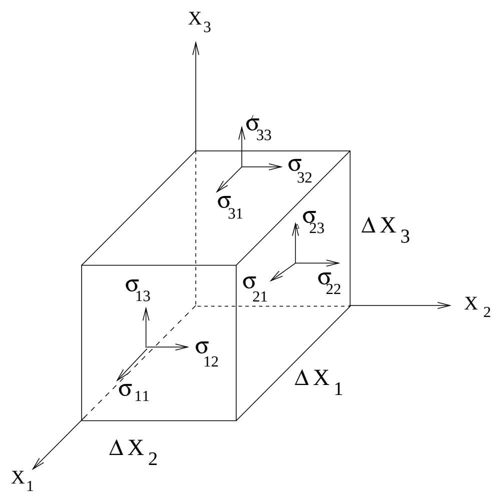


Figure 2.1: Stress Components on an Infinitesimal Element

the three tractions $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ which are acting on the x_1, x_2 and x_3 faces (It should be noted that those tractions are not necesarily normal to the faces, and they can be decomposed into a normal and shear traction if need be). In other words, stresses are nothing else than the components of tractions (stress vector), Fig. 2.2.

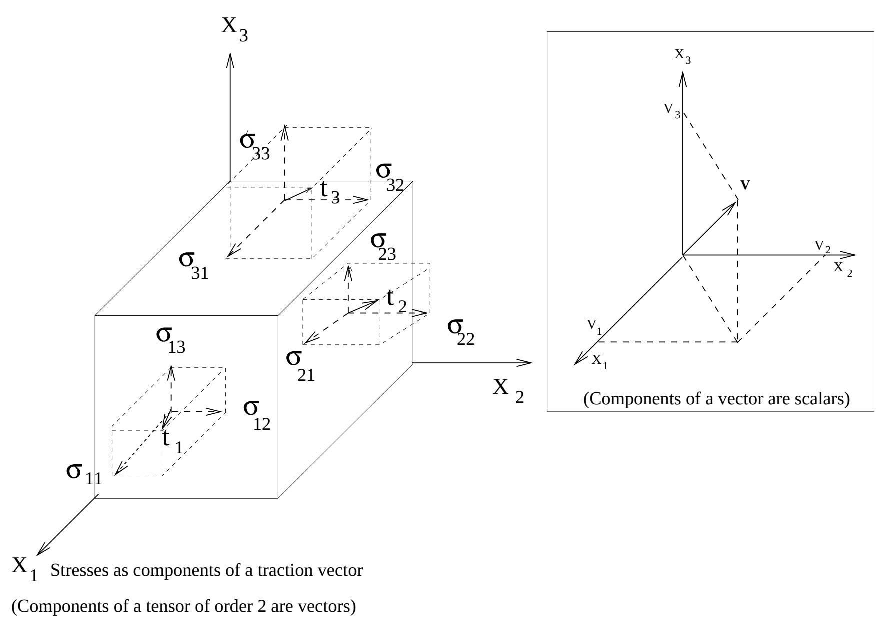


Figure 2.2: Stresses as Tensor Components

6 The state of stress at a point cannot be specified entirely by a single vector with three components; it requires the second-order tensor with all nine components.

2.2 Traction on an Arbitrary Plane; Cauchy's Stress Tensor

7 Let us now consider the problem of determining the traction acting on the surface of an oblique plane (characterized by its normal $\mathbf{n}$) in terms of the known tractions normal to the three principal axis, $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$. This will be done through the so-called Cauchy's tetrahedron shown in Fig. 2.3.

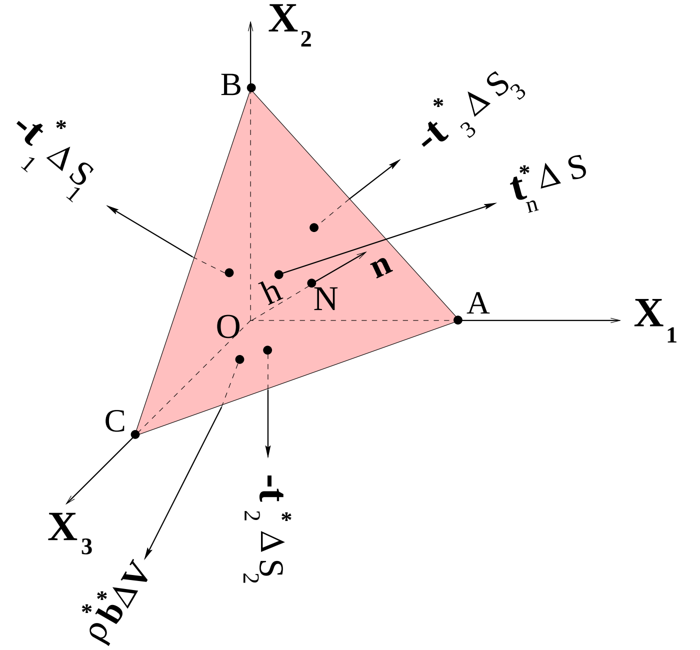


Figure 2.3: Cauchy's Tetrahedron

8 The components of the unit vector $\mathbf{n}$ are the direction cosines of its direction:

$$n_1 = \cos(\angle AON); \quad n_2 = \cos(\angle BON); \quad n_3 = \cos(\angle CON); \tag{2.3}$$

The altitude ON, of length h is a leg of the three right triangles ANO, BNO and CNO with hypothenuses OA, OB and OC. Hence

$$h = OAn_1 = OBn_2 = OCn_3 \tag{2.4}$$

9 The volume of the tetrahedron is one third the base times the altitude

$$\Delta V = \frac{1}{3}h\Delta S = \frac{1}{3}OA\Delta S_1 = \frac{1}{3}OB\Delta S_2 = \frac{1}{3}OC\Delta S_3 \tag{2.5}$$

which when combined with the preceding equation yields

$$\Delta S_1 = \Delta S n_1; \quad \Delta S_2 = \Delta S n_2; \quad \Delta S_3 = \Delta S n_3; \tag{2.6}$$

or $\Delta S_i = \Delta S n_i$.

10 In Fig. 2.3 are also shown the *average* values of the body force and of the surface tractions (thus the asterix). The negative sign appears because $\mathbf{t}_i^*$ denotes the average

traction on a surface whose outward normal points in the negative x_i direction. We seek to determine $\mathbf{t}_n^*$.

11 We invoke the **momentum principle of a collection of particles** (more about it later on) which is postulated to apply to our idealized continuous medium. This principle states that the vector sum of all external forces acting on the free body is equal to the rate of change of the total momentum[1]. The total momentum is $\int_{\Delta m} \mathbf{v} dm$. By the mean-value theorem of the integral calculus, this is equal to $\mathbf{v}^* \Delta m$ where $\mathbf{v}^*$ is average value of the velocity. Since we are considering the momentum of a given collection of particles, Δm does not change with time and $\Delta m \frac{d\mathbf{v}^*}{dt} = \rho^* \Delta V \frac{d\mathbf{v}^*}{dt}$ where ρ^* is the average density. Hence, the momentum principle yields

$$\mathbf{t}_n^* \Delta S + \rho^* \mathbf{b}^* \Delta V - \mathbf{t}_1^* \Delta S_1 - \mathbf{t}_2^* \Delta S_2 - \mathbf{t}_3^* \Delta S_3 = \rho^* \Delta V \frac{d\mathbf{v}^*}{dt} \tag{2.7}$$

Substituting for ΔV, ΔS_i from above, dividing throughout by ΔS and rearanging we obtain

$$\mathbf{t}_n^* + \frac{1}{3} h \rho^* \mathbf{b}^* = \mathbf{t}_1^* n_1 + \mathbf{t}_2^* n_2 + \mathbf{t}_3^* n_3 + \frac{1}{3} h \rho^* \frac{d\mathbf{v}}{dt} \tag{2.8}$$

and now we let $h \to 0$ and obtain

$$\boxed{\mathbf{t}_n = \mathbf{t}_1 n_1 + \mathbf{t}_2 n_2 + \mathbf{t}_3 n_3 = \mathbf{t}_i n_i} \tag{2.9}$$

We observe that we dropped the asterix as the length of the vectors approached zero.

12 It is important to note that this result was obtained without any assumption of equilibrium and that it applies as well in fluid dynamics as in solid mechanics.

13 This equation is a vector equation, and the corresponding algebraic equations for the components of $\mathbf{t}_n$ are

$$\boxed{\begin{array}{rrcl} & t_{n_1} & = & \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\ & t_{n_2} & = & \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\ & t_{n_3} & = & \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \\ \text{Indicial notation} & t_{n_i} & = & \sigma_{ji} n_j \\ \text{dyadic notation} & \mathbf{t}_n & = & \mathbf{n} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \cdot \mathbf{n} \end{array}} \tag{2.10}$$

14 We have thus established that the nine components σ_{ij} are components of the second order tensor, **Cauchy's stress tensor**.

15 Note that this stress tensor is really defined in the deformed space (Eulerian), and this issue will be revisited in Sect. 4.3.

■ Example 2-1: Stress Vectors

[1] This is really **Newton's second law** $\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$

if the stress tensor at point P is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{Bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{Bmatrix} \tag{2.11}$$

We seek to determine the traction (or stress vector) $\mathbf{t}$ passing through P and parallel to the plane ABC where $A(4,0,0)$, $B(0,2,0)$ and $C(0,0,6)$. **Solution:**
The vector normal to the plane can be found by taking the cross products of vectors AB and AC:

$$\mathbf{N} = AB \times AC = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -4 & 2 & 0 \\ -4 & 0 & 6 \end{vmatrix} \tag{2.12-a}$$

$$= 12\mathbf{e}_1 + 24\mathbf{e}_2 + 8\mathbf{e}_3 \tag{2.12-b}$$

The unit normal of N is given by

$$\mathbf{n} = \frac{3}{7}\mathbf{e}_1 + \frac{6}{7}\mathbf{e}_2 + \frac{2}{7}\mathbf{e}_3 \tag{2.13}$$

Hence the stress vector (traction) will be

$$\lfloor \tfrac{3}{7} \quad \tfrac{6}{7} \quad \tfrac{2}{7} \rfloor \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \lfloor -\tfrac{9}{7} \quad \tfrac{5}{7} \quad \tfrac{10}{7} \rfloor \tag{2.14}$$

and thus $\mathbf{t} = -\frac{9}{7}\mathbf{e}_1 + \frac{5}{7}\mathbf{e}_2 + \frac{10}{7}\mathbf{e}_3$ ■

2.3 Symmetry of Stress Tensor

16 From Fig. 2.1 the **resultant force** exerted on the positive X_1 face is

$$\lfloor \sigma_{11}\Delta X_2 \Delta X_3 \quad \sigma_{12}\Delta X_2 \Delta X_3 \quad \sigma_{13}\Delta X_2 \Delta X_3 \rfloor \tag{2.15}$$

similarly the resultant forces acting on the positive X_2 face are

$$\lfloor \sigma_{21}\Delta X_3 \Delta X_1 \quad \sigma_{22}\Delta X_3 \Delta X_1 \quad \sigma_{23}\Delta X_3 \Delta X_1 \rfloor \tag{2.16}$$

17 We now consider **moment equilibrium** ($\mathbf{M} = \mathbf{F} \times \mathbf{d}$). The stress is homogeneous, and the normal force on the opposite side is equal opposite and colinear. The moment $(\Delta X_2/2)\sigma_{31}\Delta X_1 \Delta X_2$ is likewise balanced by the moment of an equal component in the opposite face. Finally similar argument holds for σ_{32}.

18 The net moment about the X_3 axis is thus

$$M = \Delta X_1(\sigma_{12}\Delta X_2 \Delta X_3) - \Delta X_2(\sigma_{21}\Delta X_3 \Delta X_1) \tag{2.17}$$

which must be zero, hence $\sigma_{12} = \sigma_{21}$.

19 We generalize and conclude that in the absence of distributed body forces, the stress matrix is symmetric,

$$\boxed{\sigma_{ij} = \sigma_{ji}} \tag{2.18}$$

20 A more rigorous proof of the symmetry of the stress tensor will be given in Sect. 6.3.2.1.

2.3.1 Cauchy's Reciprocal Theorem

21 If we consider $\mathbf{t}_1$ as the traction vector on a plane with normal $\mathbf{n}_1$, and $\mathbf{t}_2$ the stress vector at the same point on a plane with normal $\mathbf{n}_2$, then

$$\mathbf{t}_1 = \mathbf{n}_1 \cdot \boldsymbol{\sigma} \quad \text{and} \quad \mathbf{t}_2 = \mathbf{n}_2 \boldsymbol{\sigma} \tag{2.19}$$

or in matrix form as

$$\{t_1\} = \lfloor n_1 \rfloor [\sigma] \quad \text{and} \quad \{t_2\} = \lfloor n_2 \rfloor [\sigma] \tag{2.20}$$

If we postmultiply the first equation by n_2 and the second one by n_1, by virtue of the symmetry of $[\sigma]$ we have

$$[n_1 \sigma] n_2 = [n_2 \sigma] n_1 \tag{2.21}$$

or

$$\boxed{\mathbf{t}_1 \cdot \mathbf{n}_2 = \mathbf{t}_2 \cdot \mathbf{n}_1} \tag{2.22}$$

22 In the special case of two opposite faces, this reduces to

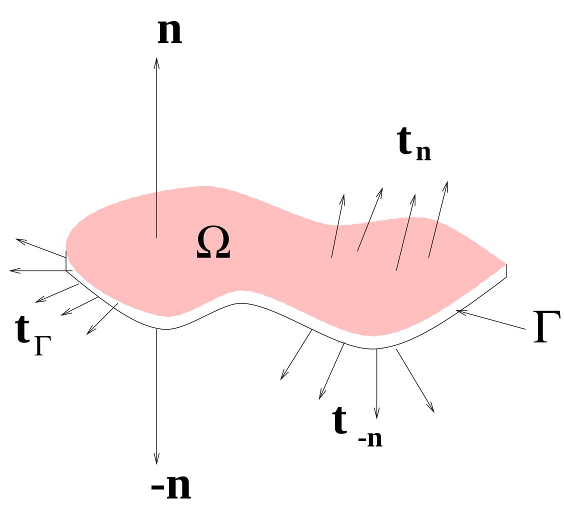


Figure 2.4: Cauchy's Reciprocal Theorem

$$\boxed{\mathbf{t}_n = -\mathbf{t}_{-n}} \tag{2.23}$$

23 We should note that this theorem is analogous to Newton's famous third law of motion *To every action there is an equal and opposite reaction.*

2.4 Principal Stresses

24 Regardless of the state of stress (as long as the stress tensor is symmetric), at a given point, it is always possible to choose a special set of axis through the point so that the shear stress components vanish when the stress components are referred to this system of axis. these special axes are called **principal axes** of the **principal stresses**.

25 To determine the principal directions at any point, we consider **n** to be a unit vector in one of the unknown directions. It has components n_i. Let λ represent the principal-stress component on the plane whose normal is **n** (note both **n** and λ are yet unknown). Since we know that there is no shear stress component on the plane perpendicular to **n**,

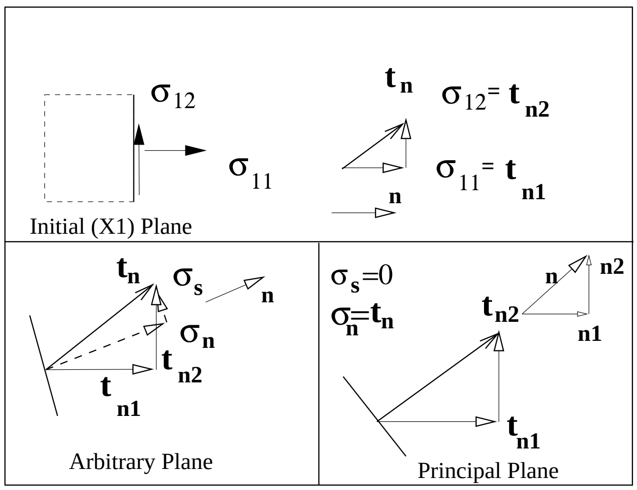


Figure 2.5: Principal Stresses

the stress vector on this plane must be parallel to **n** and

$$\mathbf{t}_n = \lambda \mathbf{n} \tag{2.24}$$

26 From Eq. 2.10 and denoting the stress tensor by $\boldsymbol{\sigma}$ we get

$$\mathbf{n}\cdot\boldsymbol{\sigma} = \lambda \mathbf{n} \tag{2.25}$$

in indicial notation this can be rewritten as

$$n_r \sigma_{rs} = \lambda n_s \tag{2.26}$$

or

$$(\sigma_{rs} - \lambda \delta_{rs}) n_r = 0 \tag{2.27}$$

in matrix notation this corresponds to

$$n\left([\boldsymbol{\sigma}] - \lambda[\mathbf{I}]\right) = 0 \tag{2.28}$$

where I corresponds to the identity matrix. We really have here a set of three homogeneous algebraic equations for the direction cosines n_i.

27 Since the direction cosines must also satisfy

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{2.29}$$

they can not all be zero. hence Eq.2.28 has solutions which are not zero if and only if the determinant of the coefficients is equal to zero, i.e

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0 \tag{2.30}$$

$$|\sigma_{rs} - \lambda\delta_{rs}| = 0 \tag{2.31}$$

$$|\boldsymbol{\sigma} - \lambda\mathbf{I}| = 0 \tag{2.32}$$

28 For a given set of the nine stress components, the preceding equation constitutes a cubic equation for the three unknown magnitudes of λ.

29 Cauchy was first to show that since the matrix is symmetric and has real elements, the roots are all real numbers.

30 The three lambdas correspond to the three principal stresses $\sigma_{(1)} > \sigma_{(2)} > \sigma_{(3)}$. When any one of them is substituted for λ in the three equations in Eq. 2.28 those equations reduce to only two independent linear equations, which must be solved together with the quadratic Eq. 2.29 to determine the direction cosines n_r^i of the normal $\mathbf{n}^i$ to the plane on which σ_i acts.

31 The three directions form a right-handed system and

$$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2 \tag{2.33}$$

32 In 2D, it can be shown that the principal stresses are given by:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{2.34}$$

2.4.1 Invariants

33 The principal stresses are physical quantities, whose values do not depend on the coordinate system in which the components of the stress were initially given. They are therefore **invariants** of the stress state.

34 When the determinant in the characteristic Eq. 2.32 is expanded, the cubic equation takes the form

$$\boxed{\lambda^3 - I_\sigma\lambda^2 - II_\sigma\lambda - III_\sigma = 0} \tag{2.35}$$

where the symbols I_σ, II_σ and III_σ denote the following scalar expressions in the stress components:

$$\begin{aligned}
I_\sigma &= \sigma_{11}+\sigma_{22}+\sigma_{33}=\sigma_{ii}=\text{tr}\,\boldsymbol{\sigma} && (2.36)\\
II_\sigma &= -(\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11})+\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2 && (2.37)\\
&= \frac{1}{2}(\sigma_{ij}\sigma_{ij}-\sigma_{ii}\sigma_{jj})=\frac{1}{2}\sigma_{ij}\sigma_{ij}-\frac{1}{2}I_\sigma^2 && (2.38)\\
&= \frac{1}{2}(\boldsymbol{\sigma}:\boldsymbol{\sigma}-I_\sigma^2) && (2.39)\\
III_\sigma &= \det\boldsymbol{\sigma}=\frac{1}{6}e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr} && (2.40)
\end{aligned}$$

35 In terms of the principal stresses, those invariants can be simplified into

$$\begin{aligned}
I_\sigma &= \sigma_{(1)}+\sigma_{(2)}+\sigma_{(3)} && (2.41)\\
II_\sigma &= -(\sigma_{(1)}\sigma_{(2)}+\sigma_{(2)}\sigma_{(3)}+\sigma_{(3)}\sigma_{(1)}) && (2.42)\\
III_\sigma &= \sigma_{(1)}\sigma_{(2)}\sigma_{(3)} && (2.43)
\end{aligned}$$

2.4.2 Spherical and Deviatoric Stress Tensors

36 If we let σ denote the mean normal stress p

$$\sigma=-p=\frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})=\frac{1}{3}\sigma_{ii}=\frac{1}{3}\text{tr}\,\boldsymbol{\sigma} \tag{2.44}$$

then the stress tensor can be written as the sum of two tensors:

Hydrostatic stress in which each normal stress is equal to $-p$ and the shear stresses are zero. The hydrostatic stress produces volume change without change in shape in an isotropic medium.

$$\sigma_{hyd}=-p\mathbf{I}=\begin{bmatrix}-p & 0 & 0\\ 0 & -p & 0\\ 0 & 0 & -p\end{bmatrix} \tag{2.45}$$

Deviatoric Stress: which causes the change in shape.

$$\sigma_{dev}=\begin{bmatrix}\sigma_{11}-\sigma & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma\end{bmatrix} \tag{2.46}$$

2.5 Stress Transformation

37 From Eq. 1.73 and 1.74, the stress transformation for the second order stress tensor is given by

$$\begin{aligned}
\overline{\sigma}_{ip} &= a_i^j a_p^q \sigma_{jq} \quad \text{in Matrix Form} \quad [\overline{\sigma}]=[A]^T[\sigma][A] && (2.47)\\
\sigma_{jq} &= a_i^j a_p^q \overline{\sigma}_{ip} \quad \text{in Matrix Form} \quad [\sigma]=[A][\overline{\sigma}][A]^T && (2.48)
\end{aligned}$$

38 For the 2D plane stress case we rewrite Eq. 1.76

$$\left\{ \begin{array}{c} \overline{\sigma}_{xx} \\ \overline{\sigma}_{yy} \\ \overline{\sigma}_{xy} \end{array} \right\} = \left[\begin{array}{ccc} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos\alpha\sin\alpha & \cos^2\alpha - \sin^2\alpha \end{array} \right] \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} \tag{2.49}$$

■ Example 2-2: Principal Stresses

The stress tensor is given at a point by

$$\boldsymbol{\sigma} = \left[\begin{array}{ccc} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{array} \right] \tag{2.50}$$

determine the principal stress values and the corresponding directions.
Solution:
From Eq.2.32 we have

$$\left| \begin{array}{ccc} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{array} \right| = 0 \tag{2.51}$$

Or upon expansion (and simplification) $(\lambda+2)(\lambda-4)(\lambda-1) = 0$, thus the roots are $\sigma_{(1)} = 4$, $\sigma_{(2)} = 1$ and $\sigma_{(3)} = -2$. We also note that those are the three **eigenvalues** of the stress tensor.

If we let $\overline{x}_1$ axis be the one corresponding to the direction of $\sigma_{(3)}$ and n_i^3 be the direction cosines of this axis, then from Eq. 2.28 we have

$$\left\{ \begin{array}{rcl} (3+2)n_1^3 + n_2^3 + n_3^3 & = & 0 \\ n_1^3 + 2n_2^3 + 2n_3^3 & = & 0 \\ n_1^3 + 2n_2^3 + 2n_3^3 & = & 0 \end{array} \right. \Rightarrow n_1^3 = 0; \quad n_2^3 = \frac{1}{\sqrt{2}}; \quad n_3^3 = -\frac{1}{\sqrt{2}} \tag{2.52}$$

Similarly If we let $\overline{x}_2$ axis be the one corresponding to the direction of $\sigma_{(2)}$ and n_i^2 be the direction cosines of this axis,

$$\left\{ \begin{array}{rcl} 2n_1^2 + n_2^2 + n_3^2 & = & 0 \\ n_1^2 - n_2^2 + 2n_3^2 & = & 0 \\ n_1^2 + 2n_2^2 - n_3^2 & = & 0 \end{array} \right. \Rightarrow n_1^2 = \frac{1}{\sqrt{3}}; \quad n_2^2 = -\frac{1}{\sqrt{3}}; \quad n_3^2 = -\frac{1}{\sqrt{3}} \tag{2.53}$$

Finally, if we let $\overline{x}_3$ axis be the one corresponding to the direction of $\sigma_{(1)}$ and n_i^1 be the direction cosines of this axis,

$$\left\{ \begin{array}{rcl} -n_1^1 + n_2^1 + n_3^1 & = & 0 \\ n_1^1 - 4n_2^1 + 2n_3^1 & = & 0 \\ n_1^1 + 2n_2^1 - 4n_3^1 & = & 0 \end{array} \right. \Rightarrow n_1^1 = -\frac{2}{\sqrt{6}}; \quad n_2^1 = -\frac{1}{\sqrt{6}}; \quad n_3^1 = -\frac{1}{\sqrt{6}} \tag{2.54}$$

Finally, we can convince ourselves that the two stress tensors have the same invariants I_σ, II_σ and III_σ. ■

■ Example 2-3: Stress Transformation

Show that the transformation tensor of direction cosines previously determined transforms the original stress tensor into the diagonal principal axes stress tensor.
Solution:

From Eq. 2.47

$$\overline{\boldsymbol{\sigma}} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \quad (2.55\text{-a})$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad (2.55\text{-b})$$

■

2.5.1 Plane Stress

39 Plane stress conditions prevail when $\sigma_{3i} = 0$, and thus we have a biaxial stress field.

40 Plane stress condition prevail in (relatively) thin plates, i.e when one of the dimensions is much smaller than the other two.

2.5.2 Mohr's Circle for Plane Stress Conditions

41 The Mohr circle will provide a graphical mean to contain the transformed state of stress $(\overline{\sigma}_{xx}, \overline{\sigma}_{yy}, \overline{\sigma}_{xy})$ at an arbitrary plane (inclined by α) in terms of the original one $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$.

42 Substituting

$$\begin{array}{rclrcl} \cos^2\alpha & = & \frac{1+\cos 2\alpha}{2} & \sin^2\alpha & = & \frac{1-\cos 2\alpha}{2} \\ \cos 2\alpha & = & \cos^2\alpha - \sin^2\alpha & \sin 2\alpha & = & 2\sin\alpha\cos\alpha \end{array} \quad (2.56)$$

into Eq. 2.49 and after some algebraic manipulation we obtain

$$\overline{\sigma}_{xx} = \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\alpha + \sigma_{xy}\sin 2\alpha \quad (2.57\text{-a})$$

$$\overline{\sigma}_{xy} = \sigma_{xy}\cos 2\alpha - \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\alpha \quad (2.57\text{-b})$$

43 Points $(\sigma_{xx}, \sigma_{xy}), (\sigma_{xx}, 0), (\sigma_{yy}, 0)$ and $[(\sigma_{xx}+\sigma_{yy})/2, 0]$ are plotted in the stress representation of Fig. 2.6. Then we observe that

$$\frac{1}{2}(\sigma_{xx}-\sigma_{yy}) = R\cos 2\beta \quad (2.58\text{-a})$$

$$\sigma_{xy} = R\sin 2\beta \quad (2.58\text{-b})$$

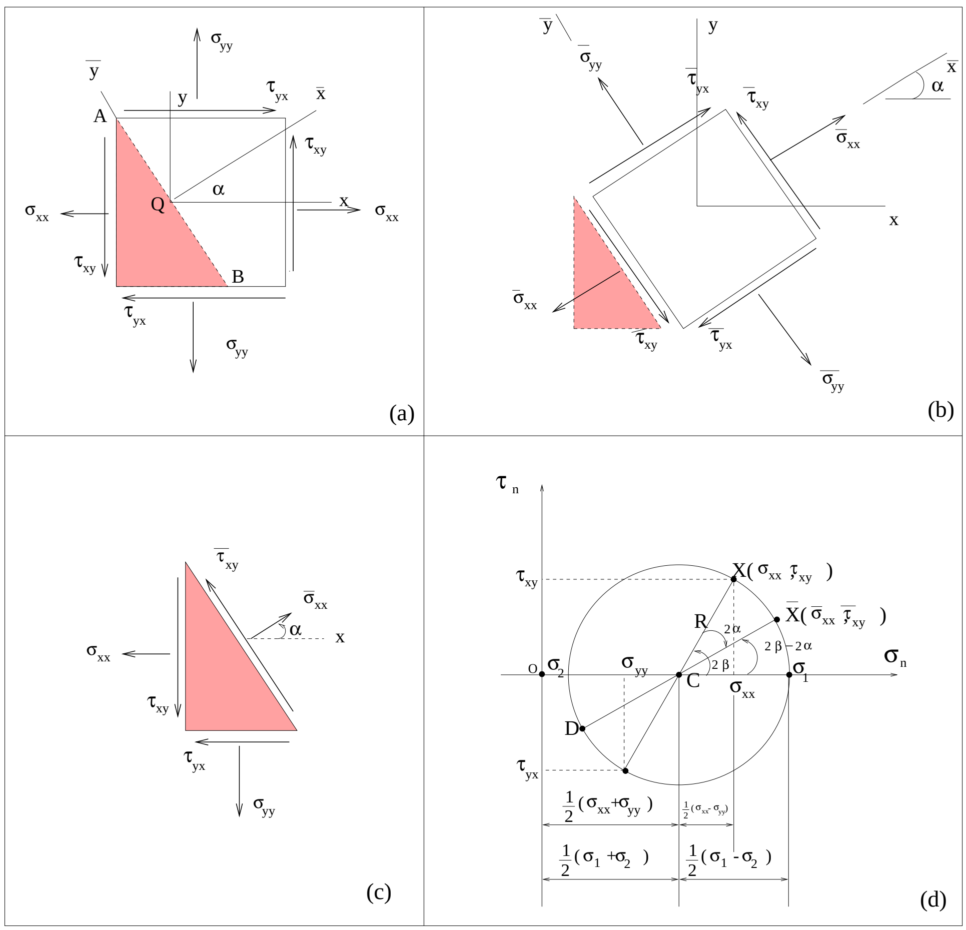


Figure 2.6: Mohr Circle for Plane Stress

where

$$R = \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \quad (2.59\text{-a})$$

$$\tan 2\beta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \quad (2.59\text{-b})$$

then after substitution and simplifiation, Eq. 2.57-a and 2.57-b would result in

$$\overline{\sigma}_{xx} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + R\cos(2\beta - 2\alpha) \quad (2.60)$$

$$\overline{\sigma}_{xy} = R\sin(2\beta - 2\alpha) \quad (2.61)$$

We observe that the form of these equations, indicates that $\overline{\sigma}_{xx}$ and $\overline{\sigma}_{xy}$ are on a circle centered at $\frac{1}{2}(\sigma_{xx} + \sigma_{yy})$ and of radius R. Furthermore, since σ_{xx}, σ_{yy}, R and β are definite numbers for a given state of stress, the previous equations provide a **graphical solution** for the evaluation of the rotated stress $\overline{\sigma}_{xx}$ and $\overline{\sigma}_{xy}$ for various angles α.

44 By eliminating the trigonometric terms, the Cartesian equation of the circle is given by

$$[\overline{\sigma}_{xx} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy})]^2 + \overline{\sigma}_{xy}^2 = R^2 \quad (2.62)$$

45 Finally, the graphical solution for the state of stresses at an inclined plane is summarized as follows

1. Plot the points $(\sigma_{xx}, 0)$, $(\sigma_{yy}, 0)$, $C : [\frac{1}{2}(\sigma_{xx} + \sigma_{yy}), 0]$, and $X : (\sigma_{xx}, \sigma_{xy})$.
2. Draw the line CX, this will be the reference line corresponding to a plane in the physical body whose normal is the positive x direction.
3. Draw a circle with center C and radius $R = CX$.
4. To determine the point that represents any plane in the physical body with normal making a counterclockwise angle α with the x direction, lay off angle 2α clockwise from CX. The terminal side $C\overline{X}$ of this angle intersects the circle in point $\overline{X}$ whose coordinates are $(\overline{\sigma}_{xx}, \overline{\sigma}_{xy})$.
5. To determine $\overline{\sigma}_{yy}$, consider the plane whose normal makes an angle $\alpha + \frac{1}{2}\pi$ with the positive x axis in the physical plane. The corresponding angle on the circle is $2\alpha + \pi$ measured clockwise from the reference line CX. This locates point D which is at the opposite end of the diameter through $\overline{X}$. The coordinates of D are $(\overline{\sigma}_{yy}, -\overline{\sigma}_{xy})$

■ Example 2-4: Mohr's Circle in Plane Stress

An element in plane stress is subjected to stresses $\sigma_{xx} = 15$, $\sigma_{yy} = 5$ and $\tau_{xy} = 4$. Using the Mohr's circle determine: a) the stresses acting on an element rotated through an angle $\theta = +40^o$ (counterclockwise); b) the principal stresses; and c) the maximum shear stresses. Show all results on sketches of properly oriented elements.

Solution:

With reference to Fig. 2.7:

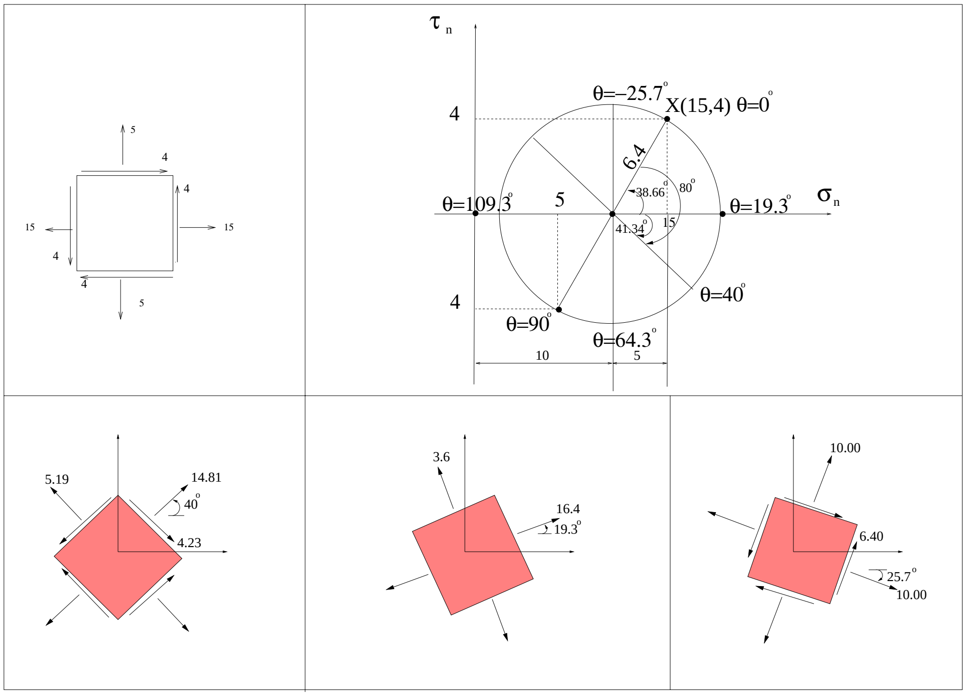


Figure 2.7: Plane Stress Mohr's Circle; Numerical Example

1. The center of the circle is located at

$$\frac{1}{2}(\sigma_{xx}+\sigma_{yy}) = \frac{1}{2}(15+5) = 10. \tag{2.63}$$

2. The radius and the angle 2β are given by

$$R = \sqrt{\frac{1}{4}(15-5)^2+4^2} = 6.403 \tag{2.64-a}$$

$$\tan 2\beta = \frac{2(4)}{15-5} = 0.8 \Rightarrow 2\beta = 38.66^o; \quad \beta = 19.33^o \tag{2.64-b}$$

3. The stresses acting on a plane at $\theta = +40^o$ are given by the point making an angle of -80^o (clockwise) with respect to point $X(15,4)$ or $-80^o + 38.66^o = -41.34^o$ with respect to the axis.

4. Thus, by inspection the stresses on the $\overline{x}$ face are

$$\overline{\sigma}_{xx} = 10 + 6.403\cos -41.34^o = \boxed{14.81} \tag{2.65-a}$$

$$\overline{\tau}_{xy} = 6.403\sin -41.34^o = \boxed{-4.23} \tag{2.65-b}$$

5. Similarly, the stresses at the face $\overline{y}$ are given by

$$\overline{\sigma}_{yy} = 10 + 6.403\cos(180^o - 41.34^o) = \boxed{5.19} \tag{2.66-a}$$

$$\overline{\tau}_{xy} = 6.403\sin(180^o - 41.34^o) = \boxed{4.23} \tag{2.66-b}$$

6. The principal stresses are simply given by

$$\sigma_{(1)} = 10 + 6.4 = \boxed{16.4} \quad (2.67\text{-a})$$
$$\sigma_{(2)} = 10 - 6.4 = \boxed{3.6} \quad (2.67\text{-b})$$

$\sigma_{(1)}$ acts on a plane defined by the angle of $+19.3^o$ clockwise from the x axis, and $\sigma_{(2)}$ acts at an angle of $\frac{38.66^o+180^o}{2} = \boxed{109.3^o}$ with respect to the x axis.

7. The maximum and minimum shear stresses are equal to the radius of the circle, i.e 6.4 at an angle of

$$\frac{90^o - 38.66^o}{2} = \boxed{25.7^0} \quad (2.68)$$

■

2.5.3 †Mohr's Stress Representation Plane

46 There can be an infinite number of planes passing through a point O, each characterized by their own normal vector along ON, Fig. 2.8. To each plane will correspond a set of σ_n and τ_n.

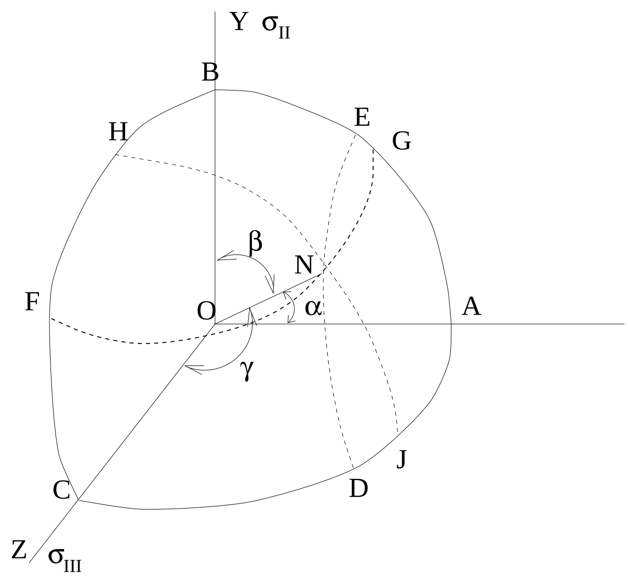


Figure 2.8: Unit Sphere in Physical Body around O

47 It can be shown that all possible sets of σ_n and τ_n which can act on the point O are within the shaded area of Fig. 2.9.

2.6 Simplified Theories; Stress Resultants

48 For many applications of continuum mechanics the problem of determining the three-dimensional stress distribution is too difficult to solve. However, in many (civil/mechanical)applications,

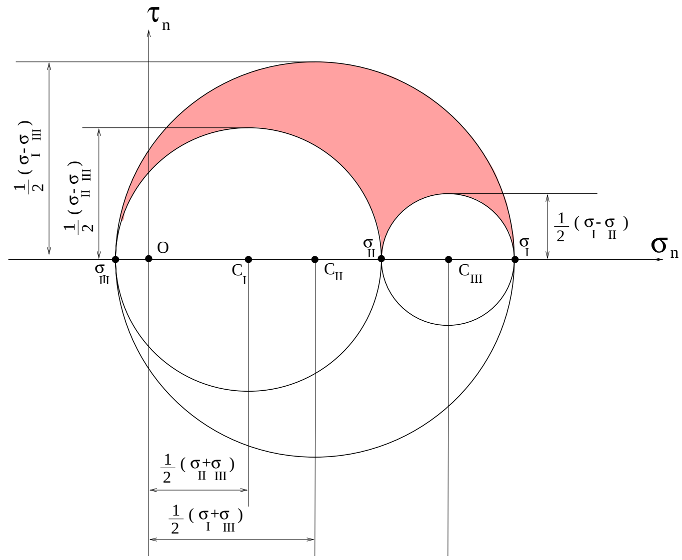


Figure 2.9: Mohr Circle for Stress in 3D

one or more dimensions is/are small compared to the others and possess certain symmetries of geometrical shape and load distribution.

49 In those cases, we may apply "**engineering theories**" for shells, plates or beams. In those problems, instead of solving for the stress components throughout the body, we solve for certain *stress resultants* (normal, shear forces, and Moments and torsions) resulting from an integration over the body. We consider separately two of those three cases.

50 Alternatively, if a continuum solution is desired, and engineering theories prove to be either too restrictive or inapplicable, we can use numerical techniques (such as **the Finite Element Method**) to solve the problem.

2.6.1 Arch

51 Fig. 2.10 illustrates the stresses acting on a differential element of a shell structure. The resulting forces in turn are shown in Fig. 2.11 and for simplification those acting per unit length of the middle surface are shown in Fig. 2.12. The net resultant forces

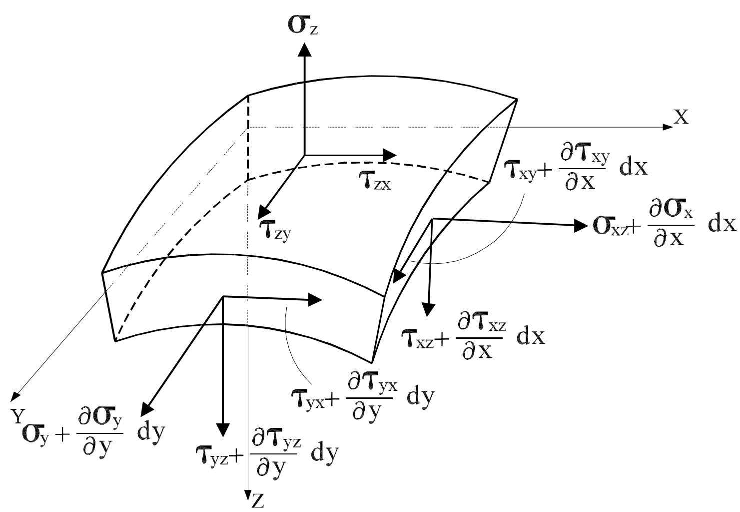


Figure 2.10: Differential Shell Element, Stresses

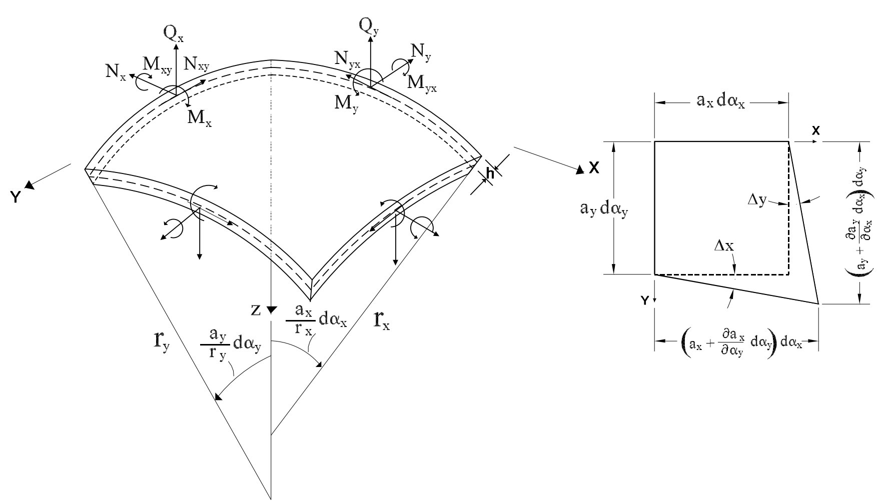


Figure 2.11: Differential Shell Element, Forces

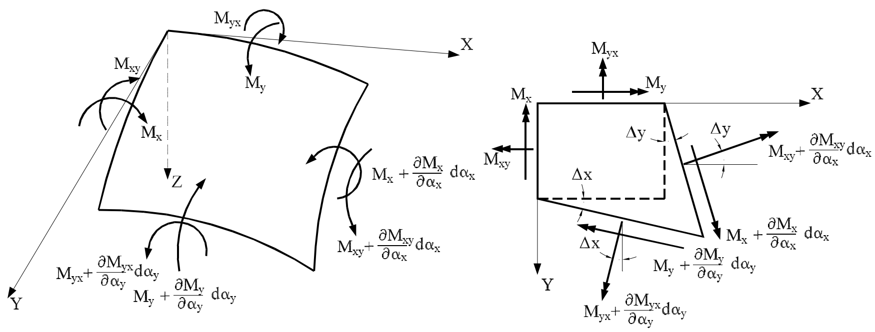


Figure 2.12: Differential Shell Element, Vectors of Stress Couples

are given by:

Membrane Force

$$\mathbf{N} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma} \left(1 - \frac{z}{r}\right) dz \qquad \left\{ \begin{array}{lcl} N_{xx} & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xx} \left(1 - \frac{z}{r_y}\right) dz \\ N_{yy} & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yy} \left(1 - \frac{z}{r_x}\right) dz \\ N_{xy} & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy} \left(1 - \frac{z}{r_y}\right) dz \\ N_{yx} & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy} \left(1 - \frac{z}{r_x}\right) dz \end{array} \right.$$

Bending Moments

$$\mathbf{M} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma} z \left(1 - \frac{z}{r}\right) dz \qquad \left\{ \begin{array}{lcl} M_{xx} & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xx} z \left(1 - \frac{z}{r_y}\right) dz \\ M_{yy} & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yy} z \left(1 - \frac{z}{r_x}\right) dz \\ M_{xy} & = & -\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy} z \left(1 - \frac{z}{r_y}\right) dz \\ M_{yx} & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy} z \left(1 - \frac{z}{r_x}\right) dz \end{array} \right. \tag{2.69}$$

Transverse Shear Forces

$$\mathbf{Q} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau} \left(1 - \frac{z}{r}\right) dz \qquad \left\{ \begin{array}{lcl} Q_x & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xz} \left(1 - \frac{z}{r_y}\right) dz \\ Q_y & = & \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{yz} \left(1 - \frac{z}{r_x}\right) dz \end{array} \right.$$

2.6.2 Plates

52 Considering an arbitrary plate, the stresses and resulting forces are shown in Fig. 2.13, and resultants *per unit width* are given by

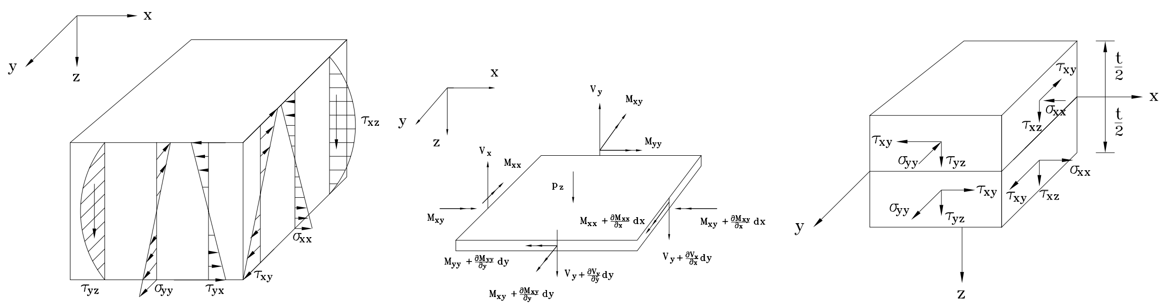


Figure 2.13: Stresses and Resulting Forces in a Plate

$$
\begin{aligned}
\textbf{Membrane Force} \quad \mathbf{N} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\sigma} dz & &\left\{ \begin{array}{lcl} N_{xx} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} dz \\ N_{yy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} dz \\ N_{xy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} dz \end{array} \right. \\
\textbf{Bending Moments} \quad \mathbf{M} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\sigma} z dz & &\left\{ \begin{array}{lcl} M_{xx} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} z dz \\ M_{yy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} z dz \\ M_{xy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} z dz \end{array} \right. \\
\textbf{Transverse Shear Forces} \quad \mathbf{V} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\tau} dz & &\left\{ \begin{array}{lcl} V_x &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz \\ V_y &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz \end{array} \right.
\end{aligned}
\qquad (2.70\text{-a})
$$

53 Note that in plate theory, we ignore the effect of the membrane forces, those in turn will be accounted for in shells.

Chapter 3

MATHEMATICAL PRELIMINARIES; Part II VECTOR DIFFERENTIATION

3.1 Introduction

1 A **field** is a function defined over a continuous region. This includes, **Scalar Field** $g(\mathbf{x})$, **Vector Field** $\mathbf{v}(\mathbf{x})$, Fig. 3.1 or **Tensor Field** $\mathbf{T}(\mathbf{x})$.

2 We first introduce the **differential vector operator** "Nabla" denoted by $\boldsymbol{\nabla}$

$$\boxed{\boldsymbol{\nabla} \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}} \tag{3.1}$$

3 We also note that there are as many ways to differentiate a vector field as there are ways of multiplying vectors, the analogy being given by Table 3.1.

Multiplication		Differentiation		Tensor Order
$\mathbf{u}\cdot\mathbf{v}$	dot	$\boldsymbol{\nabla}\cdot\mathbf{v}$	divergence	↓
$\mathbf{u}\times\mathbf{v}$	cross	$\boldsymbol{\nabla}\times\mathbf{v}$	curl	→
$\mathbf{u}\otimes\mathbf{v}$	tensor	$\boldsymbol{\nabla}\mathbf{v}$	gradient	↑

Table 3.1: Similarities Between Multiplication and Differentiation Operators

3.2 Derivative WRT to a Scalar

4 The derivative of a vector $\mathbf{p}(u)$ with respect to a scalar u, Fig. 3.2 is defined by

$$\frac{d\mathbf{p}}{du} \equiv \lim_{\Delta u \to 0} \frac{\mathbf{p}(u+\Delta u) - \mathbf{p}(u)}{\Delta u} \tag{3.2}$$

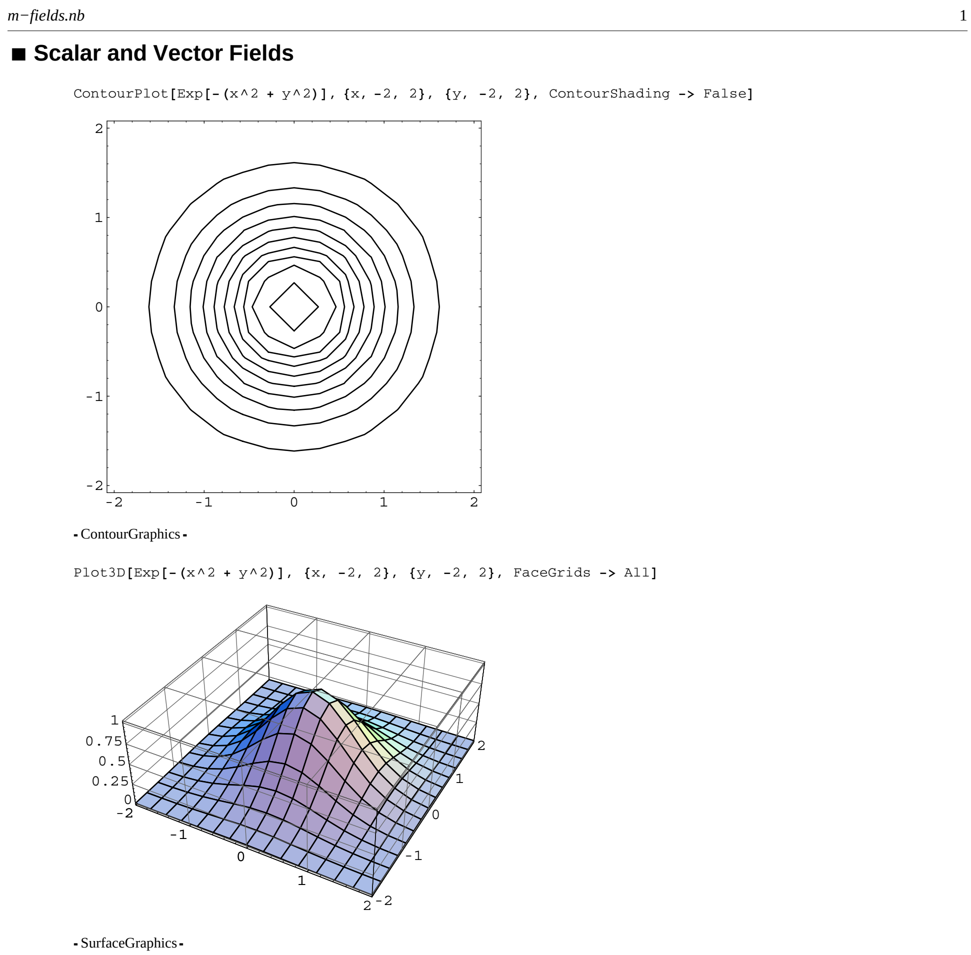


Figure 3.1: Examples of a Scalar and Vector Fields

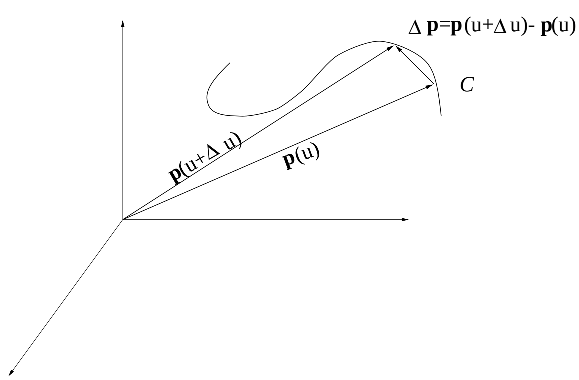


Figure 3.2: Differentiation of position vector **p**

5 If $\mathbf{p}(u)$ is a **position vector** $\mathbf{p}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, then

$$\frac{d\mathbf{p}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k} \tag{3.3}$$

is a vector along the tangent to the curve.

6 If u is the time t, then $\frac{d\mathbf{p}}{dt}$ is the velocity

7 In **differential geometry**, if we consider a curve $\mathcal{C}$ defined by the function $\mathbf{p}(u)$ then $\frac{d\mathbf{p}}{du}$ is a vector tangent ot $\mathcal{C}$, and if u is the curvilinear coordinate s measured from any point along the curve, then $\frac{d\mathbf{p}}{ds}$ is a unit tangent vector to $\mathcal{C}$ $\mathbf{T}$, Fig. 3.3. and we have the

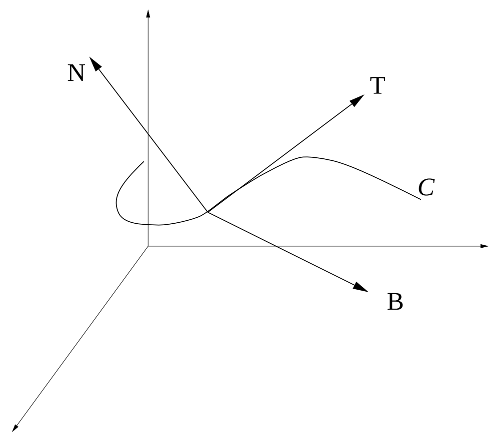


Figure 3.3: Curvature of a Curve

following relations

$$\begin{aligned}
\frac{d\mathbf{p}}{ds} &= \mathbf{T} && (3.4)\\
\frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N} && (3.5)\\
\mathbf{B} &= \mathbf{T}\times\mathbf{N} && (3.6)\\
\kappa & \quad \text{curvature} && (3.7)\\
\rho &= \frac{1}{\kappa} \text{ Radius of Curvature} && (3.8)
\end{aligned}$$

we also note that $\mathbf{p}\cdot\frac{d\mathbf{p}}{ds} = 0$ if $\left|\frac{d\mathbf{p}}{ds}\right| \neq 0$.

■ Example 3-1: Tangent to a Curve

Determine the unit vector tangent to the curve: $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$ for $t = 2$.
Solution:

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left[(t^2+1)\mathbf{i} + (4t-3)\mathbf{j} + (2t^2-6t)\mathbf{k}\right] = 2t\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k} \quad \text{(3.9-a)}$$

$$\left|\frac{d\mathbf{p}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2} \quad \text{(3.9-b)}$$

$$\mathbf{T} = \frac{2t\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}}{\sqrt{(2t)^2 + (4)^2 + (4t-6)^2}} \quad \text{(3.9-c)}$$

$$= \frac{4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(4)^2 + (4)^2 + (2)^2}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \text{ for } t = 2 \quad \text{(3.9-d)}$$

Mathematica solution is shown in Fig. 3.4 ■

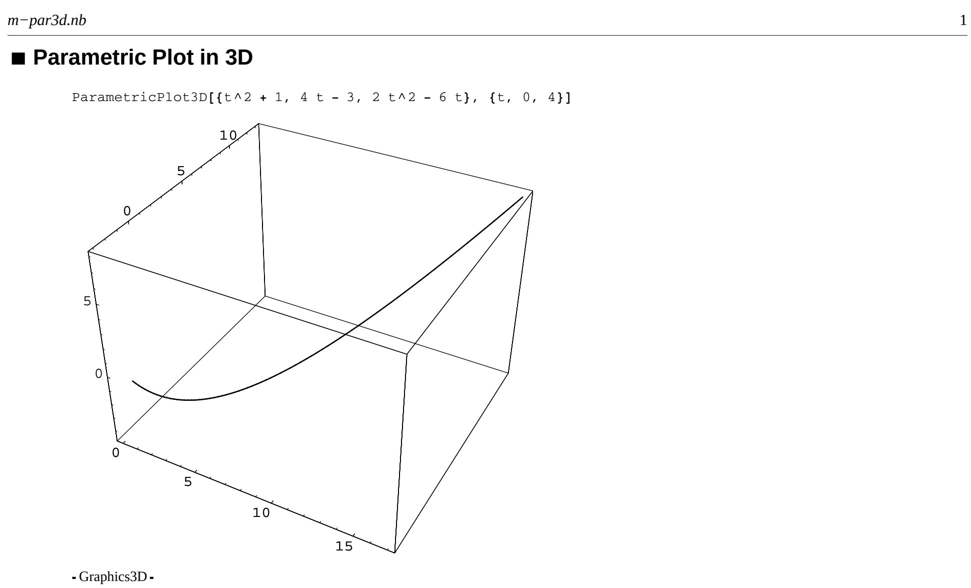


Figure 3.4: Mathematica Solution for the Tangent to a Curve in 3D

3.3 Divergence

3.3.1 Vector

8 The **divergence** of a vector field of a body $\mathcal{B}$ with boundary Ω, Fig. 3.5 is defined by considering that each point of the surface has a normal $\mathbf{n}$, and that the body is surrounded by a vector field $\mathbf{v}(\mathbf{x})$. The volume of the body is $v(\mathcal{B})$.

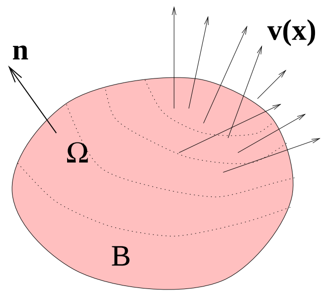


Figure 3.5: Vector Field Crossing a Solid Region

9 The divergence of the vector field is thus defined as

$$\boxed{\text{div } \mathbf{v}(\mathbf{x}) \equiv \lim_{v(\mathcal{B})\to 0} \frac{1}{v(\mathcal{B})} \int_{\Omega} \mathbf{v} \cdot \mathbf{n} dA} \tag{3.10}$$

where $\mathbf{v}.\mathbf{n}$ is often referred as the **flux** and represents the total volume of "fluid" that passes through dA in unit time, Fig. 3.6 This volume is then equal to the base of the

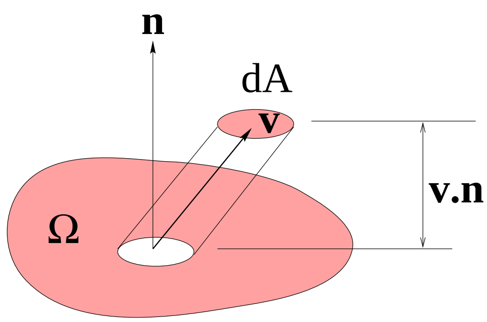


Figure 3.6: Flux Through Area dA

cylinder dA times the height of the cylinder $\mathbf{v} \cdot \mathbf{n}$. We note that the streamlines which are tangent to the boundary do not let any fluid out, while those normal to it let it out most efficiently.

10 The divergence thus measure the rate of change of a vector field.

11 The definition is clearly independent of the shape of the solid region, however we can gain an insight into the divergence by considering a rectangular parallelepiped with sides Δx_1, Δx_2, and Δx_3, and with normal vectors pointing in the directions of the coordinate axies, Fig. 3.7. If we also consider the corner closest to the origin as located at $\mathbf{x}$, then the contribution (from Eq. 3.10) of the two surfaces with normal vectors $\mathbf{e}_1$ and $-\mathbf{e}_1$ is

$$\lim_{\Delta x_1, \Delta x_2, \Delta x_3 \to 0} \frac{1}{\Delta x_1 \Delta x_2 \Delta x_3} \int_{\Delta x_2 \Delta x_3} [\mathbf{v}(\mathbf{x} + \Delta x_1 \mathbf{e}_1) \cdot \mathbf{e}_1 + \mathbf{v}(x) \cdot (-\mathbf{e}_1)] dx_2 dx_3 \tag{3.11}$$

or

$$\lim_{\Delta x_1, \Delta x_2, \Delta x_3 \to 0} \frac{1}{\Delta x_2 \Delta x_3} \int_{\Delta x_2 \Delta x_3} \frac{\mathbf{v}(\mathbf{x} + \Delta x_1 \mathbf{e}_1) - \mathbf{v}(\mathbf{x})}{\Delta x_1} \cdot \mathbf{e}_1 dx_2 dx_3 = \lim_{\Delta x_1 \to 0} \frac{\Delta \mathbf{v}}{\Delta x_1} \cdot \mathbf{e}_1 \tag{3.12-a}$$

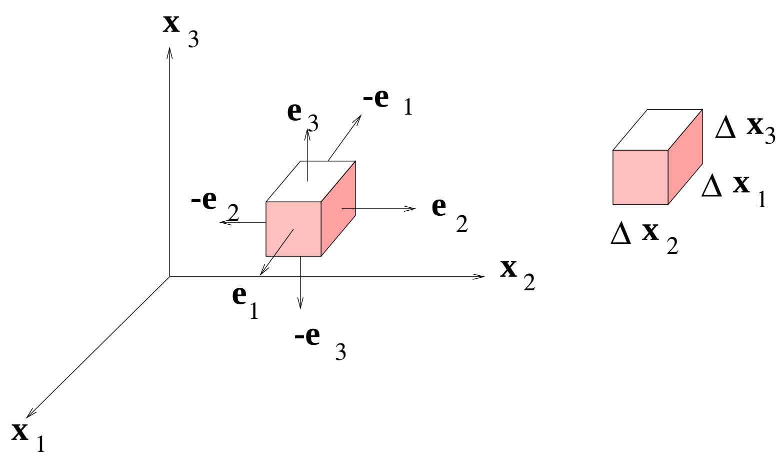


Figure 3.7: Infinitesimal Element for the Evaluation of the Divergence

$$= \frac{\partial \mathbf{v}}{\partial x_1} \cdot \mathbf{e}_1 \quad (3.12\text{-b})$$

hence, we can generalize

$$\boxed{\text{div } \mathbf{v}(\mathbf{x}) = \frac{\partial \mathbf{v}(\mathbf{x})}{\partial x_i} \cdot \mathbf{e}_i} \quad (3.13)$$

12 or alternatively

$$\text{div } \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{v} = \left(\frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \frac{\partial}{\partial x_3}\mathbf{e}_3\right) \cdot (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3) \quad (3.14)$$

$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_i}{\partial x_i} = \partial_i v_i = v_{i,i} \quad (3.15)$$

13 The divergence of a vector is a **scalar**.

14 We note that the **Laplacian Operator** is defined as

$$\boxed{\boldsymbol{\nabla}^2 F \equiv \boldsymbol{\nabla}\boldsymbol{\nabla} F = F_{,ii}} \quad (3.16)$$

■ Example 3-2: Divergence

Determine the divergence of the vector $\mathbf{A} = x^2 z \mathbf{i} - 2y^3 z^2 \mathbf{j} + xy^2 z \mathbf{k}$ at point $(1, -1, 1)$.
Solution:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (x^2 z \mathbf{i} - 2y^3 z^2 \mathbf{j} + xy^2 z \mathbf{k}) \quad (3.17\text{-a})$$

$$= \frac{\partial x^2 z}{\partial x} + \frac{\partial - 2y^3 z^2}{\partial y} + \frac{\partial xy^2 z}{\partial z} \quad (3.17\text{-b})$$

$$= \quad 2xz - 6y^2z^2 + xy^2 \qquad (3.17\text{-c})$$
$$= \quad 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = -3 \text{ at } (1,-1,1) \qquad (3.17\text{-d})$$

Mathematica solution is shown in Fig. 3.8 ■

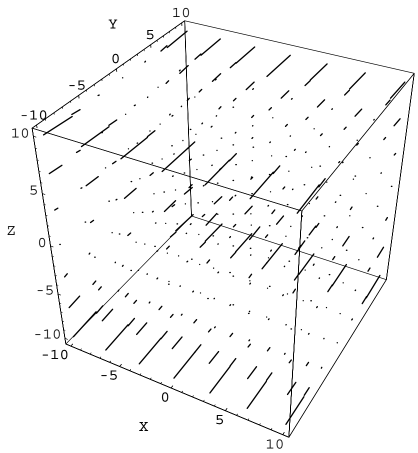
m−diver.nb 1

■ Divergence of a Vector

```
<< Calculus‘VectorAnalysis‘

V = {x^2 z, -2 y^3 z^2, x y^2 z};

Div[V, Cartesian[x, y, z]]
```

$-6z^2\,y^2 + x\,y^2 + 2\,x\,z$

```
<< Graphics‘PlotField3D‘

PlotVectorField3D[{x^2 z, -2 y^3 z^2, x y^2 z}, {x, -10, 10}, {y, -10, 10}, {z, -10, 10},
 Axes -> Automatic, AxesLabel -> {"X", "Y", "Z"}]
```



- Graphics3D -

```
Div[Curl[V, Cartesian[x, y, z]], Cartesian[x, y, z]]
```

0

Figure 3.8: Mathematica Solution for the Divergence of a Vector

3.3.2 Second-Order Tensor

15 By analogy to Eq. 3.10, the divergence of a second-order tensor field **T** is

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{T} = \text{div } \mathbf{T}(\mathbf{x}) \equiv \lim_{v(\mathcal{B})\to 0} \frac{1}{v(\mathcal{B})}\int_{\Omega} \mathbf{T}\cdot\mathbf{n}dA} \qquad (3.18)$$

which is the **vector field**

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{T} = \frac{\partial T_{pq}}{\partial x_p}\mathbf{e}_q} \qquad (3.19)$$

3.4 Gradient

3.4.1 Scalar

16 The **gradient** of a scalar field $g(\mathbf{x})$ is a vector field $\boldsymbol{\nabla} g(\mathbf{x})$ such that for any unit vector $\mathbf{v}$, the directional derivative dg/ds in the direction of $\mathbf{v}$ is given by

$$\boxed{\frac{dg}{ds} = \boldsymbol{\nabla} g \cdot \mathbf{v}} \tag{3.20}$$

where $\mathbf{v} = \frac{d\mathbf{p}}{ds}$ We note that the definition made no reference to any coordinate system. The gradient is thus a **vector invariant**.

