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Fourier Theory

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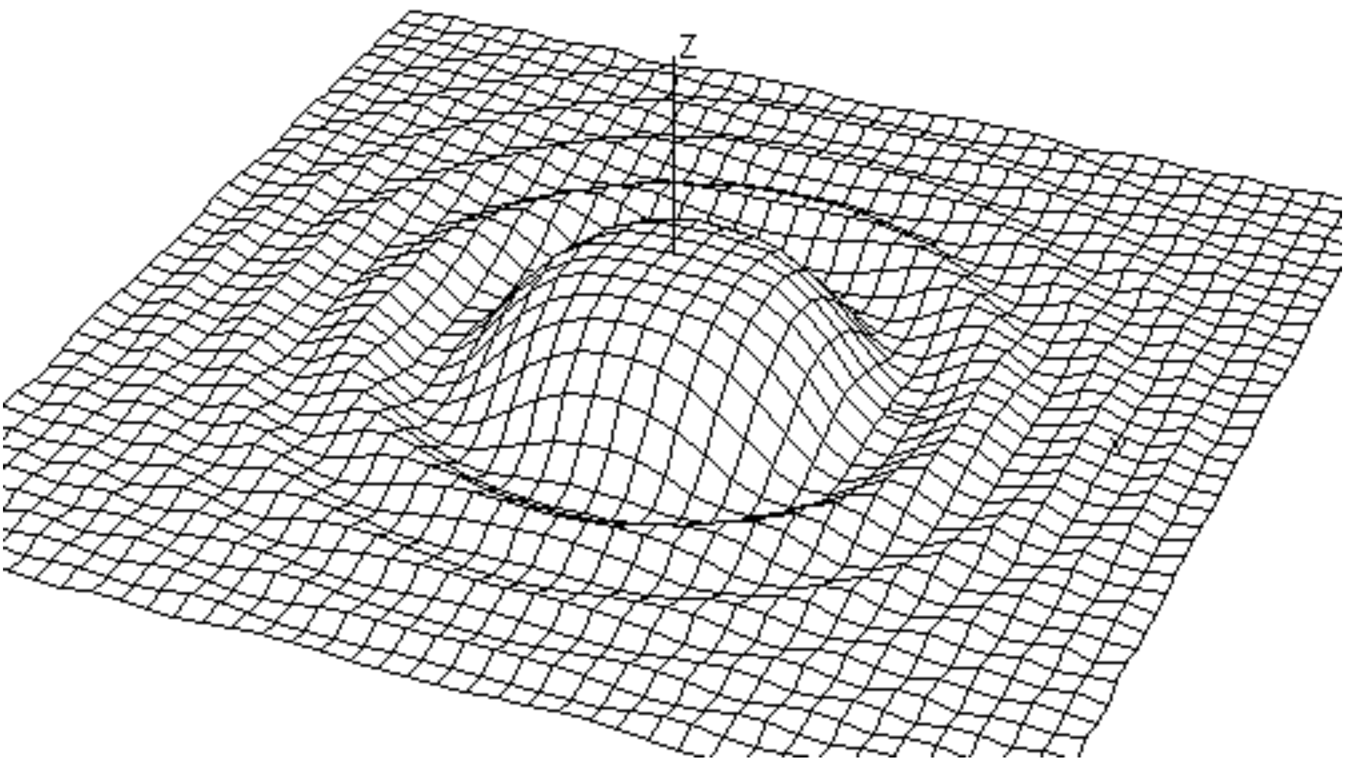


Table of Contents

1. Introduction.

Fourier theory is a branch of mathematics first invented to solve certain problems in partial differential equations. The most well-known of these equations are:

Laplace's equation, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, for $u(x, y)$ a function of two variables,

the wave equation, $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$, for $u(x, t)$ a function of two variables,

the heat equation, $\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0$, for $u(x, t)$ a function of two variables.

In the heat equation, x represents the position along the bar measured from some origin, t represents time, $u(x, t)$ the temperature at position x, time t. Fourier was initially concerned with the heat equation. Incidentally, the same equation describes the concentration of a dye diffusing in a liquid such as water. For this reason the equation is sometimes called the diffusion equation.

In the wave equation, x, represents the position along an elastic string under tension, measured from some origin, t represents time, $u(x, t)$ the displacement of the string from equilibrium at position x, time t.

In Laplace's equation, $u(x, y)$ represents the steady temperature of a flat conducting plate at the position (x, y) in the plane.

Since both the heat equation and the wave equation involve a single space variable x, we sometime refer to them as the *one dimensional heat equation* and the *one dimensional wave equation* respectively.

Laplace's equation involves two spatial variables and is therefore sometimes called the *two-dimensional laplace equation*. Laplace's equation is connected to the theory of analytic functions of a complex variable. If $f(z) = u(x, y) + iv(x, y)$, the real and imaginary parts $u(x, y)$, $v(x, y)$ satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Heat conduction and wave propagation usually occur in 3 space dimensions and are described by the following versions of Laplace's equation, the heat equation and the wave equation;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

$$\frac{\partial u}{\partial t} - \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0,$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0.$$

2. Linear differential operators.

All of the above mentioned partial differential equations can be written in the form

$$L[u] = F$$

where

$$L[u] \equiv \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{in Laplace's equation,}$$

$$L[u] \equiv \frac{\partial u}{\partial t} - \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t} - \kappa \nabla^2 u \text{ in the heat equation,}$$

and

$$L[u] \equiv \frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u \text{ in the heat equation.}$$

$L[u]$ is in each case, a *linear partial differential operator*. Linearity means that for any two functions u_1, u_2, and any two constants c_1, c_2,

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2].$$

In other words, L is linear if it preserves linear combinations of u_1, u_2. This definition generalises to

$$L[c_1 u_1 + \cdots + c_n u_n] = c_1 L[u_1] + \cdots + c_n L[u_n]$$

for any functions $u_1, \ldots, u_n$ and constants $c_1, \ldots, c_n$.

Let $u(x_1, x_2, \ldots, x_n)$ be a function of n variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. Then the most general linear partial differential operator is of the form

$$L[u] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\boldsymbol{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\boldsymbol{x})\frac{\partial u}{\partial x_i} + c(\boldsymbol{x})u$$

where $a_{ij}(\boldsymbol{x})$, $b_i(\boldsymbol{x})$, $c(\boldsymbol{x})$ are given coefficients.

The highest order partial derivative appearing is the *order* of the partial differential operator. Henceforth we will consider only second order partial differential operators of the form

$$L[u] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\boldsymbol{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\boldsymbol{x})\frac{\partial u}{\partial x_i} + c(\boldsymbol{x})u.$$

The general linear second order partial differential equation is of the form

$$L[u] = F(\boldsymbol{x})$$

where $F(\boldsymbol{x})$ is a given function. When $F(\boldsymbol{x}) \equiv 0$, the equation $L[u] = 0$ is called *homogeneous*. If $F(\boldsymbol{x}) \neq 0$, the equation $L[u] = F(\boldsymbol{x})$ is called *non-homogeneous*.

Linearity of L is essential to the success of the Fourier method. There are usually (infinitely) many solutions of a linear partial differential equation. The number of solutions may be restricted by imposing extra conditions. Often these extra conditions are given as linear equations involving u and its derivatives on the boundary of some region $\Omega \subset \mathbf{R}^n$. These equations are written as *boundary conditions*

$$B[u(\boldsymbol{x})] = \phi(\boldsymbol{x})$$

where B is a partial differential operator defined on the boundary $\partial\Omega$ of the region Ω.

As an example let $u(\boldsymbol{x}, t)$ be the temperature at $\boldsymbol{x} \in \Omega$, at time t, of a conducting body. Then u satisfies the heat equation

$$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = 0, \ \boldsymbol{x} \in \Omega, \ t > 0.$$

Suppose the temperature at the boundary is maintained at a given temperature ϕ, then

$$u(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, t), \ \boldsymbol{x} \in \partial\Omega, \ t > 0.$$

Also the initial temperature of the body is given by

$$u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega.$$

Suppose $u_1, \ldots, u_k$ satisfy the linear partial differential equations

$$L[u_j] = F_j, \ j = 1, \ldots, k$$

and boundary conditions

$$B[u_j] = \phi_j, \ j = 1, \ldots, k,$$

then the linear combination $u = c_1 u_1 + \ldots. + c_k u_k$ satisfies the linear partial differential equation

$$L[u] = c_1 F_1 + \ldots. + c_k F_k$$

and the boundary condition

$$B[u] = c_1 \phi_1 + \ldots. + c_k \phi_k .$$

This result is commonly referred to as the *principle of superposition* and it is of paramount importance for the Fourier method. It shows that by taking linear combinations of solutions of related linear partial differential equations, other solutions can be constructed for source and boundary terms F, ϕ which are linear combinations of simpler terms.

3. Separation of variables.

Example 1. To motivate Fourier series, consider a heat conducting bar of length l, insulated along its length, so that heat can flow only along the bar. The temperature $u(x,t)$ along the bar, satisfies the heat equation

$$u_t - \kappa u_{xx} = 0, \qquad 0 < x < l, t > 0$$

and boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t > 0.$$

Let the initial temperature along the bar at $t = 0$ be given by

$$u(x, 0) = f(x), \qquad 0 < x < l.$$

Assume a solution of the form

$$u(x, t) = X(x)T(t).$$

Such a solution is called a *separation-of-variables* solution. Substituting into the heat equation,

$$X(x)T'(t) = \kappa X''(x)T(t).$$

Dividing by $\kappa X(x)T(t)$, leads to

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}.$$

Since the variables x, t appear on separate sides of this equation, each side of this equation can only be equal to a constant, say λ. Then

$$T' = \kappa\lambda T, \quad \text{and} \quad X'' = \lambda X.$$

These are constant coefficient ordinary differential equations. λ is real but could be positive, negative or zero. Assume for the moment that $\lambda > 0$. The solutions are

$$T(t) = Ce^{\kappa\lambda t}, \qquad X(x) = Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}}$$

for A, B, C constants. Then $u(x, t) = X(x)T(t)$ satisfies $u(0, t) = 0$, $u(l, t) = 0$, $t > 0$, if and only if $X(0) = 0$, $X(l) = 0$. That is

$$0 = X(0) = A + B$$

$$0 = X(l) = Ae^{l\sqrt{\lambda}} + Be^{-l\sqrt{\lambda}}.$$

Eliminating A, B leads to the condition

$$e^{l\sqrt{\lambda}} - e^{-l\sqrt{\lambda}} = 0$$

or

$$\sinh\left(l\sqrt{\lambda}\right) = 0.$$

There are no values of $\lambda > 0$ which satisfy this condition. So λ cannot be positive.

Suppose that $\lambda < 0$ and let $\lambda = -\mu^2$ for some real μ . Then $\sqrt{\lambda} = i\mu$ and hence μ satisfies

$$e^{i\mu l} - e^{-i\mu l} = 0$$

or

$$\sin\left(\mu l\right) = 0.$$

This has solutions $\mu l = n\pi,\ n = \pm 1, \pm 2, \ldots$, and hence $\mu_n = \frac{n\pi}{l},\ n = \pm 1, \pm 2, \ldots$,

$$T_n(t) = e^{-\frac{\kappa n^2\pi^2 t}{l^2}},\ X_n(x) = e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}} = 2i\sin\left(\frac{n\pi x}{l}\right),\ n = 1, 2, \ldots$$

For $\lambda = 0$,

$$T' = 0, \qquad \text{and} \quad X'' = 0$$

or

$$T(t) = C, \qquad \text{and } X(x) = A + Bx.$$

The boundary conditions $u(0, t) = 0$, $u(l, t) = 0$, $t > 0$, are satisfied if and only if $X(0) = 0$, $X(l) = 0$. That is

$$0 = X(0) = A\,, \qquad 0 = X(l) = Bl$$

or

$$A = B = 0.$$

Therefore the non-trivial solutions of the heat equation

$$u_t - \kappa u_{xx} = 0, \qquad 0 < x < l, t > 0$$

satisfying boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad t > 0$$

are

$$u_n(x, t) = e^{-\frac{\kappa n^2\pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right),\ n = 1, 2, \ldots .$$

By superposition

$$u(x, t) = \sum_{n=1}^{\infty} b_n u_n(x, t) = \sum_{n=1}^{\infty} b_n e^{-\frac{\kappa n^2\pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$$

also satisfies the heat equation and the boundary conditions. For the initial condition to be satisfied

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \qquad 0 < x < l.$$

That is we must show that $f(x)$ can be represented as a series expansion of sines. We are mainly concerned wuth those functions $f(x)$ which have this property.

A series expansion such as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

is called a *Fourier series expansion*.

Example 2. Suppose the conducting bar is insulated at each end, the temperature $u(x,t)$ satisfies the same heat equation and initial condition but different boundary conditions

$$u_x(0, t) = 0, \; u_x(l, t) = 0, \quad t > 0.$$

Separation of variables $u(x, t) = X(x)T(t)$ in the heat equation leads to the same ordinary differential equations

$$T' = \kappa\lambda T, \quad \text{and} \quad X'' = \lambda X$$

and assuming for the moment that $\lambda > 0$, solutions

$$T(t) = Ce^{\kappa\lambda t}, \qquad X(x) = Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}}.$$

The boundary conditions are satisfied if and only if

$$0 = X'(0), \qquad 0 = X'(l).$$

That is

$$0 = \sqrt{\lambda}\,(A - B), \qquad 0 = \sqrt{\lambda}\left(Ae^{l\sqrt{\lambda}} - Be^{-l\sqrt{\lambda}}\right).$$

Eliminating A, B again leads to the condition, $\lambda \neq 0$,

$$e^{l\sqrt{\lambda}} - e^{-l\sqrt{\lambda}} = 0$$

or

$$\sinh\left(l\sqrt{\lambda}\right) = 0.$$

There are no values of $\lambda > 0$ which satisfy this condition. So λ cannot be positive.

Suppose that $\lambda < 0$ and let $\lambda = -\mu^2$ for some real μ. Then $\sqrt{\lambda} = i\mu$ and hence μ satisfies

$$e^{i\mu l} - e^{-i\mu l} = 0$$

or

$$\sin\left(\mu l\right) = 0.$$

This has solutions $\mu_n l = n\pi$, $n = \pm1, \pm2, \ldots$, and hence $\mu_n = \dfrac{n\pi}{l}$, $n = \pm1, \pm2, \ldots$,

$$\lambda_n = -\mu_n{}^2 = -\left(\frac{n\pi}{l}\right)^2, \; n = 1, 2, \ldots,$$

$$T_n(t) = e^{-\frac{\kappa n^2\pi^2 t}{l^2}}, \; X_n(x) = e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}} = 2\cos\left(\frac{n\pi x}{l}\right), \; n = 1, 2, \ldots$$

If $\lambda = 0$, $T(t) = C$, and $X(x) = A + Bx$. The boundary conditions $u(0, t) = 0$, $u(l, t) = 0$, $t > 0$, if and only if $X'(0) = 0$, $X'(l) = 0$. That is

$$0 = X'(0) = B, \qquad 0 = X'(l).$$

or

$$X(x) = A_0\,, \quad T(t) = C_0$$

Therefore the non-trivial solutions of the heat equation

$$u_t - \kappa u_{xx} = 0, \qquad 0 < x < l, t > 0$$

satisfying boundary conditions

$$u_x(0, t) = 0, \; u_x(l, t) = 0, \quad t > 0$$

are

$$u_n(x,t) = e^{-\frac{\kappa n^2 \pi^2 t}{l^2}} \cos\left(\frac{n\pi x}{l}\right), n = 0, 1, 2, \ldots .$$

By superposition

$$u(x,t) = \sum_{n=0}^{\infty} a_n u_n(x,t) = \sum_{n=0}^{\infty} a_n e^{-\frac{\kappa n^2 \pi^2 t}{l^2}} \cos\left(\frac{n\pi x}{l}\right)$$

also satisfies the heat equation and the boundary conditions. For the initial condition to be satisfied

$$u(x, 0) = f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \qquad 0 < x < l.$$

This is also called a Fourier series expansion.

