

INTRODUCTION TO COMPLEX ANALYSIS

W W L CHEN

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Chapter 1

COMPLEX NUMBERS

1.1. Arithmetic and Conjugates

The purpose of this chapter is to give a review of various properties of the complex numbers that we shall need in the discussion of complex analysis. As the reader is expected to be familiar with the material, all proofs have been omitted.

The equation $x^2 + 1 = 0$ has no solution $x \in \mathbb{R}$. To “solve” this equation, we have to introduce extra numbers into our number system. To do this, we define the number i by $i^2 + 1 = 0$, and then extend the field of all real numbers by adjoining the number i , which is then combined with the real numbers by the operations addition and multiplication in accordance with the Field axioms of the real number system. The numbers $a + ib$, where $a, b \in \mathbb{R}$, of the extended field are then added and multiplied in accordance with the Field axioms, suitably extended, and the restriction $i^2 + 1 = 0$. Note that the number $a + 0i$, where $a \in \mathbb{R}$, behaves like the real number a .

What we have said in the last paragraph basically amounts to the following. Consider two complex numbers $a + ib$ and $c + id$, where $a, b, c, d \in \mathbb{R}$. We have the addition and multiplication rules

$$(a + ib) + (c + id) = (a + c) + i(b + d) \quad \text{and} \quad (a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

These lead to the subtraction rule

$$(a + ib) - (c + id) = (a - c) + i(b - d),$$

and the division rule, that if $c + id \neq 0$, then

$$\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}.$$

Note the special case $a = 1$ and $b = 0$.

Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. The real number x is called the real part of z , and denoted by $x = \Re z$. The real number y is called the imaginary part of z , and denoted by $y = \Im z$. The set $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$ is called the set of all complex numbers. The complex number $\bar{z} = x - iy$ is called the conjugate of z .

It is easy to see that for every $z \in \mathbb{C}$, we have

$$\Re z = \frac{z + \bar{z}}{2} \quad \text{and} \quad \Im z = \frac{z - \bar{z}}{2i}.$$

Furthermore, if $w \in \mathbb{C}$, then

$$\overline{z + w} = \bar{z} + \bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

1.2. Polar Coordinates

Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. The real number

$$r = \sqrt{x^2 + y^2}$$

is called the modulus of z , and denoted by $|z|$. On the other hand, if $z \neq 0$, then any number $\theta \in \mathbb{R}$ satisfying the equations

$$(1) \quad x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

is called an argument of z , and denoted by $\arg z$. Hence we can write z in polar form

$$z = r(\cos \theta + i \sin \theta).$$

Note, however, that for a given $z \in \mathbb{C}$, $\arg z$ is not unique. Clearly we can add any integer multiple of 2π to θ without affecting (1). We sometimes call a real number $\theta \in \mathbb{R}$ the principal argument of z if θ satisfies the equations (1) and $-\pi < \theta \leq \pi$. The principal argument of z is usually denoted by $\text{Arg } z$.

It is easy to see that for every $z \in \mathbb{C}$, we have $|z|^2 = z\bar{z}$. Also, if $w \in \mathbb{C}$, then

$$|zw| = |z||w| \quad \text{and} \quad |z + w| \leq |z| + |w|.$$

Furthermore, if

$$z = r(\cos \theta + i \sin \theta) \quad \text{and} \quad w = s(\cos \phi + i \sin \phi),$$

where $r, s, \theta, \phi \in \mathbb{R}$ and $r, s > 0$, then

$$zw = rs(\cos(\theta + \phi) + i \sin(\theta + \phi)) \quad \text{and} \quad \frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i \sin(\theta - \phi)).$$

1.3. Rational Powers

De Moivre's theorem, that

$$(2) \quad \cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n \quad \text{for every } n \in \mathbb{N} \text{ and } \theta \in \mathbb{R},$$

is useful in finding n -th roots of complex numbers.

Suppose that $c = R(\cos \alpha + i \sin \alpha)$, where $R, \alpha \in \mathbb{R}$ and $R > 0$. Then the solutions of the equation $z^n = c$ are given by

$$z = \sqrt[n]{R} \left(\cos \frac{\alpha + 2k\pi}{n} + i \sin \frac{\alpha + 2k\pi}{n} \right), \quad \text{where } k = 0, 1, \dots, n-1.$$

Finally, we can define c^b for any $b \in \mathbb{Q}$ and non-zero $c \in \mathbb{C}$ as follows. The rational number b can be written uniquely in the form $b = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no prime factors in common. Then there are exactly q distinct numbers z satisfying $z^q = c$. We now define $c^b = z^p$, noting that the expression (2) can easily be extended to all $n \in \mathbb{Z}$. It is not too difficult to show that there are q distinct values for the rational power c^b .

PROBLEMS FOR CHAPTER 1

- Suppose that $z_0 \in \mathbb{C}$ is fixed. A polynomial $P(z)$ is said to be divisible by $z - z_0$ if there is another polynomial $Q(z)$ such that $P(z) = (z - z_0)Q(z)$.
 - Show that for every $c \in \mathbb{C}$ and $k \in \mathbb{N}$, the polynomial $c(z^k - z_0^k)$ is divisible by $z - z_0$.
 - Consider the polynomial $P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$, where $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$ are arbitrary. Show that the polynomial $P(z) - P(z_0)$ is divisible by $z - z_0$.
 - Deduce that $P(z)$ is divisible by $z - z_0$ if $P(z_0) = 0$.
 - Suppose that a polynomial $P(z)$ of degree n vanishes at n distinct values $z_1, z_2, \dots, z_n \in \mathbb{C}$, so that $P(z_1) = P(z_2) = \dots = P(z_n) = 0$. Show that $P(z) = c(z - z_1)(z - z_2)\dots(z - z_n)$, where $c \in \mathbb{C}$ is a constant.
 - Suppose that a polynomial $P(z)$ of degree n vanishes at more than n distinct values. Show that $P(z) = 0$ identically.
- Suppose that $\alpha \in \mathbb{C}$ is fixed and $|\alpha| < 1$. Show that $|z| \leq 1$ if and only if $\left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right| \leq 1$.
- Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. Express each of the following in terms of x and y :
 - $|z - 1|^3$
 - $\left| \frac{z + 1}{z - 1} \right|$
 - $\left| \frac{z + i}{1 - iz} \right|$
- Suppose that $c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\alpha \neq 0$.
 - Show that $\alpha z + \bar{\alpha}z + c = 0$ is the equation of a straight line on the plane.
 - What does the equation $z\bar{z} + \alpha z + \bar{\alpha}z + c = 0$ represent if $|\alpha|^2 \geq c$?
- Suppose that $z, w \in \mathbb{C}$. Show that $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.
- Find all the roots of the equation $(z^8 - 1)(z^3 + 8) = 0$.
- For each of the following, compute all the values and plot them on the plane:
 - $(1 + i)^{-1/2}$
 - $(-4)^{3/4}$
 - $(1 - i)^{3/8}$

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Chapter 2

FOUNDATIONS OF COMPLEX ANALYSIS

2.1. Three Approaches

We start by remarking that analysis is sometimes known as the study of the four C's: convergence, continuity, compactness and connectedness. In real analysis, we have studied convergence and continuity to some depth, but the other two concepts have been somewhat disguised. In this course, we shall try to illustrate these two latter concepts a little bit more, particularly connectedness.

Complex analysis is the study of complex valued functions of complex variables. Here we shall restrict the number of variables to one, and study complex valued functions of one complex variable. Unless otherwise stated, all functions in these notes are of the form $f : S \rightarrow \mathbb{C}$, where S is a set in \mathbb{C} .

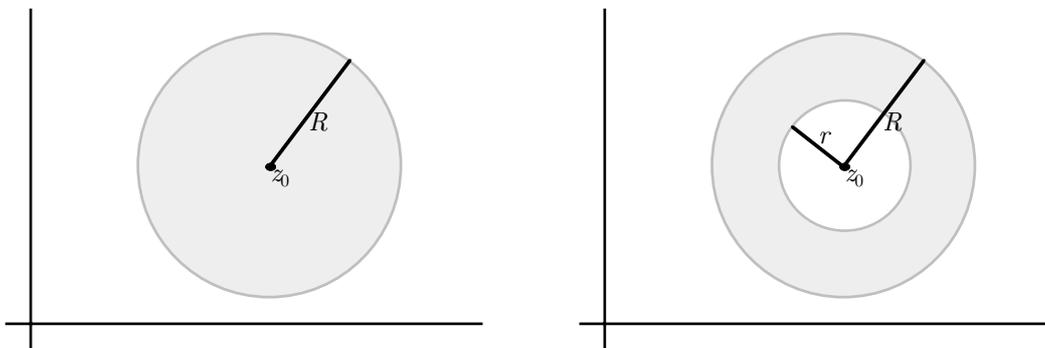
We shall study the behaviour of such functions using three different approaches. The first of these, discussed in Chapter 3 and usually attributed to Riemann, is based on differentiation and involves pairs of partial differential equations called the Cauchy-Riemann equations. The second approach, discussed in Chapters 4–11 and usually attributed to Cauchy, is based on integration and depends on a fundamental theorem known nowadays as Cauchy's integral theorem. The third approach, discussed in Chapter 16 and usually attributed to Weierstrass, is based on the theory of power series.

2.2. Point Sets in the Complex Plane

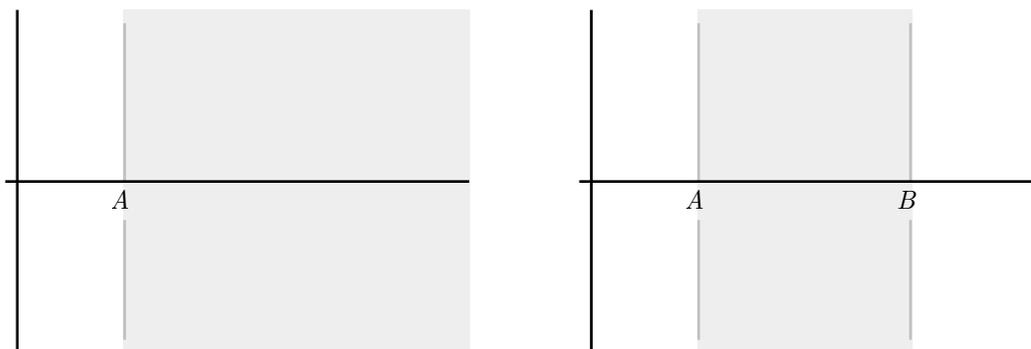
We shall study functions of the form $f : S \rightarrow \mathbb{C}$, where S is a set in \mathbb{C} . In most situations, various properties of the point sets S play a crucial role in our study. We therefore begin by discussing various types of point sets in the complex plane.

Before making any definitions, let us consider a few examples of sets which frequently occur in our subsequent discussion.

EXAMPLE 2.2.1. Suppose that $z_0 \in \mathbb{C}$, $r, R \in \mathbb{R}$ and $0 < r < R$. The set $\{z \in \mathbb{C} : |z - z_0| < R\}$ represents a disc, with centre z_0 and radius R , and the set $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ represents an annulus, with centre z_0 , inner radius r and outer radius R .



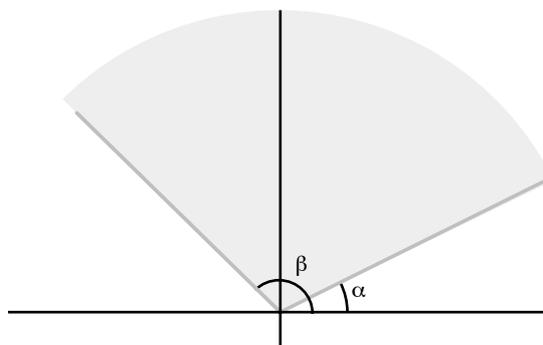
EXAMPLE 2.2.2. Suppose that $A, B \in \mathbb{R}$ and $A < B$. The set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } x > A\}$ represents a half-plane, and the set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } A < x < B\}$ represents a strip.



EXAMPLE 2.2.3. Suppose that $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha < \beta < 2\pi$. The set

$$\{z = r(\cos \theta + i \sin \theta) \in \mathbb{C} : r, \theta \in \mathbb{R} \text{ and } r > 0 \text{ and } \alpha < \theta < \beta\}$$

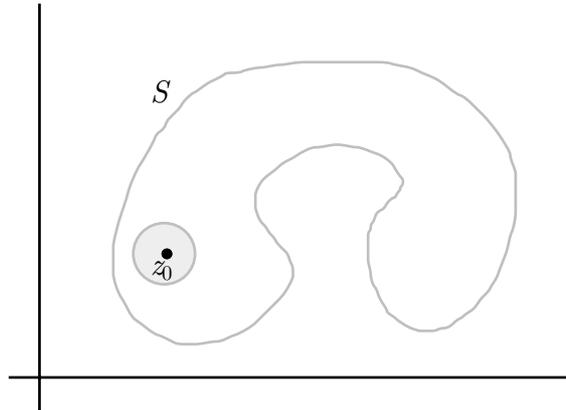
represents a sector.



We now make a number of important definitions. The reader may subsequently need to return to these definitions.

DEFINITION. Suppose that $z_0 \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$, with $\epsilon > 0$. By an ϵ -neighbourhood of z_0 , we mean a disc of the form $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$, with centre z_0 and radius $\epsilon > 0$.

DEFINITION. Suppose that S is a point set in \mathbb{C} . A point $z_0 \in S$ is said to be an interior point of S if there exists an ϵ -neighbourhood of z_0 which is contained in S . The set S is said to be open if every point of S is an interior point of S .



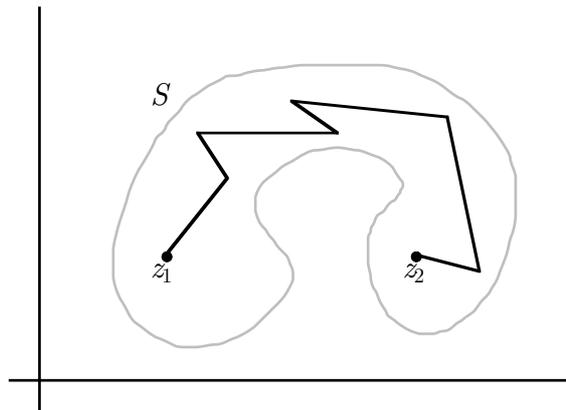
EXAMPLE 2.2.4. The sets in Examples 2.2.1–2.2.3 are open.

EXAMPLE 2.2.5. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is open.

EXAMPLE 2.2.6. The disc $\{z \in \mathbb{C} : |z - z_0| \leq R\}$ is not open.

EXAMPLE 2.2.7. The empty set \emptyset is open. Why?

DEFINITION. An open set S is said to be connected if every two points $z_1, z_2 \in S$ can be joined by the union of a finite number of line segments lying in S . An open connected set is called a domain.



REMARKS. (1) Sometimes, we say that an open set S is connected if there do not exist non-empty open sets S_1 and S_2 such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. In other words, an open connected set cannot be the disjoint union of two non-empty open sets.

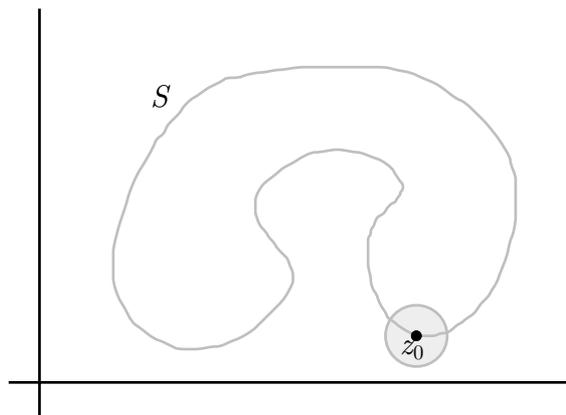
(2) In fact, it can be shown that the two definitions are equivalent.

(3) Note that we have not made any definition of connectedness for sets that are not open. In fact, the definition of connectedness for an open set given by (1) here is a special case of a much more complicated definition of connectedness which applies to all point sets.

EXAMPLE 2.2.8. The sets in Examples 2.2.1–2.2.3 are domains.

EXAMPLE 2.2.9. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is a domain.

DEFINITION. A point $z_0 \in \mathbb{C}$ is said to be a boundary point of a set S if every ϵ -neighbourhood of z_0 contains a point in S as well as a point not in S . The set of all boundary points of a set S is called the boundary of S .



EXAMPLE 2.2.10. The annulus $\{z \in \mathbb{C} : r < |z - z_0| < R\}$, where $0 < r < R$, has boundary $C_1 \cup C_2$, where

$$C_1 = \{z \in \mathbb{C} : |z - z_0| = r\} \quad \text{and} \quad C_2 = \{z \in \mathbb{C} : |z - z_0| = R\}$$

are circles, with centre z_0 and radius r and R respectively. Note that the annulus is connected and hence a domain. However, note that its boundary is made up of two separate pieces.

DEFINITION. A region is a domain together with all, some or none of its boundary points. A region which contains all its boundary points is said to be closed. For any region S , we denote by \bar{S} the closed region containing S and all its boundary points, and call \bar{S} the closure of S .

REMARK. Note that we have not made any definition of closedness for sets that are not regions. In fact, our definition of closedness for a region here is a special case of a much more complicated definition of closedness which applies to all point sets.

DEFINITION. A region S is said to be bounded or finite if there exists a real number M such that $|z| \leq M$ for every $z \in S$. A region that is closed and bounded is said to be compact.

EXAMPLE 2.2.11. The region $\{z \in \mathbb{C} : |z - z_0| \leq R\}$ is closed and bounded, hence compact. It is called the closed disc with centre z_0 and radius R .

EXAMPLE 2.2.12. The region $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leq x \leq 1\}$ is closed but not bounded.

EXAMPLE 2.2.13. The square $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leq x \leq 1 \text{ and } 0 < y < 1\}$ is bounded but not closed.

2.3. Complex Functions

In these lectures, we study complex valued functions of one complex variable. In other words, we study functions of the form $f : S \rightarrow \mathbb{C}$, where S is a set in \mathbb{C} . Occasionally, we will abuse notation and simply refer to a function by its formula, without explicitly defining the domain S . For instance, when we discuss the function $f(z) = 1/z$, we implicitly choose a set S which will not include the point $z = 0$ where the function is not defined. Also, we may occasionally wish to include the point $z = \infty$ in the domain or codomain.

We may separate the independent variable z as well as the dependent variable $w = f(z)$ into real and imaginary parts. Our usual notation will be to write $z = x + iy$ and $w = f(z) = u + iv$, where $x, y, u, v \in \mathbb{R}$. It follows that $u = u(x, y)$ and $v = v(x, y)$ can be interpreted as real valued functions of the two real variables x and y .