17 To find the components in any rectangular Cartesian coordinate system we use

$$\mathbf{v} = \frac{d\mathbf{p}}{ds} = \frac{dx_i}{ds}\mathbf{e}_i \tag{3.21-a}$$

$$\frac{dg}{ds} = \frac{\partial g}{\partial x_i}\frac{dx_i}{ds} \tag{3.21-b}$$

which can be substituted and will yield

$$\boxed{\boldsymbol{\nabla} g = \frac{\partial g}{\partial x_i}\mathbf{e}_i} \tag{3.22}$$

or

$$\boldsymbol{\nabla}\phi \equiv \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi \tag{3.23-a}$$

$$= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \tag{3.23-b}$$

and note that it defines a **vector field**.

18 The physical significance of the gradient of a scalar field is that it points in the direction in which the field is changing most rapidly (for a three dimensional surface, the gradient is pointing along the normal to the plane tangent to the surface). The length of the vector $||\boldsymbol{\nabla} g(\mathbf{x})||$ is perpendicular to the contour lines.

19 $\boldsymbol{\nabla} g(\mathbf{x}) \cdot \mathbf{n}$ gives the rate of change of the scalar field in the direction of $\mathbf{n}$.

■ **Example 3-3: Gradient of a Scalar**

Determine the gradient of $\phi = x^2yz + 4xz^2$ at point $(1, -2, -1)$ along the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Solution:

$$\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + (x^2z\mathbf{j} + (x^2y + 8xz)\mathbf{k} \tag{3.24-a}$$

$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \text{ at } (1, -2, -1) \tag{3.24-b}$$

$$\mathbf{n} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \tag{3.24-c}$$

$$\boldsymbol{\nabla}\phi\cdot\mathbf{n} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k})\cdot\left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3} \tag{3.24-d}$$

Since this last value is positive, ϕ increases along that direction. ■

■ Example 3-4: Stress Vector normal to the Tangent of a Cylinder

The stress tensor throughout a continuum is given with respect to Cartesian axes as

$$\boldsymbol{\sigma} = \begin{bmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3^2 \\ 0 & 2x_3 & 0 \end{bmatrix} \tag{3.25}$$

Determine the stress vector (or traction) at the point $P(2, 1, \sqrt{3})$ of the plane that is tangent to the cylindrical surface $x_2^2 + x_3^2 = 4$ at P, Fig. 3.9.

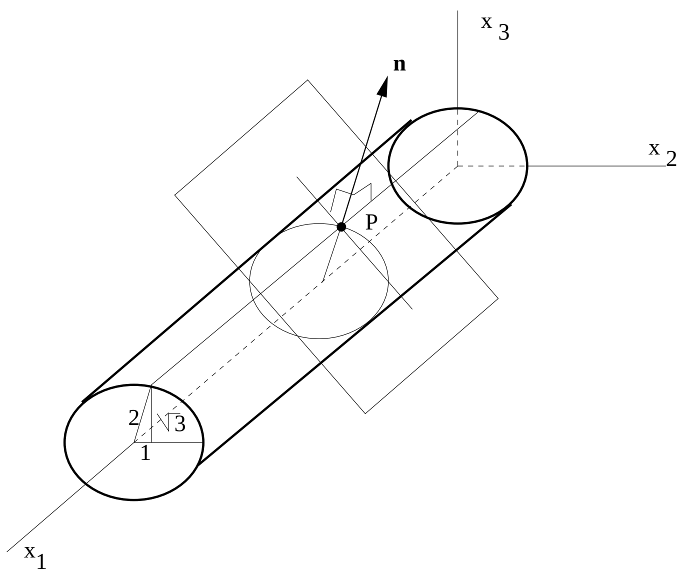


Figure 3.9: Radial Stress vector in a Cylinder

Solution:

At point P, the stress tensor is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{bmatrix} \tag{3.26}$$

The unit normal to the surface at P is given from

$$\boldsymbol{\nabla}(x_2^2 + x_3^2 - 4) = 2x_2\mathbf{2_2} + \mathbf{2x_3e_3} \tag{3.27}$$

At point P,

$$\boldsymbol{\nabla}(x_2^2 + x_3^2 - 4) = 2\mathbf{2_2} + \mathbf{2\sqrt{3}e_3} \tag{3.28}$$

and thus the unit normal at P is

$$\mathbf{n} = \frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_3 \tag{3.29}$$

Thus the traction vector will be determined from

$$\boldsymbol{\sigma} = \begin{bmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{Bmatrix} = \begin{Bmatrix} 5/2 \\ 3 \\ \sqrt{3} \end{Bmatrix} \tag{3.30}$$

or $\mathbf{t^n} = \frac{5}{2}\mathbf{e}_1 + 3\mathbf{e}_2 + \sqrt{3}\mathbf{e}_3$

3.4.2 Vector

20 We can also define the gradient of a vector field. If we consider a solid domain $\mathcal{B}$ with boundary Ω, Fig. 3.5, then the gradient of the vector field $\mathbf{v}(\mathbf{x})$ is a second order tensor defined by

$$\boxed{\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}(\mathbf{x}) \equiv \lim_{v(\mathcal{B})\to 0} \frac{1}{v(\mathcal{B})}\int_\Omega \mathbf{v}\otimes\mathbf{n}dA} \tag{3.31}$$

and with a construction similar to the one used for the divergence, it can be shown that

$$\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}(\mathbf{x}) = \frac{\partial v_i(\mathbf{x})}{\partial x_j}[\mathbf{e}_i\otimes\mathbf{e}_j] \tag{3.32}$$

where summation is implied for both i and j.

21 The components of $\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}$ are simply the various partial derivatives of the component functions with respect to the coordinates:

$$[\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{3.33}$$

$$[\mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}}] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{3.34}$$

that is $[\boldsymbol{\nabla}\mathbf{v}]_{ij}$ gives the rate of change of the ith component of $\mathbf{v}$ with respect to the jth coordinate axis.

22 Note the diference between $\mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}}$ and $\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}$. In matrix representation, one is the transpose of the other.

23 The gradient of a vector is a tensor of order 2.

24 We can interpret the gradient of a vector geometrically, Fig. 3.10. If we consider two points a and b that are near to each other (i.e Δs is very small), and let the unit vector $\mathbf{m}$ points in the direction from a to b. The value of the vector field at a is $\mathbf{v}(\mathbf{x})$ and the value of the vector field at b is $\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})$. Since the vector field changes with position in the domain, those two vectors are different both in length and orientation. If we now transport a copy of $\mathbf{v}(\mathbf{x})$ and place it at b, then we compare the differences between those two vectors. The vector connecting the heads of $\mathbf{v}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})$ is $\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})-\mathbf{v}(\mathbf{x})$, the change in vector. Thus, if we divide this change by Δs, then we get the rate of change as we move in the specified direction. Finally, taking the limit as Δs goes to zero, we obtain

$$\lim_{\Delta s\to 0}\frac{\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})-\mathbf{v}(\mathbf{x})}{\Delta s}\equiv D\mathbf{v}(\mathbf{x})\cdot\mathbf{m} \tag{3.35}$$

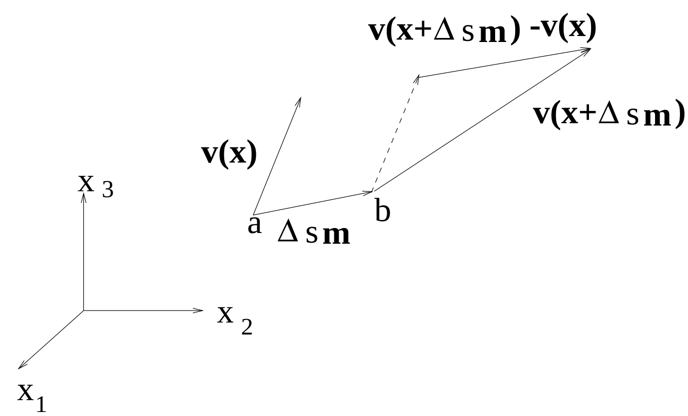


Figure 3.10: Gradient of a Vector

The quantity $D\mathbf{v}(\mathbf{x})\cdot\mathbf{m}$ is called the **directional derivative** because it gives the rate of change of the vector field as we move in the direction $\mathbf{m}$.

■ Example 3-5: Gradient of a Vector Field

Determine the gradient of the following vector field $\mathbf{v}(\mathbf{x}) = x_1x_2x_3(x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3)$.
Solution:

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}(\mathbf{x}) &= 2x_1x_2x_3[\mathbf{e}_1\otimes\mathbf{e}_1]+x_1^2x_3[\mathbf{e}_1\otimes\mathbf{e}_2]+x_1^2x_2[\mathbf{e}_1\otimes\mathbf{e}_3]\\ &\quad +x_2^2x_3[\mathbf{e}_2\otimes\mathbf{e}_1]+2x_1x_2x_3[\mathbf{e}_2\otimes\mathbf{e}_2]+x_1x_2^2[\mathbf{e}_2\otimes\mathbf{e}_3] \qquad (3.36\text{-a})\\ &\quad +x_2x_3^2[\mathbf{e}_3\otimes\mathbf{e}_1]+x_1x_3^2[\mathbf{e}_3\otimes\mathbf{e}_2]+2x_1x_2x_3[\mathbf{e}_3\otimes\mathbf{e}_3]\\ &= x_1x_2x_3\begin{bmatrix}2 & x_1/x_2 & x_1/x_3\\ x_2/x_1 & 2 & x_2/x_3\\ x_3/x_1 & x_3/x_2 & 2\end{bmatrix} \qquad (3.36\text{-b})\end{aligned}$$

■

3.4.3 Mathematica Solution

25 Mathematica solution of the two preceding examples is shown in Fig. 3.11.

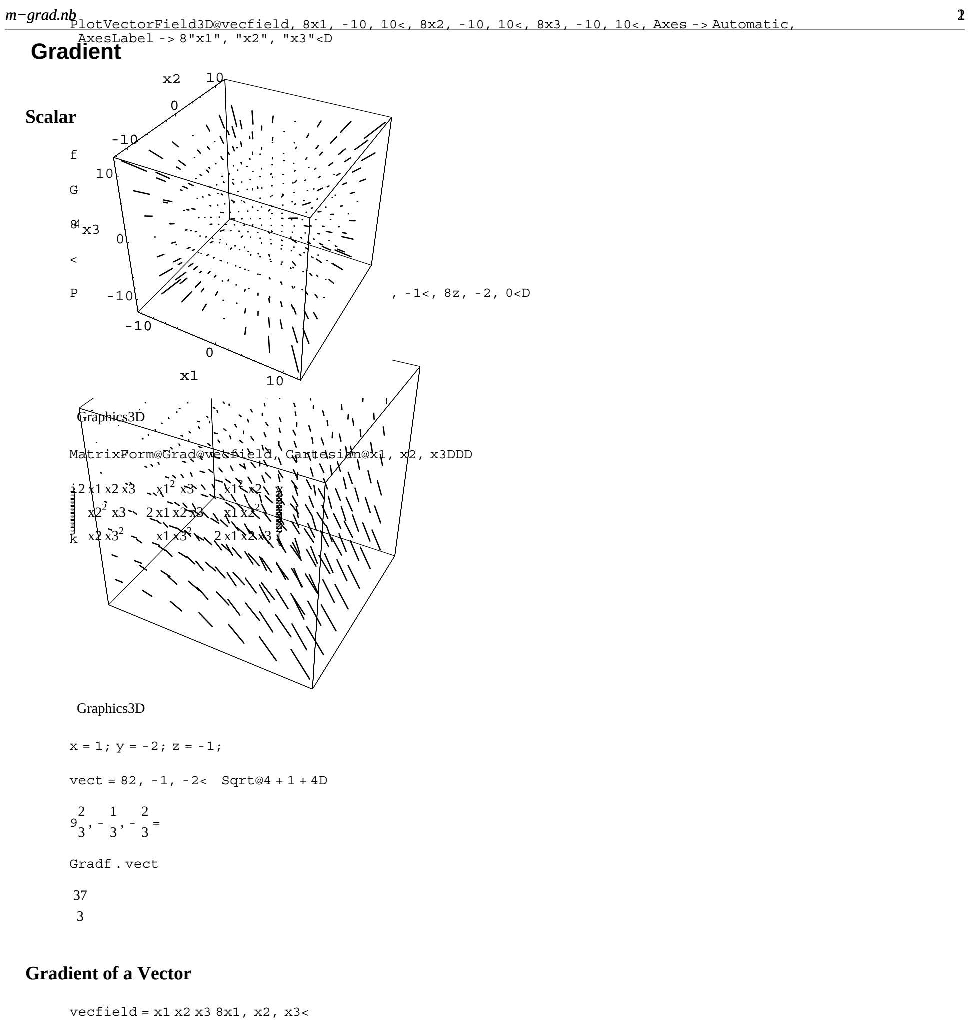


Figure 3.11: Mathematica Solution for the Gradients of a Scalar and of a Vector

3.5 Curl

26 When the vector operator ∇ operates in a manner analogous to vector multiplication, the result is a vector, curl $\mathbf{v}$ called the curl of the vector field $\mathbf{v}$ (sometimes called the rotation).

$$\text{curl } \mathbf{v} = \boldsymbol{\nabla}\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{3.37}$$

$$= \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 \tag{3.38}$$

$$= e_{ijk}\partial_j v_k \tag{3.39}$$

■ **Example 3-6: Curl of a vector**

Determine the curl of the following vector $\mathbf{A} = xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}$ at $(1, -1, 1)$.
Solution:

$$\boldsymbol{\nabla}\times\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}) \tag{3.40-a}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} \tag{3.40-b}$$

$$= \left(\frac{\partial 2yz^4}{\partial y} - \frac{\partial - 2x^2yz}{\partial z}\right)\mathbf{i} + \left(\frac{\partial xz^3}{\partial z} - \frac{\partial 2yz^4}{\partial x}\right)\mathbf{j} + \left(\frac{\partial - 2x^2yz}{\partial x} - \frac{\partial xz^3}{\partial y}\right)\mathbf{k} \tag{3.40-c}$$

$$= (2z^4 + 2x^2y)\mathbf{i} + 3xz^2\mathbf{j} - 4xyz\mathbf{k} \tag{3.40-d}$$

$$= 3\mathbf{j} + 4\mathbf{k} \text{ at } (1, -1, 1) \tag{3.40-e}$$

Mathematica solution is shown in Fig. 3.12. ■

3.6 Some useful Relations

27 Some useful relations

$$d(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot d\mathbf{B} + d\mathbf{A}\cdot\mathbf{B} \tag{3.41-a}$$

$$d(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times d\mathbf{B} + d\mathbf{A}\times\mathbf{B} \tag{3.41-b}$$

$$\boldsymbol{\nabla}(\phi + \xi) = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\xi \tag{3.41-c}$$

$$\boldsymbol{\nabla}\times(\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\times\mathbf{B} \tag{3.41-d}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} \neq \mathbf{v}\boldsymbol{\nabla} \tag{3.41-e}$$

$$\boldsymbol{\nabla}\cdot(\phi\mathbf{A}) = (\boldsymbol{\nabla}\phi)\cdot\mathbf{A} + \phi(\boldsymbol{\nabla}\times\mathbf{A}) \tag{3.41-f}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \tag{3.41-g}$$

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) \tag{3.41-h}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) \equiv \boldsymbol{\nabla}^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \text{ Laplacian Operator} \tag{3.41-i}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0 \tag{3.41-j}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = \mathbf{0} \tag{3.41-k}$$

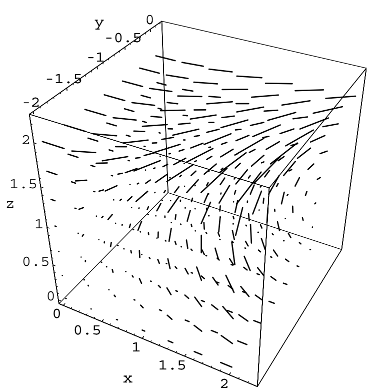
m−curl.nb 1

■ **Curl**

```
<< Calculus`VectorAnalysis`

A = {x z^3, -2 x^2 y z, 2 y z^4};

CurlOfA = Curl[A, Cartesian[x, y, z]]
```

$\{2z^4 + 2x^2 y, 3xz^2, -4xyz\}$

```
<< Graphics`PlotField3D`

PlotVectorField3D[CurlOfA, {x, 0, 2}, {y, -2, 0}, {z, 0, 2}, Axes -> Automatic, AxesLabel -> {"x", "y", "z"}]
```



- Graphics3D -

```
Div[CurlOfA, Cartesian[x, y, z]]
```

0

```
x = 1; y = -1; z = 1;

CurlOfA
```

{0, 3, 4}

Figure 3.12: Mathematica Solution for the Curl of a Vector

Chapter 4

KINEMATIC

Or on How Bodies Deform

4.1 Elementary Definition of Strain

20 We begin our detailed coverage of strain by a simplified and elementary set of definitions for the 1D and 2D cases. Following this a mathematically rigorous derivation of the various expressions for strain will follow.

4.1.1 Small and Finite Strains in 1D

21 We begin by considering an elementary case, an axial rod with initial lenght l_0, and subjected to a deformation Δl into a final deformed length of l, Fig. 4.1.

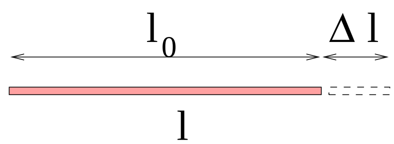


Figure 4.1: Elongation of an Axial Rod

22 We seek to quantify the deformation of the rod and even though we only have 2 variables (l_0 and l), there are different possibilities to introduce the notion of **strain**. We first define the **stretch** of the rod as

$$\lambda \equiv \frac{l}{l_0} \tag{4.1}$$

This stretch is one in the undeformed case, and greater than one when the rod is elongated.

23 Using l_0, l and λ we next introduce four possible definitions of the strain in 1D:

$$\boxed{\begin{array}{llcccl} \textbf{Engineering Strain} & \varepsilon & \equiv & \frac{l-l_0}{l_0} & = & \lambda - 1 \\ \textbf{Natural Strain} & \eta & = & \frac{l-l_0}{l} & = & 1 - \frac{1}{\lambda} \\ \textbf{Lagrangian Strain} & E & \equiv & \frac{1}{2}\left(\frac{l^2-l_0^2}{l_0^2}\right) & = & \frac{1}{2}(\lambda^2 - 1) \\ \textbf{Eulerian Strain} & E^* & \equiv & \frac{1}{2}\left(\frac{l^2-l_0^2}{l^2}\right) & = & \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right) \end{array}} \tag{4.2}$$

we note the strong analogy between the Lagrangian and the engineering strain on the one hand, and the Eulerian and the natural strain on the other.

24 The choice of which strain definition to use is related to the stress-strain relation (or constitutive law) that we will later adopt.

4.1.2 Small Strains in 2D

25 The elementary definition of strains in 2D is illustrated by Fig. 4.2 and are given by

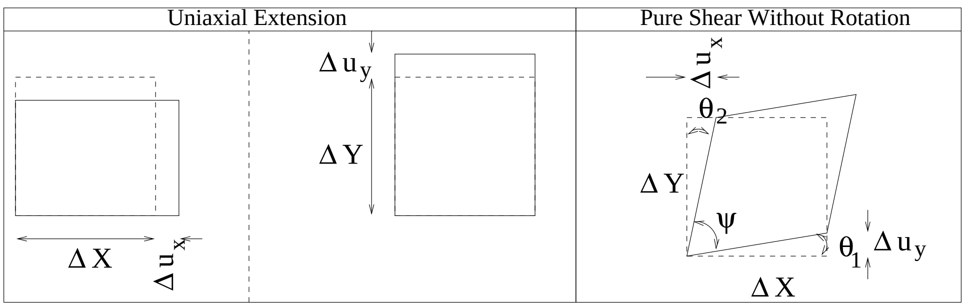


Figure 4.2: Elementary Definition of Strains in 2D

$$\varepsilon_{xx} \approx \frac{\Delta u_x}{\Delta X} \tag{4.3-a}$$

$$\varepsilon_{yy} \approx \frac{\Delta u_y}{\Delta Y} \tag{4.3-b}$$

$$\gamma_{xy} = \frac{\pi}{2} - \psi = \theta_2 + \theta_1 \tag{4.3-c}$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} \approx \frac{1}{2}\left(\frac{\Delta u_x}{\Delta Y} + \frac{\Delta u_y}{\Delta X}\right) \tag{4.3-d}$$

In the limit as both ΔX and ΔY approach zero, then

$$\boxed{\varepsilon_{xx} = \frac{\partial u_x}{\partial X}; \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial Y}; \quad \varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial Y} + \frac{\partial u_y}{\partial X}\right)} \tag{4.4}$$

We note that in the expression of the shear strain, we used $\tan\theta \approx \theta$ which is applicable as long as θ is small compared to one radian.

26 We have used capital letters to represent the coordinates in the initial state, and lower case letters for the final or current position coordinates ($x = X + u_x$). This corresponds to the Lagrangian strain representation.

4.2 Strain Tensor

27 Following the simplified (and restrictive) introduction to strain, we now turn our attention to a rigorous presentation of this important deformation tensor.

28 The presentation will proceed as follow. First, with reference to Fig. 4.3 we will derive expressions for the position and displacement vectors of a single point P from the undeformed to the deformed state. Then, we will use some of the expressions in the introduction of the strain between two points P and Q.

4.2.1 Position and Displacement Vectors; $(\mathbf{x}, \mathbf{X})$

29 We consider in Fig. 4.3 the undeformed configuration of a material continuum at time $t = 0$ together with the deformed configuration at coordinates for each configuration.

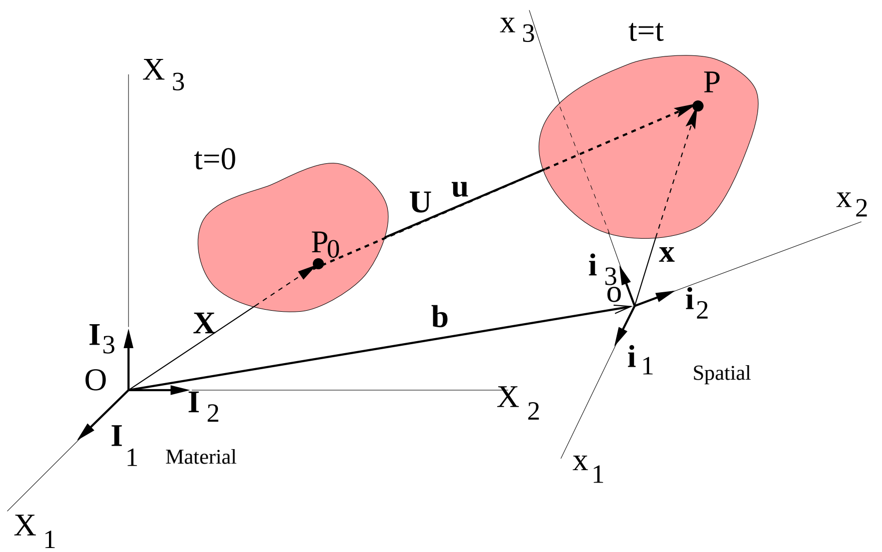


Figure 4.3: Position and Displacement Vectors

30 In the initial configuration P_0 has the **position vector**

$$\mathbf{X} = X_1\mathbf{I}_1 + X_2\mathbf{I}_2 + X_3\mathbf{I}_3 \tag{4.5}$$

which is here expressed in terms of the **material coordinates** (X_1, X_2, X_3).

31 In the deformed configuration, the particle P_0 has now moved to the new position P and has the following position vector

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{4.6}$$

which is expressed in terms of the **spatial coordinates**.

32 The relative orientation of the material axes $(OX_1X_2X_3)$ and the spatial axes $(ox_1x_2x_3)$ is specified through the direction cosines $a_{\mathbf{x}}^{\mathbf{X}}$.

33 The displacement vector $\mathbf{u}$ connecting P_0 ann P is the **displacement vector** which can be expressed in both the material or spatial coordinates

$$\mathbf{U} = U_k \mathbf{I}_k \quad (4.7\text{-a})$$
$$\mathbf{u} = u_k \mathbf{i}_k \quad (4.7\text{-b})$$

again U_k and u_k are interrelated through the direction cosines $\mathbf{i}_k = a_k^K \mathbf{I}_K$. Substituting above we obtain

$$\mathbf{u} = u_k(a_k^K \mathbf{I}_K) = U_K \mathbf{I}_K = \mathbf{U} \Rightarrow U_K = a_k^K u_k \quad (4.8)$$

34 The vector $\mathbf{b}$ relates the two origins $\mathbf{u} = \mathbf{b} + \mathbf{x} - \mathbf{X}$ or if the origins are the same (superimposed axis)

$$u_k = x_k - X_k \quad (4.9)$$

■ Example 4-1: Displacement Vectors in Material and Spatial Forms

With respect to superposed material axis X_i and spatial axes x_i, the displacement field of a continuum body is given by: $x_1 = X_1$, $x_2 = X_2 + AX_3$, and $x_3 = AX_2 + X_3$ where A is constant.

1. Determine the displacement vector components in both the material and spatial form.
2. Determine the displaced location of material particles which originally comprises the plane circular surface $X_1 = 0$, $X_2^2 + X_3^2 = 1/(1 - A^2)$ if $A = 1/2$.

Solution:

1. From Eq. 4.9 the displacement field can be written in material coordinates as

$$u_1 = x_1 - X_1 = 0 \quad (4.10\text{-a})$$
$$u_2 = x_2 - X_2 = AX_3 \quad (4.10\text{-b})$$
$$u_3 = x_3 - X_3 = AX_2 \quad (4.10\text{-c})$$

2. The displacement field can be written in matrix form as

$$\left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right\} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{array} \right] \left\{ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right\} \quad (4.11)$$

or upon inversion

$$\left\{ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right\} = \frac{1}{1 - A^2} \left[\begin{array}{ccc} 1 - A^2 & 0 & 0 \\ 0 & 1 & -A \\ 0 & -A & 1 \end{array} \right] \left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right\} \quad (4.12)$$

that is $X_1 = x_1$, $X_2 = (x_2 - Ax_3)/(1 - A^2)$, and $X_3 = (x_3 - Ax_2)/(1 - A^2)$.

3. The displacement field can be written now in spatial coordinates as

$$\begin{aligned} u_1 &= x_1 - X_1 = 0 && \text{(4.13-a)}\\ u_2 &= x_2 - X_2 = \frac{A(x_3 - Ax_2)}{1 - A^2} && \text{(4.13-b)}\\ u_3 &= x_3 - X_3 = \frac{A(x_2 - Ax_3)}{a - A^2} && \text{(4.13-c)} \end{aligned}$$

4. For the circular surface, and by direct substitution of $X_2 = (x_2 - Ax_3)/(1 - A^2)$, and $X_3 = (x_3 - Ax_2)/(1 - A^2)$ in $X_2^2 + X_3^2 = 1/(1 - A^2)$, the circular surface becomes the elliptical surface $(1 + A^2)x_2^2 - 4Ax_2x_3 + (1 + A^2)x_3^2 = (1 - A^2)$ or for $A = 1/2$, $\boxed{5x_2^2 - 8x_2x_3 + 5x_3^2 = 3}$.

■

4.2.1.1 Lagrangian and Eulerian Descriptions; $\mathbf{x}(\mathbf{X}, t), \mathbf{X}(\mathbf{x}, t)$

35 When the continuum undergoes deformation (or flow), the particles in the continuum move along various paths which can be expressed in either the material coordinates or in the spatial coordinates system giving rise to two different formulations:

Lagrangian Formulation: gives the present location x_i of the particle that occupied the point $(X_1X_2X_3)$ at time $t = 0$, and is a mapping of the initial configuration into the current one.

$$\boxed{x_i = x_i(X_1, X_2, X_3, t) \quad \text{or} \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t)} \tag{4.14}$$

Eulerian Formulation: provides a tracing of its original position of the particle that now occupies the location (x_1, x_2, x_3) at time t, and is a mapping of the current configuration into the initial one.

$$\boxed{X_i = X_i(x_1, x_2, x_3, t) \quad \text{or} \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t)} \tag{4.15}$$

and the independent variables are the coordinates x_i and t.

36 $(\mathbf{X}, t)$ and $(\mathbf{x}, t)$ are the Lagrangian and Eulerian variables respectivly.

37 If $X(x, t)$ is linear, then the deformation is said to be **homogeneous** and plane sections remain plane.

38 For both formulation to constitute a one-to-one mapping, with continuous partial derivatives, they must be the unique inverses of one another. A necessary and unique condition for the inverse functions to exist is that the determinant of the **Jacobian** should not vanish

$$|J| = \left|\frac{\partial x_i}{\partial X_i}\right| \neq 0 \tag{4.16}$$

For example, the Lagrangian description given by

$$x_1 = X_1 + X_2(e^t - 1); \quad x_2 = X_1(e^{-t} - 1) + X_2; \quad x_3 = X_3 \tag{4.17}$$

has the inverse Eulerian description given by

$$X_1 = \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}}; \quad X_2 = \frac{x_1(e^{-t} - 1) - x_2}{1 - e^t - e^{-t}}; \quad X_3 = x_3 \tag{4.18}$$

■ Example 4-2: Lagrangian and Eulerian Descriptions

The Lagrangian description of a deformation is given by $x_1 = X_1 + X_3(e^2 - 1)$, $x_2 = X_2 + X_3(e^2 - e^{-2})$, and $x_3 = e^2 X_3$ where e is a constant. Show that the jacobian does not vanish and determine the Eulerian equations describing the motion.

Solution:

The Jacobian is given by

$$\begin{vmatrix} 1 & 0 & (e^2-1) \\ 0 & 1 & (e^2 - e^{-2}) \\ 0 & 0 & e^2 \end{vmatrix} = e^2 \neq 0 \tag{4.19}$$

Inverting the equation

$$\begin{bmatrix} 1 & 0 & (e^2-1) \\ 0 & 1 & (e^2 - e^{-2}) \\ 0 & 0 & e^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & (e^{-2}-1) \\ 0 & 1 & (e^{-4} - 1) \\ 0 & 0 & e^{-2} \end{bmatrix} \Rightarrow \begin{cases} X_1 &= x_1 + (e^{-2}-1)x_3 \\ X_2 &= x_2 + (e^{-4}-1)x_3 \\ X_3 &= e^{-2}x_3 \end{cases} \tag{4.20}$$

■

4.2.2 Gradients

4.2.2.1 Deformation; $(\mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}}, \mathbf{X}\boldsymbol{\nabla}_{\mathbf{x}})$

39 Partial differentiation of Eq. 4.14 with respect to X_j produces the tensor $\partial x_i/\partial X_j$ which is the **material deformation gradient**. In symbolic notation $\partial x_i/\partial X_j$ is represented by the dyadic

$$\boxed{\mathbf{F} \equiv \mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}} = \frac{\partial \mathbf{x}}{\partial X_1}\mathbf{e}_1 + \frac{\partial \mathbf{x}}{\partial X_2}\mathbf{e}_2 + \frac{\partial \mathbf{x}}{\partial X_3}\mathbf{e}_3 = \frac{\partial x_i}{\partial X_j}} \tag{4.21}$$

The matrix form of $\mathbf{F}$ is

$$\mathcal{F} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \lfloor \frac{\partial}{\partial X_1} \quad \frac{\partial}{\partial X_2} \quad \frac{\partial}{\partial X_3} \rfloor = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \left[\frac{\partial x_i}{\partial X_j}\right] \tag{4.22}$$

40 Similarly, differentiation of Eq. 4.15 with respect to x_j produces the **spatial deformation gradient**

$$\boxed{\mathbf{H} = \mathbf{X}\boldsymbol{\nabla}_{\mathbf{x}} \equiv \frac{\partial \mathbf{X}}{\partial x_1}\mathbf{e}_1 + \frac{\partial \mathbf{X}}{\partial x_2}\mathbf{e}_2 + \frac{\partial \mathbf{X}}{\partial x_3}\mathbf{e}_3 = \frac{\partial X_i}{\partial x_j}} \tag{4.23}$$

The matrix form of $\mathbf{H}$ is

$$\mathcal{H} = \left\{ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right\} \lfloor \begin{array}{ccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{array} \rfloor = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \left[\frac{\partial X_i}{\partial x_j}\right] \tag{4.24}$$

41 The material and spatial deformation tensors are interrelated through the chain rule

$$\frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial X_k} = \delta_{ik} \tag{4.25}$$

and thus $\mathcal{F}^{-1} = \mathcal{H}$ or

$$\boxed{\mathbf{H} = \mathbf{F}^{-1}} \tag{4.26}$$

4.2.2.1.1 † Change of Area Due to Deformation 42 In order to facilitate the derivation of the **Piola-Kirchoff** stress tensor later on, we need to derive an expression for the change in area due to deformation.

43 If we consider two material element $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ emanating from $\mathbf{X}$, the rectangular area formed by them at the reference time t_0 is

$$d\mathbf{A}_0 = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} = dX_1 dX_2 \mathbf{e}_3 = dA_0\mathbf{e}_3 \tag{4.27}$$

44 At time t, $d\mathbf{X}^{(1)}$ deforms into $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ into $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$, and the new area is

$$d\mathbf{A} = \mathbf{F}d\mathbf{X}^{(1)} \times \mathbf{F}d\mathbf{X}^{(2)} = dX_1 dX_2 \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 = dA_0 \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 \tag{4.28-a}$$

$$= dA\mathbf{n} \tag{4.28-b}$$

where the orientation of the deformed area is normal to $\mathbf{F}\mathbf{e}_1$ and $\mathbf{F}\mathbf{e}_2$ which is denoted by the unit vector $\mathbf{n}$. Thus,

$$\mathbf{F}\mathbf{e}_1 \cdot dA\mathbf{n} = \mathbf{F}\mathbf{e}_2 \cdot dA\mathbf{n} = 0 \tag{4.29}$$

and recalling that $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ is equal to the determinant whose rows are components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

$$\mathbf{F}\mathbf{e}_3 \cdot d\mathbf{A} = dA_0 \underbrace{(\mathbf{F}\mathbf{e}_3 \cdot \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2)}_{\det(\mathbf{F})} \tag{4.30}$$

or

$$\mathbf{e}_3 \cdot \mathbf{F}^T\mathbf{n} = \frac{dA_0}{dA}\det(\mathbf{F}) \tag{4.31}$$

and $\mathbf{F}^T\mathbf{n}$ is in the direction of $\mathbf{e}_3$ so that

$$\mathbf{F}^T\mathbf{n} = \frac{dA_0}{dA}\det\mathbf{F}\mathbf{e}_3 \Rightarrow dA\mathbf{n} = dA_0\det(\mathbf{F})(\mathbf{F}^{-1})^T\mathbf{e}_3 \tag{4.32}$$

which implies that the deformed area has a normal in the direction of $(\mathbf{F}^{-1})^T\mathbf{e}_3$. A generalization of the preceding equation would yield

$$\boxed{dA\mathbf{n} = dA_0\det(\mathbf{F})(\mathbf{F}^{-1})^T\mathbf{n}_0} \tag{4.33}$$

4.2.2.1.2 † Change of Volume Due to Deformation 45 If we consider an infinitesimal element it has the following volume in material coordinate system:

$$d\Omega_0 = (dX_1\mathbf{e}_1 \times dX_2\mathbf{e}_2)\cdot dX_3\mathbf{e}_3 = dX_1dX_2dX_3 \tag{4.34}$$

in spatial cordiantes:

$$d\Omega = (dx_1\mathbf{e}_1 \times dx_2\mathbf{e}_2)\cdot dx_3\mathbf{e}_3 \tag{4.35}$$

If we define

$$\mathbf{F}_i = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \tag{4.36}$$

then the deformed volume will be

$$d\Omega = (\mathbf{F}_1dX_1 \times \mathbf{F}_2dX_2)\cdot\mathbf{F}_3dX_3 = (\mathbf{F}_1 \times \mathbf{F}_2\cdot\mathbf{F}_3)dX_1dX_2dX_3 \tag{4.37}$$

or

$$\boxed{d\Omega = \det\mathbf{F}d\Omega_0} \tag{4.38}$$

and J is called the **Jacobian** and is the determinant of the deformation gradient $\mathbf{F}$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \tag{4.39}$$

and thus the **Jacobian is a measure of deformation.**

46 We observe that if a material is **incompressible** than $\det\mathbf{F} = 1$.

■ Example 4-3: Change of Volume and Area

For the following deformation: $x_1 = \lambda_1X_1$, $x_2 = -\lambda_3X_3$, and $x_3 = \lambda_2X_2$, find the deformed volume for a unit cube and the deformed area of the unit square in the $X_1 - X_2$ plane.

Solution:

$$
\begin{aligned}
[\mathbf{F}] &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} && \text{(4.40-a)} \\
\det \mathbf{F} &= \lambda_1\lambda_2\lambda_3 && \text{(4.40-b)} \\
\Delta V &= \lambda_1\lambda_2\lambda_3 && \text{(4.40-c)} \\
\Delta A_0 &= 1 && \text{(4.40-d)} \\
\mathbf{n}_0 &= -\mathbf{e}_3 && \text{(4.40-e)} \\
\Delta A\mathbf{n} &= (1)(\det \mathbf{F})(\mathbf{F}^{-1})^T && \text{(4.40-f)} \\
&= \lambda_1\lambda_2\lambda_3 \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda_3} \\ 0 & \frac{1}{\lambda_2} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \lambda_1\lambda_2 \\ 0 \end{Bmatrix} && \text{(4.40-g)}
\end{aligned}
$$

$$\Delta A\mathbf{n} = \lambda_1\lambda_2\mathbf{e}_2 \tag{4.40-h}$$

■

4.2.2.2 Displacements; $(\mathbf{u}\nabla_\mathbf{X}, \mathbf{u}\nabla_\mathbf{x})$

47 We now turn our attention to the displacement vector u_i as given by Eq. 4.9. Partial differentiation of Eq. 4.9 with respect to X_j produces the **material displacement gradient**

$$\boxed{\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \quad \text{or} \quad \mathbf{J} \equiv \mathbf{u}\nabla_\mathbf{X} = \mathbf{F} - \mathbf{I}} \tag{4.41}$$

The matrix form of $\mathbf{J}$ is

$$\mathcal{J} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \lfloor \; \frac{\partial}{\partial X_1} \quad \frac{\partial}{\partial X_2} \quad \frac{\partial}{\partial X_3} \; \rfloor = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} = \left[\frac{\partial u_i}{\partial X_j}\right] \tag{4.42}$$

48 Similarly, differentiation of Eq. 4.9 with respect to x_j produces the **spatial displacement gradient**

$$\boxed{\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} \quad \text{or} \quad \mathbf{K} \equiv \mathbf{u}\nabla_\mathbf{x} = \mathbf{I} - \mathbf{H}} \tag{4.43}$$

The matrix form of $\mathbf{K}$ is

$$\mathcal{K} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \lfloor \; \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \; \rfloor = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \left[\frac{\partial u_i}{\partial x_j}\right] \tag{4.44}$$

4.2.2.3 Examples

■ Example 4-4: Material Deformation and Displacement Gradients

A displacement field is given by $\mathbf{u} = X_1X_3^2\mathbf{e}_1 + X_1^2X_2\mathbf{e}_2 + X_2^2X_3\mathbf{e}_3$, determine the material deformation gradient $\mathbf{F}$ and the material displacement gradient $\mathbf{J}$, and verify that $\mathbf{J} = \mathbf{F} - \mathbf{I}$.

Solution:

The material deformation gradient is:

$$\frac{\partial u_i}{\partial X_j} = \mathbf{J} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} = = \begin{bmatrix} \frac{\partial u_{X_1}}{\partial X_1} & \frac{\partial u_{X_1}}{\partial X_2} & \frac{\partial u_{X_1}}{\partial X_3} \\ \frac{\partial u_{X_2}}{\partial X_1} & \frac{\partial u_{X_2}}{\partial X_2} & \frac{\partial u_{X_2}}{\partial X_3} \\ \frac{\partial u_{X_3}}{\partial X_1} & \frac{\partial u_{X_3}}{\partial X_2} & \frac{\partial u_{X_3}}{\partial X_3} \end{bmatrix} \tag{4.45-a}$$

$$= \begin{bmatrix} X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & X_1^2 & 0 \\ 0 & 2X_2X_3 & X_2^2 \end{bmatrix} \tag{4.45-b}$$

Since $\mathbf{x} = \mathbf{u} + \mathbf{X}$, the displacement field is also given by

$$\mathbf{x} = \underbrace{X_1(1 + X_3^2)}_{x_1}\mathbf{e}_1 + \underbrace{X_2(1 + X_1^2)}_{x_2}\mathbf{e}_2 + \underbrace{X_3(1 + X_2^2)}_{x_3}\mathbf{e}_3 \tag{4.46}$$

and thus

$$\mathbf{F} = \mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial X_1}\mathbf{e}_1 + \frac{\partial \mathbf{x}}{\partial X_2}\mathbf{e}_2 + \frac{\partial \mathbf{x}}{\partial X_3}\mathbf{e}_3 = \frac{\partial x_i}{\partial X_j} \tag{4.47-a}$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{4.47-b}$$

$$= \begin{bmatrix} 1 + X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & 1 + X_1^2 & 0 \\ 0 & 2X_2X_3 & 1 + X_2^2 \end{bmatrix} \tag{4.47-c}$$

We observe that the two second order tensors are related by $\mathbf{J} = \mathbf{F} - \mathbf{I}$. ■

4.2.3 Deformation Tensors

49 Having derived expressions for $\frac{\partial x_i}{\partial X_j}$ and $\frac{\partial X_i}{\partial x_j}$ we now seek to determine dx^2 and dX^2 where dX and dx correspond to the distance between points P and Q in the undeformed and deformed cases respectively.

50 We consider next the initial (undeformed) and final (deformed) configuration of a continuum in which the material OX_1, X_2, X_3 and spatial coordinates $ox_1x_2x_3$ are superimposed. Neighboring particles P_0 and Q_0 in the initial configurations moved to P and Q respectively in the final one, Fig. 4.4.

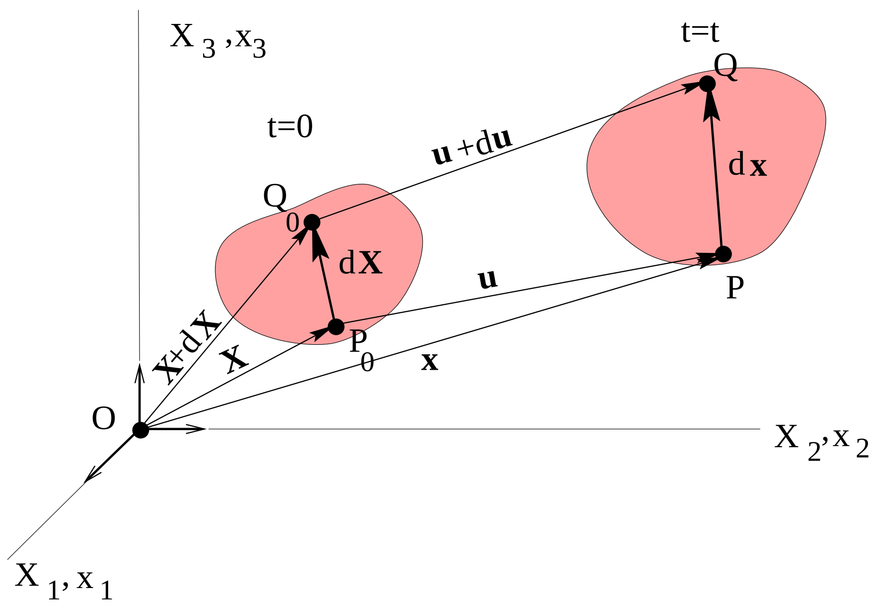


Figure 4.4: Undeformed and Deformed Configurations of a Continuum

4.2.3.1 Cauchy's Deformation Tensor; $(dX)^2$

51 The Cauchy deformation tensor, introduced by Cauchy in 1827, $\mathbf{B}^{-1}$ (alternatively denoted as $\mathbf{c}$) gives the initial square length $(dX)^2$ of an element $d\mathbf{x}$ in the deformed configuration.

52 This tensor is the inverse of the tensor $\mathbf{B}$ which will not be introduced until Sect. 4.2.6.3.

53 The square of the differential element connecting P_o and Q_0 is

$$(dX)^2 = d\mathbf{X}\cdot d\mathbf{X} = dX_i dX_i \tag{4.48}$$

however from Eq. 4.15 the distance differential dX_i is

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j \quad \text{or} \quad d\mathbf{X} = \mathbf{H}\cdot d\mathbf{x} \tag{4.49}$$

thus the squared length $(dX)^2$ in Eq. 4.48 may be rewritten as

$$\begin{aligned} (dX)^2 &= \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j = B_{ij}^{-1} dx_i dx_j & \text{(4.50-a)} \\ &= d\mathbf{x}\cdot\mathbf{B}^{-1}\cdot d\mathbf{x} & \text{(4.50-b)} \end{aligned}$$

in which the second order tensor

$$\boxed{B_{ij}^{-1} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} \quad \text{or} \quad \mathbf{B}^{-1} = \underbrace{\nabla_{\mathbf{x}}\mathbf{X}\cdot\mathbf{X}\nabla_{\mathbf{x}}}_{\mathbf{H}_c\cdot\mathbf{H}}} \tag{4.51}$$

is **Cauchy's deformation tensor**.

4.2.3.2 Green's Deformation Tensor; $(dx)^2$

54 The Green deformation tensor, introduced by Green in 1841, $\mathbf{C}$ (alternatively denoted as $\mathbf{B}^{-1}$), referred to in the undeformed configuration, gives the new square length $(dx)^2$ of the element $d\mathbf{X}$ is deformed.

55 The square of the differential element connecting P_o and Q_0 is now evaluated in terms of the spatial coordinates

$$(dx)^2 = d\mathbf{x}\cdot d\mathbf{x} = dx_i dx_i \tag{4.52}$$

however from Eq. 4.14 the distance differential dx_i is

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j \quad \text{or} \quad d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} \tag{4.53}$$

thus the squared length $(dx)^2$ in Eq. 4.52 may be rewritten as

$$(dx)^2 = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} dX_i dX_j = C_{ij} dX_i dX_j \tag{4.54-a}$$

$$= d\mathbf{X}\cdot\mathbf{C}\cdot d\mathbf{X} \tag{4.54-b}$$

in which the second order tensor

$$\boxed{C_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} \quad \text{or} \quad \mathbf{C} = \underbrace{\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{x}\cdot\mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}}}_{\mathbf{F}_c\cdot\mathbf{F}}} \tag{4.55}$$

is **Green's deformation tensor** also known as **metric tensor**, or **deformation tensor** or **right Cauchy-Green deformation tensor**.

56 Inspection of Eq. 4.51 and Eq. 4.55 yields

$$\boxed{\mathbf{C}^{-1} = \mathbf{B}^{-1} \quad \text{or} \quad \mathbf{B}^{-1} = (\mathbf{F}^{-1})^T\cdot\mathbf{F}^{-1}} \tag{4.56}$$

■ Example 4-5: Green's Deformation Tensor

A continuum body undergoes the deformation $x_1 = X_1$, $x_2 = X_2 + AX_3$, and $x_3 = X_3 + AX_2$ where A is a constant. Determine the deformation tensor $\mathbf{C}$.

Solution:

From Eq. 4.55 $\mathbf{C} = \mathbf{F}_c\cdot\mathbf{F}$ where $\mathbf{F}$ was defined in Eq. 4.21 as

$$\mathbf{F} = \frac{\partial x_i}{\partial X_j} \tag{4.57-a}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} \tag{4.57-b}$$

and thus

$$\mathbf{C} = \mathbf{F}_c\cdot\mathbf{F} \tag{4.58-a}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix} \tag{4.58-b}$$

■

4.2.4 Strains; $(dx)^2 - (dX)^2$

57 With $(dx)^2$ and $(dX)^2$ defined we can now finally introduce the concept of strain through $(dx)^2 - (dX)^2$.

4.2.4.1 Finite Strain Tensors

58 We start with the most general case of finite strains where no constraints are imposed on the deformation (small).

4.2.4.1.1 Lagrangian/Green's Tensor

59 The difference $(dx)^2 - (dX)^2$ for two neighboring particles in a continuum is used as the **measure of deformation**. Using Eqs. 4.54-a and 4.48 this difference is expressed as

$$(dx)^2 - (dX)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i dX_j = 2E_{ij}dX_i dX_j \tag{4.59-a}$$
$$= d\mathbf{X}\cdot(\mathbf{F}_c\cdot\mathbf{F} - \mathbf{I})\cdot d\mathbf{X} = 2d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X} \tag{4.59-b}$$

in which the second order tensor

$$\boxed{E_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\underbrace{\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{x}\cdot\mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}}}_{\mathbf{F}_c\cdot\mathbf{F}=\mathbf{C}} - \mathbf{I})} \tag{4.60}$$

is called the **Lagrangian (or Green's) finite strain tensor** which was introduced by Green in 1841 and St-Venant in 1844.

60 To express the Lagrangian tensor in terms of the displacements, we substitute Eq. 4.41 in the preceding equation, and after some simple algebraic manipulations, the Lagrangian finite strain tensor can be rewritten as

$$\boxed{E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\underbrace{\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} + \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}}_{\mathbf{J}+\mathbf{J}_c} + \underbrace{\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\cdot\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}}}_{\mathbf{J}_c\cdot\mathbf{J}})} \tag{4.61}$$

or:

$$E_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right] \quad (4.62\text{-a})$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) + \frac{1}{2}\left[\frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right] \quad (4.62\text{-b})$$

$$\cdots = \cdots \quad (4.62\text{-c})$$

■ **Example 4-6: Lagrangian Tensor**

Determine the Lagrangian finite strain tensor **E** for the deformation of example 4.2.3.2.
Solution:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix} \quad (4.63\text{-a})$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \quad (4.63\text{-b})$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & A^2 & 2A \\ 0 & 2A & A^2 \end{bmatrix} \quad (4.63\text{-c})$$

Note that the matrix is symmetric. ■

4.2.4.1.2 Eulerian/Almansi's Tensor

61 Alternatively, the difference $(dx)^2 - (dX)^2$ for the two neighboring particles in the continuum can be expressed in terms of Eqs. 4.52 and 4.50-b this same difference is now equal to

$$(dx)^2 - (dX)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i dx_j = 2E^*_{ij} dx_i dx_j \quad (4.64\text{-a})$$

$$= d\mathbf{x}\cdot(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H})\cdot d\mathbf{x} = 2d\mathbf{x}\cdot\mathbf{E}^*\cdot d\mathbf{x} \quad (4.64\text{-b})$$

in which the second order tensor

$$\boxed{E^*_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}^* = \frac{1}{2}(\mathbf{I} - \underbrace{\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{X}\cdot\mathbf{X}\boldsymbol{\nabla}_{\mathbf{x}}}_{\mathbf{H}_c\cdot\mathbf{H}=\mathbf{B}^{-1}})} \quad (4.65)$$

is called the **Eulerian (or Almansi) finite strain tensor**.

62 For infinitesimal strain it was introduced by Cauchy in 1827, and for finite strain by Almansi in 1911.

63 To express the Eulerian tensor in terms of the displacements, we substitute 4.43 in the preceding equation, and after some simple algebraic manipulations, the Eulerian finite strain tensor can be rewritten as

$$\boxed{E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}^* = \frac{1}{2}(\underbrace{\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} + \boldsymbol{\nabla}_\mathbf{x}\mathbf{u}}_{\mathbf{K}+\mathbf{K}_c} - \underbrace{\boldsymbol{\nabla}_\mathbf{x}\mathbf{u}\cdot\mathbf{u}\boldsymbol{\nabla}_\mathbf{x}}_{\mathbf{K}_c\cdot\mathbf{K}})} \tag{4.66}$$

64 Expanding

$$E^*_{11} = \frac{\partial u_1}{\partial x_1} - \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \tag{4.67-a}$$

$$E^*_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) - \frac{1}{2}\left[\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right] \tag{4.67-b}$$

$$\cdots = \cdots \tag{4.67-c}$$

4.2.4.2 Infinitesimal Strain Tensors; Small Deformation Theory

65 The **small deformation theory** of continuum mechanics has as basic condition the requirement that the displacement gradients be small compared to unity. The fundamental measure of deformation is the difference $(dx)^2 - (dX)^2$, which may be expressed in terms of the displacement gradients by inserting Eq. 4.61 and 4.66 into 4.59-b and 4.64-b respectively. If the displacement gradients are small, the finite strain tensors in Eq. 4.59-b and 4.64-b reduce to **infinitesimal strain tensors** and the resulting equations represent **small deformations**.

66 For instance, if we were to evaluate $\epsilon + \epsilon^2$, for $\epsilon = 10^{-3}$ and 10^{-1}, then we would obtain $0.001001 \approx 0.001$ and 0.11 respectively. In the first case ϵ^2 is "negligible" compared to ϵ, in the other it is not.

4.2.4.2.1 Lagrangian Infinitesimal Strain Tensor

67 In Eq. 4.61 if the displacement gradient components $\frac{\partial u_i}{\partial X_j}$ are each small compared to unity, then the third term are negligible and may be dropped. The resulting tensor is the **Lagrangian infinitesimal strain tensor** denoted by

$$\boxed{E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\underbrace{\mathbf{u}\boldsymbol{\nabla}_\mathbf{X} + \boldsymbol{\nabla}_\mathbf{X}\mathbf{u}}_{\mathbf{J}+\mathbf{J}_c})} \tag{4.68}$$

or:

$$E_{11} = \frac{\partial u_1}{\partial X_1} \tag{4.69-a}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) \tag{4.69-b}$$

$$\cdots = \cdots \tag{4.69-c}$$

Note the similarity with Eq. 4.4.

4.2.4.2.2 Eulerian Infinitesimal Strain Tensor

68 Similarly, inn Eq. 4.66 if the displacement gradient components $\frac{\partial u_i}{\partial x_j}$ are each small compared to unity, then the third term are negligible and may be dropped. The resulting tensor is the **Eulerian infinitesimal strain tensor** denoted by

$$\boxed{E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{E}^* = \frac{1}{2}\underbrace{(\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} + \boldsymbol{\nabla}_\mathbf{x}\mathbf{u})}_{\mathbf{K}+\mathbf{K}_c}} \tag{4.70}$$

69 Expanding

$$E^*_{11} = \frac{\partial u_1}{\partial x_1} \tag{4.71-a}$$

$$E^*_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{4.71-b}$$

$$\cdots = \cdots \tag{4.71-c}$$

4.2.4.3 Examples

■ Example 4-7: Lagrangian and Eulerian Linear Strain Tensors

A displacement field is given by $x_1 = X_1 + AX_2, x_2 = X_2 + AX_3, x_3 = X_3 + AX_1$ where A is constant. Calculate the Lagrangian and the Eulerian linear strain tensors, and compare them for the case where A is very small.

Solution:

The displacements are obtained from Eq. 4.9 $u_k = x_k - X_k$ or

$$u_1 = x_1 - X_1 = X_1 + AX_2 - X_1 = AX_2 \tag{4.72-a}$$

$$u_2 = x_2 - X_2 = X_2 + AX_3 - X_2 = AX_3 \tag{4.72-b}$$

$$u_3 = x_3 - X_3 = X_3 + AX_1 - X_3 = AX_1 \tag{4.72-c}$$

then from Eq. 4.41

$$\mathbf{J} \equiv \mathbf{u}\boldsymbol{\nabla}_\mathbf{X} = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{bmatrix} \tag{4.73}$$

From Eq. 4.68:

$$2\mathbf{E} = (\mathbf{J}+\mathbf{J}_c) = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A \\ A & 0 & 0 \\ 0 & A & 0 \end{bmatrix} \tag{4.74-a}$$

$$= \begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} \tag{4.74-b}$$

To determine the Eulerian tensor, we need the displacement u in terms of x, thus inverting the displacement field given above:

$$\left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right\} = \begin{bmatrix} 1 & A & 0 \\ 0 & 1 & A \\ A & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right\} = \frac{1}{1+A^3} \begin{bmatrix} 1 & -A & A^2 \\ A^2 & 1 & -A \\ -A & A^2 & 1 \end{bmatrix} \left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right\} \tag{4.75}$$

thus from Eq. 4.9 $u_k = x_k - X_k$ we obtain

$$u_1 = x_1 - X_1 = x_1 - \frac{1}{1+A^3}(x_1 - Ax_2 + A^2x_3) = \frac{A(A^2x_1 + x_2 - Ax_3)}{1+A^3} \tag{4.76-a}$$

$$u_2 = x_2 - X_2 = x_2 - \frac{1}{1+A^3}(A^2x_1 + x_2 - Ax_3) = \frac{A(-Ax_1 + A^2x_2 + x_3)}{1+A^3} \tag{4.76-b}$$

$$u_3 = x_3 - X_3 = x_3 - \frac{1}{1+A^3}(-Ax_1 + A^2x_2 + x_3) = \frac{A(x_1 - Ax_2 + A^2x_3)}{1+A^3} \tag{4.76-c}$$

From Eq. 4.43

$$\mathbf{K} \equiv \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} = \frac{A}{1+A^3} \begin{bmatrix} A^2 & 1 & -A \\ -A & A^2 & 1 \\ 1 & -A & A^2 \end{bmatrix} \tag{4.77}$$

Finally, from Eq. 4.66

$$2\mathbf{E}^* = \mathbf{K} + \mathbf{K}_c \tag{4.78-a}$$

$$= \frac{A}{1+A^3} \begin{bmatrix} A^2 & 1 & -A \\ -A & A^2 & 1 \\ 1 & -A & A^2 \end{bmatrix} + \frac{A}{1+A^3} \begin{bmatrix} A^2 & -A & 1 \\ 1 & A^2 & -A \\ -A & 1 & A^2 \end{bmatrix} \tag{4.78-b}$$

$$= \frac{A}{1+A^3} \begin{bmatrix} 2A^2 & 1-A & 1-A \\ 1-A & 2A^2 & 1-A \\ 1-A & 1-A & 2A^2 \end{bmatrix} \tag{4.78-c}$$

as A is very small, A^2 and higher power may be neglected with the results, then $\mathbf{E}^* \to \mathbf{E}$.
■

4.2.5 Physical Interpretation of the Strain Tensor

4.2.5.1 Small Strain

70 We finally show that the linear lagrangian tensor in small deformation E_{ij} is nothing else than the strain as was defined earlier in Eq.4.4.

71 We rewrite Eq. 4.59-b as

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = 2E_{ij}dX_i dX_j \quad (4.79\text{-a})$$

or

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = d\mathbf{X}\cdot 2\mathbf{E}\cdot d\mathbf{X} \quad (4.79\text{-b})$$

but since $dx \approx dX$ under current assumption of small deformation, then the previous equation can be rewritten as

$$\frac{\overbrace{dx - dX}^{du}}{dX} = E_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} = E_{ij}\xi_i\xi_j = \boldsymbol{\xi}\cdot\mathbf{E}\cdot\boldsymbol{\xi} \quad (4.80)$$

72 We recognize that the left hand side is nothing else than the change in length per unit original length, and is called the **normal strain** for the line element having direction cosines $\frac{dX_i}{dX}$.