4. Fourier Series.

A function $f(x)$, f: R→R is called periodic if $f(x + P) = f(x)$ for all $x \in$ R. $P > 0$ is called the period of f.

Suppose $f(x)$ is periodic with period 2π, then an important question is whether $f(x)$ has a Fourier series expansion of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right), \qquad 0 < x < 2\pi.$$

The constant term is taken as $\frac{a_0}{2}$ as a matter of convenience.
The formulas

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x$$

can be used to write the Fourier series expansion as

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

where the coefficients are

$$c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2}, \qquad n = 1, 2, \dots, \; c_0 = \frac{a_0}{2}.$$

Assuming for the moment that the 2π-periodic function f has a Fourier series expansion, the Fourier coefficients c_n will be determined using the following orthogonality property of the complex exponentials e^{inx}, $n = 0, \pm1, \pm2, \dots$,.

Lemma.

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = \begin{cases} 0; & n \neq m \\ 2\pi; & n = m \end{cases}.$$

Proof. For $n \neq m$,

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = \int_{-\pi}^{\pi} e^{i(n-m)x}\, dx$$

$$= \frac{1}{i(n-m)} \left[e^{i(n-m)\pi} - e^{-i(n-m)\pi} \right]$$

$$= \frac{2}{(n-m)} \sin(n-m)\pi = 0.$$

If $n = m$,

$$\int_{-\pi}^{\pi} e^{inx} e^{-inx}\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi.$$

Since $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$,

$$\int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx = \int_{-\pi}^{\pi} \left(\sum_{-\infty}^{\infty} c_n e^{inx} \right) e^{-imx}\, dx$$

$$= \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{-i(n-m)x}\, dx = 2\pi c_m .$$

Hence

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx , \quad m = 0, \pm 1, \pm 2, \dots ,$$

which are the Fourier coefficients of $f(x)$.

Since

$$c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2}, \qquad n = 1, 2, \dots , \ c_0 = \frac{a_0}{2} ,$$

the Fourier cosine and sine coefficients are given by

$$a_0 = 2c_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = c_n + c_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left(e^{-inx} + e^{inx} \right) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = i(c_n - c_{-n}) = \frac{i}{2\pi} \int_{-\pi}^{\pi} f(x) \left(e^{-inx} - e^{inx} \right) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

Either of the expansions

$$\sum_{-\infty}^{\infty} c_n e^{inx} , \qquad \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right),$$

is called a Fourier series expansion of $f(x)$. The first, the exponential form and the second the trigonometric form.

Notice that if $f(x)$ is an *even* periodic function, i.e. $f(-x) = f(x)$ for all x, then

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_{\pi}^{0} f(-x) \sin(-nx)\, (-dx) + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} (-f(-x) + f(x)) \sin nx\, dx = 0, \ n = 1, 2, \dots .$$

Thus the Fourier series expansion of an even function $f(x)$ has only cosine terms

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Furthermore, the coefficients a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-x)\cos(-nx)\,(-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (f(-x) + f(x))\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx, \qquad n = 0, 1, 2, \ldots .$$

Similarly if $f(x)$ is an *odd* periodic function, $f(-x) = -f(x)$, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-x)\cos(-nx)\,(-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (f(-x) + f(x))\sin nx\,dx = 0, \qquad n = 0,\ 1, 2, \ldots .$$

Thus the Fourier series expansion of an odd function $f(x)$ has only sine terms

$$\sum_{n=1}^{\infty} b_n \sin nx,$$

and the coefficients are

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-x)\sin(-nx)\,(-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (-f(-x) + f(x))\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx\,, \qquad n = 1, 2, \ldots .$$

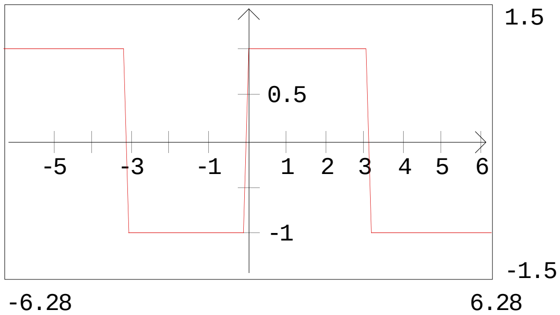
It is worth noting that for a periodic function $f(x)$, the constant term in the Fourier series

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

is the average value of $f(x)$ over a period, $-\pi < x < \pi$.

Example (a). Let $f(x)$ be periodic with period 2π,

$$f(x) = \begin{cases} -1; & -\pi < x < 0 \\ 1; & 0 < x < \pi \end{cases}.$$



This is called a squarewave function. It is obviously an odd function, hence its Fourier cosine coefficients a_n are all zero and

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^\pi 1 \sin nx\, dx$$

$$= \frac{2}{n\pi}\Big[-\cos n\pi + \cos 0\Big] = \frac{2}{n\pi}\Big[-(-1)^n + 1\Big]$$

$$= \begin{cases} 0; & n \text{ even} \\ \frac{4}{n\pi}; & n \text{ odd} \end{cases}.$$

Therefore $b_{2n} = 0$, $b_{2n-1} = \dfrac{4}{(2n-1)\pi}$, $n = 1, 2, \ldots$. The Fourier series of $f(x)$ is

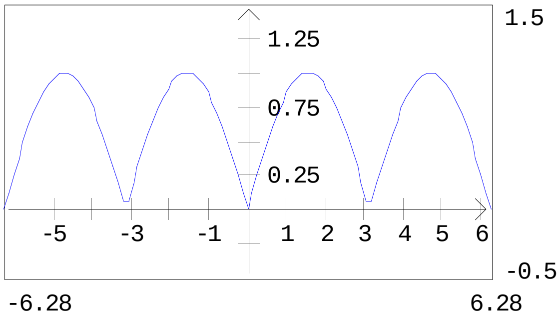
$$\sum_{n=1}^{\infty} b_n \sin nx, = \sum_{n=1}^{\infty} b_{2n-1} \sin(2n-1)x$$

$$= \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)x$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x.$$

(b). Let $f(x)$ be periodic with period 2π, $f(x) = |\sin x|$, $-\pi < x < \pi$,

$$f(x) = \begin{cases} -\sin x; & -\pi < x < 0 \\ \sin x; & 0 < x < \pi \end{cases}.$$



$f(x)$ is an even periodic function, hence $b_n = 0$ for all n.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(n+1)x + \sin((-n+1)x)\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1}(1-\cos(n+1)\pi) - \frac{1}{n-1}(1-\cos(n-1)\pi)\right]$$

$$= \begin{cases} 0; & n \text{ odd} \\ \frac{1}{\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right]; & n \text{ even} \end{cases}$$

$$= \begin{cases} 0; & n \text{ odd} \\ -\frac{4}{(n^2-1)\pi}; & n \text{ even} \end{cases}.$$

Therefore $a_{2n} = -\frac{4}{((2n)^2-1)\pi}$, $a_{2n-1} = 0$ and the Fourier series of $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx.$$

5. Bessel's inequality.

Theorem. Let f be 2π-periodic and Riemann integrable on $[-\pi, \pi]$. Then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.$$

Proof. Using $z\overline{z} = |z|^2$ for complex z,

$$0 \leq \left| f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 = \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right) \overline{\left(f(x) - \sum_{m=-N}^{N} c_m e^{imx} \right)}$$

$$= |f(x)|^2 - \sum_{n=-N}^{N} c_n e^{inx} \, \overline{f(x)} - \sum_{m=-N}^{N} \overline{c_m} \, e^{-imx} f(x) + \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} \, e^{i(n-m)x}$$

Dividing by 2π and integrating over $[-\pi, \pi]$,

$$0 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \sum_{n=-N}^{N} c_n \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} f(x) \, dx} - \sum_{m=-N}^{N} \overline{c_m} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx} f(x) \, dx$$

$$+ \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \sum_{n=-N}^{N} c_n \overline{c_n} - \sum_{m=-N}^{N} c_m \overline{c_m} + \sum_{n=-N}^{N} c_n \overline{c_n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \sum_{n=-N}^{N} |c_n|^2 \qquad \text{for any } N.$$

Letting $N \to \infty$ leads to the result.

Bessel's inequality will later be shown to be actually an equality but for now it implies that the series $\sum_{n=-\infty}^{\infty} |c_n|^2$ converges where c_n are the Fourier coefficients of the Riemann integrable function f.

Using the equations

$$c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2}, \qquad n = 1, 2, \dots, \; c_0 = \frac{a_0}{2},$$

Bessel's inequality can be written as

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^{\ 2}}{4} + \sum_{n=1}^{\infty} \left|\frac{a_n - ib_n}{2}\right|^2 + \sum_{n=1}^{\infty} \left|\frac{a_n + ib_n}{2}\right|^2$$

$$= \frac{a_0^{\ 2}}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \left[|a_n|^2 + |b_n|^2 \right]$$

$$\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx .$$

This implies that the series

$\sum_{n=1}^{\infty} |a_n|^2$, $\sum_{n=1}^{\infty} |b_n|^2$ also converge, where a_n, b_n are the Fourier cosine and sine coefficients of f.

6. Convergence results for Fourier series.

We will consider the question: For what functions f do the Fourier series

$$\sum_{-\infty}^{\infty} c_n e^{inx}\ ,\quad \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

converge?

Because we are dealing with functions, the concept of convergence must be made precise. Do we mean the numerical series converging at every $x \in [-\pi, \pi]$? Can convergence be different at different points x, indeed can we have convergence at some points and not at others and if so, which are the points of convergence? Can we have some sort of average convergence on $[-\pi, \pi]$?

A function f is called piecewise continuous on an interval $[a, b]$ if it is continuous everywhere except at finitely many points $x_1, x_2, \ldots, x_k \in [a, b]$ and the left-hand and right-hand limits of f exist at each of the points $x_1, x_2, \ldots, x_k$. The set of all piecewise continuous functions on $[a, b]$ is denoted by $PC[a, b]$.

f is called piecewise smooth if f and its derivative f' are piecewise continuous on $[a, b]$.

Consider the N^{th} partial sum of the complex Fourier series of f,

$$S_N^f(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

$$= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} f(y)\, dy \right) e^{inx}$$

$$= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-y)} \right) f(y)\, dy.$$

We define $D_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx}$. Then

$$S_N^f(x) = \int_{-\pi}^{\pi} D_N(x-y) f(y)\, dy = \int_{-\pi}^{\pi} D_N(y) f(x-y)\, dy\ .$$

[The last equality follows from a change of variable and the periodicity of the integrand $D_N(x-y)f(y)$]. The function $D_N(x)$ is called the Dirichlet kernel and

$$D_N(x) = \frac{1}{2\pi} e^{-iNx} \sum_{n=0}^{2N} e^{inx}$$

$$= \frac{1}{2\pi} e^{-iNx} \left(\frac{e^{i(2N+1)x} - 1}{e^{ix} - 1} \right)$$

$$= \frac{1}{2\pi}\left(\frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1}\right)$$

$$= \frac{1}{2\pi}\left(\frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}}\right)$$

$$= \frac{1}{2\pi}\,\frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}.$$

The question of convergence of the Fourier series reduces to question:
Does $S_N^f(x)$ converge as $N \to \infty$ for $x \in [-\pi, \pi]$? If so, to what does it converge?

We prove a preliminary result.

Lemma.

$$\int_{-\pi}^{0} D_N(x)\,dx = \int_{0}^{\pi} D_N(x)\,dx = \frac{1}{2}.$$

Proof. $D_N(x) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{\substack{n=-N\\ n\neq 0}}^{N} e^{inx} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos nx$

From the even-ness of $D_N(x)$ and integrating,

$$\int_{-\pi}^{0} D_N(x)\,dx = \int_{0}^{\pi} D_N(x)\,dx = \int_{0}^{\pi}\left[\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos nx\right]dx$$

$$= \left[\frac{x}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\frac{\sin nx}{n}\right]_0^{\pi} = \frac{1}{2}.$$

Theorem. Let f be a piecewise smooth 2π-periodic function on **R**. Then

$$\lim_{N\to\infty} S_N^f(x) = \frac{1}{2}\left[f(x^-) + f(x^+)\right]$$

for every x. Hence $\lim_{N\to\infty} S_N^f(x) = f(x)$ for each point x of continuity of f.

Proof. $$S_N^f(x) - \frac{1}{2}\left[f(x^-) + f(x^+)\right] = \int_{-\pi}^{\pi} D_N(y) f(x-y)\,dy - \frac{1}{2}\left[f(x^-) + f(x^+)\right]$$

$$= \int_{0}^{\pi} D_N(y)[f(x-y) - (f(x^-)]dy + \int_{-\pi}^{0} D_N(y)[f(x-y) - f(x^+)]dy$$

(from lemma above)

$$= \int_0^{\pi} D_N(y)[f(x-y)-f(x^-)]dy + \int_{\pi}^{0} D_N(-y)[f(x+y)-f(x^+)](-dy)$$

$$= \int_0^{\pi} D_N(y)[f(x-y)-f(x^-)+f(x+y)-f(x^+)]dy$$

$$= \int_0^{\pi} \frac{1}{2\pi} \frac{\sin\left(N+\frac{1}{2}\right)y}{\sin\left(\frac{y}{2}\right)} [f(x-y)-f(x^-)+f(x+y)-f(x^+)]dy$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin\left(N+\frac{1}{2}\right)y \left(\frac{\frac{y}{2}}{\sin\left(\frac{y}{2}\right)}\right)\left[\frac{f(x-y)-f(x^-)}{y}+\frac{f(x+y)-f(x^+)}{y}\right]dy$$

For fixed x define

$$g(y) = \left(\frac{\frac{y}{2}}{\sin\left(\frac{y}{2}\right)}\right)\left[\frac{f(x-y)-f(x^-)}{y}+\frac{f(x+y)-f(x^+)}{y}\right],$$

which is an odd piecewise continuous function for $[-\pi, \pi]$ by the condition of piecewise smoothness on f.