EXAMPLE 2.3.1. Consider the function $f : S \rightarrow \mathbb{C}$, given by $f(z) = z^2$ and where $S = \{z \in \mathbb{C} : |z| < 2\}$ is the open disc with radius 2 and centre 0. Using polar coordinates, it is easy to see that the range of the function is the open disc $f(S) = \{w \in \mathbb{C} : |w| < 4\}$ with radius 4 and centre 0.

EXAMPLE 2.3.2. Consider the function $f : \mathcal{H} \rightarrow \mathbb{C}$, where $\mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ is the upper half-plane and $f(z) = z^2$. Using polar coordinates, it is easy to see that the range of the function is the complex plane minus the non-negative real axis.

EXAMPLE 2.3.3. Consider the function $f : T \rightarrow \mathbb{C}$, where $T = \{z = x + iy \in \mathbb{C} : 1 < x < 2\}$ is a strip and $f(z) = z^2$. Let $x_0 \in (1, 2)$ be fixed, and consider the image of a point (x_0, y) on the vertical line $x = x_0$. Here we have

$$u = x_0^2 - y^2 \quad \text{and} \quad v = 2x_0y.$$

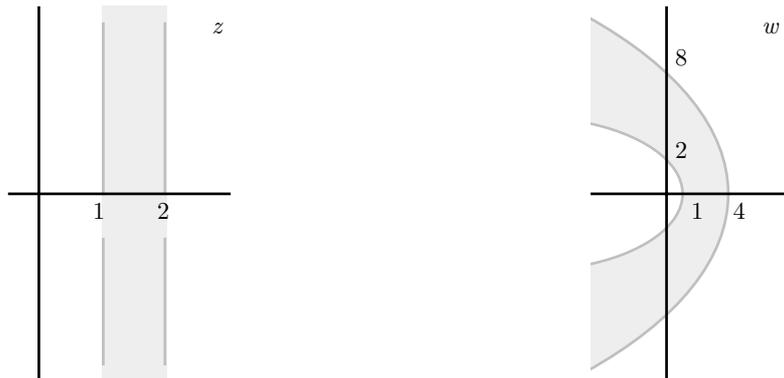
Eliminating y , we obtain the equation of a parabola

$$u = x_0^2 - \frac{v^2}{4x_0^2}$$

in the w -plane. It follows that the image of the vertical line $x = x_0$ under the function $w = z^2$ is this parabola. Now the boundary of the strip are the two lines $x = 1$ and $x = 2$. Their images under the mapping $w = z^2$ are respectively the parabolas

$$u = 1 - \frac{v^2}{4} \quad \text{and} \quad u = 4 - \frac{v^2}{16}.$$

It is easy to see that the range of the function is the part of the w -plane between these two parabolas.



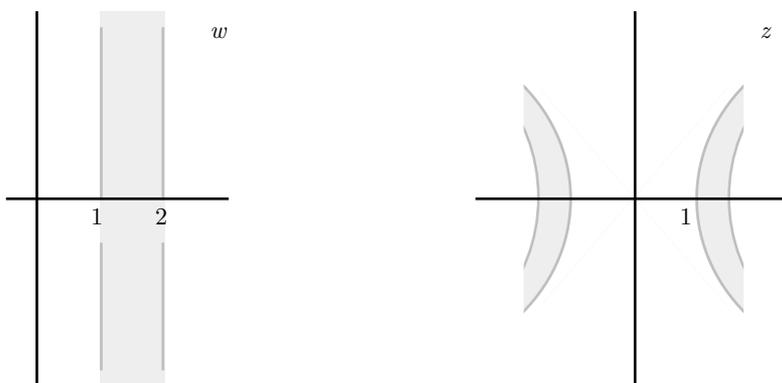
EXAMPLE 2.3.4. Consider again the function $w = z^2$. We would like to find all $z = x + iy \in \mathbb{C}$ for which $1 < \Re w < 2$. In other words, we have the restriction $1 < u < 2$, but no restriction on v . Let $u_0 \in (1, 2)$ be fixed, and consider points (x, y) in the z -plane with images on the vertical line $u = u_0$. Here we have the hyperbola

$$x^2 - y^2 = u_0.$$

The boundaries $u = 1$ and $u = 2$ are represented by the hyperbolas

$$x^2 - y^2 = 1 \quad \text{and} \quad x^2 - y^2 = 2.$$

It is easy to see that the points in question are precisely those between the two hyperbolas.



2.4. Extended Complex Plane

It is sometimes useful to extend the complex plane \mathbb{C} by the introduction of the point ∞ at infinity. Its connection with finite complex numbers can be established by setting $z + \infty = \infty + z = \infty$ for all $z \in \mathbb{C}$, and setting $z \cdot \infty = \infty \cdot z = \infty$ for all non-zero $z \in \mathbb{C}$. We can also write $\infty \cdot \infty = \infty$.

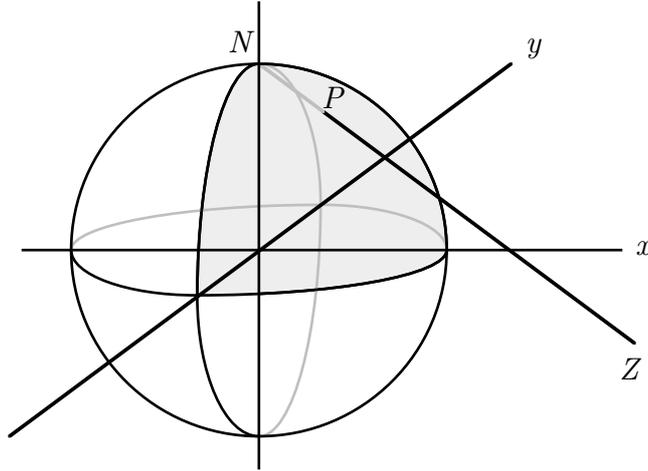
Note that it is not possible to define $\infty + \infty$ and $0 \cdot \infty$ without violating the laws of arithmetic. However, by special convention, we shall write $z/0 = \infty$ for $z \neq 0$ and $z/\infty = 0$ for $z \neq \infty$.

In the complex plane \mathbb{C} , there is no room for a point corresponding to ∞ . We can, of course, introduce an “ideal” point which we call the point at infinity. The points in \mathbb{C} , together with the point at infinity, form the extended complex plane. We decree that every straight line on the complex plane shall pass through the point at infinity, and that no half-plane shall contain the ideal point.

The main purpose of this section is to introduce a geometric model in which each point of the extended complex plane has a concrete representative. To do this, we shall use the idea of stereographic projection.

Consider a sphere of radius 1 in \mathbb{R}^3 . A typical point on this sphere will be denoted by $P(x_1, x_2, x_3)$. Note that $x_1^2 + x_2^2 + x_3^2 = 1$. Let us call the point $N(0, 0, 1)$ the north pole. The equator of this sphere is the set of all points of the form $(x_1, x_2, 0)$, where $x_1^2 + x_2^2 = 1$. Consider next the complex plane \mathbb{C} . This can be viewed as a plane in \mathbb{R}^3 . Let us position this plane in such a way that the equator of the sphere lies on this plane; in other words, our copy of the complex plane is “horizontal” and passes through the origin. We can further insist that the x -direction on our complex plane is the same as the x_1 -direction in \mathbb{R}^3 , and that the y -direction on our complex plane is the same as the x_2 -direction in \mathbb{R}^3 . Clearly a typical point $z = x + iy$ on our complex plane \mathbb{C} can be identified with the point $Z(x, y, 0)$ in \mathbb{R}^3 .

Suppose that $Z(x, y, 0)$ is on the plane. Consider the straight line that passes through Z and the north pole N . It is not too difficult to see that this straight line intersects the surface of the sphere at precisely one other point $P(x_1, x_2, x_3)$. In fact, if Z is on the equator of the sphere, then $P = Z$. If Z is on the part of the plane outside the sphere, then P is on the northern hemisphere, but is not the north pole N . If Z is on the part of the plane inside the sphere, then P is on the southern hemisphere. Check that for $Z(0, 0, 0)$, the point $P(0, 0, -1)$ is the south pole.



On the other hand, if P is any point on the sphere different from the north pole N , then a straight line passing through P and N intersects the plane at precisely one point Z . It follows that there is a pairing of all the points P on the sphere different from the north pole N and all the points on the plane. This pairing is governed by the requirement that the straight line through any pair must pass through the north pole N .

We can now visualize the north pole N as the point on the sphere corresponding to the point at infinity of the plane. The sphere is called the Riemann sphere.

2.5. Limits and Continuity

The concept of a limit in complex analysis is exactly the same as in real analysis. So, for example, we say that $f(z) \rightarrow L$ as $z \rightarrow z_0$, or

$$\lim_{z \rightarrow z_0} f(z) = L,$$

if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

This definition will be perfectly in order if the function f is defined in some open set containing z_0 , with the possible exception of z_0 itself. It follows that if z_0 is an interior point of the region S of definition of the function, our definition is in order. However, if z_0 is a boundary point of the region S of definition of the function, then we agree that the conclusion $|f(z) - L| < \epsilon$ need only hold for those $z \in S$ satisfying $0 < |z - z_0| < \delta$.

Similarly, we say that a function $f(z)$ is continuous at z_0 if $f(z) \rightarrow f(z_0)$ as $z \rightarrow z_0$. A similar qualification on z applies if z_0 is a boundary point of the region S of definition of the function. We also say that a function is continuous in a region if it is continuous at every point of the region.

Note that for a function to be continuous in a region, it is enough to have continuity at every point of the region. Hence the choice of δ may depend on a point z_0 in question. If δ can be chosen independently of z_0 , then we have some uniformity as well. To be precise, we make the following definition.

DEFINITION. A function $f(z)$ is said to be uniformly continuous in a region S if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ for every $z_1, z_2 \in S$ satisfying $|z_1 - z_2| < \delta$.

REMARK. Note that if we fix z_2 to be a point z_0 and write z for z_1 , then we require $|f(z) - f(z_0)| < \epsilon$ for every $z \in S$ satisfying $|z - z_0| < \delta$. In other words, δ cannot depend on z_0 .

EXAMPLE 2.5.1. Consider the punctured disc $S = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The function $f(z) = 1/z$ is continuous in S but not uniformly continuous in S . To see this, note first of all that continuity follows from the simple observation that the function z is continuous and non-zero in S . To show that the function is not uniformly continuous in S , it suffices to show that there exists $\epsilon > 0$ such that for every $\delta > 0$, there exist $z_1, z_2 \in S$ such that

$$|z_1 - z_2| < \delta \quad \text{and} \quad \left| \frac{1}{z_1} - \frac{1}{z_2} \right| \geq \epsilon.$$

Let $\epsilon = 1$. For every $\delta > 0$, choose $n \in \mathbb{N}$ such that $n > \delta^{-1/2}$, and let

$$z_1 = \frac{1}{n} \quad \text{and} \quad z_2 = \frac{1}{n+1}.$$

Clearly $z_1, z_2 \in S$. It is easy to see that

$$|z_1 - z_2| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{1}{n(n+1)} < \delta \quad \text{and} \quad \left| \frac{1}{z_1} - \frac{1}{z_2} \right| = 1.$$

PROBLEMS FOR CHAPTER 2

- For each of the following functions, find $f(z+3)$, $f(1/z)$ and $f(f(z))$:
 - $f(z) = z - 1$
 - $f(z) = z^2$
 - $f(z) = 1/z$
 - $f(z) = \frac{1-z}{3+z}$
- Which of the sets below are domains?
 - $\{z : 0 < |z| < 1\}$
 - $\{z : \Im z < 3|z|\}$
 - $\{z : |z-1| \leq |z+1|\}$
 - $\{z : |z^2 - 1| < 1\}$
 - $\{z : 0 < \Re z \leq 1\}$
- Find the image of the strip $\{z : |\Re z| < 1\}$ and of the disc $\{z : |z| < 1\}$ under each of the following mappings:
 - $w = (1+i)z + 1$
 - $w = 2z^2$
 - $w = z^{-1}$
 - $w = \frac{z+1}{z-1}$
- A function $f(z)$ is said to be an isometry if $|f(z_1) - f(z_2)| = |z_1 - z_2|$ for every $z_1, z_2 \in \mathbb{C}$; in other words, if it preserves distance.
 - Suppose that $f(z)$ is an isometry. Show that for every $a, b \in \mathbb{C}$ with $|a| = 1$, the function $g(z) = af(z) + b$ is also an isometry.
 - Show that the function

$$h(z) = \frac{f(z) - f(0)}{f(1) - f(0)}$$

is an isometry with $h(0) = 0$ and $h(1) = 1$.

INTRODUCTION TO COMPLEX ANALYSIS

W W L CHEN

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Chapter 3

COMPLEX DIFFERENTIATION

3.1. Introduction

Suppose that $D \subseteq \mathbb{C}$ is a domain. A function $f : D \rightarrow \mathbb{C}$ is said to be differentiable at $z_0 \in D$ if the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, we write

$$(1) \quad f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and call $f'(z_0)$ the derivative of f at z_0 .

If $z \neq z_0$, then

$$f(z) = \left(\frac{f(z) - f(z_0)}{z - z_0} \right) (z - z_0) + f(z_0).$$

It follows from (1) and the arithmetic of limits that if $f'(z_0)$ exists, then $f(z) \rightarrow f(z_0)$ as $z \rightarrow z_0$, so that f is continuous at z_0 . In other words, differentiability at z_0 implies continuity at z_0 .

Note that the argument here is the same as in the case of a real valued function of a real variable. In fact, the similarity in argument extends to the arithmetic of limits. Indeed, if the functions $f : D \rightarrow \mathbb{C}$ and $g : D \rightarrow \mathbb{C}$ are both differentiable at $z_0 \in D$, then both $f + g$ and fg are differentiable at z_0 , and

$$(f + g)'(z_0) = f'(z_0) + g'(z_0) \quad \text{and} \quad (fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

If the extra condition $g'(z_0) \neq 0$ holds, then f/g is differentiable at z_0 , and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

One can also establish the Chain rule for differentiation as in real analysis. More precisely, suppose that the function f is differentiable at z_0 and the function g is differentiable at $w_0 = f(z_0)$. Then the function $g \circ f$ is differentiable at $z = z_0$, and

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0).$$

EXAMPLE 3.1.1. Consider the function $f(z) = \bar{z}$, where for every $z \in \mathbb{C}$, \bar{z} denotes the complex conjugate of z . Suppose that $z_0 \in \mathbb{C}$. Then

$$(2) \quad \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}.$$

If $z - z_0 = h$ is real and non-zero, then (2) takes the value 1. On the other hand, if $z - z_0 = ik$ is purely imaginary, then (2) takes the value -1 . It follows that this function is not differentiable anywhere in \mathbb{C} , although its real and imaginary parts are rather well behaved.

3.2. The Cauchy-Riemann Equations

If we use the notation

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h},$$

then in Example 3.1.1, we have examined the behaviour of the ratio

$$\frac{f(z+h) - f(z)}{h}$$

first as $h \rightarrow 0$ through real values and then through imaginary values. Indeed, for the derivative to exist, it is essential that these two limiting processes produce the same limit $f'(z)$. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. If h is real, then the two limiting processes above correspond to

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \rightarrow 0} \frac{v(x+h, y) - v(x, y)}{h} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

and

$$\lim_{h \rightarrow 0} \frac{f(z+ih) - f(z)}{ih} = \lim_{h \rightarrow 0} \frac{u(x, y+h) - u(x, y)}{ih} + i \lim_{h \rightarrow 0} \frac{v(x, y+h) - v(x, y)}{ih} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

respectively. Equating real and imaginary parts, we obtain

$$(3) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note that while the existence of the derivative in real analysis is a mild smoothness condition, the existence of the derivative in complex analysis leads to a pair of partial differential equations.

DEFINITION. The partial differential equations (3) are called the Cauchy-Riemann equations.

We have proved the following result.

THEOREM 3A. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that $f'(z)$ exists. Then the four partial derivatives in (3) exist, and the Cauchy-Riemann equations (3) hold. Furthermore, we have

$$(4) \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{and} \quad f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

A natural question to ask is whether the Cauchy-Riemann equations are sufficient to guarantee the existence of the derivative. We shall show next that we require also the continuity of the partial derivatives in (3).

THEOREM 3B. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that the four partial derivatives in (3) are continuous and satisfy the Cauchy-Riemann equations (3) at z_0 . Then f is differentiable at z_0 , and the derivative $f'(z_0)$ is given by the equations (4) evaluated at z_0 .

PROOF. Write $z_0 = x_0 + iy_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{(u(x, y) - u(x_0, y_0)) + i(v(x, y) - v(x_0, y_0))}{z - z_0}.$$

We can write

$$u(x, y) - u(x_0, y_0) = (x - x_0) \left(\frac{\partial u}{\partial x} \right)_{z_0} + (y - y_0) \left(\frac{\partial u}{\partial y} \right)_{z_0} + |z - z_0| \epsilon_1(z)$$

and

$$v(x, y) - v(x_0, y_0) = (x - x_0) \left(\frac{\partial v}{\partial x} \right)_{z_0} + (y - y_0) \left(\frac{\partial v}{\partial y} \right)_{z_0} + |z - z_0| \epsilon_2(z).$$

If the four partial derivatives in (3) are continuous at z_0 , then

$$\lim_{z \rightarrow z_0} \epsilon_1(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow z_0} \epsilon_2(z) = 0.$$

In view of the Cauchy-Riemann equations (3), we have

$$\begin{aligned} & (u(x, y) - u(x_0, y_0)) + i(v(x, y) - v(x_0, y_0)) \\ &= (x - x_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + (y - y_0) \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)) \\ &= (x - x_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + (y - y_0) \left(-\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)) \\ &= (x - x_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + i(y - y_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)) \\ &= (z - z_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)). \end{aligned}$$

Hence

$$\frac{f(z) - f(z_0)}{z - z_0} = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + \left(\frac{|z - z_0|}{z - z_0} \right) (\epsilon_1(z) + i \epsilon_2(z)) \rightarrow \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0}$$

as $z \rightarrow z_0$, giving the desired results. \circ

3.3. Analytic Functions

In the previous section, we have shown that differentiability in complex analysis leads to a pair of partial differential equations. Now partial differential equations are seldom of interest at a single point, but rather in a region. It therefore seems reasonable to make the following definition.

DEFINITION. A function f is said to be analytic at a point $z_0 \in \mathbb{C}$ if it is differentiable at every z in some ϵ -neighbourhood of the point z_0 . The function f is said to be analytic in a region if it is analytic at every point in the region. The function f is said to be entire if it is analytic in \mathbb{C} .