73 With reference to Fig. 4.5 we consider two cases: normal and shear strain.

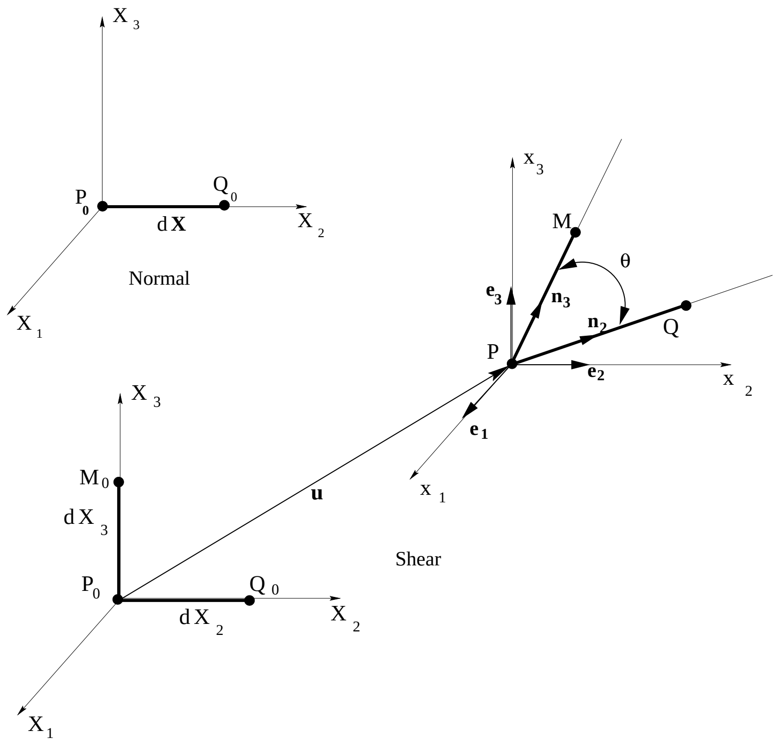


Figure 4.5: Physical Interpretation of the Strain Tensor

Normal Strain: When Eq. 4.80 is applied to the differential element P_0Q_0 which lies along the X_2 axis, the result will be the normal strain because since $\frac{dX_1}{dX} = \frac{dX_3}{dX} = 0$ and $\frac{dX_2}{dX} = 1$. Therefore, Eq. 4.80 becomes (with $u_i = x_i - X_i$):

$$\boxed{\frac{dx - dX}{dX} = E_{22} = \frac{\partial u_2}{\partial X_2}} \quad (4.81)$$

Likewise for the other 2 directions. Hence the diagonal terms of the linear strain tensor represent normal strains in the coordinate system.

Shear Strain: For the diagonal terms E_{ij} we consider the two line elements originally located along the X_2 and the X_3 axes before deformation. After deformation, the original right angle between the lines becomes the angle θ. From Eq. 4.96 ($du_i = \left(\frac{\partial u_i}{\partial X_j}\right)_{P_0} dX_j$) a first order approximation gives the unit vector at P in the direction of Q, and M as:

$$\mathbf{n}_2 = \frac{\partial u_1}{\partial X_2}\mathbf{e}_1 + \mathbf{e}_2 + \frac{\partial u_3}{\partial X_2}\mathbf{e}_3 \quad (4.82\text{-a})$$

$$\mathbf{n}_3 = \frac{\partial u_1}{\partial X_3}\mathbf{e}_1 + \frac{\partial u_2}{\partial X_3}\mathbf{e}_2 + \mathbf{e}_3 \quad (4.82\text{-b})$$

and from the definition of the dot product:

$$\cos\theta = \mathbf{n}_2 \cdot \mathbf{n}_3 = \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \quad (4.83)$$

or neglecting the higher order term

$$\boxed{\cos\theta = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} = 2E_{23}} \quad (4.84)$$

74 Finally taking the change in right angle between the elements as $\gamma_{23} = \pi/2 - \theta$, and recalling that for small strain theory γ_{23} is very small it follows that

$$\boxed{\gamma_{23} \approx \sin\gamma_{23} = \sin(\pi/2 - \theta) = \cos\theta = 2E_{23}.} \quad (4.85)$$

Therefore the off diagonal terms of the linear strain tensor represent one half of the angle change between two line elements originally at right angles to one another. These components are called the **shear strains**.

74 The **Engineering shear strain** is defined as one half the tensorial shear strain, and the resulting tensor is written as

$$\boxed{E_{ij} = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \varepsilon_{33} \end{bmatrix}} \quad (4.86)$$

75 We note that a similar development paralleling the one just presented can be made for the linear Eulerian strain tensor (where the straight lines and right angle will be in the deformed state).

4.2.5.2 Finite Strain; Stretch Ratio

76 The simplest and most useful measure of the extensional strain of an infinitesimal element is the **stretch** or **stretch ratio** as $\frac{dx}{dX}$ which may be defined at point P_0 in the

undeformed configuration or at P in the deformed one (Refer to the original definition given by Eq, 4.1).

77 Hence, from Eq. 4.54-a, and Eq. 4.60 the squared stretch at P_0 for the line element along the unit vector $\mathbf{m} = \frac{d\mathbf{X}}{dX}$ is given by

$$\Lambda^2_{\mathbf{m}} \equiv \left(\frac{dx}{dX}\right)^2_{P_0} = C_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} \quad \text{or} \quad \Lambda^2_{\mathbf{m}} = \mathbf{m}\cdot\mathbf{C}\cdot\mathbf{m} \tag{4.87}$$

Thus for an element originally along X_2, Fig. 4.5, $\mathbf{m} = \mathbf{e}_2$ and therefore $dX_1/dX = dX_3/dX = 0$ and $dX_2/dX = 1$, thus Eq. 4.87 (with Eq. **??**) yields

$$\Lambda^2_{\mathbf{e}_2} = C_{22} = 1 + 2E_{22} \tag{4.88}$$

and similar results can be obtained for $\Lambda^2_{\mathbf{e}_1}$ and $\Lambda^2_{\mathbf{e}_3}$.

78 Similarly from Eq. 4.50-b, the reciprocal of the squared stretch for the line element at P along the unit vector $\mathbf{n} = \frac{d\mathbf{x}}{dx}$ is given by

$$\frac{1}{\lambda^2_{\mathbf{n}}} \equiv \left(\frac{dX}{dx}\right)^2_{P} = B^{-1}_{ij}\frac{dx_i}{dx}\frac{dx_j}{dx} \quad \text{or} \quad \frac{1}{\lambda^2_{\mathbf{n}}} = \mathbf{n}\cdot\mathbf{B}^{-1}\cdot\mathbf{n} \tag{4.89}$$

Again for an element originally along X_2, Fig. 4.5, we obtain

$$\frac{1}{\lambda^2_{\mathbf{e}_2}} = 1 - 2E^*_{22} \tag{4.90}$$

79 we note that in general $\Lambda_{\mathbf{e}_2} \neq \lambda_{\mathbf{e}_2}$ since the element originally along the X_2 axis will not be along the x_2 after deformation. Furthermore Eq. 4.87 and 4.89 show that in the matrices of rectangular cartesian components the diagonal elements of both $\mathbf{C}$ and $\mathbf{B}^{-1}$ must be positive, while the elements of $\mathbf{E}$ must be greater than $-\frac{1}{2}$ and those of $\mathbf{E}^*$ must be greater than $+\frac{1}{2}$.

80 The unit extension of the element is

$$\frac{dx - dX}{dX} = \frac{dx}{dX} - 1 = \Lambda_{\mathbf{m}} - 1 \tag{4.91}$$

and for the element P_0Q_0 along the X_2 axis, the **unit extension** is

$$\frac{dx - dX}{dX} = E_{(2)} = \Lambda_{\mathbf{e}_2} - 1 = \sqrt{1 + 2E_{22}} - 1 \tag{4.92}$$

for small deformation theory $E_{22} << 1$, and

$$\frac{dx - dX}{dX} = E_{(2)} = (1 + 2E_{22})^{\frac{1}{2}} - 1 \simeq 1 + \frac{1}{2}2E_{22} - 1 \simeq E_{22} \tag{4.93}$$

which is identical to Eq. 4.81.

81 For the two differential line elements of Fig. 4.5, the change in angle $\gamma_{23} = \frac{\pi}{2} - \theta$ is given in terms of both $\Lambda_{\mathbf{e}_2}$ and $\Lambda_{\mathbf{e}_3}$ by

$$\sin\gamma_{23} = \frac{2E_{23}}{\Lambda_{\mathbf{e}_2}\Lambda_{\mathbf{e}_3}} = \frac{2E_{23}}{\sqrt{1 + 2E_{22}}\sqrt{1 + 2E_{33}}} \tag{4.94}$$

Again, when deformations are small, this equation reduces to Eq. 4.85.

4.2.6 Linear Strain and Rotation Tensors

82 Strain components are quantitative measures of certain type of relative displacement between neighboring parts of the material. A solid material will resist such relative displacement giving rise to internal stresses.

83 Not all kinds of relative motion give rise to strain (and stresses). If a body moves as a **rigid body**, the rotational part of its motion produces relative displacement. Thus the general problem is to express the strain in terms of the displacements by separating off that part of the displacement distribution which does not contribute to the strain.

4.2.6.1 Small Strains

84 From Fig. 4.6 the displacements of two neighboring particles are represented by the vectors u^{P_0} and u^{Q_0} and the vector

$$du_i = u_i^{Q_0} - u_i^{P_0} \quad \text{or} \quad d\mathbf{u} = \mathbf{u}^{Q_0} - \mathbf{u}^{P_0} \tag{4.95}$$

is called the **relative displacement vector** of the particle originally at Q_0 with respect to the one originally at P_0.

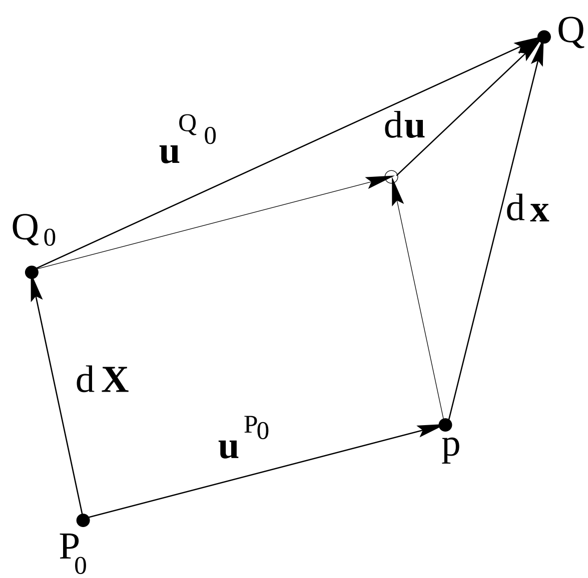


Figure 4.6: Relative Displacement $d\mathbf{u}$ of Q relative to P

4.2.6.1.1 Lagrangian Formulation

85 Neglecting higher order terms, and through a Taylor expansion

$$du_i = \left(\frac{\partial u_i}{\partial X_j}\right)_{P_0} dX_j \quad \text{or} \quad d\mathbf{u} = (\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}})_{P_0}\, d\mathbf{X} \tag{4.96}$$

86 We also define a **unit relative displacement vector** du_i/dX where dX is the magnitude of the differential distance dX_i, or $dX_i = \xi_i dX$, then

$$\frac{du_i}{dX} = \frac{\partial u_i}{\partial X_j}\frac{dX_j}{dX} = \frac{\partial u_i}{\partial X_j}\xi_j \quad \text{or} \quad \frac{d\mathbf{u}}{dX} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}}\cdot\boldsymbol{\xi} = \mathbf{J}\cdot\boldsymbol{\xi} \tag{4.97}$$

87 The material displacement gradient $\frac{\partial u_i}{\partial X_j}$ can be decomposed uniquely into a symmetric and an antisymetric part, we rewrite the previous equation as

$$du_i = \left[\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}_{E_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)}_{W_{ij}}\right] dX_j \tag{4.98-a}$$

or

$$d\mathbf{u} = \left[\underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} + \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\right)}_{\mathbf{E}} + \underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} - \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\right)}_{\mathbf{W}}\right]\cdot d\mathbf{X} \tag{4.98-b}$$

or

$$\mathbf{E} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{\partial u_2}{\partial X_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{4.99}$$

We thus introduce the **linear lagrangian rotation tensor**

$$\boxed{W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} - \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\right)} \tag{4.100}$$

in matrix form:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} - \frac{\partial u_3}{\partial X_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} - \frac{\partial u_3}{\partial X_2}\right) & 0 \end{bmatrix} \tag{4.101}$$

88 In a displacement for which E_{ij} is zero in the vicinity of a point P_0, the relative displacement at that point will be an infinitesimal **rigid body rotation**. It can be shown that this rotation is given by the **linear Lagrangian rotation vector**

$$\boxed{w_i = \frac{1}{2}\epsilon_{ijk}W_{kj} \quad \text{or} \quad \mathbf{w} = \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{X}}\times\mathbf{u}} \tag{4.102}$$

or

$$\mathbf{w} = -W_{23}\mathbf{e}_1 - W_{31}\mathbf{e}_2 - W_{12}\mathbf{e}_3 \tag{4.103}$$

4.2.6.1.2 Eulerian Formulation

89 The derivation in an Eulerian formulation parallels the one for Lagrangian formulation. Hence,

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \quad \text{or} \quad d\mathbf{u} = \mathbf{K}\cdot d\mathbf{x} \tag{4.104}$$

90 The **unit relative displacement vector** will be

$$du_i = \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dx} = \frac{\partial u_i}{\partial x_j}\eta_j \quad \text{or} \quad \frac{d\mathbf{u}}{dx} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}\cdot\boldsymbol{\eta} = \mathbf{K}\cdot\beta \tag{4.105}$$

91 The decomposition of the Eulerian displacement gradient $\frac{\partial u_i}{\partial x_j}$ results in

$$du_i = \left[\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}_{E^*_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}}\right] dx_j \tag{4.106-a}$$

or

$$d\mathbf{u} = \left[\underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}+\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\right)}_{\mathbf{E}^*} + \underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}-\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\right)}_{\boldsymbol{\Omega}}\right]\cdot d\mathbf{x} \tag{4.106-b}$$

or

$$\mathbf{E} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{4.107}$$

92 We thus introduced the **linear Eulerian rotation tensor**

$$\boxed{w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \boldsymbol{\Omega} = \frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}-\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\right)} \tag{4.108}$$

in matrix form:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) & 0 \end{bmatrix} \tag{4.109}$$

and the **linear Eulerian rotation vector** will be

$$\boxed{\omega_i = \frac{1}{2}\epsilon_{ijk}\omega_{kj} \quad \text{or} \quad \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{x}}\times\mathbf{u}} \tag{4.110}$$

4.2.6.2 Examples

■ Example 4-8: Relative Displacement along a specified direction

A displacement field is specified by $\mathbf{u} = X_1^2X_2\mathbf{e}_1 + (X_2 - X_3^2)\mathbf{e}_2 + X_2^2X_3\mathbf{e}_3$. Determine the relative displacement vector $d\mathbf{u}$ in the direction of the $-X_2$ axis at $P(1,2,-1)$. Determine the relative displacements $\mathbf{u}_{Q_i} - \mathbf{u}_P$ for $Q_1(1,1,-1)$, $Q_2(1,3/2,-1)$, $Q_3(1,7/4,-1)$ and $Q_4(1,15/8,-1)$ and compute their directions with the direction of $d\mathbf{u}$.
Solution:
From Eq. 4.41, $\mathbf{J} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}}$ or

$$\frac{\partial u_i}{\partial X_j} = \begin{bmatrix} 2X_1X_2 & X_1^2 & 0 \\ 0 & 1 & -2X_3 \\ 0 & 2X_2X_3 & X_2^2 \end{bmatrix} \tag{4.111}$$

thus from Eq. 4.96 $d\mathbf{u} = (\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}})_P\, d\mathbf{X}$ in the direction of $-X_2$ or

$$\{d\mathbf{u}\} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \\ 4 \end{Bmatrix} \tag{4.112}$$

By direct calculation from $\mathbf{u}$ we have

$$\mathbf{u}_P = 2\mathbf{e}_1 + \mathbf{e}_2 - 4\mathbf{e}_3 \tag{4.113-a}$$
$$\mathbf{u}_{Q_1} = \mathbf{e}_1 - \mathbf{e}_3 \tag{4.113-b}$$

thus

$$\mathbf{u}_{Q_1} - \mathbf{u}_P = -\mathbf{e}_1 - \mathbf{e}_2 + 3\mathbf{e}_3 \tag{4.114-a}$$
$$\mathbf{u}_{Q_2} - \mathbf{u}_P = \frac{1}{2}(-\mathbf{e}_1 - \mathbf{e}_2 + 3.5\mathbf{e}_3) \tag{4.114-b}$$
$$\mathbf{u}_{Q_3} - \mathbf{u}_P = \frac{1}{4}(-\mathbf{e}_1 - \mathbf{e}_2 + 3.75\mathbf{e}_3) \tag{4.114-c}$$
$$\mathbf{u}_{Q_4} - \mathbf{u}_P = \frac{1}{8}(-\mathbf{e}_1 - \mathbf{e}_2 + 3.875\mathbf{e}_3) \tag{4.114-d}$$

and it is clear that as Q_i approaches P, the direction of the relative displacements of the two particles approaches the limiting direction of $d\mathbf{u}$. ■

■ Example 4-9: Linear strain tensor, linear rotation tensor, rotation vector

Under the restriction of small deformation theory $\mathbf{E} = \mathbf{E}^*$, a displacement field is given by $\mathbf{u} = (x_1 - x_3)^2\mathbf{e}_1 + (x_2 + x_3)^2\mathbf{e}_2 - x_1x_2\mathbf{e}_3$. Determine the linear strain tensor, the linear rotation tensor and the rotation vector at point $P(0,2,-1)$.

Solution:

the matrix form of the displacement gradient is

$$[\frac{\partial u_i}{\partial x_j}] = \begin{bmatrix} 2(x_1-x_3) & 0 & -2(x_1-x_3) \\ 0 & 2(x_2+x_3) & 2(x_2+x_3) \\ -x_2 & -x_1 & 0 \end{bmatrix} \quad (4.115\text{-a})$$

$$\left[\frac{\partial u_i}{\partial x_j}\right]_P = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix} \quad (4.115\text{-b})$$

Decomposing this matrix into symmetric and antisymmetric components give:

$$[E_{ij}] + [w_{ij}] = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad (4.116)$$

and from Eq. Eq. 4.103

$$\mathbf{w} = -W_{23}\mathbf{e}_1 - W_{31}\mathbf{e}_2 - W_{12}\mathbf{e}_3 = -1\mathbf{e}_1 \quad (4.117)$$

4.2.6.3 Finite Strain; Polar Decomposition

93 When the displacement gradients are finite, then we no longer can decompose $\frac{\partial u_i}{\partial X_j}$ (Eq. 4.96) or $\frac{\partial u_i}{\partial x_j}$ (Eq. 4.104) into a unique sum of symmetric and skew parts (pure strain and pure rotation).

94 Thus in this case, rather than having an **additive** decomposition, we will have a **multiplicative** decomposition.

95 we call this a **polar decomposition** and it should decompose the deformation gradient in the product of two tensors, one of which represents a rigid-body rotation, while the other is a symmetric positive-definite tensor.

96 We apply this decomposition to the deformation gradient **F**:

$$\boxed{F_{ij} \equiv \frac{\partial x_i}{\partial X_j} = R_{ik}U_{kj} = V_{ik}R_{kj} \quad \text{or} \quad \mathbf{F} = \mathbf{R}\cdot\mathbf{U} = \mathbf{V}\cdot\mathbf{R}} \quad (4.118)$$

where **R** is the **orthogonal rotation tensor**, and **U** and **V** are positive symmetric tensors known as the **right stretch tensor** and the **left stretch tensor** respectively.

97 The interpretation of the above equation is obtained by inserting the above equation into $dx_i = \frac{\partial x_i}{\partial X_j} dX_j$

$$dx_i = R_{ik}U_{kj}dX_j = V_{ik}R_{kj}dX_j \quad \text{or} \quad d\mathbf{x} = \mathbf{R}\cdot\mathbf{U}\cdot d\mathbf{X} = \mathbf{V}\cdot\mathbf{R}\cdot d\mathbf{X} \quad (4.119)$$

and we observe that in the first form the deformation consists of a sequential stretching (by **U**) and rotation (**R**) to be followed by a rigid body displacement to **x**. In the second case, the orders are reversed, we have first a rigid body translation to **x**, followed by a rotation (**R**) and finally a stretching (by **V**).

98 To determine the stretch tensor from the deformation gradient

$$\mathbf{F}^T\mathbf{F} = (\mathbf{RU})^T(\mathbf{RU}) = \mathbf{U}^T\mathbf{R}^T\mathbf{RU} = \mathbf{U}^T\mathbf{U} \tag{4.120}$$

Recalling that $\mathbf{R}$ is an orthonormal matrix, and thus $\mathbf{R}^T = \mathbf{R}^{-1}$ then we can compute the various tensors from

$$\boxed{\begin{aligned} \mathbf{U} &= \sqrt{\mathbf{F}^T\mathbf{F}} & (4.121)\\ \mathbf{R} &= \mathbf{FU}^{-1} & (4.122)\\ \mathbf{V} &= \mathbf{FR}^T & (4.123) \end{aligned}}$$

99 It can be shown that

$$\boxed{\mathbf{U} = \mathbf{C}^{1/2} \text{ and } \mathbf{V} = \mathbf{B}^{1/2}} \tag{4.124}$$

■ Example 4-10: Polar Decomposition I

Given $x_1 = X_1$, $x_2 = -3X_3$, $x_3 = 2X_2$, find the deformation gradient $\mathbf{F}$, the right stretch tensor $\mathbf{U}$, the rotation tensor $\mathbf{R}$, and the left stretch tensor $\mathbf{V}$.

Solution:

From Eq. 4.22

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \tag{4.125}$$

From Eq. 4.121

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \tag{4.126}$$

thus

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{4.127}$$

From Eq. 4.122

$$\mathbf{R} = \mathbf{FU}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.128}$$

Finally, from Eq. 4.123

$$\mathbf{V} = \mathbf{FR}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{4.129}$$

■

■ **Example 4-11: Polar Decomposition II**

For the following deformation: $x_1 = \lambda_1 X_1$, $x_2 = -\lambda_3 X_3$, and $x_3 = \lambda_2 X_2$, find the rotation tensor.

Solution:

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \tag{4.130}$$

$$[\mathbf{U}]^2 = [\mathbf{F}]^T[\mathbf{F}] \tag{4.131}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{4.132}$$

$$[\mathbf{U}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{4.133}$$

$$[\mathbf{R}] = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.134}$$

Thus we note that $\mathbf{R}$ corresponds to a 90^o rotation about the $\mathbf{e}_1$ axis. ■

■ **Example 4-12: Polar Decomposition III**

In[4]:=
```
{v1, v2, v3} = N[Eigenvectors[CST], 4]
```

Determine U and U^{-1} with respect to the e_i basis

Pola

Out[4]=

In[10]:=
```
U_e = N[vnormalized . Ueigen . vnormalized, 3]
```

Given x_1

matrix U

obtain th

In[5]:=

Out[10]=

$$\begin{pmatrix} 0.707 & 0.707 & 0. \\ 0.707 & 2.12 & 0. \\ 0. & 0. & 1. \end{pmatrix}$$

Deterr

In[6]:=
```
vnormalized = GramSchmidt[{v3, -v2, v1}]
```

In[11]:=
```
U_einverse = N[Inverse[%], 3]
```

In[1]:=

Out[6]=

Out[11]=

$$\begin{pmatrix} 2.12 & -0.707 & 0. \\ -0.707 & 0.707 & 0. \\ 0. & 0. & 1. \end{pmatrix}$$

Out[1]=

In[7]:=
```
CSTeigen = Chop[N[vnormalized . CST . vnormalized, 4]]
```

Determine R with respect to the e_i basis

Solve

Out[7]=

In[12]:=
```
R = N[F . %, 3]
```

In[2]:=

Out[12]=

$$\begin{pmatrix} 0.707 & 0.707 & 0. \\ -0.707 & 0.707 & 0. \\ 0. & 0. & 1. \end{pmatrix}$$

Deterr

Out[2]=

In[8]:=
```
Ueigen = N[Sqrt[CSTeigen], 4]
```

Out[8]=

$$\begin{pmatrix} 2.414 & 0 & 0 \\ 0 & 0.4142 & 0 \\ 0 & 0 & 1. \end{pmatrix}$$

Deterr

In[3]:=
```
N[Eigenvalues[CST]]
```

In[9]:=
```
Ueigenminus1 = Inverse[Ueigen]
```

Out[3]= {1., 0.171573, 5.82843}

Out[9]=

$$\begin{pmatrix} 0.414214 & 0. & 0. \\ 0. & 2.41421 & 0. \\ 0. & 0. & 1. \end{pmatrix}$$

■

4.2.7 Summary and Discussion

100 From the above, we deduce the following observations:

1. If both the displacement gradients and the displacements themselves are small, then $\frac{\partial u_i}{\partial X_j} \approx \frac{\partial u_i}{\partial x_j}$ and thus the Eulerian and the Lagrangian infinitesimal strain tensors may be taken as equal $E_{ij} = E^*_{ij}$.
2. If the displacement gradients are small, but the displacements are large, we should use the Eulerian infinitesimal representation.
3. If the displacements gradients are large, but the displacements are small, use the Lagrangian finite strain representation.
4. If both the displacement gradients and the displacements are large, use the Eulerian finite strain representation.

4.2.8 †Explicit Derivation

101 If the derivations in the preceding section was perceived as too complex through a first reading, this section will present a "gentler" approach to essentially the same results albeit in a less "elegant" mannser. The previous derivation was carried out using indicial notation, in this section we repeat the derivation using explicitly.

102 Similarities between the two approaches is facilitated by Table 4.2.

103 Considering two points A and B in a 3D solid, the distance between them is ds

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{4.135}$$

As a result of deformation, point A moves to A', and B to B' the distance between the two points is ds', Fig. 12.7.

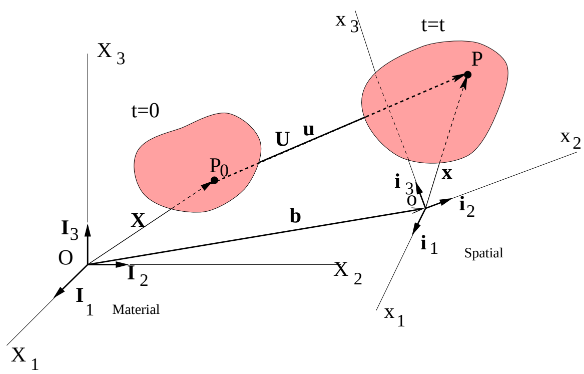
$$ds'^2 = dx'^2 + dy'^2 + dz'^2 \tag{4.136}$$

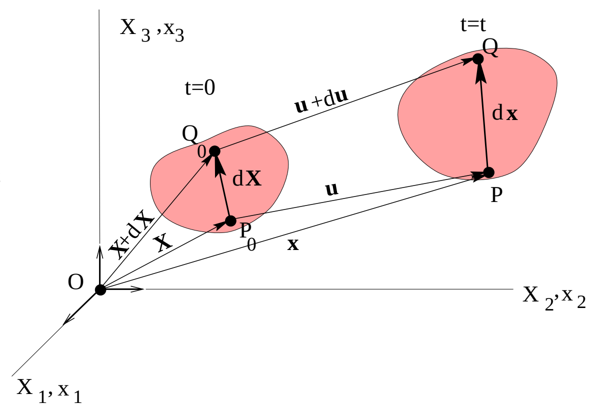
104 The displacement of point A to A' is given by

$$u = x' - x \Rightarrow dx' = du + dx \tag{4.137-a}$$
$$v = y' - y \Rightarrow dy' = dv + dy \tag{4.137-b}$$
$$w = z' - z \Rightarrow dz' = dw + dz \tag{4.137-c}$$

105 Substituting these equations into Eq. 4.136, we obtain

$$ds'^2 = \underbrace{dx^2 + dy^2 + dz^2}_{ds^2} + 2dudx + 2dvdy + 2dwdz + du^2 + dv^2 + dw^2 \tag{4.138}$$





	LAGRANGIAN **Material**	**EULERIAN** **Spatial**
Position Vector	$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$	$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$
	GRADIENTS	
Deformation	$\mathbf{F} = \mathbf{x}\boldsymbol{\nabla}_\mathbf{X} \equiv \frac{\partial x_i}{\partial X_j}$	$\mathbf{H} = \mathbf{X}\boldsymbol{\nabla}_\mathbf{x} \equiv \frac{\partial X_i}{\partial x_j}$
	$\mathbf{H} = \mathbf{F}^{-1}$	
Displacement	$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij}$ or $\mathbf{J} = \mathbf{u}\boldsymbol{\nabla}_\mathbf{X} = \mathbf{F} - \mathbf{I}$	$\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j}$ or $\mathbf{K} \equiv \mathbf{u}\boldsymbol{\nabla}_\mathbf{x} = \mathbf{I} - \mathbf{H}$
	TENSOR	
Deformation	$dX^2 = d\mathbf{x}{\cdot}\mathbf{B}^{-1}{\cdot}d\mathbf{x}$	$dx^2 = d\mathbf{X}{\cdot}\mathbf{C}{\cdot}d\mathbf{X}$
	Cauchy $B^{-1}_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}$ or $\mathbf{B}^{-1} = \boldsymbol{\nabla}_\mathbf{x}\mathbf{X}{\cdot}\mathbf{X}\boldsymbol{\nabla}_\mathbf{x} = \mathbf{H}_c{\cdot}\mathbf{H}$	Green $C_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}$ or $\mathbf{C} = \boldsymbol{\nabla}_\mathbf{X}\mathbf{x}{\cdot}\mathbf{x}\boldsymbol{\nabla}_\mathbf{X} = \mathbf{F}_c{\cdot}\mathbf{F}$
	$\mathbf{C}^{-1} = \mathbf{B}^{-1}$	
	STRAINS	
	Lagrangian	**Eulerian/Almansi**
Finite Strain	$dx^2 - dX^2 = d\mathbf{X}{\cdot}2\mathbf{E}{\cdot}d\mathbf{X}$	$dx^2 - dX^2 = d\mathbf{x}{\cdot}2\mathbf{E}^*{\cdot}d\mathbf{x}$
	$E_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$ or $\mathbf{E} = \frac{1}{2}(\underbrace{\boldsymbol{\nabla}_\mathbf{X}\mathbf{x}{\cdot}\mathbf{x}\boldsymbol{\nabla}_\mathbf{X}}_{\mathbf{F}_c\cdot\mathbf{F}} - \mathbf{I})$	$E^*_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right)$ or $\mathbf{E}^* = \frac{1}{2}(\mathbf{I} - \underbrace{\boldsymbol{\nabla}_\mathbf{x}\mathbf{X}{\cdot}\mathbf{X}\boldsymbol{\nabla}_\mathbf{x}}_{\mathbf{H}_c\cdot\mathbf{H}})$
	$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$ or $\mathbf{E} = \frac{1}{2}\underbrace{(\mathbf{u}\boldsymbol{\nabla}_\mathbf{X} + \boldsymbol{\nabla}_\mathbf{X}\mathbf{u} + \boldsymbol{\nabla}_\mathbf{X}\mathbf{u}{\cdot}\mathbf{u}\boldsymbol{\nabla}_\mathbf{X})}_{\mathbf{J}+\mathbf{J}_c+\mathbf{J}_c\cdot\mathbf{J}}$	$E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$ or $\mathbf{E}^* = \frac{1}{2}\underbrace{(\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} + \boldsymbol{\nabla}_\mathbf{x}\mathbf{u} - \boldsymbol{\nabla}_\mathbf{x}\mathbf{u}{\cdot}\mathbf{u}\boldsymbol{\nabla}_\mathbf{x})}_{\mathbf{K}+\mathbf{K}_c-\mathbf{K}_c\cdot\mathbf{K}}$
Small Deformation	$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$ $\mathbf{E} = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{X} + \boldsymbol{\nabla}_\mathbf{X}\mathbf{u}) = \frac{1}{2}(\mathbf{J} + \mathbf{J}_c)$	$E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\mathbf{E}^* = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} + \boldsymbol{\nabla}_\mathbf{x}\mathbf{u}) = \frac{1}{2}(\mathbf{K} + \mathbf{K}_c)$
	ROTATION TENSORS	
Small deformation	$[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)]dX_j$ $[\underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{X} + \boldsymbol{\nabla}_\mathbf{X}\mathbf{u})}_{\mathbf{E}} + \underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{X} - \boldsymbol{\nabla}_\mathbf{X}\mathbf{u})}_{\mathbf{W}}]{\cdot}d\mathbf{X}$	$\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dx_j$ $[\underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} + \boldsymbol{\nabla}_\mathbf{x}\mathbf{u})}_{\mathbf{E}^*} + \underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} - \boldsymbol{\nabla}_\mathbf{x}\mathbf{u})}_{\boldsymbol{\Omega}}]{\cdot}d\mathbf{x}$
Finite Strain	$\mathbf{F} = \mathbf{R}{\cdot}\mathbf{U} = \mathbf{V}{\cdot}\mathbf{R}$	
	STRESS TENSORS	
	Piola-Kirchoff	**Cauchy**
First Second	$\mathbf{T}_0 = (\det\mathbf{F})\mathbf{T}\left(\mathbf{F}^{-1}\right)^T$ $\tilde{\mathbf{T}} = (\det\mathbf{F})\left(\mathbf{F}^{-1}\right)\mathbf{T}\left(\mathbf{F}^{-1}\right)^T$	

Table 4.1: Summary of Major Equations

Tensorial	**Explicit**
$X_1, X_2, X_3,\ dX$	$x, y, z,\ ds$
$x_1, x_2, x_3,\ dx$	$x', y', z',\ ds'$
u_1, u_2, u_3	u, v, w
E_{ij}	ε_{ij}

Table 4.2: Tensorial *vs* Explicit Notation

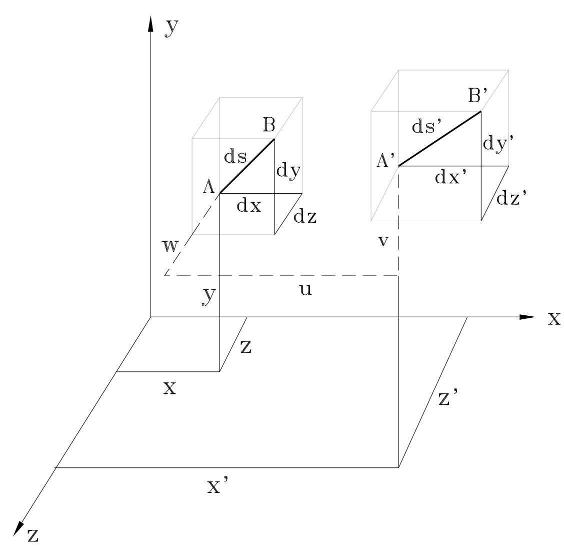


Figure 4.7: Strain Definition

106 From the chain rule of differrentiation

$$
\begin{aligned}
du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz && \text{(4.139-a)}\\
dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz && \text{(4.139-b)}\\
dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz && \text{(4.139-c)}
\end{aligned}
$$

107 Substituting this equation into the preceding one yields the **finite strains**

$$
\begin{aligned}
ds'^2 - ds^2 &= 2\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]\right\}dx^2\\
&+ 2\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\right\}dy^2\\
&+ 2\left\{\frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]\right\}dz^2\\
&+ 2\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)dxdy\\
&+ 2\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right)dxdz\\
&+ 2\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)dydz && \text{(4.140-a)}
\end{aligned}
$$

108 We observe that $ds'^2 - ds^2$ is zero if there is no relative displacement between A and B (i.e. rigid body motion), otherwise the solid is strained. Hence $ds'^2 - ds^2$ can be selected as an appropriate measure of the deformation of the solid, and we define the strain components as

$$
ds'^2 - ds^2 = 2\varepsilon_{xx}dx^2 + 2\varepsilon_{yy}dy^2 + 2\varepsilon_{zz}dz^2 + 4\varepsilon_{xy}dxdy + 4\varepsilon_{xz}dxdz + 4\varepsilon_{yz}dydz \quad (4.141)
$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \quad (4.142)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \quad (4.143)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \quad (4.144)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \quad (4.145)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \quad (4.146)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \quad (4.147)$$

or

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \quad (4.148)$$

From this equation, we note that:

1. We define the **engineering shear strain** as

$$\gamma_{ij} = 2\varepsilon_{ij} \quad (i \neq j) \quad (4.149)$$

2. If the strains are given, then these strain-displacements provide a system of (6) nonlinear partial differential equation in terms of the unknown displacements (3).

3. ε_{ik} is the **Green-Lagrange strain tensor**.

4. The strains have been expressed in terms of the coordinates x, y, z in the undeformed state, i.e. in the **Lagrangian coordinate** which is the preferred one in structural mechanics.

5. Alternatively we could have expressed $ds'^2 - ds^2$ in terms of coordinates in the deformed state, i.e. **Eulerian coordinates** x', y', z', and the resulting strains are referred to as the **Almansi strain** which is the preferred one in fluid mechanics.

6. In most cases the deformations are small enough for the quadratic term to be dropped, the resulting equations reduce to

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad (4.150)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad (4.151)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \quad (4.152)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (4.153)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad (4.154)$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad (4.155)$$

or

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,k} + u_{k,i}\right) \quad (4.156)$$

which is called the **Cauchy strain**

109 In finite element, the strain is often expressed through the **linear operator L**

$$\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u} \quad (4.157)$$

or

$$\underbrace{\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix}}_{\boldsymbol{\varepsilon}} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}}_{\mathbf{u}} \quad (4.158)$$

4.2.9 Compatibility Equation

110 If $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ then we have six differential equations (in 3D the strain tensor has a total of 9 terms, but due to symmetry, there are 6 independent ones) for determining (upon integration) three unknowns displacements u_i. Hence the system is overdetermined, and there must be some linear relations between the strains.

111 It can be shown (through appropriate successive differentiation of the strain expression) that the compatibility relation for strain reduces to:

$$\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0. \text{ or } \boldsymbol{\nabla}_{\mathbf{x}} \times \mathbf{L} \times \boldsymbol{\nabla}_{\mathbf{x}} = 0 \quad (4.159)$$

There are 81 equations in all, but only six are distinct

$$
\begin{aligned}
\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} &= 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} &\quad (4.160\text{-a})\\
\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} &= 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} &\quad (4.160\text{-b})\\
\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} &= 2\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} &\quad (4.160\text{-c})\\
\frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} &\quad (4.160\text{-d})\\
\frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} &\quad (4.160\text{-e})\\
\frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right) &= \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} &\quad (4.160\text{-f})
\end{aligned}
$$

In 2D, this results in (by setting $i = 2$, $j = 1$ and $l = 2$):

$$
\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2}} \quad (4.161)
$$

(recall that $2\varepsilon_{12} = \gamma_{12}$.)

112 When he compatibility equation is written in term of the stresses, it yields:

$$
\frac{\partial^2 \sigma_{11}}{\partial x_2^2} - \nu\frac{\partial \sigma_{22}{}^2}{\partial x_2^2} + \frac{\partial^2 \sigma_{22}}{\partial x_1^2} - \nu\frac{\partial^2 \sigma_{11}}{\partial x_1^2} = 2\left(1+\nu\right)\frac{\partial^2 \sigma_{21}}{\partial x_1 \partial x_2} \quad (4.162)
$$

■ Example 4-13: Strain Compatibility

For the following strain field

$$
\begin{bmatrix}
-\frac{X_2}{X_1^2+X_2^2} & \frac{X_1}{2(X_1^2+X_2^2)} & 0\\
\frac{X_1}{2(X_1^2+X_2^2)} & 0 & 0\\
0 & 0 & 0
\end{bmatrix} \quad (4.163)
$$

does there exist a single-valued continuous displacement field?
Solution:

$$
\begin{aligned}
\frac{\partial E_{11}}{\partial X_2} &= -\frac{(X_1^2+X_2^2) - X_2(2X_2)}{(X_1^2+X_2^2)^2} = \frac{X_2^2 - X_1^2}{(X_1^2+X_2^2)^2} &\quad (4.164\text{-a})\\
2\frac{\partial E_{12}}{\partial X_1} &= \frac{(X_1^2+X_2^2) - X_1(2X_1)}{(X_1^2+X_2^2)^2} = \frac{X_2^2 - X_1^2}{(X_1^2+X_2^2)^2} &\quad (4.164\text{-b})\\
\frac{\partial E_{22}}{\partial X_1^2} &= 0 &\quad (4.164\text{-c})
\end{aligned}
$$

$$\Rightarrow \frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \checkmark \tag{4.164-d}$$

Actually, it can be easily verified that the unique displacement field is given by

$$u_1 = \arctan \frac{X_2}{X_1}; \quad u_2 = 0; \quad u_3 = 0 \tag{4.165}$$

to which we could add the rigid body displacement field (if any). ■

4.3 Lagrangian Stresses; Piola Kirchoff Stress Tensors

113 In Sect. 2.2 the discussion of stress applied to the deformed configuration dA (using spatial coordiantes**x**), that is the one where equilibrium must hold. The deformed configuration being the natural one in which to characterize stress. Hence we had

$$d\mathbf{f} = \mathbf{t}dA \tag{4.166-a}$$
$$\mathbf{t} = \mathbf{Tn} \tag{4.166-b}$$

(note the use of $\mathbf{T}$ instead of $\boldsymbol{\sigma}$). Hence the Cauchy stress tensor was really defined in the Eulerian space.

114 However, there are certain advantages in referring all quantities back to the undeformed configuration (Lagrangian) of the body because often that configuration has geometric features and symmetries that are lost through the deformation.

115 Hence, if we were to define the strain in material coordinates (in terms of $\mathbf{X}$), we need also to express the stress as a function of the material point $\mathbf{X}$ in material coordinates.

4.3.1 First

116 The first Piola-Kirchoff stress tensor $\mathbf{T}_0$ is defined in the undeformed geometry in such a way that it results in the **same total force** as the traction in the deformed configuration (where Cauchy's stress tensor was defined). Thus, we define

$$d\mathbf{f} \equiv \mathbf{t}_0 dA_0 \tag{4.167}$$

where $\mathbf{t}_0$ is a **pseudo-stress vector** in that being based on the undeformed area, it does not describe the actual intensity of the force, however it has the same direction as Cauchy's stress vector $\mathbf{t}$.

117 The first Piola-Kirchoff stress tensor (also known as **Lagrangian Stress Tensor**) is thus the linear transformation $\mathbf{T}_0$ such that

$$\mathbf{t}_0 = \mathbf{T}_0\mathbf{n}_0 \tag{4.168}$$

and for which

$$d\mathbf{f} = \mathbf{t}_0 dA_0 = \mathbf{t}dA \Rightarrow \mathbf{t}_0 = \frac{dA}{dA_0}\mathbf{t} \tag{4.169}$$

using Eq. 4.166-b and 4.168 the preceding equation becomes

$$\mathbf{T}_0\mathbf{n}_0 = \frac{dA}{dA_0}\mathbf{T}\mathbf{n} = \frac{\mathbf{T}dA\mathbf{n}}{dA_0} \tag{4.170}$$

and using Eq. 4.33 $dA\mathbf{n} = dA_0(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^T\mathbf{n}_0$ we obtain

$$\mathbf{T}_0\mathbf{n}_0 = \mathbf{T}(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^T\mathbf{n}_0 \tag{4.171}$$

the above equation is true for all $\mathbf{n}_0$, therefore

$$\boxed{\begin{aligned} \mathbf{T}_0 &= (\det\mathbf{F})\mathbf{T}\left(\mathbf{F}^{-1}\right)^T && (4.172)\\ \mathbf{T} &= \frac{1}{(\det\mathbf{F})}\mathbf{T}_0\mathbf{F}^T \quad \text{or} \quad T_{ij} = \frac{1}{(\det\mathbf{F})}(T_0)_{im}F_{jm} && (4.173) \end{aligned}}$$

and we note that this first Piola-Kirchoff stress tensor is **not symmetric** in general.

118 To determine the corresponding stress vector, we solve for $\mathbf{T}_0$ first, then for dA_0 and $\mathbf{n}_0$ from $dA_0\mathbf{n}_0 = \frac{1}{\det\mathbf{F}}\mathbf{F}^T\mathbf{n}$ (assuming unit area dA), and finally $\mathbf{t}_0 = \mathbf{T}_0\mathbf{n}_0$.

4.3.2 Second

119 The second Piola-Kirchoff stress tensor, $\tilde{\mathbf{T}}$ is formulated differently. Instead of the actual force $d\mathbf{f}$ on dA, it gives the force $d\tilde{\mathbf{f}}$ related to the force $d\mathbf{f}$ in the same way that a material vector $d\mathbf{X}$ at $\mathbf{X}$ is related by the deformation to the corresponding spatial vector $d\mathbf{x}$ at $\mathbf{x}$. Thus, if we let

$$d\tilde{\mathbf{f}} = \tilde{\mathbf{t}}dA_0 \tag{4.174-a}$$

and

$$d\mathbf{f} = \mathbf{F}d\tilde{\mathbf{f}} \tag{4.174-b}$$

where $d\tilde{\mathbf{f}}$ is the pseudo differential force which transforms, under the deformation gradient $\mathbf{F}$, the (actual) differential force $d\mathbf{f}$ at the deformed position (note similarity with $d\mathbf{x} = \mathbf{F}d\mathbf{X}$). Thus, the pseudo vector $\mathbf{t}$ is in general in a differnt direction than that of the Cauchy stress vector $\mathbf{t}$.

120 The second Piola-Kirchoff stress tensor is a linear transformation $\tilde{\mathbf{T}}$ such that

$$\boxed{\tilde{\mathbf{t}} = \tilde{\mathbf{T}}n_0} \tag{4.175}$$

thus the preceding equations can be combined to yield

$$d\mathbf{f} = \mathbf{F}\tilde{\mathbf{T}}n_0 dA_0 \tag{4.176}$$

we also have from Eq. 4.167 and 4.168

$$d\mathbf{f} = \mathbf{t}_0 dA_0 = \mathbf{T}_0\mathbf{n}_0 dA_0 \tag{4.177}$$

and comparing the last two equations we note that

$$\boxed{\tilde{\mathbf{T}} = \mathbf{F}^{-1}\mathbf{T}_0} \tag{4.178}$$

which gives the relationship between the first Piola-Kirchoff stress tensor $\mathbf{T}_0$ and the second Piola-Kirchoff stress tensor $\tilde{\mathbf{T}}$.

121 Finally the relation between the second Piola-Kirchoff stress tensor and the Cauchy stress tensor can be obtained from the preceding equation and Eq. 4.172

$$\boxed{\tilde{\mathbf{T}} = (\det \mathbf{F}) \left(\mathbf{F}^{-1}\right) \mathbf{T} \left(\mathbf{F}^{-1}\right)^T} \tag{4.179}$$

and we note that this second Piola-Kirchoff stress tensor is always symmetric (if the Cauchy stress tensor is symmetric).

122 To determine the corresponding stress vector, we solve for $\tilde{\mathbf{T}}$ first, then for dA_0 and $\mathbf{n}_0$ from $dA_0\mathbf{n}_0 = \frac{1}{\det \mathbf{F}}\mathbf{F}^T\mathbf{n}$ (assuming unit area dA), and finally $\tilde{\mathbf{t}} = \tilde{\mathbf{T}}\mathbf{n}_0$.

■ **Example 4-14: Piola-Kirchoff Stress Tensors**

■

■

4.4 Hydrostatic and Deviatoric Strain

93 The lagrangian and Eulerian **linear** strain tensors can each be split into **spherical** and **deviator** tensor as was the case for the stresses. Hence, if we define

$$\frac{1}{3}e = \frac{1}{3}\text{tr }\mathbf{E} \tag{4.180}$$

then the components of the strain deviator $\mathbf{E}'$ are given by

$$\boxed{E'_{ij} = E_{ij} - \frac{1}{3}e\delta_{ij} \quad \text{or} \quad \mathbf{E}' = \mathbf{E} - \frac{1}{3}e\mathbf{1}} \tag{4.181}$$

We note that $\mathbf{E}'$ measures the change in shape of an element, while the **spherical** or **hydrostatic** strain $\frac{1}{3}e\mathbf{1}$ represents the volume change.

4.5 Principal Strains, Strain Invariants, Mohr Circle

94 Determination of the principal strains ($E_{(3)} < E_{(2)} < E_{(1)}$, strain invariants and the Mohr circle for strain parallel the one for stresses (Sect. 2.4) and will not be repeated

■ Second Piola−Kirchoff Stress Tensor

The defo... ess tensor is given by ... calculate the respective ...

```
MatrixForm[Transpose[F] . n / detF]
```

$$\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

Thus $n = e_2$ and using $t_0 = T_0\, n_0$ we obtain

{{0, 0, 0}, {0, 25/4, 0}, {0, 0, 0}}

■ F tensor

```
t01st = MatrixForm[Tfirst . {0, 1, 0}]
```

$$\begin{pmatrix} 0 \\ 0 \\ 25 \end{pmatrix}$$

(0 0 0)

We note that this vector is in the same direction as the Cauchy stress vector, its magnitude is one fourth of that of the Cauchy stress vector, because the undeformed area is 4 times that of the deformed area

■ Cuchy stress vector

■ Pseudo−Stress vector associated with the Second Piola−Kirchoff stress tensor

Can be obtained from t=CST n

```
F = {{1/2, 0, 0}, {0, 0, -1/2}, {0, 4, 0}}
```

$\{\{\frac{1}{2}, 0, 0\}, \{0, 0, -\frac{1}{2}\}, \{0, 4, 0\}\}$

```
t0second = MatrixForm[Tsecond . {0, 1, 0}]
```

$$\begin{pmatrix} 0 \\ \frac{25}{4} \\ 0 \end{pmatrix}$$

We see that this pseudo stress vector is in a different direction from that of the Cauchy stress vector (and we note that the tensor F transforms e_2 into e_3).

■ Pseudo−Stress vector associated with the First Piola−Kirchoff stress tensor

■ First Piola−Kirchoff Stress Tensor

For a unit area in the deformed state in the e_3 direction, its undeformed area $dA_0\, n_0$ is given by $dA_0\, \boldsymbol{n}_0 = \frac{F^T\, n}{\det F}$

```
detF = Det[F]
```

1

```
n = {0, 0, 1}
```

{0, 0, 1}

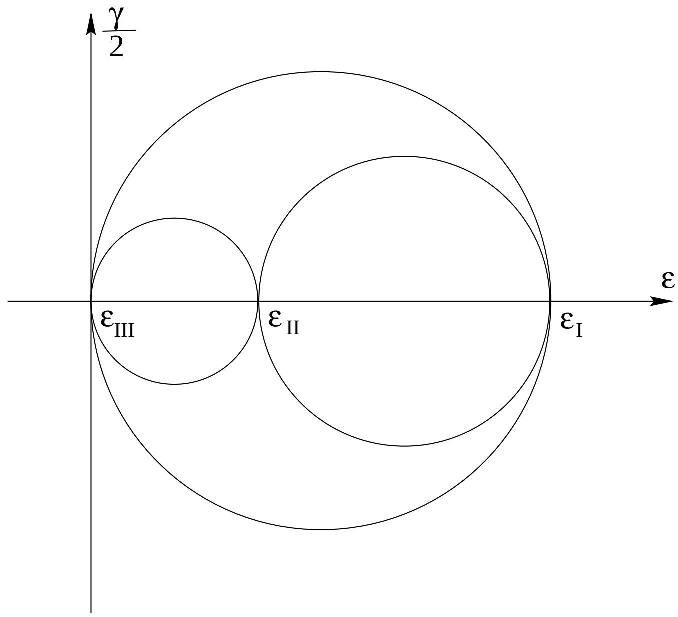


Figure 4.8: Mohr Circle for Strain

here.

$$\boxed{\lambda^3 - I_E\lambda^2 - II_E\lambda - III_E = 0} \quad (4.182)$$

where the symbols I_E, II_E and III_E denote the following scalar expressions in the strain components:

$$
\begin{aligned}
I_E &= E_{11} + E_{22} + E_{33} = E_{ii} = \text{tr } \mathbf{E} && (4.183)\\
II_E &= -(E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11}) + E_{23}^2 + E_{31}^2 + E_{12}^2 && (4.184)\\
&= \frac{1}{2}(E_{ij}E_{ij} - E_{ii}E_{jj}) = \frac{1}{2}E_{ij}E_{ij} - \frac{1}{2}I_E^2 && (4.185)\\
&= \frac{1}{2}(\mathbf{E} : \mathbf{E} - I_E^2) && (4.186)\\
III_E &= \det\mathbf{E} = \frac{1}{6}e_{ijk}e_{pqr}E_{ip}E_{jq}E_{kr} && (4.187)
\end{aligned}
$$

95 In terms of the principal strains, those invariants can be simplified into

$$
\begin{aligned}
I_E &= E_{(1)} + E_{(2)} + E_{(3)} && (4.188)\\
II_E &= -(E_{(1)}E_{(2)} + E_{(2)}E_{(3)} + E_{(3)}E_{(1)}) && (4.189)\\
III_E &= E_{(1)}E_{(2)}E_{(3)} && (4.190)
\end{aligned}
$$

96 The Mohr circle uses the **Engineering shear strain** definition of Eq. 4.86, Fig. 4.8

■ Example 4-15: Strain Invariants & Principal Strains

Determine the planes of principal strains for the following strain tensor

$$\begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.191}$$

Solution:

The strain invariants are given by

$$I_E = E_{ii} = 2 \tag{4.192-a}$$
$$II_E = \frac{1}{2}(E_{ij}E_{ij} - E_{ii}E_{jj}) = -1 + 3 = +2 \tag{4.192-b}$$
$$III_E = |E_{ij}| = -3 \tag{4.192-c}$$

The principal strains by

$$E_{ij} - \lambda\delta_{ij} = \begin{bmatrix} 1-\lambda & \sqrt{3} & 0 \\ \sqrt{3} & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \tag{4.193-a}$$
$$= (1-\lambda)\left(\lambda - \frac{1+\sqrt{13}}{2}\right)\left(\lambda - \frac{1-\sqrt{13}}{2}\right) \tag{4.193-b}$$
$$E_{(1)} = \lambda_{(1)} = \frac{1+\sqrt{13}}{2} = 2.3 \tag{4.193-c}$$
$$E_{(2)} = \lambda_{(2)} = 1 \tag{4.193-d}$$
$$E_{(3)} = \lambda_{(3)} = \frac{1-\sqrt{13}}{2} = -1.3 \tag{4.193-e}$$

The eigenvectors for $E_{(1)} = \frac{1+\sqrt{13}}{2}$ give the principal directions $\mathbf{n}^{(1)}$:

$$\begin{bmatrix} 1-\frac{1+\sqrt{13}}{2} & \sqrt{3} & 0 \\ \sqrt{3} & -\frac{1+\sqrt{13}}{2} & 0 \\ 0 & 0 & 1-\frac{1+\sqrt{13}}{2} \end{bmatrix} \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} \left(1-\frac{1+\sqrt{13}}{2}\right)n_1^{(1)} + \sqrt{3}n_2^{(1)} \\ \sqrt{3}n_1^{(1)} - \left(\frac{1+\sqrt{13}}{2}\right)n_2^{(1)} \\ \left(1-\frac{1+\sqrt{13}}{2}\right)n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.194}$$

which gives

$$n_1^{(1)} = \frac{1+\sqrt{13}}{2\sqrt{3}}n_2^{(1)} \tag{4.195-a}$$
$$n_3^{(1)} = 0 \tag{4.195-b}$$
$$\mathbf{n}^{(1)}\cdot\mathbf{n}^{(1)} = \left(\frac{1+2\sqrt{13}+13}{12}+1\right)\left(n_2^{(1)}\right)^2 = 1 \Rightarrow n_2^1 = 0.8; \tag{4.195-c}$$
$$\Rightarrow \mathbf{n}^{(1)} = \lfloor\ 0.8 \quad 0.6 \quad 0\ \rfloor \tag{4.195-d}$$

For the second eigenvector $\lambda_{(2)} = 1$:

$$\begin{bmatrix} 1-1 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{Bmatrix} n_1^{(2)} \\ n_2^{(2)} \\ n_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} \sqrt{3}n_2^{(2)} \\ \sqrt{3}n_1^{(2)} - n_2^{(2)} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.196}$$

which gives (with the requirement that $\mathbf{n}^{(2)}\cdot\mathbf{n}^{(2)} = 1$)

$$\mathbf{n}^{(2)} = \lfloor\ 0 \quad 0 \quad 1\ \rfloor \tag{4.197}$$

Finally, the third eigenvector can be obrained by the same manner, but more easily from

$$\mathbf{n}^{(3)} = \mathbf{n}^{(1)}\times\mathbf{n}^{(2)} = \det\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.6\mathbf{e}_1 - 0.8\mathbf{e}_2 \tag{4.198}$$

Therefore

$$a_i^j = \left\{ \begin{array}{c} \mathbf{n}^{(1)} \\ \mathbf{n}^{(2)} \\ \mathbf{n}^{(3)} \end{array} \right\} = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix} \tag{4.199}$$

and this results can be checked via

$$[\mathbf{a}][\mathbf{E}][\mathbf{a}]^T = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0 & 0.6 \\ 0.6 & 0 & -0.8 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2.3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1.3 \end{bmatrix} \tag{4.200}$$

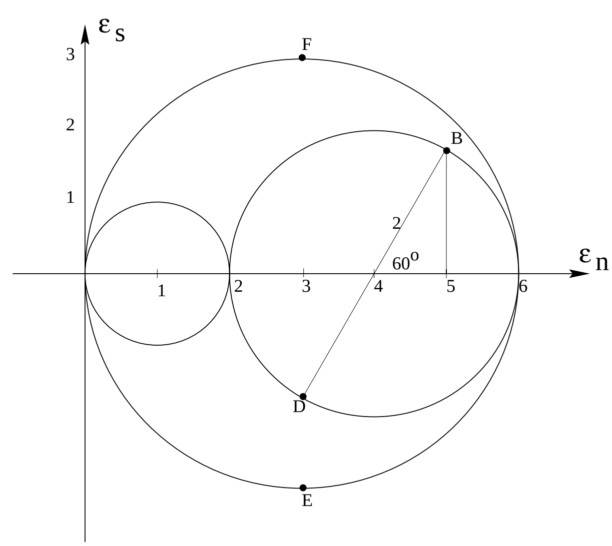
■

■ Example 4-16: Mohr's Circle

Construct the Mohr's circle for the following plane strain case:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & \sqrt{3} \\ 0 & \sqrt{3} & 3 \end{bmatrix} \tag{4.201}$$

Solution:



■

We note that since $E_{(1)} = 0$ is a principal value for plane strain, ttwo of the circles are drawn as shown.

4.6 Initial or Thermal Strains

97 Initial (or thermal strain) in 2D:

$$\varepsilon_{ij} = \underbrace{\begin{bmatrix} \alpha\Delta T & 0 \\ 0 & \alpha\Delta T \end{bmatrix}}_{\text{Plane Stress}} = \underbrace{(1+\nu)\begin{bmatrix} \alpha\Delta T & 0 \\ 0 & \alpha\Delta T \end{bmatrix}}_{\text{Plane Strain}} \tag{4.202}$$

note there is no shear strains caused by thermal expansion.

4.7 † Experimental Measurement of Strain

98 Typically, the transducer to measure strains in a material is the strain gage. The most common type of strain gage used today for stress analysis is the bonded resistance strain gage shown in Figure 4.9.

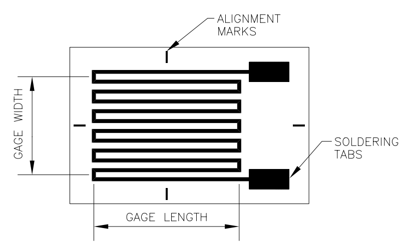


Figure 4.9: Bonded Resistance Strain Gage

99 These gages use a grid of fine wire or a metal foil grid encapsulated in a thin resin backing. The gage is glued to the carefully prepared test specimen by a thin layer of epoxy. The epoxy acts as the carrier matrix to transfer the strain in the specimen to the strain gage. As the gage changes in length, the tiny wires either contract or elongate depending upon a tensile or compressive state of stress in the specimen. The cross sectional area will increase for compression and decrease in tension. Because the wire has an electrical resistance that is proportional to the inverse of the cross sectional area, $R\,\alpha\,\frac{1}{A}$, a measure of the change in resistance can be converted to arrive at the strain in the material.

100 Bonded resistance strain gages are produced in a variety of sizes, patterns, and resistance. One type of gage that allows for the complete state of strain at a point in a plane to be determined is a strain gage rosette. It contains three gages aligned radially from a common point at different angles from each other, as shown in Figure 4.10. The strain transformation equations to convert from the three strains a t any angle to the strain at a point in a plane are:

$$\epsilon_a = \epsilon_x \cos^2\theta_a + \epsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a \tag{4.203}$$

$$\epsilon_b = \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b \tag{4.204}$$
$$\epsilon_c = \epsilon_x \cos^2\theta_c + \epsilon_y \sin^2\theta_c + \gamma_{xy} \sin\theta_c \cos\theta_c \tag{4.205}$$

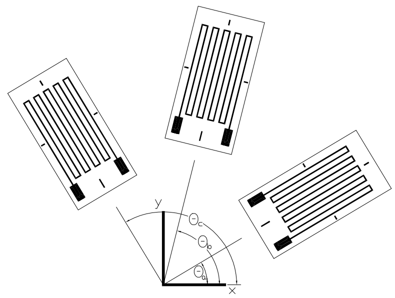


Figure 4.10: Strain Gage Rosette

101 When the measured strains ϵ_a, ϵ_b, and ϵ_c, are measured at their corresponding angles from the reference axis and substituted into the above equations the state of strain at a point may be solved, namely, ϵ_x, ϵ_y, and γ_{xy}. In addition the principal strains may then be computed by Mohr's circle or the principal strain equations.

102 Due to the wide variety of styles of gages, many factors must be considered in choosing the right gage for a particular application. Operating temperature, state of strain, and stability of installation all influence gage selection. Bonded resistance strain gages are well suited for making accurate and practical strain measurements because of their high sensitivity to strains, low cost, and simple operation.

103 The measure of the change in electrical resistance when the strain gage is strained is known as the gage factor. The gage factor is defined as the fractional change in resistance divided by the fractional change in length along the axis of the gage. $GF = \frac{\frac{\Delta R}{R}}{\frac{\Delta L}{L}}$ Common gage factors are in the range of 1.5-2 for most resistive strain gages.

104 Common strain gages utilize a grid pattern as opposed to a straight length of wire in order to reduce the gage length. This grid pattern causes the gage to be sensitive to deformations transverse to the gage length. Therefore, corrections for transverse strains should be computed and applied to the strain data. Some gages come with the tranverse correction calculated into the gage factor. The transverse sensitivity factor, K_t, is defined as the transverse gage factor divided by the longitudinal gage factor. $K_t = \frac{GF_{transverse}}{GF_{longitudinal}}$ These sensitivity values are expressed as a percentage and vary from zero to ten percent.

105 A final consideration for maintaining accurate strain measurement is temperature compensation. The resistance of the gage and the gage factor will change due to the variation of resistivity and strain sensitivity with temperature. Strain gages are produced with different temperature expansion coefficients. In order to avoid this problem, the expansion coefficient of the strain gage should match that of the specimen. If no large temperature change is expected this may be neglected.

106 The change in resistance of bonded resistance strain gages for most strain measurements is very small. From a simple calculation, for a strain of 1 $\mu\epsilon$ ($\mu = 10^{-6}$) with

a 120 Ω gage and a gage factor of 2, the change in resistance produced by the gage is $\Delta R = 1 \times 10^{-6} \times 120 \times 2 = 240 \times 10^{-6}\Omega$. Furthermore, it is the fractional change in resistance that is important and the number to be measured will be in the order of a couple of μ ohms. For large strains a simple multi-meter may suffice, but in order to acquire sensitive measurements in the $\mu\Omega$ range a Wheatstone bridge circuit is necessary to amplify this resistance. The Wheatstone bridge is described next.