Then

$$S_N^f(x) - \frac{1}{2}\left[f(x^-)+f(x^+)\right] = \frac{1}{\pi}\int_0^{\pi} \sin\left(N+\frac{1}{2}\right)y\ g(y)dy$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(Ny)\left\{\cos\left(\frac{y}{2}\right)g(y)\right\}dy + \frac{1}{\pi}\int_0^{\pi}\cos(Ny)\left\{\sin\left(\frac{y}{2}\right)g(y)\right\}dy$$

The last two terms are the Fourier sine and cosine coefficients B_N, A_N of $\frac{1}{2}\cos\left(\frac{y}{2}\right)g(y), \frac{1}{2}\sin\left(\frac{y}{2}\right)g(y)$ respectively. By Bessel's inequality, $B_N, A_N \to 0$ as $N \to \infty$.

Therefore

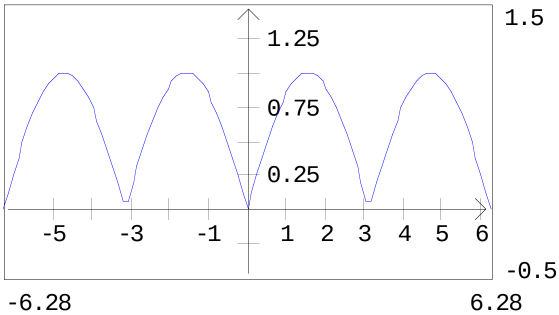
$$S_N^f(x) - \frac{1}{2}\left[f(x^-)+f(x^+)\right] = B_N + A_N \to 0, \text{ as } N \to \infty.$$

This result, that the partial sums of the Fourier series of a piecewise smooth 2π–periodic function converges pointwise to the mean of the left and right hand limits at x. If x is a point of continuity of f, then the partial sums converge pointwise to $f(x)$.

Fourier series provide a useful method for summing certain numerical series.

Example. The Fourier series of the continuous periodic function $f(x) = |\sin x|$, $x \in [-\pi, \pi]$, is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nx$$



Because $x = 0$ is a point of continuity of f, the Fourier series converges at $x = 0$ to $f(0) = 0$,

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

At $x = \frac{\pi}{2}$, the Fourier series converges to

$$f\left(\frac{\pi}{2}\right) = 1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos(n\pi)$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = -\left(\frac{\pi-2}{4}\right)$$

7. Differentiation and Integration of Fourier Series.

Fourier series can be differentiated term-by-term but the question is does the resulting series converge and if so, to what does it converge? Similar concerns apply to the series resulting from the term-by-term integration of a Fourier series.

Theorem. Let a_n, b_n, c_n be the Fourier coefficients of a 2π-periodic piecewise smooth function f and a_n', b_n', c_n' be the Fourier coefficients of f'. Then

$$a_n' = nb_n, \qquad b_n' = -na_n, \qquad c_n' = inc_n.$$

Proof. By integration by parts,

$$c_n' = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} f(x) e^{-inx} \Big|_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, (-ine^{-inx})dx$$

$$= in\, \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = inc_n.$$

A similar proof works for $a_n' = nb_n, \quad b_n' = -na_n$.

Theorem. Let f be 2π-periodic, piecewise smooth, with piecewise smooth derivative f'. Then the Fourier series of f' is

$$\sum_{-\infty}^{\infty} inc_n e^{inx} = \sum_{n=1}^{\infty} \left(nb_n \cos nx - na_n \sin nx \right)$$

and converge for each x where $f'(x)$ exists. If f' is not continuous at x, then the series above converge to $\frac{1}{2}[f'(x-) + f'(x+)]$.

Proof. This result follows by combining the previous two theorems.

Integration of Fourier series is not so straightforward since the anti-derivative of a periodic function need not be periodic. For example, $f(x) = 1$ is periodic but its anti-derivative $F(x) = x$ is not. However, since all but the constant term of a Fourier series has a periodic anti-derivative, the following result is true.

Theorem. Let f be 2π-periodic, piecewise continuous with Fourier coefficients a_n, b_n, c_n and let $F(x) = \int_0^x f(t)\, dt$. If $c_0 = \frac{1}{2} a_0 = 0$, then for all x,

$$F(x) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{c_n}{in} e^{inx} = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\left(\frac{-b_n}{n}\right) \cos nx + \left(\frac{a_n}{n}\right) \sin nx \right)$$

where $C_0 = \frac{A_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx$ the average value of F on $[-\pi, \pi]$.
If $c_0 \neq 0$, then the series above converges to $F(x) - c_0 x$.

Proof. Since $f(x)$ is piecewise continuous, $F(x) = \int_0^x f(t)\,dt$ is continuous.

If $c_0 = 0$, $F(x)$ is 2π-periodic since

$$F(x + 2\pi) - F(x) = \int_0^{x+2\pi} f(t)\,dt - \int_0^x f(t)\,dt = \int_x^{x+2\pi} f(t)\,dt = \int_{-\pi}^{\pi} f(t)\,dt = 2\pi c_0 = 0.$$

Therefore the Fourier series of F converges pointwise at each $x \in [-\pi, \pi]$ to $F(x)$. Since $f(x) = F'(x)$, by the two previous theorems,

$$a_n = nB_n, \qquad b_n = -nA_n, \quad c_n = inC_n,$$

where a_n, b_n, c_n are the Fourier coefficients of f and A_n, B_n, C_n are the Fourier coefficients of F. If $n \neq 0$, this implies that

$$A_n = -\frac{b_n}{n}, \quad B_n = \frac{a_n}{n}, \qquad C_n = \frac{c_n}{in}.$$

The constant $C_0 = \frac{A_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx$ is the constant term in the Fourier series of

$F(x) = \int_0^x f(t)\,dt.$

If $c_0 \neq 0$, $f(x) - c_0$ has zero mean value on $[-\pi, \pi]$ and therefore its zeroth Fourier coefficient is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - c_0)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx - c_0 = c_0 - c_0 = 0.$$

Applying the result just obtained to $f(x) - c_0$ and its anti-derivative $F(x) - c_0 x$ completes the theorem.

Integrating and differentiating known Fourier series using the above results is a useful way of obtaining new Fourier series.

Example. The 2π-periodic function $f:\mathbf{R}\to\mathbf{R}$, $f(x) = x(\pi - |x|)$, $x \in [-\pi, \pi]$, is continuous with piecewise smooth derivative. Therefore its Fourier series converges at each x to $f(x)$. The Fourier series is by the table #17

$$f(x) = x(\pi - |x|) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$$

By the above results,

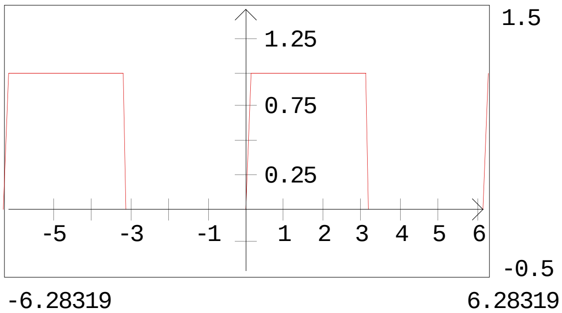
$$f'(x) = \pi - 2|x| = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x$$

or

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

which agrees with table #2.

Example. $f(x) = \begin{cases} 0; & -\pi < x < 0 \\ 1; & 0 < x < \pi \end{cases}$



is piecewise continuous with Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

(table #7). Since $c_0 = \frac{1}{2} \neq 0$, $F(x) = \int_0^x f(t)\,dt. = \begin{cases} 0; & -\pi < x < 0 \\ x; & 0 < x < \pi \end{cases}$.

Therefore

$$F(x) - \frac{x}{2} = \frac{A_0}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

Since $\frac{A_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{\pi}{4}$, $F(x) - \frac{x}{2} = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x.$

But $F(x) - \frac{x}{2} = \begin{cases} -\frac{x}{2}; & -\pi < x < 0 \\ \frac{x}{2}; & 0 < x < \pi \end{cases} = \frac{|x|}{2}$

Therefore

$$\frac{|x|}{2} = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

or

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x.$$

This result agrees with the previous example

8. Half-range Fourier series.

It is often convenient to represent a given function as a Fourier series which contains only cosine terms or only sine terms, as in the initial examples.

If $f(x)$ is piecewise continuous on $[0, \pi]$, it can be extended to $[-\pi, 0]$ as either an even function or as an odd function.

Then f is periodically extended to the whole real line as a 2π-periodic function by $f(x + 2\pi) = f(x)$, $x \in \mathbf{R}$.

Let $f(x)$ be a piecewise smooth function given on the interval $[0, \pi]$ and extended to $[-\pi, 0]$ as an even function. That is, $f(x) = f(-x)$, $x \in [-\pi, 0]$. Then the periodic extension to $\mathbf{R}$, is piecewise smooth and has Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx.$$

Similarly, extending $f(x)$ to $[-\pi, 0]$ as an odd function and periodically to $\mathbf{R}$, the extension has Fourier series

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

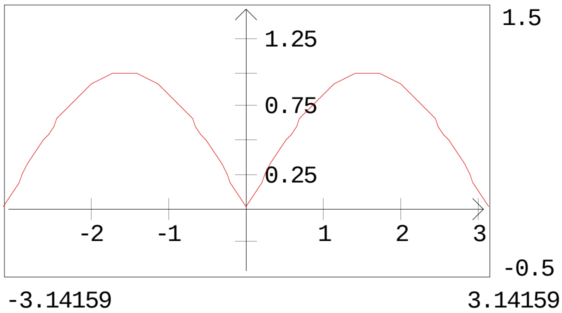
The Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, $\sum_{n=1}^{\infty} b_n \sin nx$ are known as half-range series for $f(x)$, $x \in [0, \pi]$.

Example. Let $f(x) = \sin x$, $x \in [0, \pi]$. Extend f to $[-\pi, 0]$ as an even function.

That is

$$f(x) = \begin{cases} \sin(-x) = -\sin x; & -\pi < x < 0 \\ \sin x; & 0 < x < \pi \end{cases}$$

$$= |\sin x|$$



The half-range Fourier series of f is given by

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n^2 - 1} \cos 2nx \quad \text{(table \#8).}$$

9. General Intervals.

The theory of Fourier series can be expanded to include functions which have arbitrary period. Let $f:\mathbf{R}\to\mathbf{R}$ have period $2l > 0$. That is $f(x + 2l) = f(x)$, $x \in \mathbf{R}$. Also define $\phi:\mathbf{R}\to\mathbf{R}$ by $\phi(x) = f\left(\frac{lx}{\pi}\right)$. Then

$$\phi(x + 2\pi) = f\left(\frac{l(x + 2\pi)}{\pi}\right) = f\left(\frac{lx}{\pi} + 2l\right) = f\left(\frac{lx}{\pi}\right) = \phi(x)$$

So ϕ has period 2π. Applying the earlier results to ϕ, results in the Fourier series

$$\phi(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin nx\,dx, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)\,e^{-inx}\,dx.$$

The change of variable $y = \frac{lx}{\pi}$ leads to

$$a_n = \frac{1}{l}\int_{-l}^{l} \phi\left(\frac{\pi y}{l}\right)\cos\left(\frac{n\pi y}{l}\right)dy = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi y}{l}\right)dy,$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \phi\left(\frac{\pi y}{l}\right)\sin\left(\frac{n\pi y}{l}\right)dy = \frac{1}{l}\int_{-l}^{l} f(y)\sin\left(\frac{n\pi y}{l}\right)dy,$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} \phi\left(\frac{\pi y}{l}\right)e^{-\frac{in\pi y}{l}}\,dy = \frac{1}{2l}\int_{-l}^{l} f(y)\,e^{-\frac{in\pi y}{l}}\,dy.$$

Therefore

$$f(x) = \phi\left(\frac{\pi x}{l}\right) = \sum_{-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{l}}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right)$$

is the Fourier series of a $2l$-periodic function f.

It follows from the above calculations that if $f(x)$ is an even function on $[-l, l]$, its Fourier series contains no sine terms and is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right),$$

with

$$a_n = \frac{2}{l}\int_0^l f(\xi)\cos\left(\frac{n\pi\xi}{l}\right)d\xi$$

Likewise if $f(x)$ is an odd function on $[-l, l]$, its Fourier series contains no cosine terms and is of the form

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

with

$$b_n = \frac{2}{l}\int_0^l f(\xi)\sin\left(\frac{n\pi\xi}{l}\right)d\xi.$$

A function $f(x)$ defined on an interval $[0, l]$ can be extended to $[-l, l]$ either as an even or an odd function and the extended periodically with period $2l$ to the real line. The Fourier series of such functions are half-range expansions on $[0, l]$ and contain no sine terms in the case of an even extension or no cosine terms for an odd extension.

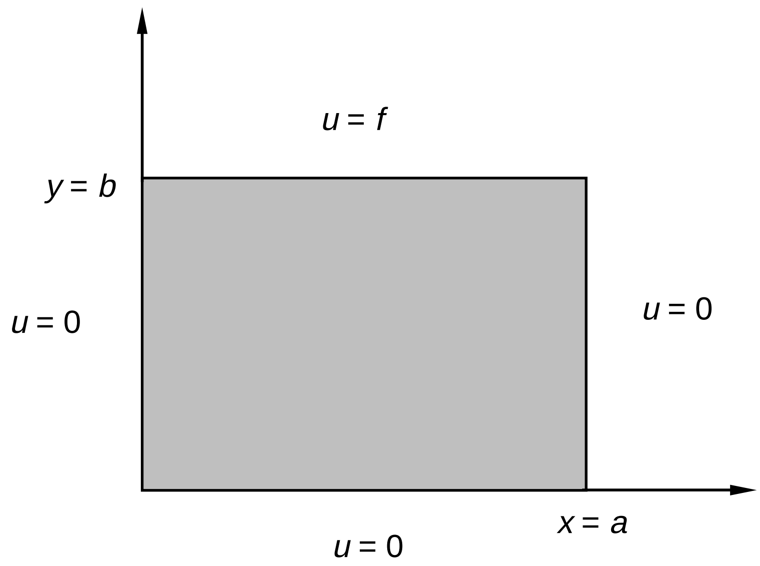
10. Application to Laplace's equation.

Example 1. Let Ω be a rectangle $0 \le x \le a$, $0 \le y \le b$, and consider the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 \le x \le a, \; 0 \le y \le b$$

$$u(x, 0) = 0, \; u(x, b) = f(x), \; 0 \le x \le a$$

$$u(0, y) = u(a, y) = 0, \; 0 \le y \le b.$$



Assuming a separation of variables solution of the form $u(x, y) = X(x)Y(y)$ and substituting into Laplaces equation,

$$X''Y + XY'' = 0$$

or

$$-\frac{X''}{X} = \frac{Y''}{Y} = \text{constant} = \lambda$$

For $\lambda > 0$, $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$, hence

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x, \;\; Y(y) = C\cosh\sqrt{\lambda}y + D\sinh\sqrt{\lambda}y.$$

To satisfy the boundary conditions $u(0, y) = u(a, y) = 0, \; 0 \le y \le b$, requires that

$$X(0) = A = 0, \qquad X(a) = B\sin\left(a\sqrt{\lambda}\right) = 0.$$

Therefore $a\sqrt{\lambda} = n\pi$, $\qquad \lambda = \left(\frac{n\pi}{a}\right)^2$, $n = 1, 2, \ldots$, and $X(x) = B\sin\left(\frac{n\pi x}{a}\right)$

To also satisfy the boundary condition $u(x, 0) = 0, \; 0 \le x \le a$, requires that

$Y(0) = C = 0$ and therefore $Y(y) = D \sinh\left(\frac{n\pi y}{a}\right)$. The separation of variables solutions for $\lambda > 0$ are

$$u(x, y) = X(x)Y(y) = BD \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right), \quad n = 1, 2, \dots ,$$

for an arbitrary constant BD.