EXAMPLE 3.3.1. Consider the function $f(z) = |z|^2$. In our usual notation, we clearly have

$$u = x^2 + y^2 \quad \text{and} \quad v = 0.$$

The Cauchy-Riemann equations

$$2x = 0 \quad \text{and} \quad 2y = 0$$

can only be satisfied at $z = 0$. It follows that the function is differentiable only at the point $z = 0$, and is therefore analytic nowhere.

EXAMPLE 3.3.2. The function $f(z) = z^2$ is entire.

EXAMPLE 3.3.3. Suppose that the function f is analytic in a domain D . Suppose further that f has constant real part u . Then clearly

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0.$$

Since f is analytic in D , it is differentiable at every point in D , and so the Cauchy-Riemann equations hold in D . It follows that

$$\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

Hence f must have constant imaginary part v , and so f must be constant in D .

EXAMPLE 3.3.4. Suppose that the function f is analytic in a domain D . Suppose further that f has constant imaginary part v . A similar argument shows that f must have constant real part u . Hence f must be constant in D .

EXAMPLE 3.3.5. Suppose that the function f is analytic in a domain D . Suppose further that f has constant modulus. In other words, $u^2 + v^2 = C$ for some non-negative real number C . Differentiating this with respect to x and to y , we obtain respectively

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0.$$

In view of the Cauchy-Riemann equations, these can be written as

$$2u \frac{\partial u}{\partial x} - 2v \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad 2v \frac{\partial u}{\partial x} + 2u \frac{\partial u}{\partial y} = 0.$$

In matrix notation, these become

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note now that

$$\det \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2 = C.$$

If $C > 0$, then we must have the unique solution

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

so that the real part u is constant. It then follows from Example 3.3.3 that f is constant in D . On the other hand, if $C = 0$, then clearly $u = v = 0$, so that $f = 0$ in D .

3.4. Introduction to Special Functions

In this section, we shall generalize various functions that we have studied in real analysis to the complex domain. Consider first of all the exponential function. It seems reasonable to extend the property $e^{x_1+x_2} = e^{x_1}e^{x_2}$ for real variables to complex values of the variables to obtain

$$e^z = e^{x+iy} = e^x e^{iy}, \quad \text{where } x, y \in \mathbb{R}.$$

This suggests the following definition.

DEFINITION. Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. Then the exponential function e^z is defined for every $z \in \mathbb{C}$ by

$$(5) \quad e^z = e^x(\cos y + i \sin y).$$

If we write $e^z = u(x, y) + iv(x, y)$, then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

It is easy to check that the Cauchy-Riemann equations are satisfied for every $z \in \mathbb{C}$, so that e^z is an entire function. Furthermore, it follows from (4) that

$$\frac{d}{dz} e^z = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y) = e^z,$$

so that e^z is its own derivative. On the other hand, note that for every $y_1, y_2 \in \mathbb{R}$, we have

$$e^{i(y_1+y_2)} = \cos(y_1 + y_2) + i \sin(y_1 + y_2) = (\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) = e^{iy_1} e^{iy_2}.$$

Furthermore, if $x_1, x_2 \in \mathbb{R}$, then

$$e^{x_1+x_2} e^{i(y_1+y_2)} = (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2}) = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}).$$

Writing $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we deduce the addition formula

$$e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

Finally, note that

$$|e^z| = |e^x(\cos y + i \sin y)| = e^x |\cos y + i \sin y| = e^x.$$

Since e^x is never zero, it follows that the exponential function e^z is non-zero for every $z \in \mathbb{C}$.

Next, we turn our attention to the trigonometric functions. Note first of all that if $z = x + iy$, where $x, y \in \mathbb{R}$, then $iz = -y + ix$. Replacing z in (5) by iz and by $-iz$ gives respectively

$$e^{iz} = e^{-y}(\cos x + i \sin x) \quad \text{and} \quad e^{-iz} = e^y(\cos x - i \sin x).$$

The special case $y = 0$ gives respectively

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x.$$

It follows that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This suggests the following definition.

DEFINITION. Suppose that $z \in \mathbb{C}$. Then the trigonometric functions $\cos z$ and $\sin z$ are defined in terms of the exponential function by

$$(6) \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Since the exponential function is an entire function, it follows easily from (6) that both $\cos z$ and $\sin z$ are entire functions. Furthermore, it can easily be deduced from (6) that

$$\frac{d}{dz} \cos z = -\sin z \quad \text{and} \quad \frac{d}{dz} \sin z = \cos z.$$

We can define the functions $\tan z$, $\cot z$, $\sec z$ and $\operatorname{cosec} z$ in terms of the functions $\cos z$ and $\sin z$ as in real variables. However, note that these four functions are not entire. Also, we can deduce from (6) the formulas

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \quad \text{and} \quad \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

and a host of other trigonometric identities that we know hold for real variables.

Finally, we turn our attention to the hyperbolic functions. These are defined as in real analysis.

DEFINITION. Suppose that $z \in \mathbb{C}$. Then the hyperbolic functions $\cosh z$ and $\sinh z$ are defined in terms of the exponential function by

$$(7) \quad \cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Since the exponential function is an entire function, it follows easily from (7) that both $\cosh z$ and $\sinh z$ are entire functions. Furthermore, it can easily be deduced from (7) that

$$\frac{d}{dz} \cosh z = \sinh z \quad \text{and} \quad \frac{d}{dz} \sinh z = \cosh z.$$

We can define the functions $\tanh z$, $\coth z$, $\operatorname{sech} z$ and $\operatorname{cosech} z$ in terms of the functions $\cosh z$ and $\sinh z$ as in real variables. However, note that these four functions are not entire. Also, we can deduce from (7) a host of hyperbolic identities that we know hold for real variables. Note also that comparing (6) and (7), we obtain

$$\cosh z = \cos iz \quad \text{and} \quad \sinh z = -i \sin iz.$$

3.5. Periodicity and its Consequences

One of the fundamental differences between real and complex analysis is that the exponential function is periodic in \mathbb{C} .

DEFINITION. A function f is periodic in \mathbb{C} if there is some fixed non-zero $\omega \in \mathbb{C}$ such that the identity $f(z + \omega) = f(z)$ holds for every $z \in \mathbb{C}$. Any constant $\omega \in \mathbb{C}$ with this property is called a period of f .

THEOREM 3C. *The exponential function e^z is periodic in \mathbb{C} with period $2\pi i$. Furthermore, any period $\omega \in \mathbb{C}$ of e^z is of the form $\omega = 2\pi ki$, where $k \in \mathbb{Z}$ is non-zero.*

PROOF. The first assertion follows easily from the observation

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1.$$

Suppose now that $\omega \in \mathbb{C}$. Clearly $e^{z+\omega} = e^z$ implies $e^\omega = 1$. Write $\omega = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. Then

$$e^\alpha (\cos \beta + i \sin \beta) = 1.$$

Taking modulus, we conclude that $e^\alpha = 1$, so that $\alpha = 0$. It then follows that $\cos \beta + i \sin \beta = 1$. Equating real and imaginary parts, we conclude that $\cos \beta = 1$ and $\sin \beta = 0$, so that $\beta = 2\pi k$, where $k \in \mathbb{Z}$. The second assertion follows. \circ

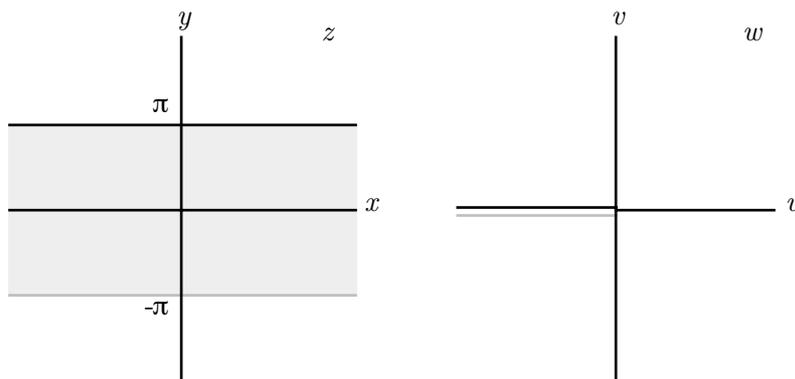
Consider now the mapping $w = e^z$. By (5), we have $w = e^x (\cos y + i \sin y)$, where $x, y \in \mathbb{R}$. It follows that

$$|w| = e^x \quad \text{and} \quad \arg w = y + 2\pi k,$$

where $k \in \mathbb{Z}$. Usually we make the choice $\arg w = y$, with the restriction that $-\pi < y \leq \pi$. This restriction means that z lies on the horizontal strip

$$(8) \quad \mathcal{R}_0 = \{z \in \mathbb{C} : -\infty < x < \infty, -\pi < y \leq \pi\}.$$

The restriction $-\pi < \arg w \leq \pi$ can also be indicated on the complex w -plane by a cut along the negative real axis. The upper edge of the cut, corresponding to $\arg w = \pi$, is regarded as part of the cut w -plane. The lower edge of the cut, corresponding to $\arg w = -\pi$, is not regarded as part of the cut w -plane.



It is easy to check that the function $\exp : \mathcal{R}_0 \rightarrow \mathbb{C} \setminus \{0\}$, defined for every $z \in \mathcal{R}_0$ by $\exp(z) = e^z$, is one-to-one and onto.

REMARK. The region \mathcal{R}_0 is usually known as a fundamental region of the exponential function. In fact, it is easy to see that every set of the type

$$(9) \quad \mathcal{R}_k = \{z \in \mathbb{C} : -\infty < x < \infty, (2k - 1)\pi < y \leq (2k + 1)\pi\},$$

where $k \in \mathbb{Z}$, has this same property as \mathcal{R}_0 .

Let us return to the function $\exp : \mathcal{R}_0 \rightarrow \mathbb{C} \setminus \{0\}$. Since it is one-to-one and onto, there is an inverse function.

DEFINITION. The function $\text{Log} : \mathbb{C} \setminus \{0\} \rightarrow \mathcal{R}_0$, defined by $\text{Log}(w) = z \in \mathcal{R}_0$, where $\exp(z) = w$, is called the principal logarithmic function.

Suppose that $z = x + iy$ and $w = u + iv$, where $x, y, u, v \in \mathbb{R}$. Suppose further that we impose the restriction $-\pi < y \leq \pi$. If $w = \exp(z)$, then it follows from (5) that $u = e^x \cos y$ and $v = e^x \sin y$, and so

$$|w| = (u^2 + v^2)^{1/2} = e^x \quad \text{and} \quad y = \text{Arg}(w),$$

where $\text{Arg}(w)$ denotes the principal argument of w . It follows that

$$x = \log |w| \quad \text{and} \quad y = \text{Arg}(w).$$

Hence

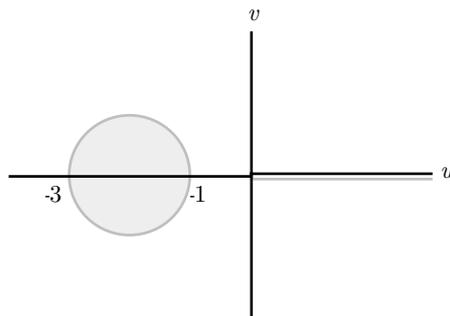
$$(10) \quad \text{Log}(w) = \log |w| + i \text{Arg}(w).$$

In many practical situations, we usually try to define

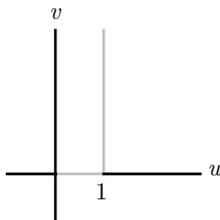
$$\log w = \log |w| + i \arg w,$$

where the argument is chosen in order to make the logarithmic function continuous in its domain of definition, if this is at all possible. The following three examples show that great care needs to be taken in the study of such “many valued functions”.

EXAMPLE 3.5.1. Consider the logarithmic function in the disc $\{w : |w+2| < 1\}$, an open disc of radius 1 and centred at the point $w = -2$. Note that this disc crosses the cut on the w -plane along the negative real axis discussed earlier. In this case, we may restrict the argument to satisfy, for example, $0 \leq \arg w < 2\pi$. The logarithmic function defined in this way is then continuous in the disc $\{w : |w+2| < 1\}$.

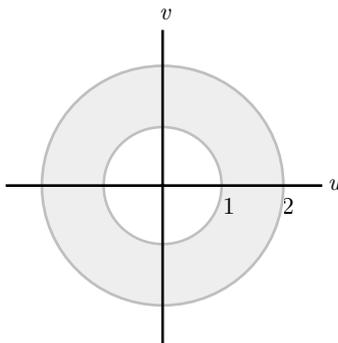


EXAMPLE 3.5.2. Consider the region P obtained from the w -plane by removing both the line segment $\{u + iv : 0 \leq u \leq 1, v = 0\}$ and the half-line $\{u + iv : u = 1, v > 0\}$, as shown below.



Suppose that we wish to define the logarithmic function to be continuous in this region P . One way to do this is to restrict the argument to the range $\pi < \arg w \leq 3\pi$ for any $w \in P$ satisfying $u \geq 1$, and to the range $0 < \arg w \leq 2\pi$ for any $w \in P$ satisfying $u < 1$.

EXAMPLE 3.5.3. Consider the annulus $\{w : 1 < |w| < 2\}$. It is impossible to define the logarithmic function to be continuous in this annulus. Heuristically, if one goes round the annulus once, the argument has to change by 2π if it varies continuously. If we return to the original starting point after going round once, the argument cannot therefore be the same.



It should now be quite clear that we cannot expect to have

$$\text{Log}(w_1 w_2) = \text{Log}(w_1) + \text{Log}(w_2),$$

or even

$$\log w_1 w_2 = \log w_1 + \log w_2.$$

Instead, we have

$$\log w_1 w_2 = \log w_1 + \log w_2 + 2\pi i k \quad \text{for some } k \in \mathbb{Z}.$$

Let us return to the principal logarithmic function $\text{Log} : \mathbb{C} \setminus \{0\} \rightarrow \mathcal{R}_0$. Recall (10). We have

$$\text{Log}(z) = \log |z| + i \text{Arg}(z).$$

Recall from real analysis that for any $t \in \mathbb{R}$, the equation $\tan \theta = t$ has a unique solution θ satisfying $-\pi/2 < \theta < \pi/2$. This solution is denoted by $\tan^{-1} t$ and satisfies

$$\frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2}.$$

It is not difficult to show that if we write

$$(11) \quad v(x, y) = \begin{cases} -\tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2} & \text{if } y < 0, \\ -\tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0, \\ -\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2} & \text{if } y > 0, \end{cases}$$

then $\text{Arg}(z) = v(x, y)$. Hence $\text{Log}(z) = u(x, y) + iv(x, y)$, where

$$(12) \quad u(x, y) = \frac{1}{2} \log(x^2 + y^2).$$

It now follows from (12) that

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},$$

and from (11) that

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}.$$

Clearly the Cauchy-Riemann equations are satisfied, and so

$$\frac{d}{dz} \text{Log}(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

Power functions are defined in terms of the exponential and logarithmic functions. Given $z, a \in \mathbb{C}$, we write $z^a = e^{a \log z}$. Naturally, the precise value depends on the logarithmic function that is chosen, and care again must be exercised for these “many valued functions”.

3.6. Laplace’s Equation and Harmonic Conjugates

We have shown that for any function $f = u + iv$, the existence of the derivative f' leads to the Cauchy-Riemann equations. More precisely, we have

$$(13) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Furthermore,

$$(14) \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Suppose now that the second derivative f'' also exists. Then f' satisfies the Cauchy-Riemann equations. The Cauchy-Riemann equations corresponding to the expression (14) are

$$(15) \quad \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = -\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right).$$

Substituting (13) into (15), we obtain

$$(16) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

We also obtain

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

DEFINITION. A continuous function $\phi(x, y)$ that satisfies Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in a domain $D \subseteq \mathbb{C}$ is said to be harmonic in D .

We have in fact proved the following result.

THEOREM 3D. Suppose that $f = u + iv$, where u and v are real valued. Suppose further that $f''(z)$ exists in a domain $D \subseteq \mathbb{C}$. Then u and v both satisfy Laplace's equation and are harmonic in D .

DEFINITION. Two harmonic functions u and v in a domain $D \subseteq \mathbb{C}$ are said to be harmonic conjugates in D if they satisfy the Cauchy-Riemann equations.

The remainder of this chapter is devoted to a discussion on finding harmonic conjugates. We shall illustrate the following theorem by discussing the special case when $D = \mathbb{C}$.

THEOREM 3E. Suppose that a function u is real valued and harmonic in a domain $D \subseteq \mathbb{C}$. Then there exists a real valued function v which satisfies the following conditions:

- (a) The functions u and v satisfy the Cauchy-Riemann equations in D .
- (b) The function $f = u + iv$ is analytic in D .
- (c) The function v is harmonic in D .

Clearly, parts (b) and (c) follow from part (a). We shall now indicate a proof of part (a) in the special case $D = \mathbb{C}$, and shall omit reference to this domain.

Suppose that u is real valued and harmonic. Then we need to find a real valued function v such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Let $X_0 + iY_0 \in D$ be chosen and fixed. Integrating the second of these with respect to x , we obtain

$$(17) \quad v(X, y) = -\int_{X_0}^X \frac{\partial u}{\partial y}(x, y) dx + c(y),$$

where $c(y)$ is some function depending at most on y . Differentiating with respect to y , we obtain

$$\frac{\partial v}{\partial y}(X, y) = -\frac{\partial}{\partial y} \int_{X_0}^X \frac{\partial u}{\partial y}(x, y) dx + c'(y).$$

Clearly the first of the Cauchy-Riemann equations requires

$$\frac{\partial u}{\partial x}(X, y) = -\frac{\partial}{\partial y} \int_{X_0}^X \frac{\partial u}{\partial y}(x, y) dx + c'(y).$$

Changing the order of differentiation and integration, we obtain

$$\frac{\partial u}{\partial x}(X, y) = - \int_{X_0}^X \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) (x, y) dx + c'(y) = - \int_{X_0}^X \frac{\partial^2 u}{\partial y^2} (x, y) dx + c'(y).$$

Since u is harmonic, we obtain

$$\frac{\partial u}{\partial x}(X, y) = \int_{X_0}^X \frac{\partial^2 u}{\partial x^2} (x, y) dx + c'(y) = \frac{\partial u}{\partial x}(X, y) - \frac{\partial u}{\partial x}(X_0, y) + c'(y),$$

so that

$$c'(y) = \frac{\partial u}{\partial x}(X_0, y).$$

Integrating with respect to y , we obtain

$$(18) \quad c(Y) = \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy + c,$$

where c is an absolute constant. On the other hand, (17) can be rewritten in the form

$$(19) \quad v(X, Y) = - \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + c(Y).$$

Combining (18) and (19), we obtain

$$(20) \quad v(X, Y) = - \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy + c.$$

It is easy to check that this function v satisfies the Cauchy-Riemann equations. Indeed, we have

$$\frac{\partial}{\partial X} v(X, Y) = - \frac{\partial}{\partial X} \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + \frac{\partial}{\partial X} \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy = - \frac{\partial u}{\partial y}(X, Y).$$

On the other hand, we have

$$\begin{aligned} \frac{\partial}{\partial Y} v(X, Y) &= - \frac{\partial}{\partial Y} \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + \frac{\partial}{\partial Y} \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy = - \int_{X_0}^X \frac{\partial^2 u}{\partial y^2}(x, Y) dx + \frac{\partial u}{\partial x}(X_0, Y) \\ &= \int_{X_0}^X \frac{\partial^2 u}{\partial x^2}(x, Y) dx + \frac{\partial u}{\partial x}(X_0, Y) = \frac{\partial u}{\partial x}(X, Y) - \frac{\partial u}{\partial x}(X_0, Y) + \frac{\partial u}{\partial x}(X_0, Y) = \frac{\partial u}{\partial x}(X, Y). \end{aligned}$$

This completes our sketched proof.