4.7.1 Wheatstone Bridge Circuits

107 Due to their outstanding sensitivity, Wheatstone bridge circuits are very advantageous for the measurement of resistance, inductance, and capacitance. Wheatstone bridges are widely used for strain measurements. A Wheatstone bridge is shown in Figure 4.11. It consists of 4 resistors arranged in a diamond orientation. An input DC voltage, or excitation voltage, is applied between the top and bottom of the diamond and the output voltage is measured across the middle. When the output voltage is zero, the bridge is said to be balanced. One or more of the legs of the bridge may be a resistive transducer, such as a strain gage. The other legs of the bridge are simply completion resistors with resistance equal to that of the strain gage(s). As the resistance of one of the legs changes, by a change in strain from a resistive strain gage for example, the previously balanced bridge is now unbalanced. This unbalance causes a voltage to appear across the middle of the bridge. This induced voltage may be measured with a voltmeter or the resistor in the opposite leg may be adjusted to re-balance the bridge. In either case the change in resistance that caused the induced voltage may be measured and converted to obtain the engineering units of strain.

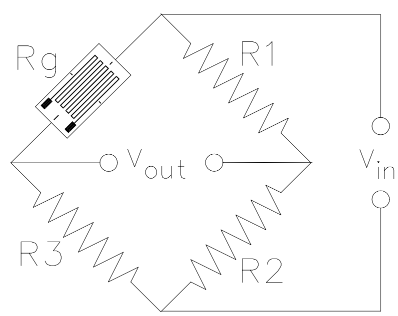


Figure 4.11: Quarter Wheatstone Bridge Circuit

4.7.2 Quarter Bridge Circuits

108 If a strain gage is oriented in one leg of the circuit and the other legs contain fixed resistors as shown in Figure 4.11, the circuit is known as a quarter bridge circuit. The circuit is balanced when $\frac{R_1}{R_2} = \frac{R_{gage}}{R_3}$. When the circuit is unbalanced $V_{out} = V_{in}(\frac{R_1}{R_1+R_2} - \frac{R_{gage}}{R_{gage}+R_3})$.

109 Wheatstone bridges may also be formed with two or four legs of the bridge being composed of resistive transducers and are called a half bridge and full bridge respectively.

Depending upon the type of application and desired results, the equations for these circuits will vary as shown in Figure 4.12. Here E_0 is the output voltage in mVolts, E is the excitation voltage in Volts, ϵ is strain and ν is Poisson's ratio.

110 In order to illustrate how to compute a calibration factor for a particular experiment, suppose a single active gage in uniaxial compression is used. This will correspond to the upper Wheatstone bridge configuration of Figure 4.12. The formula then is

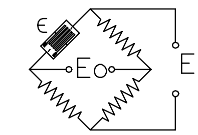
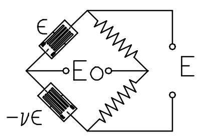
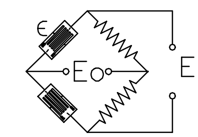
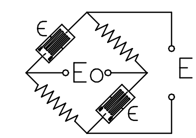
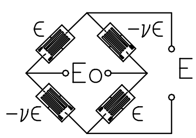
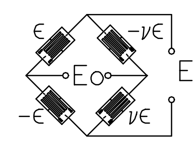
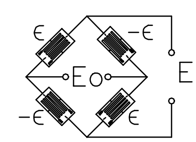
Bridge	Description	Eo/E in mV/V, ϵ in microStrain
	Single active gage in uniaxial compression or tension.	$\frac{E}{Eo} = \frac{F\epsilon(10^{-3})}{(4+2F\epsilon(10^{-6}))}$
	Two active gages in uniaxial stress field. One "Poisson Gage" & one gage aligned with max–imum principal strain.	$\frac{E}{Eo} = \frac{F\epsilon(1+\nu)(10^{-3})}{(4+2F\epsilon(1-\nu)(10^{-6}))}$
	Two active gages with equal and opposite. Common for bending beam test.	$\frac{E}{Eo} = \frac{F\epsilon(10^{-3})}{2}$
	Two active gages with equal strains of the same sign. Bending cancellation arrangement.	$\frac{E}{Eo} = \frac{F\epsilon(10^{-3})}{(2+F\epsilon(10^{-6}))}$
	Four active gages in uniaxial stress field. Two "Poisson Gages" & two aligned with maximum principal strain. Column test.	$\frac{E}{Eo} = \frac{F\epsilon(1+\nu)(10^{-3})}{(2+F\epsilon(1-\nu)(10^{-6}))}$
	Four active gages in uniaxial stress field. Two "Poisson Gages" & two aligned with maxium principal strain. Beam Test	$\frac{E}{Eo} = \frac{F\epsilon(1+\nu)(10^{-3})}{2}$
	Four active gages with pairs subjected to equal and opposite strains. Typical of a beam in bending or a shaft in torsion.	$\frac{E}{Eo} = F\epsilon(10^{-3})$

Figure 4.12: Wheatstone Bridge Configurations

$$\frac{E_0}{E} = \frac{F\epsilon(10^{-3})}{4 + 2F\epsilon(10^{-6})} \tag{4.206}$$

111 The extra term in the denominator $2F\epsilon(10^{-6})$ is a correction factor for non-linearity. Because this term is quite small compared to the other term in the denominator it will be ignored. For most measurements a gain is necessary to increase the output voltage from the Wheatstone bridge. The gain relation for the output voltage may be written as $V = GE_0(10^3)$, where V is now in Volts. so Equation 4.206 becomes

$$\begin{aligned} \frac{V}{EG(10^3)} &= \frac{F\epsilon(10^{-3})}{4} \\ \frac{\epsilon}{V} &= \frac{4}{FEG} \end{aligned} \tag{4.207}$$

112 Here, Equation 4.207 is the calibration factor in units of strain per volt. For common values where $F = 2.07$, $G = 1000$, $E = 5$, the calibration factor is simply $\frac{4}{(2.07)(1000)(5)}$ or 386.47 microstrain per volt.

Chapter 5

MATHEMATICAL PRELIMINARIES; Part III VECTOR INTEGRALS

5.1 Integral of a Vector

20 The integral of a vector $\mathbf{R}(u) = R_1(u)\mathbf{e}_1 + R_2(u)\mathbf{e}_2 + R_3(u)\mathbf{e}_3$ is defined as

$$\int \mathbf{R}(u)du = \mathbf{e}_1 \int R_1(u)du + \mathbf{e}_2 \int R_2(u)du + \mathbf{e}_3 \int R_3(u)du \tag{5.1}$$

if a vector $\mathbf{S}(u)$ exists such that $\mathbf{R}(u) = \frac{d}{du}(\mathbf{S}(u))$, then

$$\int \mathbf{R}(u)du = \int \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + c \tag{5.2}$$

5.2 Line Integral

21 Given $\mathbf{r}(u) = x(u)\mathbf{e}_1 + y(u)\mathbf{e}_2 + z(u)\mathbf{e}_3$ where $\mathbf{r}(u)$ is a position vector defining a curve $\mathcal{C}$ connecting point P_1 to P_2 where $u = u_1$ and $u = u_2$ respectively, anf given $A(x, y, z) = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ being a vectorial function defined and continuous along $\mathcal{C}$, then the integral of the tangential component of $\mathbf{A}$ along $\mathcal{C}$ from P_1 to P_2 is given by

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{r} = \int_{\mathcal{C}} A_1dx + A_2dy + A_3dz \tag{5.3}$$

If $\mathbf{A}$ were a force, then this integral would represent the corresponding work.

22 If the contour is closed, then we define the **contour integral** as

$$\oint_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{r} = \int_{\mathcal{C}} A_1dx + A_2dy + A_3dz \tag{5.4}$$

23 It can be shown that if $\mathbf{A} = \boldsymbol{\nabla}\phi$ then

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} \qquad \text{is independent of the path } \mathcal{C} \text{ connecting } P_1 \text{ to } P_2 \tag{5.5-a}$$

$$\oint_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{r} = 0 \qquad \text{along a closed contour line} \tag{5.5-b}$$

5.3 Integration by Parts

24 The integration by part formula is

$$\boxed{\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b v(x)u'(x)dx} \tag{5.6}$$

5.4 Gauss; Divergence Theorem

25 The divergence theorem (also known as Ostrogradski's Theorem) comes repeatedly in solid mechanics and can be stated as follows:

$$\boxed{\int_\Omega \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}d\Omega = \int_\Gamma \mathbf{v}.\mathbf{n}d\Gamma \quad \text{or} \quad \int_\Omega v_{i,i}d\Omega = \int_\Gamma v_i n_i d\Gamma} \tag{5.7}$$

That is the integral of the outer normal component of a vector over a closed surface (which is the **volume flux**) is equal to the integral of the divergence of the vector over the volume bounded by the closed surface.

26 For 2D-1D transformations, we have

$$\boxed{\int_A \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{q}dA = \oint_s \mathbf{q}^T\mathbf{n}ds} \tag{5.8}$$

27 This theorem is sometime refered to as Green's theorem in space.

5.5 Stoke's Theorem

28 Stoke's theorem states that

$$\boxed{\oint_{\mathcal{C}} \mathbf{A}\boldsymbol{\cdot}d\mathbf{r} = \int\int_S (\boldsymbol{\nabla}\times\mathbf{A})\boldsymbol{\cdot}\mathbf{n}dS = \int\int_{\mathcal{S}} (\boldsymbol{\nabla}\times\mathbf{A})\boldsymbol{\cdot}d\mathbf{S}} \tag{5.9}$$

where $\mathcal{S}$ is an open surface with two faces confined by $\mathcal{C}$

5.6 Green; Gradient Theorem

29 Green's theorem in plane is a special case of Stoke's theorem.

$$\boxed{\oint (Rdx + Sdy) = \int_\Gamma \left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right) dxdy} \tag{5.10}$$

■ Example 5-1: Physical Interpretation of the Divergence Theorem

Provide a physical interpretation of the Divergence Theorem.

Solution:

A fluid has a velocity field $\mathbf{v}(x, y, z)$ and we first seek to determine the net inflow per unit time per unit volume in a parallelepiped centered at $P(x, y, z)$ with dimensions $\Delta x, \Delta y, \Delta z$, Fig. 5.1-a.

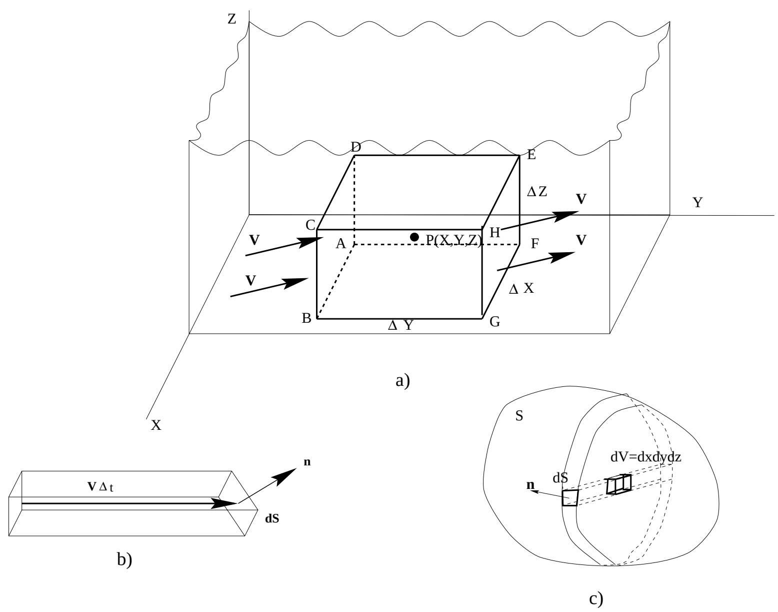


Figure 5.1: Physical Interpretation of the Divergence Theorem

$$v_x\big|_{x,y,z} \approx v_x \tag{5.11-a}$$

$$v_x\Big|_{x-\Delta x/2,y,z} \approx v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x \quad \text{AFED} \tag{5.11-b}$$

$$v_x\Big|_{x+\Delta x/2,y,z} \approx v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x \quad \text{GHCB} \tag{5.11-c}$$

The net inflow per unit time across the x planes is

$$\Delta V_x = \left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\Delta z - \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\Delta z \tag{5.12-a}$$

$$= \frac{\partial v_x}{\partial x}\Delta x\Delta y\Delta z \tag{5.12-b}$$

Similarly

$$\Delta V_y = \frac{\partial v_y}{\partial y}\Delta x\Delta y\Delta z \tag{5.13-a}$$

$$\Delta V_z = \frac{\partial v_z}{\partial z}\Delta x \Delta y \Delta z \tag{5.13-b}$$

Hence, the total increase per unit volume and unit time will be given by

$$\frac{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z} = \text{div } \mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{v} \tag{5.14}$$

Furthermore, if we consider the total of fluid crossing dS during Δt, Fig. 5.1-b, it will be given by $(\mathbf{v}\Delta t)\cdot\mathbf{n}dS = \mathbf{v}\cdot\mathbf{n}dS\Delta t$ or the volume of fluid crossing dS per unit time is $\mathbf{v}\cdot\mathbf{n}dS$.

Thus for an arbitrary volume, Fig. 5.1-c, the total amount of fluid crossing a closed surface S per unit time is $\int_S \mathbf{v}\cdot\mathbf{n}dS$. But this is equal to $\int_V \boldsymbol{\nabla}\cdot\mathbf{v}dV$ (Eq. 5.14), thus

$$\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_V \boldsymbol{\nabla}\cdot\mathbf{v}dV \tag{5.15}$$

which is the divergence theorem. ■

Chapter 6

FUNDAMENTAL LAWS of CONTINUUM MECHANICS

6.1 Introduction

20 We have thus far studied the stress tensors (Cauchy, Piola Kirchoff), and several other tensors which describe strain at a point. In general, those tensors will vary from point to point and represent a **tensor field**.

21 We have also obtained only one differential equation, that was the compatibility equation.

22 In this chapter, we will derive additional differential equations governing the way stress and deformation vary at a point and with time. They will apply to any continuous medium, and yet we will not have enough equations to determine unknown tensor field. For that we need to wait for the next chapter where constitututive laws relating stress and strain will be introduced. Only with constitutive equations and boundary and initial conditions would we be able to obtain a well defined mathematical problem to solve for the stress and deformation distribution or the displacement or velocity fields.

23 In this chapter we shall derive differential equations expressing locally the conservation of mass, momentum and energy. These differential equations of balance will be derived from integral forms of the equation of balance expressing the fundamental postulates of continuum mechanics.

6.1.1 Conservation Laws

24 Conservation laws constitute a fundamental component of classical physics. A conservation law establishes a balance of a scalar or tensorial quantity in voulme V bounded by a surface S. In its most general form, such a law may be expressed as

$$\underbrace{\frac{d}{dt}\int_V \mathcal{A} dV}_{\text{Rate of variation}} + \underbrace{\int_S \boldsymbol{\alpha} dS}_{\text{Exchange by Diffusion}} = \underbrace{\int_V \mathbf{A} dV}_{\text{Source}} \tag{6.1}$$

where $\mathcal{A}$ is the volumetric density of the quantity of interest (mass, linear momentum, energy, ...) **a**, **A** is the rate of volumetric density of what is provided from the outside, and $\boldsymbol{\alpha}$ is the rate of surface density of what is lost through the surface S of V and will be a function of the normal to the surface **n**.

25 Hence, we read the previous equation as: The input quantity (provided by the right hand side) is equal to what is lost across the boundary, and to modify $\mathcal{A}$ which is the quantity of interest. The dimensions of various quantities are given by

$$\begin{aligned} \dim(\mathbf{a}) &= \dim(\mathcal{A}L^{-3}) && \text{(6.2-a)} \\ \dim(\boldsymbol{\alpha}) &= \dim(\mathcal{A}L^{-2}t^{-1}) && \text{(6.2-b)} \\ \dim(\mathbf{A}) &= \dim(\mathcal{A}L^{-3}t^{-1}) && \text{(6.2-c)} \end{aligned}$$

26 Hence this chapter will apply the previous conservation law to mass, momentum, and energy. the resulting differential equations will provide additional interesting relation with regard to the imcompressibiltiy of solids (important in classical hydrodynamics and plasticity theories), equilibrium and symmetry of the stress tensor, and the first law of thermodynamics.

27 The enunciation of the preceding three conservation laws plus the second law of thermodynamics, constitute what is commonly known as the **fundamental laws of continuum mechanics**.

6.1.2 Fluxes

28 Prior to the enunciation of the first conservation law, we need to define the concept of flux across a bounding surface.

29 The **flux** across a surface can be graphically defined through the consideration of an imaginary surface fixed in space with continuous "medium" flowing through it. If we assign a positive side to the surface, and take **n** in the positive sense, then the volume of "material" flowing through the infinitesimal surface area dS in time dt is equal to the volume of the cylinder with base dS and slant height vdt parallel to the velocity vector **v**, Fig. 6.1 (If $\mathbf{v}\cdot\mathbf{n}$ is negative, then the flow is in the negative direction). Hence, we define the volume flux as

$$\boxed{\text{Volume Flux } = \int_S \mathbf{v}\cdot\mathbf{n}dS = \int_S v_j n_j dS} \quad (6.3)$$

where the last form is for rectangular cartesian components.

30 We can generalize this definition and define the following fluxes per unit area through dS:

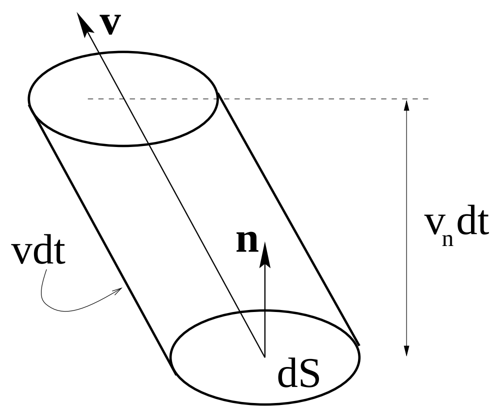


Figure 6.1: Flux Through Area dS

$$\begin{aligned}
\text{Mass Flux} &= \int_S \rho \mathbf{v}\cdot\mathbf{n} dS = \int_S \rho v_j n_j dS && (6.4)\\
\text{Momentum Flux} &= \int_S \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS = \int_S \rho v_k v_j n_j dS && (6.5)\\
\text{Kinetic Energy Flux} &= \int_S \frac{1}{2}\rho v^2(\mathbf{v}\cdot\mathbf{n}) dS = \int_S \frac{1}{2}\rho v_i v_i v_j n_j dS && (6.6)\\
\text{Heat flux} &= \int_S \mathbf{q}\cdot\mathbf{n} dS = \int_S q_j n_j dS && (6.7)\\
\text{Electric flux} &= \int_S \mathbf{J}\cdot\mathbf{n} dS = \int_S J_j n_j dS && (6.8)
\end{aligned}$$

6.2 Conservation of Mass; Continuity Equation

6.2.1 Spatial Form

31 If we consider an arbitrary volume V, fixed in space, and bounded by a surface S. If a continuous medium of density ρ fills the volume at time t, then the total mass in V is

$$M = \int_V \rho(\mathbf{x}, t) dV \tag{6.9}$$

where $\rho(\mathbf{x}, t)$ is a continuous function called the **mass density**. We note that this spatial form in terms of $\mathbf{x}$ is most common in fluid mechanics.

32 The rate of increase of the total mass in the volume is

$$\frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t} dV \tag{6.10}$$

33 The **Law of conservation of mass** requires that the mass of a specific portion of the continuum remains constant. Hence, if no mass is created or destroyed inside V, then the preceding equation must eqaul the **inflow of mass** (of **flux**) through the surface. The outflow is equal to $\mathbf{v}\cdot\mathbf{n}$, thus the inflow will be equal to $-\mathbf{v}\cdot\mathbf{n}$.

$$\int_S (-\rho v_n) dS = -\int_S \rho \mathbf{v}\cdot\mathbf{n} dS = -\int_V \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) dV \tag{6.11}$$

must be equal to $\frac{\partial M}{\partial t}$. Thus

$$\int_V \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \mathbf{v})\right] dV = 0 \tag{6.12}$$

since the integral must hold for any arbitrary choice of dV, then we obtain

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \mathbf{v}) \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \tag{6.13}$$

34 The chain rule will in turn give

$$\frac{\partial(\rho v_i)}{\partial x_i} = \rho \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \rho}{\partial x_i} \tag{6.14}$$

35 It can be shown that the rate of change of the density in the neighborhood of a particle instantaneously at $\mathbf{x}$ by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v}\boldsymbol{\cdot}\nabla\rho = \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} \tag{6.15}$$

where the first term gives the local rate of change of the density in the neighborhood of the place of $\mathbf{x}$, while the second term gives the **convective rate of change** of the density in the neighborhood of a particle as it moves to a place having a different density. The first term vanishes in a steady flow, while the second term vanishes in a uniform flow.

36 Upon substitution in the last three equations, we obtain the continuity equation

$$\boxed{\frac{d\rho}{dt} + \rho \frac{\partial v_i}{\partial x_i} = 0 \quad \text{or} \quad \frac{d\rho}{dt} + \rho \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v} = 0} \tag{6.16}$$

The vector form is independent of any choice of coordinates. This equation shows that the divergence of the velocity vector field equals $(-1/\rho)(d\rho/dt)$ and measures the rate of flow of material away from the particle and is equal to the unit rate of decrease of density ρ in the neighborhood of the particle.

37 If the material is incompressible, so that the density in the neighborhood of each material particle remains constant as it moves, then the continuity equation takes the simpler form

$$\boxed{\frac{\partial v_i}{\partial x_i} = 0 \quad \text{or} \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v} = 0} \tag{6.17}$$

this is the **condition of incompressibility**

6.2.2 Material Form

38 If material coordinates $\mathbf{X}$ are used, the conservation of mass, and using Eq. 4.38 ($dV = |J| dV_0$), implies

$$\int_{V_0} \rho(\mathbf{X}, t_0) dV_0 = \int_V \rho(\mathbf{x}, t) dV = \int_{V_0} \rho(\mathbf{x}, t)|J| dV_0 \tag{6.18}$$

or

$$\int_{V_0} [\rho_0 - \rho|J|] dV_0 = 0 \tag{6.19}$$

and for an arbitrary volume dV_0, the integrand must vanish. If we also suppose that the initial density ρ_0 is everywhere positive in V_0 (no empty spaces), and at time $t = t_0$, $J = 1$, then we can write

$$\rho J = \rho_0 \tag{6.20}$$

or

$$\boxed{\frac{d}{dt}(\rho J) = 0} \tag{6.21}$$

which is the **continuity equation due to Euler**, or the **Lagrangian differential form** of the continuity equation.

39 We note that this is the same equation as Eq. 6.16 which was expressed in spatial form. Those two equations can be derived one from the other.

40 The more commonly used form if the continuity equation is Eq. 6.16.

6.3 Linear Momentum Principle; Equation of Motion

6.3.1 Momentum Principle

41 The momentum principle states that *the time rate of change of the total momentum of a given set of particles equals the vector sum of all external forces acting on the particles of the set, provided Newton's Third Law applies.* The continuum form of this principle is a basic **postulate** of continuum mechanics.

$$\int_S \mathbf{t} dS + \int_V \rho \mathbf{b} dV = \frac{d}{dt} \int_V \rho \mathbf{v} dV \tag{6.22}$$

Then we substitute $t_i = T_{ij} n_j$ and apply the divergence theorm to obtain

$$\int_V \left(\frac{\partial T_{ij}}{\partial x_j} + \rho b_i \right) dV = \int_V \rho \frac{dv_i}{dt} dV \tag{6.23-a}$$

$$\int_V \left[\frac{\partial T_{ij}}{\partial x_j} + \rho b_i - \rho \frac{dv_i}{dt} \right] dV = 0 \tag{6.23-b}$$

or for an arbitrary volume

$$\boxed{\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho \frac{dv_i}{dt} \quad \text{or} \quad \boldsymbol{\nabla} \mathbf{T} + \rho \mathbf{b} = \rho \frac{dv}{dt}} \tag{6.24}$$

which is **Cauchy's (first) equation of motion**, or **the linear momentum principle**, or more simply **equilibrium equation**.

42 When expanded in 3D, this equation yields:

$$\begin{aligned}
\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}+\rho b_1 &= 0\\
\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}+\rho b_2 &= 0\\
\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}+\rho b_3 &= 0
\end{aligned} \tag{6.25-a}$$

43 We note that these equations could also have been derived from the free body diagram shown in Fig. 6.2 with the assumption of **equilibrium** (via Newton's second law) considering an infinitesimal element of dimensions $dx_1 \times dx_2 \times dx_3$. Writing the summation of forces, will yield

$$\boxed{T_{ij,j}+\rho b_i = 0} \tag{6.26}$$

where ρ is the density, b_i is the body force (including inertia).

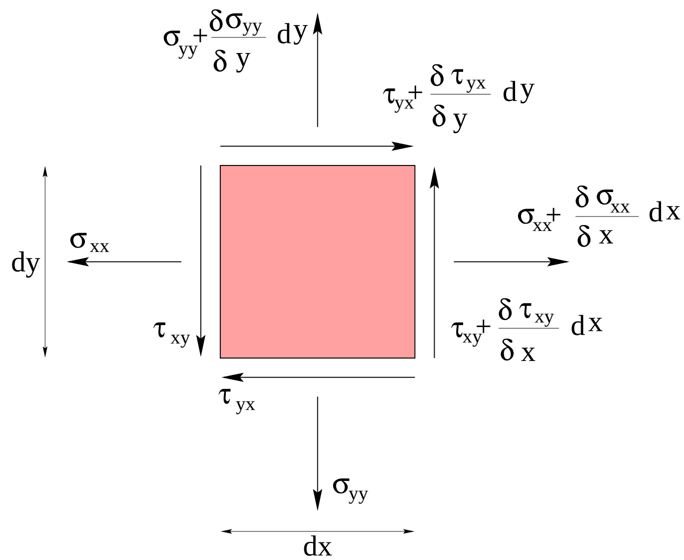


Figure 6.2: Equilibrium of Stresses, Cartesian Coordinates

■ Example 6-1: Equilibrium Equation

In the absence of body forces, does the following stress distribution

$$\begin{bmatrix} x_2^2+\nu(x_1^2-x_x^2) & -2\nu x_1 x_2 & 0\\ -2\nu x_1 x_2 & x_1^2+\nu(x_2^2-x_1^2) & 0\\ 0 & 0 & \nu(x_1^2+x_2^2)\end{bmatrix} \tag{6.27}$$

where ν is a constant, satisfy equilibrium?

Solution:

$$\frac{\partial T_{1j}}{\partial x_j} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 2\nu x_1 - 2\nu x_1 = 0\surd \tag{6.28-a}$$

$$\frac{\partial T_{2j}}{\partial x_j} = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = -2\nu x_2 + 2\nu x_2 = 0\surd \tag{6.28-b}$$

$$\frac{\partial T_{3j}}{\partial x_j} = \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0\surd \tag{6.28-c}$$

Therefore, equilibrium is satisfied. ■

6.3.2 Moment of Momentum Principle

44 The moment of momentum principle states that *the time rate of change of the total moment of momentum of a given set of particles equals the vector sum of the moments of all external forces acting on the particles of the set.*

45 Thus, in the absence of **distributed couples** (this theory of Cosserat will not be covered in this course) we postulate the same principle for a continuum as

$$\boxed{\int_S (\mathbf{r}\times\mathbf{t})dS + \int_V (\mathbf{r}\times\rho\mathbf{b})dV = \frac{d}{dt}\int_V (\mathbf{r}\times\rho\mathbf{v})dV} \tag{6.29}$$

6.3.2.1 Symmetry of the Stress Tensor

46 We observe that the preceding equation does not furnish any new differential equation of motion. If we substitute $\mathbf{t}_n = \mathbf{T}\mathbf{n}$ and the symmetry of the tensor is assumed, then the linear momentum principle (Eq. 6.24) is satisfied.

47 Alternatively, we may start by using Eq. 1.18 ($c_i = \varepsilon_{ijk}a_jb_k$) to express the cross product in indicial form and substitute above:

$$\int_S (\varepsilon_{rmn}x_m t_n)dS + \int_V (\varepsilon_{rmn}x_m b_n \rho)dV = \frac{d}{dt}\int_V (\varepsilon_{rmn}x_m \rho v_n)dV \tag{6.30}$$

we then substitute $t_n = T_{jn}n_j$, and apply Gauss theorem to obtain

$$\int_V \varepsilon_{rmn}\left[\frac{\partial x_m T_{jn}}{\partial x_j} + x_m\rho b_n\right]dV = \int_V \varepsilon_{rmn}\frac{d}{dt}(x_m v_n)\rho dV \tag{6.31}$$

but since $dx_m/dt = v_m$, this becomes

$$\int_V \varepsilon_{rmn}\left[x_m\left(\frac{\partial T_{jn}}{\partial x_j} + \rho b_n\right) + \delta_{mj}T_{jn}\right]dV = \int_V \varepsilon_{rmn}\left(v_m v_n + x_m\frac{dv_n}{dt}\right)\rho dV \tag{6.32}$$

but $\varepsilon_{rmn}v_m v_n = 0$ since $v_m v_n$ is symmetric in the indeces mn while ε_{rmn} is antisymmetric, and the last term on the right cancels with the first term on the left, and finally with $\delta_{mj}T_{jn} = T_{mn}$ we are left with

$$\int_V \varepsilon_{rmn}T_{mn}dV = 0 \tag{6.33}$$

or for an arbitrary volume V,

$$\boxed{\varepsilon_{rmn}T_{mn} = 0} \tag{6.34}$$

at each point, and this yields

$$\boxed{\begin{array}{lllll} \text{for } r & = & 1 & T_{23} - T_{32} & = & 0 \\ \text{for } r & = & 2 & T_{31} - T_{13} & = & 0 \\ \text{for } r & = & 3 & T_{12} - T_{21} & = & 0 \end{array}} \tag{6.35}$$

establishing the symmetry of the stress matrix without any assumption of equilibrium or of uniformity of stress distribution as was done in Sect. 2.3.

48 The symmetry of the stress matrix is **Cauchy's second law of motion** (1827).

6.4 Conservation of Energy; First Principle of Thermodynamics

49 The first principle of thermodynamics relates the work done on a (closed) system and the heat transfer into the system to the change in energy of the system. We shall assume that the only energy transfers to the system are by mechanical work done on the system by surface traction and body forces, by heat transfer through the boundary.

6.4.1 Spatial Gradient of the Velocity

50 We define **L** as the **spatial gradient of the velocity** and in turn this gradient can be decomposed into a symmetric **rate of deformation tensor D** (or **stretching tensor**) and a skew-symmeteric tensor **W** called the **spin tensor** or **vorticity tensor**[1].

$$L_{ij} = v_{i,j} \quad \text{or} \quad \mathbf{L} = \mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}} \tag{6.36}$$

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \tag{6.37}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}) \quad \text{and} \quad \mathbf{W} = \frac{1}{2}(\mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}} - \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}) \tag{6.38}$$

this term will be used in the derivation of the first principle.

6.4.2 First Principle

51 If mechanical quantities only are considered, the **principle of conservation of energy** for the continuum may be derived directly from the equation of motion given by

[1]Note similarity with Eq. 4.106-b.

Eq. 6.24. This is accomplished by taking the integral over the volume V of the scalar product between Eq. 6.24 and the velocity v_i.

$$\int_V v_i T_{ji,j} dV + \int_V \rho b_i v_i dV = \int_V \rho v_i \frac{dv_i}{dt} dV \tag{6.39}$$

If we consider the right hand side

$$\int_V \rho v_i \frac{dv_i}{dt} dV = \frac{d}{dt}\int_V \frac{1}{2}\rho v_i v_i dV = \frac{d}{dt}\int_V \frac{1}{2}\rho v^2 dV = \frac{dK}{dt} \tag{6.40}$$

which represents the time rate of change of the **kinetic energy** K in the continuum.

52 Also we have $v_i T_{ji,j} = (v_i T_{ji})_{,j} - v_{i,j} T_{ji}$ and from Eq. 6.37 we have $v_{i,j} = L_{ij} + W_{ij}$. It can be shown that since W_{ij} is skew-symmetric, and $\mathbf{T}$ is symmetric, that $T_{ij}W_{ij} = 0$, and thus $T_{ij}L_{ij} = T_{ij}D_{ij}$. $\mathbf{T}\ddot{\mathbf{D}}$ is called the **stress power**.

53 If we consider thermal processes, the rate of increase of total heat into the continuum is given by

$$Q = -\int_S q_i n_i dS + \int_V \rho r dV \tag{6.41}$$

Q has the dimension of power, that is ML^2T^{-3}, and the SI unit is the Watt (W). $\mathbf{q}$ is the **heat flux** per unit area by conduction, its dimension is MT^{-3} and the corresponding SI unit is Wm^{-2}. Finally, r is the **radiant heat constant** per unit mass, its dimension is $MT^{-3}L^{-4}$ and the corresponding SI unit is Wm^{-6}.

54 We thus have

$$\frac{dK}{dt} + \int_V D_{ij}T_{ij} dV = \int_V (v_i T_{ji})_{,j} dV + \int_V \rho v_i b_i dV + Q \tag{6.42}$$

55 We next convert the first integral on the right hand side to a surface integral by the divergence theorem ($\int_V \boldsymbol{\nabla}\cdot\mathbf{v} dV = \int_S \mathbf{v}.\mathbf{n} dS$) and since $t_i = T_{ij}n_j$ we obtain

$$\frac{dK}{dt} + \int_V D_{ij}T_{ij} dV = \int_S v_i t_i dS + \int_V \rho v_i b_i dV + Q \tag{6.43}$$

$$\frac{dK}{dt} + \frac{dU}{dt} = \frac{dW}{dt} + Q \tag{6.44}$$

this equation relates the time rate of change of total mechanical energy of the continuum on the left side to the rate of work done by the surface and body forces on the right hand side.

56 If both mechanical and non mechanical energies are to be considered, the first principle states that *the time rate of change of the kinetic plus the internal energy is equal to the sum of the rate of work plus all other energies supplied to, or removed from the continuum per unit time (heat, chemical, electromagnetic, etc.).*

57 For a thermomechanical continuum, it is customary to express the time rate of change of internal energy by the integral expression

$$\frac{dU}{dt} = \frac{d}{dt}\int_V \rho u dV \tag{6.45}$$

where u is the internal energy per unit mass or **specific internal energy**. We note that U appears only as a differential in the first principle, hence if we really need to evaluate this quantity, we need to have a reference value for which U will be null. The dimension of U is one of energy dim $U = ML^2T^{-2}$, and the SI unit is the Joule, similarly dim $u = L^2T^{-2}$ with the SI unit of Joule/Kg.

58 In terms of energy integrals, the first principle can be rewritten as

$$\overbrace{\underbrace{\frac{d}{dt}\int_V \frac{1}{2}\rho v_i v_i dV}_{\frac{dK}{dt}} + \underbrace{\frac{d}{dt}\int_V \rho u dV}_{\frac{dU}{dt}}}^{\text{Rate of increae}} = \underbrace{\overbrace{\int_S t_i v_i dS}^{\text{Exchange}} + \overbrace{\int_V \rho v_i b_i dV}^{\text{Source}}}_{\frac{dW}{dt}} + \underbrace{\overbrace{\int_V \rho r dV}^{\text{Source}} - \overbrace{\int_S q_i n_i dS}^{\text{Exchange}}}_{Q} \quad (6.46)$$

we apply Gauss theorem to convert the surface integral, collect terms and use the fact that dV is arbitrary to obtain

$$\rho\frac{du}{dt} = \mathbf{T}:\mathbf{D} + \rho r - \boldsymbol{\nabla}\cdot\mathbf{q} \quad (6.47)$$

or

$$\rho\frac{du}{dt} = T_{ij}D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j} \quad (6.48)$$

59 This equation expresses the rate of change of **internal energy** as the sum of the **stress power** plus the **heat** added to the continuum.

60 In ideal elasticity, heat transfer is considered insignificant, and all of the input work is assumed converted into internal energy in the form of recoverable stored elastic strain energy, which can be recovered as work when the body is unloaded.

61 In general, however, the major part of the input work into a deforming material is not recoverably stored, but dissipated by the deformation process causing an increase in the body's temperature and eventually being conducted away as heat.

6.5 Equation of State; Second Principle of Thermodynamics

62 The complete characterization of a thermodynamic system is said to describe the **state** of a system (here a continuum). This description is specified, in general, by several thermodynamic and kinematic **state variables**. A change in time of those state variables constitutes a **thermodynamic process**. Usually state variables are not all independent, and functional relationships exist among them through **equations of state**. Any state variable which may be expressed as a single valued function of a set of other state variables is known as a **state function**.

63 The first principle of thermodynamics can be regarded as an expression of the interconvertibility of heat and work, maintaining an energy balance. It places no restriction on the direction of the process. In classical mechanics, kinetic and potential energy can be easily transformed from one to the other in the absence of friction or other dissipative mechanism.

64 The first principle leaves unanswered the question of the extent to which conversion process is **reversible** or **irreversible**. If thermal processes are involved (friction) dissipative processes are irreversible processes, and it will be up to the second principle of thermodynamics to put limits on the direction of such processes.

6.5.1 Entropy

65 The basic criterion for irreversibility is given by the **second principle of thermodynamics** through the statement on the limitation of **entropy production**. This law postulates the existence of two distinct state functions: θ the **absolute temperature** and S the **entropy** with the following properties:

1. θ is a positive quantity.
2. Entropy is an extensive property, i.e. the total entropy is in a system is the sum of the entropies of its parts.

66 Thus we can write

$$ds = ds^{(e)} + ds^{(i)} \tag{6.49}$$

where $ds^{(e)}$ is the increase due to interaction with the exterior, and $ds^{(i)}$ is the internal increase, and

$$\begin{aligned} ds^{(e)} &> 0 \text{ irreversible process} && \text{(6.50-a)} \\ ds^{(i)} &= 0 \text{ reversible process} && \text{(6.50-b)} \end{aligned}$$

67 Entropy expresses a variation of energy associated with a variation in the temperature.

6.5.1.1 Statistical Mechanics

68 In statistical mechanics, entropy is related to the probability of the occurrence of that state among all the possible states that could occur. It is found that changes of states are more likely to occur in the direction of greater **disorder** when a system is left to itself. Thus *increased entropy means increased disorder.*

69 Hence Boltzman's principle postulates that entropy of a state is proportional to the logarithm of its probability, and for a gas this would give

$$S = kN[\ln V + \frac{3}{2}\ln\theta] + C \tag{6.51}$$

where S is the total entropy, V is volume, θ is absolute temperature, k is Boltzman's constant, and C is a constant and N is the number of molecules.

6.5.1.2 Classical Thermodynamics

70 In a reversible process (more about that later), the change in **specific entropy** s is given by

$$ds = \left(\frac{dq}{\theta}\right)_{rev} \tag{6.52}$$

71 If we consider an ideal gas governed by

$$pv = R\theta \tag{6.53}$$

where R is the gas constant, and assuming that the specific energy u is only a function of temperature θ, then the first principle takes the form

$$du = dq - pdv \tag{6.54}$$

and for constant volume this gives

$$du = dq = c_v d\theta \tag{6.55}$$

wher c_v is the specific heat at constant volume. The assumption that $u = u(\theta)$ implies that c_v is a function of θ only and that

$$du = c_v(\theta) d\theta \tag{6.56}$$

72 Hence we rewrite the first principle as

$$dq = c_v(\theta) d\theta + R\theta \frac{dv}{v} \tag{6.57}$$

or division by θ yields

$$\boxed{s - s_0 = \int_{p_0,v_0}^{p,v} \frac{dq}{\theta} = \int_{\theta_0}^{\theta} c_v(\theta) \frac{d\theta}{\theta} + R \ln \frac{v}{v_0}} \tag{6.58}$$

which gives the change in entropy for any reversible process in an ideal gas. In this case, entropy is a state function which returns to its initial value whenever the temperature returns to its initial value that is p and v return to their initial values.

6.5.2 Clausius-Duhem Inequality

73 We restate the definition of entropy as heat divided by temperature, and write the second principle

$$\underbrace{\frac{d}{dt}\int_V \rho s}_{\text{Rate of Entropy Increase}} = \underbrace{\int_V \rho \frac{r}{\theta} dV}_{\text{Sources}} - \underbrace{\int_S \frac{\mathbf{q}}{\theta} \cdot \mathbf{n} dS}_{\text{Exchange}} + \underbrace{\Gamma}_{\text{Internal production}} ; \quad \Gamma \geq 0 \tag{6.59}$$

$$\frac{dS}{dt} = \frac{Q}{\theta} + \Gamma; \quad \Gamma \geq 0 \tag{6.60}$$

$\Gamma = 0$ for reversible processes, and $\Gamma > 0$ in irreversible ones. The dimension of $S = \int_v \rho s dV$ is one of energy divided by temperature or $L^2 M T^{-2} \theta^{-1}$, and the SI unit for entropy is Joule/Kelvin.

74 The second principle postulates that *the time rate of change of total entropy S in a continuum occupying a volume V is always greater or equal than the sum of the entropy influx through the continuum surface plus the entropy produced internally by body sources.*

75 The previous inequality holds for any arbitrary volume, thus after transformation of the surface integral into a volume integral, we obtain the following local version of the Clausius-Duhem inequality which must holds at every point

$$\boxed{\underbrace{\rho\frac{ds}{dt}}_{\text{Rate of Entropy Increase}} \geq \underbrace{\frac{\rho r}{\theta}}_{\text{Sources}} - \underbrace{\boldsymbol{\nabla}\cdot\frac{\mathbf{q}}{\theta}}_{\text{Exchange}}} \tag{6.61}$$

76 We next seek to express the Clausius-Duhem inequality in terms of the stress tensor,

$$\boldsymbol{\nabla}\cdot\frac{\mathbf{q}}{\theta} = \frac{1}{\theta}\boldsymbol{\nabla}\cdot\mathbf{q} - \mathbf{q}\cdot\boldsymbol{\nabla}\frac{1}{\theta} = \frac{1}{\theta}\boldsymbol{\nabla}\cdot\mathbf{q} - \frac{1}{\theta^2}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \tag{6.62}$$

thus

$$\rho\frac{ds}{dt} \geq -\frac{1}{\theta}\boldsymbol{\nabla}\cdot\mathbf{q} + \frac{1}{\theta^2}\mathbf{q}\cdot\boldsymbol{\nabla}\theta + \frac{\rho r}{\theta} \tag{6.63}$$

but since θ is always positive,

$$\rho\theta\frac{ds}{dt} \geq -\boldsymbol{\nabla}\cdot\mathbf{q} + \rho r + \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \tag{6.64}$$

where $-\boldsymbol{\nabla}\cdot\mathbf{q} + \rho r$ is the heat input into V and appeared in the first principle Eq. 6.47

$$\rho\frac{du}{dt} = \mathbf{T}:\mathbf{D} + \rho r - \boldsymbol{\nabla}\cdot\mathbf{q} \tag{6.65}$$

hence, substituting, we obtain

$$\boxed{\mathbf{T}:\mathbf{D} - \rho\left(\frac{du}{dt} - \theta\frac{ds}{dt}\right) - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0} \tag{6.66}$$

6.6 Balance of Equations and Unknowns

77 In the preceding sections several equations and unknowns were introduced. Let us count them. for both the coupled and uncoupled cases.

		Coupled	Uncoupled
$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$	Continuity Equation	1	1
$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\frac{dv_i}{dt}$	Equation of motion	3	3
$\rho\frac{du}{dt} = T_{ij}D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j}$	Energy equation	1	
Total number of equations		**5**	**4**

78 Assuming that the body forces b_i and distributed heat sources r are prescribed, then we have the following unknowns:

		Coupled	Uncoupled
Density	ρ	1	1
Velocity (or displacement)	v_i (u_i)	3	3
Stress components	T_{ij}	6	6
Heat flux components	q_i	3	-
Specific internal energy	u	1	-
Entropy density	s	1	-
Absolute temperature	θ	1	-
Total number of unknowns		**16**	**10**

and in addition the Clausius-Duhem inequality $\frac{ds}{dt} \geq \frac{r}{\theta} - \frac{1}{\rho}\text{div}\,\frac{\mathbf{q}}{\theta}$ which governs entropy production must hold.

79 We thus need an additional $16 - 5 = 11$ additional equations to make the system determinate. These will be later on supplied by:

6	constitutive equations
3	temperature heat conduction
2	thermodynamic equations of state
11	**Total number of additional equations**

80 The next chapter will thus discuss constitutive relations, and a subsequent one will separately discuss thermodynamic equations of state.

81 We note that for the uncoupled case

1. The energy equation is essentially the integral of the equation of motion.
2. The 6 missing equations will be entirely supplied by the constitutive equations.
3. The temperature field is regarded as known, or at most, the heat-conduction problem must be solved separately and independently from the mechanical problem.

6.7 † Elements of Heat Transfer

82 One of the relations which we will need is the one which relates temperature to heat flux. This constitutive realtion will be discussed in the next chapter under Fourrier's law.

83 However to place the reader in the right frame of reference to understand Fourrier's law, this section will provide some elementary concepts of heat transfer.

84 There are three fundamental modes of heat transfer:

Conduction: takes place when a temperature gradient exists within a material and is governed by Fourier's Law, Fig. 6.3 on Γ_q:

$$q_x = -k_x \frac{\partial T}{\partial x} \tag{6.67}$$

$$q_y = -k_y \frac{\partial T}{\partial y} \tag{6.68}$$

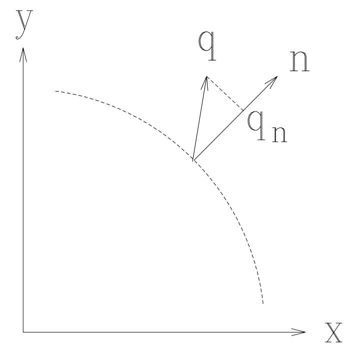


Figure 6.3: Flux vector

where $T = T(x, y)$ is the temperature field in the medium, q_x and q_y are the componenets of the heat flux (W/m^2 or Btu/h-ft^2), k is the thermal conductivity (W/m.^{o}C or Btu/h-ft-^{o}F) and $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ are the temperature gradients along the x and y respectively. Note that heat flows from "hot" to "cool" zones, hence the negative sign.

Convection: heat transfer takes place when a material is exposed to a moving fluid which is at different temperature. It is governed by the Newton's Law of Cooling

$$q = h(T - T_\infty) \quad \text{on} \quad \Gamma_c \tag{6.69}$$

where q is the convective heat flux, h is the convection heat transfer coefficient or *film coefficient* (W/m^2.^{o}C or Btu/h-ft^2.^{o}F). It depends on various factors, such as whether convection is natural or forced, laminar or turbulent flow, type of fluid, and geometry of the body; T and T_∞ are the surface and fluid temperature, respectively. This mode is considered as part of the boundary condition.

Radiation: is the energy transferred between two separated bodies at different temperatures by means of electromagnetic waves. The fundamental law is the Stefan-Boltman's Law of Thermal Radiation for black bodies in which the flux is proportional to the fourth power of the absolute temperature., which causes the problem to be nonlinear. This mode will not be covered.

6.7.1 Simple 2D Derivation

85 If we consider a unit thickness, 2D differential body of dimensions dx by dy, Fig. 6.4 then

1. Rate of heat generation/sink is
$$I_2 = Q dx dy \tag{6.70}$$

2. Heat flux across the boundary of the element is shown in Fig. **??** (note similarity

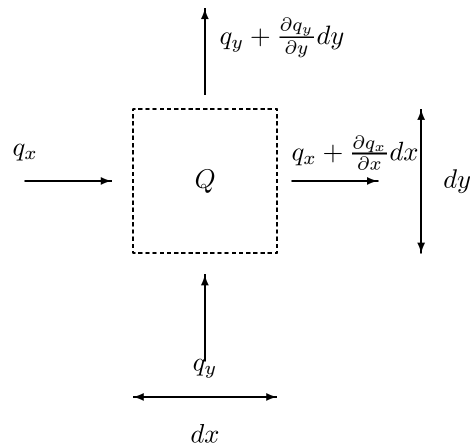


Figure 6.4: Flux Through Sides of Differential Element

with equilibrium equation)

$$I_1 = \left[\left(q_x + \frac{\partial q_x}{\partial x} dx \right) - q_x dx \right] dy + \left[\left(q_y + \frac{\partial q_y}{\partial y} dy \right) - q_y dy \right] dx = \frac{\partial q_x}{\partial x} dxdy + \frac{\partial q_y}{\partial y} dydx \quad (6.71)$$

3. Change in stored energy is

$$I_3 = c\rho \frac{d\phi}{dt}.dxdy \quad (6.72)$$

where we define the *specific heat* c as the amount of heat required to raise a unit mass by one degree.

86 From the first law of thermodaynamics, energy produced I_2 plus the net energy across the boundary I_1 must be equal to the energy absorbed I_3, thus

$$I_1 + I_2 - I_3 = 0 \quad (6.73\text{-a})$$

$$\underbrace{\frac{\partial q_x}{\partial x} dxdy + \frac{\partial q_y}{\partial y} dydx}_{I_1} + \underbrace{Qdxdy}_{I_2} - \underbrace{c\rho \frac{d\phi}{dt} dxdy}_{I_3} = 0 \quad (6.73\text{-b})$$

6.7.2 †Generalized Derivation

87 The amount of flow per unit time into an element of volume Ω and surface Γ is

$$I_1 = \int_\Gamma \mathbf{q}(-\mathbf{n}) d\Gamma = \int_\Gamma \mathbf{D} \boldsymbol{\nabla} \phi.\mathbf{n} d\Gamma \quad (6.74)$$

where $\mathbf{n}$ is the unit exterior normal to Γ, Fig. 6.5

Figure 6.5: *Flow through a surface Γ

88 Using the divergence theorem

$$\int_\Gamma \mathbf{vn}d\Gamma = \int_\Omega \text{div } \mathbf{v}d\Omega \tag{6.75}$$

Eq. 6.74 transforms into

$$I_1 = \int_\Omega \text{div } (\mathbf{D}\boldsymbol{\nabla}\phi)d\Omega \tag{6.76}$$

89 Furthermore, if the instantaneous volumetric rate of "heat" generation or removal at a point x, y, z inside Ω is $Q(x, y, z, t)$, then the total amount of heat/flow produced per unit time is

$$I_2 = \int_\Omega Q(x, y, z, t)d\Omega \tag{6.77}$$

90 Finally, we define the *specific heat* of a solid c as the amount of heat required to raise a unit mass by one degree. Thus if $\Delta\phi$ is a temperature change which occurs in a mass m over a time Δt, then the corresponding amount of heat that was added must have been $cm\Delta\phi$, or

$$I_3 = \int_\Omega \rho c\Delta\phi d\Omega \tag{6.78}$$

where ρ is the density, Note that another expression of I_3 is $\Delta t(I_1 + I_2)$.

91 The balance equation, or conservation law states that the energy produced I_2 plus the net energy across the boundary I_1 must be equal to the energy absorbed I_3, thus

$$I_1 + I_2 - I_3 = 0 \tag{6.79-a}$$

$$\int_\Omega \left(\text{div } (\mathbf{D}\boldsymbol{\nabla}\phi) + Q - \rho c\frac{\Delta\phi}{\Delta t}\right) d\Omega = 0 \tag{6.79-b}$$

but since t and Ω are both arbitrary, then

$$\text{div } (\mathbf{D}\boldsymbol{\nabla}\phi) + Q - \rho c\frac{\partial\phi}{\partial t} = 0 \tag{6.80}$$

or

$$\boxed{\text{div } (\mathbf{D}\boldsymbol{\nabla}\phi) + Q = \rho c\frac{\partial\phi}{\partial t}} \tag{6.81}$$

This equation can be rewritten as

$$\boxed{\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + Q = \rho c\frac{\partial\phi}{\partial t}} \tag{6.82}$$

1. Note the similarity between this last equation, and the equation of equilibrium

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \rho b_x = \rho m\frac{\partial^2 u_x}{\partial t^2} \tag{6.83-a}$$

$$\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x} + \rho b_y = \rho m\frac{\partial^2 u_y}{\partial t^2} \tag{6.83-b}$$

2. For steady state problems, the previous equation does not depend on t, and for 2D problems, it reduces to

$$\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial\phi}{\partial y}\right)\right]+Q=0 \tag{6.84}$$

3. For steady state isotropic problems,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+\frac{Q}{k}=0 \tag{6.85}$$

   which is *Poisson's equation* in 3D.

4. If the heat input $Q = 0$, then the previous equation reduces to

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{6.86}$$

   which is an **Elliptic** (or *Laplace*) equation. Solutions of Laplace equations are termed *harmonic functions* (right hand side is zero) which is why Eq. 6.84 is refered to as the *quasi-harmonic* equation.

5. If the function depends only on x and t, then we obtain

$$\rho c\frac{\partial\phi}{\partial t}=\frac{\partial}{\partial x}\left(k_x\frac{\partial\phi}{\partial x}\right)+Q \tag{6.87}$$

   which is a **parabolic** (or Heat) equation.

Chapter 7

CONSTITUTIVE EQUATIONS; Part I LINEAR

ceiinosssttuu
Hooke, 1676
Ut tensio sic vis
Hooke, 1678

7.1 † Thermodynamic Approach

7.1.1 State Variables

20 The method of local state postulates that the thermodynamic **state of a continuum** at a given point and instant is completely defined by several **state variables** (also known as **thermodynamic or independent variables**). A change in time of those state variables constitutes a **thermodynamic process**. Usually state variables are not all independent, and functional relationships exist among them through **equations of state**. Any state variable which may be expressed as a single valued function of a set of other state variables is known as a **state function**.

21 The time derivatives of these variables are not involved in the definition of the state, this postulate implies that any evolution can be considered as a succession of equilibrium states (therefore ultra rapid phenomena are excluded).

22 The **thermodynamic state** is specified by $n+1$ variables $\nu_1, \nu_2, \cdots, \nu_n$ and s where ν_i are the **thermodynamic substate variables** and s the specific entropy. The former have mechanical (or electromagnetic) dimensions, but are otherwise left arbitrary in the general formulation. In ideal elasticity we have nine substate variables the components of the strain or deformation tensors.

23 The **basic assumption of thermodynamics** is that in addition to the n substate variables, just one additional dimensionally independent scalar paramer suffices to determine the specific internal energy u. This assumes that there exists a **caloric equation**

of state

$$\boxed{u = u(s, \boldsymbol{\nu}, \mathbf{X})} \tag{7.1}$$

24 In general the internal energy u can not be experimentally measured but rather its derivative.

25 For instance we can define the **thermodynamic temperature** θ and the **thermodynamic "tension"** τ_j as

$$\boxed{\theta \equiv \left(\frac{\partial u}{\partial s}\right)_{\nu}; \qquad \tau_j \equiv \left(\frac{\partial u}{\partial \nu_j}\right)_{s,\nu_{i(i\neq j)}}; \qquad j = 1, 2, \cdots, n} \tag{7.2}$$

where the subscript outside the parenthesis indicates that the variables are held constant.

26 By extension $A_i = -\rho\tau_i$ would be the **thermodynamic "force"** and its dimension depends on the one of ν_i.

7.1.2 Gibbs Relation

27 From the chain rule we can express

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial s}\right)_{\nu}\frac{ds}{dt} + \tau_p\frac{d\nu_p}{dt} \tag{7.3}$$

28 substituting into Clausius-Duhem inequality of Eq. 6.66

$$\mathbf{T}{:}\mathbf{D} - \rho\left(\frac{du}{dt} - \theta\frac{ds}{dt}\right) - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0 \tag{7.4}$$

we obtain

$$\mathbf{T}{:}\mathbf{D} + \rho\frac{ds}{dt}\left[\theta - \left(\frac{\partial u}{\partial s}\right)_{\nu}\right] + A_p\frac{d\nu_p}{dt} - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0 \tag{7.5}$$

but the second principle must be satisfied for all possible evolution and in particular the one for which $\mathbf{D} = \mathbf{0}$, $\frac{d\nu_p}{dt} = 0$ and $\boldsymbol{\nabla}\theta = \mathbf{0}$ for any value of $\frac{ds}{dt}$ thus the coefficient of $\frac{ds}{dt}$ is zero or

$$\theta = \left(\frac{\partial u}{\partial s}\right)_{\nu} \tag{7.6}$$

thus

$$\boxed{\mathbf{T}{:}\mathbf{D} + A_p\frac{d\nu_p}{dt} - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0} \tag{7.7}$$

and Eq. 7.3 can be rewritten as

$$\frac{du}{dt} = \theta\frac{ds}{dt} + \tau_p\frac{d\nu_p}{dt} \tag{7.8}$$

and if we adopt the differential notation, we obtain **Gibbs relation**

$$\boxed{du = \theta ds + \tau_p d\nu_p} \tag{7.9}$$

29 For fluid, the Gibbs relation takes the form

$$du = \theta ds - pdv; \quad \text{and} \quad \theta \equiv \left(\frac{\partial u}{\partial s}\right)_v; \qquad -p \equiv \left(\frac{\partial u}{\partial v}\right)_s \tag{7.10}$$

where p is the thermodynamic pressure; and the thermodynamic tension conjugate to the specific volume v is $-p$, just as θ is conjugate to s.

7.1.3 Thermal Equation of State

30 From the caloric equation of state, Eq. 7.1, and the the definitions of Eq. 7.2 it follows that the temperature and the thermodynamic tensions are functions of the thermodynamic state:

$$\theta = \theta(s, \boldsymbol{\nu}); \qquad \tau_j = \tau_j(s, \boldsymbol{\nu}) \tag{7.11}$$

we assume the first one to be invertible

$$s = s(\theta, \boldsymbol{\nu}) \tag{7.12}$$

and substitute this into Eq. 7.1 to obtain an alternative form of the caloric equation of state with corresponding **thermal equations of state** (obtained by simple substitution).

$$u = u(\theta, \boldsymbol{\nu}, bX) \quad \leftarrow \tag{7.13}$$
$$\tau_i = \tau_i(\theta, \boldsymbol{\nu}, \mathbf{X}) \tag{7.14}$$
$$\nu_i = \nu_i(\theta, \boldsymbol{\theta}, \mathbf{X}) \tag{7.15}$$

31 The thermal equations of state resemble stress-strain relations, but some caution is necessary in interpreting the tesnisons as stresses and the ν_j as strains.

7.1.4 Thermodynamic Potentials

32 Based on the assumed existence of a caloric equation of state, four thermodynamic potentials are introduced, Table 7.1. Those potentials are derived through the **Legendre-**

Potential		**Relation to u**	**Independent Variables**
Internal energy	u	u	s, ν_j
Helmholtz free energy	Ψ	$\Psi = u - s\theta$	$\theta, \nu_j \quad \leftarrow$
Enthalpy	h	$h = u - \tau_j \nu_j$	s, τ_j
Free enthalpy	g	$g = u - s\theta - \tau_j \nu_j$	θ, τ_j

Table 7.1: Thermodynamic Potentials

Fenchel transformation on the basis of selected state variables best suited for a given problem.

33 By means of the preceding equations, any one of the potentials can be expressed in terms of any of the four choices of state variables listed in Table 7.1.

34 In any actual or hypothetical change obeying the equations of state, we have

$$du = \theta ds + \tau_j d\nu_j \tag{7.16-a}$$
$$d\Psi = -sd\theta + \tau_j d\nu_j \quad \leftarrow \tag{7.16-b}$$
$$dh = \theta ds - \nu_j d\tau_j \tag{7.16-c}$$
$$dg = -sd\theta - \nu_j d\tau_j \tag{7.16-d}$$

and from these differentials we obtain the following partial derivative expressions

$$\theta = \left(\frac{\partial u}{\partial s}\right)_{\boldsymbol{\nu}}; \qquad \tau_j = \left(\frac{\partial u}{\partial \nu_j}\right)_{s,\nu_{i(i\neq j)}} \tag{7.17-a}$$
$$s = -\left(\frac{\partial \Psi}{\partial \theta}\right)_{\boldsymbol{\nu}}; \qquad \tau_j = \left(\frac{\partial \Psi}{\partial \nu_j}\right)_{\theta} \quad \leftarrow \tag{7.17-b}$$
$$\theta = \left(\frac{\partial h}{\partial s}\right)_{\boldsymbol{\tau}}; \qquad \nu_j = -\left(\frac{\partial h}{\partial \tau_j}\right)_{s,\nu_{i(i\neq j)}} \tag{7.17-c}$$
$$= -\left(\frac{\partial g}{\partial \theta}\right)_{\boldsymbol{\tau}}; \qquad \nu_j = -\left(\frac{\partial g}{\partial \tau_j}\right)_{\theta} \tag{7.17-d}$$

where the **free energy** Ψ is the portion of the internal energy available for doing work at constant temperature, the **enthalpy** h (as defined here) is the portion of the internal energy that can be released as heat when the thermodynamic tensions are held constant.

7.1.5 Elastic Potential or Strain Energy Function

35 Green defined an elastic material as one for which a strain-energy function exists. Such a material is called **Green-elastic** or **hyperelastic** if there exists an **elastic potential function** W or **strain energy function**, a scalar function of one of the strain or deformation tensors, whose derivative with respect to a strain component determines the corresponding stress component.

36 For the fully recoverable case of isothermal deformation with reversible heat conduction we have

$$\tilde{T}_{IJ} = \rho_0 \left(\frac{\partial \Psi}{\partial E_{IJ}}\right)_{\theta} \tag{7.18}$$

hence $W = \rho_0 \Psi$ is an elastic potential function for this case, while $W = \rho_0 u$ is the potential for adiabatic isentropic case ($s =$ constant).

37 Hyperelasticity ignores thermal effects and assumes that the elastic potential function always exists, it is a function of the strains alone and is purely mechanical

$$\tilde{T}_{IJ} = \frac{\partial W(\mathbf{E})}{\partial E_{IJ}} \tag{7.19}$$

and $W(\mathbf{E})$ is the **strain energy per unit undeformed volume**. If the displacement gradients are small compared to unity, then we obtain

$$\boxed{T_{ij} = \frac{\partial W}{\partial E_{ij}}} \tag{7.20}$$

which is written in terms of Cauchy stress T_{ij} and small strain E_{ij}.

38 We assume that the elastic potential is represented by a power series expansion in the small-strain components.

$$W = c_0 + c_{ij}E_{ij} + \frac{1}{2}c_{ijkm}E_{ij}E_{km} + \frac{1}{3}c_{ijkmnp}E_{ij}E_{km}E_{np} + \cdots \tag{7.21}$$

where c_0 is a constant and $c_{ij}, c_{ijkm}, c_{ijkmnp}$ denote tensorial properties required to maintain the invariant property of W. Physically, the second term represents the energy due to residual stresses, the third one refers to the **strain energy** which corresponds to linear elastic deformation, and the fourth one indicates nonlinear behavior.

39 Neglecting terms higher than the second degree in the series expansion, then W is quadratic in terms of the strains

$$\begin{aligned} W = \; & c_0 + c_1E_{11} + c_2E_{22} + c_3E_{33} + 2c_4E_{23} + 2c_5E_{31} + 2c_6E_{12} \\ & +\tfrac{1}{2}c_{1111}E_{11}^2 + c_{1122}E_{11}E_{22} + c_{1133}E_{11}E_{33} + 2c_{1123}E_{11}E_{23} + 2c_{1131}E_{11}E_{31} + 2c_{1112}E_{11}E_{12} \\ & +\tfrac{1}{2}c_{2222}E_{22}^2 + c_{2233}E_{22}E_{33} + 2c_{2223}E_{22}E_{23} + 2c_{2231}E_{22}E_{31} + 2c_{2212}E_{22}E_{12} \\ & +\tfrac{1}{2}c_{3333}E_{33}^2 + 2c_{3323}E_{33}E_{23} + 2c_{3331}E_{33}E_{31} + 2c_{3312}E_{33}E_{12} \\ & +2c_{2323}E_{23}^2 + 4c_{2331}E_{23}E_{31} + 4c_{2312}E_{23}E_{12} \\ & +2c_{3131}E_{31}^2 + 4c_{3112}E_{31}E_{12} \\ & +2c_{1212}E_{12}^2 \end{aligned} \tag{7.22}$$

we require that W vanish in the unstrained state, thus $c_0 = 0$.