For $\lambda < 0$, let $\lambda = -\mu$, $\mu > 0$. Then $X'' - \mu X = 0$, $Y'' + \mu Y = 0$, hence

$$X(x) = A \cosh\sqrt{\mu}x + B \sinh\sqrt{\mu}x, \quad Y(y) = C \cos\sqrt{\mu}y + D \sin\sqrt{\mu}y .$$

To satisfy the boundary conditions $u(0, y) = u(a, y) = 0$, $0 \le y \le b$, requires that

$$X(0) = A = 0, \qquad X(a) = B \sinh\left(a\sqrt{\lambda}\right) = 0.$$

But $B = 0$ gives a trivial solution and $\sinh\left(a\sqrt{\lambda}\right) \neq 0$ for $\lambda < 0$. So there is only the trivial separation of variables solution $u = 0$, for $\lambda < 0$.

Finally suppose that $\lambda = 0$, then $X'' = 0$, $Y'' = 0$, hence

$$X(x) = A + Bx, \qquad Y(y) = C + Dy.$$

To satisfy the boundary conditions $u(0, y) = u(a, y) = 0$, $0 \le y \le b$, requires that

$$X(0) = A = 0, \qquad X(a) = Ba = 0.$$

That is, $A = B = 0$ and the only separation of variables solution for $\lambda = 0$ is the trivial solution.

Summarising thus far, there is a sequence of non-trivial solutions of the form

$$u_n(x, y) = B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right), \qquad n = 1, 2, \dots ,$$

where B_n are the constants BD for each $n = 1, 2, \dots$. By superposition, a solution is also given by

$$u(x, y) = \sum_{n=1}^{\infty} B_n u_n(x, y)$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right).$$

The boundary condition $u(x, b) = f(x)$, $0 \le x \le a$, implies that

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)$$

Since $\phi(x)$, $0 \le x \le a$, has a Fourier half-range series $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)$, where

$$b_n = \frac{2}{a}\int_0^a f(\xi) \sin\left(\frac{n\pi\xi}{a}\right) d\xi = B_n \sinh\left(\frac{n\pi b}{a}\right)$$

Therefore

$$B_n = \frac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)} = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(\xi) \sin\left(\frac{n\pi\xi}{a}\right) d\xi$$

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

Suppose all four sides of the rectangle have non-zero boundary conditions, then we can break up the problem into four sub-problems each similar to the one we have just solved. Each of these subproblems will have zero boundary conditions on three of the four sides and can be solved as above. Then the solution to the boundary-value problem is the sum of the four sub-problems by superposition.

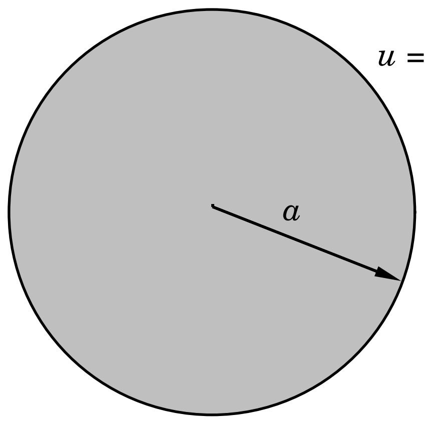
Example 2. We seek a solution to Laplaces' equation on a circle, satisfying Dirichlet boundary conditions. It is natural to choose Ω a circle centre the origin, radius a, and Laplaces' equation in polar coordinates,

$$\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad r < a,\ 0 \le \theta < 2\pi,$$

subject to boundary conditions

$$u(a, \theta) = f(\theta), \quad 0 \le \theta < 2\pi,$$

where ϕ is a continuous function on $0 \le \theta < 2\pi$.



Using separation of variables, assume that

$$u(r,\ \theta) = R(r)\Theta(\theta)$$

and substitute into the differential equation to obtain

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

or separating variables,

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \text{constant} = \lambda^2 \text{ (say)},$$

which leads to the two families of ordinary differential equations

$$r^2R'' + rR' - \lambda^2R = 0,\ \Theta'' + \lambda^2\Theta = 0.$$

Consider the case $\lambda = 0$; then the solutions of the above equations are

$$\Theta = A_0 + B_0\theta,$$

$$R = C_0 + D_0 \log r$$

Since solutions u must be periodic in θ, and bounded for $r \le a$, we conclude that $B_0 = 0, D_0 = 0$.

In the case $\lambda \neq 0$ the solutions of the above equations are

$$\Theta = A_\lambda \cos \lambda\theta + B_\lambda \sin \lambda\theta$$

$$R = C_\lambda r^\lambda + D_\lambda r^{-\lambda}$$

where the coefficients depend on λ. Again because solutions are periodic in θ with period 2π, we conclude that $\lambda = n = 1,2,3,..,$ and because solutions are bounded for $r \le a, D_n = 0, n = 1,2,3,...$ By superposition we can combine these solutions to obtain

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n r^n \left(A_n \cos n\theta + B_n \sin n\theta \right)$$

We may as well incorporate the constant C_n into A_n and B_n, hence

$$u(r,\ \theta) = \sum_{n=0}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta \right)$$

At the boundary $r = a$,

$$f(\theta) = u(a,\ \theta) = \sum_{n=0}^{\infty} a^n \left(A_n \cos n\theta + B_n \sin n\theta \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta \right)$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt \ , \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt$$

are the Fourier coefficients of the function f. Equating coefficients of cos $n\theta$, sin $n\theta$, in these two expressions for $f(\theta)$, we obtain that

$$A_n = \frac{a_n}{a^n} \, , \ B_n = \frac{b_n}{a^n}$$

and hence

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(a_n \cos n\theta + b_n \sin n\theta \right)$$

This series converges *uniformly* for $0 \le \theta < 2\pi, r < a$.

We can get a closed form expression for u by substituting the formulae for the Fourier coefficients as follows.

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \, dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \int_0^{2\pi} f(t) \left(\cos nt \cos n\theta + \sin nt \sin n\theta \right) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(t) \, dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \int_0^{2\pi} f(t) \cos n(t - \theta) \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(t) \, dt + \frac{1}{\pi} \int_0^{2\pi} f(t) \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(t - \theta) \, dt$$

where the interchange of the order of summation and integration is allowed by the uniform convergence of the series.

Therefore $$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(t - \theta) \right] dt.$$

Using the exponential form, $2 \cos x = e^{ix} + e^{-ix}$ the term in the square brackets reduces to two infinite geometric series whose sum is

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(t - \theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos(t - \theta) + r^2}$$

(exercise). Therefore

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{a^2 - r^2}{a^2 - 2ar \cos(t - \theta) + r^2} dt,$$

which is *Poissons' integral formula for the circle*.

11. Sturm-Liouville problems and orthogonal functions.

The functions $\sin nx$, $\cos nx$, e^{inx}, e^{-inx}, $n = 0, 1, 2, \ldots$, are examples of orthogonal functions on $[-\pi, \pi]$ Their orthogonality properties follow from the fact that they solutions of linear second order ordinary differential equations.

For example $\sin nx$, $\cos nx$, e^{inx} satisfy

$$-u''(x) = \lambda u(x),$$

$$u(-\pi) = u(\pi), \qquad u'(-\pi) = u'(\pi)$$

for $\lambda = n^2$. The ordinary differential equation together with the boundary conditions is called a *boundary-value problem.*

The set $\{e^{inx}, n = 0, \pm 1, \pm 2, \ldots\}$ form a basis for the vector space $L_2[-\pi, \pi]$ consisting of functions $f : [-\pi, \pi] \to \mathbf{R}$ for which $\int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty$. A set of functions $\{\phi_n(x);\ n = 1, 2, \ldots\}$ is said to be a basis for $L_2[-\pi, \pi]$ if for any function $f \in L_2[-\pi, \pi]$, there is a unique set of scalars c_n, $n = 1, 2, \ldots$ such that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right|^2 dx = 0.$$

Then we say that f has the expansion $f = \sum_{n=1}^{\infty} c_n \phi_n$.

Furthermore, if the set $\{\phi_n(x);\ n = 1, 2, \ldots\}$ is orthonormal, that is

$$\int_{-\pi}^{\pi} \phi_n(x)\, \overline{\phi_m(x)}\, dx = \begin{cases} 0; & n \neq m \\ 1; & n = m \end{cases}$$

then the coefficients c_n are given by

$$c_n = \int_{-\pi}^{\pi} f(x)\, \overline{\phi_n(x)}\, dx, \qquad n = 1, 2, \ldots .$$

The interval $[-\pi, \pi]$ can be replaced by $[a, b]$ and the theory of Fourier series generalised to the expansion of arbitrary functions $f \in L_2[a, b]$ in terms of an orthonormal basis $\{\phi_n(x);\ n = 1, 2, \ldots\}$ consisting of functions which are solutions of a boundary-value problem for certain second order linear ordinary differential equations.

Let $p(x)$, $q(x)$, $w(x)$ be real continuous functions on the interval $[a, b]$. Let $p(x)$ be continuous and $p(x) > 0$, $q(x) \geq 0$, $w(x) \geq 0$ on $[a, b]$. Let α_0, α_1, β_0, β_1, be real constants.

A *Sturm-Liouville problem* is a boundary value problem

$$-\big(p(x)u'\big)' + q(x)u = \lambda w(x)u, \quad x \in [a, b]$$

$$\alpha_0 u(a) + \alpha_1 u'(a) = 0,$$

$$\beta_0 u(b) + \beta_1 u'(b) = 0.$$

The value of λ for which this boundary-value problem has non-trivial solutions are called *eigenvalues* of the boundary-value problem and the corresponding solutions $u(x)$ are called *eigenfunctions*. It can be shown that the eigenvalues form a countable set $\{\lambda_n; n = 1, 2, \ldots\}$ and the corresponding eigenfunctions $\{\phi_n(x); n = 1, 2, \ldots\}$ are orthonormal with respect to the weighted inner-product

$$\langle f; g\rangle_w = \int_a^b w(x)\, f(x)\, \overline{g(x)}\, dx.$$

Let the vector space $L_2^w[a, b]$ consist of functions $f:[a, b] \to \mathbf{R}$ for which $\|f\|^2 = \int_a^b w(x)\, |f(x)|^2\, dx < \infty$. A set of functions $\{\phi_n(x); n = 1, 2, \ldots\}$ is said to be a basis for $L_2^w[a, b]$ if for any function $f \in L_2^w[a, b]$, there is a unique set of scalars c_n, $n = 1, 2, \ldots$ such that

$$\lim_{N\to\infty} \int_a^b w(x) \left| f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right|^2 dx = 0.$$

Let the linear Sturm-Liouville differential operator L be defined as

$$L[f] = \frac{1}{w(x)} \left[-\left(p(x) f' \right)' + q(x) f \right].$$

Let $f(x)$, $g(x)$ be C^2 functions on $[a, b]$. Then

$$\langle L[f]; g\rangle_w = \int_a^b \left[\left(p(x) f' \right)' + q(x) f \right] \overline{g(x)}\, dx$$

$$= -p(x) f' \overline{g} \Big|_a^b + \int_a^b \left(p(x) f' \overline{g}' + q(x) f \overline{g} \right) dx$$

Also

$$\langle f; L[g]\rangle_w = \int_a^b f(x)\, \overline{\left[\left(p(x) g' \right)' + q(x) g \right]}\, dx$$

$$= -p(x) f \overline{g'} \Big|_a^b + \int_a^b \left(p(x) f' \overline{g'} + q(x) f \overline{g} \right) dx.$$

Subtracting these two expressions,

$$\langle L[f]; g\rangle_w - \langle f; L[g]\rangle_w = -p(x) \left(f' \overline{g} - f \overline{g'} \right) \Big|_a^b.$$

This is Lagrange's identity.

Lemma. If f, g are C^2 functions on $[a, b]$ which satisfy

$$\alpha_0 f(a) + \alpha_1 f'(a) = 0, \qquad \alpha_0 g(a) + \alpha_1 g'(a) = 0,$$

$$\beta_0 f(b) + \beta_1 f'(b) = 0, \qquad \beta_0 g(b) + \beta_1 g'(b) = 0,$$

then

$$<L[f]\,;g>_w \; = \; <f\,;L[g]>_w \quad .$$

Proof. Suppose $\alpha_0 \neq 0$, $\beta_0 \neq 0$, then by Lagrange's identity,

$$<L[f]\,;g>_w - <f\,;L[g]>_w \; = -p(x)\left(f'\,\overline{g} - f\,\overline{g'} \right)\Big|_a^b$$

$$= -p(b)\left[f(b)\left(-\frac{\beta_1}{\beta_0}\,\overline{g'(b)} \right) - \left(-\frac{\beta_1}{\beta_0} f'(b) \right)\overline{g'(b)} \right]$$

$$+\, p(a)\left[f(a)\left(-\frac{\alpha_1}{\alpha_0}\,\overline{g'(a)} \right) - \left(-\frac{\alpha_1}{\alpha_0} f'(a) \right)\overline{g'(a)} \right]$$

$$= 0.$$

Similarly if $\alpha_1 \neq 0$, $\beta_1 \neq 0$ or $\alpha_1 \neq 0$, $\beta_0 \neq 0$ or $\alpha_0 \neq 0$, $\beta_1 \neq 0$, the result follows with minor changes to the argument.

–

Lemma. The eigenvalues of the Sturm-Liouville operator L are real.

Proof. Let λ be an eigenvalue with corresponding eigenfunction ϕ. Then ϕ satisfies

$$L[\phi] = \frac{1}{w(x)}\left[-\left(p(x)\phi' \right)' + q(x)\phi \right] = \lambda\phi,$$

$$\alpha_0 \phi(a) + \alpha_1 \phi'(a) = 0,$$

$$\beta_0 \phi(b) + \beta_1 \phi'(b) = 0.$$

Then

$$\lambda\, \|\phi\|_w^2 = \lambda <\phi\,;\phi>_w \; = \; <\lambda\phi\,;\phi>_w$$

$$= \; <L[\phi]\,;\phi>_w \; = \; <\phi\,;L[\phi]>_w \quad \text{(above lemma)}$$

$$= \; <\phi\,;\lambda\phi>_w \; = \; \overline{\lambda} <\phi\,;\phi>_w \; = \; \overline{\lambda}\, \|\phi\|_w^2.$$

Therefore

$$(\lambda - \overline{\lambda})\|\phi\|_w^2 = 0.$$

Since $\phi \neq 0$, $\lambda = \overline{\lambda}$ and λ is real.