In practice, we may use the following technique. Suppose that u is a real valued harmonic function in a domain D . Write

$$(21) \quad g(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$

Then the Cauchy-Riemann equations for g are

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right),$$

which clearly hold. It follows that g is analytic in D . Suppose now that u is the real part of an analytic function f in D . Then $f'(z)$ agrees with the right hand side of (21) in view of (3) and (4). Hence $f' = g$

in D . The question here, of course, is to find this function f . If we are successful, then the imaginary part v of f is a harmonic conjugate of the harmonic function u .

EXAMPLE 3.6.1. Consider the function $u(x, y) = x^3 - 3xy^2$. It is easily checked that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

so that u is harmonic in \mathbb{C} . Using $X_0 = Y_0 = 0$ in (20), we obtain

$$v(X, Y) = - \int_0^X \frac{\partial u}{\partial y}(x, Y) dx + \int_0^Y \frac{\partial u}{\partial x}(0, y) dy + c = 6 \int_0^X xY dx - 3 \int_0^Y y^2 dy + c = 3X^2Y - Y^3 + c,$$

where c is any arbitrary constant. On the other hand, we can write

$$g(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = 3(x^2 - y^2) + 6ixy = 3(x^2 + 2ixy - y^2) = 3(x + iy)^2 = 3z^2.$$

It follows that u is the real part of an analytic function f in \mathbb{C} such that $f'(z) = g(z)$ for every $z \in \mathbb{C}$. The function $f(z) = z^3 + C$ satisfies this requirement for any arbitrary constant C . Note that the imaginary part of f is $3x^2y - y^3 + c$, where c is the imaginary part of C .

EXAMPLE 3.6.2. Consider the function $u(x, y) = e^x \sin y$. It is easily checked that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

so that u is harmonic in \mathbb{C} . Using $X_0 = Y_0 = 0$ in (20), we obtain

$$\begin{aligned} v(X, Y) &= - \int_0^X \frac{\partial u}{\partial y}(x, Y) dx + \int_0^Y \frac{\partial u}{\partial x}(0, y) dy + c = - \int_0^X e^x \cos Y dx + \int_0^Y \sin y dy + c \\ &= \cos Y - e^X \cos Y - \cos Y + 1 + c = c' - e^X \cos Y, \end{aligned}$$

where c' is any arbitrary constant. On the other hand, we can write

$$g(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = e^x \sin y - ie^x \cos y = -ie^x(\cos y + i \sin y) = -ie^z.$$

It follows that u is the real part of an analytic function f in \mathbb{C} such that $f'(z) = g(z)$ for every $z \in \mathbb{C}$. The function $f(z) = C - ie^z$ satisfies this requirement for any arbitrary constant C . Note that the imaginary part of f is $c' - e^x \cos y$, where c' is the imaginary part of C .

PROBLEMS FOR CHAPTER 3

1. a) Suppose that $P(z) = (z - z_1)(z - z_2) \dots (z - z_k)$, where $z_1, z_2, \dots, z_k \in \mathbb{C}$. Show that

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_k} \quad \text{for every } z \in \mathbb{C} \setminus \{z_1, z_2, \dots, z_k\}.$$

- b) Suppose further that $\Re z_j < 0$ for every $j = 1, \dots, k$, and that $\Re z \geq 0$. Show in this case that $\Re(z - z_j)^{-1} > 0$ for every $j = 1, \dots, k$, and deduce that $P'(z) \neq 0$.

[REMARK: Polynomials all of whose roots have negative real parts are called Hurwitz polynomials. We have shown here that the derivative of a non-constant Hurwitz polynomial is also a Hurwitz polynomial.]

2. For each of the following functions $f(z)$, determine whether the Cauchy-Riemann equations are satisfied:
- a) $f(z) = x^2 - y^2 - 2ixy$ b) $f(z) = \log(x^2 + y^2) + 2i \cot^{-1}(x/y)$
 c) $f(z) = x^3 - 3y^2 + 2x + i(3x^2y - y^3 + 2y)$ d) $f(z) = \log(x^2 - y^2) + 2i \tan^{-1}(y/x)$
3. Show that a real valued analytic function is constant.
4. We are required to define an analytic function $f(z)$ such that $f(x + iy) = e^x f(iy)$ for every $x, y \in \mathbb{R}$ and $f(0) = 1$. Suppose that for every $y \in \mathbb{R}$, we write $f(iy) = c(y) + is(y)$, where $c(y), s(y) \in \mathbb{R}$ for every $y \in \mathbb{R}$.
- a) Show by the Cauchy-Riemann equations that $c'(y) = -s(y)$ and $s'(y) = c(y)$ for every $y \in \mathbb{R}$.
 b) For every $y \in \mathbb{R}$, write $g(y) = (c(y) - \cos y)^2 + (s(y) - \sin y)^2$. Show that $g'(y) = 0$ for every $y \in \mathbb{R}$. Deduce that $g(y) = 0$ for every $y \in \mathbb{R}$.
 c) Comment on the above.
5. a) Suppose that $P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$, where $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$ are constants. Show that for every $k = 0, 1, \dots, n$, we have

$$a_k = \frac{P^{(k)}(0)}{k!}.$$

- b) Apply the result to the polynomial $(1 + z)^n = c_0 + c_1z + c_2z^2 + \dots + c_nz^n$ and show that for every $k = 0, 1, \dots, n$, we have

$$c_k = \frac{n!}{k!(n-k)!}.$$

6. a) Show that for every $z \in \mathbb{C}$, we have $e^{iz} = \cos z + i \sin z$.
 b) Show that for every $z, w \in \mathbb{C}$, we have

$$\cos(z + w) + i \sin(z + w) = (\cos z + i \sin z)(\cos w + i \sin w)$$

and

$$\cos(z + w) - i \sin(z + w) = (\cos z - i \sin z)(\cos w - i \sin w).$$

- c) Express $\sin(z + w)$ and $\cos(z + w)$ in terms of $\sin z, \sin w, \cos z$ and $\cos w$.
7. Suppose that $a_1, a_2, \dots, a_n \in \mathbb{C}$ are distinct, and consider the polynomial

$$Q(z) = (z - z_1)(z - z_2) \dots (z - z_n).$$

Suppose further that $P(z)$ is a polynomial of degree less than n . Follow the steps below to show that there exist $a_1, a_2, \dots, a_n \in \mathbb{C}$ such that

$$\frac{P(z)}{Q(z)} = \frac{a_1}{z - z_1} + \frac{a_2}{z - z_2} + \dots + \frac{a_n}{z - z_n}.$$

- a) We shall first of all show that the expression above is possible by multiplying it by $Q(z)$ and then determining a_1, a_2, \dots, a_n so that the resulting equation between polynomials of degree less than n holds when $z = z_1, z_2, \dots, z_n$.
 [HINT: Recall Problem 1 in Chapter 1.]
 b) Show that for every $k = 1, \dots, n$, we have

$$a_k = \lim_{z \rightarrow z_k} (z - z_k) \frac{P(z)}{Q(z)} = \frac{P(z_k)}{Q'(z_k)}.$$

[HINT: Note that $Q(z_k) = 0$ for every $k = 1, \dots, n$.]

-
8. Suppose that $a \in \mathbb{C}$ is non-zero. Show that for any fixed choice of value for $\log a$, the function $f(z) = a^z = e^{z \log a}$ satisfies $f'(z) = f(z) \log a$.
9. For each expression below, compute all possible values and plot their positions in the complex plane:
- | | |
|----------------|------------------------|
| a) $\log(-i)$ | b) $\log(1+i)$ |
| c) $(-i)^{-i}$ | d) i^2 |
| e) $2^{\pi i}$ | f) $(1+i)^i(1+i)^{-i}$ |
10. For each of the following equations, find all solutions:
- | | |
|------------------------------|-----------------------|
| a) $\text{Log}(z) = \pi i/3$ | b) $e^z = 2i$ |
| c) $\sin z = i$ | d) $\sin z = -\cos z$ |
| e) $\tan^2 z = -1$ | |
11. For each of the functions below, determine whether the function is harmonic. If so, find also its harmonic conjugate:
- | | |
|---------------------------|--|
| a) $x^2 - y^2 + y$ | b) $e^x \sin y$ |
| c) $x^3 - y^3$ | d) $xe^x \cos y - ye^x \sin y$ |
| e) $3x^2y - y^3 + xy$ | f) $x^4 - 6x^2y^2 + y^4 + x^3y - xy^3$ |
| g) $e^{x^2-y^2} \sin 2xy$ | |
12. a) Suppose that the functions $f(z)$ and $g(z)$ both satisfy the Cauchy-Riemann equations at a particular point $z \in \mathbb{C}$. Show that the functions $f(z) + g(z)$ and $f(z)g(z)$ also satisfy the Cauchy-Riemann equations at the point z .
- b) Show that the constant function and the function $f(z) = z$ both satisfy the Cauchy-Riemann equations everywhere in \mathbb{C} .
- c) Deduce that every polynomial $P(z)$ with complex coefficients satisfies the Cauchy-Riemann equations everywhere in \mathbb{C} .
13. A real valued function $u(x, y)$ which is continuous and satisfies the inequality $u_{xx} + u_{yy} \geq 0$ in a region D is said to be subharmonic in D . Show that $u = |f(z)|^2$ is subharmonic in any region where $f(z)$ is analytic.

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 1

COMPLEX NUMBERS

1.1. Arithmetic and Conjugates

The purpose of this chapter is to give a review of various properties of the complex numbers that we shall need in the discussion of complex analysis. As the reader is expected to be familiar with the material, all proofs have been omitted.

The equation $x^2 + 1 = 0$ has no solution $x \in \mathbb{R}$. To “solve” this equation, we have to introduce extra numbers into our number system. To do this, we define the number i by $i^2 + 1 = 0$, and then extend the field of all real numbers by adjoining the number i , which is then combined with the real numbers by the operations addition and multiplication in accordance with the Field axioms of the real number system. The numbers $a + ib$, where $a, b \in \mathbb{R}$, of the extended field are then added and multiplied in accordance with the Field axioms, suitably extended, and the restriction $i^2 + 1 = 0$. Note that the number $a + 0i$, where $a \in \mathbb{R}$, behaves like the real number a .

What we have said in the last paragraph basically amounts to the following. Consider two complex numbers $a + ib$ and $c + id$, where $a, b, c, d \in \mathbb{R}$. We have the addition and multiplication rules

$$(a + ib) + (c + id) = (a + c) + i(b + d) \quad \text{and} \quad (a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

These lead to the subtraction rule

$$(a + ib) - (c + id) = (a - c) + i(b - d),$$

and the division rule, that if $c + id \neq 0$, then

$$\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}.$$

Note the special case $a = 1$ and $b = 0$.

Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. The real number x is called the real part of z , and denoted by $x = \Re z$. The real number y is called the imaginary part of z , and denoted by $y = \Im z$. The set $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$ is called the set of all complex numbers. The complex number $\bar{z} = x - iy$ is called the conjugate of z .

It is easy to see that for every $z \in \mathbb{C}$, we have

$$\Re z = \frac{z + \bar{z}}{2} \quad \text{and} \quad \Im z = \frac{z - \bar{z}}{2i}.$$

Furthermore, if $w \in \mathbb{C}$, then

$$\overline{z + w} = \bar{z} + \bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

1.2. Polar Coordinates

Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. The real number

$$r = \sqrt{x^2 + y^2}$$

is called the modulus of z , and denoted by $|z|$. On the other hand, if $z \neq 0$, then any number $\theta \in \mathbb{R}$ satisfying the equations

$$(1) \quad x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

is called an argument of z , and denoted by $\arg z$. Hence we can write z in polar form

$$z = r(\cos \theta + i \sin \theta).$$

Note, however, that for a given $z \in \mathbb{C}$, $\arg z$ is not unique. Clearly we can add any integer multiple of 2π to θ without affecting (1). We sometimes call a real number $\theta \in \mathbb{R}$ the principal argument of z if θ satisfies the equations (1) and $-\pi < \theta \leq \pi$. The principal argument of z is usually denoted by $\text{Arg } z$.

It is easy to see that for every $z \in \mathbb{C}$, we have $|z|^2 = z\bar{z}$. Also, if $w \in \mathbb{C}$, then

$$|zw| = |z||w| \quad \text{and} \quad |z + w| \leq |z| + |w|.$$

Furthermore, if

$$z = r(\cos \theta + i \sin \theta) \quad \text{and} \quad w = s(\cos \phi + i \sin \phi),$$

where $r, s, \theta, \phi \in \mathbb{R}$ and $r, s > 0$, then

$$zw = rs(\cos(\theta + \phi) + i \sin(\theta + \phi)) \quad \text{and} \quad \frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i \sin(\theta - \phi)).$$

1.3. Rational Powers

De Moivre's theorem, that

$$(2) \quad \cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n \quad \text{for every } n \in \mathbb{N} \text{ and } \theta \in \mathbb{R},$$

is useful in finding n -th roots of complex numbers.

Suppose that $c = R(\cos \alpha + i \sin \alpha)$, where $R, \alpha \in \mathbb{R}$ and $R > 0$. Then the solutions of the equation $z^n = c$ are given by

$$z = \sqrt[n]{R} \left(\cos \frac{\alpha + 2k\pi}{n} + i \sin \frac{\alpha + 2k\pi}{n} \right), \quad \text{where } k = 0, 1, \dots, n-1.$$

Finally, we can define c^b for any $b \in \mathbb{Q}$ and non-zero $c \in \mathbb{C}$ as follows. The rational number b can be written uniquely in the form $b = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no prime factors in common. Then there are exactly q distinct numbers z satisfying $z^q = c$. We now define $c^b = z^p$, noting that the expression (2) can easily be extended to all $n \in \mathbb{Z}$. It is not too difficult to show that there are q distinct values for the rational power c^b .

PROBLEMS FOR CHAPTER 1

- Suppose that $z_0 \in \mathbb{C}$ is fixed. A polynomial $P(z)$ is said to be divisible by $z - z_0$ if there is another polynomial $Q(z)$ such that $P(z) = (z - z_0)Q(z)$.
 - Show that for every $c \in \mathbb{C}$ and $k \in \mathbb{N}$, the polynomial $c(z^k - z_0^k)$ is divisible by $z - z_0$.
 - Consider the polynomial $P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$, where $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$ are arbitrary. Show that the polynomial $P(z) - P(z_0)$ is divisible by $z - z_0$.
 - Deduce that $P(z)$ is divisible by $z - z_0$ if $P(z_0) = 0$.
 - Suppose that a polynomial $P(z)$ of degree n vanishes at n distinct values $z_1, z_2, \dots, z_n \in \mathbb{C}$, so that $P(z_1) = P(z_2) = \dots = P(z_n) = 0$. Show that $P(z) = c(z - z_1)(z - z_2)\dots(z - z_n)$, where $c \in \mathbb{C}$ is a constant.
 - Suppose that a polynomial $P(z)$ of degree n vanishes at more than n distinct values. Show that $P(z) = 0$ identically.
- Suppose that $\alpha \in \mathbb{C}$ is fixed and $|\alpha| < 1$. Show that $|z| \leq 1$ if and only if $\left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right| \leq 1$.
- Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. Express each of the following in terms of x and y :
 - $|z - 1|^3$
 - $\left| \frac{z + 1}{z - 1} \right|$
 - $\left| \frac{z + i}{1 - iz} \right|$
- Suppose that $c \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ with $\alpha \neq 0$.
 - Show that $\alpha z + \bar{\alpha}z + c = 0$ is the equation of a straight line on the plane.
 - What does the equation $z\bar{z} + \alpha z + \bar{\alpha}z + c = 0$ represent if $|\alpha|^2 \geq c$?
- Suppose that $z, w \in \mathbb{C}$. Show that $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.
- Find all the roots of the equation $(z^8 - 1)(z^3 + 8) = 0$.
- For each of the following, compute all the values and plot them on the plane:
 - $(1 + i)^{-1/2}$
 - $(-4)^{3/4}$
 - $(1 - i)^{3/8}$

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 2

FOUNDATIONS OF COMPLEX ANALYSIS

2.1. Three Approaches

We start by remarking that analysis is sometimes known as the study of the four C's: convergence, continuity, compactness and connectedness. In real analysis, we have studied convergence and continuity to some depth, but the other two concepts have been somewhat disguised. In this course, we shall try to illustrate these two latter concepts a little bit more, particularly connectedness.

Complex analysis is the study of complex valued functions of complex variables. Here we shall restrict the number of variables to one, and study complex valued functions of one complex variable. Unless otherwise stated, all functions in these notes are of the form $f : S \rightarrow \mathbb{C}$, where S is a set in \mathbb{C} .