40 We next apply Eq. 7.20 to the quadratic expression of W and obtain for instance

$$T_{12} = \frac{\partial W}{\partial E_{12}} = 2c_6 + c_{1112}E_{11} + c_{2212}E_{22} + c_{3312}E_{33} + c_{1212}E_{12} + c_{1223}E_{23} + c_{1231}E_{31} \tag{7.23}$$

if the stress must also be zero in the unstrained state, then $c_6 = 0$, and similarly all the coefficients in the first row of the quadratic expansion of W. Thus the elastic potential function is a **homogeneous quadratic** function of the strains and we obtain **Hooke's law**

7.2 Experimental Observations

41 We shall discuss two experiments which will yield the elastic **Young's modulus**, and then the **bulk modulus**. In the former, the simplicity of the experiment is surrounded by the intriguing character of Hooke, and in the later, the bulk modulus is mathematically related to the Green deformation tensor $\mathbf{C}$, the deformation gradient $\mathbf{F}$ and the Lagrangian strain tensor $\mathbf{E}$.

7.2.1 Hooke's Law

42 Hooke's Law is determined on the basis of a very simple experiment in which a uniaxial force is applied on a specimen which has one dimension much greater than the other two (such as a rod). The elongation is measured, and then the stress is plotted in terms of the strain (elongation/length). The slope of the line is called **Young's modulus**.

43 Hooke anticipated some of the most important discoveries and inventions of his time but failed to carry many of them through to completion. He formulated the theory of planetary motion as a problem in mechanics, and grasped, but did not develop mathematically, the fundamental theory on which Newton formulated the law of gravitation.

His most important contribution was published in 1678 in the paper *De Potentia Restitutiva*. It contained results of his experiments with elastic bodies, and was the first paper in which the elastic properties of material was discussed.

> "*Take a wire string of 20, or 30, or 40 ft long, and fasten the upper part thereof to a nail, and to the other end fasten a Scale to receive the weights: Then with a pair of compasses take the distance of the bottom of the scale from the ground or floor underneath, and set down the said distance, then put inweights into the said scale and measure the several stretchings of the said string, and set them down. Then compare the several stretchings of the said string, and you will find that they will always bear the same proportions one to the other that the weights do that made them*".

This became **Hooke's Law**

$$\boxed{\sigma = E\varepsilon} \tag{7.24}$$

44 Because he was concerned about patent rights to his invention, he did not publish his law when first discovered it in 1660. Instead he published it in the form of an anagram "*ceiinosssttuu*" in 1676 and the solution was given in 1678. *Ut tensio sic vis* (at the time the two symbols u and v were employed interchangeably to denote either the vowel u or the consonant v), i.e. *extension varies directly with force.*

7.2.2 Bulk Modulus

45 If, instead of subjecting a material to a uniaxial state of stress, we now subject it to a hydrostatic pressure p and measure the change in volume ΔV.

46 From the summary of Table 4.1 we know that:

$$V = (\det \mathbf{F})V_0 \tag{7.25-a}$$

$$\det \mathbf{F} = \sqrt{\det \mathbf{C}} = \sqrt{\det[\mathbf{I} + 2\mathbf{E}]} \tag{7.25-b}$$

therefore,

$$\frac{V + \Delta V}{V} = \sqrt{\det[\mathbf{I} + 2\mathbf{E}]} \tag{7.26}$$

we can expand the determinant of the tensor det$[\mathbf{I}+2\mathbf{E}]$ to find

$$\det[\mathbf{I}+2\mathbf{E}] = 1 + 2I_E + 4II_E + 8III_E \tag{7.27}$$

but for small strains, $I_E \gg II_E \gg III_E$ since the first term is linear in $\mathbf{E}$, the second is quadratic, and the third is cubic. Therefore, we can approximate det$[\mathbf{I}+2\mathbf{E}] \approx 1+2I_E$, hence we define the **volumetric dilatation** as

$$\boxed{\frac{\Delta V}{V} \equiv e \approx I_E = \text{tr } \mathbf{E}} \tag{7.28}$$

this quantity is readily measurable in an experiment.

7.3 Stress-Strain Relations in Generalized Elasticity

7.3.1 Anisotropic

47 From Eq. 7.22 and 7.23 we obtain the stress-strain relation for homogeneous anisotropic material

$$\boxed{\underbrace{\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{Bmatrix}}_{T_{ij}} = \underbrace{\begin{bmatrix} c_{1111} & c_{1112} & c_{1133} & c_{1112} & c_{1123} & c_{1131} \\ & c_{2222} & c_{2233} & c_{2212} & c_{2223} & c_{2231} \\ & & c_{3333} & c_{3312} & c_{3323} & c_{3331} \\ & & & c_{1212} & c_{1223} & c_{1231} \\ & \text{SYM.} & & & c_{2323} & c_{2331} \\ & & & & & c_{3131} \end{bmatrix}}_{c_{ijkm}} \underbrace{\begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12}(\gamma_{12}) \\ 2E_{23}(\gamma_{23}) \\ 2E_{31}(\gamma_{31}) \end{Bmatrix}}_{E_{km}}} \tag{7.29}$$

which is **Hooke's law** for small strain in linear elasticity.

48 We also observe that for symmetric c_{ij} we retrieve **Clapeyron formula**

$$\boxed{W = \frac{1}{2}T_{ij}E_{ij}} \tag{7.30}$$

49 In general the elastic moduli c_{ij} relating the cartesian components of stress and strain depend on the orientation of the coordinate system with respect to the body. If the form of elastic potential function W and the values c_{ij} are independent of the orientation, the material is said to be **isotropic**, if not it is **anisotropic**.

50 c_{ijkm} is a fourth order tensor resulting with $3^4 = 81$ terms.

$$\left[\begin{array}{ccc}\left(\begin{array}{ccc}c_{1,1,1,1} & c_{1,1,1,2} & c_{1,1,1,3}\\ c_{1,1,2,1} & c_{1,1,2,2} & c_{1,1,2,3}\\ c_{1,1,3,1} & c_{1,1,3,2} & c_{1,1,3,3}\\ c_{2,1,1,1} & c_{2,1,1,2} & c_{2,1,1,3}\\ c_{2,1,2,1} & c_{2,1,2,2} & c_{2,1,2,3}\\ c_{2,1,3,1} & c_{2,1,3,2} & c_{2,1,3,3}\\ c_{3,1,1,1} & c_{3,1,1,2} & c_{3,1,1,3}\\ c_{3,1,2,1} & c_{3,1,2,2} & c_{3,1,2,3}\\ c_{3,1,3,1} & c_{3,1,3,2} & c_{3,1,3,3}\end{array}\right) & \left(\begin{array}{ccc}c_{1,2,1,1} & c_{1,2,1,2} & c_{1,2,1,3}\\ c_{1,2,2,1} & c_{1,2,2,2} & c_{1,2,2,3}\\ c_{1,2,3,1} & c_{1,2,3,2} & c_{1,2,3,3}\\ c_{2,2,1,1} & c_{2,2,1,2} & c_{2,2,1,3}\\ c_{2,2,2,1} & c_{2,2,2,2} & c_{2,2,2,3}\\ c_{2,2,3,1} & c_{2,2,3,2} & c_{2,2,3,3}\\ c_{3,2,1,1} & c_{3,2,1,2} & c_{3,2,1,3}\\ c_{3,2,2,1} & c_{3,2,2,2} & c_{3,2,2,3}\\ c_{3,2,3,1} & c_{3,2,3,2} & c_{3,2,3,3}\end{array}\right) & \left(\begin{array}{ccc}c_{1,3,1,1} & c_{1,3,1,2} & c_{1,3,1,3}\\ c_{1,3,2,1} & c_{1,3,2,2} & c_{1,3,2,3}\\ c_{1,3,3,1} & c_{1,3,3,2} & c_{1,3,3,3}\\ c_{2,3,1,1} & c_{2,3,1,2} & c_{2,3,1,3}\\ c_{2,3,2,1} & c_{2,3,2,2} & c_{2,3,2,3}\\ c_{2,3,3,1} & c_{2,3,3,2} & c_{2,3,3,3}\\ c_{3,3,1,1} & c_{3,3,1,2} & c_{3,3,1,3}\\ c_{3,3,2,1} & c_{3,3,2,2} & c_{3,3,2,3}\\ c_{3,3,3,1} & c_{3,3,3,2} & c_{3,3,3,3}\end{array}\right)\end{array}\right] \tag{7.31}$$

But the matrix must be symmetric thanks to Cauchy's second law of motion (i.e symmetry of both the stress and the strain), and thus for **anisotropic** material we will have a symmetric 6 by 6 matrix with $\frac{(6)(6+1)}{2} = 21$ independent coefficients.

51 By means of coordinate transformation we can relate the material properties in one coordinate system (old) x_i, to a new one $\overline{x}_i$, thus from Eq. 1.27 ($\overline{v}_j = a_j^p v_p$) we can rewrite

$$W = \frac{1}{2}c_{rstu}E_{rs}E_{tu} = \frac{1}{2}c_{rstu}a_i^r a_j^s a_k^t a_m^u \overline{E}_{ij}\overline{E}_{km} = \frac{1}{2}c_{ijkm}\overline{E}_{ij}\overline{E}_{km} \tag{7.32}$$

thus we deduce

$$c_{ijkm} = a_i^r a_j^s a_k^t a_m^u c_{rstu} \tag{7.33}$$

that is the fourth order tensor of material constants in old coordinates may be transformed into a new coordinate system through an eighth-order tensor $a_i^r a_j^s a_k^t a_m^u$

7.3.2 Monotropic Material

52 A **plane of elastic symmetry** exists at a point where the elastic constants have the same values for every pair of coordinate systems which are the reflected images of one another with respect to the plane. The axes of such coordinate systems are referred to as "equivalent elastic directions".

53 If we assume $\overline{x}_1 = x_1$, $\overline{x}_2 = x_2$ and $\overline{x}_3 = -x_3$, then the transformation $\overline{x}_i = a_i^j x_j$ is defined through

$$a_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{7.34}$$

where the negative sign reflects the symmetry of the mirror image with respect to the x_3 plane.

54 We next substitute in Eq.7.33, and as an example we consider $c_{1123} = a_1^r a_1^s a_2^t a_3^u c_{rstu} = a_1^1 a_1^1 a_2^2 a_3^3 c_{1123} = (1)(1)(1)(-1)c_{1123} = -c_{1123}$, obviously, this is not possible, and the only way the relation can remanin valid is if $c_{1123} = 0$. We note that all terms in c_{ijkl} with the index 3 occurring an odd number of times will be equal to zero. Upon substitution,

we obtain

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & 0 & 0 \\ & c_{2222} & c_{2233} & c_{2212} & 0 & 0 \\ & & c_{3333} & c_{3312} & 0 & 0 \\ & & & c_{1212} & 0 & 0 \\ & \text{SYM.} & & & c_{2323} & c_{2331} \\ & & & & & c_{3131} \end{bmatrix} \tag{7.35}$$

we now have 13 nonzero coefficients.

7.3.3 Orthotropic Material

55 If the material possesses three mutually perpendicular planes of elastic symmetry, (that is symmetric with respect to two planes x_2 and x_3), then the transformation $x_i = a_i^j x_j$ is defined through

$$a_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{7.36}$$

where the negative sign reflects the symmetry of the mirror image with respect to the x_3 plane. Upon substitution in Eq.7.33 we now would have

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{1212} & 0 & 0 \\ & \text{SYM.} & & & c_{2323} & 0 \\ & & & & & c_{3131} \end{bmatrix} \tag{7.37}$$

We note that in here all terms of c_{ijkl} with the indices 3 and 2 occuring an odd number of times are again set to zero.

56 Wood is usually considered an orthotropic material and will have 9 nonzero coefficients.

7.3.4 Transversely Isotropic Material

57 A material is transversely isotropic if there is a preferential direction normal to all but one of the three axes. If this axis is x_3, then rotation about it will require that

$$a_i^j = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.38}$$

substituting Eq. 7.33 into Eq. 7.41, using the above transformation matrix, we obtain

$$c_{1111} = (\cos^4\theta)c_{1111} + (\cos^2\theta\sin^2\theta)(2c_{1122} + 4c_{1212}) + (\sin^4\theta)c_{2222} \tag{7.39-a}$$

$$c_{1122} = (\cos^2\theta\sin^2\theta)c_{1111} + (\cos^4\theta)c_{1122} - 4(\cos^2\theta\sin^2\theta)c_{1212} + (\sin^4\theta)c_{2211} \tag{7.39-b}$$

$$+(\sin^2\theta\cos^2\theta)c_{2222} \tag{7.39-c}$$

$$c_{1133} = (\cos^2\theta)c_{1133} + (\sin^2\theta)c_{2233} \tag{7.39-d}$$

$$c_{2222} = (\sin^4\theta)c_{1111} + (\cos^2\theta\sin^2\theta)(2c_{1122} + 4c_{1212}) + (\cos^4\theta)c_{2222} \quad (7.39\text{-e})$$
$$c_{1212} = (\cos^2\theta\sin^2\theta)c_{1111} - 2(\cos^2\theta\sin^2\theta)c_{1122} - 2(\cos^2\theta\sin^2\theta)c_{1212} + (\cos^4\theta)c_{1212} \quad (7.39\text{-f})$$
$$+(\sin^2\theta\cos^2\theta)c_{2222} + \sin^4\theta c_{1212} \quad (7.39\text{-g})$$
$$\vdots$$

But in order to respect our initial assumption about symmetry, these results require that

$$c_{1111} = c_{2222} \quad (7.40\text{-a})$$
$$c_{1133} = c_{2233} \quad (7.40\text{-b})$$
$$c_{2323} = c_{3131} \quad (7.40\text{-c})$$
$$c_{1212} = \frac{1}{2}(c_{1111} - c_{1122}) \quad (7.40\text{-d})$$

yielding

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & \frac{1}{2}(c_{1111} - c_{1122}) & 0 & 0 \\ & \text{SYM.} & & & c_{2323} & 0 \\ & & & & & c_{3131} \end{bmatrix} \quad (7.41)$$

we now have 5 nonzero coefficients.

58 It should be noted that very few natural or man-made materials are truly orthotropic (certain crystals as topaz are), but a number are transversely isotropic (laminates, shist, quartz, roller compacted concrete, etc...).

7.3.5 Isotropic Material

59 An isotropic material is symmetric with respect to every plane and every axis, that is the elastic properties are identical in all directions.

60 To mathematically characterize an isotropic material, we require coordinate transformation with rotation about x_2 and x_1 axes in addition to all previous coordinate transformations. This process will enforce symmetry about all planes and all axes.

61 The rotation about the x_2 axis is obtained through

$$a_i^j = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad (7.42)$$

we follow a similar procedure to the case of transversely isotropic material to obtain

$$c_{1111} = c_{3333} \quad (7.43\text{-a})$$
$$c_{3131} = \frac{1}{2}(c_{1111} - c_{1133}) \quad (7.43\text{-b})$$

62 next we perform a rotation about the x_1 axis

$$a_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{7.44}$$

it follows that

$$c_{1122} = c_{1133} \tag{7.45-a}$$
$$c_{3131} = \frac{1}{2}(c_{3333} - c_{1133}) \tag{7.45-b}$$
$$c_{2323} = \frac{1}{2}(c_{2222} - c_{2233}) \tag{7.45-c}$$

which will finally give

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & a & 0 & 0 \\ & \text{SYM.} & & & b & 0 \\ & & & & & c \end{bmatrix} \tag{7.46}$$

with $a = \frac{1}{2}(c_{1111} - c_{1122})$, $b = \frac{1}{2}(c_{2222} - c_{2233})$, and $c = \frac{1}{2}(c_{3333} - c_{1133})$.

63 If we denote $c_{1122} = c_{1133} = c_{2233} = \lambda$ and $c_{1212} = c_{2323} = c_{3131} = \mu$ then from the previous relations we determine that $c_{1111} = c_{2222} = c_{3333} = \lambda + 2\mu$, or

$$c_{ijkm} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{SYM.} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \tag{7.47}$$
$$= \lambda\delta_{ij}\delta_{km} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{kj}) \tag{7.48}$$

and we are thus left with only two independent non zero coefficients λ and μ which are called **Lame's constants**.

64 Substituting the last equation into Eq. 7.29,

$$T_{ij} = [\lambda\delta_{ij}\delta_{km} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{kj})]E_{km} \tag{7.49}$$

Or in terms of λ and μ, **Hooke's Law** for an isotropic body is written as

$$T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij} \quad \text{or} \quad \mathbf{T} = \lambda\mathbf{I}_E + 2\mu\mathbf{E} \tag{7.50}$$
$$E_{ij} = \frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}T_{kk}\right) \quad \text{or} \quad \mathbf{E} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{I}_T + \frac{1}{2\mu}\mathbf{T} \tag{7.51}$$

65 It should be emphasized that Eq. 7.47 is written in terms of the **Engineering strains** (Eq. 7.29) that is $\gamma_{ij} = 2E_{ij}$ for $i \neq j$. On the other hand the preceding equations are written in terms of the **tensorial strains** E_{ij}

7.3.5.1 Engineering Constants

66 The stress-strain relations were expressed in terms of Lame's parameters which can not be readily measured experimentally. As such, in the following sections we will reformulate those relations in terms of "engineering constants" (Young's and the bulk's modulus). This will be done for both the isotropic and transversely isotropic cases.

7.3.5.1.1 Isotropic Case

7.3.5.1.1.1 Young's Modulus

67 In order to avoid certain confusion between the strain **E** and the elastic constant E, we adopt the usual engineering notation $T_{ij} \rightarrow \sigma_{ij}$ and $E_{ij} \rightarrow \varepsilon_{ij}$

68 If we consider a simple uniaxial state of stress in the x_1 direction, then from Eq. 7.51

$$\varepsilon_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma \qquad (7.52\text{-a})$$

$$\varepsilon_{22} = \varepsilon_{33} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)}\sigma \qquad (7.52\text{-b})$$

$$0 = \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} \qquad (7.52\text{-c})$$

69 Yet we have the elementary relations in terms engineering constants E **Young's modulus** and ν **Poisson's ratio**

$$\varepsilon_{11} = \frac{\sigma}{E} \qquad (7.53\text{-a})$$

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} \qquad (7.53\text{-b})$$

then it follows that

$$\boxed{\begin{aligned} \frac{1}{E} &= \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}; \nu = \frac{\lambda}{2(\lambda + \mu)} && (7.54) \\ \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)}; \mu = G = \frac{E}{2(1+\nu)} && (7.55) \end{aligned}}$$

70 Similarly in the case of pure shear in the x_1x_3 and x_2x_3 planes, we have

$$\sigma_{21} = \sigma_{12} = \tau \quad \text{all other} \quad \sigma_{ij} = 0 \qquad (7.56\text{-a})$$

$$2\varepsilon_{12} = \frac{\tau}{G} \qquad (7.56\text{-b})$$

and the μ is equal to the **shear modulus** G.

71 Hooke's law for isotropic material in terms of engineering constants becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right) \text{ or } \boldsymbol{\sigma} = \frac{E}{1+\nu}\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\mathbf{I}_\varepsilon\right) \quad (7.57)$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \text{ or } \boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathbf{I}_\sigma \quad (7.58)$$

72 When the strain equation is expanded in 3D cartesian coordinates it would yield:

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}(2\varepsilon_{xy})\\ \gamma_{yz}(2\varepsilon_{yz})\\ \gamma_{zx}(2\varepsilon_{zx})\end{array}\right\} = \frac{1}{E}\left[\begin{array}{cccccc}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{array}\right]\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{array}\right\} \quad (7.59)$$

73 If we invert this equation, we obtain

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{array}\right\} = \left[\begin{array}{cc}\frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc}1-\nu & \nu & \nu\\ \nu & 1-\nu & \nu\\ \nu & \nu & 1-\nu\end{array}\right] & 0\\ 0 & G\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\end{array}\right]\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}(2\varepsilon_{xy})\\ \gamma_{yz}(2\varepsilon_{yz})\\ \gamma_{zx}(2\varepsilon_{zx})\end{array}\right\} \quad (7.60)$$

7.3.5.1.1.2 Bulk's Modulus; Volumetric and Deviatoric Strains

74 We can express the trace of the stress I_σ in terms of the **volumetric** strain I_ε From Eq. 7.50

$$\sigma_{ii} = \lambda\delta_{ii}\varepsilon_{kk} + 2\mu\varepsilon_{ii} = (3\lambda + 2\mu)\varepsilon_{ii} \equiv 3K\varepsilon_{ii} \quad (7.61)$$

or

$$K = \lambda + \frac{2}{3}\mu \quad (7.62)$$

75 We can provide a complement to the volumetric part of the constitutive equations by substracting the trace of the stress from the stress tensor, hence we define the **deviatoric** stress and strains as as

$$\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} - \frac{1}{3}(\text{tr }\boldsymbol{\sigma})\mathbf{I} \quad (7.63)$$

$$\boldsymbol{\varepsilon}' \equiv \boldsymbol{\varepsilon} - \frac{1}{3}(\text{tr }\boldsymbol{\varepsilon})\mathbf{I} \quad (7.64)$$

and the corresponding constitutive relation will be

$$\boldsymbol{\sigma} = Ke\mathbf{I} + 2\mu\boldsymbol{\varepsilon}' \quad (7.65)$$

$$\boldsymbol{\varepsilon} = \frac{p}{3K}\mathbf{I} + \frac{1}{2\mu}\boldsymbol{\sigma}' \quad (7.66)$$

where $p \equiv \frac{1}{3}\text{tr }(\boldsymbol{\sigma})$ is the pressure, and $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - p\mathbf{I}$ is the stress deviator.

7.3.5.1.1.3 Restriction Imposed on the Isotropic Elastic Moduli

76 We can rewrite Eq. 7.20 as

$$dW = T_{ij}dE_{ij} \tag{7.67}$$

but since dW is a scalar invariant (energy), it can be expressed in terms of volumetric (hydrostatic) and deviatoric components as

$$dW = -pde + \sigma'_{ij}dE'_{ij} \tag{7.68}$$

substituting $p = -Ke$ and $\sigma'_{ij} = 2GE'_{ij}$, and integrating, we obtain the following expression for the **isotropic strain energy**

$$W = \frac{1}{2}Ke^2 + GE'_{ij}E'_{ij} \tag{7.69}$$

and since positive work is required to cause any deformation $W > 0$ thus

$$\lambda + \frac{2}{3}G \equiv K > 0 \tag{7.70-a}$$

$$G > 0 \tag{7.70-b}$$

ruling out $K = G = 0$, we are left with

$$\boxed{E > 0; \quad -1 < \nu < \frac{1}{2}} \tag{7.71}$$

77 The isotropic strain energy function can be alternatively expressed as

$$W = \frac{1}{2}\lambda e^2 + GE_{ij}E_{ij} \tag{7.72}$$

78 From Table 7.2, we observe that $\nu = \frac{1}{2}$ implies $G = \frac{E}{3}$, and $\frac{1}{K} = 0$ or elastic **incompressibility**.

	λ, μ	E, ν	μ, ν	E, μ	K, ν
λ	λ	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{2\mu\nu}{1-2\nu}$	$\frac{\mu(E-2\mu)}{3\mu-E}$	$\frac{3K\nu}{1+\nu}$
μ	μ	$\frac{E}{2(1+\nu)}$	μ	μ	$\frac{3K(1-2\nu)}{2(1+\nu)}$
K	$\lambda + \frac{2}{3}\mu$	$\frac{E}{3(1-2\nu)}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{\mu E}{3(3\mu-E)}$	K
E	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	E	$2\mu(1+\nu)$	E	$3K(1-2\nu)$
ν	$\frac{\lambda}{2(\lambda+\mu)}$	ν	ν	$\frac{E}{2\mu} - 1$	ν

Table 7.2: Conversion of Constants for an Isotropic Elastic Material

79 The elastic properties of selected materials is shown in Table 7.3.

Material	E (MPa)	ν
A316 Stainless Steel	196,000	0.3
A5 Aluminum	68,000	0.33
Bronze	61,000	0.34
Plexiglass	2,900	0.4
Rubber	2	$\rightarrow$0.5
Concrete	60,000	0.2
Granite	60,000	0.27

Table 7.3: Elastic Properties of Selected Materials at 20^{0c}

7.3.5.1.2 Transversly Isotropic Case

80 For transversely isotropic, we can express the stress-strain relation in tems of

$$\begin{aligned}
\varepsilon_{xx} &= a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{13}\sigma_{zz} \\
\varepsilon_{yy} &= a_{12}\sigma_{xx} + a_{11}\sigma_{yy} + a_{13}\sigma_{zz} \\
\varepsilon_{zz} &= a_{13}(\sigma_{xx} + \sigma_{yy}) + a_{33}\sigma_{zz} \\
\gamma_{xy} &= 2(a_{11} - a_{12})\tau_{xy} \\
\gamma_{yz} &= a_{44}\tau_{xy} \\
\gamma_{xz} &= a_{44}\tau_{xz}
\end{aligned} \tag{7.73}$$

and

$$a_{11} = \frac{1}{E}; \qquad a_{12} = -\frac{\nu}{E}; \qquad a_{13} = -\frac{\nu'}{E'}; \qquad a_{33} = -\frac{1}{E'}; \qquad a_{44} = -\frac{1}{\mu'} \tag{7.74}$$

where E is the Young's modulus in the plane of isotropy and E' the one in the plane normal to it. ν corresponds to the transverse contraction in the plane of isotropy when tension is applied in the plane; ν' corresponding to the transverse contraction in the plane of isotropy when tension is applied normal to the plane; μ' corresponding to the shear moduli for the plane of isotropy and any plane normal to it, and μ is shear moduli for the plane of isotropy.

7.3.5.2 Special 2D Cases

81 Often times one can make simplifying assumptions to reduce a 3D problem into a 2D one.

7.3.5.2.1 Plane Strain

82 For problems involving a long body in the z direction with no variation in load or geometry, then $\varepsilon_{zz} = \gamma_{yz} = \gamma_{xz} = \tau_{xz} = \tau_{yz} = 0$. Thus, replacing into Eq. 5.2 we obtain

$$\left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ \nu & \nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{array}\right] \left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{array}\right\} \tag{7.75}$$

7.3.5.2.2 Axisymmetry

83 In solids of revolution, we can use a polar coordinate sytem and

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \quad (7.76\text{-a})$$
$$\varepsilon_{\theta\theta} = \frac{u}{r} \quad (7.76\text{-b})$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \quad (7.76\text{-c})$$
$$\varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \quad (7.76\text{-d})$$

84 The constitutive relation is again analogous to 3D/plane strain

$$\left\{\begin{array}{c}\sigma_{rr}\\ \sigma_{zz}\\ \sigma_{\theta\theta}\\ \tau_{rz}\end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{cccc}1-\nu & \nu & \nu & 0\\ \nu & 1-\nu & \nu & 0\\ \nu & \nu & 1-\nu & 0\\ \nu & \nu & 1-\nu & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \gamma_{rz}\end{array}\right\} \quad (7.77)$$

7.3.5.2.3 Plane Stress

85 If the longitudinal dimension in z direction is much smaller than in the x and y directions, then $\tau_{yz} = \tau_{xz} = \sigma_{zz} = \gamma_{xz} = \gamma_{yz} = 0$ throughout the thickness. Again, substituting into Eq. 5.2 we obtain:

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{array}\right\} = \frac{1}{1-\nu^2}\left[\begin{array}{ccc}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} \quad (7.78\text{-a})$$
$$\varepsilon_{zz} = -\frac{1}{1-\nu}\nu(\varepsilon_{xx}+\varepsilon_{yy}) \quad (7.78\text{-b})$$

7.4 Linear Thermoelasticity

86 If thermal effects are accounted for, the components of the linear strain tensor E_{ij} may be considered as the sum of

$$E_{ij} = E_{ij}^{(T)} + E_{ij}^{(\Theta)} \quad (7.79)$$

where $E_{ij}^{(T)}$ is the contribution from the stress field, and $E_{ij}^{(\Theta)}$ the contribution from the temperature field.

87 When a body is subjected to a temperature change $\Theta-\Theta_0$ with respect to the reference state temperature, the strain componenet of an elementary volume of an unconstrained isotropic body are given by

$$E_{ij}^{(\Theta)} = \alpha(\Theta-\Theta_0)\delta_{ij} \quad (7.80)$$

where α is the **linear coefficient of thermal expansion**.

88 Inserting the preceding two equation into Hooke's law (Eq. 7.51) yields

$$\boxed{E_{ij} = \frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}T_{kk}\right) + \alpha(\Theta-\Theta_0)\delta_{ij}} \tag{7.81}$$

which is known as **Duhamel-Neumann** relations.

89 If we invert this equation, we obtain the **thermoelastic constitutive equation**:

$$\boxed{T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij} - (3\lambda+2\mu)\alpha\delta_{ij}(\Theta-\Theta_0)} \tag{7.82}$$

90 Alternatively, if we were to consider the derivation of the Green-elastic hyperelastic equations, (Sect. 7.1.5), we required the constants c_1 to c_6 in Eq. 7.22 to be zero in order that the stress vanish in the unstrained state. If we accounted for the temperature change $\Theta-\Theta_0$ with respect to the reference state temperature, we would have $c_k = -\beta_k(\Theta-\Theta_0)$ for $k = 1$ to 6 and would have to add like terms to Eq. 7.22, leading to

$$T_{ij} = -\beta_{ij}(\Theta-\Theta_0) + c_{ijrs}E_{rs} \tag{7.83}$$

for linear theory, we suppose that β_{ij} is independent from the strain and c_{ijrs} independent of temperature change with respect to the natural state. Finally, for isotropic cases we obtain

$$T_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} - \beta_{ij}(\Theta-\Theta_0)\delta_{ij} \tag{7.84}$$

which is identical to Eq. 7.82 with $\beta = \frac{E\alpha}{1-2\nu}$. Hence

$$\boxed{T^{\Theta}_{ij} = \frac{E\alpha}{1-2\nu}} \tag{7.85}$$

91 In terms of deviatoric stresses and strains we have

$$\boxed{T'_{ij} = 2\mu E'_{ij} \quad \text{and} \quad E'_{ij} = \frac{T'_{ij}}{2\mu}} \tag{7.86}$$

and in terms of volumetric stress/strain:

$$\boxed{p = -Ke + \beta(\Theta-\Theta_0) \quad \text{and} \quad e = \frac{p}{K} + 3\alpha(\Theta-\Theta_0)} \tag{7.87}$$

7.5 Fourrier Law

92 Consider a solid through which there is a *flow* **q** of heat (or some other quantity such as mass, chemical, etc...)

93 The rate of transfer per unit area is **q**

94 The direction of flow is in the direction of maximum "potential" (temperature in this case, but could be, piezometric head, or ion concentration) decreases (Fourrier, Darcy, Fick...).

$$\mathbf{q} = \left\{ \begin{array}{c} q_x \\ q_y \\ q_z \end{array} \right\} = -\mathbf{D} \left\{ \begin{array}{c} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{array} \right\} = -\mathbf{D}\boldsymbol{\nabla}\phi \tag{7.88}$$

D is a three by three (symmetric) **constitutive/conductivity** matrix
The conductivity can be either

Isotropic

$$\mathbf{D} = k \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \tag{7.89}$$

Anisotropic

$$\mathbf{D} = \left[\begin{array}{ccc} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{array} \right] \tag{7.90}$$

Orthotropic

$$\mathbf{D} = \left[\begin{array}{ccc} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{zz} \end{array} \right] \tag{7.91}$$

Note that for flow through porous media, Darcy's equation is only valid for laminar flow.

7.6 Updated Balance of Equations and Unknowns

95 In light of the new equations introduced in this chapter, it would be appropriate to revisit our balance of equations and unknowns.

		Coupled	Uncoupled
$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$	Continuity Equation	1	1
$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\frac{dv_i}{dt}$	Equation of motion	3	3
$\rho\frac{du}{dt} = T_{ij}D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j}$	Energy equation	1	
$\mathbf{T} = \lambda\mathbf{I}_E + 2\mu\mathbf{E}$	Hooke's Law	6	6
$\mathbf{q} = -\mathbf{D}\boldsymbol{\nabla}\phi$	Heat Equation (Fourrier)	3	
$\Theta = \Theta(s, \boldsymbol{\nu})$; $\tau_j = \tau_j(s, \boldsymbol{\nu})$	Equations of state	2	
Total number of equations		**16**	**10**

and we repeat our list of unknowns

		Coupled	Uncoupled
Density	ρ	1	1
Velocity (or displacement)	v_i (u_i)	3	3
Stress components	T_{ij}	6	6
Heat flux components	q_i	3	-
Specific internal energy	u	1	-
Entropy density	s	1	-
Absolute temperature	Θ	1	-
Total number of unknowns		**16**	**10**

and in addition the Clausius-Duhem inequality $\frac{ds}{dt} \geq \frac{r}{\Theta} - \frac{1}{\rho}\text{div}\,\frac{\mathbf{q}}{\Theta}$ which governs entropy production must hold.

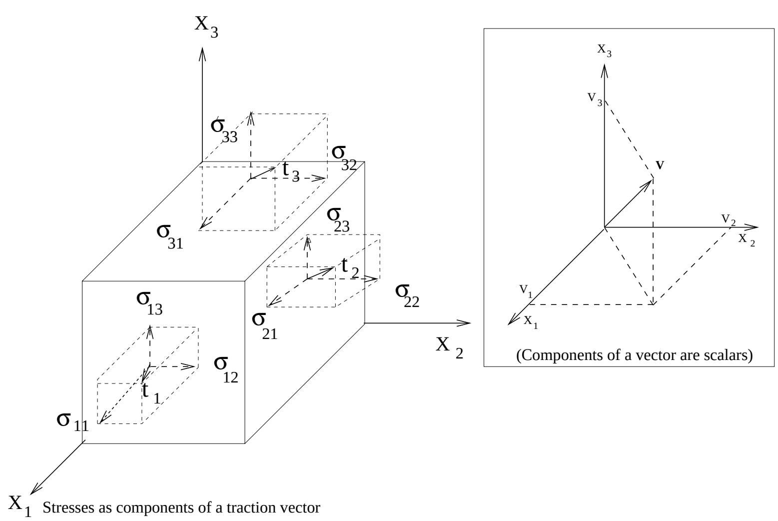
96 Hence we now have as many equations as unknowns and are (almost) ready to pose and solve problems in continuum mechanics.

Chapter 8

INTERMEZZO

In light of the lengthy and rigorous derivation of the fundamental equations of Continuum Mechanics in the preceding chapter, the reader may be at a loss as to what are the most important ones to remember.

Hence, since the complexity of some of the derivation may have eclipsed the final results, this handout seeks to summarize the most fundamental relations which you should always remember.



Stresses as components of a traction vector

(Components of a tensor of order 2 are vectors)

Stress Vector/Tensor $t_i = T_{ij} n_j$ (8.1-a)

Strain Tensor

$$E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}\right) \tag{8.1-b}$$

$$= \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \varepsilon_{33} \end{bmatrix} \tag{8.1-c}$$

Engineering Strain

$$\gamma_{23} \approx \sin\gamma_{23} = \sin(\pi/2 - \theta) = \cos\theta = 2E_{23} \tag{8.1-d}$$

Equilibrium

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\frac{dv_i}{dt} \tag{8.1-e}$$

Boundary Conditions $\Gamma = \Gamma_u + \Gamma_t$ (8.1-f)

Energy Potential $T_{ij} = \dfrac{\partial W}{\partial E_{ij}}$ (8.1-g)

Hooke's Law $T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}$ (8.1-h)

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{array}\right\} = \left[\begin{array}{cccccc}\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G}\end{array}\right] \left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{array}\right\} \quad (8.1\text{-i})$$

Plane Stress $\sigma_{zz} = 0; \quad \varepsilon_{zz} \neq 0$ (8.1-j)

Plane Strain $\varepsilon_{zz} = 0; \quad \sigma_{zz} \neq 0$ (8.1-k)

Part II

ELASTICITY/SOLID MECHANICS

Chapter 9

BOUNDARY VALUE PROBLEMS in ELASTICITY

9.1 Preliminary Considerations

20 All problems in elasticity require three basic components:

3 Equations of Motion (Equilibrium): i.e. Equations relating the applied tractions and body forces to the stresses (3)

$$\frac{\partial T_{ij}}{\partial X_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{9.1}$$

6 Stress-Strain relations: (Hooke's Law)

$$\mathbf{T} = \lambda \mathbf{I}_E + 2\mu \mathbf{E} \tag{9.2}$$

6 Geometric (kinematic) equations: i.e. Equations of geometry of deformation relating displacement to strain (6)

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}) \tag{9.3}$$

21 Those 15 equations are written in terms of 15 unknowns: 3 displacement u_i, 6 stress components T_{ij}, and 6 strain components E_{ij}.

22 In addition to these equations which describe what is happening inside the body, we must describe what is happening on the surface or boundary of the body. These extra conditions are called **boundary conditions**.

9.2 Boundary Conditions

23 In describing the boundary conditions (B.C.), we must note that:

1. Either we know the displacement but not the traction, or we know the traction and not the corresponding displacement. We can never know both *a priori*.

2. Not all boundary conditions specifications are acceptable. For example we can not apply tractions to the entire surface of the body. Unless those tractions are specially prescribed, they may not necessarily satisfy equilibrium.

24 Properly specified boundary conditions result in **well-posed** boundary value problems, while improperly specified boundary conditions will result in **ill-posed** boundary value problem. Only the former can be solved.

25 Thus we have two types of boundary conditions in terms of *known* quantitites, Fig. 9.1:

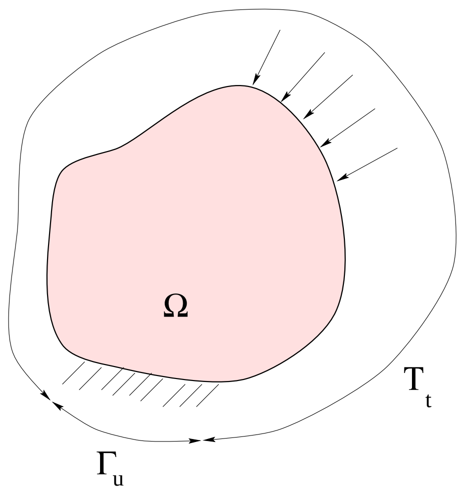


Figure 9.1: Boundary Conditions in Elasticity Problems

Displacement boundary conditions along Γ_u with the three components of u_i prescribed on the boundary. The displacement is decomposed into its cartesian (or curvilinear) components, i.e. u_x, u_y

Traction boundary conditions along Γ_t with the three traction components $t_i = n_j T_{ij}$ prescribed at a boundary where the unit normal is $\mathbf{n}$. The traction is decomposed into its normal and shear(s) components, i.e t_n, t_s.

Mixed boundary conditions where displacement boundary conditions are prescribed on a part of the bounding surface, while traction boundary conditions are prescribed on the remainder.

We note that at some points, traction may be specified in one direction, and displacement at another. Displacement and tractions can never be specified at the same point in the same direction.

26 Various terms have been associated with those boundary conditions in the litterature, those are suumarized in Table 9.1.

27 Often time we take advantage of symmetry not only to simplify the problem, but also to properly define the appropriate boundary conditions, Fig. 9.2.

$\mathbf{u}$, Γ_u	$\mathbf{t}$, Γ_t
Dirichlet	Neuman
Field Variable	Derivative(s) of Field Variable
Essential	Non-essential
Forced	**Natural**
Geometric	Static

Table 9.1: Boundary Conditions in Elasticity

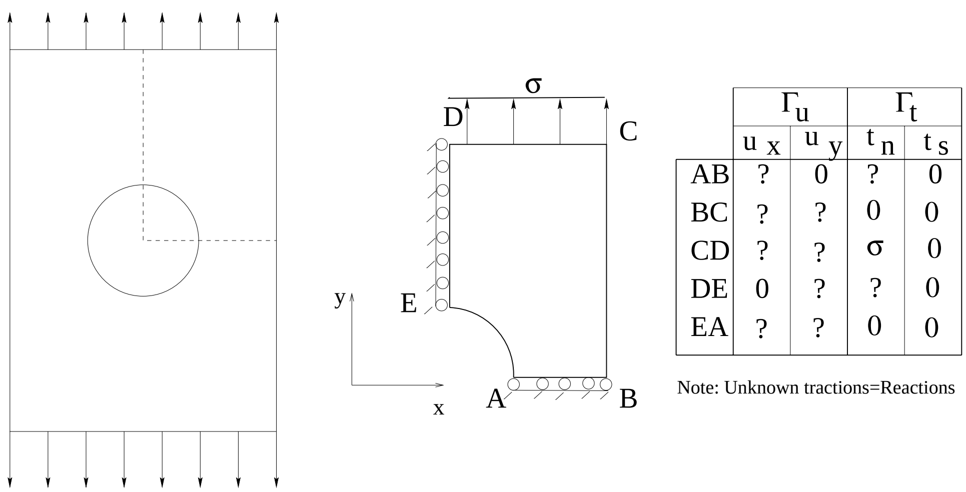


Figure 9.2: Boundary Conditions in Elasticity Problems

9.3 Boundary Value Problem Formulation

28 Hence, the boundary value formulation is suumarized by

$$\frac{\partial T_{ij}}{\partial X_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad \text{in} \quad \Omega \tag{9.4}$$

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} + \boldsymbol{\nabla}_\mathbf{x}\mathbf{u}) \tag{9.5}$$

$$\mathbf{T} = \lambda \mathbf{I}_E + 2\mu\mathbf{E} \quad \text{in} \quad \Omega \tag{9.6}$$

$$\mathbf{u} = \overline{\mathbf{u}} \quad \text{in} \quad \Gamma_u \tag{9.7}$$

$$\mathbf{t} = \overline{\mathbf{t}} \quad \text{in} \quad \Gamma_t \tag{9.8}$$

and is illustrated by Fig. 9.3. This is now a **well posed problem**.

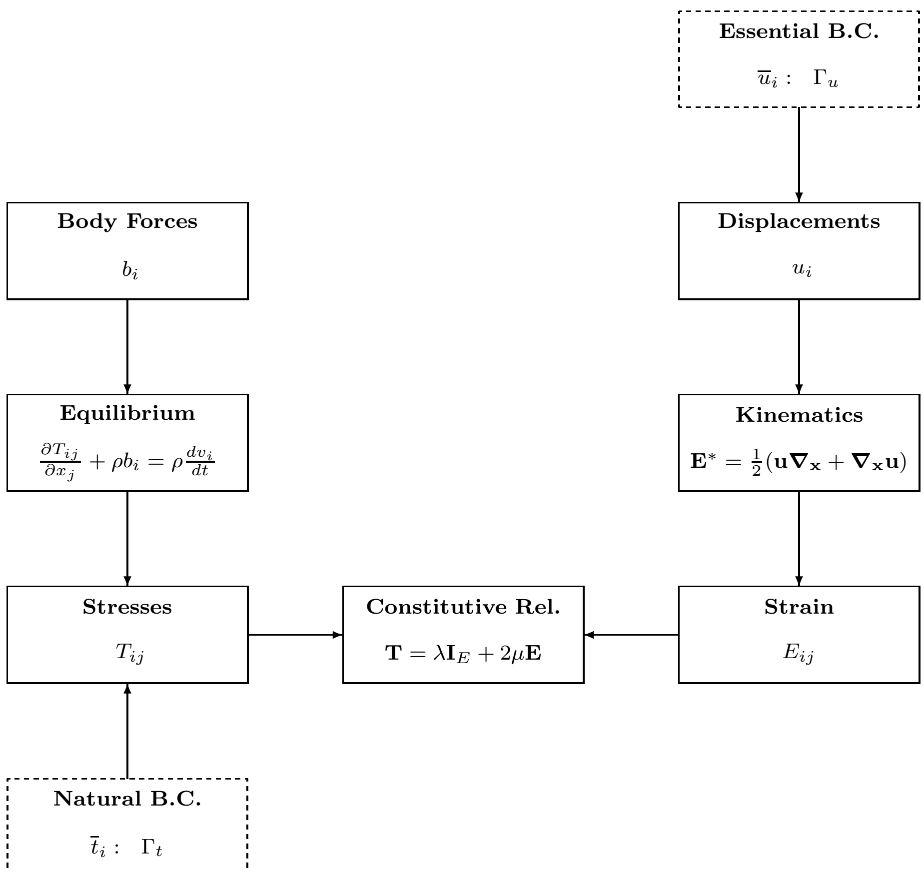


Figure 9.3: Fundamental Equations in Solid Mechanics

9.4 Compacted Forms

29 Solving a boundary value problem with 15 unknowns through 15 equations is a formidable task. Hence, there are numerous methods to reformulate the problem in terms of fewer

unknows.

9.4.1 Navier-Cauchy Equations

30 One such approach is to substitute the displacement-strain relation into Hooke's law (resulting in stresses in terms of the gradient of the displacement), and the resulting equation into the equation of motion to obtain three second-order partial differential equations for the three displacement components known as **Navier's Equation**

$$(\lambda+\mu)\frac{\partial^2 u_k}{\partial X_i \partial X_k} + \mu\frac{\partial^2 u_i}{\partial X_k \partial X_k} + \rho b_i = \rho\frac{\partial^2 u_i}{\partial t^2} \quad (9.9)$$

or

$$(\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mu\boldsymbol{\nabla}^2\mathbf{u} + \rho\mathbf{b} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} \quad (9.10)$$

$$(9.11)$$

9.4.2 Beltrami-Mitchell Equations

31 Whereas Navier-Cauchy equation was expressed in terms of the gradient of the displacement, we can follow a similar approach and write a single equation in term of the gradient of the tractions.

$$\boldsymbol{\nabla}^2 T_{ij} + \frac{1}{1+\nu}T_{pp,ij} = -\frac{\nu}{1-\nu}\delta_{ij}\boldsymbol{\nabla}\cdot(\rho\mathbf{b}) - \rho(b_{i,j}+b_{j,i}) \quad (9.12)$$

or

$$T_{ij,pp} + \frac{1}{1+\nu}T_{pp,ij} = -\frac{\nu}{1-\nu}\delta_{ij}\rho b_{p,p} - \rho(b_{i,j}+b_{j,i}) \quad (9.13)$$

9.4.3 Ellipticity of Elasticity Problems

9.5 Strain Energy and Extenal Work

32 For the isotropic Hooke's law, we saw that there always exist a strain energy function W which is positive-definite, homogeneous quadratic function of the strains such that, Eq. 7.20

$$T_{ij} = \frac{\partial W}{\partial E_{ij}} \quad (9.14)$$

hence it follows that

$$W = \frac{1}{2}T_{ij}E_{ij} \quad (9.15)$$

33 The external work done by a body in equilibrium under body forces b_i and surface traction t_i is equal to $\int_\Omega \rho b_i u_i d\Omega + \int_\Gamma t_i u_i d\Gamma$. Substituting $t_i = T_{ij}n_j$ and applying Gauss theorem, the second term becomes

$$\int_\Gamma T_{ij}n_j u_i d\Gamma = \int_\Omega (T_{ij}u_i)_{,j} d\Omega = \int_\Omega (T_{ij,j}u_i + T_{ij}u_{i,j})d\Omega \quad (9.16)$$

but $T_{ij}u_{i,j} = T_{ij}(E_{ij} + \Omega_{ij}) = T_{ij}E_{ij}$ and from equilibrium $T_{ij,j} = -\rho b_i$, thus

$$\int_\Omega \rho b_i u_i d\Omega + \int_\Gamma t_i u_i d\Gamma = \int_\Omega \rho b_i u_i d\Omega + \int_\Omega (T_{ij}E_{ij} - \rho b_i u_i) d\Omega \tag{9.17}$$

or

$$\boxed{\underbrace{\int_\Omega \rho b_i u_i d\Omega + \int_\Gamma t_i u_i d\Gamma}_{\text{External Work}} = 2 \underbrace{\int_\Omega \frac{T_{ij}E_{ij}}{2} d\Omega}_{\text{Internal Strain Energy}}} \tag{9.18}$$

that is *For an elastic system, the total strain energy is one half the work done by the external forces acting through their displacements* u_i.

9.6 Uniqueness of the Elastostatic Stress and Strain Field

34 Because the equations of linear elasticity are linear equations, the principles of superposition may be used to obtain additional solutions from those established. Hence, given two sets of solution $T_{ij}^{(1)}$, $u_i^{(1)}$, and $T_{ij}^{(2)}$, $u_i^{(2)}$, then $T_{ij} = T_{ij}^{(2)} - T_{ij}^{(1)}$, and $u_i = u_i^{(2)} - u_i^{(1)}$ with $b_i = b_i^{(2)} - b_i^{(1)} = 0$ must also be a solution.

35 Hence for this "difference" solution, Eq. 9.18 would yield $\int_\Gamma t_i u_i d\Gamma = 2\int_\Omega u^* d\Omega$ but the left hand side is zero because $t_i = t_i^{(2)} - t_i^{(1)} = 0$ on Γ_u, and $u_i = u_i^{(2)} - u_i^{(1)} = 0$ on Γ_t, thus $\int_\Omega u^* d\Omega = 0$.

36 But u^* is positive-definite and continuous, thus the integral can vanish if and only if $u^* = 0$ everywhere, and this is only possible if $E_{ij} = 0$ everywhere so that

$$\boxed{E_{ij}^{(2)} = E_{ij}^{(1)} \Rightarrow T_{ij}^{(2)} = Tij^{(1)}} \tag{9.19}$$

hence, there can not be two different stress and strain fields corresponding to the same externally imposed body forces and boundary conditions[1] and satisfying the linearized elastostatic Eqs 9.1, 9.14 and 9.3.

9.7 Saint Venant's Principle

37 This famous **principle** of Saint Venant was enunciated in 1855 and is of great importance in applied elasticity where it is often invoked to justify certain "simplified" solutions to complex problem.

> In elastostatics, if the boundary tractions on a part Γ_1 of the boundary Γ are replaced by a statically equivalent traction distribution, the effects on the stress distribution in the body are negligible at points whose distance from Γ_1 is large compared to the maximum distance between points of Γ_1.

[1] This theorem is attributed to Kirchoff (1858).

38 For instance the analysis of the problem in Fig. 9.4 can be greatly simplified if the tractions on Γ_1 are replaced by a concentrated statically equivalent force.

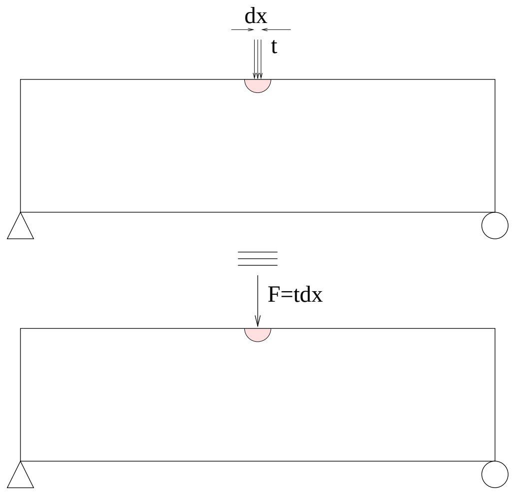


Figure 9.4: St-Venant's Principle

9.8 Cylindrical Coordinates

39 So far all equations have been written in either vector, indicial, or engineering notation. The last two were so far restricted to an othonormal cartesian coordinate system.

40 We now rewrite some of the fundamental relations in **cylindrical** coordinate system, Fig. 9.5, as this would enable us to analytically solve some simple problems of great practical usefulness (torsion, pressurized cylinders, ...). This is most often achieved by reducing the dimensionality of the problem from 3 to 2 or even to 1.

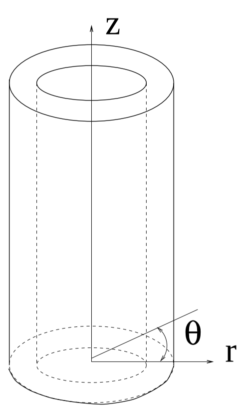


Figure 9.5: Cylindrical Coordinates

9.8.1 Strains

41 With reference to Fig. 9.6, we consider the displacement of point P to P^*. the

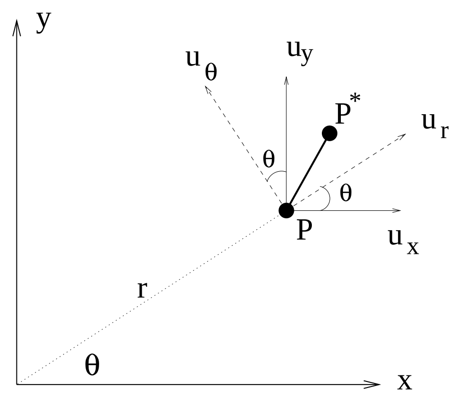


Figure 9.6: Polar Strains

displacements can be expressed in cartesian coordinates as u_x, u_y, or in polar coordinates as u_r, u_θ. Hence,

$$u_x = u_r \cos\theta - u_\theta \sin\theta \quad (9.20\text{-a})$$
$$u_y = u_r \sin\theta + u_\theta \cos\theta \quad (9.20\text{-b})$$

substituting into the strain definition for ε_{xx} (for small displacements) we obtain

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u_x}{\partial r}\frac{\partial r}{\partial x} \quad (9.21\text{-a})$$
$$\frac{\partial u_x}{\partial \theta} = \frac{\partial u_r}{\partial \theta}\cos\theta - u_r\sin\theta - \frac{\partial u_\theta}{\partial \theta}\sin\theta - u_\theta\cos\theta \quad (9.21\text{-b})$$
$$\frac{\partial u_x}{\partial r} = \frac{\partial u_r}{\partial r}\cos\theta - \frac{\partial u_\theta}{\partial r}\sin\theta \quad (9.21\text{-c})$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \quad (9.21\text{-d})$$
$$\frac{\partial r}{\partial x} = \cos\theta \quad (9.21\text{-e})$$
$$\varepsilon_{xx} = \left(-\frac{\partial u_r}{\partial \theta}\cos\theta + u_r\sin\theta + \frac{\partial u_\theta}{\partial \theta}\sin\theta + u_\theta\cos\theta\right)\frac{\sin\theta}{r} + \left(\frac{\partial u_r}{\partial r}\cos\theta - \frac{\partial u_\theta}{\partial r}\sin\theta\right)\cos\theta \quad (9.21\text{-f})$$

Noting that as $\theta \to 0$, $\varepsilon_{xx} \to \varepsilon_{rr}$, $\sin\theta \to 0$, and $\cos\theta \to 1$, we obtain

$$\varepsilon_{rr} = \varepsilon_{xx}|_{\theta\to 0} = \frac{\partial u_r}{\partial r} \quad (9.22)$$

42 Similarly, if $\theta \to \pi/2$, $\varepsilon_{xx} \to \varepsilon_{\theta\theta}$, $\sin\theta \to 1$, and $\cos\theta \to 0$. Hence,

$$\varepsilon_{\theta\theta} = \varepsilon_{xx}|_{\theta\to\pi/2} = \frac{1}{r}\partial u_\theta \partial\theta + \frac{u_r}{r} \quad (9.23)$$

finally, we may express ε_{xy} as a function of u_r, u_θ and θ and noting that $\varepsilon_{xy} \rightarrow \varepsilon_{r\theta}$ as $\theta \rightarrow 0$, we obtain

$$\varepsilon_{r\theta} = \frac{1}{2}\left[\varepsilon_{xy}|_{\theta\rightarrow 0} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \tag{9.24}$$

43 In summary, and with the addition of the z components (not explicitely derived), we obtain

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{9.25}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \tag{9.26}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{9.27}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_theta}{r}\right] \tag{9.28}$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right] \tag{9.29}$$

$$\varepsilon_{rz} = \frac{1}{2}\left[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right] \tag{9.30}$$

9.8.2 Equilibrium

44 Whereas the equilibrium equation as given In Eq. 6.24 was obtained from the linear momentum principle (without any reference to the notion of equilibrium of forces), its derivation (as mentioned) could have been obtained by equilibrium of forces considerations. This is the approach which we will follow for the polar coordinate system with respect to Fig. 9.7.

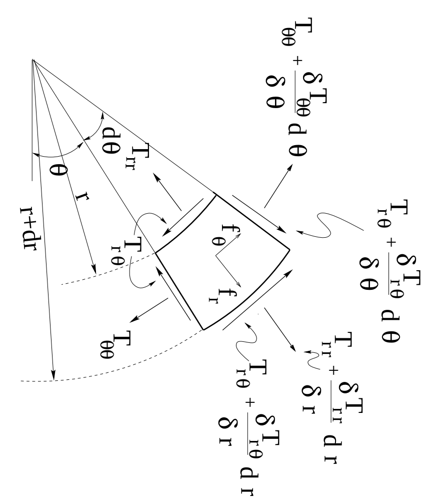


Figure 9.7: Stresses in Polar Coordinates

45 Summation of forces parallel to the radial direction through the center of the element with unit thickness in the z direction yields:

$$\left(T_{rr}+\frac{\partial T_{rr}}{\partial r}dr\right)(r+dr)d\theta - T_{rr}(rd\theta) \quad (9.31\text{-a})$$

$$-\left(T_{\theta\theta}+\frac{\partial T_{\theta\theta}}{\partial\theta}+T_{\theta\theta}\right)dr\sin\frac{d\theta}{2}$$

$$+\left(T_{\theta r}+\frac{\partial T_{\theta r}}{\partial\theta}d\theta - T_{\theta r}\right)dr\cos\frac{d\theta}{2}+f_r r dr d\theta = 0 \quad (9.31\text{-b})$$

we approximate $\sin(d\theta/2)$ by $d\theta/2$ and $\cos(d\theta/2)$ by unity, divide through by $rdrd\theta$,

$$\frac{1}{r}T_{rr}+\frac{\partial T_{rr}}{\partial r}\left(1+\frac{dr}{r}\right)-\frac{T_{\theta\theta}}{r}-\frac{\partial T_{\theta\theta}}{\partial\theta}\frac{d\theta}{dr}+\frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta}+f_r=0 \quad (9.32)$$

46 Similarly we can take the summation of forces in the θ direction. In both cases if we were to drop the dr/r and $d\theta/r$ in the limit, we obtain

$$\boxed{\begin{aligned}\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta}+\frac{1}{r}(T_{rr}-T_{\theta\theta})+f_r &= 0 \quad (9.33)\\ \frac{\partial T_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{1}{r}(T_{r\theta}-T_{\theta r})+f_\theta &= 0 \quad (9.34)\end{aligned}}$$

47 It is often necessary to express cartesian stresses in terms of polar stresses and vice versa. This can be done through the following relationships

$$\begin{bmatrix}T_{xx} & T_{xy}\\ T_{xy} & T_{yy}\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}T_{rr} & T_{r\theta}\\ T_{r\theta} & T_{\theta\theta}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^T \quad (9.35)$$

yielding

$$\begin{aligned}T_{xx} &= T_{rr}\cos^2\theta+T_{\theta\theta}\sin^2\theta-T_{r\theta}\sin 2\theta \quad (9.36\text{-a})\\ T_{yy} &= T_{rr}\sin^2\theta+T_{\theta\theta}\cos^2\theta+T_{r\theta}\sin 2\theta \quad (9.36\text{-b})\\ T_{xy} &= (T_{rr}-T_{\theta\theta})\sin\theta\cos\theta+T_{r\theta}(\cos^2\theta-\sin^2\theta) \quad (9.36\text{-c})\end{aligned}$$

(recalling that $\sin^2\theta = 1/2\sin 2\theta$, and $\cos^2\theta = 1/2(1+\cos 2\theta)$).

9.8.3 Stress-Strain Relations

48 In orthogonal curvilinear coordinates, the physical components of a tensor at a point are merely the Cartesian components in a local coordinate system at the point with its axes tangent to the coordinate curves. Hence,

$$\boxed{\begin{aligned}T_{rr} &= \lambda e+2\mu\varepsilon_{rr} \quad (9.37)\\ T_{\theta\theta} &= \lambda e+2\mu\varepsilon_{\theta\theta} \quad (9.38)\\ T_{r\theta} &= 2\mu\varepsilon_{r\theta} \quad (9.39)\\ T_{zz} &= \nu(T_{rr}+T_{\theta\theta}) \quad (9.40)\end{aligned}}$$

with $e = \varepsilon_{rr} + \varepsilon_{\theta\theta}$. alternatively,

$$E_{rr} = \frac{1}{E}\left[(1-\nu^2)T_{rr} - \nu(1+\nu)T_{\theta\theta}\right] \quad (9.41)$$

$$E_{\theta\theta} = \frac{1}{E}\left[(1-\nu^2)T_{\theta\theta} - \nu(1+\nu)T_{rr}\right] \quad (9.42)$$

$$E_{r\theta} = \frac{1+\nu}{E}T_{r\theta} \quad (9.43)$$

$$E_{rz} = E_{\theta z} = E_{zz} = 0 \quad (9.44)$$

9.8.3.1 Plane Strain

49 For Plane strain problems, from Eq. 7.75:

$$\left\{\begin{array}{c}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{r\theta}\end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc}(1-\nu) & \nu & 0\\ \nu & (1-\nu) & 0\\ \nu & \nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \gamma_{r\theta}\end{array}\right\} \quad (9.45)$$

and $\varepsilon_{zz} = \gamma_{rz} = \gamma_{\theta z} = \tau_{rz} = \tau_{\theta z} = 0$.

50 Inverting,

$$\left\{\begin{array}{c}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \gamma_{r\theta}\end{array}\right\} = \frac{1}{E}\left[\begin{array}{ccc}1-\nu^2 & -\nu(1+\nu) & 0\\ -\nu(1+\nu) & 1-\nu^2 & 0\\ \nu & \nu & 0\\ 0 & 0 & 2(1+\nu\end{array}\right]\left\{\begin{array}{c}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{r\theta}\end{array}\right\} \quad (9.46)$$

9.8.3.2 Plane Stress

51 For plane stress problems, from Eq. 7.78-a

$$\left\{\begin{array}{c}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \tau_{r\theta}\end{array}\right\} = \frac{E}{1-\nu^2}\left[\begin{array}{ccc}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \gamma_{r\theta}\end{array}\right\} \quad (9.47\text{-a})$$

$$\varepsilon_{zz} = -\frac{1}{1-\nu}\nu(\varepsilon_{rr} + \varepsilon_{\theta\theta}) \quad (9.47\text{-b})$$

and $\tau_{rz} = \tau_{\theta z} = \sigma_{zz} = \gamma_{rz} = \gamma_{\theta z} = 0$

52 Inverting

$$\left\{\begin{array}{c}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \gamma_{r\theta}\end{array}\right\} = \frac{1}{E}\left[\begin{array}{ccc}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{array}\right]\left\{\begin{array}{c}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \tau_{r\theta}\end{array}\right\} \quad (9.48\text{-a})$$

Chapter 10

SOME ELASTICITY PROBLEMS

20 Practical solutions of two-dimensional boundary-value problem in simply connected regions can be accomplished by numerous techniques. Those include: a) Finite-difference approximation of the differential equation, b) Complex function method of Muskhelisvili (most useful in problems with stress concentration), c) Variational methods (which will be covered in subsequent chapters), d) Semi-inverse methods, and e) Airy stress functions.

21 Only the last two methods will be discussed in this chapter.

10.1 Semi-Inverse Method

22 Often a solution to an elasticity problem may be obtained without seeking simulateneous solutions to the equations of motion, Hooke's Law and boundary conditions. One may attempt to seek solutions by making certain assumptions or guesses about the components of strain stress or displacement while leaving enough freedom in these assumptions so that the equations of elasticity be satisfied.

23 If the assumptions allow us to satisfy the elasticity equations, then by the uniqueness theorem, we have succeeded in obtaining the solution to the problem.