–

Lemma. Let $\phi(x)$, $\psi(x)$ be eigenfunctions of the Sturm-Liouville boundary-value problem with corresponding eigenvalues λ, μ respectively. Then If $\lambda \neq \mu$, $\phi(x)$, $\psi(x)$ are orthogonal on $[a, b]$ with respect to the weighted inner product $<f\,; g>_w$.

Proof.

$$\begin{cases} L[\phi] = \lambda\phi \\ \alpha_0\phi(a) + \alpha_1\phi'(a) = 0 \\ \beta_0\phi(b) + \beta_1\phi'(b) = 0 \end{cases} \qquad \begin{cases} L[\psi] = \lambda\psi \\ \alpha_0\psi(a) + \alpha_1\psi'(a) = 0 \\ \beta_0\psi(b) + \beta_1\psi'(b) = 0 \end{cases}$$

From a previous lemma,

$$<L[\phi]\,;\psi>_w \;= <\phi\,; L[\psi]>_w \quad ,$$

$$<\lambda\phi\,;\psi>_w \;= <\phi\,;\mu\psi>_w \quad ,$$

$$(\lambda-\mu)<\phi\,;\psi>_w \;= 0.$$

Since $\lambda \neq \mu$, $<\phi\,;\psi>_w \;= 0$.

These results can be used to prove the following result:

Theorem. The eigenfunctions of a regular Sturm-Liouville problem are a countable set $\{\phi_n(x); n = 1, 2, \dots\}$ and form an orthonormal basis for $L_2^w[a, b]$. That is, each function $f \in L_2^w[a, b]$ has a series expansion $f = \sum_{n=1}^{\infty} c_n\phi_n$ where the $c_n = <f\,;\phi_n>_w$, $n = 1, 2, \dots$, and convergence is in the sense that

$$\lim_{N\to\infty} \int_a^b w(x) \left| f(x) - \sum_{n=1}^{N} c_n\phi_n(x) \right|^2 dx \;= 0,$$

or

$$\lim_{N\to\infty} \| f - \sum_{n=1}^{N} c_n\phi_n \|_w^2 = 0.$$

If $f(x) \in C^2[a, b]$ and satisfies the boundary conditions $\alpha_0 f(a) + \alpha_1 f'(a) = 0$, $\beta_0 f(b) + \beta_1 f'(b) = 0$, the series $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$ converges uniformly on $[a, b]$. That is

$$\lim_{N\to\infty} \max_{x\in[a;\,b]} \left| f(x) - \sum_{n=1}^{N} c_n\phi_n(x) \right| = 0.$$

12. Bessel Functions.

Consider the heat equation on a disc $0 \le r < a,\ 0 \le \theta < 2\pi$,,

$$\frac{\partial u}{\partial t} - \kappa \Delta u(r,\ \theta) = \frac{\partial u}{\partial t} - \kappa\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right) = 0,$$

$0 \le r < a,\ 0 \le \theta < 2\pi,\ t > 0$, where $u(r,\ \theta,\ t)$ is the temperature, $\kappa > 0$ a constant.
Let the initial temperature at $t = 0$, be

$$u(r,\ \theta,\ 0) = f(r,\ \theta),$$

$0 \le r < a,\ 0 \le \theta < 2\pi$, and boundary condition at $r = a$,

$$u(a,\ \theta,\ t) = 0,$$

$0 \le \theta < 2\pi,\ t > 0.$

Assuming a solution of the form $u(r,\ \theta,\ t) = R(r)\ \Theta(\theta)\ T(t)$, substituting in the heat equation,

$$R\ \Theta\ T' - \kappa\left(R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T \right) = 0,$$

$$\frac{T'}{\kappa T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = \text{constant} = -\ \lambda^2 \text{ (say)},$$

$$T' + \lambda^2\kappa T = 0, \qquad \frac{r^2R''}{R} + \frac{rR'}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = \text{constant} = \nu^2 \text{ (say)},$$

$$\Theta'' + \nu^2\Theta = 0, \qquad r^2R'' + rR' + (\lambda^2 r^2 - \nu^2)R = 0.$$

The equations for T, Θ are familiar but the equation for R is Bessel's equation and has non-constant coefficients. We can write it in the Sturm-Liouville form,

$$-\left(rR' \right)' + \frac{\nu^2}{r}R = \lambda^2 rR,$$

where $p(r) = r \ge 0,\ q(r) = \dfrac{\nu^2}{r} > 0,\ w(r) = r \ge 0.$
A change of variable $\rho = \lambda r$ in Bessel's equation results in

$$\rho^2\frac{d^2S}{d\rho^2} + \rho\frac{dS}{d\rho} + (\rho^2 - \nu^2)S = 0,$$

where $S(\rho) = R(r) = R\left(\dfrac{\rho}{\lambda}\right)$. We denote a solution of this Bessel's equation as $S(\rho) = J_\nu(\rho)$ and then a solution of the original form of Bessel's equation is $R(r) = S(\lambda r) = J_\nu(\lambda r)$. A second solution of Bessel's equation is $Y_\nu(\lambda r)$, a Bessel function of the second kind, and therefore the general solution is

$$R(r) = AJ_\nu(\lambda r) + BY_\nu(\lambda r)\,.$$

If $\nu \neq 0, 1, 2, \ldots$, then $J_{-\nu}(\lambda r)$ is also a solution and $\left\{ J_\nu(\lambda r) \,;\, J_{-\nu}(\lambda r) \right\}$ is a linearly independent set of solutions to Bessel's equation. The general solution is then

$$R(r) = AJ_\nu(\lambda r) + BJ_{-\nu}(\lambda r)$$

for constants A, B. The solution $J_\nu(\lambda r)$ is called a Bessel's function of the first kind and a series representation can be found

$$J_\nu(\lambda r) = (\lambda r)^\nu \left[1 - \frac{\lambda^2 r^2}{2(2+2\nu)} + \frac{\lambda^4 r^4}{2.4.(2+2\nu)(4+2\nu)} + \cdots \right].$$

For $\nu > 0$, $J_\nu(0) = 0$, $J_0(0) = 1$. $Y_\nu(\lambda r)$, $J_{-\nu}(\lambda r)$ are unbounded as $r \to \infty$, hence the only bounded solutions $R(r)$ occur when $B = 0$.

The Bessel functions $J_\nu(\lambda r)$ are oscillatory for $r > 0$. For $\nu > 0$, let the zeros of $J_\nu(\rho)$ be $\{\rho_{\nu m} \,;\, m = 0, 1, \ldots\}$, where $\rho_{\nu 0} = 0$ for all $\nu > 0$.

Then $R(r) = J_\nu(\lambda r)$ solves the Sturm-Liouville problem

$$-(rR'(r))' + \left(\frac{\nu^2}{r}\right) R(r) = \lambda^2\, rR(r),$$

$$R(0) = J_\nu(0) = 0, \qquad R(a) = J_\nu(\lambda a) = 0,$$

if $\lambda a = \rho_{\nu m}$, $m = 1, 2, \ldots$, for each $\nu > 0$.

The Bessel functions $\left\{ J_\nu\left(\frac{\rho_{\nu m} r}{a}\right) \right\}_{m=1}^{\infty}$, are orthogonal on the interval $(0, a)$, with respect to the weighted inner product $<f\,;g>_w = \int_0^a r\, f(r)\, \overline{g(r)}\, dr$. That is,

$$\int_0^a r\, J_\nu\left(\frac{\rho_{\nu m} r}{a}\right) J_\nu\left(\frac{\rho_{\nu l} r}{a}\right) dr = 0, \text{ if } m \neq l.$$

Returning to the heat equation on the disc $0 \le r < a$, $0 \le \theta < 2\pi$, the solutions of the form $u(r, \theta, t) = R(r)\, \Theta(\theta)\, T(t)$ are given by

$$R(r) = AJ_\nu(\lambda r) + BY_\nu(\lambda r),$$

$$\Theta(\theta) = C \cos(\nu\theta) + D \sin(\nu\theta),$$

$$T(t) = E\, e^{-\lambda^2 \kappa t}.$$

For $\Theta(\theta)$ to have period 2π, $\nu = n = 0, 1, 2, \ldots$.

For $R(r)$ to be bounded at $r = 0$, $B = 0$.

For $u(a, \theta, t) = 0$, $R(a) = J_n(\lambda a) = 0$ or $\lambda a = \rho_{nm}$, $n = 0, 1, 2, \dots$, $m = 1, 2, \dots$, the zeros of $J_n(\rho)$. That is $\lambda_{nm} = \dfrac{\lambda_{nm}}{a}$, $n = 0, 1, 2, \dots$, $m = 1, 2, \dots$.

By superposition, a solution of the heat equation on the disc satisfying the boundary condition $u(a, \theta, t) = 0$, is given by

$$u(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \exp\left(-\frac{\rho_{nm}{}^{2}\kappa t}{a^2}\right) J_n\left(\frac{\rho_{nm} r}{a}\right) \left(C_{nm} \cos(n\theta) + D_{nm} \sin(n\theta) \right).$$

The orthogonality relations

$$\int_0^a r\, J_n\left(\frac{\rho_{nm} r}{a}\right) J_n\left(\frac{\rho_{nl} r}{a}\right) dr = 0, \text{ if } m \neq l, \quad \text{and} \quad \int_{-\pi}^{\pi} \cos(n\theta) \sin(k\theta)\, d\theta, \text{ if } n \neq k,$$

imply the orthogonality of $\left\{ J_n\left(\frac{\rho_{nm} r}{a}\right)\cos(n\theta) \,;\, J_n\left(\frac{\rho_{nm} r}{a}\right)\sin(n\theta) \right\}_{n=0;m=1}^{\infty;\infty}$ on the rectangle $(r, \theta) \in (0, a)\times(-\pi, \pi)$ with respect to the inner product

$$<f; g> = \int_0^a \int_{-\pi}^{\pi} r\, f(r, \theta)\, g(r, \theta)\, d\theta\, dr.$$

That is,

$$\int_0^a \int_{-\pi}^{\pi} r \left(J_n\left(\frac{\rho_{nm} r}{a}\right)\cos(n\theta) \right)\left(J_k\left(\frac{\rho_{kl} r}{a}\right)\cos(k\theta) \right) d\theta\, dr = \begin{cases} 0\,; & n \neq k \\ 0\,; & n = k \text{ and } m \neq l \\ \pi \displaystyle\int_0^a r\, J_k\left(\frac{\rho_{kl} r}{a}\right)^2 dr; & n = k \text{ and } m = l \end{cases}$$

$$\int_0^a \int_{-\pi}^{\pi} r \left(J_n\left(\frac{\rho_{nm} r}{a}\right)\sin(n\theta) \right)\left(J_k\left(\frac{\rho_{kl} r}{a}\right)\sin(k\theta) \right) d\theta\, dr = \begin{cases} 0\,; & n \neq k \\ 0\,; & n = k \text{ and } m \neq l \\ \pi \displaystyle\int_0^a r\, J_k\left(\frac{\rho_{kl} r}{a}\right)^2 dr; & n = k \text{ and } m = l \end{cases}$$

$$\int_0^a \int_{-\pi}^{\pi} r \left(J_n\left(\frac{\rho_{nm} r}{a}\right)\cos(n\theta) \right)\left(J_k\left(\frac{\rho_{kl} r}{a}\right)\sin(k\theta) \right) d\theta\, dr = 0 \text{ for all } n, m, k \text{ and } l.$$

The coefficients C_{nm}, D_{nm} are determined using these orthogonality properties. For example,

$$\int_0^a \int_{-\pi}^{\pi} r\, J_k\!\left(\frac{\rho_{kl} r}{a}\right) \cos(k\,\theta)\, f(r,\,\theta)\; d\theta\; dr = C_{kl}\;\pi \int_0^a r\, J_k\!\left(\frac{\rho_{kl} r}{a}\right)^2 dr,$$

hence

$$C_{kl} = \frac{1}{\pi \displaystyle\int_0^a r\, J_k\!\left(\frac{\rho_{kl} r}{a}\right)^2 dr} \int_0^a \int_{-\pi}^{\pi} r\, J_k\!\left(\frac{\rho_{kl} r}{a}\right) \cos(k\,\theta)\, f(r,\,\theta)\; d\theta\; dr.$$

Similarly,

$$D_{kl} = \frac{1}{\pi \displaystyle\int_0^a r\, J_k\!\left(\frac{\rho_{kl} r}{a}\right)^2 dr} \int_0^a \int_{-\pi}^{\pi} r\, J_k\!\left(\frac{\rho_{kl} r}{a}\right) \sin(k\,\theta)\, f(r,\,\theta)\; d\theta\, dr,$$

$k = 0, 1, 2, \ldots$, $l = 1, 2, \ldots$.

13. Fourier Transforms.

A function $f : \mathbf{R}\to\mathbf{R}$ is said to be integrable on $\mathbf{R}$ if $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. We call the class of all such functions $L^1(\mathbf{R})$. Similarly, $f : \mathbf{R}\to\mathbf{R}$ is said to be square integrable on $\mathbf{R}$ if $\int_{-\infty}^{\infty} |f(x)|^2dx < \infty$. and we call the class of all such functions $L^2(\mathbf{R})$.

Examples. Let $f : \mathbf{R}\to\mathbf{R}$ and $g : \mathbf{R}\to\mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^{-\frac{2}{3}} ; 0 < x < 1 \\ 0 ;\ \text{otherwise} \end{cases} ,$$

$$g(x) = \begin{cases} x^{-\frac{2}{3}};\ x > 1 \\ 0 ; \text{otherwise} \end{cases} .$$

Then

$$\int_{-\infty}^{\infty} |f(x)|dx = \int_0^1 x^{-\frac{2}{3}}dx = 3x^{\frac{1}{3}} \Big|_0^1 = 3 < \infty$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2dx = \int_0^1 x^{-\frac{4}{3}}dx = -3x^{-\frac{1}{3}} \Big|_0^1 = \infty.$$

Therefore $f \in L^1(\mathbf{R})$ but $f \notin L^2(\mathbf{R})$. On the other hand,

$$\int_{-\infty}^{\infty} |g(x)|dx = \int_1^{\infty} x^{-\frac{2}{3}}dx = 3x^{\frac{1}{3}} \Big|_1^{\infty} = \infty$$

and

$$\int_{-\infty}^{\infty} |g(x)|^2dx = \int_0^1 x^{-\frac{4}{3}}dx = -3x^{-\frac{1}{3}} \Big|_1^{\infty} = 3 < \infty.$$

Therefore $g \notin L^1(\mathbf{R})$ but $g \in L^2(\mathbf{R})$.

Given two function $f \in L^1(\mathbf{R})$ and $g \in L^1(\mathbf{R})$, the product fg is not necessarily in $L^1(\mathbf{R})$. Counter-examples are given by

$$f(x) = g(x) = \begin{cases} x^{-\frac{2}{3}} ; 0 < x < 1 \\ 0 ;\ \text{otherwise} \end{cases} .$$

There is a product $*$ for which $f*g \in L^1(\mathbf{R})$ whenever $f \in L^1(\mathbf{R})$ and $g \in L^1(\mathbf{R})$. We define the convolution product

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy.$$

Lemma. If $f \in L^1(\mathbf{R})$ and $g \in L^1(\mathbf{R})$, then $f*g \in L^1(\mathbf{R})$.