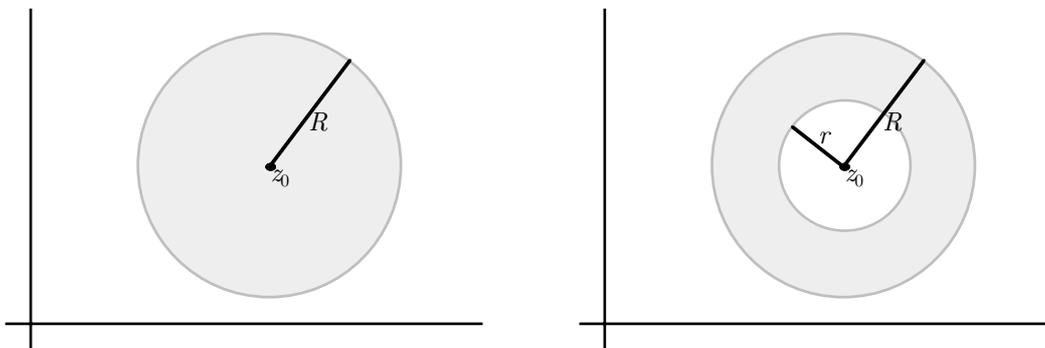
We shall study the behaviour of such functions using three different approaches. The first of these, discussed in Chapter 3 and usually attributed to Riemann, is based on differentiation and involves pairs of partial differential equations called the Cauchy-Riemann equations. The second approach, discussed in Chapters 4–11 and usually attributed to Cauchy, is based on integration and depends on a fundamental theorem known nowadays as Cauchy's integral theorem. The third approach, discussed in Chapter 16 and usually attributed to Weierstrass, is based on the theory of power series.

2.2. Point Sets in the Complex Plane

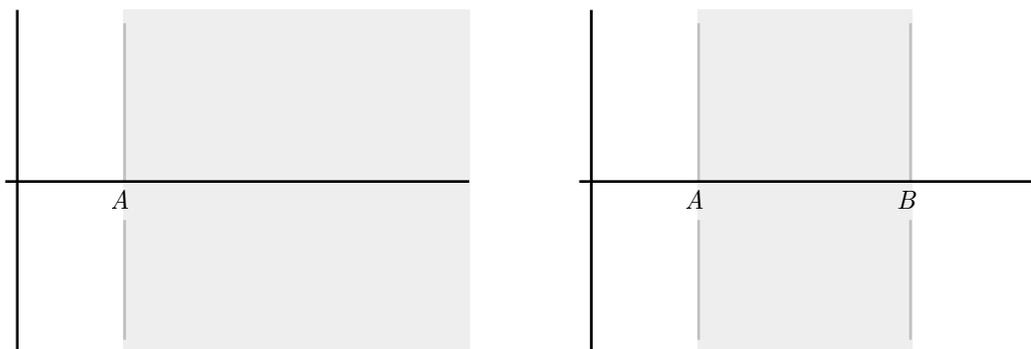
We shall study functions of the form $f : S \rightarrow \mathbb{C}$, where S is a set in \mathbb{C} . In most situations, various properties of the point sets S play a crucial role in our study. We therefore begin by discussing various types of point sets in the complex plane.

Before making any definitions, let us consider a few examples of sets which frequently occur in our subsequent discussion.

EXAMPLE 2.2.1. Suppose that $z_0 \in \mathbb{C}$, $r, R \in \mathbb{R}$ and $0 < r < R$. The set $\{z \in \mathbb{C} : |z - z_0| < R\}$ represents a disc, with centre z_0 and radius R , and the set $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ represents an annulus, with centre z_0 , inner radius r and outer radius R .



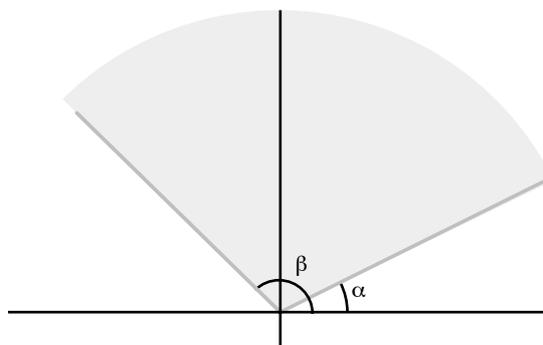
EXAMPLE 2.2.2. Suppose that $A, B \in \mathbb{R}$ and $A < B$. The set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } x > A\}$ represents a half-plane, and the set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } A < x < B\}$ represents a strip.



EXAMPLE 2.2.3. Suppose that $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha < \beta < 2\pi$. The set

$$\{z = r(\cos \theta + i \sin \theta) \in \mathbb{C} : r, \theta \in \mathbb{R} \text{ and } r > 0 \text{ and } \alpha < \theta < \beta\}$$

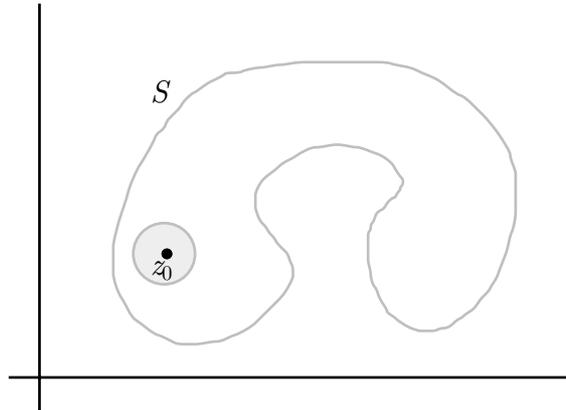
represents a sector.



We now make a number of important definitions. The reader may subsequently need to return to these definitions.

DEFINITION. Suppose that $z_0 \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$, with $\epsilon > 0$. By an ϵ -neighbourhood of z_0 , we mean a disc of the form $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$, with centre z_0 and radius $\epsilon > 0$.

DEFINITION. Suppose that S is a point set in \mathbb{C} . A point $z_0 \in S$ is said to be an interior point of S if there exists an ϵ -neighbourhood of z_0 which is contained in S . The set S is said to be open if every point of S is an interior point of S .



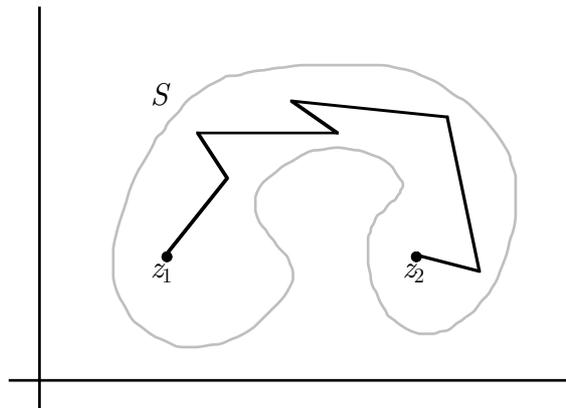
EXAMPLE 2.2.4. The sets in Examples 2.2.1–2.2.3 are open.

EXAMPLE 2.2.5. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is open.

EXAMPLE 2.2.6. The disc $\{z \in \mathbb{C} : |z - z_0| \leq R\}$ is not open.

EXAMPLE 2.2.7. The empty set \emptyset is open. Why?

DEFINITION. An open set S is said to be connected if every two points $z_1, z_2 \in S$ can be joined by the union of a finite number of line segments lying in S . An open connected set is called a domain.



REMARKS. (1) Sometimes, we say that an open set S is connected if there do not exist non-empty open sets S_1 and S_2 such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. In other words, an open connected set cannot be the disjoint union of two non-empty open sets.

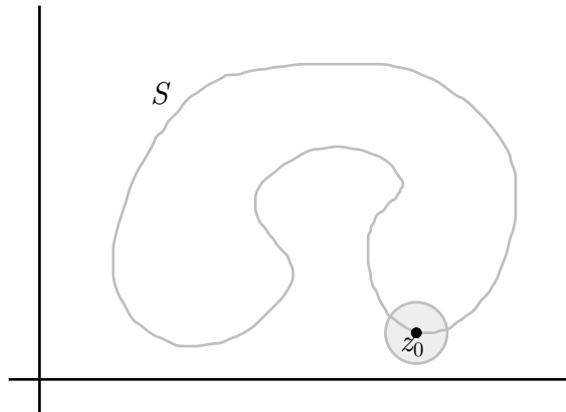
(2) In fact, it can be shown that the two definitions are equivalent.

(3) Note that we have not made any definition of connectedness for sets that are not open. In fact, the definition of connectedness for an open set given by (1) here is a special case of a much more complicated definition of connectedness which applies to all point sets.

EXAMPLE 2.2.8. The sets in Examples 2.2.1–2.2.3 are domains.

EXAMPLE 2.2.9. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is a domain.

DEFINITION. A point $z_0 \in \mathbb{C}$ is said to be a boundary point of a set S if every ϵ -neighbourhood of z_0 contains a point in S as well as a point not in S . The set of all boundary points of a set S is called the boundary of S .



EXAMPLE 2.2.10. The annulus $\{z \in \mathbb{C} : r < |z - z_0| < R\}$, where $0 < r < R$, has boundary $C_1 \cup C_2$, where

$$C_1 = \{z \in \mathbb{C} : |z - z_0| = r\} \quad \text{and} \quad C_2 = \{z \in \mathbb{C} : |z - z_0| = R\}$$

are circles, with centre z_0 and radius r and R respectively. Note that the annulus is connected and hence a domain. However, note that its boundary is made up of two separate pieces.

DEFINITION. A region is a domain together with all, some or none of its boundary points. A region which contains all its boundary points is said to be closed. For any region S , we denote by \bar{S} the closed region containing S and all its boundary points, and call \bar{S} the closure of S .

REMARK. Note that we have not made any definition of closedness for sets that are not regions. In fact, our definition of closedness for a region here is a special case of a much more complicated definition of closedness which applies to all point sets.

DEFINITION. A region S is said to be bounded or finite if there exists a real number M such that $|z| \leq M$ for every $z \in S$. A region that is closed and bounded is said to be compact.

EXAMPLE 2.2.11. The region $\{z \in \mathbb{C} : |z - z_0| \leq R\}$ is closed and bounded, hence compact. It is called the closed disc with centre z_0 and radius R .

EXAMPLE 2.2.12. The region $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leq x \leq 1\}$ is closed but not bounded.

EXAMPLE 2.2.13. The square $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leq x \leq 1 \text{ and } 0 < y < 1\}$ is bounded but not closed.

2.3. Complex Functions

In these lectures, we study complex valued functions of one complex variable. In other words, we study functions of the form $f : S \rightarrow \mathbb{C}$, where S is a set in \mathbb{C} . Occasionally, we will abuse notation and simply refer to a function by its formula, without explicitly defining the domain S . For instance, when we discuss the function $f(z) = 1/z$, we implicitly choose a set S which will not include the point $z = 0$ where the function is not defined. Also, we may occasionally wish to include the point $z = \infty$ in the domain or codomain.

We may separate the independent variable z as well as the dependent variable $w = f(z)$ into real and imaginary parts. Our usual notation will be to write $z = x + iy$ and $w = f(z) = u + iv$, where $x, y, u, v \in \mathbb{R}$. It follows that $u = u(x, y)$ and $v = v(x, y)$ can be interpreted as real valued functions of the two real variables x and y .

EXAMPLE 2.3.1. Consider the function $f : S \rightarrow \mathbb{C}$, given by $f(z) = z^2$ and where $S = \{z \in \mathbb{C} : |z| < 2\}$ is the open disc with radius 2 and centre 0. Using polar coordinates, it is easy to see that the range of the function is the open disc $f(S) = \{w \in \mathbb{C} : |w| < 4\}$ with radius 4 and centre 0.

EXAMPLE 2.3.2. Consider the function $f : \mathcal{H} \rightarrow \mathbb{C}$, where $\mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ is the upper half-plane and $f(z) = z^2$. Using polar coordinates, it is easy to see that the range of the function is the complex plane minus the non-negative real axis.

EXAMPLE 2.3.3. Consider the function $f : T \rightarrow \mathbb{C}$, where $T = \{z = x + iy \in \mathbb{C} : 1 < x < 2\}$ is a strip and $f(z) = z^2$. Let $x_0 \in (1, 2)$ be fixed, and consider the image of a point (x_0, y) on the vertical line $x = x_0$. Here we have

$$u = x_0^2 - y^2 \quad \text{and} \quad v = 2x_0y.$$

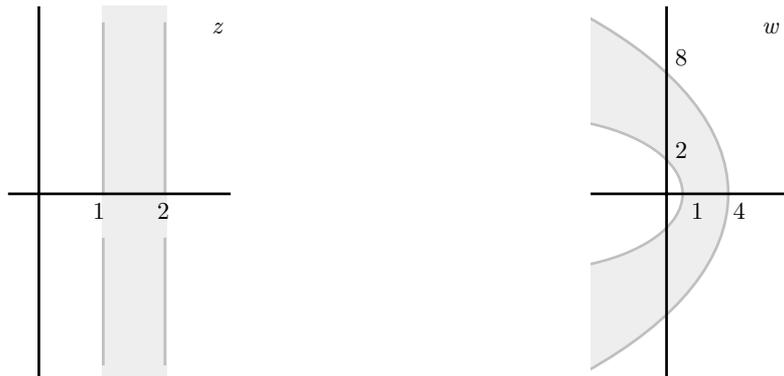
Eliminating y , we obtain the equation of a parabola

$$u = x_0^2 - \frac{v^2}{4x_0^2}$$

in the w -plane. It follows that the image of the vertical line $x = x_0$ under the function $w = z^2$ is this parabola. Now the boundary of the strip are the two lines $x = 1$ and $x = 2$. Their images under the mapping $w = z^2$ are respectively the parabolas

$$u = 1 - \frac{v^2}{4} \quad \text{and} \quad u = 4 - \frac{v^2}{16}.$$

It is easy to see that the range of the function is the part of the w -plane between these two parabolas.



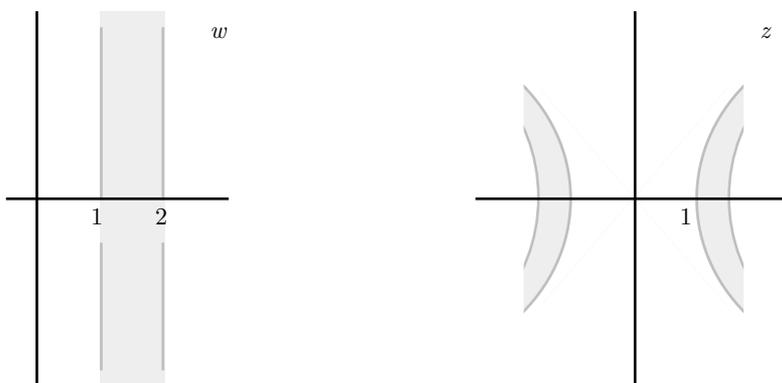
EXAMPLE 2.3.4. Consider again the function $w = z^2$. We would like to find all $z = x + iy \in \mathbb{C}$ for which $1 < \Re w < 2$. In other words, we have the restriction $1 < u < 2$, but no restriction on v . Let $u_0 \in (1, 2)$ be fixed, and consider points (x, y) in the z -plane with images on the vertical line $u = u_0$. Here we have the hyperbola

$$x^2 - y^2 = u_0.$$

The boundaries $u = 1$ and $u = 2$ are represented by the hyperbolas

$$x^2 - y^2 = 1 \quad \text{and} \quad x^2 - y^2 = 2.$$

It is easy to see that the points in question are precisely those between the two hyperbolas.



2.4. Extended Complex Plane

It is sometimes useful to extend the complex plane \mathbb{C} by the introduction of the point ∞ at infinity. Its connection with finite complex numbers can be established by setting $z + \infty = \infty + z = \infty$ for all $z \in \mathbb{C}$, and setting $z \cdot \infty = \infty \cdot z = \infty$ for all non-zero $z \in \mathbb{C}$. We can also write $\infty \cdot \infty = \infty$.

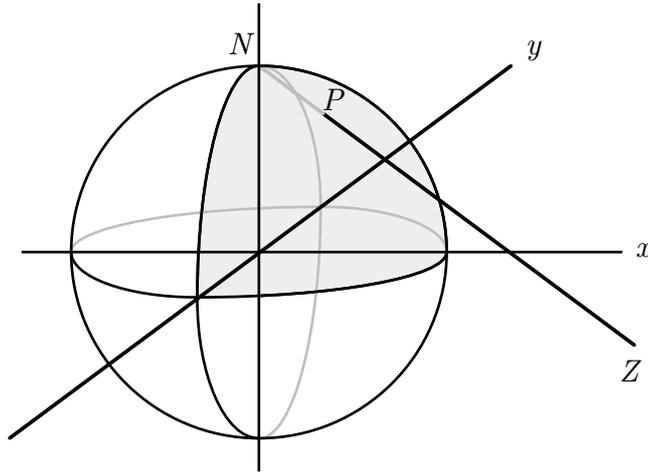
Note that it is not possible to define $\infty + \infty$ and $0 \cdot \infty$ without violating the laws of arithmetic. However, by special convention, we shall write $z/0 = \infty$ for $z \neq 0$ and $z/\infty = 0$ for $z \neq \infty$.

In the complex plane \mathbb{C} , there is no room for a point corresponding to ∞ . We can, of course, introduce an “ideal” point which we call the point at infinity. The points in \mathbb{C} , together with the point at infinity, form the extended complex plane. We decree that every straight line on the complex plane shall pass through the point at infinity, and that no half-plane shall contain the ideal point.

The main purpose of this section is to introduce a geometric model in which each point of the extended complex plane has a concrete representative. To do this, we shall use the idea of stereographic projection.

Consider a sphere of radius 1 in \mathbb{R}^3 . A typical point on this sphere will be denoted by $P(x_1, x_2, x_3)$. Note that $x_1^2 + x_2^2 + x_3^2 = 1$. Let us call the point $N(0, 0, 1)$ the north pole. The equator of this sphere is the set of all points of the form $(x_1, x_2, 0)$, where $x_1^2 + x_2^2 = 1$. Consider next the complex plane \mathbb{C} . This can be viewed as a plane in \mathbb{R}^3 . Let us position this plane in such a way that the equator of the sphere lies on this plane; in other words, our copy of the complex plane is “horizontal” and passes through the origin. We can further insist that the x -direction on our complex plane is the same as the x_1 -direction in \mathbb{R}^3 , and that the y -direction on our complex plane is the same as the x_2 -direction in \mathbb{R}^3 . Clearly a typical point $z = x + iy$ on our complex plane \mathbb{C} can be identified with the point $Z(x, y, 0)$ in \mathbb{R}^3 .

Suppose that $Z(x, y, 0)$ is on the plane. Consider the straight line that passes through Z and the north pole N . It is not too difficult to see that this straight line intersects the surface of the sphere at precisely one other point $P(x_1, x_2, x_3)$. In fact, if Z is on the equator of the sphere, then $P = Z$. If Z is on the part of the plane outside the sphere, then P is on the northern hemisphere, but is not the north pole N . If Z is on the part of the plane inside the sphere, then P is on the southern hemisphere. Check that for $Z(0, 0, 0)$, the point $P(0, 0, -1)$ is the south pole.