24 This method was employed by Saint-Venant in his treatment of the torsion problem, hence it is often referred to as the **Saint-Venant semi-inverse method**.

10.1.1 Example: Torsion of a Circular Cylinder

25 Let us consider the elastic deformation of a cylindrical bar with circular cross section of radius a and length L twisted by equal and opposite end moments M_1, Fig. 10.1.

26 From symmetry, it is reasonable to assume that the motion of each cross-sectional plane is a rigid body rotation about the x_1 axis. Hence, for a small rotation angle θ, the displacement field will be given by:

$$\mathbf{u} = (\theta\mathbf{e}_1)\times\mathbf{r} = (\theta\mathbf{e}_1)\times(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = \theta(x_2\mathbf{e}_3 - x_3\mathbf{e}_2) \tag{10.1}$$

or

$$u_1 = 0; \qquad u_2 = -\theta x_3; \qquad u_3 = \theta x_2 \tag{10.2}$$

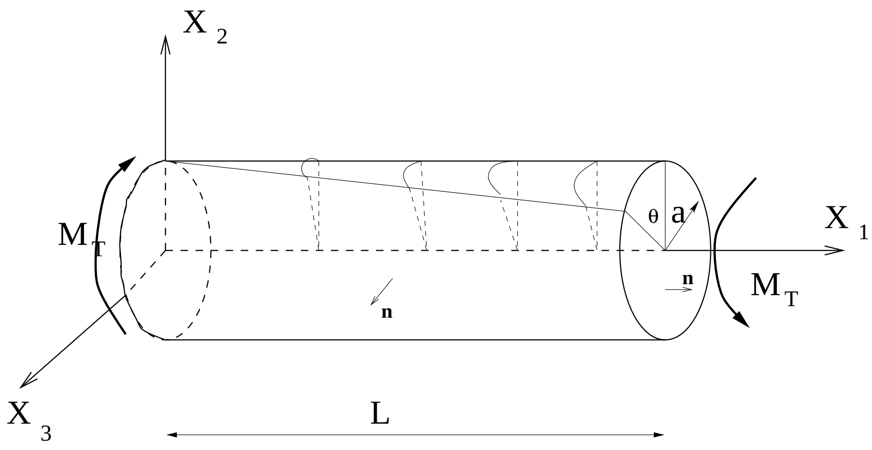


Figure 10.1: Torsion of a Circular Bar

where $\theta = \theta(x_1)$.

27 The corresponding strains are given by

$$E_{11} = E_{22} = E_{33} = 0 \quad (10.3\text{-a})$$

$$E_{12} = -\frac{1}{2} x_3 \frac{\partial \theta}{\partial x_1} \quad (10.3\text{-b})$$

$$E_{13} = \frac{1}{2} x_2 \frac{\partial \theta}{\partial x_1} \quad (10.3\text{-c})$$

28 The non zero stress components are obtained from Hooke's law

$$T_{12} = -\mu x_3 \frac{\partial \theta}{\partial x_1} \quad (10.4\text{-a})$$

$$T_{13} = \mu x_2 \frac{\partial \theta}{\partial x_1} \quad (10.4\text{-b})$$

29 We need to check that this state of stress satisfies equilibrium $\partial T_{ij}/\partial x_j = 0$. The first one $j = 1$ is identically satisfied, whereas the other two yield

$$-\mu x_3 \frac{d^2 \theta}{dx_1^2} = 0 \quad (10.5\text{-a})$$

$$\mu x_2 \frac{d^2 \theta}{dx_1^2} = 0 \quad (10.5\text{-b})$$

thus,

$$\frac{d\theta}{dx_1} \equiv \theta' = \text{constant} \quad (10.6)$$

Physically, this means that equilibrium is only satisfied if the increment in angular rotation (twist per unit length) is a constant.

30 We next determine the corresponding surface tractions. On the lateral surface we have a unit normal vector $\mathbf{n} = \frac{1}{a}(x_2\mathbf{e}_2 + x_3\mathbf{e}_3)$, therefore the surface traction on the lateral surface is given by

$$\{\mathbf{t}\} = [\mathbf{T}]\{\mathbf{n}\} = \frac{1}{a}\begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & 0 \\ T_{31} & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{1}{a}\begin{Bmatrix} x_2T_{12} \\ 0 \\ 0 \end{Bmatrix} \tag{10.7}$$

31 Substituting,

$$\mathbf{t} = \frac{\mu}{a}(-x_2x_3\theta' + x_2x_3\theta')\mathbf{e}_1 = \mathbf{0} \tag{10.8}$$

which is in agreement with the fact that the bar is twisted by end moments only, the lateral surface is traction free.

32 On the face $x_1 = L$, we have a unit normal $\mathbf{n} = \mathbf{e}_1$ and a surface traction

$$\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \tag{10.9}$$

this distribution of surface traction on the end face gives rise to the following resultants

$$R_1 = \int T_{11}dA = 0 \tag{10.10-a}$$

$$R_2 = \int T_{21}dA = \mu\theta'\int x_3dA = 0 \tag{10.10-b}$$

$$R_3 = \int T_{31}dA = \mu\theta'\int x_2dA = 0 \tag{10.10-c}$$

$$M_1 = \int(x_2T_{31} - x_3T_{21})dA = \mu\theta'\int(x_2^2 + x_3^2)dA = \mu\theta' J \tag{10.10-d}$$

$$M_2 = M_3 = 0 \tag{10.10-e}$$

We note that $\int(x_2^2 + x_3^3)^2dA$ is the **polar moment of inertia** of the cross section and is equal to $J = \pi a^4/2$, and we also note that $\int x_2dA = \int x_3dA = 0$ because the area is symmetric with respect to the axes.

33 From the last equation we note that

$$\theta' = \frac{M}{\mu J} \tag{10.11}$$

which implies that the shear modulus μ can be determined froma simple torsion experiment.

34 Finally, in terms of the twisting couple M, the stress tensor becomes

$$[\mathbf{T}] = \begin{bmatrix} 0 & -\frac{Mx_3}{J} & \frac{Mx_2}{J} \\ -\frac{Mx_3}{J} & 0 & 0 \\ \frac{Mx_2}{J} & 0 & 0 \end{bmatrix} \tag{10.12}$$

10.2 Airy Stress Functions

10.2.1 Cartesian Coordinates; Plane Strain

35 If the deformation of a cylindrical body is such that there is no axial components of the displacement and that the other components do not depend on the axial coordinate,

then the body is said to be in a state of plane strain. If $\mathbf{e}_3$ is the direction corresponding to the cylindrical axis, then we have

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0 \tag{10.13}$$

and the strain components corresponding to those displacements are

$$E_{11} = \frac{\partial u_1}{\partial x_1} \tag{10.14-a}$$
$$E_{22} = \frac{\partial u_2}{\partial x_2} \tag{10.14-b}$$
$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{10.14-c}$$
$$E_{13} = E_{23} = E_{33} = 0 \tag{10.14-d}$$

and the non-zero stress components are $T_{11}, T_{12}, T_{22}, T_{33}$ where

$$T_{33} = \nu(T_{11} + T_{22}) \tag{10.15}$$

36 Considering a static stress field with no body forces, the equilibrium equations reduce to:

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = 0 \tag{10.16-a}$$
$$\frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} = 0 \tag{10.16-b}$$
$$\frac{\partial T_{33}}{\partial x_1} = 0 \tag{10.16-c}$$

we note that since $T_{33} = T_{33}(x_1, x_2)$, the last equation is always satisfied.

37 Hence, it can be easily verified that for any arbitrary scalar variable Φ, if we compute the stress components from

$$T_{11} = \frac{\partial^2 \Phi}{\partial x_2^2} \tag{10.17}$$
$$T_{22} = \frac{\partial^2 \Phi}{\partial x_1^2} \tag{10.18}$$
$$T_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \tag{10.19}$$

then the first two equations of equilibrium are automatically satisfied. This function Φ is called **Airy stress function**.

38 However, if stress components determined this way are **statically admissible** (i.e. they satisfy equilibrium), they are not necessarily **kinematically admissible** (i.e. satisfy compatibility equations).

39 To ensure compatibility of the strain components, we obtain the strains components in terms of Φ from Hooke's law, Eq. 5.1 and Eq. 10.15.

$$E_{11} = \frac{1}{E}\left[(1-\nu^2)T_{11} - \nu(1+\nu)T_{22}\right] = \frac{1}{E}\left[(1-\nu^2)\frac{\partial^2\Phi}{\partial x_2^2} - \nu(1+\nu)\frac{\partial^2\Phi}{\partial x_1^2}\right] \quad (10.20\text{-a})$$

$$E_{22} = \frac{1}{E}\left[(1-\nu^2)T_{22} - \nu(1+\nu)T_{11}\right] = \frac{1}{E}\left[(1-\nu^2)\frac{\partial^2\Phi}{\partial x_1^2} - \nu(1+\nu)\frac{\partial^2\Phi}{\partial x_2^2}\right] \quad (10.20\text{-b})$$

$$E_{12} = \frac{1}{E}(1+\nu)T_{12} = -\frac{1}{E}(1+\nu)\frac{\partial^2\Phi}{\partial x_1 \partial x_2} \quad (10.20\text{-c})$$

40 For plane strain problems, the only compatibility equation, 4.159, that is not automatically satisfied is

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} \quad (10.21)$$

thus we obtain the following equation governing the scalar function Φ

$$(1-\nu)\left(\frac{\partial^4\Phi}{\partial x_1^4} + 2\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\Phi}{\partial x_1^4}\right) = 0 \quad (10.22)$$

or

$$\boxed{\frac{\partial^4\Phi}{\partial x_1^4} + 2\frac{\partial^4\Phi}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\Phi}{\partial x_1^4} = 0 \;\text{ or }\; \nabla^4\Phi = 0} \quad (10.23)$$

Hence, any function which satisfies the preceding equation will satisfy **both** equilibrium and kinematic and is thus an acceptable elasticity solution.

41 We can also obtain from the Hooke's law, the compatibility equation 10.21, and the equilibrium equations the following

$$\boxed{\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(T_{11}+T_{22}) = 0 \;\text{ or }\; \nabla^2(T_{11}+T_{22}) = 0} \quad (10.24)$$

42 Any polynomial of degree three or less in x and y satisfies the biharmonic equation (Eq. 10.23). A systematic way of selecting coefficients begins with

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}x^m y^n \quad (10.25)$$

43 The stresses will be given by

$$T_{xx} = \sum_{m=0}^{\infty}\sum_{n=2}^{\infty} n(n-1)C_{mn}x^m y^{n-2} \quad (10.26\text{-a})$$

$$T_{yy} = \sum_{m=2}^{\infty}\sum_{n=0}^{\infty} m(m-1)C_{mn}x^{m-1} y^{n} \quad (10.26\text{-b})$$

$$T_{xy} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mnC_{mn}x^{m-1} y^{n-1} \quad (10.26\text{-c})$$

44 Substituting into Eq. 10.23 and regrouping we obtain

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}[(m+2)(m+1)m(m-1)C_{m+2,n-2}+2m(m-1)n(n-1)C_{mn}+(n+2)(n+1)n(n-1)C_{m-2,n+2}]x^{m-2}y^{n-2} = \tag{10.27}$$

but since the equation must be identically satisfied for all x and y, the term in bracket must be equal to zero.

$$(m+2)(m+1)m(m-1)C_{m+2,n-2}+2m(m-1)n(n-1)C_{mn}+(n+2)(n+1)n(n-1)C_{m-2,n+2} = 0 \tag{10.28}$$

Hence, the recursion relation establishes relationships among groups of three alternate coefficients which can be selected from

$$\begin{bmatrix} 0 & 0 & C_{02} & C_{03} & \boxed{C_{04}} & C_{05} & C_{06} & \cdots \\ 0 & C_{11} & C_{12} & C_{13} & C_{14} & \underline{C_{15}} & \cdots & \\ C_{20} & C_{21} & \boxed{C_{22}} & C_{23} & C_{24} & \cdots & & \\ C_{30} & C_{31} & C_{32} & \underline{C_{33}} & \cdots & & & \\ \boxed{C_{40}} & C_{41} & C_{42} & \cdots & & & & \\ C50 & \underline{C_{51}} & \cdots & & & & & \\ C_{60} & & & & & & & \cdots \end{bmatrix} \tag{10.29}$$

For example if we consider $m = n = 2$, then

$$(4)(3)(2)(1)C_{40} + (2)(2)(1)(2)(1)C_{22} + (4)(3)(2)(1)C_{04} = 0 \tag{10.30}$$

or $3C_{40} + C_{22} + 3C_{04} = 0$

10.2.1.1 Example: Cantilever Beam

45 We consider the homogeneous fourth-degree polynomial

$$\Phi_4 = C_{40}x^4 + C_{31}x^3y + C_{22}x^2y^2 + C_{13}xy^3 + C_{04}y^4 \tag{10.31}$$

with $3C_{40} + C_{22} + 3C_{04} = 0$,

46 The stresses are obtained from Eq. 10.26-a-10.26-c

$$T_{xx} = 2C_{22}x^2 + 6C_{13}xy + 12C_{04}y^2 \tag{10.32-a}$$
$$T_{yy} = 12C_{40}x^2 + 6C_{31}xy + 2C_{22}y^2 \tag{10.32-b}$$
$$T_{xy} = -3C_{31}x^2 - 4C_{22}xy - 3C_{13}y^2 \tag{10.32-c}$$

These can be used for the end-loaded cantilever beam with width b along the z axis, depth $2a$ and length L.

47 If all coefficients except C_{13} are taken to be zero, then

$$T_{xx} = 6C_{13}xy \tag{10.33-a}$$
$$T_{yy} = 0 \tag{10.33-b}$$
$$T_{xy} = -3C_{13}y^2 \tag{10.33-c}$$

48 This will give a parabolic shear traction on the loaded end (correct), but also a uniform shear traction $T_{xy} = -3C_{13}a^2$ on top and bottom. These can be removed by superposing uniform shear stress $T_{xy} = +3C_{13}a^2$ corresponding to $\Phi_2 = -3C_{13}a^2xy$. Thus

$$T_{xy} = 3C_{13}(a^2 - y^2) \tag{10.34}$$

note that $C_{20} = C_{02} = 0$, and $C_{11} = -3C_{13}a^2$.

49 The constant C_{13} is determined by requiring that

$$P = b\int_{-a}^{a} -T_{xy}dy = -3bC_{13}\int_{-a}^{a}(a^2 - y^2)dy \tag{10.35}$$

hence

$$C_{13} = -\frac{P}{4a^3b} \tag{10.36}$$

and the solution is

$$\begin{aligned}
\Phi &= \frac{3P}{4ab}xy - \frac{P}{4a^3b}xy^3 && \text{(10.37-a)}\\
T_{xx} &= -\frac{3P}{2a^3b}xy && \text{(10.37-b)}\\
T_{xy} &= -\frac{3P}{4a^3b}(a^2 - y^2) && \text{(10.37-c)}\\
T_{yy} &= 0 && \text{(10.37-d)}
\end{aligned}$$

50 We observe that the second moment of area for the rectangular cross section is $I = b(2a)^3/12 = 2a^3b/3$, hence this solution agrees with the elementary beam theory solution

$$\begin{aligned}
\Phi &= C_{11}xy + C_{13}xy^3 = \frac{3P}{4ab}xy - \frac{P}{4a^3b}xy^3 && \text{(10.38-a)}\\
T_{xx} &= -\frac{P}{I}xy = -M\frac{y}{I} = -\frac{M}{S} && \text{(10.38-b)}\\
T_{xy} &= -\frac{P}{2I}(a^2 - y^2) && \text{(10.38-c)}\\
T_{yy} &= 0 && \text{(10.38-d)}
\end{aligned}$$

10.2.2 Polar Coordinates

10.2.2.1 Plane Strain Formulation

51 In polar coordinates, the strain components in plane strain are, Eq. 9.46

$$\begin{aligned}
E_{rr} &= \frac{1}{E}\left[(1 - \nu^2)T_{rr} - \nu(1 + \nu)T_{\theta\theta}\right] && \text{(10.39-a)}\\
E_{\theta\theta} &= \frac{1}{E}\left[(1 - \nu^2)T_{\theta\theta} - \nu(1 + \nu)T_{rr}\right] && \text{(10.39-b)}
\end{aligned}$$

$$E_{r\theta} = \frac{1+\nu}{E}T_{r\theta} \quad (10.39\text{-c})$$
$$E_{rz} = E_{\theta z} = E_{zz} = 0 \quad (10.39\text{-d})$$

and the equations of equilibrium are

$$\frac{1}{r}\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} - \frac{T_{\theta\theta}}{r} = 0 \quad (10.40\text{-a})$$
$$\frac{1}{r^2}\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \quad (10.40\text{-b})$$

52 Again, it can be easily verified that the equations of equilibrium are identically satisfied if

$$T_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \quad (10.41)$$
$$T_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} \quad (10.42)$$
$$T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right) \quad (10.43)$$

53 In order to satisfy the compatibility conditions, the cartesian stress components must also satisfy Eq. 10.24. To derive the equivalent expression in cylindrical coordinates, we note that $T_{11} + T_{22}$ is the first scalar invariant of the stress tensor, therefore

$$T_{11} + T_{22} = T_{rr} + T_{\theta\theta} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial r^2} \quad (10.44)$$

54 We also note that in cylindrical coordinates, the Laplacian operator takes the following form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad (10.45)$$

55 Thus, the function Φ must satisfy the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) = 0 \text{ or } \nabla^4 = 0 \quad (10.46)$$

10.2.2.2 Axially Symmetric Case

56 If Φ is a function of r only, we have

$$T_{rr} = \frac{1}{r}\frac{d\Phi}{dr}; \quad T_{\theta\theta} = \frac{d^2\Phi}{dr^2}; \quad T_{r\theta} = 0 \quad (10.47)$$

and

$$\frac{d^4\Phi}{dr^4} + \frac{2}{r}\frac{d^3\Phi}{dr^3} - \frac{1}{r^2}\frac{d^2\Phi}{dr^2} + \frac{1}{r^3}\frac{d\Phi}{dr} = 0 \quad (10.48)$$

57 The general solution to this problem; using Mathematica:

```
DSolve[phi''''[r]+2 phi'''[r]/r-phi''[r]/r^2+phi'[r]/r^3==0,phi[r],r]
```

$$\Phi = A\ln r + Br^2\ln r + Cr^2 + D \tag{10.49}$$

58 The corresponding stress field is

$$\boxed{\begin{aligned} T_{rr} &= \frac{A}{r^2} + B(1+2\ln r) + 2C && (10.50)\\ T_{\theta\theta} &= -\frac{A}{r^2} + B(3+2\ln r) + 2C && (10.51)\\ T_{r\theta} &= 0 && (10.52)\end{aligned}}$$

and the strain components are (from Sect. 9.8.1)

$$\boxed{\begin{aligned} E_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{E}\left[\frac{(1+\nu)A}{r^2} + (1-3\nu-4\nu^2)B + 2(1-\nu-2\nu^2)B\ln r + 2(1-\nu-2\nu^2)C\right] && (10\\ E_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \frac{1}{E}\left[-\frac{(1+\nu)A}{r^2} + (3-\nu-4\nu^2)B + 2(1-\nu-2\nu^2)B\ln r + 2(1-\nu-2\nu^2)C\right] && (10\\ E_{r\theta} &= 0 && (10\end{aligned}}$$

59 Finally, the displacement components can be obtained by integrating the above equations

$$\boxed{\begin{aligned} u_r &= \frac{1}{E}\left[-\frac{(1+\nu)A}{r} - (1+\nu)Br + 2(1-\nu-2\nu^2)r\ln rB + 2(1-\nu-2\nu^2)rC\right] && (10.56)\\ u_\theta &= \frac{4r\theta B}{E}(1-\nu^2) && (10.57)\end{aligned}}$$

10.2.2.3 Example: Thick-Walled Cylinder

60 If we consider a circular cylinder with internal and external radii a and b respectively, subjected to internal and external pressures p_i and p_o respectively, Fig. 10.2, then the boundary conditions for the plane strain problem are

$$\begin{aligned} T_{rr} &= -p_i \quad \text{at} \quad r = a && (10.58\text{-a})\\ T_{rr} &= -p_o \quad \text{at} \quad r = b && (10.58\text{-b})\end{aligned}$$

61 These Boundary conditions can be easily shown to be satisfied by the following stress field

$$\begin{aligned} T_{rr} &= \frac{A}{r^2} + 2C && (10.59\text{-a})\\ T_{\theta\theta} &= -\frac{A}{r^2} + 2C && (10.59\text{-b})\\ T_{r\theta} &= 0 && (10.59\text{-c})\end{aligned}$$

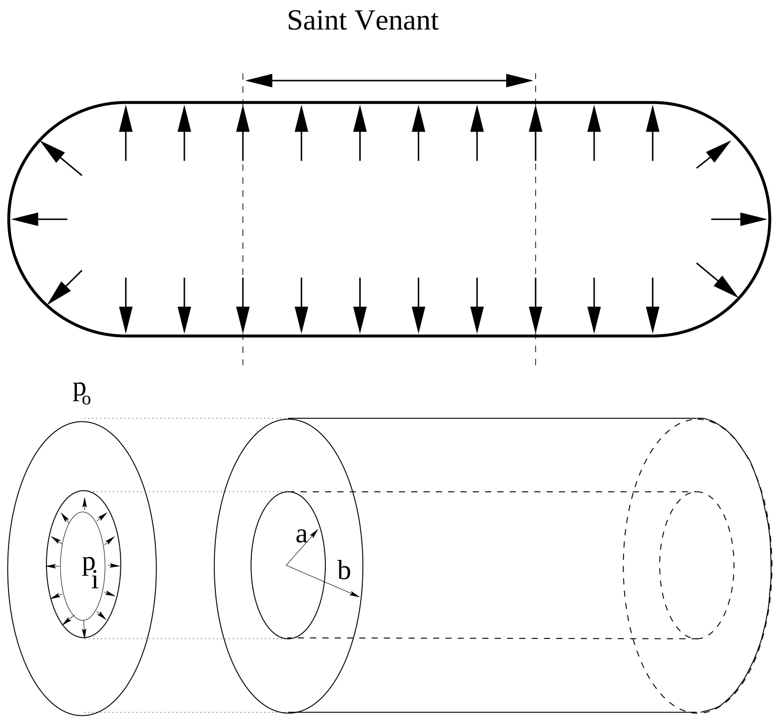


Figure 10.2: Pressurized Thick Tube

These equations are taken from Eq. 10.50, 10.51 and 10.52 with $B = 0$ and therefore represent a possible state of stress for the plane strain problem.

62 We note that if we take $B \neq 0$, then $u_\theta = \frac{4r\theta B}{E}(1-\nu^2)$ and this is not acceptable because if we were to start at $\theta = 0$ and trace a curve around the origin and return to the same point, than $\theta = 2\pi$ and the displacement would then be different.

63 Applying the boundary condition we find that

$$T_{rr} = -p_i \frac{(b^2/r^2)-1}{(b^2/a^2)-1} - p_0 \frac{1-(a^2/r^2)}{1-(a^2/b^2)} \quad (10.60)$$

$$T_{\theta\theta} = p_i \frac{(b^2/r^2)+1}{(b^2/a^2)-1} - p_0 \frac{1+(a^2/r^2)}{1-(a^2/b^2)} \quad (10.61)$$

$$T_{r\theta} = 0 \quad (10.62)$$

64 We note that if only the internal pressure p_i is acting, then T_{rr} is always a compressive stress, and $T_{\theta\theta}$ is always positive.

65 If the cylinder is thick, then the strains are given by Eq. 10.53, 10.54 and 10.55. For a very thin cylinder in the axial direction, then the strains will be given by

$$E_{rr} = \frac{du}{dr} = \frac{1}{E}(T_{rr} - \nu T_{\theta\theta}) \quad (10.63\text{-a})$$

$$E_{\theta\theta} = \frac{u}{r} = \frac{1}{E}(T_{\theta\theta} - \nu T_{rr}) \quad (10.63\text{-b})$$

$$E_{zz} = \frac{dw}{dz} = \frac{\nu}{E}(T_{rr} + T_{\theta\theta}) \quad (10.63\text{-c})$$

$$E_{r\theta} = \frac{(1+\nu)}{E}T_{r\theta} \quad (10.63\text{-d})$$

66 It should be noted that applying Saint-Venant's principle the above solution is only valid away from the ends of the cylinder.

10.2.2.4 Example: Hollow Sphere

67 We consider next a hollow sphere with internal and xternal radii a_i and a_o respectively, and subjected to internal and external pressures of p_i and p_o, Fig. 10.3.

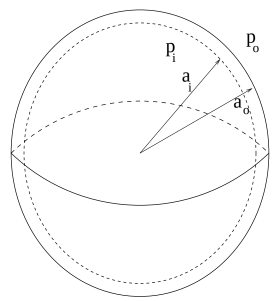


Figure 10.3: Pressurized Hollow Sphere

68 With respect to the spherical ccordinates (r, θ, ϕ), it is clear due to the spherical symmetry of the geometry and the loading that each particle of the elastic sphere will expereince only a radial displacement whose magnitude depends on r only, that is

$$u_r = u_r(r), \quad u_\theta = u_\phi = 0 \quad (10.64)$$

10.2.2.5 Example: Stress Concentration due to a Circular Hole in a Plate

69 Analysing the infinite plate under uniform tension with a circular hole of diameter a, and subjected to a uniform stress σ_0, Fig. 10.4.

70 The peculiarity of this problem is that the far-field boundary conditions are better expressed in cartesian coordinates, whereas the ones around the hole should be written in polar coordinate system.

71 First we select a stress function which satisfies the biharmonic Equation (Eq. 10.23), and the far-field boundary conditions. From St Venant principle, away from the hole, the boundary conditions are given by:

$$T_{xx} = \sigma_0; \qquad T_{yy} = T_{xy} = 0 \quad (10.65)$$

Recalling (Eq. 10.19) that $T_{xx} = \frac{\partial^2\Phi}{\partial y^2}$, this would would suggest a stress function Φ of the form $\Phi = \sigma_0 y^2$. Alternatively, the presence of the circular hole would suggest a polar representation of Φ. Thus, substituting $y = r\sin\theta$ would result in $\Phi = \sigma_0 r^2 \sin^2\theta$.

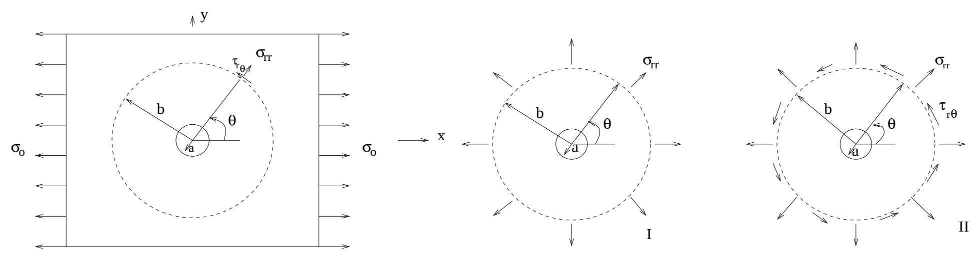


Figure 10.4: Circular Hole in an Infinite Plate

72 Since $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, we could simplify the stress function into

$$\Phi = f(r)\cos 2\theta \tag{10.66}$$

Substituting this function into the biharmonic equation (Eq. 10.46) yields

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right) = 0 \tag{10.67-a}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{4f}{r^2}\right) = 0 \tag{10.67-b}$$

73 The general solution of this ordinary linear fourth order differential equation is

$$f(r) = Ar^2 + Br^4 + C\frac{1}{r^2} + D \tag{10.68}$$

thus the stress function becomes

$$\Phi = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D\right)\cos 2\theta \tag{10.69}$$

Using Eq. 10.41-10.43, the stresses are given by

$$T_{rr} = \frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = -\left(2A+\frac{6C}{r^4}+\frac{4D}{r^2}\right)\cos 2\theta \tag{10.70-a}$$

$$T_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = \left(2A+12Br^2+\frac{6C}{r^4}\right)\cos 2\theta \tag{10.70-b}$$

$$T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = \left(2A+6Br^2-\frac{6C}{r^4}-\frac{2D}{r^2}\right)\sin 2\theta \tag{10.70-c}$$

74 Next we seek to solve for the four constants of integration by applying the boundary conditions. We will identify two sets of boundary conditions:

1. Outer boundaries: around an infinitely large circle of radius b inside a plate subjected to uniform stress σ_0, the stresses in polar coordinates are obtained from Eq. 9.35

$$\begin{bmatrix} T_{rr} & T_{r\theta} \\ T_{r\theta} & T_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \tag{10.71}$$

yielding (recalling that $\sin^2\theta = 1/2\sin 2\theta$, and $\cos^2\theta = 1/2(1+\cos 2\theta)$).

$$(T_{rr})_{r=b} = \sigma_0\cos^2\theta = \frac{1}{2}\sigma_0(1+\cos 2\theta) \quad (10.72\text{-a})$$

$$(T_{r\theta})_{r=b} = \frac{1}{2}\sigma_0\sin 2\theta \quad (10.72\text{-b})$$

$$(T_{\theta\theta})_{r=b} = \frac{\sigma_0}{2}(1-\cos 2\theta) \quad (10.72\text{-c})$$

For reasons which will become apparent later, it is more convenient to decompose the state of stress given by Eq. 10.72-a and 10.72-b, into state I and II:

$$(T_{rr})^I_{r=b} = \frac{1}{2}\sigma_0 \quad (10.73\text{-a})$$

$$(T_{r\theta})^I_{r=b} = 0 \quad (10.73\text{-b})$$

$$(T_{rr})^{II}_{r=b} = \frac{1}{2}\sigma_0\cos 2\theta \quad (10.73\text{-c})$$

$$(T_{r\theta})^{II}_{r=b} = \frac{1}{2}\sigma_0\sin 2\theta \quad (10.73\text{-d})$$

Where state I corresponds to a thick cylinder with external pressure applied on $r = b$ and of magnitude $\sigma_0/2$. This problem has already been previously solved. Hence, only the last two equations will provide us with boundary conditions.

2. Around the hole: the stresses should be equal to zero:

$$(T_{rr})_{r=a} = 0 \quad (10.74\text{-a})$$

$$(T_{r\theta})_{r=a} = 0 \quad (10.74\text{-b})$$

75 Upon substitution in Eq. 10.70-a the four boundary conditions (Eq. 10.73-c, 10.73-d, 10.74-a, and 10.74-b) become

$$-\left(2A+\frac{6C}{b^4}+\frac{4D}{b^2}\right) = \frac{1}{2}\sigma_0 \quad (10.75\text{-a})$$

$$\left(2A+6Bb^2-\frac{6C}{b^4}-\frac{2D}{b^2}\right) = \frac{1}{2}\sigma_0 \quad (10.75\text{-b})$$

$$-\left(2A+\frac{6C}{a^4}+\frac{4D}{a^2}\right) = 0 \quad (10.75\text{-c})$$

$$\left(2A+6Ba^2-\frac{6C}{a^4}-\frac{2D}{a^2}\right) = 0 \quad (10.75\text{-d})$$

76 Solving for the four unknowns, and taking $\frac{a}{b} = 0$ (i.e. an infinite plate), we obtain:

$$A = -\frac{\sigma_0}{4}; \qquad B = 0; \qquad C = -\frac{a^4}{4}\sigma_0; \qquad D = \frac{a^2}{2}\sigma_0 \quad (10.76)$$

77 To this solution, we must superimpose the one of a thick cylinder subjected to a uniform radial traction $\sigma_0/2$ on the outer surface, and with b much greater than a. These

stresses were derived in Eqs. 10.60 and 10.61 yielding for this problem (carefull about the sign)

$$T_{rr} = \frac{\sigma_0}{2}\left(1 - \frac{a^2}{r^2}\right) \quad (10.77\text{-a})$$

$$T_{\theta\theta} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) \quad (10.77\text{-b})$$

Thus, upon substitution into Eq. 10.70-a, we obtain

$$T_{rr} = \frac{\sigma_0}{2}\left(1 - \frac{a^2}{r^2}\right) + \left(1 + 3\frac{a^4}{r^4} - \frac{4a^2}{r^2}\right)\frac{1}{2}\sigma_0 \cos 2\theta \quad (10.78\text{-a})$$

$$T_{\theta\theta} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\frac{1}{2}\sigma_0 \cos 2\theta \quad (10.78\text{-b})$$

$$T_{r\theta} = -\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\frac{1}{2}\sigma_0 \sin 2\theta \quad (10.78\text{-c})$$

78 We observe that as $r \rightarrow \infty$, both T_{rr} and $T_{r\theta}$ are equal to the values given in Eq. 10.72-a and 10.72-b respectively.

79 Alternatively, at the edge of the hole when $r = a$ we obtain $T_{rr} = T_{r\theta} = 0$ and

$$\boxed{(T_{\theta\theta})_{r=a} = \sigma_0(1 - 2\cos 2\theta)} \quad (10.79)$$

which for $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ gives a stress concentration factor (SCF) of 3. For $\theta = 0$ and $\theta = \pi$, $T_{\theta\theta} = -\sigma_0$.

Chapter 11

THEORETICAL STRENGTH OF PERFECT CRYSTALS

`This chapter (taken from the author's lecture notes in Fracture Mechanics) is of primary interest to students in Material Science.`

11.1 Introduction

20 In Eq. **??** we showed that around a circular hole in an infinite plate under uniform traction, we do have a **stress concentration factor** of 3.

21 Following a similar approach (though with curvilinear coordinates), it can be shown that if we have an elliptical hole, Fig. **??**, we would have

$$\boxed{(\sigma_{\beta\beta})^{\beta=0,\pi}_{\alpha=\alpha_0} = \sigma_0 \left(1 + 2\frac{a}{b}\right)} \tag{11.1}$$

We observe that for $a = b$, we recover the stress concentration factor of 3 of a circular hole, and that for a degenerated ellipse, i.e a crack there is an infinite stress. Alternatively,

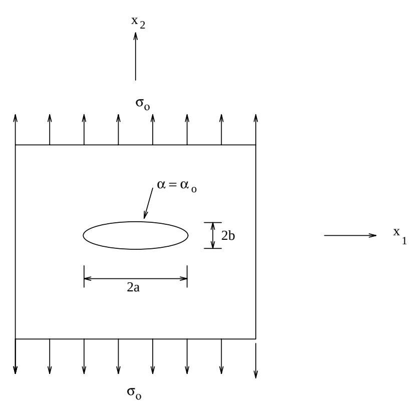


Figure 11.1: Elliptical Hole in an Infinite Plate

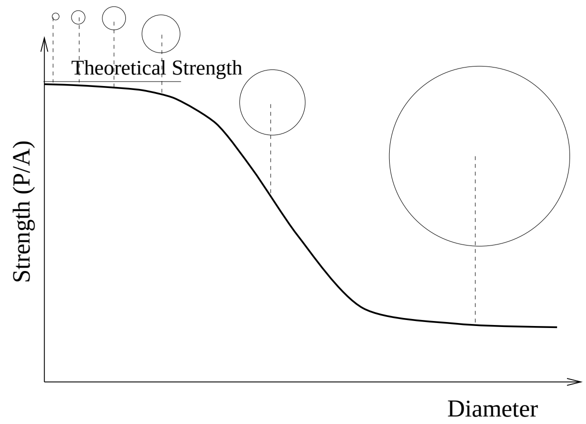


Figure 11.2: Griffith's Experiments

the stress can be expressed in terms of ρ, the radius of curvature of the ellipse,

$$\boxed{(\sigma_{\beta\beta})_{\alpha=\alpha_0}^{\beta=0,\pi} = \sigma_0\left(1+2\sqrt{\frac{a}{\rho}}\right)} \tag{11.2}$$

From this equation, we note that the stress concentration factor is inversely proportional to the radius of curvature of an opening.

22 This equation, derived by Inglis, shows that if $a = b$ we recover the factor of 3, and the stress concentration factor increase as the ratio a/b increases. In the limit, as $b = 0$ we would have a crack resulting in an infinite stress concentration factor, or a **stress singularity**.

23 Around 1920, Griffith was exploring the theoretical strength of solids by performing a series of experiments on glass rods of various diameters.

24 He observed that the tensile strength (σ^t) of glass decreased with an increase in diameter, and that for a diameter $\phi \approx \frac{1}{10{,}000}$ in., $\sigma_t = 500,000$ psi; furthermore, by extrapolation to "zero" diameter he obtained a theoretical maximum strength of approximately 1,600,000 psi, and on the other hand for very large diameters the asymptotic values was around 25,000 psi.

$$\begin{array}{rccccccc}
\text{Area} & A_1 & < & A_2 & < & A_3 & < & A_4 \\
\text{Failure Load} & P_1 & < & P_2 & < & P_3 & > & P_4 \\
\text{Failure Strength } (P/A) & \sigma_1^t & > & \sigma_2^t & > & \sigma_3^t & > & \sigma_4^t
\end{array} \tag{11.3}$$

Furthermore, as the diameter was further reduced, the failure strength asymptotically approached a limit which will be shown later to be the **theoretical strength** of glass, Fig. 11.2.

25 Clearly, one would have expected the failure strength to be constant, yet it was not. So Griffith was confronted with two questions:

1. What is this apparent theoretical strength, can it be derived?
2. Why is there a size effect for the actual strength?

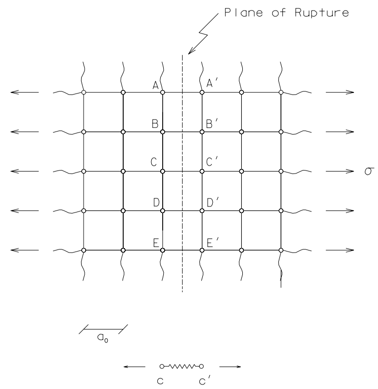


Figure 11.3: Uniformly Stressed Layer of Atoms Separated by a_0

The answers to those two questions are essential to establish a **link between Mechanics and Materials**.

26 In the next sections we will show that the theoretical strength is related to the force needed to break a bond linking adjacent atoms, and that the size effect is caused by the size of imperfections inside a solid.

11.2 Theoretical Strength

27 We start, [**?**] by exploring the energy of interaction between two adjacent atoms at equilibrium separated by a distance a_0, Fig. 11.3. The total energy which must be supplied to separate atom C from C' is

$$U_0 = 2\gamma \tag{11.4}$$

where γ is the **surface energy**[1], and the factor of 2 is due to the fact that upon separation, we have two distinct surfaces.

11.2.1 Ideal Strength in Terms of Physical Parameters

28 We shall first derive an expression for the ideal strength in terms of physical parameters, and in the next section the strength will be expressed in terms of engineering ones.

Solution I: Force being the derivative of energy, we have $F = \frac{dU}{da}$, thus $F = 0$ at $a = a_0$, Fig. 11.4, and is maximum at the inflection point of the $U_0 - a$ curve. Hence, the slope of the force displacement curve is the stiffness of the atomic spring and should be related to E. If we let $x = a - a_0$, then the strain would be equal to $\varepsilon = \frac{x}{a_0}$.

[1]From watching raindrops and bubbles it is obvious that liquid water has surface tension. When the surface of a liquid is extended (soap bubble, insect walking on liquid) work is done against this tension, and energy is stored in the new surface. When insects walk on water it sinks until the surface energy just balances the decrease in its potential energy. For solids, the chemical bonds are stronger than for liquids, hence the surface energy is stronger. The reason why we do not notice it is that solids are too rigid to be distorted by it. Surface energy γ is expressed in J/m^2 and the surface energies of water, most solids, and diamonds are approximately .077, 1.0, and 5.14 respectively.

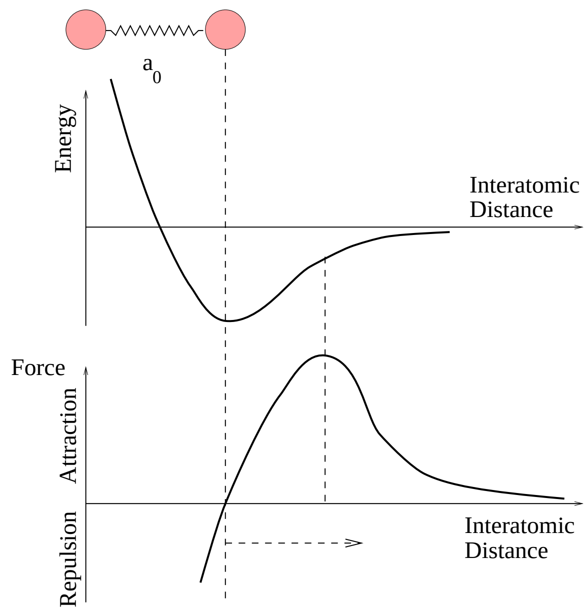


Figure 11.4: Energy and Force Binding Two Adjacent Atoms

Furthermore, if we define the stress as $\sigma = \frac{F}{a_0^2}$, then the $\sigma - \varepsilon$ curve will be as shown in Fig. 11.5.

From this diagram, it would appear that the sine curve would be an adequate approximation to this relationship. Hence,

$$\sigma = \sigma_{max}^{theor} \sin 2\pi \frac{x}{\lambda} \tag{11.5}$$

and the maximum stress σ_{max}^{theor} would occur at $x = \frac{\lambda}{4}$. The energy required to separate two atoms is thus given by the area under the sine curve, and from Eq. 11.4, we would have

$$2\gamma = U_0 = \int_0^{\frac{\lambda}{2}} \sigma_{max}^{theor} \sin\left(2\pi\frac{x}{\lambda}\right) dx \tag{11.6}$$

$$= \frac{\lambda}{2\pi}\sigma_{max}^{theor}[-\cos(\frac{2\pi x}{\lambda})] |_0^{\frac{\lambda}{2}} \tag{11.7}$$

$$= \frac{\lambda}{2\pi}\sigma_{max}^{theor}[-\overbrace{\cos(\frac{2\pi\lambda}{2\lambda})}^{-1} + \overbrace{\cos(0)}^{1}] \tag{11.8}$$

$$\Rightarrow \lambda = \frac{2\gamma\pi}{\sigma_{max}^{theor}} \tag{11.9}$$

Also for very small displacements (small x) $\sin x \approx x$, thus Eq. 11.5 reduces to

$$\sigma \approx \sigma_{max}^{theor}\frac{2\pi x}{\lambda} \approx \frac{Ex}{a_0} \tag{11.10}$$

elliminating x,

$$\sigma_{max}^{theor} \approx \frac{E}{a_0}\frac{\lambda}{2\pi} \tag{11.11}$$

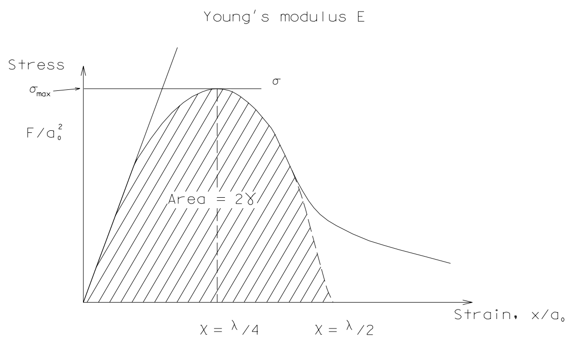


Figure 11.5: Stress Strain Relation at the Atomic Level

Substituting for λ from Eq. 11.9, we get

$$\boxed{\sigma_{max}^{theor} \approx \sqrt{\frac{E\gamma}{a_0}}} \tag{11.12}$$

Solution II: For two layers of atoms a_0 apart, the strain energy per unit area due to σ (for linear elastic systems) is

$$\left.\begin{array}{lcl} U & = & \frac{1}{2}\sigma\varepsilon a_o \\ \sigma & = & E\varepsilon \end{array}\right\} U = \frac{\sigma^2 a_o}{2E} \tag{11.13}$$

If γ is the surface energy of the solid per unit area, then the total surface energy of two new fracture surfaces is 2γ.

For our theoretical strength, $U = 2\gamma \Rightarrow \frac{(\sigma_{max}^{theor})^2 a_0}{2E} = 2\gamma$ or $\sigma_{max}^{theor} = 2\sqrt{\frac{\gamma E}{a_0}}$

Note that here we have assumed that the material obeys Hooke's Law up to failure, since this is seldom the case, we can simplify this approximation to:

$$\boxed{\sigma_{max}^{theor} = \sqrt{\frac{E\gamma}{a_0}}} \tag{11.14}$$

which is the same as Equation 11.12

Example: As an example, let us consider steel which has the following properties: $\gamma = 1\frac{J}{m^2}$; $E = 2 \times 10^{11}\frac{N}{m^2}$; and $a_0 \approx 2 \times 10^{-10}$ m. Thus from Eq. 11.12 we would have:

$$\sigma_{max}^{theor} \approx \sqrt{\frac{(2 \times 10^{11})(1)}{2 \times 10^{-10}}} \tag{11.15}$$

$$\approx 3.16 \times 10^{10}\frac{N}{m^2} \tag{11.16}$$

$$\approx \frac{E}{6} \tag{11.17}$$

Thus this would be the ideal theoretical strength of steel.

11.2.2 Ideal Strength in Terms of Engineering Parameter

29 We note that the force to separate two atoms drops to zero when the distance between them is $a_0 + a$ where a_0 corresponds to the origin and a to $\frac{\lambda}{2}$. Thus, if we take $a = \frac{\lambda}{2}$ or $\lambda = 2a$, combined with Eq. 11.11 would yield

$$\sigma_{max}^{theor} \approx \frac{E}{a_0}\frac{a}{\pi} \tag{11.18}$$

30 Alternatively combining Eq. 11.9 with $\lambda = 2a$ gives

$$a \approx \frac{\gamma\pi}{\sigma_{max}^{theor}} \tag{11.19}$$

Combining those two equations will give

$$\gamma \approx \frac{E}{a_0}\left(\frac{a}{\pi}\right)^2 \tag{11.20}$$

31 However, since as a first order approximation $a \approx a_0$ then the surface energy will be

$$\gamma \approx \frac{Ea_0}{10} \tag{11.21}$$

This equation, combined with Eq. 11.12 will finally give

$$\boxed{\sigma_{max}^{theor} \approx \frac{E}{\sqrt{10}}} \tag{11.22}$$

which is an approximate expression for the theoretical maximum strength in terms of E.

11.3 Size Effect; Griffith Theory

32 In his quest for an explanation of the size effect, Griffith came across Inglis's paper, and his "strike of genius" was to assume that *strength is reduced due to the presence of internal flaws*. Griffith postulated that the theoretical strength can only be reached at the point of highest stress concentration, and accordingly the far-field applied stress will be much smaller.

33 Hence, assuming an elliptical imperfection, and from equation 11.2

$$\sigma_{max}^{theor} = \sigma_{cr}^{act}\left(1 + 2\sqrt{\frac{a}{\rho}}\right) \tag{11.23}$$

σ is the stress at the tip of the ellipse which is caused by a (lower) far field stress σ_{cr}^{act}. Asssuming $\rho \approx a_0$ and since $2\sqrt{\frac{a}{a_0}} \gg 1$, for an ideal plate under tension with only one single elliptical flaw the strength may be obtained from

$$\sigma_{max}^{theor} = 2\sigma_{cr}^{act}\sqrt{\frac{a}{a_0}} \tag{11.24}$$

hence, equating with Eq. 11.12, we obtain

$$\boxed{\sigma_{max}^{theor} = \underbrace{2\sigma_{cr}^{act}\sqrt{\frac{a}{a_o}}}_{\text{Macro}} = \underbrace{\sqrt{\frac{E\gamma}{a_0}}}_{\text{Micro}}} \tag{11.25}$$

From this very important equation, we observe that

1. The left hand side is based on a linear elastic solution of a macroscopic problem solved by Inglis.
2. The right hand side is based on the theoretical strength derived from the sinusoidal stress-strain assumption of the interatomic forces, and finds its roots in micro-physics.

Finally, this equation would give (at fracture)

$$\boxed{\sigma_{cr}^{act} = \sqrt{\frac{E\gamma}{4a}}} \tag{11.26}$$

As an example, let us consider a flaw with a size of $2a = 5,000a_0$

$$\left.\begin{array}{lll}\sigma_{cr}^{act} & = & \sqrt{\frac{E\gamma}{4a}} \\ \gamma & = & \frac{Ea_0}{10}\end{array}\right\} \left.\begin{array}{lll}\sigma_{cr}^{act} & = & \sqrt{\frac{E^2}{40}\frac{a_o}{a}} \\ \frac{a}{a_0} & = & 2,500\end{array}\right\} \sigma_{cr}^{act} = \sqrt{\frac{E^2}{100{,}000}} = \frac{E}{100\sqrt{10}} \tag{11.27}$$

Thus if we set a flaw size of $2a = 5,000a_0$ in $\gamma \approx \frac{Ea_0}{10}$ this is enough to lower the theoretical fracture strength from $\frac{E}{\sqrt{10}}$ to a critical value of magnitude $\frac{E}{100\sqrt{10}}$, or a factor of 100.

As an example

$$\left.\begin{array}{lll}\sigma_{max}^{theor} & = & 2\sigma_{cr}^{act}\sqrt{\frac{a}{a_o}} \\ a & = & 10^{-6}m = 1\mu \\ a_o & = & 1\mathring{A} = \rho = 10^{-10}m\end{array}\right\} \sigma_{max}^{theor} = 2\sigma_{cr}^{act}\sqrt{\frac{10^{-6}}{10^{-10}}} = 200\sigma_{cr}^{act} \tag{11.28}$$

Therefore at failure

$$\left.\begin{array}{lll}\sigma_{cr}^{act} & = & \frac{\sigma_{max}^{theor}}{200} \\ \sigma_{max}^{theor} & = & \frac{E}{10}\end{array}\right\} \sigma_{cr}^{act} \approx \frac{E}{2,000} \tag{11.29}$$

which can be attained. For instance for steel $\frac{E}{2{,}000} = \frac{30{,}000}{2{,}000} = 15$ ksi

Chapter 12

BEAM THEORY

`This chapter is adapted from the Author's lecture notes in Structural Analysis.`

12.1 Introduction

20 In the preceding chapters we have focused on the behavior of a *continuum*, and the 15 equations and 15 variables we introduced, were all derived for an *infinitesimal* element.

21 In practice, few problems can be solved analytically, and even with computer it is quite difficult to view every object as a three dimensional one. That is why we introduced the 2D simplification (plane stress/strain), or 1D for axially symmetric problems. In the preceding chapter we saw a few of those solutions.

22 Hence, to widen the scope of application of the fundamental theory developed previously, we could either resort to numerical methods (such as the finite difference, finite element, or boundary elements), or we could further simplify the problem.

23 Solid bodies, in general, have certain peculiar geometric features amenable to a reduction from three to fewer dimensions. If one dimension of the **structural element**[1] under consideration is much greater or smaller than the other three, than we have a beam, or a plate respectively. If the plate is curved, then we have a shell.

24 For those structural elements, it is customary to consider as internal variables the **resultant** of the stresses as was shown in Sect. **??**.

25 Hence, this chapter will focus on a brief introduction to beam theory. This will however be preceded by an introduction to Statics as the internal forces would also have to be in equilibrium with the external ones.

26 Beam theory is perhaps the most successful theory in all of structural mechanics, and it forms the basis of **structural analysis** which is so dear to Civil and Mechanical engineers.

[1] So far we have restricted ourselves to a continuum, in this chapter we will consider a structural element.

12.2 Statics

12.2.1 Equilibrium

27 Any structural element, or part of it, must satisfy equilibrium.

28 Summation of forces and moments, **in a static system** must be equal to zero[2].

29 In a 3D cartesian coordinate system there are a total of 6 **independent** equations of equilibrium:

$$\begin{array}{ccccccc} \Sigma F_x & = & \Sigma F_y & = & \Sigma F_z & = & 0 \\ \Sigma M_x & = & \Sigma M_y & = & \Sigma M_z & = & 0 \end{array} \tag{12.1}$$

30 In a 2D cartesian coordinate system there are a total of 3 independent equations of equilibrium:

$$\Sigma F_x = \Sigma F_y = \Sigma M_z = 0 \tag{12.2}$$

31 All the externally applied forces on a structure must be in equilibrium. **Reactions** are accordingly determined.

32 For reaction calculations, the externally applied load may be reduced to an equivalent force[3].

33 Summation of the moments can be taken with respect to **any** arbitrary point.

34 Whereas forces are represented by a vector, moments are also vectorial quantities and are represented by a curved arrow or a double arrow vector.

35 Not all equations are applicable to all structures, Table 12.1

Structure Type	**Equations**					
Beam, no axial forces		ΣF_y				ΣM_z
2D Truss, Frame, Beam	ΣF_x	ΣF_y				ΣM_z
Grid			ΣF_z	ΣM_x	ΣM_y	
3D Truss, Frame	ΣF_x	ΣF_y	ΣF_z	ΣM_x	ΣM_y	ΣM_z
Alternate Set						
Beams, no axial Force	ΣM_z^A	ΣM_z^B				
2 D Truss, Frame, Beam	ΣF_x	ΣM_z^A	ΣM_z^B			
	ΣM_z^A	ΣM_z^B	ΣM_z^C			

Table 12.1: Equations of Equilibrium

36 The three conventional equations of equilibrium in 2D: $\Sigma F_x, \Sigma F_y$ and ΣM_z can be replaced by the independent moment equations ΣM_z^A, ΣM_z^B, ΣM_z^C provided that A, B, and C **are not colinear**.

[2] In a dynamic system $\Sigma \mathbf{F} = m\mathbf{a}$ where m is the mass and $\mathbf{a}$ is the acceleration.
[3] However for internal forces (shear and moment) we must use the actual load distribution.

37 It is always preferable to **check** calculations by another equation of equilibrium.

38 Before you write an equation of equilibrium,

1. Arbitrarily decide which is the **+ve** direction
2. Assume a direction for the unknown quantities
3. The right hand side of the equation should be zero

If your reaction is negative, then it will be in a direction opposite from the one assumed.

39 Summation of external forces is equal and **opposite** to the internal ones (more about this below). Thus the net force/moment is equal to zero.

40 The external forces give rise to the (non-zero) shear and moment diagram.

12.2.2 Reactions

41 In the analysis of structures, it is often easier to start by determining the reactions.

42 Once the reactions are determined, internal forces (shear and moment) are determined next; finally, internal stresses and/or deformations (deflections and rotations) are determined last.

43 Depending on the type of structures, there can be different types of support conditions, Fig. 12.1.

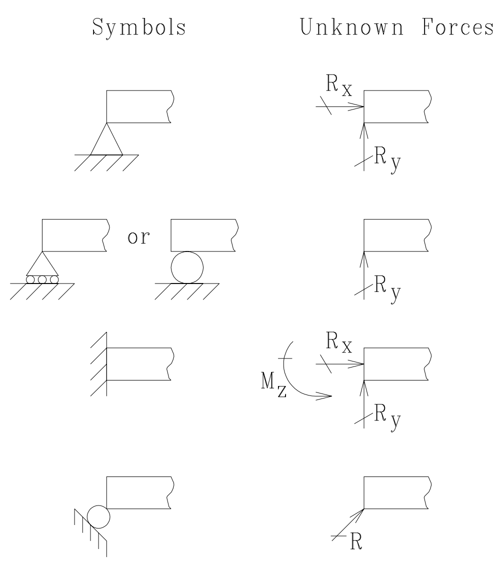


Figure 12.1: Types of Supports

Roller: provides a restraint in only one direction in a 2D structure, in 3D structures a roller may provide restraint in one or two directions. A roller will allow rotation.

Hinge: allows rotation but no displacements.

Fixed Support: will prevent rotation and displacements in all directions.

12.2.3 Equations of Conditions

44 If a structure has an **internal hinge** (which may connect two or more substructures), then this will provide an additional equation ($\Sigma M = 0$ at the hinge) which can be exploited to determine the reactions.

45 Those equations are often exploited in trusses (where each connection is a hinge) to determine reactions.

46 In an **inclined roller** support with S_x and S_y horizontal and vertical projection, then the reaction R would have, Fig. 12.2.

$$\boxed{\frac{R_x}{R_y} = \frac{S_y}{S_x}} \tag{12.3}$$

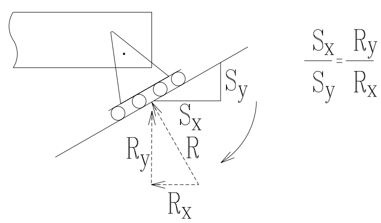


Figure 12.2: Inclined Roller Support

12.2.4 Static Determinacy

47 In statically determinate structures, reactions depend only on the geometry, boundary conditions and loads.

48 If the reactions can not be determined simply from the equations of static equilibrium (and equations of conditions if present), then the reactions of the structure are said to be **statically indeterminate**.

49 The **degree of static indeterminacy** is equal to the difference between the number of reactions and the number of equations of equilibrium (plus the number of equations of conditions if applicable), Fig. 12.3.

50 Failure of one support in a statically determinate system results in the collapse of the structures. Thus a statically indeterminate structure is **safer** than a statically determinate one.

51 For statically indeterminate structures, reactions depend also on the material properties (e.g. Young's and/or shear modulus) and element cross sections (e.g. length, area, moment of inertia).

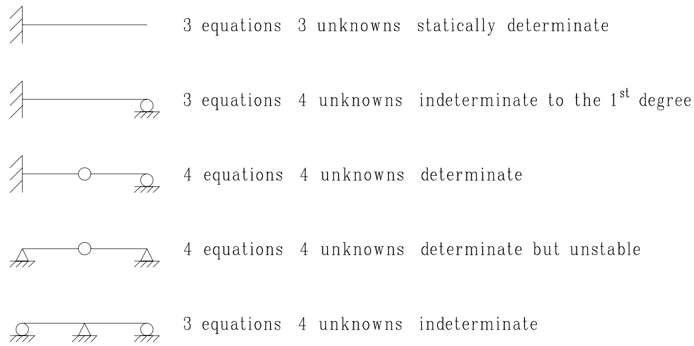


Figure 12.3: Examples of Static Determinate and Indeterminate Structures

12.2.5 Geometric Instability

52 The stability of a structure is determined not only by the number of reactions but also by their arrangement.

53 Geometric instability will occur if:

1. All **reactions are parallel** and a non-parallel load is applied to the structure.
2. All **reactions are concurrent**, Fig. 12.4.

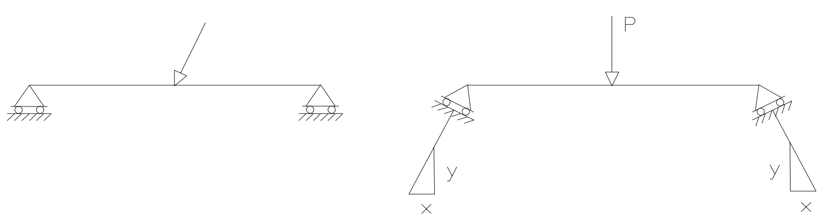


Figure 12.4: Geometric Instability Caused by Concurrent Reactions

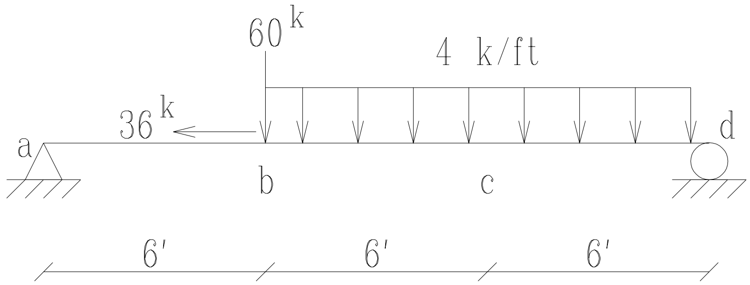
3. The number of reactions is smaller than the number of equations of equilibrium, that is a **mechanism** is present in the structure.

54 Mathematically, this can be shown if the **determinant** of the equations of equilibrium is equal to zero (or the equations are inter-dependent).

12.2.6 Examples

■ **Example 12-1: Simply Supported Beam**

Determine the reactions of the simply supported beam shown below.



Solution:
The beam has 3 reactions, we have 3 equations of static equilibrium, hence it is statically determinate.

$$\begin{array}{rl}(+\rightarrow)\ \Sigma F_x = 0; & \Rightarrow\ R_{ax} - 36\text{ k} = 0\\ (+\uparrow)\ \Sigma F_y = 0; & \Rightarrow\ R_{ay} + R_{dy} - 60\text{ k} - (4)\text{ k/ft}(12)\text{ ft} = 0\\ (+\circlearrowleft)\ \Sigma M_z^c = 0; & \Rightarrow\ 12R_{ay} - 6R_{dy} - (60)(6) = 0\end{array}$$

or through matrix inversion (on your calculator)

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 12 & -6\end{bmatrix}\begin{Bmatrix}R_{ax}\\ R_{ay}\\ R_{dy}\end{Bmatrix} = \begin{Bmatrix}36\\ 108\\ 360\end{Bmatrix} \Rightarrow \begin{Bmatrix}R_{ax}\\ R_{ay}\\ R_{dy}\end{Bmatrix} = \begin{Bmatrix}36\text{ k}\\ 56\text{ k}\\ 52\text{ k}\end{Bmatrix}$$

Alternatively we could have used another set of equations:

$$\begin{array}{ll}(+\circlearrowleft)\ \Sigma M_z^a = 0; & (60)(6) + (48)(12) - (R_{dy})(18) = 0 \ \Rightarrow\ R_{dy} = \boxed{52\text{ k}\uparrow}\\ (+\circlearrowleft)\ \Sigma M_z^d = 0; & (R_{ay})(18) - (60)(12) - (48)(6) = 0 \ \Rightarrow\ R_{ay} = \boxed{56\text{ k}\uparrow}\end{array}$$

Check:

$$(+\uparrow)\ \Sigma F_y = 0;\,;56 - 52 - 60 - 48 = 0\surd$$

■

12.3 Shear & Moment Diagrams

12.3.1 Design Sign Conventions

55 Before we derive the Shear-Moment relations, let us *arbitrarily* define a sign convention.

56 The sign convention adopted here, is the one commonly used for design purposes[4]. With reference to Fig. 12.5

Load Positive along the beam's local y axis (assuming a right hand side convention), that is positive upward.

Axial: tension positive.

[4]Note that this sign convention is the opposite of the one commonly used in Europe!

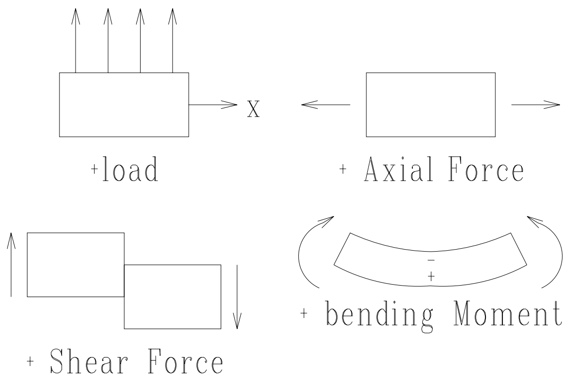


Figure 12.5: Shear and Moment Sign Conventions for Design

Flexure A positive moment is one which causes tension in the lower fibers, and compression in the upper ones. For frame members, a positive moment is one which causes tension along the inner side.

Shear A positive shear force is one which is "up" on a negative face, or "down" on a positive one. Alternatively, a pair of positive shear forces will cause clockwise rotation.