Proof.

$$\int_{-\infty}^{\infty} |(f*g)(x)|dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \right| dx$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)\, g(y)|\, dydx$$

$$\leq \int_{-\infty}^{\infty} |f(x-y)| \int_{-\infty}^{\infty} |g(y)|\, dydx$$

$$\leq \int_{-\infty}^{\infty} |f(x)|\, dx \int_{-\infty}^{\infty} |g(y)|\, dx < \infty.$$

The Fourier transform of a function $f \in L^1(\mathbf{R})$ is defined to be

$$f^\wedge(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx,$$

where $\xi \in \mathbf{R}$. It is easy to see that the Fourier transform of a function $f \in L^1(\mathbf{R})$ is a bounded continuous function on $\mathbf{R}$. The boundedness follows easily from

$$|f^\wedge(\xi)| = \left| \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \right| \leq \int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

The continuity follows from the following.

Theorem. (Riemann-Lebesgue). If $f \in L^1(\mathbf{R})$, then $f^\wedge \in C(\mathbf{R})$ and $f^\wedge(\xi) \to 0$ as $|\xi| \to \infty$.

Proof. Let $\xi \in \mathbf{R}$, $\xi \neq 0$. Then

$$f^\wedge(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = -\, e^{\pi i} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

$$= -\int_{-\infty}^{\infty} f(x) e^{-i\xi\left(x + \frac{\pi}{\xi}\right)} dx = -\int_{-\infty}^{\infty} f\left(x - \frac{\pi}{\xi}\right) e^{-i\xi x} dx.$$

Therefore

$$2|f^\wedge(\xi)| = \left| \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx - \int_{-\infty}^{\infty} f\left(x - \frac{\pi}{\xi}\right) e^{-i\xi x} dx \right|$$

$$\leq \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{\xi}\right) \right| |e^{-i\xi x}|\, dx$$

$$\leq \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{\xi}\right) \right| dx \to 0 \text{ as } |\xi| \to \infty.$$

To prove continuity of $f^\wedge$, let $\varepsilon > 0$ be given and $a > 0$ chosen such that $\int_{|x|>a} |f(x)| dx < \frac{\varepsilon}{4}$ and $\delta > 0$ chosen such that $2a\delta \int_{|x|<a} |f(x)| dx < \varepsilon$. Then for $|\eta| < \delta$,

$$\left| f^\wedge(\xi + \eta) - f^\wedge(\xi) \right| = \left| \int_{-\infty}^{\infty} \left(f(x) e^{-i(\xi + \eta)x} - f(x) e^{-i\xi x} \right) dx \right|$$

$$\leq \int_{-\infty}^{\infty} |f(x)| \cdot |e^{-i\xi x} (e^{-i\eta x} - 1)| \, dx = 2 \int_{-\infty}^{\infty} |f(x)| \left| \sin\left(\frac{\eta x}{2}\right) \right| dx$$

$$\leq 2 \int_{|x|>a} |f(x)| \left| \sin\left(\frac{\eta x}{2}\right) \right| dx + 2 \int_{|x|<a} |f(x)| \left| \sin\left(\frac{\eta x}{2}\right) \right| dx$$

$$\leq 2 \int_{|x|>a} |f(x)| \, dx + 2 \int_{|x|<a} |f(x)| \left| \frac{\eta x}{2} \right| dx$$

$$\leq \frac{\varepsilon}{2} + a\delta \int_{|x|<a} |f(x)| dx \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $f^\wedge(\xi)$ is uniformly continuous on **R.**

We define the linear transformation $\mathbf{F}: f \to f^\wedge$ defined by

$$(\mathbf{F}f)(\xi) = f^\wedge(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx.$$

$\mathbf{F}$ is called the Fourier integral transformation and $f^\wedge$ the Fourier transform of $f \in L^1(\mathbf{R})$.

Properties of Fourier transforms.

1. Translation (a). Let $f : \mathbf{R} \to \mathbf{R}$ and $h \in \mathbf{R}$. The translate of f by h is the function $\tau_h f$ defined by

$$(\tau_h f)(x) = f(x - h), \; x \in \mathbf{R}.$$

The Fourier transform of $\tau_h f$ is

$$\begin{aligned}\left(\tau_h f\right)\hat{}(\xi) &= \int_{-\infty}^{\infty} (\tau_h)(x)e^{-i\xi x}dx \\ &= \int_{-\infty}^{\infty} f(x-h)e^{-i\xi x}dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-i\xi(x+h)}dx \\ &= e^{-i\xi h}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx \\ &= e^{-i\xi h}f\hat{}(\xi).\end{aligned}$$

(b). For real c,

$$\begin{aligned}\left(e^{icx}f(x)\right)\hat{} &= \int_{-\infty}^{\infty} e^{-i\xi x}\left(e^{icx}f(x)\right)dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-i\xi(x-c)}dx \\ &= f\hat{}(\xi - c) = \left(\tau_c f\hat{}\right)(\xi).\end{aligned}$$

2. Dilation. Let $\lambda \in \mathbf{R}$, $\lambda > 0$, the dilation of f by λ is defined as $\delta_\lambda f$ where

$$(\delta_\lambda f)(x) = \lambda^{-\frac{1}{2}}f(\lambda^{-1}x), \; x \in \mathbf{R}.$$

The Fourier transform of $\delta_\lambda f$ is

$$\begin{aligned}\left(\delta_\lambda f\right)\hat{}(\xi) &= \int_{-\infty}^{\infty} (\delta_\lambda f)(x)e^{-i\xi x}dx \\ &= \lambda^{-\frac{1}{2}}\int_{-\infty}^{\infty} f(\lambda^{-1}x)\,e^{-i\xi x}dx \\ &= \lambda^{-\frac{1}{2}}\int_{-\infty}^{\infty} f(y)\,e^{-i\xi\lambda y}\,\lambda dy \\ &= \lambda^{\frac{1}{2}}\hat{f}(\lambda\xi) = \left(\delta_{\lambda^{-1}}\hat{f}\right)(\xi)\end{aligned}$$

3. Differentiation. Let f and $f' \in L^1(\mathbf{R})$ and denote by D the differential operator $D = \frac{\partial}{\partial x}$.
The Fourier transform of Df is

$$\big(Df\big)^\wedge(\xi) = \int_{-\infty}^{\infty} (Df)(x)e^{-i\xi x}dx$$

$$= f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\big(De^{-i\xi x}\big)dx$$

$$= (i\xi)f^\wedge(\xi).$$

By induction $\big(D^k f\big)^\wedge = (i\xi)^k f^\wedge$, $k = 1, 2, \dots$.

4. Multiplication. We denote by ∂ the differential operator $\partial = \frac{\partial}{\partial \xi}$. If f and $xf \in L^1(\mathbf{R})$ then

$$\big(\partial f^\wedge\big)(\xi) = \frac{\partial}{\partial \xi}\left(\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\right)$$

$$= \int_{-\infty}^{\infty} (-ix)f(x)e^{-i\xi x}dx.$$

Therefore $\big(-ixf\big)^\wedge = \partial f^\wedge$. By induction it follows that $\partial^k f^\wedge = \big((-ix)^k f\big)^\wedge$, $k = 1, 2, \dots$.

5. Convolution. Let $f, g \in L^1(\mathbf{R})$. Then $f * g \in L^1(\mathbf{R})$ and has Fourier transform

$$(f*g)^\wedge(\xi) = \int_{-\infty}^{\infty} (f*g)(x)e^{-i\xi x}dx$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy\right)e^{-i\xi x}dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\,g(y)\,e^{-i\xi(x+y)}dx\,dy$$

$$= \left(\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx\right)\left(\int_{-\infty}^{\infty} g(y)e^{-i\xi y}dy\right)$$

$$= f^\wedge(\xi)g^\wedge(\xi).$$

Example 1. Let $f(x) = e^{-\frac{x^2}{2}}$. Then $f \in L^1(\mathbf{R})$ and has Fourier transform

$$f^\wedge(\xi) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}e^{-i\xi x}dx$$

$$\frac{\partial f^{\wedge}}{\partial \xi}(\xi) = \frac{\partial}{\partial \xi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-i\xi x} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} (-ix) e^{-i\xi x} dx$$

$$= i \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(e^{-\frac{x^2}{2}}\right) e^{-i\xi x} dx$$

$$= ie^{-i\xi x} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} - i \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \frac{\partial}{\partial x}\left(e^{-i\xi x} \right) dx$$

$$= -\xi \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-i\xi x} dx$$

$$= -\xi f^{\wedge}(\xi).$$

Therefore $f^{\wedge}$ satisfies a first order ordinary differential equation with solution $f^{\wedge}(\xi) = ce^{-\frac{\xi^2}{2}}$.

Since $f^{\wedge}(0) = c = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$,

$$\left(e^{-\frac{x^2}{2}}\right)^{\wedge} = \sqrt{2\pi}\, e^{-\frac{\xi^2}{2}}.$$

Example 2. Let $f(x) = \begin{cases} 1\ ; & |x| \le a \\ 0\ ; & \text{otherwise} \end{cases} \equiv \chi_a(x) \in L^1(\mathbf{R})$ and has Fourier transform

$$\left(\chi_a\right)^{\wedge}(\xi) = \int_{-\infty}^{\infty} \chi_a(x)\, e^{-i\xi x} dx = \int_{-a}^{a} e^{-i\xi x} dx$$

$$= -\frac{1}{i\xi} e^{-i\xi x} \Big|_{-a}^{a} = -\frac{1}{i\xi} \left(e^{-i\xi a} - e^{-i\xi a} \right)$$

$$= \frac{2 \sin a\xi}{\xi}.$$

14. Inverse Fourier Transforms.

The Fourier transform of $f \in L^1(\mathbf{R})$ is continuous, bounded and $|f^\wedge(\xi)| \to 0$ as $|\xi| \to \infty$. However $f^\wedge$ is not neccesarily in $L^1(\mathbf{R})$. Assuming $f \in L^1(\mathbf{R})$, f piecewise smooth, and $f^\wedge \in L^1(\mathbf{R})$, define

$$f_a(x) = \frac{1}{2\pi}\int_{-a}^{a} f^\wedge(\xi)\, e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\left(\int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy\right) e^{i\xi x} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-a}^{a} e^{i\xi(x-y)} d\xi\right) f(y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{ia(x-y)} - e^{-ia(x-y)}}{i(x-y)}\right) f(y)\, dy$$

$$= \int_{-\infty}^{\infty} D_a(x-y)\, f(y)\, dy = \int_{-\infty}^{\infty} D_a(y)\, f(x-y)\, dy$$

$$= \left(D_a * f\right)(x)$$

where $D_a(x) = \dfrac{\sin ax}{\pi x}$.

Now

$$f_a(x) - \tfrac{1}{2}\left[f(x^-) + f(x^+)\right] = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x-y)\,\frac{\sin ay}{y}\, dy - \tfrac{1}{2}\left[f(x^-) + f(x^+)\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[f(x-y) - f(x^-)\right]\frac{\sin ay}{y}\, dy$$
$$+ \frac{1}{\pi}\int_{-\infty}^{0}\left[f(x-y) - f(x^+)\right]\frac{\sin ay}{y}\, dy \qquad \left(\text{since } \int_{0}^{\infty}\frac{\sin ay}{y}\, dy = \frac{\pi}{2}\right)$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[f(x-y) - f(x^-)\right]\frac{\sin ay}{y}\, dy$$
$$+ \frac{1}{\pi}\int_{0}^{\infty}\left[f(x+y) - f(x^+)\right]\frac{\sin ay}{y}\, dy$$

In the second integral,

$$\frac{1}{\pi}\int_0^\infty \left[f(x+y) - f(x^+)\right]\frac{\sin ay}{y}\,dy$$

$$= \frac{1}{\pi}\int_0^K \left[f(x+y) - f(x^+)\right]\frac{\sin ay}{y}\,dy + \frac{1}{\pi}\int_K^\infty \left[f(x+y) - f(x^+)\right]\frac{\sin ay}{y}\,dy$$

If $K \geq 1$,

$$\left|\int_K^\infty f(x+y)\,\frac{\sin ay}{y}\,dy\right| \leq \int_K^\infty \left|f(x+y)\right|dy$$

and

$$\int_K^\infty f(x^+)\,\frac{\sin ay}{y}\,dy = f(x^+)\int_K^\infty \frac{\sin ay}{y}\,dy.$$

Since $\int_0^\infty |f(x)|dx$, $\int_0^\infty \frac{\sin ay}{y}\,dy$ are both convergent integrals, $\int_K^\infty \frac{\sin ay}{y}\,dy \to 0$,

$\int_K^\infty \left|f(x+y)\right|dy \to 0$ as $K \to \infty$.

For the integrals over $[0, K]$,

$$\int_0^K \left[f(x+y) - f(x^+)\right]\frac{\sin ay}{y}\,dy = \int_{-\infty}^\infty \left(\frac{e^{iay} - e^{-iay}}{2i}\right)g(y)\,dy$$

$$= \frac{1}{2i}\left[g^\wedge(-a) - g^\wedge(a)\right]$$

where $g(y) = \begin{cases} \dfrac{f(x+y) - f(x^+)}{y} & ; 0 < y < K \\ 0 ; & \text{otherwise} \end{cases}$.

Since f is piecewise smooth, $f'(x^+)$ exists for all $x \in \mathbf{R}$ and $\lim\limits_{y \to 0^+} g(y) = f'(x^+)$. Therefore g is bounded on $[0, K]$ and hence $g \in L^1(\mathbf{R})$. By the Riemann-Lebesgue lemma, $g^\wedge$ exists, is continuous, $g^\wedge(\pm a) \to 0$ as $a \to \infty$ and therefore

$$\int_0^K \left[f(x+y) - f(x^+)\right]\frac{\sin ay}{y}\,dy \to 0 \text{ as } a \to \infty$$

for $K \geq 1$. A virtually identical argument works for the first integral.

Therefore

$$\left| f_a(x) - \tfrac{1}{2}\left[f(x^-) + f(x^+)\right] \right| \to 0 \text{ as } a \to \infty$$

or

$$\lim_{a\to\infty} f_a(x) = \lim_{a\to\infty} \frac{1}{2\pi}\int_{-a}^{a} f^\wedge(\xi)\,e^{i\xi x}d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f^\wedge(\xi)\,e^{i\xi x}d\xi = \tfrac{1}{2}\left[f(x^-) + f(x^+)\right].$$

Summarising

Theorem. Let $f \in L^1(\mathbf{R})$ and let f be piecewise smooth on $\mathbf{R}$. Then for every $x \in \mathbf{R}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f^\wedge(\xi)\, e^{\,i\xi x} d\xi = \tfrac{1}{2}\left[f(x^-) + f(x^+) \right].$$

If x is a point of continuity of f, then $\frac{1}{2\pi}\int_{-\infty}^{\infty} f^\wedge(\xi)\, e^{\,i\xi x} d\xi = f(x)$.