On the other hand, if P is any point on the sphere different from the north pole N , then a straight line passing through P and N intersects the plane at precisely one point Z . It follows that there is a pairing of all the points P on the sphere different from the north pole N and all the points on the plane. This pairing is governed by the requirement that the straight line through any pair must pass through the north pole N .

We can now visualize the north pole N as the point on the sphere corresponding to the point at infinity of the plane. The sphere is called the Riemann sphere.

2.5. Limits and Continuity

The concept of a limit in complex analysis is exactly the same as in real analysis. So, for example, we say that $f(z) \rightarrow L$ as $z \rightarrow z_0$, or

$$\lim_{z \rightarrow z_0} f(z) = L,$$

if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

This definition will be perfectly in order if the function f is defined in some open set containing z_0 , with the possible exception of z_0 itself. It follows that if z_0 is an interior point of the region S of definition of the function, our definition is in order. However, if z_0 is a boundary point of the region S of definition of the function, then we agree that the conclusion $|f(z) - L| < \epsilon$ need only hold for those $z \in S$ satisfying $0 < |z - z_0| < \delta$.

Similarly, we say that a function $f(z)$ is continuous at z_0 if $f(z) \rightarrow f(z_0)$ as $z \rightarrow z_0$. A similar qualification on z applies if z_0 is a boundary point of the region S of definition of the function. We also say that a function is continuous in a region if it is continuous at every point of the region.

Note that for a function to be continuous in a region, it is enough to have continuity at every point of the region. Hence the choice of δ may depend on a point z_0 in question. If δ can be chosen independently of z_0 , then we have some uniformity as well. To be precise, we make the following definition.

DEFINITION. A function $f(z)$ is said to be uniformly continuous in a region S if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ for every $z_1, z_2 \in S$ satisfying $|z_1 - z_2| < \delta$.

REMARK. Note that if we fix z_2 to be a point z_0 and write z for z_1 , then we require $|f(z) - f(z_0)| < \epsilon$ for every $z \in S$ satisfying $|z - z_0| < \delta$. In other words, δ cannot depend on z_0 .

EXAMPLE 2.5.1. Consider the punctured disc $S = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The function $f(z) = 1/z$ is continuous in S but not uniformly continuous in S . To see this, note first of all that continuity follows from the simple observation that the function z is continuous and non-zero in S . To show that the function is not uniformly continuous in S , it suffices to show that there exists $\epsilon > 0$ such that for every $\delta > 0$, there exist $z_1, z_2 \in S$ such that

$$|z_1 - z_2| < \delta \quad \text{and} \quad \left| \frac{1}{z_1} - \frac{1}{z_2} \right| \geq \epsilon.$$

Let $\epsilon = 1$. For every $\delta > 0$, choose $n \in \mathbb{N}$ such that $n > \delta^{-1/2}$, and let

$$z_1 = \frac{1}{n} \quad \text{and} \quad z_2 = \frac{1}{n+1}.$$

Clearly $z_1, z_2 \in S$. It is easy to see that

$$|z_1 - z_2| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{1}{n(n+1)} < \delta \quad \text{and} \quad \left| \frac{1}{z_1} - \frac{1}{z_2} \right| = 1.$$

PROBLEMS FOR CHAPTER 2

- For each of the following functions, find $f(z+3)$, $f(1/z)$ and $f(f(z))$:
 - $f(z) = z - 1$
 - $f(z) = z^2$
 - $f(z) = 1/z$
 - $f(z) = \frac{1-z}{3+z}$
- Which of the sets below are domains?
 - $\{z : 0 < |z| < 1\}$
 - $\{z : \Im z < 3|z|\}$
 - $\{z : |z-1| \leq |z+1|\}$
 - $\{z : |z^2 - 1| < 1\}$
 - $\{z : 0 < \Re z \leq 1\}$
- Find the image of the strip $\{z : |\Re z| < 1\}$ and of the disc $\{z : |z| < 1\}$ under each of the following mappings:
 - $w = (1+i)z + 1$
 - $w = 2z^2$
 - $w = z^{-1}$
 - $w = \frac{z+1}{z-1}$
- A function $f(z)$ is said to be an isometry if $|f(z_1) - f(z_2)| = |z_1 - z_2|$ for every $z_1, z_2 \in \mathbb{C}$; in other words, if it preserves distance.
 - Suppose that $f(z)$ is an isometry. Show that for every $a, b \in \mathbb{C}$ with $|a| = 1$, the function $g(z) = af(z) + b$ is also an isometry.
 - Show that the function

$$h(z) = \frac{f(z) - f(0)}{f(1) - f(0)}$$

is an isometry with $h(0) = 0$ and $h(1) = 1$.

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 3

COMPLEX DIFFERENTIATION

3.1. Introduction

Suppose that $D \subseteq \mathbb{C}$ is a domain. A function $f : D \rightarrow \mathbb{C}$ is said to be differentiable at $z_0 \in D$ if the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, we write

$$(1) \quad f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and call $f'(z_0)$ the derivative of f at z_0 .

If $z \neq z_0$, then

$$f(z) = \left(\frac{f(z) - f(z_0)}{z - z_0} \right) (z - z_0) + f(z_0).$$

It follows from (1) and the arithmetic of limits that if $f'(z_0)$ exists, then $f(z) \rightarrow f(z_0)$ as $z \rightarrow z_0$, so that f is continuous at z_0 . In other words, differentiability at z_0 implies continuity at z_0 .

Note that the argument here is the same as in the case of a real valued function of a real variable. In fact, the similarity in argument extends to the arithmetic of limits. Indeed, if the functions $f : D \rightarrow \mathbb{C}$ and $g : D \rightarrow \mathbb{C}$ are both differentiable at $z_0 \in D$, then both $f + g$ and fg are differentiable at z_0 , and

$$(f + g)'(z_0) = f'(z_0) + g'(z_0) \quad \text{and} \quad (fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

If the extra condition $g'(z_0) \neq 0$ holds, then f/g is differentiable at z_0 , and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

One can also establish the Chain rule for differentiation as in real analysis. More precisely, suppose that the function f is differentiable at z_0 and the function g is differentiable at $w_0 = f(z_0)$. Then the function $g \circ f$ is differentiable at $z = z_0$, and

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0).$$

EXAMPLE 3.1.1. Consider the function $f(z) = \bar{z}$, where for every $z \in \mathbb{C}$, \bar{z} denotes the complex conjugate of z . Suppose that $z_0 \in \mathbb{C}$. Then

$$(2) \quad \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}.$$

If $z - z_0 = h$ is real and non-zero, then (2) takes the value 1. On the other hand, if $z - z_0 = ik$ is purely imaginary, then (2) takes the value -1 . It follows that this function is not differentiable anywhere in \mathbb{C} , although its real and imaginary parts are rather well behaved.

3.2. The Cauchy-Riemann Equations

If we use the notation

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h},$$

then in Example 3.1.1, we have examined the behaviour of the ratio

$$\frac{f(z+h) - f(z)}{h}$$

first as $h \rightarrow 0$ through real values and then through imaginary values. Indeed, for the derivative to exist, it is essential that these two limiting processes produce the same limit $f'(z)$. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. If h is real, then the two limiting processes above correspond to

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \rightarrow 0} \frac{v(x+h, y) - v(x, y)}{h} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

and

$$\lim_{h \rightarrow 0} \frac{f(z+ih) - f(z)}{ih} = \lim_{h \rightarrow 0} \frac{u(x, y+h) - u(x, y)}{ih} + i \lim_{h \rightarrow 0} \frac{v(x, y+h) - v(x, y)}{ih} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

respectively. Equating real and imaginary parts, we obtain

$$(3) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note that while the existence of the derivative in real analysis is a mild smoothness condition, the existence of the derivative in complex analysis leads to a pair of partial differential equations.

DEFINITION. The partial differential equations (3) are called the Cauchy-Riemann equations.

We have proved the following result.

THEOREM 3A. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that $f'(z)$ exists. Then the four partial derivatives in (3) exist, and the Cauchy-Riemann equations (3) hold. Furthermore, we have

$$(4) \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{and} \quad f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

A natural question to ask is whether the Cauchy-Riemann equations are sufficient to guarantee the existence of the derivative. We shall show next that we require also the continuity of the partial derivatives in (3).

THEOREM 3B. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that the four partial derivatives in (3) are continuous and satisfy the Cauchy-Riemann equations (3) at z_0 . Then f is differentiable at z_0 , and the derivative $f'(z_0)$ is given by the equations (4) evaluated at z_0 .

PROOF. Write $z_0 = x_0 + iy_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{(u(x, y) - u(x_0, y_0)) + i(v(x, y) - v(x_0, y_0))}{z - z_0}.$$

We can write

$$u(x, y) - u(x_0, y_0) = (x - x_0) \left(\frac{\partial u}{\partial x} \right)_{z_0} + (y - y_0) \left(\frac{\partial u}{\partial y} \right)_{z_0} + |z - z_0| \epsilon_1(z)$$

and

$$v(x, y) - v(x_0, y_0) = (x - x_0) \left(\frac{\partial v}{\partial x} \right)_{z_0} + (y - y_0) \left(\frac{\partial v}{\partial y} \right)_{z_0} + |z - z_0| \epsilon_2(z).$$

If the four partial derivatives in (3) are continuous at z_0 , then

$$\lim_{z \rightarrow z_0} \epsilon_1(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow z_0} \epsilon_2(z) = 0.$$

In view of the Cauchy-Riemann equations (3), we have

$$\begin{aligned} & (u(x, y) - u(x_0, y_0)) + i(v(x, y) - v(x_0, y_0)) \\ &= (x - x_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + (y - y_0) \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)) \\ &= (x - x_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + (y - y_0) \left(-\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)) \\ &= (x - x_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + i(y - y_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)) \\ &= (z - z_0) \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + |z - z_0| (\epsilon_1(z) + i \epsilon_2(z)). \end{aligned}$$

Hence

$$\frac{f(z) - f(z_0)}{z - z_0} = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0} + \left(\frac{|z - z_0|}{z - z_0} \right) (\epsilon_1(z) + i \epsilon_2(z)) \rightarrow \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)_{z_0}$$

as $z \rightarrow z_0$, giving the desired results. \circ

3.3. Analytic Functions

In the previous section, we have shown that differentiability in complex analysis leads to a pair of partial differential equations. Now partial differential equations are seldom of interest at a single point, but rather in a region. It therefore seems reasonable to make the following definition.

DEFINITION. A function f is said to be analytic at a point $z_0 \in \mathbb{C}$ if it is differentiable at every z in some ϵ -neighbourhood of the point z_0 . The function f is said to be analytic in a region if it is analytic at every point in the region. The function f is said to be entire if it is analytic in \mathbb{C} .

EXAMPLE 3.3.1. Consider the function $f(z) = |z|^2$. In our usual notation, we clearly have

$$u = x^2 + y^2 \quad \text{and} \quad v = 0.$$

The Cauchy-Riemann equations

$$2x = 0 \quad \text{and} \quad 2y = 0$$

can only be satisfied at $z = 0$. It follows that the function is differentiable only at the point $z = 0$, and is therefore analytic nowhere.

EXAMPLE 3.3.2. The function $f(z) = z^2$ is entire.

EXAMPLE 3.3.3. Suppose that the function f is analytic in a domain D . Suppose further that f has constant real part u . Then clearly

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0.$$

Since f is analytic in D , it is differentiable at every point in D , and so the Cauchy-Riemann equations hold in D . It follows that

$$\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

Hence f must have constant imaginary part v , and so f must be constant in D .

EXAMPLE 3.3.4. Suppose that the function f is analytic in a domain D . Suppose further that f has constant imaginary part v . A similar argument shows that f must have constant real part u . Hence f must be constant in D .

EXAMPLE 3.3.5. Suppose that the function f is analytic in a domain D . Suppose further that f has constant modulus. In other words, $u^2 + v^2 = C$ for some non-negative real number C . Differentiating this with respect to x and to y , we obtain respectively

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0.$$

In view of the Cauchy-Riemann equations, these can be written as

$$2u \frac{\partial u}{\partial x} - 2v \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad 2v \frac{\partial u}{\partial x} + 2u \frac{\partial u}{\partial y} = 0.$$

In matrix notation, these become

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note now that

$$\det \begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2 + v^2 = C.$$

If $C > 0$, then we must have the unique solution

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

so that the real part u is constant. It then follows from Example 3.3.3 that f is constant in D . On the other hand, if $C = 0$, then clearly $u = v = 0$, so that $f = 0$ in D .

3.4. Introduction to Special Functions

In this section, we shall generalize various functions that we have studied in real analysis to the complex domain. Consider first of all the exponential function. It seems reasonable to extend the property $e^{x_1+x_2} = e^{x_1}e^{x_2}$ for real variables to complex values of the variables to obtain

$$e^z = e^{x+iy} = e^x e^{iy}, \quad \text{where } x, y \in \mathbb{R}.$$

This suggests the following definition.

DEFINITION. Suppose that $z = x + iy$, where $x, y \in \mathbb{R}$. Then the exponential function e^z is defined for every $z \in \mathbb{C}$ by

$$(5) \quad e^z = e^x(\cos y + i \sin y).$$

If we write $e^z = u(x, y) + iv(x, y)$, then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

It is easy to check that the Cauchy-Riemann equations are satisfied for every $z \in \mathbb{C}$, so that e^z is an entire function. Furthermore, it follows from (4) that

$$\frac{d}{dz} e^z = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y) = e^z,$$

so that e^z is its own derivative. On the other hand, note that for every $y_1, y_2 \in \mathbb{R}$, we have

$$e^{i(y_1+y_2)} = \cos(y_1 + y_2) + i \sin(y_1 + y_2) = (\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) = e^{iy_1} e^{iy_2}.$$

Furthermore, if $x_1, x_2 \in \mathbb{R}$, then

$$e^{x_1+x_2} e^{i(y_1+y_2)} = (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2}) = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2}).$$

Writing $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we deduce the addition formula

$$e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

Finally, note that

$$|e^z| = |e^x(\cos y + i \sin y)| = e^x |\cos y + i \sin y| = e^x.$$

Since e^x is never zero, it follows that the exponential function e^z is non-zero for every $z \in \mathbb{C}$.

Next, we turn our attention to the trigonometric functions. Note first of all that if $z = x + iy$, where $x, y \in \mathbb{R}$, then $iz = -y + ix$. Replacing z in (5) by iz and by $-iz$ gives respectively

$$e^{iz} = e^{-y}(\cos x + i \sin x) \quad \text{and} \quad e^{-iz} = e^y(\cos x - i \sin x).$$

The special case $y = 0$ gives respectively

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x.$$

It follows that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This suggests the following definition.

DEFINITION. Suppose that $z \in \mathbb{C}$. Then the trigonometric functions $\cos z$ and $\sin z$ are defined in terms of the exponential function by

$$(6) \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Since the exponential function is an entire function, it follows easily from (6) that both $\cos z$ and $\sin z$ are entire functions. Furthermore, it can easily be deduced from (6) that

$$\frac{d}{dz} \cos z = -\sin z \quad \text{and} \quad \frac{d}{dz} \sin z = \cos z.$$

We can define the functions $\tan z$, $\cot z$, $\sec z$ and $\operatorname{cosec} z$ in terms of the functions $\cos z$ and $\sin z$ as in real variables. However, note that these four functions are not entire. Also, we can deduce from (6) the formulas

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \quad \text{and} \quad \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

and a host of other trigonometric identities that we know hold for real variables.

Finally, we turn our attention to the hyperbolic functions. These are defined as in real analysis.

DEFINITION. Suppose that $z \in \mathbb{C}$. Then the hyperbolic functions $\cosh z$ and $\sinh z$ are defined in terms of the exponential function by

$$(7) \quad \cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Since the exponential function is an entire function, it follows easily from (7) that both $\cosh z$ and $\sinh z$ are entire functions. Furthermore, it can easily be deduced from (7) that

$$\frac{d}{dz} \cosh z = \sinh z \quad \text{and} \quad \frac{d}{dz} \sinh z = \cosh z.$$

We can define the functions $\tanh z$, $\coth z$, $\operatorname{sech} z$ and $\operatorname{cosech} z$ in terms of the functions $\cosh z$ and $\sinh z$ as in real variables. However, note that these four functions are not entire. Also, we can deduce from (7) a host of hyperbolic identities that we know hold for real variables. Note also that comparing (6) and (7), we obtain

$$\cosh z = \cos iz \quad \text{and} \quad \sinh z = -i \sin iz.$$

3.5. Periodicity and its Consequences

One of the fundamental differences between real and complex analysis is that the exponential function is periodic in \mathbb{C} .

DEFINITION. A function f is periodic in \mathbb{C} if there is some fixed non-zero $\omega \in \mathbb{C}$ such that the identity $f(z + \omega) = f(z)$ holds for every $z \in \mathbb{C}$. Any constant $\omega \in \mathbb{C}$ with this property is called a period of f .

THEOREM 3C. *The exponential function e^z is periodic in \mathbb{C} with period $2\pi i$. Furthermore, any period $\omega \in \mathbb{C}$ of e^z is of the form $\omega = 2\pi ki$, where $k \in \mathbb{Z}$ is non-zero.*

PROOF. The first assertion follows easily from the observation

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1.$$

Suppose now that $\omega \in \mathbb{C}$. Clearly $e^{z+\omega} = e^z$ implies $e^\omega = 1$. Write $\omega = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. Then

$$e^\alpha (\cos \beta + i \sin \beta) = 1.$$

Taking modulus, we conclude that $e^\alpha = 1$, so that $\alpha = 0$. It then follows that $\cos \beta + i \sin \beta = 1$. Equating real and imaginary parts, we conclude that $\cos \beta = 1$ and $\sin \beta = 0$, so that $\beta = 2\pi k$, where $k \in \mathbb{Z}$. The second assertion follows. \circ

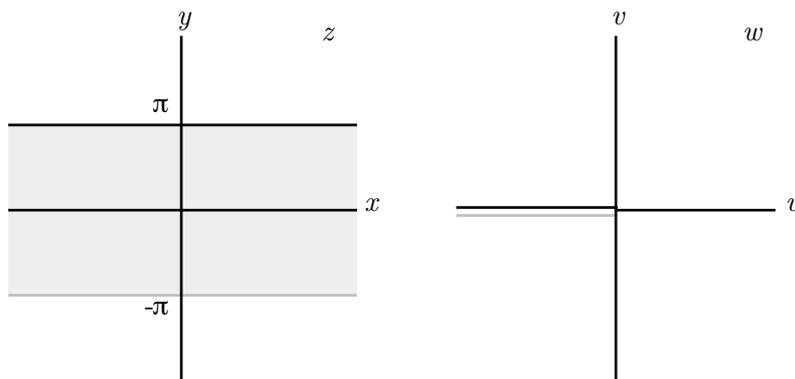
Consider now the mapping $w = e^z$. By (5), we have $w = e^x (\cos y + i \sin y)$, where $x, y \in \mathbb{R}$. It follows that

$$|w| = e^x \quad \text{and} \quad \arg w = y + 2\pi k,$$

where $k \in \mathbb{Z}$. Usually we make the choice $\arg w = y$, with the restriction that $-\pi < y \leq \pi$. This restriction means that z lies on the horizontal strip

$$(8) \quad \mathcal{R}_0 = \{z \in \mathbb{C} : -\infty < x < \infty, -\pi < y \leq \pi\}.$$

The restriction $-\pi < \arg w \leq \pi$ can also be indicated on the complex w -plane by a cut along the negative real axis. The upper edge of the cut, corresponding to $\arg w = \pi$, is regarded as part of the cut w -plane. The lower edge of the cut, corresponding to $\arg w = -\pi$, is not regarded as part of the cut w -plane.