12.3.2 Load, Shear, Moment Relations

57 Let us derive the basic relations between load, shear and moment. Considering an infinitesimal length dx of a beam subjected to a positive load[5] $w(x)$, Fig. 12.6. The

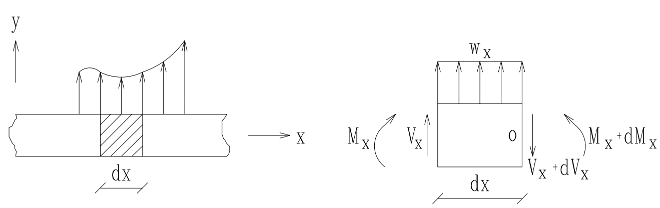


Figure 12.6: Free Body Diagram of an Infinitesimal Beam Segment

infinitesimal section must also be in equilibrium.

58 There are no axial forces, thus we only have two equations of equilibrium to satisfy $\Sigma F_y = 0$ and $\Sigma M_z = 0$.

59 Since dx is infinitesimally small, the small variation in load along it can be neglected, therefore we assume $w(x)$ to be constant along dx.

60 To denote that a small change in shear and moment occurs over the length dx of the element, we add the differential quantities dV_x and dM_x to V_x and M_x on the right face.

[5]In this derivation, as in all other ones we should assume all quantities to be positive.

61 Next considering the first equation of equilibrium

$$(+\uparrow)\, \Sigma F_y = 0 \Rightarrow V_x + w_x dx - (V_x + dV_x) = 0$$

or

$$\boxed{\frac{dV}{dx} = w(x)} \tag{12.4}$$

The slope of the shear curve at any point along the axis of a member is given by the load curve at that point.

62 Similarly

$$(+\curvearrowleft)\, \Sigma M_o = 0 \Rightarrow M_x + V_x dx - w_x dx\frac{dx}{2} - (M_x + dM_x) = 0$$

Neglecting the dx^2 term, this simplifies to

$$\boxed{\frac{dM}{dx} = V(x)} \tag{12.5}$$

The slope of the moment curve at any point along the axis of a member is given by the shear at that point.

63 Alternative forms of the preceding equations can be obtained by integration

$$V = \int w(x)dx \tag{12.6}$$

$$\Delta V_{21} = V_{x_2} - V_{x_1} = \int_{x_1}^{x_2} w(x)dx \tag{12.7}$$

The change in shear between 1 and 2, ΔV_{21}, is equal to the area under the load between x_1 and x_2.

and

$$M = \int V(x)dx \tag{12.8}$$

$$\Delta M_{21} = M_2 - M_1 = \int_{x_1}^{x_2} V(x)dx \tag{12.9}$$

The change in moment between 1 and 2, ΔM_{21}, is equal to the area under the shear curve between x_1 and x_2.

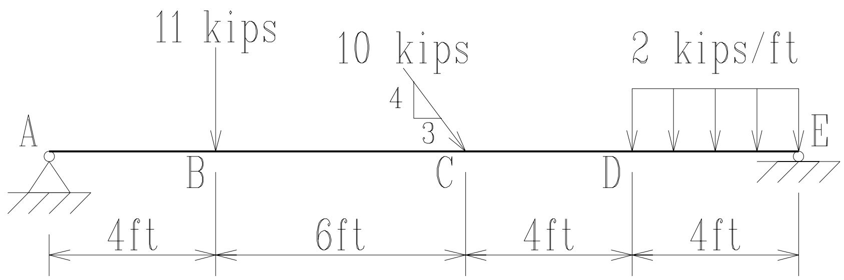
64 Note that we still need to have V_1 and M_1 in order to obtain V_2 and M_2 respectively.

65 It can be shown that the equilibrium of forces and of moments equations are nothing else than the three dimensional linear momentum $\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\frac{dv_i}{dt}$ and moment of momentum $\int_S (\mathbf{r}\times\mathbf{t})dS + \int_V (\mathbf{r}\times\rho\mathbf{b})dV = \frac{d}{dt}\int_V (\mathbf{r}\times\rho\mathbf{v})dV$ equations satisfied *on the average over the cross section.*

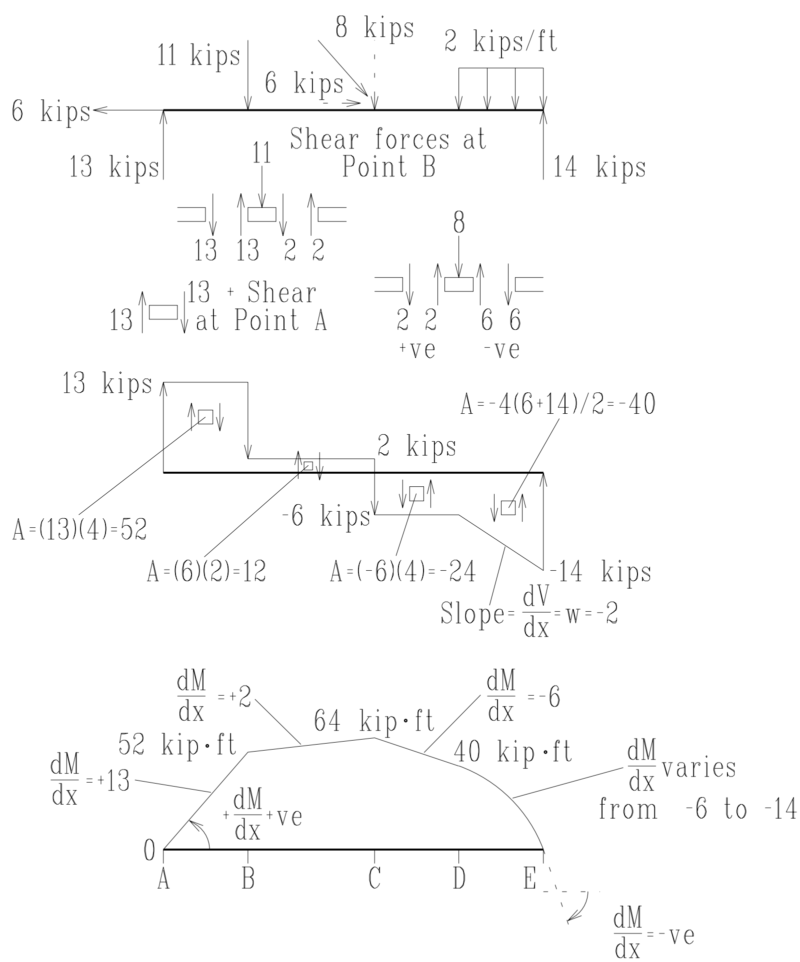
12.3.3 Examples

■ **Example 12-2: Simple Shear and Moment Diagram**

Draw the shear and moment diagram for the beam shown below



Solution:
The free body diagram is drawn below



Reactions are determined from the equilibrium equations

$$
\begin{array}{rll}
(+ \rightarrow) \Sigma F_x = 0; & \Rightarrow & -R_{A_x} + 6 = 0 \Rightarrow R_{A_x} = 6 \text{ k} \\
(+ \circlearrowright) \Sigma M_A = 0; & \Rightarrow & (11)(4) + (8)(10) + (4)(2)(14+2) - R_{F_y}(18) = 0 \Rightarrow R_{F_y} = 14 \text{ k} \\
(+ \uparrow) \Sigma F_y = 0; & \Rightarrow & R_{A_y} - 11 - 8 - (4)(2) + 14 = 0 \Rightarrow R_{A_y} = 13 \text{ k}
\end{array}
$$

Shear are determined next.

1. At A the shear is equal to the reaction and is positive.
2. At B the shear drops (negative load) by 11 k to 2 k.
3. At C it drops again by 8 k to −6 k.
4. It stays constant up to D and then it decreases (constant negative slope since the load is uniform and negative) by 2 k per linear foot up to −14 k.
5. As a check, −14 k is also the reaction previously determined at F.

Moment is determined last:

1. The moment at A is zero (hinge support).
2. The change in moment between A and B is equal to the area under the corresponding shear diagram, or $\Delta M_{B-A} = (13)(4) = 52$.
3. etc...

■

12.4 Beam Theory

12.4.1 Basic Kinematic Assumption; Curvature

66 Fig.12.7 shows portion of an originally straight beam which has been bent to the radius ρ by end couples M. support conditions, Fig. 12.1. It is *assumed* that **plane cross-sections normal to the length of the unbent beam remain plane after the beam is bent**.

67 Except for the neutral surface all other longitudinal fibers either lengthen or shorten, thereby creating a longitudinal strain ε_x. Considering a segment EF of length dx at a distance y from the neutral axis, its original length is

$$EF = dx = \rho d\theta \tag{12.10}$$

and

$$d\theta = \frac{dx}{\rho} \tag{12.11}$$

68 To evaluate this strain, we consider the deformed length $E'F'$

$$E'F' = (\rho - y)d\theta = \rho d\theta - y d\theta = dx - y\frac{dx}{\rho} \tag{12.12}$$

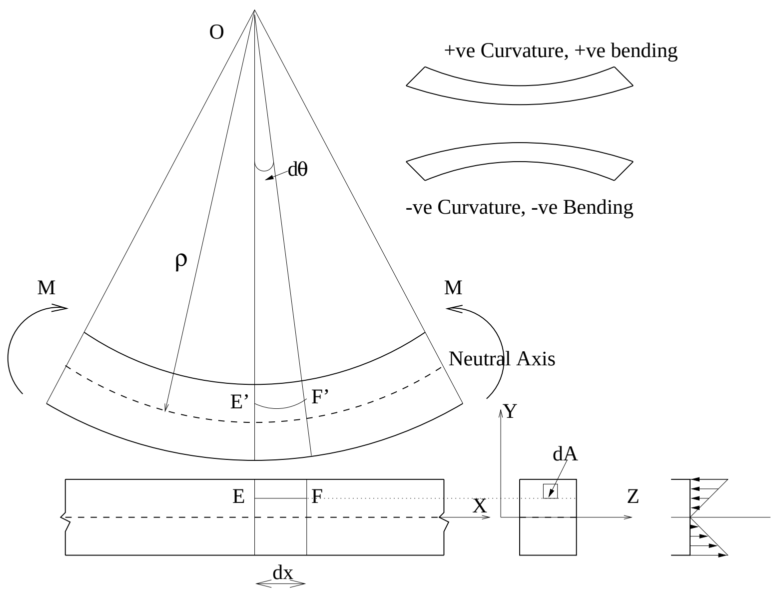


Figure 12.7: Deformation of a Beam under Pure Bending

The strain is now determined from:

$$\varepsilon_x = \frac{E'F' - EF}{EF} = \frac{dx - y\frac{dx}{\rho} - dx}{dx} \tag{12.13}$$

or after simplification

$$\boxed{\varepsilon_x = -\frac{y}{\rho}} \tag{12.14}$$

where y is measured from the axis of rotation (neutral axis). Thus strains are proportional to the distance from the neutral axis.

69 ρ (Greek letter *rho*) is the **radius of curvature**. In some textbook, the **curvature** κ (Greek letter *kappa*) is also used where

$$\kappa = \frac{1}{\rho} \tag{12.15}$$

thus,

$$\boxed{\varepsilon_x = -\kappa y} \tag{12.16}$$

70 It should be noted that Galileo (1564-1642) was the first one to have made a contribution to beam theory, yet he failed to make the right assumption for the planar cross section. This crucial assumption was made later on by Jacob Bernoulli (1654-1705), who did not make it quite right. Later Leonhard Euler (1707-1783) made significant contributions to the theory of beam deflection, and finally it was Navier (1785-1836) who clarified the issue of the kinematic hypothesis.

12.4.2 Stress-Strain Relations

71 So far we considered the kinematic of the beam, yet later on we will need to consider equilibrium in terms of the stresses. Hence we need to relate strain to stress.

72 For linear elastic material Hooke's law states

$$\boxed{\sigma_x = E\varepsilon_x} \tag{12.17}$$

where E is **Young's Modulus**.

73 Combining Eq. with equation 12.16 we obtain

$$\boxed{\sigma_x = -E\kappa y} \tag{12.18}$$

12.4.3 Internal Equilibrium; Section Properties

74 Just as external forces acting on a structure must be in equilibrium, the internal forces must also satisfy the equilibrium equations.

75 The internal forces are determined by *slicing* the beam. The internal forces on the "cut" section must be in equilibrium with the external forces.

12.4.3.1 $\Sigma F_x = 0$; Neutral Axis

76 The first equation we consider is the summation of axial forces.

77 Since there are no external axial forces (unlike a column or a beam-column), the internal axial forces must be in equilibrium.

$$\Sigma F_x = 0 \Rightarrow \int_A \sigma_x dA = 0 \tag{12.19}$$

where σ_x was given by Eq. 12.18, substituting we obtain

$$\int_A \sigma_x dA = -\int_A E\kappa y dA = 0 \tag{12.20-a}$$

But since the curvature κ and the modulus of elasticity E are constants, we conclude that

$$\boxed{\int_A y dA = 0} \tag{12.21}$$

or the first moment of the cross section with respect to the z axis is zero. Hence we conclude that the **neutral axis passes through the centroid of the cross section**.

12.4.3.2 $\Sigma M = 0$; Moment of Inertia

78 The second equation of internal equilibrium which must be satisfied is the summation of moments. However contrarily to the summation of axial forces, we now have an external moment to account for, the one from the moment diagram at that particular location where the beam was sliced, hence

$$\Sigma M_z = 0; \curvearrowleft +\text{ve}; \underbrace{M}_{\text{Ext.}} = \underbrace{-\int_A \sigma_x y dA}_{\text{Int.}} \tag{12.22}$$

where dA is an differential area a distance y from the neutral axis.

79 Substituting Eq. 12.18

$$\left.\begin{array}{rcl} M & = & -\int_A \sigma_x y dA \\ \sigma_x & = & -E\kappa y \end{array}\right\} M = \kappa E \int_A y^2 dA \tag{12.23}$$

80 We now pause and define the section moment of inertia with respect to the z axis as

$$\boxed{I \stackrel{\text{def}}{=} \int_A y^2 dA} \tag{12.24}$$

and section modulus as

$$\boxed{S \stackrel{\text{def}}{=} \frac{I}{c}} \tag{12.25}$$

12.4.4 Beam Formula

81 We now have the ingredients in place to derive one of the most important equations in structures, the beam formula. This formula will be extensively used for **design** of structural components.

82 We merely substitute Eq. 12.24 into 12.23,

$$\left.\begin{array}{rcl} M & = & \kappa E \int_A y^2 dA \\ I & = & \int_a y^2 dA \end{array}\right\} \boxed{\frac{M}{EI} = \kappa = \frac{1}{\rho}} \tag{12.26}$$

which shows that the *curvature of the longitudinal axis of a beam is proportional to the bending moment M and inversely proportional to EI which we call* **flexural rigidity**.

83 Finally, inserting Eq. 12.18 above, we obtain

$$\left.\begin{array}{rcl} \sigma_x & = & -E\kappa y \\ \kappa & = & \frac{M}{EI} \end{array}\right\} \boxed{\sigma_x = -\frac{My}{I}} \tag{12.27}$$

Hence, for a positive y (above neutral axis), and a positive moment, we will have compressive stresses above the neutral axis.

84 Alternatively, the maximum fiber stresses can be obtained by combining the preceding equation with Equation 12.25

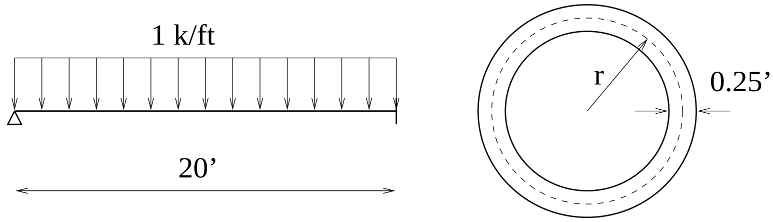
$$\boxed{\sigma_x = -\frac{M}{S}} \tag{12.28}$$

12.4.5 Limitations of the Beam Theory

12.4.6 Example

■ **Example 12-3: Design Example**

A 20 ft long, uniformly loaded, beam is simply supported at one end, and rigidly connected at the other. The beam is composed of a steel tube with thickness $t = 0.25$ in. Select the radius such that $\sigma_{max} \leq 18$ ksi, and $\Delta_{max} \leq L/360$.



Solution:

1. Steel has $E = 29,000$ ksi, and from above $M_{max} = \frac{wL^2}{8}$, $\Delta_{max} = \frac{wL^4}{185EI}$, and $I = \pi r^3 t$.
2. The maximum moment will be

$$M_{max} = \frac{wL^2}{8} = \frac{(1)\text{ k/ft}(20)^2\text{ ft}^2}{8} = 50\text{ k.ft} \tag{12.29}$$

3. We next seek a relation between maximum deflection and radius

$$\left.\begin{array}{rcl} \Delta_{max} & = & \frac{wL^4}{185EI} \\ I & = & \pi r^3 t \end{array}\right\} \begin{array}{rcl} \Delta & = & \frac{wL^4}{185E\pi r^3 t} \\ & = & \frac{(1)\text{ k/ft}(20)^4\text{ ft}^4(12)^3\text{ in}^3/\text{ ft}^3}{(185)(29{,}000)\text{ ksi}(3.14)r^3(0.25)\text{ in}} \\ & = & \frac{65.65}{r^3} \end{array} \tag{12.30}$$

4. Similarly for the stress

$$\left.\begin{array}{rcl} \sigma & = & \frac{M}{S} \\ S & = & \frac{I}{r} \\ I & = & \pi r^3 t \end{array}\right\} \begin{array}{rcl} \sigma & = & \frac{M}{\pi r^2 t} \\ & = & \frac{(50)\text{ k.ft}(12)\text{ in/ft}}{(3.14)r^2(0.25)\text{ in}} \\ & = & \frac{764}{r^2} \end{array} \tag{12.31}$$

5. We now set those two values equal to their respective maximum

$$\Delta_{max} = \frac{L}{360} = \frac{(20)\text{ ft}(12)\text{ in/ft}}{360} = 0.67\text{ in} = \frac{65.65}{r^3} \Rightarrow r = \sqrt[3]{\frac{65.65}{0.67}} = 4.61\text{ in} \tag{12.32-a}$$

$$\sigma_{max} = (18)\text{ ksi} = \frac{764}{r^2} \Rightarrow r = \sqrt{\frac{764}{18}} = \boxed{6.51\text{ in}} \tag{12.32-b}$$

■

Chapter 13

VARIATIONAL METHODS

`Abridged section from author's lecture notes in finite elements.`

20 Variational methods provide a powerful method to solve complex problems in continuum mechanics (and other fields as well).

21 As shown in Appendix C, there is a duality between the **strong form**, in which a differential equation (or Euler's equation) is exactly satisfied at every point (such as in **Finite Differences**), and the **weak form** where the equation is satisfied in an averaged sense (as in **finite elements**).

22 Since only few problems in continuum mechanics can be solved analytically, we often have to use numerical techniques, Finite Elements being one of the most powerful and flexible one.

23 At the core of the finite element formulation are the variational formulations (or energy based methods) which will be discussed in this chapter.

24 For illustrative examples, we shall use beams, but the methods is obviously applicable to 3D continuum.

13.1 Preliminary Definitions

25 Work is defined as the product of a force and displacement

$$W \stackrel{\text{def}}{=} \int_a^b \mathbf{F}.d\mathbf{s} \tag{13.1-a}$$

$$dW = F_x dx + F_y dy \tag{13.1-b}$$

26 Energy is a quantity representing the ability or capacity to perform work.

27 The change in energy is proportional to the amount of work performed. Since only the change of energy is involved, any datum can be used as a basis for measure of energy. Hence energy is neither created nor consumed.

28 The first principle of thermodynamics (Eq. 6.44), states

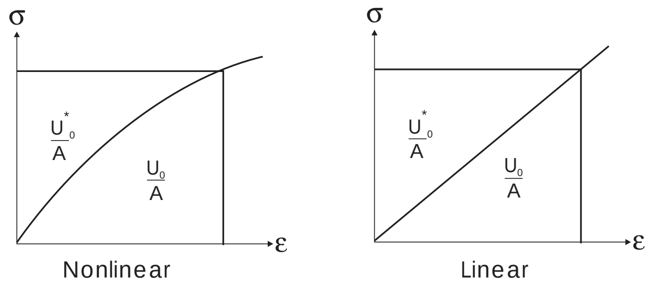


Figure 13.1: *Strain Energy and Complementary Strain Energy

> The time-rate of change of the total energy (i.e., sum of the kinetic energy and the internal energy) is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time:

$$\boxed{\frac{d}{dt}(K+U) = W_e + H} \tag{13.2}$$

where K is the kinetic energy, U the internal strain energy, W the external work, and H the heat input to the system.

29 For an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), the above relation simplifies to:

$$\boxed{W_e = U} \tag{13.3}$$

13.1.1 Internal Strain Energy

30 The **strain energy density** of an arbitrary material is defined as, Fig. 13.1

$$\boxed{U_0 \stackrel{\text{def}}{=} \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma}{:}d\boldsymbol{\varepsilon}} \tag{13.4}$$

31 The **complementary strain energy** density is defined

$$\boxed{U_0^* \stackrel{\text{def}}{=} \int_0^{\boldsymbol{\sigma}} \boldsymbol{\varepsilon}{:}d\boldsymbol{\sigma}} \tag{13.5}$$

32 **The strain energy** itself is equal to

$$U \stackrel{\text{def}}{=} \int_\Omega U_0 d\Omega \tag{13.6}$$

$$U^* \stackrel{\text{def}}{=} \int_\Omega U_0^* d\Omega \tag{13.7}$$

33 To obtain a general form of the internal strain energy, we first define a stress-strain relationship accounting for both initial strains and stresses

$$\boldsymbol{\sigma} = \mathbf{D}{:}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \boldsymbol{\sigma}_0 \tag{13.8}$$

where $\mathbf{D}$ is the constitutive matrix (Hooke's Law); $\boldsymbol{\epsilon}$ is the strain vector due to the displacements $\mathbf{u}$; $\boldsymbol{\epsilon}_0$ is the initial strain vector; $\boldsymbol{\sigma}_0$ is the initial stress vector; and $\boldsymbol{\sigma}$ is the stress vector.

34 The initial strains and stresses are the result of conditions such as heating or cooling of a system or the presence of pore pressures in a system.

35 The strain energy U for a linear elastic system is obtained by substituting

$$\boldsymbol{\sigma} = \mathbf{D}{:}\boldsymbol{\varepsilon} \tag{13.9}$$

with Eq. 13.4 and 13.8

$$\boxed{U = \frac{1}{2}\int_\Omega \boldsymbol{\varepsilon}^T{:}\mathbf{D}{:}\boldsymbol{\varepsilon}\, d\Omega - \int_\Omega \boldsymbol{\varepsilon}^T{:}\mathbf{D}{:}\boldsymbol{\varepsilon}_0 d\Omega + \int_\Omega \boldsymbol{\varepsilon}^T{:}\boldsymbol{\sigma}_0 d\Omega} \tag{13.10}$$

where Ω is the volume of the system.

36 Considering *uniaxial stresses*, in the absence of initial strains and stresses, and **for linear elastic systems**, Eq. 13.10 reduces to

$$\boxed{U = \frac{1}{2}\int_\Omega \varepsilon \underbrace{E\varepsilon}_{\sigma} d\Omega} \tag{13.11}$$

37 When this relation is applied to various one dimensional structural elements it leads to

Axial Members:

$$\left.\begin{array}{l} U = \displaystyle\int_\Omega \frac{\varepsilon\sigma}{2} d\Omega \\ \sigma = \frac{P}{A} \\ \varepsilon = \frac{P}{AE} \\ d\Omega = Adx \end{array}\right\} \boxed{U = \frac{1}{2}\int_0^L \frac{P^2}{AE} dx} \tag{13.12}$$

Flexural Members:

$$\left.\begin{array}{l} U = \frac{1}{2}\displaystyle\int_\Omega \varepsilon \underbrace{E\varepsilon}_{\sigma} \\ \sigma_x = \frac{M_z y}{I_z} \\ \varepsilon = \frac{M_z y}{EI_z} \\ d\Omega = dAdx \\ \displaystyle\int_A y^2 dA = I_z \end{array}\right\} \boxed{U = \frac{1}{2}\int_0^L \frac{M^2}{EI_z} dx} \tag{13.13}$$

13.1.2 External Work

38 External work W performed by the applied loads on an arbitrary system is defined as

$$\boxed{W_e \stackrel{\text{def}}{=} \int_\Omega \mathbf{u}^T \cdot \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}^T \cdot \hat{\mathbf{t}} d\Gamma} \tag{13.14}$$

where $\mathbf{b}$ is the body force vector; $\hat{\mathbf{t}}$ is the applied surface traction vector; and Γ_t is that portion of the boundary where $\hat{\mathbf{t}}$ is applied, and $\mathbf{u}$ is the displacement.

39 For point loads and moments, the external work is

$$\boxed{W_e = \int_0^{\Delta_f} P d\Delta + \int_0^{\theta_f} M d\theta} \tag{13.15}$$

40 For **linear elastic systems**, $(P = K\Delta)$ we have for point loads

$$\left.\begin{array}{rcl} P & = & K\Delta \\ W_e & = & \int_0^{\Delta_f} P d\Delta \end{array}\right\} W_e = K\int_0^{\Delta_f} \Delta d\Delta = \frac{1}{2}K\Delta_f^2 \tag{13.16}$$

When this last equation is combined with $P_f = K\Delta_f$ we obtain

$$\boxed{W_e = \frac{1}{2}P_f\Delta_f} \tag{13.17}$$

where K is the **stiffness** of the structure.

41 Similarly for an applied moment we have

$$\boxed{W_e = \frac{1}{2}M_f\theta_f} \tag{13.18}$$

13.1.3 Virtual Work

42 We *define* the virtual work done by the load on a body during a small, admissible (continuous and satisfying the boundary conditions) change in displacements.

$$\text{Internal Virtual Work} \quad \delta W_i \stackrel{\text{def}}{=} -\int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} d\Omega \tag{13.19}$$

$$\text{External Virtual Work} \quad \delta W_e \stackrel{\text{def}}{=} \int_{\Gamma_t} \hat{\mathbf{t}} \cdot \delta\mathbf{u} d\Gamma + \int_\Omega \mathbf{b} \cdot \delta\mathbf{u} d\Omega \tag{13.20}$$

where all the terms have been previously defined and $\mathbf{b}$ is the body force vector.

43 Note that the virtual quantity (displacement or force) is one that we will *approximate/guess* as long as it meets some admissibility requirements.

13.1.3.1 Internal Virtual Work

44 Next we shall derive a displacement based expression of δU for each type of one dimensional structural member. It should be noted that the Virtual Force method would yield analogous ones but based on forces rather than displacements.

45 Two sets of solutions will be given, the first one is independent of the material stress strain relations, and the other assumes a linear elastic stress strain relation.

Elastic Systems In this set of formulation, we derive expressions of the virtual strain energies which are independent of the material constitutive laws. Thus δU will be left in terms of forces and displacements.

Axial Members:

$$\left.\begin{array}{l}\delta U = \int_0^L \sigma\delta\varepsilon d\Omega \\ d\Omega = Adx\end{array}\right\} \boxed{\delta U = A\int_0^L \sigma\delta\varepsilon dx} \tag{13.21}$$

Flexural Members:

$$\left.\begin{array}{l}\delta U = \int \sigma_x\delta\varepsilon_x d\Omega \\ M = \int_A \sigma_x y dA \Rightarrow \frac{M}{y} = \int_A \sigma_x dA \\ \delta\phi = \frac{\delta\varepsilon}{y} \Rightarrow \delta\phi y = \delta\varepsilon \\ d\Omega = \int_0^L \int_A dAdx\end{array}\right\} \boxed{\delta U = \int_0^L M\delta\phi dx} \tag{13.22}$$

Linear Elastic Systems Should we have a linear elastic material ($\sigma = E\varepsilon$) then:

Axial Members:

$$\left.\begin{array}{l}\delta U = \int \sigma\delta\varepsilon d\Omega \\ \sigma_x = E\varepsilon_x = E\frac{du}{dx} \\ \delta\varepsilon = \frac{d(\delta u)}{dx} \\ d\Omega = Adx\end{array}\right\} \boxed{\delta U = \int_0^L \underbrace{E\frac{du}{dx}}_{\text{``}\sigma\text{''}} \underbrace{\frac{d(\delta u)}{dx}}_{\text{``}\delta\varepsilon\text{''}} \underbrace{Adx}_{d\Omega}} \tag{13.23}$$

Flexural Members:

$$\left.\begin{array}{l}\delta U = \int \sigma_x\delta\varepsilon_x d\Omega \\ \left.\begin{array}{l}\sigma_x = \frac{My}{I_z} \\ M = \frac{d^2v}{dx^2}EI_z\end{array}\right\} \sigma_x = \underbrace{\frac{d^2v}{dx^2}}_{\kappa} Ey \\ \delta\varepsilon_x = \frac{\delta\sigma_x}{E} = \frac{d^2(\delta v)}{dx^2}y \\ d\Omega = dAdx\end{array}\right\} \delta U = \int_0^L \int_A \frac{d^2v}{dx^2}Ey\frac{d^2(\delta v)}{dx^2}ydAdx \tag{13.24}$$

or:

$$\left.\begin{array}{l}\text{Eq. 13.24} \\ \int_A y^2 dA = I_z\end{array}\right\} \boxed{\delta U = \int_0^L \underbrace{EI_z\frac{d^2v}{dx^2}}_{\text{``}\sigma\text{''}} \underbrace{\frac{d^2(\delta v)}{dx^2}}_{\text{``}\delta\varepsilon\text{''}} dx} \tag{13.25}$$

13.1.3.2 External Virtual Work δW

46 For concentrated forces (and moments):

$$\boxed{\delta W = \int \delta\Delta q dx + \sum_i (\delta\Delta_i) P_i + \sum_i (\delta\theta_i) M_i} \tag{13.26}$$

where: $\delta\Delta_i$ = virtual displacement.

13.1.4 Complementary Virtual Work

47 We define the complementary virtual work done by the load on a body during a small, admissible (continuous and satisfying the boundary conditions) change in displacements.

$$\text{Complementary Internal Virtual Work} \quad \delta W_i^* \stackrel{\text{def}}{=} -\int_\Omega \boldsymbol{\varepsilon}{:}\delta\boldsymbol{\sigma} d\Omega \tag{13.27}$$

$$\text{Complementary External Virtual Work} \quad \delta W_e^* \stackrel{\text{def}}{=} \int_{\Gamma_u} \hat{\mathbf{u}}{\cdot}\delta\mathbf{t} d\Gamma \tag{13.28}$$

13.1.5 Potential Energy

48 The potential of external work W in an arbitrary system is defined as

$$\boxed{\mathcal{W}_e \stackrel{\text{def}}{=} \int_\Omega \mathbf{u}^T{\cdot}\mathbf{b}\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{u}^T{\cdot}\hat{\mathbf{t}}\mathrm{d}\Gamma + \mathbf{u}{\cdot}\mathbf{P}} \tag{13.29}$$

where $\mathbf{u}$ are the displacements, $\mathbf{b}$ is the body force vector; $\hat{\mathbf{t}}$ is the applied surface traction vector; Γ_t is that portion of the boundary where $\hat{\mathbf{t}}$ is applied, and $\mathbf{P}$ are the applied nodal forces.

49 Note that the potential of the external work ($\mathcal{W}$) is different from the external work itself (W)

50 The potential energy of a system is defined as

$$\Pi \stackrel{\text{def}}{=} U - \mathcal{W}_e \tag{13.30}$$

$$= \int_\Omega U_0 d\Omega - \left(\int_\Omega \mathbf{u}{\cdot}\mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}{\cdot}\hat{\mathbf{t}} d\Gamma + \mathbf{u}{\cdot}\mathbf{P} \right) \tag{13.31}$$

51 Note that in the potential the full load is always acting, and through the displacements of its points of application it does work but loses an equivalent amount of potential, this explains the negative sign.

13.2 Principle of Virtual Work and Complementary Virtual Work

52 The principles of Virtual Work and Complementary Virtual Work relate *force* systems which satisfy the requirements of *equilibrium*, and *deformation* systems which satisfy the

requirement of *compatibility*:

1. In any application the force system could either be the actual set of *external* loads $d\mathbf{p}$ or some *virtual* force system which happens to satisfy the condition of *equilibrium* $\delta\overline{\mathbf{p}}$. This set of external forces will induce internal actual forces $d\boldsymbol{\sigma}$ or internal hypothetical forces $\delta\overline{\boldsymbol{\sigma}}$ compatible with the externally applied load.
2. Similarly the deformation could consist of either the actual joint deflections $d\mathbf{u}$ and compatible internal deformations $d\boldsymbol{\varepsilon}$ of the structure, or some *hypothetical* external and internal deformation $\delta\overline{\mathbf{u}}$ and $\delta\overline{\boldsymbol{\varepsilon}}$ which satisfy the conditions of *compatibility*.

53 Thus we may have 2 possible combinations, Table 13.1: where: d corresponds to the

	Force		Deformation		Formulation
	External	Internal	External	Internal	
1	$\delta\overline{\mathbf{p}}$	$\delta\overline{\boldsymbol{\sigma}}$	$d\mathbf{u}$	$d\boldsymbol{\varepsilon}$	δU^*
2	$d\mathbf{p}$	$d\boldsymbol{\sigma}$	$\delta\overline{\mathbf{u}}$	$\delta\overline{\boldsymbol{\varepsilon}}$	δU

Table 13.1: Possible Combinations of Real and Hypothetical Formulations

actual, and δ (with an overbar) to the hypothetical values.

13.2.1 Principle of Virtual Work

54 Derivation of the principle of virtual work starts with the assumption of that forces are in equilibrium and satisfaction of the static boundary conditions.

55 The Equation of equilibrium (Eq. 6.26) which is rewritten as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \tag{13.32}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + b_y = 0 \tag{13.33}$$

where $\mathbf{b}$ representing the body force. In matrix form, this can be rewritten as

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{array} \right\} + \left\{ \begin{array}{c} b_x \\ b_y \end{array} \right\} = 0 \tag{13.34}$$

or

$$\boxed{\mathbf{L}^T\boldsymbol{\sigma} + \mathbf{b} = 0} \tag{13.35}$$

Note that this equation can be generalized to 3D.

56 The surface Γ of the solid can be decomposed into two parts Γ_t and Γ_u where tractions and displacements are respectively specified.

$$\Gamma = \Gamma_t + \Gamma_u \tag{13.36-a}$$

$$\mathbf{t} = \hat{\mathbf{t}} \quad \text{on } \Gamma_t \quad \text{Natural B.C.} \tag{13.36-b}$$

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Gamma_u \quad \text{Essential B.C.} \tag{13.36-c}$$

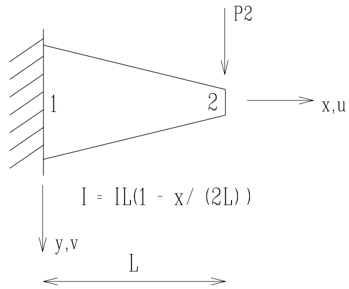


Figure 13.2: Tapered Cantilivered Beam Analysed by the Vitual Displacement Method

Equations 13.35 and 13.36-b constitute a statically admissible stress field.

57 The *principle of virtual work* (or more specifically of virtual displacement) can be stated as

> *A deformable system is in equilibrium if the sum of the external virtual work and the internal virtual work is zero for virtual displacements* $\delta\mathbf{u}$ *which are kinematically admissible.*

The major governing equations are summarized

$$\underbrace{\int_\Omega \delta\boldsymbol{\varepsilon}^T{:}\boldsymbol{\sigma}d\Omega}_{-\delta W_i} - \underbrace{\int_\Omega \delta\mathbf{u}^T{\cdot}\mathbf{b}d\Omega - \int_{\Gamma_t}\delta\mathbf{u}^T{\cdot}\hat{\mathbf{t}}d\Gamma}_{-\delta W_e} = 0 \quad (13.37)$$

$$\delta\boldsymbol{\varepsilon} = \mathbf{L}{:}\delta\mathbf{u} \quad \text{in} \quad \Omega \quad (13.38)$$

$$\delta\mathbf{u} = 0 \quad \text{on} \quad \Gamma_u \quad (13.39)$$

58 Note that the principle is independent of material properties, and that the primary unknowns are the displacements.

■ Example 13-1: Tapered Cantiliver Beam, Virtual Displacement

Analyse the problem shown in Fig. 13.2, by the virtual displacement method.
Solution:

1. For this flexural problem, we must apply the expression of the virtual internal strain energy as derived for beams in Eq. 13.25. And the solutions must be expressed in terms of the displacements which in turn must satisfy the essential boundary conditions.

   The *approximate* solutions proposed to this problem are

$$v = \left(1-\cos\frac{\pi x}{2l}\right)v_2 \quad (13.40)$$

$$v = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]v_2 \quad (13.41)$$

2. These equations do indeed satisfy the essential B.C. (i.e kinematic), but for them to also satisfy equilibrium they must satisfy the principle of virtual work.

3. Using the virtual displacement method we evaluate the displacements v_2 from three different combination of virtual and actual displacement:

Solution	Total	Virtual
1	Eqn. 13.40	Eqn. 13.41
2	Eqn. 13.40	Eqn. 13.40
3	Eqn. 13.41	Eqn. 13.41

Where actual and virtual values for the two assumed displacement fields are given below.

	Trigonometric (Eqn. 13.40)	Polynomial (Eqn. 13.41)
v	$\left(1-\cos\frac{\pi x}{2l}\right)v_2$	$\left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right]v_2$
δv	$\left(1-\cos\frac{\pi x}{2l}\right)\delta v_2$	$\left[3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3\right]\delta v_2$
v''	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2l}v_2$	$\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2$
$\delta v''$	$\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2l}\delta v_2$	$\left[\frac{6}{L^2}-\frac{12x}{L^3}\right]\delta v_2$

$$\delta U = \int_0^L \delta v'' E I_z v'' dx \qquad (13.42)$$
$$\delta W = P_2 \delta v_2 \qquad (13.43)$$

Solution 1:

$$\begin{aligned}\delta U &= \int_0^L \frac{\pi^2}{4L^2}\cos\left(\frac{\pi x}{2l}\right)v_2\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)\delta v_2 E I_1\left(1-\frac{x}{2L}\right)dx \\ &= \frac{3\pi E I_1}{2L^3}\left[1-\frac{10}{\pi}+\frac{16}{\pi^2}\right]v_2\delta v_2 \\ &= P_2\delta v_2 \end{aligned} \qquad (13.44)$$

which yields:

$$v_2 = \frac{P_2 L^3}{2.648 E I_1} \qquad (13.45)$$

Solution 2:

$$\begin{aligned}\delta U &= \int_0^L \frac{\pi^4}{16L^4}\cos^2\left(\frac{\pi x}{2l}\right)v_2\delta v_2 E I_1\left(1-\frac{x}{2l}\right)dx \\ &= \frac{\pi^4 E I_1}{32L^3}\left(\frac{3}{4}+\frac{1}{\pi^2}\right)v_2\delta v_2 \\ &= P_2\delta v_2 \end{aligned} \qquad (13.46)$$

which yields:

$$v_2 = \frac{P_2 L^3}{2.57 E I_1} \qquad (13.47)$$

Solution 3:

$$\begin{aligned}\delta U &= \int_0^L \left(\frac{6}{L^2}-\frac{12x}{L^3}\right)^2\left(1-\frac{x}{2l}\right)EI_1\delta v_2 v_2 dx\\ &= \frac{9EI}{L^3}v_2\delta v_2\\ &= P_2\delta v_2 \end{aligned} \tag{13.48}$$

which yields:

$$v_2=\frac{P_2L^3}{9EI} \tag{13.49}$$

■

13.2.2 Principle of Complementary Virtual Work

59 Derivation of the principle of complementary virtual work starts from the assumption of a *kinematicaly admissible displacements* and satisfaction of the essential boundary conditions.

60 Whereas we have previously used the vector notation for the principle of virtual work, we will now use the tensor notation for this derivation.

61 The kinematic condition (strain-displacement):

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{13.50}$$

62 The essential boundary conditions are expressed as

$$u_i=\hat{u} \text{ on } \Gamma_u \tag{13.51}$$

63 The *principle of virtual complementary work* (or more specifically of virtual force) which can be stated as

> A deformable system satisfies all kinematical requirements if the sum of the external complementary virtual work and the internal complementary virtual work is zero for all statically admissible virtual stresses $\delta\sigma_{ij}$.

The major governing equations are summarized

$$\underbrace{\int_\Omega \varepsilon_{ij}\delta\sigma_{ij}d\Omega}_{-\delta W_i^*}-\underbrace{\int_{\Gamma_u}\hat{u}_i\delta t_i d\Gamma}_{\delta W_e^*} = 0 \tag{13.52}$$

$$\delta\sigma_{ij,j}=0 \text{ in } \Omega \tag{13.53}$$

$$\delta t_i=0 \text{ on } \Gamma_t \tag{13.54}$$

64 Note that the principle is independent of material properties, and that the primary unknowns are the stresses.

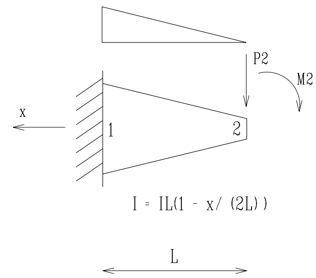


Figure 13.3: Tapered Cantilevered Beam Analysed by the Virtual Force Method

65 Expressions for the complimentary virtual work in beams are given in Table 13.3

■ Example 13-2: Tapered Cantilivered Beam; Virtual Force

"Exact" solution of previous problem using principle of virtual work with virtual force.

$$\boxed{\underbrace{\int_0^L \delta M \frac{M}{EI_z} dx}_{\text{Internal}} = \underbrace{\delta P \Delta}_{\text{External}}} \tag{13.55}$$

Note: This represents the internal virtual strain energy and external virtual work written in terms of *forces* and should be compared with the similar expression derived in Eq. 13.25 written in terms of displacements:

$$\delta U^* = \int_0^L \underbrace{EI_z \frac{d^2 v}{dx^2}}_{\sigma} \underbrace{\frac{d^2(\delta v)}{dx^2}}_{\delta\varepsilon} dx \tag{13.56}$$

Here: δM and δP are the virtual forces, and $\frac{M}{EI_z}$ and Δ are the actual displacements. See Fig. 13.3 If $\delta P = 1$, then $\delta M = x$ and $M = P_2 x$ or:

$$\begin{aligned} (1)\Delta &= \int_0^L x \frac{P_2 x}{EI_1(.5 + \frac{x}{L})} dx \\ &= \frac{P_2}{EI_1} \int_0^L \frac{x^2}{\frac{L+x}{2l}} dx \\ &= \frac{P_2 2L}{EI_1} \int_0^L \frac{x^2}{L+x} dx \end{aligned} \tag{13.57}$$

From *Mathematica* we note that:

$$\int_0^0 \frac{x^2}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2 \ln(a+bx)\right] \tag{13.58}$$

Thus substituting $a = L$ and $b = 1$ into Eqn. 13.58, we obtain:

$$\Delta = \frac{2P_2L}{EI_1}\left[\frac{1}{2}(L+x)^2 - 2L(L+x) + L^2 \ln(L+x)\right] |_0^L$$

$$
\begin{aligned}
&= \frac{2P_2L}{EI_1}\left[2L^2-4L^2+L^2\ln 2L-\frac{L^2}{2}+2L^2+L^2\log L\right] \\
&= \frac{2P_2L}{EI_1}\left[L^2(\ln 2-\frac{1}{2})\right] \\
&= \frac{P_2L^3}{2.5887EI_1} \qquad (13.59)
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\theta &= \int_0^L \frac{M(1)}{EI_1\left(.5+\frac{x}{L}\right)} = \frac{2ML}{EI_1}\int_0^L\frac{1}{L+x} = \frac{2ML}{EI_1}\ln(L+x)\mid_0^L \\
&= \frac{2ML}{EI_1}(\ln 2L-\ln L) = \frac{2ML}{EI_1}\ln 2 = \frac{ML}{.721EI_1} \qquad (13.60)
\end{aligned}
$$

■

13.3 Potential Energy

13.3.1 Derivation

66 From section ??, if U_0 is a potential function, we take its differential

$$dU_0 = \frac{\partial U_0}{\partial \varepsilon_{ij}} d\varepsilon_{ij} \qquad (13.61\text{-a})$$

$$dU_0^* = \frac{\partial U_0}{\partial \sigma_{ij}} d\sigma_{ij} \qquad (13.61\text{-b})$$

67 However, from Eq. 13.4

$$
\begin{aligned}
U_0 &= \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \qquad (13.62\text{-a}) \\
dU_0 &= \sigma_{ij} d\varepsilon_{ij} \qquad (13.62\text{-b})
\end{aligned}
$$

thus,

$$
\boxed{
\begin{aligned}
\frac{\partial U_0}{\partial \varepsilon_{ij}} &= \sigma_{ij} \qquad (13.63) \\
\frac{\partial U_0^*}{\partial \sigma_{ij}} &= \varepsilon_{ij} \qquad (13.64)
\end{aligned}
}
$$

68 We now define the variation of the strain energy density at a point[1]

$$\delta U_0 = \frac{\partial U}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} = \sigma_{ij}\delta\varepsilon_{ij} \qquad (13.65)$$

69 Applying the principle of virtual work, Eq. 13.37, it can be shown that

[1]Note that the variation of strain energy density is, $\delta U_0 = \sigma_{ij}\delta\varepsilon_{ij}$, and the variation of the strain energy itself is $\delta U = \int_\Omega \delta U_0 d\Omega$.

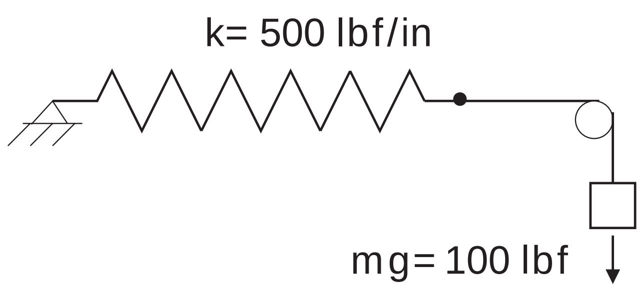


Figure 13.4: Single DOF Example for Potential Energy

$$\begin{aligned} \delta\Pi &= 0 && (13.66)\\ \Pi &\stackrel{\rm def}{=} U - \mathcal{W}_e && (13.67)\\ &= \int_\Omega U_0 d\Omega - \left(\int_\Omega \mathbf{u}\cdot\mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}\cdot\hat{\mathbf{t}} d\Gamma + \mathbf{u}\cdot\mathbf{P}\right) && (13.68) \end{aligned}$$

70 We have thus derived the principle of stationary value of the potential energy:

> Of all kinematically admissible deformations (displacements satisfying the essential boundary conditions), the actual deformations (those which correspond to stresses which satisfy equilibrium) are the ones for which the total potential energy assumes a stationary value.

71 For problems involving multiple degrees of freedom, it results from calculus that

$$\delta\Pi = \frac{\partial\Pi}{\partial\Delta_1}\delta\Delta_1 + \frac{\partial\Pi}{\partial\Delta_2}\delta\Delta_2 + \ldots + \frac{\partial\Pi}{\partial\Delta_n}\delta\Delta_n \quad (13.69)$$

72 It can be shown that the minimum potential energy yields a *lower bound* prediction of displacements.

73 As an illustrative example (adapted from Willam, 1987), let us consider the single dof system shown in Fig. 13.4. The strain energy U and potential of the external work $\mathcal{W}$ are given by

$$U = \frac{1}{2}u(Ku) = 250u^2 \quad (13.70\text{-a})$$

$$\mathcal{W}_e = mgu = 100u \quad (13.70\text{-b})$$

Thus the total potential energy is given by

$$\Pi = 250u^2 - 100u \quad (13.71)$$

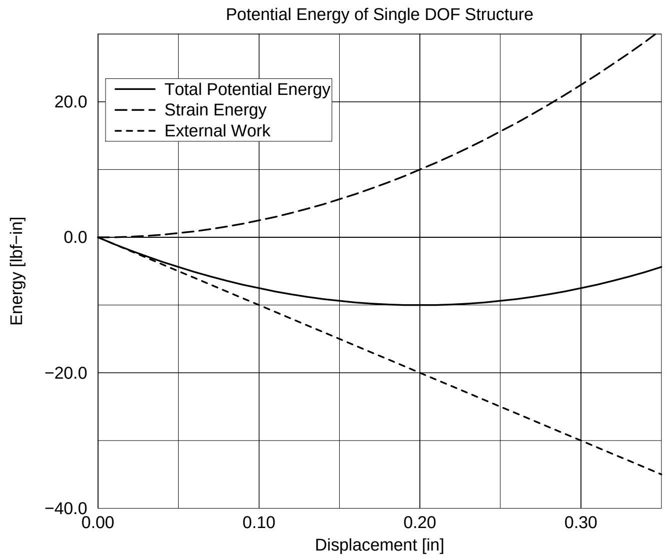


Figure 13.5: Graphical Representation of the Potential Energy

and will be stationary for

$$\partial\Pi = \frac{d\Pi}{du} = 0 \Rightarrow 500u - 100 = 0 \Rightarrow \boxed{u = 0.2 \text{ in}} \tag{13.72}$$

Substituting, this would yield

$$\begin{array}{rcccl} U &=& 250(0.2)^2 &=& 10 \text{ lbf-in} \\ \mathcal{W} &=& 100(0.2) &=& 20 \text{ lbf-in} \\ \Pi &=& 10 - 20 &=& -10 \text{ lbf-in} \end{array} \tag{13.73}$$

Fig. 13.5 illustrates the two components of the potential energy.

13.3.2 Rayleigh-Ritz Method

74 Continuous systems have infinite number of degrees of freedom, those are the displacements at every point within the structure. Their behavior can be described by the Euler Equation, or the partial differential equation of equilibrium. However, only the simplest problems have an exact solution which (satisfies equilibrium, and the boundary conditions).

75 An *approximate* method of solution is the Rayleigh-Ritz method which is based on the principle of virtual displacements. In this method we *approximate* the displacement field by a function

$$u_1 \approx \sum_{i=1}^{n} c_i^1 \phi_i^1 + \phi_0^1 \tag{13.74-a}$$

$$u_2 \approx \sum_{i=1}^{n} c_i^2 \phi_i^2 + \phi_0^2 \tag{13.74-b}$$

$$u_3 \approx \sum_{i=1}^{n} c_i^3 \phi_i^3 + \phi_0^3 \tag{13.74-c}$$

where c_i^j denote undetermined parameters, and ϕ are appropriate functions of positions.

76 ϕ should satisfy three conditions

1. Be continuous.
2. Must be *admissible*, i.e. satisfy the essential boundary conditions (the natural boundary conditions are included already in the variational statement. However, if ϕ also satisfy them, then better results are achieved).
3. Must be independent and complete (which means that the exact displacement and their derivatives that appear in Π can be arbitrary matched if enough terms are used. Furthermore, lowest order terms must also be included).

In general ϕ is a polynomial or trigonometric function.

77 We determine the parameters c_i^j by requiring that the principle of virtual work for arbitrary variations δc_i^j. or

$$\delta\Pi(u_1, u_2, u_3) = \sum_{i=1}^{n} \left(\frac{\partial \Pi}{\partial c_i^1} \delta c_i^1 + \frac{\partial \Pi}{\partial c_i^2} \delta c_i^2 + \frac{\partial \Pi}{\partial c_i^3} \delta c_i^3 \right) = 0 \tag{13.75}$$

for arbitrary and independent variations of δc_i^1, δc_i^2, and δc_i^3, thus it follows that

$$\boxed{\frac{\partial \Pi}{\partial c_i^j} = 0 \qquad i = 1, 2, \cdots, n; j = 1, 2, 3} \tag{13.76}$$

Thus we obtain a total of $3n$ linearly independent simultaneous equations. From these displacements, we can then determine strains and stresses (or internal forces). Hence we have replaced a problem with an infinite number of d.o.f by one with a finite number.

78 Some general observations

1. c_i^j can either be a set of coefficients with no physical meanings, or variables associated with nodal generalized displacements (such as deflection or displacement).
2. If the coordinate functions ϕ satisfy the above requirements, then the solution converges to the exact one if n increases.
3. For increasing values of n, the previously computed coefficients remain unchanged.
4. Since the strains are computed from the approximate displacements, strains and stresses are generally less accurate than the displacements.
5. The equilibrium equations of the problem are satisfied only in the energy sense $\delta\Pi = 0$ and not in the differential equation sense (i.e. in the weak form but not in the strong one). Therefore the displacements obtained from the approximation generally do not satisfy the equations of equilibrium.

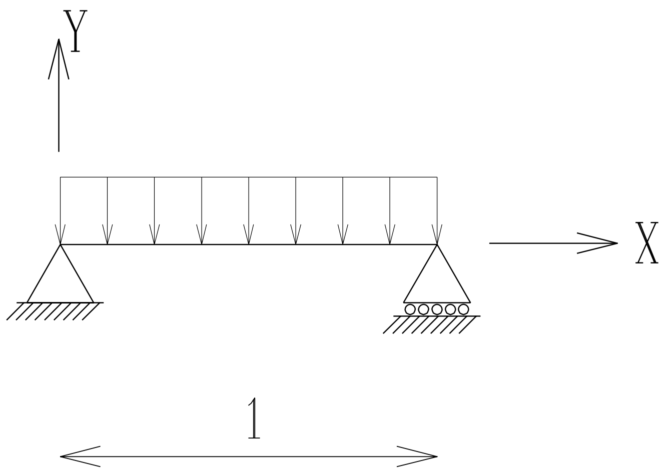


Figure 13.6: Uniformly Loaded Simply Supported Beam Analyzed by the Rayleigh-Ritz Method

6. Since the continuous system is approximated by a finite number of coordinates (or d.o.f.), then the approximate system is stiffer than the actual one, and the displacements obtained from the Ritz method converge to the exact ones from below.

■ **Example 13-3: Uniformly Loaded Simply Supported Beam; Polynomial Approximation**

For the uniformly loaded beam shown in Fig. 13.6
let us assume a solution given by the following infinite series:

$$v = a_1 x(L-x) + a_2 x^2 (L-x)^2 + \dots \tag{13.77}$$

for this particular solution, let us retain only the first term:

$$v = a_1 x(L-x) \tag{13.78}$$

We observe that:

1. Contrarily to the previous example problem the geometric B.C. are immediately satisfied at both $x = 0$ and $x = L$.
2. We can keep v in terms of a_1 and take $\frac{\partial \Pi}{\partial a_1} = 0$ (If we had left v in terms of a_1 and a_2 we should then take both $\frac{\partial \Pi}{\partial a_1} = 0$, and $\frac{\partial \Pi}{\partial a_2} = 0$).
3. Or we can solve for a_1 in terms of $v_{\max}(@x = \frac{L}{2})$ and take $\frac{\partial \Pi}{\partial v_{\max}} = 0$.

$$\Pi = U - \mathcal{W} = \int_o^L \frac{M^2}{2EI_z} dx - \int_0^L w v(x) dx \tag{13.79}$$

Recalling that: $\frac{M}{EI_z} = \frac{d^2v}{dx^2}$, the above simplifies to:

$$\begin{aligned}
\Pi &= \int_0^L \left[\frac{EI_z}{2} \left(\frac{d^2v}{dx^2} \right)^2 - wv(x) \right] dx & (13.80)\\
&= \int_0^L \left[\frac{EI_z}{2} (-2a_1)^2 - a_1 wx(L-x) \right] dx \\
&= \frac{EI_z}{2} 4a_1^2 L - a_1 w \frac{L^3}{2} + a_1 w \frac{L^3}{3} \\
&= 2a_1^2 EI_z L - \frac{a_1 w L^3}{6} & (13.81)
\end{aligned}$$

If we now take $\frac{\partial \Pi}{\partial a_1} = 0$, we would obtain:

$$\begin{aligned}
4a_1 EI_z l - \frac{wL^3}{6} &= 0 \\
a_1 &= \frac{wL^2}{24EI_z} & (13.82)
\end{aligned}$$

Having solved the displacement field in terms of a_1, we now determine $v_{\max}$ at $\frac{L}{2}$:

$$\begin{aligned}
v &= \underbrace{\frac{wL^4}{24EI_z}}_{a_1} \left(\frac{x}{L} - \frac{x^2}{L^2} \right) \\
&= \frac{wL^4}{96EI_z} & (13.83)
\end{aligned}$$

This is to be compared with the exact value of $v_{\max}^{exact} = \frac{5}{384}\frac{wL^4}{EI_z} = \frac{wL^4}{76.8EI_z}$ which constitutes $\approx 17\%$ error.

Note: If two terms were retained, then we would have obtained: $a_1 = \frac{wL^2}{24EI_z}$ & $a_2 = \frac{w}{24EI_z}$ and $v_{\max}$ would be equal to $v_{\max}^{exact}$. (Why?) ■

13.4 Summary

79 Summary of Virtual work methods, Table 13.2.

	Starts with	Ends with	In terms of virtual	Solve for
Virtual Work U	KAD	SAS	Displacement/strains	Displacement
Complimentary Virtual Work U^*	SAS	KAD	Forces/Stresses	Displacement

KAD: Kinematically Admissible Dispacements
SAS: Statically Admissible Stresses

Table 13.2: Comparison of Virtual Work and Complementary Virtual Work

80 A summary of the various methods introduced in this chapter is shown in Fig. 13.7.

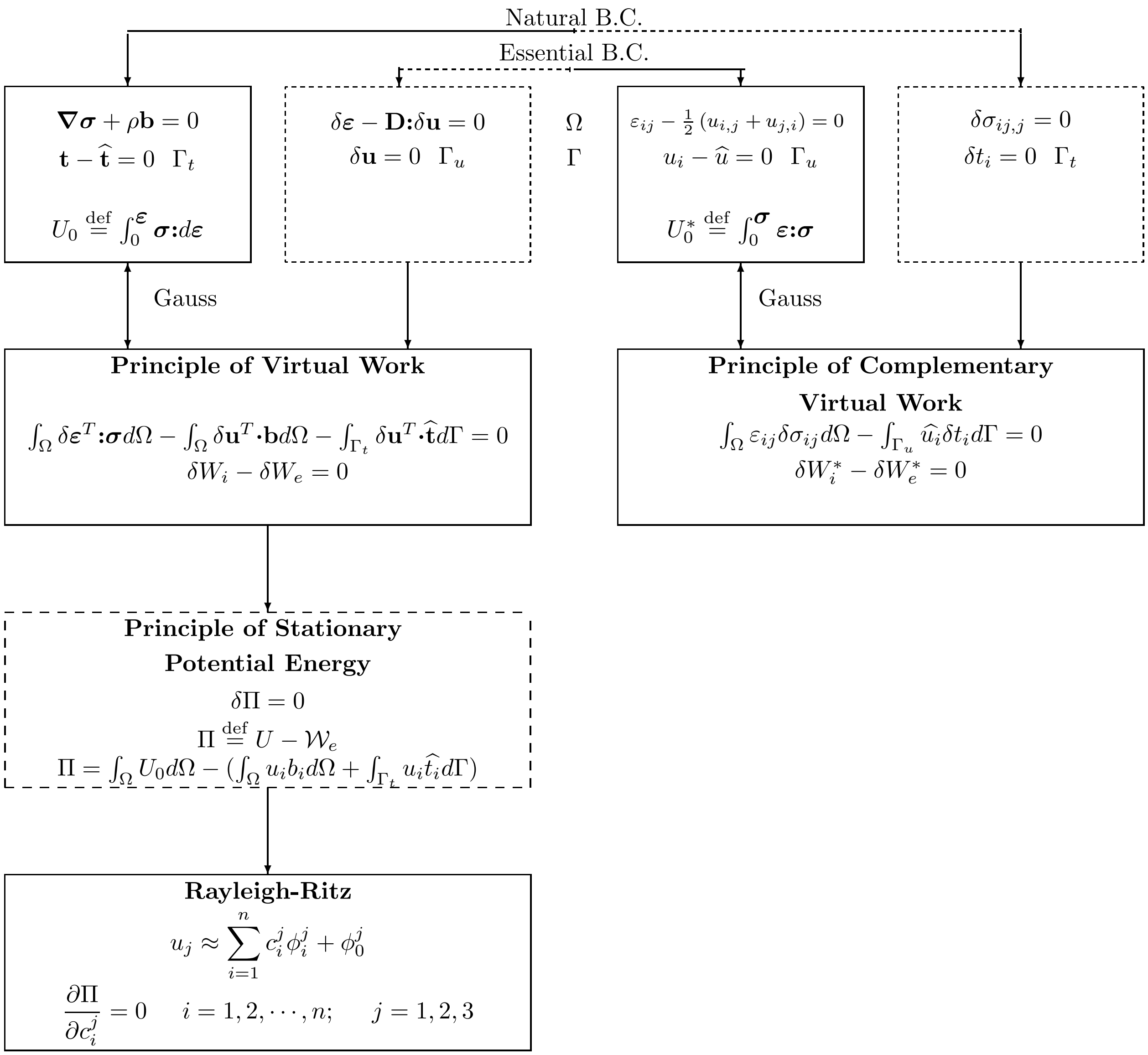


Figure 13.7: Summary of Variational Methods

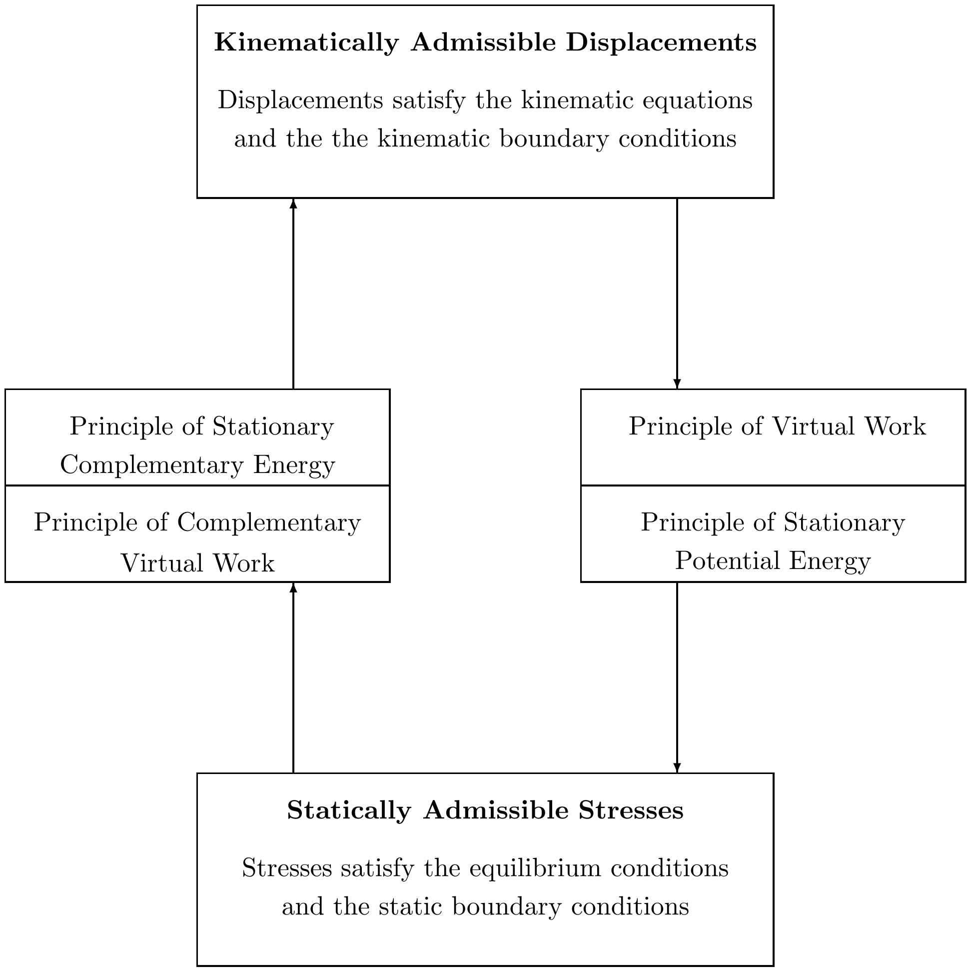


Figure 13.8: Duality of Variational Principles

81 The duality between the two variational principles is highlighted by Fig. 13.8, where beginning with kinematically admissible displacements, the principle of virtual work provides statically admissible solutions. Similarly, for statically admissible stresses, the principle of complementary virtual work leads to kinematically admissible solutions.