For $\phi \in L^1(\mathbf{R})$, we call $\check{\phi}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\xi)\, e^{\,i\xi x} d\xi$ the *inverse Fourier transform* of ϕ and

F^{-1} defined by $\left(\mathrm{F}^{-1}\phi \right)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\xi)\, e^{\,i\xi x} d\xi$ in called the *inverse Fourier transformation*.

If ϕ is the Fourier transform of a piecewise smooth function $f \in L^1(\mathbf{R})$, that is $\phi = f^\wedge$, then we define $f(x) = \tfrac{1}{2}\left[f(x^-) + f(x^+) \right]$ at a point of discontinuity of f.

Then $f(x) = \check{\phi}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^\wedge(\xi)\, e^{\,i\xi x} d\xi$. That is $\mathrm{F}^{-1}\mathrm{F}f = f$.

Example. The inverse Fourier transform is useful in computing Fourier transforms.
Since $\mathrm{F}\left(e^{-a|x|} \right) = \dfrac{2a}{a^2 + \xi^2}$, it follows that $\mathrm{F}^{-1}\mathrm{F}\left(e^{-a|x|} \right) = \mathrm{F}^{-1}\left(\dfrac{2a}{a^2 + \xi^2} \right)$ or

$e^{-a|x|} = \mathrm{F}^{-1}\left(\dfrac{2a}{a^2 + \xi^2} \right)$. Interchanging the roles of x and ξ, multiplying by 2π, leads to

$\mathrm{F}\left(\dfrac{2a}{a^2 + x^2} \right) = 2\pi\, e^{-a|\xi|}$ or $\mathrm{F}\left(\dfrac{1}{a^2 + x^2} \right) = \dfrac{\pi}{a} e^{-a|\xi|}$. In a similar fashion, every Fourier transform pair defines a dual pair using the inverse Fourier transform.

15. Applications to Differential Equations.

1. The wave equation. The equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, -\infty < x < \infty, t > 0,$$

describes the vertical vibrations of an infinite stretched elastic string, where $u(x, t)$ is the vertical displacement of the string from its rest position at position x, time t. Let the initial displacement and velocity be given as

$$u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, t) = g(x), -\infty < x < \infty.$$

We take Fourier transforms of the wave equation and the initial conditions with respect to the x variable and denote by $u^\wedge(\xi, t)$ the Fourier transform $\mathrm{F}\left(u(x, t) \right)$. The using the derivative properties of the Fourier Transform,

$$\frac{\partial^2 u^\wedge}{\partial t^2} - (ci\xi)^2 u^\wedge = 0, -\infty < \xi < \infty, t > 0,$$

or

$$\frac{\partial^2 u^\wedge}{\partial t^2} + (c\xi)^2 u^\wedge = 0.$$

Then

$$u^\wedge(\xi, t) = A(\xi)\cos(c\xi t) + B(\xi)\sin(c\xi t).$$

When $t = 0$, $u^\wedge(\xi, t) = A(\xi) = f^\wedge(\xi)$, $\frac{\partial u^\wedge}{\partial t}(\xi, t) = g^\wedge(\xi) = c\xi B(\xi)$. Therefore

$$u^\wedge(\xi, t) = f^\wedge(\xi)\cos(c\xi t) + \frac{g^\wedge(\xi)}{2c}\,\frac{2\sin(c\xi t)}{\xi}$$

$$= f^\wedge(\xi)\left(\frac{e^{i(ct)\xi} + e^{-i(ct)\xi}}{2}\right) + \frac{1}{2c}\left(\chi_{ct} * g \right)^\wedge(\xi).$$

Using the translation property and the convolution theorem,

$$u(x, t) = \frac{1}{2}\left[f(x + ct) + f(x - ct) \right] + \frac{1}{2c}\int_{-\infty}^{\infty} \chi_{ct}(x - ct)\, g(y)\, dy$$

$$= \frac{1}{2}\left[f(x + ct) + f(x - ct) \right] + \frac{1}{2c}\int_{|x - ct| < ct} g(y)\, dy$$

$$= \frac{1}{2}\left[f(x + ct) + f(x - ct) \right] + \frac{1}{2c}\int_{x - ct}^{x + ct} g(y)\, dy.$$

This is d'Alemberts solution to the one-dimensional wave equation.

2. Laplace's Equation.

Consider Laplace's equation in two variables on the upper half-plane $y > 0$. Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad -\infty < x < \infty, y > 0.$$

Let the boundary condition

$$u(x, 0) = f(x), \qquad -\infty < x < \infty$$

be given for a function $f \in L^1(\mathbf{R})$. Then taking the Fourier transform in the variable x,

$$(i\xi)^2 u^\wedge(\xi, y) + \frac{\partial^2 u^\wedge}{\partial y^2}(\xi, y) = 0, \qquad -\infty < x < \infty, y > 0.$$

or

$$\frac{\partial^2 u^\wedge}{\partial y^2}(\xi, y) - \xi^2 u^\wedge(\xi, y) = 0,$$

which has solutions

$$u^\wedge(\xi, y) = A(\xi)e^{\xi y} + B(\xi)e^{-\xi y}.$$

We need two conditions to determne the functions $A(\xi), B(\xi)$. In addition to $u(x, 0) = f(x)$, $-\infty < x < \infty$, let $\frac{\partial u}{\partial y}(x, 0) = g(x)$, $-\infty < x < \infty$ for some function $g \in L^1(\mathbf{R})$. We will find g such that the solution $u(x, y)$ is bounded for $y > 0$, in fact such that $u(x, y) \to 0$ as $y \to \infty$. Taking transforms,

$$u^\wedge(\xi, 0) = f^\wedge(\xi), \qquad \frac{\partial u^\wedge}{\partial y}(\xi, 0) = g^\wedge(\xi),$$

or

$$A(\xi) + B(\xi) = f^\wedge(\xi), \ \xi A(\xi) - \xi B(\xi) = g^\wedge(\xi).$$

Solving for $A(\xi), B(\xi)$,

$$A(\xi) = \tfrac{1}{2}\left(f^\wedge(\xi) + \frac{g^\wedge(\xi)}{\xi}\right), B(\xi) = \tfrac{1}{2}\left(f^\wedge(\xi) - \frac{g^\wedge(\xi)}{\xi}\right),$$

and

$$u^\wedge(\xi, y) = \tfrac{1}{2}\left(f^\wedge(\xi) + \frac{g^\wedge(\xi)}{\xi}\right)e^{\xi y} + \tfrac{1}{2}\left(f^\wedge(\xi) - \frac{g^\wedge(\xi)}{\xi}\right)e^{-\xi y}.$$

For $\xi > 0$, $u^\wedge(\xi, y) \to 0$ as $y \to 0$ if and only if $f^\wedge(\xi) + \frac{g^\wedge(\xi)}{\xi} = 0$, and for $\xi < 0$, $u^\wedge(\xi, y) \to 0$ as $y \to 0$ if and only if $f^\wedge(\xi) - \frac{g^\wedge(\xi)}{\xi} = 0$. Therefore

$$g^\wedge(\xi) = \begin{cases} -\xi f^\wedge(\xi) \ ; \ \xi > 0 \\ \ \ \xi f^\wedge(\xi) \ ; \ \xi < 0 \end{cases}$$

$$= -|\xi| f^\wedge(\xi).$$

Hence

$$u^\wedge(\xi, y) = \begin{cases} f^\wedge(\xi)e^{-\xi y} \; ; \; \xi > 0 \\ f^\wedge(\xi)e^{\xi y} \; ; \; \xi < 0 \end{cases}$$

$$= f^\wedge(\xi)e^{-\xi|y|}.$$

Since $e^{-\xi|y|} = \left(\dfrac{y}{\pi(x^2+y^2)}\right)^\wedge$, by the convolution theorem,

$$u(x, y) = \left(\frac{y}{\pi(x^2+y^2)} * f\right) = \int_{-\infty}^{\infty} \frac{y}{\pi((x-s)^2+y^2)} f(s)\, ds.$$

3. The Heat Equation.

The heat equation,

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \qquad -\infty < x < \infty,\ t > 0$$

for $u(x, t)$ a function of two variables, with initial condition

$$u(x, t) = f(x), -\infty < x < \infty$$

for $f \in L^1(\mathbf{R}), f$ continuous and bounded, can be solved using Fourier transforms. Taking transforms in the variable x,

$$\frac{\partial u^\wedge}{\partial t}(\xi, t) - \kappa\,(i\xi)^2 u^\wedge(\xi, t) = 0,$$

or

$$\frac{\partial u^\wedge}{\partial t}(\xi, t) + \kappa\,\xi^2 u^\wedge(\xi, t) = 0.$$

Solving this first order ordinary differential equation,

$$u^\wedge(\xi, t) = u^\wedge(\xi, 0)e^{-\kappa\xi^2 t} = f^\wedge(\xi)e^{-\kappa\xi^2 t}.$$

Now $\left(e^{-\frac{x^2}{2}}\right)^\wedge = \sqrt{2\pi}\, e^{-\frac{\xi^2}{2}}$ and using the dilation property of Fourier transforms

$$\left(\lambda^{-\frac{1}{2}} e^{-\frac{(\lambda^{-1}x)^2}{2}}\right)^\wedge = \sqrt{2\pi}\, \lambda^{\frac{1}{2}} e^{-\frac{(\lambda\xi)^2}{2}}$$

$$e^{-\frac{(\lambda\xi)^2}{2}} = \frac{1}{\lambda\sqrt{2\pi}} \left(e^{-\frac{(\lambda^{-1}x)^2}{2}}\right)^\wedge.$$

Let $\dfrac{\lambda^2}{2} = \kappa t$ or $\lambda = \sqrt{2\kappa t}$, then

$$e^{-\kappa\xi^2 t} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\kappa t}}\left(e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2\kappa t}}\right)^2}\right)^\wedge$$

$$= \frac{1}{\sqrt{4\pi\kappa t}}\left(e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2\kappa t}}\right)^2}\right)^\wedge.$$

Therefore from the convolution theorem, since

$$u^\wedge(\xi, t) = f^\wedge(\xi)\ \frac{1}{\sqrt{4\pi\kappa t}}\left(e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2\kappa t}}\right)^2}\right)^\wedge$$

$$u(x, t) = \left(f * \frac{1}{\sqrt{4\pi\kappa t}}\ e^{-\frac{1}{2}\left(\frac{x}{\sqrt{2\kappa t}}\right)^2}\right)(x)$$

$$= \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t}} f(y)\,dy.$$

The function $h(x, t) = \dfrac{e^{-\frac{x^2}{4\kappa t}}}{\sqrt{4\pi\kappa t}}$, is called the heat kernel. We can then write

$$u(x, t) = \int_{-\infty}^{\infty} h(x-y, t)\, f(y)\,dy.$$

16. Plancherels' and Parsevals' Identities.

By the convolution theorem, for $f, g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$,

$$(f * g)^\wedge = f^\wedge g^\wedge.$$

Therefore

$$f * g = (f^\wedge g^\wedge)^\vee$$

or

$$\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} f^\wedge(\xi)\, g^\wedge(\xi)\, d\xi.$$

Set $x = 0$,

$$\int_{-\infty}^{\infty} f(-y)\, g(y)\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^\wedge(\xi)\, g^\wedge(\xi)\, d\xi$$

Replacing $g(x)$ by $\overline{g(-x)}$, the Fourier transform $g^\wedge(\xi)$ is replaced by $\overline{g^\wedge(\xi)}$, hence

$$\int_{-\infty}^{\infty} f(-y)\, \overline{g(-y)}\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^\wedge(\xi)\, \overline{g^\wedge(\xi)}\, d\xi$$

or

$$\int_{-\infty}^{\infty} f(y)\, \overline{g(y)}\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^\wedge(\xi)\, \overline{g^\wedge(\xi)}\, d\xi.$$

This is *Plancherels' identity*. When $f = g$ we obtain *Parsevals' identity*,

$$\int_{-\infty}^{\infty} |f(y)|^2\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f^\wedge(\xi)|^2 d\xi.$$

Examples.

1. Let $f(x) = \chi_a(x) = \begin{cases} 1; & |x| < a \\ 0; & \text{otherwise} \end{cases}$.

Then $f^\wedge(\xi) = \dfrac{2 \sin a\xi}{\xi}$, and by Parsevals' identity,

$$\int_{-\infty}^{\infty} |\chi_a(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\sin a\xi}{\xi} \right|^2 d\xi$$

$$\int_{-\infty}^{\infty} \left(\frac{2 \sin a\xi}{\xi} \right)^2 d\xi = 4\pi a$$

or

$$\int_{-\infty}^{\infty} \left(\frac{\sin a\xi}{\xi} \right)^2 d\xi = \pi a.$$

2. Let $f(x) = e^{-a|x|}$, $f\hat{\ }(\xi) = \dfrac{2a}{a^2+\xi^2}$ and by Parsevals' identity

$$\int_{-\infty}^{\infty} \left(e^{-a|x|}\right)^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2a}{a^2+\xi^2} \right)^2 d\xi$$

$$\frac{1}{a} = \frac{2a^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{(a^2+\xi^2)^2} \, d\xi$$

or

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+\xi^2)^2} \, d\xi = \frac{\pi}{2a^3} \, .$$

17. Band Limited Functions and Shannon's Sampling Theorem.

The Fourier transform variable has the role of frequency and $f^\wedge(\xi)$ is referred to as the frequency representation of $f(x)$.

If $f^\wedge(\xi) = 0$ for $|\xi| > \xi_c > 0$, then $f(x)$ is called a *band-limited function* and ξ_c is called the *cut-off frequency*. Many functions from science and technology, are band-limited. For example, human hearing is assumed to be limited to frequencies below about 20 kHz. Therefore the acoustic signals recorded on compact discs are limited to a bandwidth of 22 kHz.

A first step in the processing of signals, is *sampling*. A signal represented by a continuous function $f(x)$, is replaced by its samples at regular intervals, $\{f(nL), n = 0, \pm1, \pm2, \ldots\}$. Shannons' Theorem shows that it is possible to exactly recover the band-limited continuous function $f(x)$ from knowledge of its samples, provided the sampling interval L, is sufficiently short.