It is easy to check that the function $\exp : \mathcal{R}_0 \rightarrow \mathbb{C} \setminus \{0\}$, defined for every $z \in \mathcal{R}_0$ by $\exp(z) = e^z$, is one-to-one and onto.

REMARK. The region \mathcal{R}_0 is usually known as a fundamental region of the exponential function. In fact, it is easy to see that every set of the type

$$(9) \quad \mathcal{R}_k = \{z \in \mathbb{C} : -\infty < x < \infty, (2k - 1)\pi < y \leq (2k + 1)\pi\},$$

where $k \in \mathbb{Z}$, has this same property as \mathcal{R}_0 .

Let us return to the function $\exp : \mathcal{R}_0 \rightarrow \mathbb{C} \setminus \{0\}$. Since it is one-to-one and onto, there is an inverse function.

DEFINITION. The function $\text{Log} : \mathbb{C} \setminus \{0\} \rightarrow \mathcal{R}_0$, defined by $\text{Log}(w) = z \in \mathcal{R}_0$, where $\exp(z) = w$, is called the principal logarithmic function.

Suppose that $z = x + iy$ and $w = u + iv$, where $x, y, u, v \in \mathbb{R}$. Suppose further that we impose the restriction $-\pi < y \leq \pi$. If $w = \exp(z)$, then it follows from (5) that $u = e^x \cos y$ and $v = e^x \sin y$, and so

$$|w| = (u^2 + v^2)^{1/2} = e^x \quad \text{and} \quad y = \text{Arg}(w),$$

where $\text{Arg}(w)$ denotes the principal argument of w . It follows that

$$x = \log |w| \quad \text{and} \quad y = \text{Arg}(w).$$

Hence

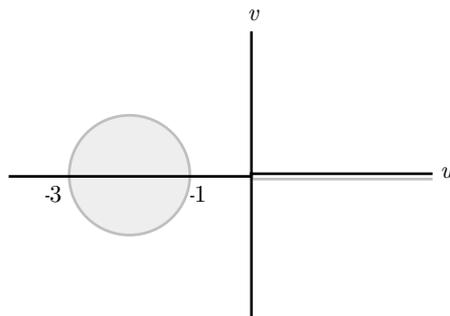
$$(10) \quad \text{Log}(w) = \log |w| + i \text{Arg}(w).$$

In many practical situations, we usually try to define

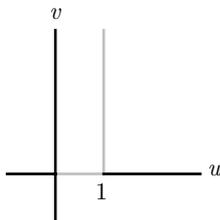
$$\log w = \log |w| + i \arg w,$$

where the argument is chosen in order to make the logarithmic function continuous in its domain of definition, if this is at all possible. The following three examples show that great care needs to be taken in the study of such “many valued functions”.

EXAMPLE 3.5.1. Consider the logarithmic function in the disc $\{w : |w+2| < 1\}$, an open disc of radius 1 and centred at the point $w = -2$. Note that this disc crosses the cut on the w -plane along the negative real axis discussed earlier. In this case, we may restrict the argument to satisfy, for example, $0 \leq \arg w < 2\pi$. The logarithmic function defined in this way is then continuous in the disc $\{w : |w+2| < 1\}$.

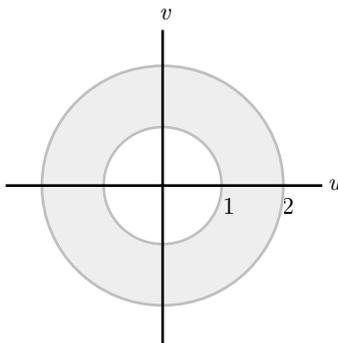


EXAMPLE 3.5.2. Consider the region P obtained from the w -plane by removing both the line segment $\{u + iv : 0 \leq u \leq 1, v = 0\}$ and the half-line $\{u + iv : u = 1, v > 0\}$, as shown below.



Suppose that we wish to define the logarithmic function to be continuous in this region P . One way to do this is to restrict the argument to the range $\pi < \arg w \leq 3\pi$ for any $w \in P$ satisfying $u \geq 1$, and to the range $0 < \arg w \leq 2\pi$ for any $w \in P$ satisfying $u < 1$.

EXAMPLE 3.5.3. Consider the annulus $\{w : 1 < |w| < 2\}$. It is impossible to define the logarithmic function to be continuous in this annulus. Heuristically, if one goes round the annulus once, the argument has to change by 2π if it varies continuously. If we return to the original starting point after going round once, the argument cannot therefore be the same.



It should now be quite clear that we cannot expect to have

$$\operatorname{Log}(w_1 w_2) = \operatorname{Log}(w_1) + \operatorname{Log}(w_2),$$

or even

$$\log w_1 w_2 = \log w_1 + \log w_2.$$

Instead, we have

$$\log w_1 w_2 = \log w_1 + \log w_2 + 2\pi i k \quad \text{for some } k \in \mathbb{Z}.$$

Let us return to the principal logarithmic function $\operatorname{Log} : \mathbb{C} \setminus \{0\} \rightarrow \mathcal{R}_0$. Recall (10). We have

$$\operatorname{Log}(z) = \log |z| + i \operatorname{Arg}(z).$$

Recall from real analysis that for any $t \in \mathbb{R}$, the equation $\tan \theta = t$ has a unique solution θ satisfying $-\pi/2 < \theta < \pi/2$. This solution is denoted by $\tan^{-1} t$ and satisfies

$$\frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2}.$$

It is not difficult to show that if we write

$$(11) \quad v(x, y) = \begin{cases} -\tan^{-1}\left(\frac{x}{y}\right) - \frac{\pi}{2} & \text{if } y < 0, \\ -\tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0, \\ -\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2} & \text{if } y > 0, \end{cases}$$

then $\text{Arg}(z) = v(x, y)$. Hence $\text{Log}(z) = u(x, y) + iv(x, y)$, where

$$(12) \quad u(x, y) = \frac{1}{2} \log(x^2 + y^2).$$

It now follows from (12) that

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},$$

and from (11) that

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}.$$

Clearly the Cauchy-Riemann equations are satisfied, and so

$$\frac{d}{dz} \text{Log}(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

Power functions are defined in terms of the exponential and logarithmic functions. Given $z, a \in \mathbb{C}$, we write $z^a = e^{a \log z}$. Naturally, the precise value depends on the logarithmic function that is chosen, and care again must be exercised for these “many valued functions”.

3.6. Laplace’s Equation and Harmonic Conjugates

We have shown that for any function $f = u + iv$, the existence of the derivative f' leads to the Cauchy-Riemann equations. More precisely, we have

$$(13) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Furthermore,

$$(14) \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Suppose now that the second derivative f'' also exists. Then f' satisfies the Cauchy-Riemann equations. The Cauchy-Riemann equations corresponding to the expression (14) are

$$(15) \quad \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = -\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right).$$

Substituting (13) into (15), we obtain

$$(16) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

We also obtain

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

DEFINITION. A continuous function $\phi(x, y)$ that satisfies Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in a domain $D \subseteq \mathbb{C}$ is said to be harmonic in D .

We have in fact proved the following result.

THEOREM 3D. Suppose that $f = u + iv$, where u and v are real valued. Suppose further that $f''(z)$ exists in a domain $D \subseteq \mathbb{C}$. Then u and v both satisfy Laplace's equation and are harmonic in D .

DEFINITION. Two harmonic functions u and v in a domain $D \subseteq \mathbb{C}$ are said to be harmonic conjugates in D if they satisfy the Cauchy-Riemann equations.

The remainder of this chapter is devoted to a discussion on finding harmonic conjugates. We shall illustrate the following theorem by discussing the special case when $D = \mathbb{C}$.

THEOREM 3E. Suppose that a function u is real valued and harmonic in a domain $D \subseteq \mathbb{C}$. Then there exists a real valued function v which satisfies the following conditions:

- (a) The functions u and v satisfy the Cauchy-Riemann equations in D .
- (b) The function $f = u + iv$ is analytic in D .
- (c) The function v is harmonic in D .

Clearly, parts (b) and (c) follow from part (a). We shall now indicate a proof of part (a) in the special case $D = \mathbb{C}$, and shall omit reference to this domain.

Suppose that u is real valued and harmonic. Then we need to find a real valued function v such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Let $X_0 + iY_0 \in D$ be chosen and fixed. Integrating the second of these with respect to x , we obtain

$$(17) \quad v(X, y) = -\int_{X_0}^X \frac{\partial u}{\partial y}(x, y) dx + c(y),$$

where $c(y)$ is some function depending at most on y . Differentiating with respect to y , we obtain

$$\frac{\partial v}{\partial y}(X, y) = -\frac{\partial}{\partial y} \int_{X_0}^X \frac{\partial u}{\partial y}(x, y) dx + c'(y).$$

Clearly the first of the Cauchy-Riemann equations requires

$$\frac{\partial u}{\partial x}(X, y) = -\frac{\partial}{\partial y} \int_{X_0}^X \frac{\partial u}{\partial y}(x, y) dx + c'(y).$$

Changing the order of differentiation and integration, we obtain

$$\frac{\partial u}{\partial x}(X, y) = - \int_{X_0}^X \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) (x, y) dx + c'(y) = - \int_{X_0}^X \frac{\partial^2 u}{\partial y^2} (x, y) dx + c'(y).$$

Since u is harmonic, we obtain

$$\frac{\partial u}{\partial x}(X, y) = \int_{X_0}^X \frac{\partial^2 u}{\partial x^2} (x, y) dx + c'(y) = \frac{\partial u}{\partial x}(X, y) - \frac{\partial u}{\partial x}(X_0, y) + c'(y),$$

so that

$$c'(y) = \frac{\partial u}{\partial x}(X_0, y).$$

Integrating with respect to y , we obtain

$$(18) \quad c(Y) = \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy + c,$$

where c is an absolute constant. On the other hand, (17) can be rewritten in the form

$$(19) \quad v(X, Y) = - \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + c(Y).$$

Combining (18) and (19), we obtain

$$(20) \quad v(X, Y) = - \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy + c.$$

It is easy to check that this function v satisfies the Cauchy-Riemann equations. Indeed, we have

$$\frac{\partial}{\partial X} v(X, Y) = - \frac{\partial}{\partial X} \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + \frac{\partial}{\partial X} \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy = - \frac{\partial u}{\partial y}(X, Y).$$

On the other hand, we have

$$\begin{aligned} \frac{\partial}{\partial Y} v(X, Y) &= - \frac{\partial}{\partial Y} \int_{X_0}^X \frac{\partial u}{\partial y}(x, Y) dx + \frac{\partial}{\partial Y} \int_{Y_0}^Y \frac{\partial u}{\partial x}(X_0, y) dy = - \int_{X_0}^X \frac{\partial^2 u}{\partial y^2}(x, Y) dx + \frac{\partial u}{\partial x}(X_0, Y) \\ &= \int_{X_0}^X \frac{\partial^2 u}{\partial x^2}(x, Y) dx + \frac{\partial u}{\partial x}(X_0, Y) = \frac{\partial u}{\partial x}(X, Y) - \frac{\partial u}{\partial x}(X_0, Y) + \frac{\partial u}{\partial x}(X_0, Y) = \frac{\partial u}{\partial x}(X, Y). \end{aligned}$$

This completes our sketched proof.

In practice, we may use the following technique. Suppose that u is a real valued harmonic function in a domain D . Write

$$(21) \quad g(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$

Then the Cauchy-Riemann equations for g are

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right),$$

which clearly hold. It follows that g is analytic in D . Suppose now that u is the real part of an analytic function f in D . Then $f'(z)$ agrees with the right hand side of (21) in view of (3) and (4). Hence $f' = g$

in D . The question here, of course, is to find this function f . If we are successful, then the imaginary part v of f is a harmonic conjugate of the harmonic function u .

EXAMPLE 3.6.1. Consider the function $u(x, y) = x^3 - 3xy^2$. It is easily checked that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

so that u is harmonic in \mathbb{C} . Using $X_0 = Y_0 = 0$ in (20), we obtain

$$v(X, Y) = - \int_0^X \frac{\partial u}{\partial y}(x, Y) dx + \int_0^Y \frac{\partial u}{\partial x}(0, y) dy + c = 6 \int_0^X xY dx - 3 \int_0^Y y^2 dy + c = 3X^2Y - Y^3 + c,$$

where c is any arbitrary constant. On the other hand, we can write

$$g(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = 3(x^2 - y^2) + 6ixy = 3(x^2 + 2ixy - y^2) = 3(x + iy)^2 = 3z^2.$$

It follows that u is the real part of an analytic function f in \mathbb{C} such that $f'(z) = g(z)$ for every $z \in \mathbb{C}$. The function $f(z) = z^3 + C$ satisfies this requirement for any arbitrary constant C . Note that the imaginary part of f is $3x^2y - y^3 + c$, where c is the imaginary part of C .

EXAMPLE 3.6.2. Consider the function $u(x, y) = e^x \sin y$. It is easily checked that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

so that u is harmonic in \mathbb{C} . Using $X_0 = Y_0 = 0$ in (20), we obtain

$$\begin{aligned} v(X, Y) &= - \int_0^X \frac{\partial u}{\partial y}(x, Y) dx + \int_0^Y \frac{\partial u}{\partial x}(0, y) dy + c = - \int_0^X e^x \cos Y dx + \int_0^Y \sin y dy + c \\ &= \cos Y - e^X \cos Y - \cos Y + 1 + c = c' - e^X \cos Y, \end{aligned}$$

where c' is any arbitrary constant. On the other hand, we can write

$$g(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = e^x \sin y - ie^x \cos y = -ie^x(\cos y + i \sin y) = -ie^z.$$

It follows that u is the real part of an analytic function f in \mathbb{C} such that $f'(z) = g(z)$ for every $z \in \mathbb{C}$. The function $f(z) = C - ie^z$ satisfies this requirement for any arbitrary constant C . Note that the imaginary part of f is $c' - e^x \cos y$, where c' is the imaginary part of C .

PROBLEMS FOR CHAPTER 3

1. a) Suppose that $P(z) = (z - z_1)(z - z_2) \dots (z - z_k)$, where $z_1, z_2, \dots, z_k \in \mathbb{C}$. Show that

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_k} \quad \text{for every } z \in \mathbb{C} \setminus \{z_1, z_2, \dots, z_k\}.$$

- b) Suppose further that $\Re z_j < 0$ for every $j = 1, \dots, k$, and that $\Re z \geq 0$. Show in this case that $\Re(z - z_j)^{-1} > 0$ for every $j = 1, \dots, k$, and deduce that $P'(z) \neq 0$.

[REMARK: Polynomials all of whose roots have negative real parts are called Hurwitz polynomials. We have shown here that the derivative of a non-constant Hurwitz polynomial is also a Hurwitz polynomial.]

2. For each of the following functions $f(z)$, determine whether the Cauchy-Riemann equations are satisfied:
- a) $f(z) = x^2 - y^2 - 2ixy$ b) $f(z) = \log(x^2 + y^2) + 2i \cot^{-1}(x/y)$
 c) $f(z) = x^3 - 3y^2 + 2x + i(3x^2y - y^3 + 2y)$ d) $f(z) = \log(x^2 - y^2) + 2i \tan^{-1}(y/x)$
3. Show that a real valued analytic function is constant.
4. We are required to define an analytic function $f(z)$ such that $f(x + iy) = e^x f(iy)$ for every $x, y \in \mathbb{R}$ and $f(0) = 1$. Suppose that for every $y \in \mathbb{R}$, we write $f(iy) = c(y) + is(y)$, where $c(y), s(y) \in \mathbb{R}$ for every $y \in \mathbb{R}$.
- a) Show by the Cauchy-Riemann equations that $c'(y) = -s(y)$ and $s'(y) = c(y)$ for every $y \in \mathbb{R}$.
 b) For every $y \in \mathbb{R}$, write $g(y) = (c(y) - \cos y)^2 + (s(y) - \sin y)^2$. Show that $g'(y) = 0$ for every $y \in \mathbb{R}$. Deduce that $g(y) = 0$ for every $y \in \mathbb{R}$.
 c) Comment on the above.
5. a) Suppose that $P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$, where $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$ are constants. Show that for every $k = 0, 1, \dots, n$, we have

$$a_k = \frac{P^{(k)}(0)}{k!}.$$

- b) Apply the result to the polynomial $(1 + z)^n = c_0 + c_1z + c_2z^2 + \dots + c_nz^n$ and show that for every $k = 0, 1, \dots, n$, we have

$$c_k = \frac{n!}{k!(n-k)!}.$$

6. a) Show that for every $z \in \mathbb{C}$, we have $e^{iz} = \cos z + i \sin z$.
 b) Show that for every $z, w \in \mathbb{C}$, we have

$$\cos(z + w) + i \sin(z + w) = (\cos z + i \sin z)(\cos w + i \sin w)$$

and

$$\cos(z + w) - i \sin(z + w) = (\cos z - i \sin z)(\cos w - i \sin w).$$

- c) Express $\sin(z + w)$ and $\cos(z + w)$ in terms of $\sin z, \sin w, \cos z$ and $\cos w$.
7. Suppose that $a_1, a_2, \dots, a_n \in \mathbb{C}$ are distinct, and consider the polynomial

$$Q(z) = (z - z_1)(z - z_2) \dots (z - z_n).$$

Suppose further that $P(z)$ is a polynomial of degree less than n . Follow the steps below to show that there exist $a_1, a_2, \dots, a_n \in \mathbb{C}$ such that

$$\frac{P(z)}{Q(z)} = \frac{a_1}{z - z_1} + \frac{a_2}{z - z_2} + \dots + \frac{a_n}{z - z_n}.$$

- a) We shall first of all show that the expression above is possible by multiplying it by $Q(z)$ and then determining a_1, a_2, \dots, a_n so that the resulting equation between polynomials of degree less than n holds when $z = z_1, z_2, \dots, z_n$.
 [HINT: Recall Problem 1 in Chapter 1.]
 b) Show that for every $k = 1, \dots, n$, we have

$$a_k = \lim_{z \rightarrow z_k} (z - z_k) \frac{P(z)}{Q(z)} = \frac{P(z_k)}{Q'(z_k)}.$$

[HINT: Note that $Q(z_k) = 0$ for every $k = 1, \dots, n$.]