82 Finally, Table 13.3 summarizes some of the major equations associated with one dimensional rod elements.

	U	Virtual Displacement δU		Virtual Force δU^*	
		General	Linear	General	Linear
Axial	$\frac{1}{2}\int_0^L \frac{P^2}{AE}dx$	$\int_0^L \sigma\delta\varepsilon dx$	$\int_0^L \underbrace{E\frac{du}{dx}}_{\sigma}\underbrace{\frac{d(\delta u)}{dx}}_{\delta\varepsilon}\underbrace{Adx}_{d\Omega}$	$\int_0^L \delta\sigma\varepsilon dx$	$\int_0^L \underbrace{\delta P}_{\delta\sigma}\underbrace{\frac{P}{AE}}_{\varepsilon}dx$
Flexure	$\frac{1}{2}\int_0^L \frac{M^2}{EI_z}dx$	$\int_0^L M\delta\phi dx$	$\int_0^L \underbrace{EI_z\frac{d^2v}{dx^2}}_{\sigma}\underbrace{\frac{d^2(\delta v)}{dx^2}}_{\delta\varepsilon}dx$	$\int_0^L \delta M\phi dx$	$\int_0^L \underbrace{\delta M}_{\delta\sigma}\underbrace{\frac{M}{EI_z}}_{\varepsilon}dx$

	W	Virtual Displacement δW	Virtual Force δW^*
P	$\Sigma_i \frac{1}{2}P_i\Delta_i$	$\Sigma_i P_i\delta\Delta_i$	$\Sigma_i \delta P_i\Delta_i$
M	$\Sigma_i \frac{1}{2}M_i\theta_i$	$\Sigma_i M_i\delta\theta_i$	$\Sigma_i \delta M_i\theta_i$
w	$\int_0^L w(x)v(x)dx$	$\int_0^L w(x)\delta v(x)dx$	$\int_0^L \delta w(x)v(x)dx$

Table 13.3: Summary of Variational Terms Associated with One Dimensional Elements

Chapter 14

INELASTICITY (incomplete)

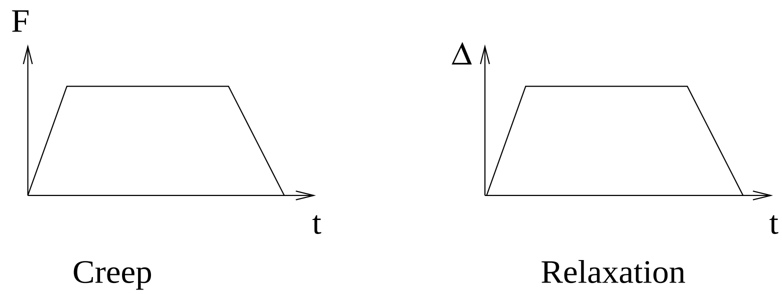


Figure 14.1: test

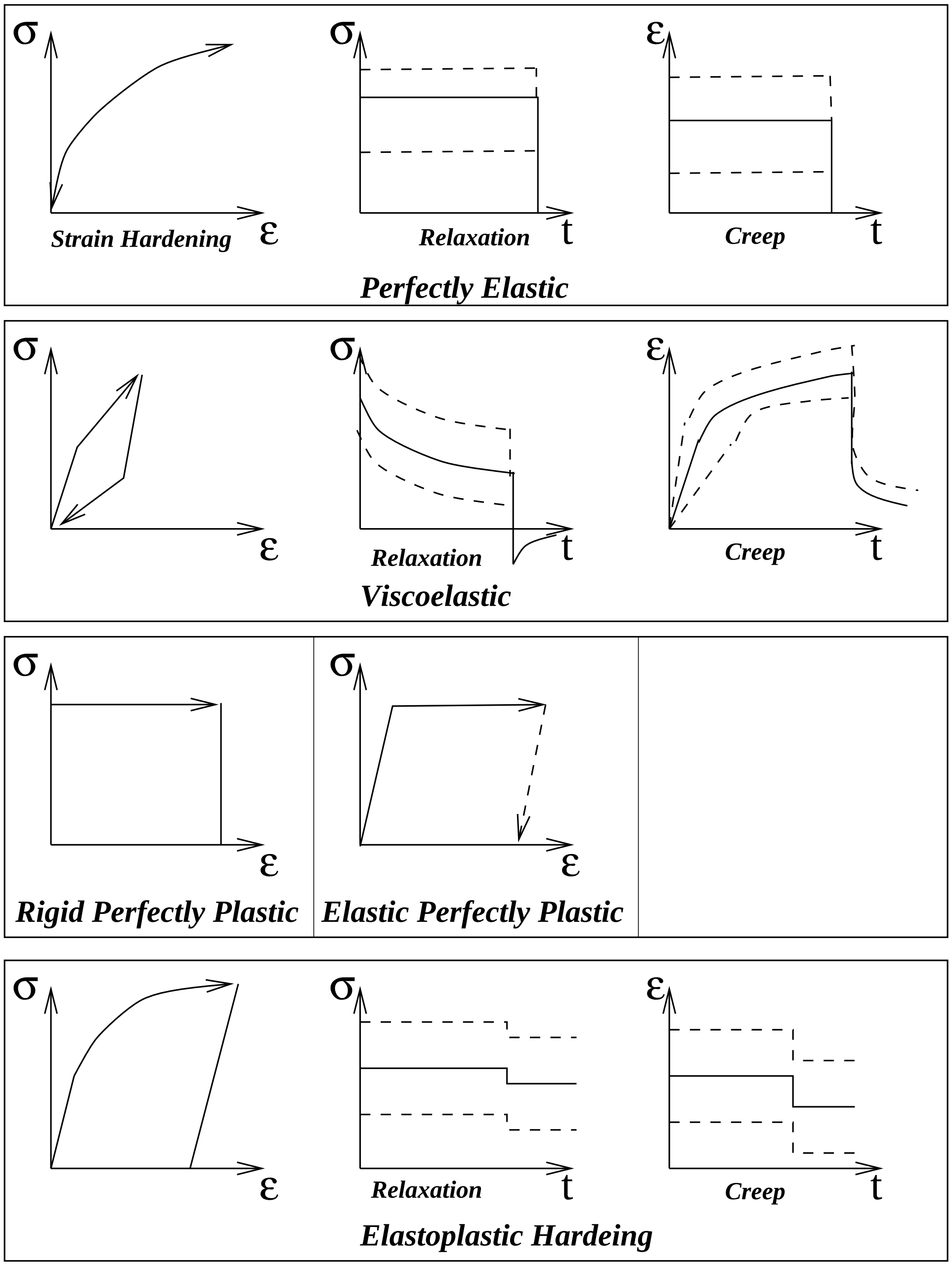


Figure 14.2: mod1

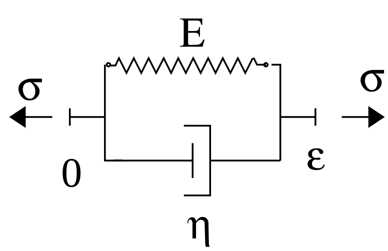


Figure 14.3: v-kv

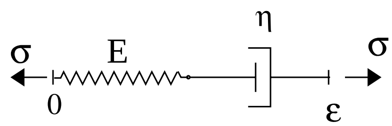


Figure 14.4: visfl

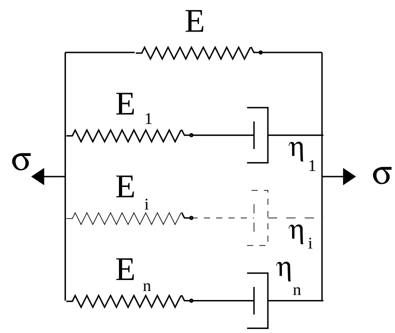


Figure 14.5: visfl

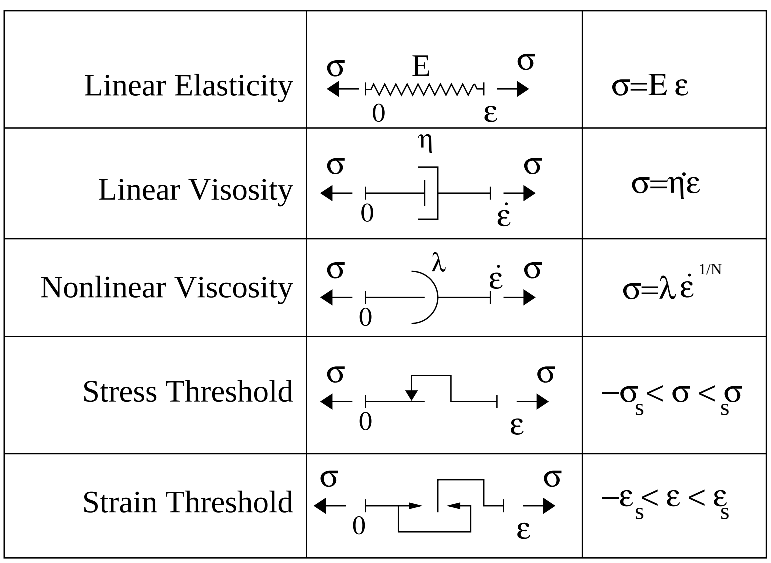


Figure 14.6: comp

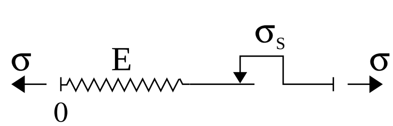


Figure 14.7: epp

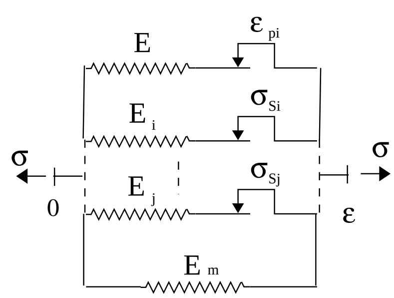
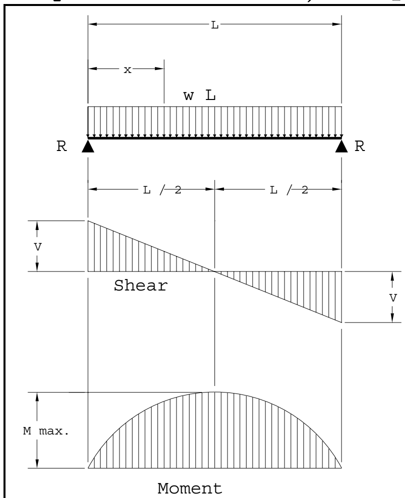


Figure 14.8: ehs

Appendix A

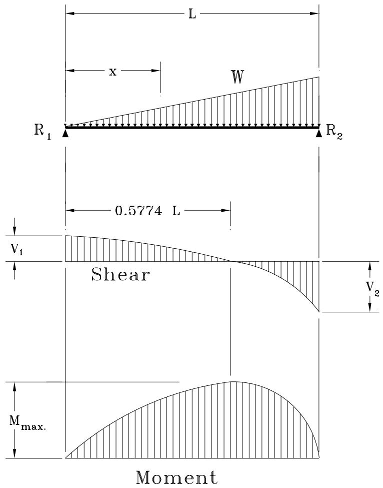
SHEAR, MOMENT and DEFLECTION DIAGRAMS for BEAMS

Adapted from [?] **1) Simple Beam; uniform Load**



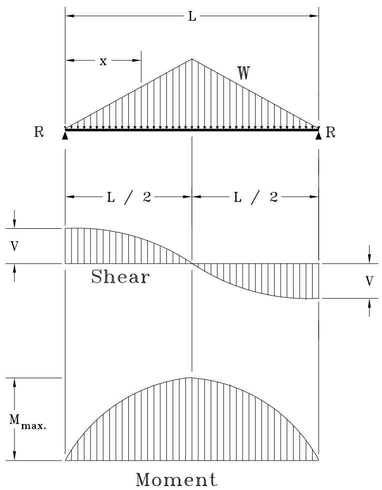
$$\begin{aligned} R &= V \\ V_x &= w\left(\frac{L}{2} - x\right) \\ \text{at center} \quad M_{max} &= \frac{wL^2}{8} \\ M_x &= \frac{wx}{2}(L-x) \\ \Delta_{max} &= \frac{5}{384}\frac{wL^4}{EI} \\ \Delta_x &= \frac{wx}{24EI}(L^3 - 2Lx^2 + x^3) \end{aligned}$$

2) Simple Beam; Unsymmetric Triangular Load



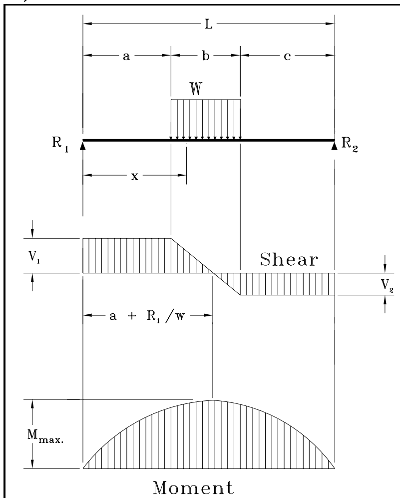
$$
\begin{aligned}
& R_1 = V_1 &=& \frac{W}{3} \\
\text{Max} \quad & R_2 = V_2 &=& \frac{2W}{3} \\
& V_x &=& \frac{W}{3} - \frac{Wx^2}{L^2} \\
\text{at } x = .577L \quad & M_{max} &=& .1283WL \\
& M_x &=& \frac{Wx}{3L^2}(L^2 - x^2) \\
\text{at } x = .5193L \quad & \Delta_{max} &=& .01304\frac{WL^3}{EI} \\
& \Delta_x &=& \frac{Wx^3}{180EIL^2}(3x^4 - 10L^2x^2 + 7L^4)
\end{aligned}
$$

3) Simple Beam; Symmetric Triangular Load



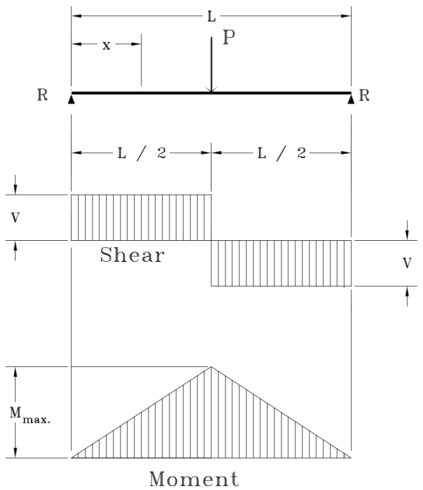
$$
\begin{aligned}
& R = V &=& \frac{W}{2} \\
\text{for } x < \tfrac{L}{2} \quad & V_x &=& \frac{W}{2L^2}(L^2 - 4x^2) \\
\text{at center} \quad & M_{max} &=& \frac{WL}{6} \\
\text{for } x < \tfrac{L}{2} \quad & M_x &=& Wx\left(\frac{1}{2} - \frac{2}{3}\frac{x^2}{L^2}\right) \\
\text{for } x < \tfrac{L}{2} \quad & \Delta_x &=& \frac{Wx}{480EIL^2}(5L^2 - 4x^2)^2 \\
& \Delta_{max} &=& \frac{WL^3}{60EI}
\end{aligned}
$$

4) Simple Beam; Uniform Load Partially Distributed



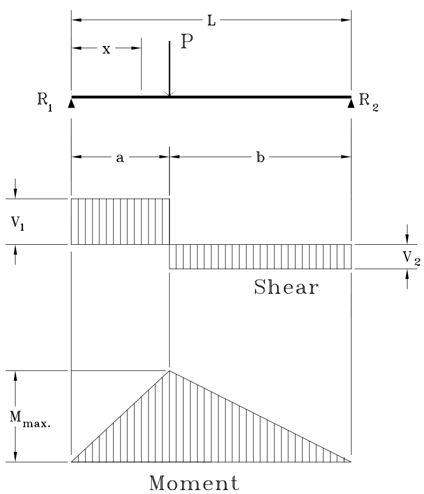
$$
\begin{aligned}
\text{Max when } a < c \quad & R_1 = V_1 &=& \frac{wb}{2L}(2c + b) \\
\text{Max when } a > c \quad & R_2 = V_2 &=& \frac{wb}{2L}(2a + b) \\
\text{when } a < x < a + b \quad & V_x &=& R_1 - w(x - a) \\
\text{when } x < a \quad & M_x &=& R_1 x \\
\text{when } a < x < a + b \quad & M_x &=& R_1 x - \frac{w}{2}(x - a)^2 \\
\text{when } a + b < x \quad & M_x &=& R_2(L - x) \\
\text{at } x = a + \tfrac{R_1}{w} \quad & M_{max} &=& R_1\left(a + \frac{R_1}{2w}\right)
\end{aligned}
$$

5) Simple Beam; Concentrated Load at Center



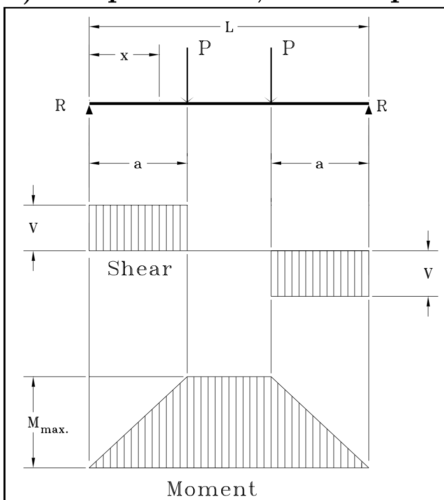
$$
\begin{array}{rlcl}
\text{max} & R_1 = V_1 & = & \frac{wa}{2L}(2L-a) \\
 & R = V & = & 2P \\
\text{at } x = \frac{L}{2} & M_{max} & = & \frac{PL}{4} \\
\text{when } x < \frac{L}{2} & M_x & = & \frac{Px}{2} \\
\text{when} x < \frac{L}{2} & \Delta_x & = & \frac{Px}{48EI}(3L^2 - 4x^2) \\
\text{at } x = \frac{L}{2} & \Delta_{max} & = & \frac{PL^3}{48EI}
\end{array}
$$

6) Simple Beam; Concentrated Load at Any Point



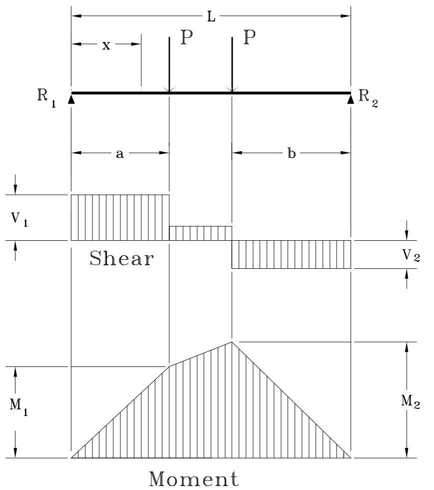
$$
\begin{array}{rlcl}
\text{max when } a < b & R_1 = V_1 & = & \frac{Pb}{L} \\
\text{max when } a > b & R_2 = V_2 & = & \frac{Pa}{L} \\
\text{at } x = a & M_{max} & = & \frac{Pab}{L} \\
\text{when } x < a & M_x & = & \frac{Pbx}{L} \\
\text{at } x = a & \Delta_a & = & \frac{Pa^2b^2}{3EIL} \\
\text{when } x < a & \Delta_x & = & \frac{Pbx}{6EIL}(L^2 - b^2 - x^2) \\
\text{at } x = \sqrt{\frac{a(a+2b)}{3}} \text{ \& } a > b & \Delta_{max} & = & \frac{Pab(a+2b)\sqrt{3a(a+2b)}}{27EIL}
\end{array}
$$

7) Simple Beam; Two Equally Concentrated Symmetric Loads



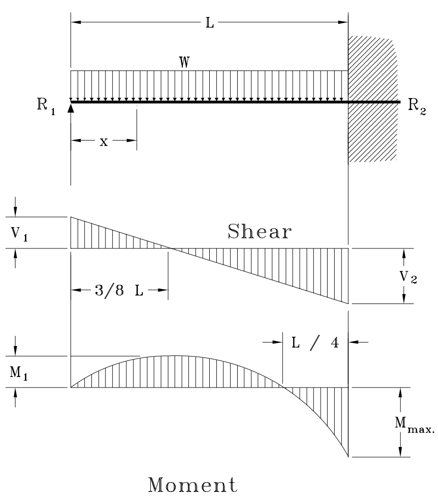
$$
\begin{array}{rlcl}
 & R = V & = & P \\
 & M_{max} & = & Pa \\
 & \Delta_{max} & = & \frac{Pa}{24EI}(3L^2 - 4a^2) \\
\text{when } x < a & \Delta_x & = & \frac{Px}{6EI}(3La - 3a^2 - x^2) \\
\text{when } a < x < L - a & \Delta_x & = & \frac{Pa}{6EI}(3Lx - 3x^2 - a^2)
\end{array}
$$

8) Simple Beam; Two Equally Concentrated Unsymmetric Loads



max when $a < b$	$R_1 = V_1$	$= \frac{P}{L}(L - a + b)$
max when $b < a$	$R_2 = V_2$	$= \frac{P}{L}(L - b + a)$
when $a < x < L - b$	V_x	$= \frac{P}{L}(b - a)$
max when $b < a$	M_1	$= R_1 a$
max when $a < b$	M_2	$= R_2 b$
when $x < a$	M_x	$= R_1 x$
when $a < x < L - b$	M_x	$= R_1 x - P(x - a)$

9) Cantilevered Beam, Uniform Load



$$R_1 = V_1 = \frac{3}{8}wL$$

$$R_2 = V_2 = \frac{5}{8}wL$$

$$V_x = R_1 - wx$$

$$M_{max} = \frac{wL^2}{8}$$

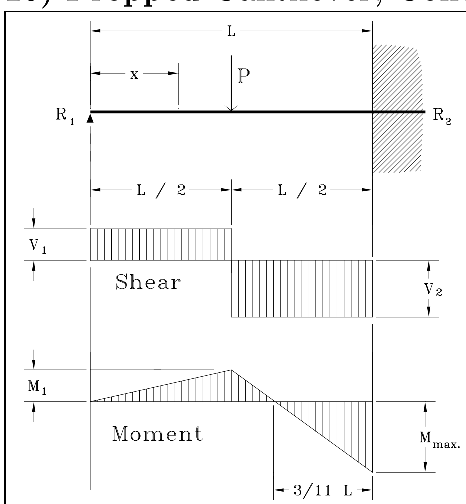
at $x = \frac{3}{8}L$: $$M_1 = \frac{9}{128}wL^2$$

$$M_x = R_1 x - \frac{wx^2}{2}$$

$$\Delta_x = \frac{wx}{48EI}(L^3 - 3Lx^{+}2x^3)$$

at $x = .4215L$: $$\Delta_{max} = \frac{wL^4}{185EI}$$

10) Propped Cantilever, Concentrated Load at Center



$$R_1 = V_1 = \frac{5P}{16}$$

$$R_2 = V_2 = \frac{11P}{16}$$

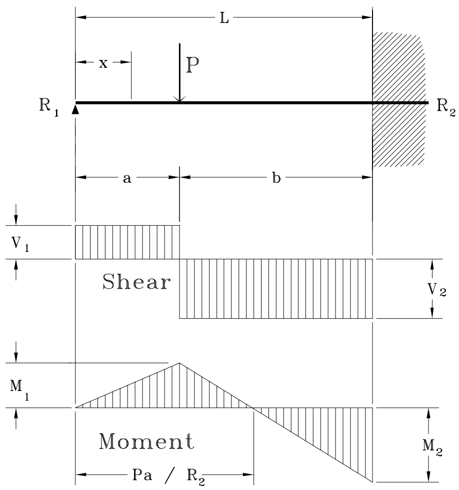
at $x = L$: $$M_{max} = \frac{3PL}{16}$$

when $x < \frac{L}{2}$: $$M_x = \frac{5Px}{16}$$

when $\frac{L}{2} < x$: $$M_x = P\left(\frac{L}{2} - \frac{11x}{16}\right)$$

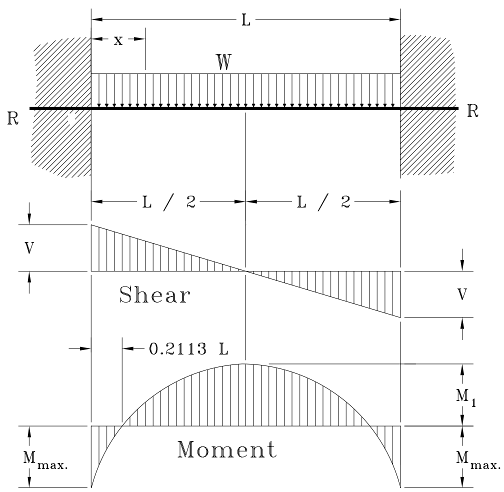
at $x = .4472L$: $$\Delta_{max} = .009317\frac{PL^3}{EI}$$

11) Propped Cantilever; Concentrated Load



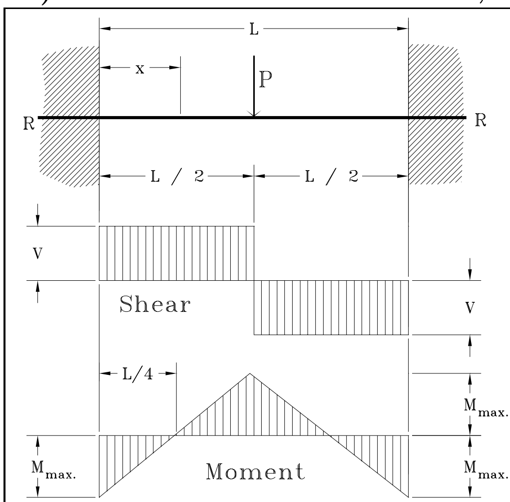
$$
\begin{aligned}
& R_1 = V_1 &=& \frac{Pb^2}{2L^3}(a+2L) \\
& R_2 = V_2 &=& \frac{Pa}{2L^3}(3L^2-a^2) \\
\text{at } x=a \quad & M_1 &=& R_1 a \\
\text{at } x=L \quad & M_2 &=& \frac{Pab}{2L^2}(a+L) \\
\text{at } x=a \quad & \Delta_a &=& \frac{Pa^2b^3}{12EIL^3}(3L+a) \\
\text{when } a<.414L \text{ at } x = L\frac{L^2+a^2}{3L^2-a^2} \quad & \Delta_{max} &=& \frac{Pa}{3EI}\frac{(L^2-a^2)^3}{(3L^2-a^2)^2} \\
\text{when } .414L<a \text{ at } x = L\sqrt{\frac{a}{2L+a}} \quad & \Delta_{max} &=& \frac{Pab^2}{6EI}\sqrt{\frac{a}{2L+a2}}
\end{aligned}
$$

12) Beam Fixed at Both Ends, Uniform Load



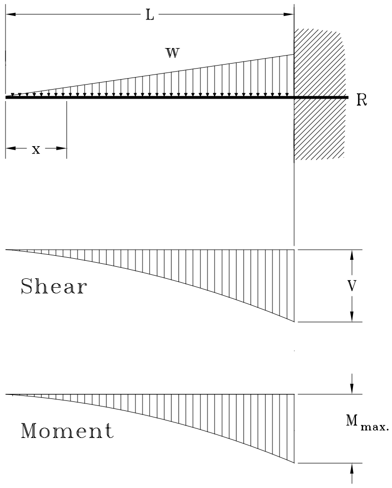
$$
\begin{aligned}
& R = V &=& \frac{wL}{2} \\
& V_x &=& w\left(\frac{L}{2}-x\right) \\
\text{at } x=0 \text{ and } x=L \quad & M_{max} &=& \frac{wL^2}{12} \\
\text{at } x=\tfrac{L}{2} \quad & M &=& \frac{wL^2}{24} \\
\text{at } x=\tfrac{L}{2} \quad & \Delta_{max} &=& \frac{wL^4}{384EI} \\
& \Delta_x &=& \frac{wx^2}{24EI}(L-x)^2
\end{aligned}
$$

13) Beam Fixed at Both Ends; Concentrated Load



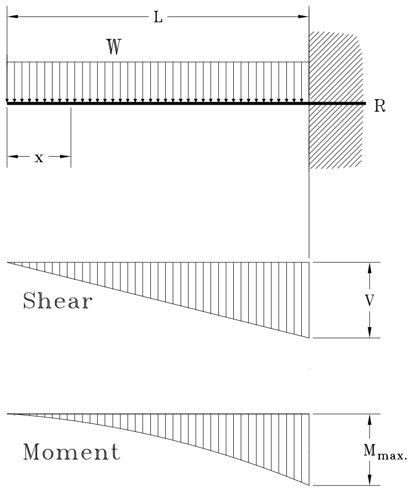
$$
\begin{aligned}
& R = V &=& \frac{P}{2} \\
\text{at } x=\tfrac{L}{2} \quad & M_{max} &=& \frac{PL}{8} \\
\text{when } x<\tfrac{L}{2} \quad & M_x &=& \frac{P}{8}(4x-L) \\
\text{at } x=\tfrac{L}{2} \quad & \Delta_{max} &=& \frac{PL^3}{192EI} \\
\text{when } x<\tfrac{L}{2} \quad & \Delta_x &=& \frac{Px^2}{48EI}(3L-4x)
\end{aligned}
$$

14) Cantilever Beam; Triangular Unsymmetric Load



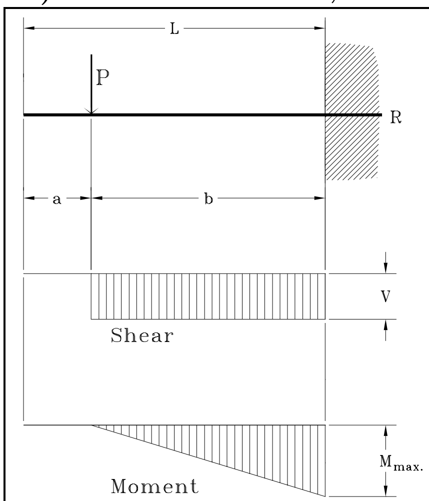
$$
\begin{array}{lll}
 & R = V & = \frac{8}{3}W \\
 & V_x & = W\frac{x^2}{L^2} \\
\text{at } x = L & M_{max} & = \frac{WL}{3} \\
 & M_x & = \frac{Wx^2}{3L^2} \\
 & \Delta_x & = \frac{W}{60EIL^2}(x^5 - 5L^2x + 4L^5) \\
\text{at } x = 0 & \Delta_{max} & = \frac{WL^3}{15EI}
\end{array}
$$

15) Cantilever Beam; Uniform Load



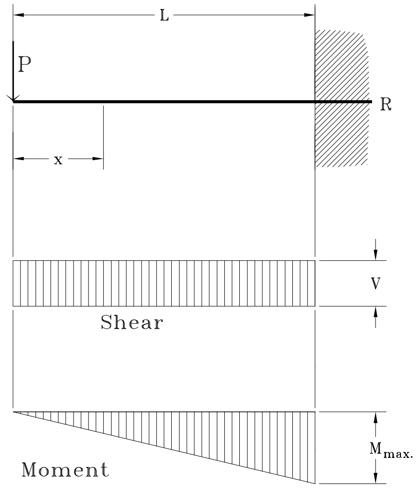
$$
\begin{array}{lll}
 & R = V & = wL \\
 & V_x & = wx \\
 & M_x & = \frac{wx^2}{2} \\
\text{at } x = L & M_{max} & = \frac{wL^2}{2} \\
 & \Delta_x & = \frac{w}{24EI}(x^4 - 4L^3x + 3L^4) \\
\text{at } x = 0 & \Delta_{max} & = \frac{wL^4}{8EI}
\end{array}
$$

16) Cantilever Beam; Point Load



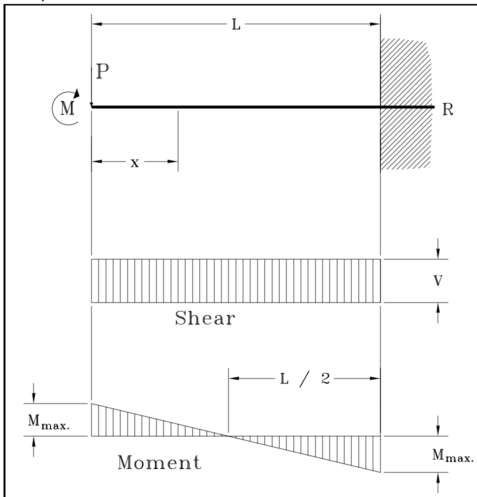
$$
\begin{array}{lll}
 & R = V & = P \\
\text{at } x = L & M_{max} & = Pb \\
\text{when } a < x & M_x & = P(x-a) \\
\text{at } x = 0 & \Delta_{max} & = \frac{Pb^2}{6EI}(3L - b) \\
\text{at } x = a & \Delta_a & = \frac{Pb^3}{3EI} \\
\text{when } x < a & \Delta_x & = \frac{Pb^2}{6EI}(3L - 3x - b) \\
\text{when } a < x & \Delta_x & = \frac{P(L-x)^2}{6EI}(3b - L + x)
\end{array}
$$

17) Cantilever Beam; Point Load at Free End



$$R = V = P$$
$$\text{at } x = L \quad M_{max} = PL$$
$$M_x = Px$$
$$\text{at } x = 0 \quad \Delta_{max} = \frac{PL^3}{3EI}$$
$$\Delta_x = \frac{P}{6EI}(2L^3 - 3L^2x + x^3)$$

18) Cantilever Beam; Concentrated Force and Moment at Free End

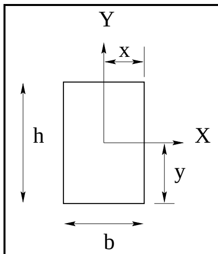


$$R = V = P$$
$$M_x = P\left(\frac{L}{2} - x\right)$$
$$\text{at } x = 0 \text{ and } x = L \quad M_{max} = \frac{PL}{2}$$
$$\text{at } x = 0 \quad \Delta_{max} = \frac{PL^3}{12EI}$$
$$\Delta_x = \frac{P(L-x)^2}{12EI}((L + 2x)$$

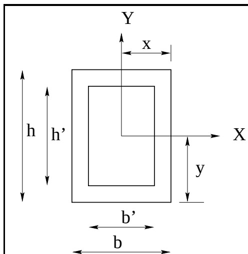
Appendix B

SECTION PROPERTIES

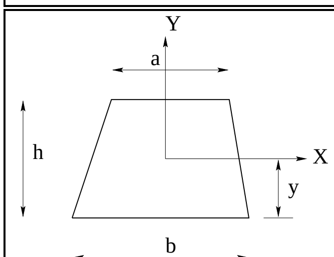
Section properties for selected sections are shown in Table B.1.



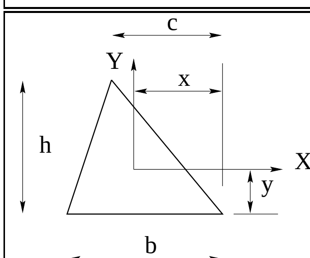
$$A = bh$$
$$x = \frac{b}{2}$$
$$y = \frac{h}{2}$$
$$I_x = \frac{bh^3}{12}$$
$$I_y = \frac{hb^3}{12}$$



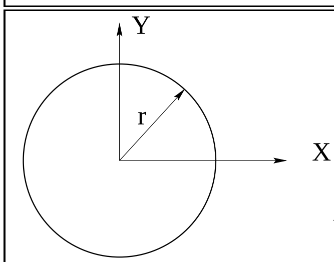
$$A = bh - b'h'$$
$$x = \frac{b}{2}$$
$$y = \frac{h}{2}$$
$$I_x = \frac{bh^3 - b'h'^3}{12}$$
$$I_y = \frac{hb^3 - h'b'^3}{12}$$



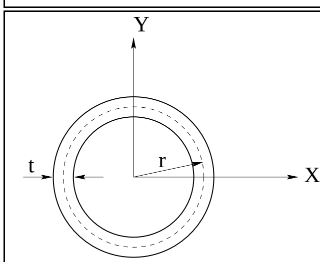
$$A = \frac{h(a+b)}{2}$$
$$y = \frac{h(2a+b)}{3(a+b)}$$
$$I_x = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}$$



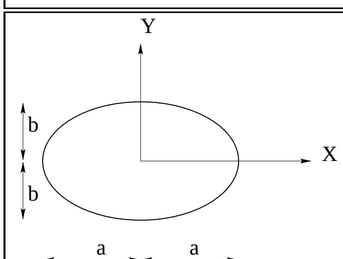
$$A = \frac{bh}{2}$$
$$x = \frac{b+c}{3}$$
$$y = \frac{h}{3}$$
$$I_x = \frac{bh^3}{36}$$
$$I_y = \frac{bh}{36}(b^2 - bc + c^2)$$



$$A = \pi r^2 = \frac{\pi d^2}{4}$$
$$I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$



$$A = 2\pi r t = \pi d t$$
$$I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$$



$$A = \pi ab$$
$$I_x = \frac{\pi ab^3}{3}$$
$$I_y = \frac{\pi ba^3}{4}$$

Table B.1: Section Properties

Appendix C

MATHEMATICAL PRELIMINARIES; Part IV VARIATIONAL METHODS

`Abridged section from author's lecture notes in finite elements.`

C.1 Euler Equation

20 The fundamental problem of the calculus of variation[1] is to find a function $u(x)$ such that

$$\Pi = \int_a^b F(x, u, u')dx \tag{3.1}$$

is stationary. Or,

$$\boxed{\delta\Pi = 0} \tag{3.2}$$

where δ indicates the *variation*

21 We define $u(x)$ to be a function of x in the interval (a, b), and F to be a known function (such as the energy density).

22 We define the *domain* of a functional as the collection of admissible functions belonging to a class of functions in function space rather than a region in coordinate space (as is the case for a function).

23 We seek the function $u(x)$ which extremizes Π.

24 Letting $\tilde{u}$ to be a family of neighbouring paths of the extremizing function $u(x)$ and we assume that at the end points $x = a, b$ they coincide. We define $\tilde{u}$ as the sum of the extremizing path and some arbitrary variation, Fig. C.1.

$$\tilde{u}(x, \varepsilon) = u(x) + \varepsilon\eta(x) = u(x) + \delta u(x) \tag{3.3}$$

[1]Differential calculus involves a function of one or more variable, whereas variational calculus involves a function of a function, or a functional.

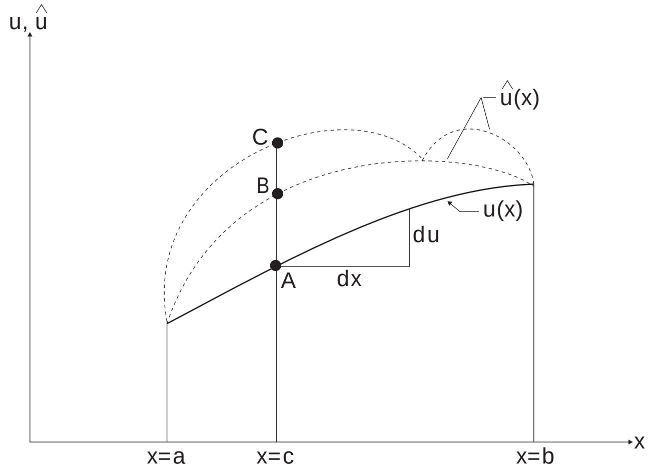


Figure C.1: Variational and Differential Operators

where ε is a small parameter, and $\delta u(x)$ is the *variation* of $u(x)$

$$\delta u = \tilde{u}(x,\varepsilon) - u(x) \tag{3.4-a}$$
$$= \varepsilon\eta(x) \tag{3.4-b}$$

and $\eta(x)$ is twice differentiable, has undefined amplitude, and $\eta(a) = \eta(b) = 0$. We note that $\tilde{u}$ coincides with u if $\varepsilon = 0$

25 The variational operator δ and the differential calculus operator d have clearly different meanings. du is associated with a neighboring point at a distance dx, however δu is a small *arbitrary* change in u for a given x (there is no associated δx).

26 For boundaries where u is specified, its variation must be zero, and it is arbitrary elsewhere. The variation δu of u is said to undergo a *virtual* change.

27 To solve the variational problem of extremizing Π, we consider

$$\Pi(u + \varepsilon\eta) = \Phi(\varepsilon) = \int_a^b F(x, u + \varepsilon\eta, u' + \varepsilon\eta')dx \tag{3.5}$$

28 Since $\tilde{u} \to u$ as $\varepsilon \to 0$, the necessary condition for Π to be an extremum is

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0 \tag{3.6}$$

29 From Eq. 3.3 and applying the chain rule with $\varepsilon = 0$, $\tilde{u} = u$, we obtain

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_a^b \left(\eta\frac{\partial F}{\partial u} + \eta'\frac{\partial F}{\partial u'}\right) dx = 0 \tag{3.7}$$

30 It can be shown (through integration by part and the fundamental lemma of the

calculus of variation) that this would lead to

$$\boxed{\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} = 0} \tag{3.8}$$

31 This differential equation is called the **Euler equation** associated with Π and is a necessary condition for $u(x)$ to extremize Π.

32 Generalizing for a functional Π which depends on two field variables, $u = u(x, y)$ and $v = v(x, y)$

$$\Pi = \int\int F(x, y, u, v, u_{,x}, u_{,y}, v_{,x}, v_{,y}, \cdots, v_{,yy})dxdy \tag{3.9}$$

There would be as many Euler equations as dependent field variables

$$\left\{\begin{array}{rcl} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial u_{,xx}} + \frac{\partial^2}{\partial x\partial y}\frac{\partial F}{\partial u_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial u_{,yy}} & = & 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\frac{\partial F}{\partial v_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial v_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial v_{,xx}} + \frac{\partial^2}{\partial x\partial y}\frac{\partial F}{\partial v_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial v_{,yy}} & = & 0 \end{array}\right. \tag{3.10}$$

33 We note that the Functional and the corresponding Euler Equations, Eq. 3.1 and 3.8, or Eq. 3.9 and 3.10 describe the same problem.

34 The Euler equations usually correspond to the governing differential equation and are referred to as the **strong form** (or classical form).

35 The functional is referred to as the **weak form** (or generalized solution). This classification stems from the fact that equilibrium is enforced in an average sense over the body (and the field variable is differentiated m times in the weak form, and $2m$ times in the strong form).

36 Euler equations are differential equations which can not always be solved by exact methods. An alternative method consists in bypassing the Euler equations and go directly to the variational statement of the problem to the solution of the Euler equations.

37 Finite Element formulation are based on the weak form, whereas the formulation of Finite Differences are based on the strong form.

38 Finally, we still have to define $\delta\Pi$

$$\left.\begin{array}{rcl} \delta F & = & \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \\ \delta\Pi & = & \int_a^b \delta F dx \end{array}\right\} \delta\Pi = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) dx \tag{3.11}$$

As above, integration by parts of the second term yields

$$\boxed{\delta\Pi = \int_a^b \delta u \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right) dx} \tag{3.12}$$

39 We have just shown that finding the stationary value of Π by setting $\delta\Pi = 0$ is equivalent to finding the extremal value of Π by setting $\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}$ equal to zero.

40 Similarly, it can be shown that as with second derivatives in calculus, the second variation $\delta^2\Pi$ can be used to characterize the extremum as either a minimum or maximum.

41 Revisiting the integration by parts of the second term in Eq. 3.7, we obtain

$$\boxed{\int_a^b \eta' \frac{\partial F}{\partial u'} dx = \eta \left. \frac{\partial F}{\partial u'} \right|_a^b - \int_a^b \eta \frac{d}{dx} \frac{\partial F}{\partial u'} dx} \quad (3.13)$$

We note that

1. Derivation of the Euler equation required $\eta(a) = \eta(b) = 0$, thus this equation is a statement of the *essential* (or forced) boundary conditions, where $u(a) = u(b) = 0$.
2. If we left η arbitrary, then it would have been necessary to use $\frac{\partial F}{\partial u'} = 0$ at $x = a$ and b. These are the *natural* boundary conditions.

42 For a problem with, one field variable, in which the highest derivative in the governing differential equation is of order $2m$ (or simply m in the corresponding functional), then we have

Essential (or Forced, or geometric) boundary conditions, involve derivatives of order zero (the field variable itself) through m-1. Trial displacement functions are explicitely required to satisfy this B.C. Mathematically, this corresponds to *Dirichlet boundary-value problems.*

Nonessential (or Natural, or static) boundary conditions, involve derivatives of order m and up. This B.C. is implied by the satisfaction of the variational statement but not explicitly stated in the functional itself. Mathematically, this corresponds to *Neuman boundary-value problems.*

These boundary conditions were already introduced, albeit in a less formal way, in Table 9.1.

43 Table C.1 illustrates the boundary conditions associated with some problems

Problem	Axial Member Distributed load	Flexural Member Distributed load
Differential Equation	$AE\frac{d^2u}{dx^2} + q = 0$	$EI\frac{d^4w}{dx^4} - q = 0$
m	1	2
Essential B.C. $[0, m-1]$	u	w, $\frac{dw}{dx}$
Natural B.C. $[m, 2m-1]$	$\frac{du}{dx}$ or $\sigma_x = Eu_{,x}$	$\frac{d^2w}{dx^2}$ and $\frac{d^3w}{dx^3}$ or $M = EIw_{,xx}$ and $V = EIw_{,xxx}$

Table C.1: Essential and Natural Boundary Conditions

■ Example C-1: Extension of a Bar

The total potential energy Π of an axial member of length L, modulus of elasticity E, cross sectional area A, fixed at left end and subjected to an axial force P at the right one is given by

$$\Pi = \int_0^L \frac{EA}{2} \left(\frac{du}{dx} \right)^2 dx - Pu(L) \quad (3.14)$$

Determine the Euler Equation by requiring that Π be a minimum.
Solution:

Solution I The first variation of Π is given by

$$\delta\Pi = \int_0^L \frac{EA}{2} 2 \left(\frac{du}{dx}\right) \delta \left(\frac{du}{dx}\right) dx - P\delta u(L) \tag{3.15}$$

Integrating by parts we obtain

$$\begin{aligned} \delta\Pi &= \int_0^L -\frac{d}{dx}\left(EA\frac{du}{dx}\right)\delta u dx + EA\frac{du}{dx}\delta u\Big|_0^L - P\delta u(L) & \text{(3.16-a)} \\ &= -\int_0^L \delta u \underbrace{\frac{d}{dx}\left(EA\frac{du}{dx}\right)} dx + \left[\underbrace{\left(EA\frac{du}{dx}\right)\Big|_{x=L} - P}\right]\delta u(L) \\ &= -\underbrace{\left(EA\frac{du}{dx}\right)}\Big|_{x=0} \delta u(0) & \text{(3.16-b)} \end{aligned}$$

The last term is zero because of the specified essential boundary condition which implies that $\delta u(0) = 0$. Recalling that δ in an arbitrary operator which can be assigned any value, we set the coefficients of δu between $(0, L)$ and those for δu at $x = L$ equal to zero separately, and obtain

Euler Equation:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0 \quad 0 < x < L \tag{3.17}$$

Natural Boundary Condition:

$$EA\frac{du}{dx} - P = 0 \quad \text{at} \quad x = L \tag{3.18}$$

Solution II We have

$$F(x, u, u') = \frac{EA}{2}\left(\frac{du}{dx}\right)^2 \tag{3.19}$$

(note that since P is an applied load at the end of the member, it does not appear as part of $F(x, u, u')$ To evaluate the Euler Equation from Eq. 3.8, we evaluate

$$\frac{\partial F}{\partial u} = 0 \;\; \& \;\; \frac{\partial F}{\partial u'} = EAu' \tag{3.20-a}$$

Thus, substituting, we obtain

$$\begin{aligned} \frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} &= 0 \;\; \text{Euler Equation} & \text{(3.21-a)} \\ \frac{d}{dx}\left(EA\frac{du}{dx}\right) &= 0 \;\; \text{B.C.} & \text{(3.21-b)} \end{aligned}$$

■

■ **Example C-2: Flexure of a Beam**

The total potential energy of a beam is given by

$$\Pi = \int_0^L \left(\frac{1}{2}M\kappa - pw\right)dx = \int_0^L \left(\frac{1}{2}(EIw'')w'' - pw\right)dx \tag{3.22}$$

Derive the first variational of Π.
Solution:
Extending Eq. 3.11, and integrating by part twice

$$\begin{aligned}
\delta\Pi &= \int_0^L \delta F dx = \int_0^L \left(\frac{\partial F}{\partial w''}\delta w'' + \frac{\partial F}{\partial w}\delta w\right)dx && \text{(3.23-a)}\\
&= = \int_0^L (EIw''\delta w'' - p\delta w)dx && \text{(3.23-b)}\\
&= (EIw''\delta w')|_0^L - \int_0^L \left[(EIw'')'\delta w' - p\delta w\right]dx && \text{(3.23-c)}\\
&= (EIw''\delta w')|_0^L - \left[(EIw'')'\delta w\right]|_0^L + \int_0^L \left[(EIw'')'' + p\right]\delta w dx = 0 && \text{(3.23-d)}
\end{aligned}$$

Or

$$(EIw'')'' = -p \quad \text{for all x}$$

which is the governing differential equation of beams and

Essential		Natural
$\delta w' = 0$	or	$EIw'' = -M = 0$
$\delta w = 0$	or	$(EIw'')' = -V = 0$

at $x = 0$ and $x = L$ ■

Appendix D

MID TERM EXAM

Continuum Mechanics
LMC/DMX/EPFL
Prof. Saouma
Exam I (Closed notes), March 27, 1998
3 Hours

There are 19 problems worth a total of 63 points. Select any problems you want as long as the total number of corresponding points is equal to or larger than 50.

---

1. (2 pts) Write in matrix form the following 3rd order tensor D_{ijk} in $\mathbf{R}^2$ space. i, j, k range from 1 to 2.
2. (2 pts) Solve for $E_{ij}a_i$ in indicial notation.
3. (4 pts) if the stress tensor at point P is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 10 & -2 & 0 \\ -2 & 4 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

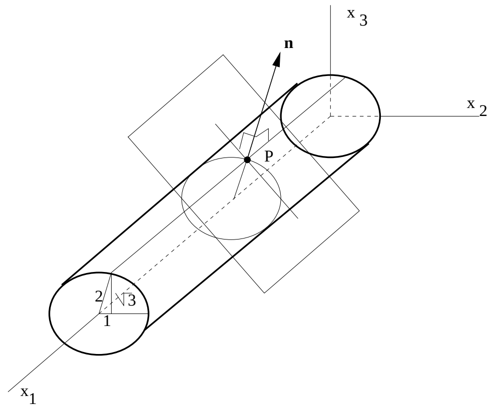
   determine the traction (or stress vector) $\mathbf{t}$ on the plane passing through P and parallel to the plane ABC where $A(6,0,0)$, $B(0,4,0)$ and $C(0,0,2)$.
4. (5 pts) For a plane stress problem charaterized by the following stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

   use Mohr's circle to determine the principal stresses, and show on an appropriate figure the orientation of those principal stresses.
5. (4 pts) The stress tensor throughout a continuum is given with respect to Cartesian axes as

$$\boldsymbol{\sigma} = \begin{bmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3^2 \\ 0 & 2x_3^2 & 0 \end{bmatrix}$$

   (a) Determine the stress vector (or traction) at the point $P(2, 1, \sqrt{3})$ of the plane that is tangent to the cylindrical surface $x_2^2 + x_3^2 = 4$ at P,



(b) Are the stresses in equlibrium, explain.

6. (2 pts) A displacement field is given by $\mathbf{u} = X_1X_3^2\mathbf{e}_1 + X_1^2X_2\mathbf{e}_2 + X_2^2X_3\mathbf{e}_3$, determine the material deformation gradient $\mathbf{F}$ and the material displacement gradient $\mathbf{J}$, and verify that $\mathbf{J} = \mathbf{F} - \mathbf{I}$.
7. (4 pts) A continuum body undergoes the deformation $x_1 = X_1 + AX2$, $x_2 = X_2 + AX_3$, and $x_3 = X_3 + AX_1$ where A is a constant. Determine: 1) Deformation (or Green) tensor $\mathbf{C}$; and 2) Lagrangian tensor $\mathbf{E}$.
8. (4 pts) Linear and finite strain tensors can be decomposed into the sum or product of two other tensors.
   (a) Which strain tensor can be decomposed into a sum, and which other one into a product.
   (b) Why is such a decomposition performed?
9. (2 pts) Why do we have a condition imposed on the strain field (compatibility equation)?
10. (6 pts) Stress tensors:
   (a) When shall we use the Piola-Kirchoff stress tensors?
   (b) What is the difference between Cauchy, first and second Piola-Kirchoff stress tensors?
   (c) In which coordinate system is the Cauchy and Piola-Kirchoff stress tensors expressed?
11. (2 pts) What is the difference between the tensorial and engineering strain ($E_{ij}, \gamma_{ij}, i \neq j$) ?
12. (3 pts) In the absence of body forces, does the following stress distribution

$$\begin{bmatrix} x_2^2 + \nu(x_1^2 - x_x^2) & -2\nu x_1 x_2 & 0 \\ -2\nu x_1 x_2 & x_1^2 + \nu(x_2^2 - x_1^2) & 0 \\ 0 & 0 & \nu(x_1^2 + x_2^2) \end{bmatrix}$$

   where ν is a constant, satisfy equilibrium in the X_1 direction?
13. (2 pts) From which principle is the symmetry of the stress tensor derived?
14. (2 pts) How is the First principle obtained from the equation of motion?
15. (4 pts) What are the 1) 15 Equations; and 2) 15 Unknowns in a thermoelastic formulation.

16. (2 pts) What is free energy Ψ?
17. (2 pts) What is the relationship between strain energy and strain?
18. (5 pts) If a plane of elastic symmetry exists in an anisotropic material,

$$\left\{\begin{array}{c} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{array}\right\} = \left[\begin{array}{cccccc} c_{1111} & c_{1112} & c_{1133} & c_{1112} & c_{1123} & c_{1131} \\ & c_{2222} & c_{2233} & c_{2212} & c_{2223} & c_{2231} \\ & & c_{3333} & c_{3312} & c_{3323} & c_{3331} \\ & & & c_{1212} & c_{1223} & c_{1231} \\ & \text{SYM.} & & & c_{2323} & c_{2331} \\ & & & & & c_{3131} \end{array}\right] \left\{\begin{array}{c} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12}(\gamma_{12}) \\ 2E_{23}(\gamma_{23}) \\ 2E_{31}(\gamma_{31}) \end{array}\right\}$$

then,

$$a_i^j = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$$

show that under these conditions c_{1131} is equal to zero.

19. (6 pts) The state of stress at a point of structural steel is given by

$$\mathbf{T} = \left[\begin{array}{ccc} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{M}Pa$$

with $E = 207$ GPa, $\mu = 80$ GPa, and $\nu = 0.3$.

(a) Determine the engineering strain components
(b) If a five centimer cube of structural steel is subjected to this stress tensor, what would be the change in volume?

Appendix E

MATHEMATICA ASSIGNMENT and SOLUTION

Connect to *Mathematica* using the following procedure:

1. login on an HP workstation
2. Open a shell (window)
3. Type `xhost+`
4. type `rlogin mxsg1`
5. On the newly opened shell, enter your password first, and then type `setenv DISPLAY xxx:0.0` where `xxx` is the workstation name which should appear on a small label on the workstation itself.
6. Type `mathematica &`

and then solve the following problems:

1. The state of stress through a continuum is given with respect to the cartesian axes $Ox_1x_2x_3$ by

$$T_{ij} = \begin{bmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{bmatrix} \text{MPa}$$

   Determine the stress vector at point $P(1, 1, \sqrt{3})$ of the plane that is tangent to the cylindrical surface $x_2^2 + x_3^2 = 4$ at P.

2. For the following stress tensor

$$T_{ij} = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

   (a) Determine directly the three invariants I_σ, II_σ and III_σ of the following stress tensor
   (b) Determine the principal stresses and the principal stress directions.
   (c) Show that the transformation tensor of direction cosines transforms the original stress tensor into the diagonal principal axes stress tensor.
   (d) Recompute the three invariants from the principal stresses.

(e) Split the stress tensor into its spherical and deviator parts.

(f) Show that the first invariant of the deviator is zero.

3. The Lagrangian description of a deformation is given by $x_1 = X_1 + X_3(e^2 - 1)$, $x_2 = X_2 + X_3(e^2 - e^{-2}$, and $x_3 = e^2 X_3$ where e is a constant. SHow that the Jacobian J does not vanish and determine the Eulerian equations describing this motion.

4. A displacement field is given by $\mathbf{u} = X_1 X_3^2 \mathbf{e}_1 + X_1^2 X_2 \mathbf{e}_2 + X_2^2 X_3 \mathbf{e}_3$. Determine independently the material deformation gradient $\mathbf{F}$ and the material displacement gradient $\mathbf{J}$ and verify that $\mathbf{J} = \mathbf{F} - \mathbf{I}$.

5. A continuum body undergoes the deformation $x_1 = X_1$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$ where A is a constant. Compute the deformation tensor $\mathbf{C}$ and use this to determine the Lagrangian finite strain tensor $\mathbf{E}$.

6. A continuum body undergoes the deformation $x_1 = X_1 + AX_2$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$ where A is a constant.

(a) Compute the deformation tensor $\mathbf{C}$

(b) Use the computed $\mathbf{C}$ to determine the Lagrangian finite strain tensor $\mathbf{E}$.

(c) COmpute the Eulerian strain tensor $\mathbf{E}^*$ and compare with $\mathbf{E}$ for very small values of A.

7. A continuum body undergoes the deformation $x_1 = X_1 + 2X_2$, $x_2 = X_2$, $x_3 = X_3$

(a) Determine the Green's deformation tensor $\mathbf{C}$

(b) Determine the principal values of $\mathbf{C}$ and the corresponding principal directions.

(c) Determine the right stretch tensor $\mathbf{U}$ and $\mathbf{U}^{-1}$ with respect to the principal directions.

(d) Determine the right stretch tensor $\mathbf{U}$ and $\mathbf{U}^{-1}$ with respect to the $\mathbf{e}_i$ basis.

(e) Determine the orthogonal rotation tensor $\mathbf{R}$ with respect to the $\mathbf{e}_i$ basis.

8. A continuum body undergoes the deformation $x_1 = 4X_1$, $x_2 = -\frac{1}{2}X_2$, $x_3 = -\frac{1}{2}X_3$ and the Cauchy stress tensor for this body is

$$T_{ij} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

(a) Determine the corresponding first Piola-Kirchoff stress tensor.

(b) Determine the corresponding second Piola-Kirchoff stress tensor.

(c) Determine the pseudo stress vector associated with the first Piola-Kirchoff stress tensor on the $\mathbf{e}_1$ plane in the deformed state.

(d) Determine the pseudo stress vector associated with the second Piola-Kirchoff stress tensor on the $\mathbf{e}_1$ plane in the deformed state.

9. Show that in the case of isotropy, the anisotropic stress-strain relation

$$c_{ijkm}^{\text{Aniso}} = \begin{bmatrix} c_{1111} & c_{1112} & c_{1133} & c_{1112} & c_{1123} & c_{1131} \\ & c_{2222} & c_{2233} & c_{2212} & c_{2223} & c_{2231} \\ & & c_{3333} & c_{3312} & c_{3323} & c_{3331} \\ & & & c_{1212} & c_{1223} & c_{1231} \\ & \text{SYM.} & & & c_{2323} & c_{2331} \\ & & & & & c_{3131} \end{bmatrix}$$

reduces to

$$c^{\rm iso}_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & a & 0 & 0 \\ & \text{SYM.} & & & b & 0 \\ & & & & & c \end{bmatrix}$$

with $a = \frac{1}{2}(c_{1111} - c_{1122})$, $b = \frac{1}{2}(c_{2222} - c_{2233})$, and $c = \frac{1}{2}(c_{3333} - c_{1133})$.

10. Determine the stress tensor at a point where the Lagrangian strain tensor is given by

$$E_{ij} = \begin{bmatrix} 30 & 50 & 20 \\ 50 & 40 & 0 \\ 20 & 0 & 30 \end{bmatrix} \times 10^{-6}$$

and the material is steel with $\lambda = 119.2$ GPa and $\mu = 79.2$ GPa.

11. Determine the strain tensor at a point where the Cauchy stress tensor is given by

$$T_{ij} = \begin{bmatrix} 100 & 42 & 6 \\ 42 & -2 & 0 \\ 6 & 0 & 15 \end{bmatrix} \text{MPa}$$

with $E = 207$ GPa, $\mu = 79.2$ GPa, and $\nu = 0.30$

12. Determine the thermally induced stresses in a constrained body for a rise in temerature of 50^{oF}, $\alpha = 5.6 \times 10^{-6} / \ ^{0F}$

13. Show that the inverse of

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy}(2\varepsilon_{xy}) \\ \gamma_{yz}(2\varepsilon_{yz}) \\ \gamma_{zx}(2\varepsilon_{zx}) \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \tag{5.1}$$

is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} & 0 \\ 0 & G \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy}(2\varepsilon_{xy}) \\ \gamma_{yz}(2\varepsilon_{yz}) \\ \gamma_{zx}(2\varepsilon_{zx}) \end{Bmatrix} \tag{5.2}$$

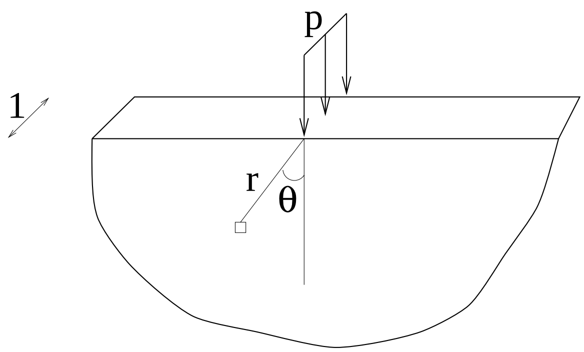
and then derive the relations between stresses in terms of strains, and strains in terms of stress, for plane stress and plane strain.

14. Show that the function $\Phi = f(r)\cos 2\theta$ satisfies the biharmonic equation $\nabla(\nabla\Phi) = 0$ Note: You must `<<Calculus‘VectorAnalysis‘`, define Φ, and `SetCoordinates[Cylindrical[r,θ,z]]`, and finally use the `Laplacian` (or `Biharmonic`) functions.

15. Solve for

$$\begin{bmatrix} T_{rr} & T_{r\theta} \\ T_{r\theta} & T_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \tag{5.3}$$

16. If a point load p is applied on a semi-infinite medium



show that for $\Phi = -\frac{p}{\pi} r\theta \sin\theta$ we have the following stress tensors:

$$\begin{bmatrix} -\frac{2p}{\pi}\frac{\cos\theta}{r} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{2p\cos^3\theta}{\pi r} & -\frac{2p\sin\theta\cos^2\theta}{\pi r} \\ -\frac{2p\sin\theta\cos^2\theta}{\pi r} & -\frac{2p\sin^2\theta\cos\theta}{\pi r} \end{bmatrix} \tag{5.4}$$

Determine the maximum principal stress at an y arbitrary point, (contour) plot the magnitude of this stress below p. Note that `D[Φ,r], D[Φ,{θ,2}]` would give the first and second derivatives of Φ with respect to r and θ respectively.