Consider the functions $\hat{\phi}_n(\xi)$ given by

$$\hat{\phi}_n(\xi) = \begin{cases} 0; & |\xi| > \xi_c \\ \dfrac{1}{\sqrt{2\xi_c}} e^{-\frac{in\pi\xi}{\xi_c}} & ; |\xi| \le \xi_c \end{cases}$$

The inverse Fourier transforms of $\hat{\phi}_n(\xi)$ are given by

$$\phi_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi}\, \hat{\phi}_n(\xi)\, d\xi$$

$$2\pi\, \phi_n(x) = \int_{-\xi_c}^{\xi_c} e^{ix\xi} \frac{1}{\sqrt{2\xi_c}} e^{-\frac{in\pi\xi}{\xi_c}}\, d\xi$$

$$= \frac{1}{\sqrt{2\xi_c}} \int_{-\xi_c}^{\xi_c} e^{i\xi\left(x - \frac{n\pi}{\xi_c}\right)} d\xi$$

$$= \frac{1}{\sqrt{2\xi_c}} \left[\frac{e^{i\xi\left(x - \frac{n\pi}{\xi_c}\right)}}{i\left(x - \frac{n\pi}{\xi_c}\right)} \right]_{-\xi_c}^{\xi_c}$$

$$= \sqrt{2\xi_c} \left(\frac{e^{i(\xi_c x - n\pi)} - e^{-i(\xi_c x - n\pi)}}{2i(\xi_c x - n\pi)} \right)$$

$$= \sqrt{2\xi_c}\ \frac{\sin(\xi_c x - n\pi)}{\xi_c x - n\pi}$$

$$= \sqrt{\frac{2\pi}{L}}\ \frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)} \quad \text{where} \quad L = \frac{\pi}{\xi_c}.$$

Consider the inner products

$$\langle \hat{\phi}_n ; \hat{\phi}_m \rangle = \int_{-\infty}^{\infty} \hat{\phi}_n(\xi)\ \overline{\hat{\phi}_m(\xi)}\ d\xi$$

$$= \frac{1}{2\xi_c} \int_{-\xi_c}^{\xi_c} e^{-\frac{in\pi\xi}{\xi_c}} \left(\overline{e^{-\frac{im\pi\xi}{\xi_c}}} \right) d\xi$$

$$= \frac{1}{2\xi_c} \int_{-\xi_c}^{\xi_c} e^{\frac{i(m-n)\pi\xi}{\xi_c}}\ d\xi$$

$$= \begin{cases} 0;\ m \neq n \\ 1;\ m = n \end{cases}.$$

So the functions $\{\hat{\phi}_n(\xi)\ ;\ n = 0, \pm 1, \pm 2, \ldots\}$, form an orthogonal set in $L^2(-\xi_c, \xi_c)$. Since $f\hat{\ }(\xi)$ is band limited to $|\xi| \le \xi_c$, it has a Fourier series

$$f\hat{\ }(\xi) = \sum_{n=-\infty}^{\infty} c_n\ \hat{\phi}_n(\xi)$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{\sqrt{2\xi_c}} e^{-\frac{in\pi\xi}{\xi_c}}\ d\xi$$

where the Fourier coefficients c_n are given by

$$c_n = \langle \hat{f}\ ;\ \hat{\phi}_n \rangle = \int_{-\xi_c}^{\xi_c} f\hat{\ }(\xi) \frac{1}{\sqrt{2\xi_c}} e^{\frac{in\pi\xi}{\xi_c}}\ d\xi.$$

By Plancherel's theorem

$$c_n = \langle \hat{f}\ ;\ \hat{\phi}_n \rangle = 2\pi \langle f\ ;\ \phi_n \rangle$$

so

$$f\hat{\ }(\xi) = \sum_{n=-\infty}^{\infty} c_n\ \hat{\phi}_n(\xi) = \sum_{n=-\infty}^{\infty} 2\pi \langle f\ ;\ \phi_n \rangle \hat{\phi}_n(\xi).$$

Taking inverse Fourier transforms,

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f\ ;\ \phi_n \rangle 2\pi\, \phi_n(x)$$

$$= \sum_{n=-\infty}^{\infty} \langle f\ ;\ \phi_n \rangle \sqrt{\frac{2\pi}{L}}\ \frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)}.$$

The samples at intervals of length $L = \dfrac{\pi}{\xi_c}$ are

$$f(kL) = \sum_{n=-\infty}^{\infty} \langle f\,;\,\phi_n \rangle \sqrt{\frac{2\pi}{L}}\;\frac{\sin \xi_c(kL - nL)}{\xi_c(kL - nL)}$$

$$= \sum_{n=-\infty}^{\infty} \langle f\,;\,\phi_n \rangle \sqrt{\frac{2\pi}{L}}\;\frac{\sin(k-n)\pi}{(k-n)\pi}$$

$$= \langle f\,;\,\phi_k \rangle \sqrt{\frac{2\pi}{L}}\;.$$

So

$$\langle f\,;\,\phi_k \rangle = f(kL)\sqrt{\frac{L}{2\pi}}\,, \qquad k = 0, \pm 1, \pm 2, \ldots\,.$$

Finally therefore

$$f(x) = \sum_{n=-\infty}^{\infty} \langle f\,;\,\phi_n \rangle \sqrt{\frac{2\pi}{L}}\;\frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)}$$

$$= \sum_{n=-\infty}^{\infty} f(nL)\,\frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)}\,.$$

Summarising we have Shannon's theorem,

Theorem. Let $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ be continuous and band limited to $|\xi| \le \xi_c$. Then

$$f(x) = \sum_{n=-\infty}^{\infty} f(nL)\,\frac{\sin \xi_c(x - nL)}{\xi_c(x - nL)} \qquad \text{where } L = \frac{\pi}{\xi_c}\;.$$

The relationship $\omega L = 2\pi$, where ω is the frequency of sampling in cycles per unit length, shows that from Shannon's theorem, to reconstruct a band-limited function, it suffices to sample at a frequency $\omega = \dfrac{2\pi}{L} = 2\xi_c$, twice the cut-off frequency.

18. Heisenberg's Inequality.

Let $f \in L^2(\mathbf{R})$, $xf \in L^2(\mathbf{R})$. Then the quantity

$$\Delta_a f \equiv \left(\frac{\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2 \, dx}{\int_{-\infty}^{\infty} |f(x)|^2 \, dx} \right)$$

is called the *dispersion about the point* $x = a$ of f. The reasoning behind the definition is that if $f(x)$ is concentrated near $x = a$, then $\Delta_a f$ is smaller than when f is not close to zero far from $x = a$.

Example. Consider the characteristic function

$$\chi_b(x) = \begin{cases} 1 ; & |x| \le b \\ 0 ; & \text{otherwise} \end{cases}$$

which has Fourier transform

$$(\chi_b)^\wedge(\xi) = \frac{2 \sin b\xi}{\xi}.$$

Notice that χ_b is concentrated near $x = 0$ for small b. The dispersion about the origin is

$$\Delta_0 \chi_b = \left(\frac{\int_{-b}^{b} x^2 dx}{\int_{-b}^{b} dx} \right) = \frac{b^2}{3}$$

and it clear that the dispersion increases as b increases.

Notice that the Fourier transform $\hat{\chi}_b$ does not have a finite dispersion about the origin, since

$$\int_{-\infty}^{\infty} \xi^2 |\hat{\chi}_b(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} \xi^2 \left(\frac{2 \sin(b\xi)}{\xi} \right)^2 d\xi$$
$$= 4 \int_{-\infty}^{\infty} \sin^2(b\xi) \, d\xi = \infty.$$

This indicates that $\hat{\chi}_b$ is spread out away from $x = 0$.

The following result shows that there is a type of inverse relationship between the dispersion of a function and that of its Fourier transform.

Theorem. Let $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then for all $a, \alpha \in \mathbf{R}$,

$$\left(\Delta_a f\right)\left(\Delta_\alpha f^\wedge\right) = \left(\frac{\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx}{\int_{-\infty}^{\infty}|f(x)|^2\,dx}\right)\left(\frac{\int_{-\infty}^{\infty}\xi^2|f^\wedge(\xi)|^2 d\xi}{\int_{-\infty}^{\infty}|f^\wedge(\xi)|^2 d\xi}\right) \geq \frac{1}{4}$$

and equality holds if and only if $f(x) = c\,e^{-kx^2}$ for constants $c \in \mathbf{R}$ and $k > 0$.

Proof. We firstly prove the result for $a = \alpha = 0$. $\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx$ and $\int_{-\infty}^{\infty}\xi^2|f^\wedge(\xi)|^2 d\xi$ are both assumed finite since otherwise the result is trivial.
Let $f^*(x) \equiv ((i\xi f^\wedge(\xi))^\vee$ or $(f^*)^\wedge(\xi) = i\xi f^\wedge(\xi)$. Then $f^* \in L^2(\mathbf{R})$, and

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty}\xi^2|f^\wedge(\xi)|^2 d\xi\right) = \left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty}|(i\xi)f^\wedge(\xi)|^2 d\xi\right)$$

$$= \left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty}|(f^*)^\wedge(\xi)|^2 d\xi\right)$$

$$= 2\pi\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty}|f^*(x)|^2\,dx\right) \quad \text{(by Parseval's identity).}$$

Since

$$\left[\int_{-\infty}^{\infty}\frac{1}{2}x\left(f^*(x)\,\overline{f(x)} + \overline{f^*(x)}\,f(x)\right)dx\right]^2 = \left[\int_{-\infty}^{\infty} x\,\mathrm{Re}\left(\overline{f^*(x)}\,f(x)\right)dx\right]^2$$

$$= \left[\mathrm{Re}\int_{-\infty}^{\infty} x\,f(x)\,\overline{f^*(x)}\,dx\right]^2 \leq \left|\int_{-\infty}^{\infty} x\,f(x)\,\overline{f^*(x)}\,dx\right|^2$$

$$\leq \left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty}|f^*(x)|^2\,dx\right)$$

(by the Cauchy-Schwartz inequality).

We will show that

$$-\int_{-\infty}^{\infty} x\left(f^*(x)\,\overline{f(x)} + \overline{f^*(x)}\,f(x)\right)dx = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

from which the result follows, for then

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty}\xi^2|f^\wedge(\xi)|^2 d\xi\right) = 2\pi\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty}|f^*(x)|^2\,dx\right)$$

$$\geq 2\pi\left[\int_{-\infty}^{\infty}\frac{1}{2}x\left(f^*(x)\,\overline{f(x)} + \overline{f^*(x)}\,f(x)\right)dx\right]^2$$

$$= \frac{\pi}{2}\left[\int_{-\infty}^{\infty} |f(x)|^2\, dx\right]^2 = \frac{\pi}{2}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |f\hat{}(\xi)|^2 d\xi\right)$$

$$= \frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |f\hat{}(\xi)|^2 d\xi\right).$$

To complete the proof, we assume that f is continuous and piecewise smooth, This assumption can be removed since functions in $L^1(\mathbf{R})$ are the uniform limit of such functions.

Then from the property of Fourier transforms,

$$f^*(x) = f'(x)$$

wherever the derivative exists. Then for any interval $[a, b]$,

$$b\,|f(b)|^2 - a\,|f(a)|^2 = \int_a^b \frac{d}{dx}\left(x\,|f(x)|^2\right) dx$$

$$= \int_a^b \left(x\,f'(x)\,\overline{f(x)} + x\,f(x)\,\overline{f'(x)} + |f(x)|^2\right) dx$$

$$= \int_a^b x\left(f^*(x)\,\overline{f(x)} + \overline{f^*(x)}\,f(x)\right) dx + \int_a^b |f(x)|^2\, dx$$

The assumption $f \in L^2(\mathbf{R})$ implies that $b\,|f(b)|^2 \to 0$ as $b \to \infty$ and $a\,|f(a)|^2 \to 0$ as $a \to -\infty$ since otherwise $|f(x)| > c\ |x|^{-1/2}$ as $|x| \to \infty$, which is not integrable. Taking the limit as $b \to \infty$ and $a \to -\infty$,

$$0 = \int_{-\infty}^{\infty} x\left(f^*(x)\,\overline{f(x)} + \overline{f^*(x)}\,f(x)\right) dx + \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

as required.

As for the case of equality in Heisenberg's inequality, this holds if and only if $f(x)\,\overline{f^*(x)}$ is real and $f^*(x) = \mathrm{K}\,x\,f(x)$ for some complex constant K. That is,

$$f(x)\,\overline{f^*(x)} = f(x)\,\overline{\mathrm{K}\,x\,f(x)} = x\,|f(x)|^2\,\overline{\mathrm{K}} \quad \text{is real.}$$

Therefore K is real.

The differential equation

$$f'(x) = f^*(x) = \mathrm{K}\,x\,f(x)$$

has solutions of the form

$$f(x) = c\,e^{-\frac{\mathrm{K}x^2}{2}},\ c \text{ any real constant,}$$

and $f(x) = c\, e^{-\frac{Kx^2}{2}} \in L^2(\mathbf{R})$ if and only if K > 0. Therefore equality holds in Heisenberg's inequality only if $f(x) = c\, e^{-kx^2}$ for constants $c \in \mathbf{R}$ and $k > 0$.

Conversely, let $f(x) = e^{-\frac{Kx^2}{2}}$ for constant K > 0. Then

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} \xi^2|f\^{}(\xi)|^2 d\xi\right) = 2\pi\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |f^*(x)|^2\, dx\right)$$

$$= 2\pi\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |\mathrm{K}\, x\, f(x)|^2\, dx\right)$$

$$= 2\pi\, \mathrm{K}^2\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\, dx\right)^2$$

$$= 2\pi\, \mathrm{K}^2\left(\int_{-\infty}^{\infty} x^2 e^{-\mathrm{K}x^2}\, dx\right)^2$$

$$= 2\pi\, \mathrm{K}^2\left[\int_{-\infty}^{\infty} \left(\frac{x}{-2\mathrm{K}}\right)\left(-2\mathrm{K}x e^{-\mathrm{K}x^2}\right) dx\right]^2$$

$$= 2\pi\, \mathrm{K}^2\left[\int_{-\infty}^{\infty} \left(\frac{x}{-2\mathrm{K}}\right)\left(-2\mathrm{K}x e^{-\mathrm{K}x^2}\right) dx\right]^2$$

$$= 2\pi\, \mathrm{K}^2\left[\int_{-\infty}^{\infty} \left(\frac{x}{-2\mathrm{K}}\right)\frac{d}{dx}\left(e^{-\mathrm{K}x^2}\right) dx\right]^2$$

$$= 2\pi\, \mathrm{K}^2\left[\int_{-\infty}^{\infty} \left(\frac{1}{2\mathrm{K}}\right) e^{-\mathrm{K}x^2}\, dx\right]^2 \qquad \text{(integration by parts)}$$

$$= \frac{\pi}{2}\left[\int_{-\infty}^{\infty} e^{-\mathrm{K}x^2}\, dx\right]^2$$

$$= \frac{\pi}{2\mathrm{K}}\left[\int_{-\infty}^{\infty} e^{-t^2}\, dt\right]^2 = \frac{\pi^2}{2\mathrm{K}}.$$

Whereas,

$$\frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |f\^{}(\xi)|^2 d\xi\right) = \frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right) 2\pi \left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)$$

$$= \frac{\pi}{2}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)^2$$

$$= \frac{\pi}{2}\left[\int_{-\infty}^{\infty} e^{-Kx^2}\,dx\right]^2$$

and equality holds.

The case of $a \neq 0,\ \alpha \neq 0$, follows by observing that $F(x) = e^{-i\alpha x} f(x + a)$ satisfies the same hypotheses as $f(x)$ and $\Delta_a f = \Delta_0 F$ and $\Delta_\alpha f^\wedge = \Delta_0 F^\wedge$ for any $a \neq 0,\ \alpha \neq 0$.

As a consequence of the inequality $\left(\Delta_a f\right)\left(\Delta_\alpha f^\wedge\right) \geq \frac{1}{4}$, we see that it is impossible for both $\Delta_a f$ and $\Delta_\alpha f^\wedge$ to be simultaneously small. That is, if one of $\Delta_a f$ or $\Delta_\alpha f^\wedge$ is very small then the other must be large.