-
8. Suppose that $a \in \mathbb{C}$ is non-zero. Show that for any fixed choice of value for $\log a$, the function $f(z) = a^z = e^{z \log a}$ satisfies $f'(z) = f(z) \log a$.
9. For each expression below, compute all possible values and plot their positions in the complex plane:
- | | |
|----------------|------------------------|
| a) $\log(-i)$ | b) $\log(1+i)$ |
| c) $(-i)^{-i}$ | d) i^2 |
| e) $2^{\pi i}$ | f) $(1+i)^i(1+i)^{-i}$ |
10. For each of the following equations, find all solutions:
- | | |
|------------------------------|-----------------------|
| a) $\text{Log}(z) = \pi i/3$ | b) $e^z = 2i$ |
| c) $\sin z = i$ | d) $\sin z = -\cos z$ |
| e) $\tan^2 z = -1$ | |
11. For each of the functions below, determine whether the function is harmonic. If so, find also its harmonic conjugate:
- | | |
|---------------------------|--|
| a) $x^2 - y^2 + y$ | b) $e^x \sin y$ |
| c) $x^3 - y^3$ | d) $xe^x \cos y - ye^x \sin y$ |
| e) $3x^2y - y^3 + xy$ | f) $x^4 - 6x^2y^2 + y^4 + x^3y - xy^3$ |
| g) $e^{x^2-y^2} \sin 2xy$ | |
12. a) Suppose that the functions $f(z)$ and $g(z)$ both satisfy the Cauchy-Riemann equations at a particular point $z \in \mathbb{C}$. Show that the functions $f(z) + g(z)$ and $f(z)g(z)$ also satisfy the Cauchy-Riemann equations at the point z .
- b) Show that the constant function and the function $f(z) = z$ both satisfy the Cauchy-Riemann equations everywhere in \mathbb{C} .
- c) Deduce that every polynomial $P(z)$ with complex coefficients satisfies the Cauchy-Riemann equations everywhere in \mathbb{C} .
13. A real valued function $u(x, y)$ which is continuous and satisfies the inequality $u_{xx} + u_{yy} \geq 0$ in a region D is said to be subharmonic in D . Show that $u = |f(z)|^2$ is subharmonic in any region where $f(z)$ is analytic.

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 4

COMPLEX INTEGRALS

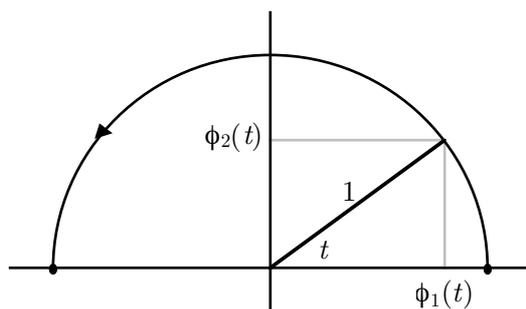
4.1. Curves in the Complex Plane

Integration of functions of a complex variable is carried out over curves in \mathbb{C} and leads to many important results useful in pure and applied mathematics. In this section, we give a brief introduction to curves in \mathbb{C} . We are interested in complex valued functions of the form

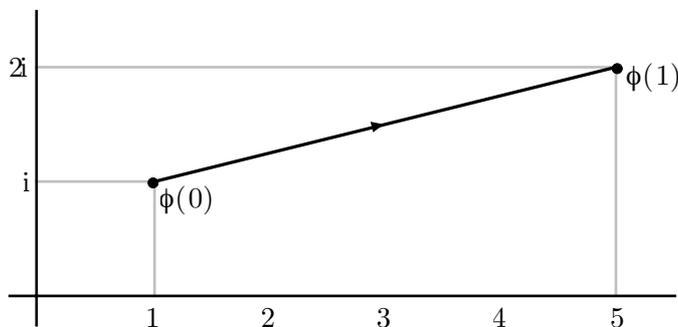
$$\phi(t) = \phi_1(t) + i\phi_2(t),$$

where the functions ϕ_1 and ϕ_2 are real valued and defined on some closed interval $[A, B]$ in \mathbb{R} . The functions ϕ_1 and ϕ_2 are called the real and imaginary parts of the function ϕ respectively.

EXAMPLE 4.1.1. The function $\phi(t) = e^{it}$, defined on the interval $[0, \pi]$, represents the upper half of a circle centred at the origin and of radius 1. As t varies from 0 to π , $\phi(t)$ follows this half-circle in an anticlockwise direction. Also $\phi_1(t) = \cos t$ and $\phi_2(t) = \sin t$.



EXAMPLE 4.1.2. The function $\phi(t) = (4t + 1) + i(t + 1)$, defined on the interval $[0, 1]$, represents a line segment from the point $1 + i$ to the point $5 + 2i$.



The functions ϕ_1 and ϕ_2 are real valued functions of a real variable, and we have already studied continuity, differentiability and integrability of such functions. We can now extend these definitions to the function ϕ .

We say that ϕ is continuous at t_0 if both ϕ_1 and ϕ_2 are continuous at t_0 . We also say that ϕ is continuous in an interval if both ϕ_1 and ϕ_2 are continuous in the interval. It is simple to show that the arithmetic of limits, as applied to continuity, holds.

We say that ϕ is differentiable at t_0 if both ϕ_1 and ϕ_2 are differentiable at t_0 , and write

$$\phi'(t_0) = \phi_1'(t_0) + i\phi_2'(t_0).$$

It is simple to show that the arithmetic of derivatives holds.

We also have the Chain rule: Suppose that f is analytic at the point $z_0 = \phi(t_0)$, and that ϕ is differentiable at t_0 . Then the complex valued function $\psi(t) = f(\phi(t))$ is differentiable at t_0 , and

$$\psi'(t_0) = f'(z_0)\phi'(t_0).$$

We say that ϕ is integrable over the interval $[A, B]$ if both ϕ_1 and ϕ_2 are integrable over $[A, B]$, and write

$$\int_A^B \phi(t) dt = \int_A^B \phi_1(t) dt + i \int_A^B \phi_2(t) dt.$$

Many rules of integration for real valued functions can be carried over to this case. For example, if ϕ is continuous in $[A, B]$, then there exists a function Φ satisfying $\Phi' = \phi$, and the Fundamental theorem of integral calculus can be generalized to

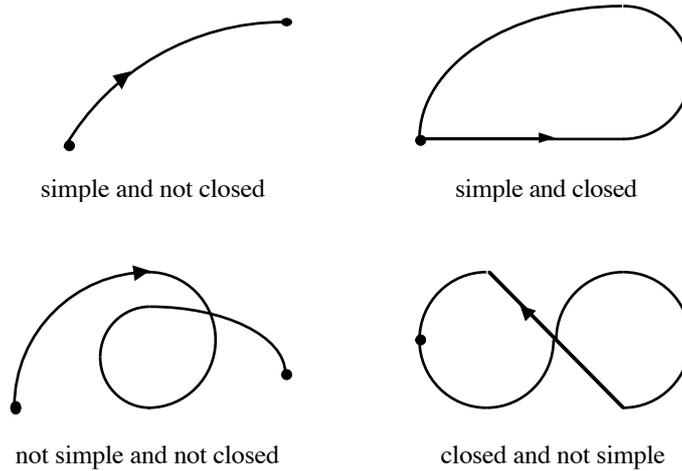
$$\int_A^B \phi(t) dt = \Phi(B) - \Phi(A) \quad \text{and} \quad \frac{d}{dt} \int_A^t \phi(\tau) d\tau = \phi(t).$$

DEFINITION. A complex valued function $\zeta : [A, B] \rightarrow \mathbb{C}$ is called a curve. The curve is said to be continuous if ζ is continuous in $[A, B]$, and differentiable if ζ is also differentiable in $[A, B]$. The set $\zeta([A, B])$ is called the trace of the curve. The point $\zeta(A)$ is called the initial point of the curve, and the point $\zeta(B)$ is called the terminal point of the curve.

REMARKS. (1) Of course, we can only have continuity and differentiability from the right at A and from the left at B .

(2) Usually, we do not distinguish between the curve and the function ζ , and simply refer to the curve ζ .

DEFINITION. A curve $\zeta : [A, B] \rightarrow \mathbb{C}$ is said to be simple if $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, with the possible exception that $\zeta(A) = \zeta(B)$. A curve $\zeta : [A, B] \rightarrow \mathbb{C}$ is said to be closed if $\zeta(A) = \zeta(B)$.



4.2. Contour Integrals

DEFINITION. A curve $\zeta : [A, B] \rightarrow \mathbb{C}$ is said to be an arc if ζ is differentiable in $[A, B]$ and ζ' is continuous in $[A, B]$.

EXAMPLE 4.2.1. The unit circle is a simple closed arc, since we can describe it by $\zeta : [0, 2\pi] \rightarrow \mathbb{C}$, given by $\zeta(t) = e^{it} = \cos t + i \sin t$. It is easy to check that $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, the only exception being $\zeta(0) = \zeta(2\pi)$. Furthermore, $\zeta'(t) = -\sin t + i \cos t$ is continuous in $[0, 2\pi]$.

DEFINITION. Suppose that C is an arc given by the function $\zeta : [A, B] \rightarrow \mathbb{C}$. A complex valued function f is said to be continuous on the arc C if the function $\psi(t) = f(\zeta(t))$ is continuous in $[A, B]$. In this case, the integral of f on C is defined to be

$$(1) \quad \int_C f(z) dz = \int_A^B f(\zeta(t)) \zeta'(t) dt.$$

REMARKS. (1) Note that (1) can be obtained by the formal substitution $z = \zeta(t)$ and $dz = \zeta'(t) dt$.

(2) If we describe the arc C in the opposite direction from $t = B$ to $t = A$, this opposite arc can be designated by $-C$. Since

$$\int_B^A f(\zeta(t)) \zeta'(t) dt = - \int_A^B f(\zeta(t)) \zeta'(t) dt,$$

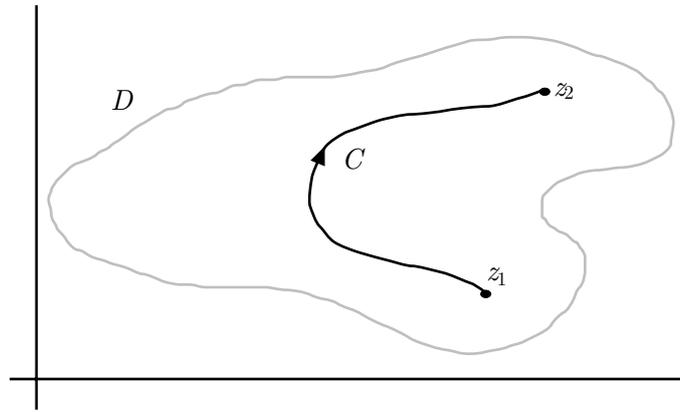
we have

$$\int_{-C} f(z) dz = - \int_C f(z) dz.$$

(3) Integration of functions on arcs is a linear operation. More precisely, suppose that f and g are continuous on the arc C . Then for any $\alpha, \beta \in \mathbb{C}$, we have

$$\int_C (\alpha f(z) + \beta g(z)) dz = \alpha \int_C f(z) dz + \beta \int_C g(z) dz.$$

(4) Suppose that F is analytic in a domain D and has a continuous derivative $f = F'$ in D . Suppose further that C , defined by $\zeta : [A, B] \rightarrow \mathbb{C}$, is an arc lying in D , with initial point z_1 and terminal point z_2 .



Define $\psi : [A, B] \rightarrow \mathbb{C}$ by $\psi(t) = F(\zeta(t))$. Then by the Fundamental theorem of integral calculus applied to the function $\psi'(t) = F'(\zeta(t))\zeta'(t) = f(\zeta(t))\zeta'(t)$, we have

$$\int_C f(z) dz = \int_A^B f(\zeta(t))\zeta'(t) dt = F(\zeta(B)) - F(\zeta(A)) = F(z_2) - F(z_1).$$

We can extend the case of one arc to the case of a finite number of arcs joined together.

DEFINITION. Suppose that $A_1 < B_1 = A_2 < B_2 = \dots = A_k < B_k$ are real numbers, and that for every $j = 1, \dots, k$, C_j is an arc given by the function $\zeta_j : [A_j, B_j] \rightarrow \mathbb{C}$. Suppose further that $\zeta_j(B_j) = \zeta_{j+1}(A_{j+1})$ for every $j = 1, \dots, k-1$. Then

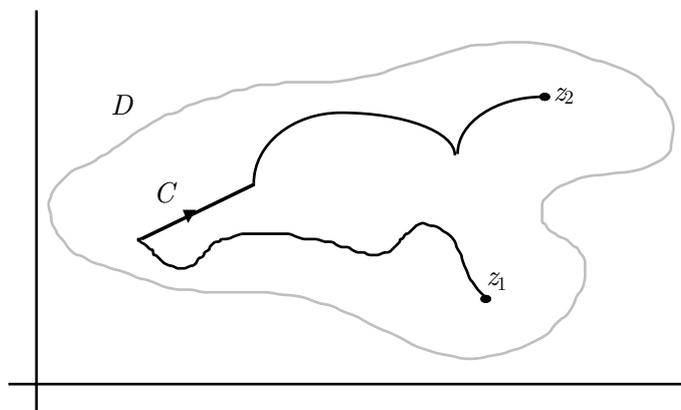
$$C = C_1 \cup C_2 \cup \dots \cup C_k$$

is called a contour. The point $\zeta_1(A_1)$ is called the initial point of the contour C , and the point $\zeta_k(B_k)$ is called the terminal point of the contour C . A complex valued function f is said to be continuous on the contour C if it is continuous on the arc C_j for every $j = 1, \dots, k$. In this case, the integral of f on C is defined to be

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \dots + \int_{C_k} f(z) dz.$$

The following result follows immediately from this definition and Remark (4) above.

THEOREM 4A. Suppose that F is analytic in a domain D and has a continuous derivative $f = F'$ in D . Suppose further that C is a contour lying in D , with initial point z_1 and terminal point z_2 .



Then

$$(2) \quad \int_C f(z) dz = F(z_2) - F(z_1).$$

REMARKS. (1) Note that the right hand side of (2) is independent of the contour C . It follows that under the hypotheses of Theorem 4A, we have

$$(3) \quad \int_C f(z) dz = 0$$

for any closed contour C in D .

(2) Naturally, we would like to extend (3) to all analytic functions f in D and all closed contours C in D . Note, however, the restrictive nature of the hypotheses of Theorem 4A in this case. In many situations, no analytic functions F satisfying $F' = f$ may be at hand. Consider, for example, the function $f(z) = \cos z^2$.

EXAMPLE 4.2.2. Consider the contour C given by $\zeta : [A, B] \rightarrow \mathbb{C}$, where $\zeta(t) = e^{it}$. Then

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} = \frac{1}{2\pi i} \int_A^B \frac{\zeta'(t)}{\zeta(t)} dt = \frac{1}{2\pi i} \int_A^B \frac{ie^{it}}{e^{it}} dt = \frac{B - A}{2\pi}.$$

Suppose that $A = 0$ and $B = 2k\pi$, so that the contour “winds” round the origin k times in the anticlockwise direction. In this case, we have

$$\frac{1}{2\pi i} \int_C \frac{dz}{z} = k.$$

Note, however, that in this case, the initial point and terminal point of the contour are the same. Yet, (3) does not hold. Clearly the function $1/z$ is analytic in the domain $D = \{z : 1/2 < |z| < 3/2\}$ and the contour C lies in D . However, we cannot find an analytic function F in D such that $F'(z) = 1/z$ in D . The logarithmic function $\log z$ appears to be a candidate; however, it is not possible to define $\log z$ to be continuous in this annulus D . See Example 3.5.3.

EXAMPLE 4.2.3. Suppose that C is any contour in \mathbb{C} with initial point z_1 and terminal point z_2 . Then

$$\int_C e^z dz = e^{z_2} - e^{z_1} \quad \text{and} \quad \int_C \cos z dz = \sin z_2 - \sin z_1.$$

These follow from Theorem 4A since the entire functions e^z and $\sin z$ satisfy

$$\frac{d}{dz}e^z = e^z \quad \text{and} \quad \frac{d}{dz}\sin z = \cos z.$$

EXAMPLE 4.2.4. Suppose that f is a polynomial in z with coefficients in \mathbb{C} . Then

$$\int_C f(z) dz = 0$$

for any closed contour C in \mathbb{C} . This is a special case of (3). For any such polynomial f , it is easy to find a polynomial F in z with coefficients in \mathbb{C} and such that $F'(z) = f(z)$ in \mathbb{C} .

4.3. Inequalities for Contour Integrals

Suppose that $\phi : [A, B] \rightarrow \mathbb{C}$ is continuous in $[A, B]$. Let

$$I = \left| \int_A^B \phi(t) dt \right|.$$

If $I = 0$, then clearly

$$(4) \quad \left| \int_A^B \phi(t) dt \right| \leq \int_A^B |\phi(t)| dt.$$

If $I > 0$, then there exists a real number θ such that

$$\int_A^B \phi(t) dt = Ie^{i\theta},$$

so that

$$(5) \quad I = \int_A^B e^{-i\theta} \phi(t) dt = \int_A^B \Re[e^{-i\theta} \phi(t)] dt + i \int_A^B \Im[e^{-i\theta} \phi(t)] dt.$$

Since I is real, the last integral on the right hand side of (5) must be 0. It follows that

$$I = \int_A^B \Re[e^{-i\theta} \phi(t)] dt.$$

On the other hand, clearly

$$\Re[e^{-i\theta} \phi(t)] \leq |e^{-i\theta} \phi(t)| = |\phi(t)|$$

for every $t \in [A, B]$, and so it follows from the theory of real integration that

$$I \leq \int_A^B |\phi(t)| dt.$$

Hence the inequality (4) always holds.

Consider now an arc C given by the function $\zeta : [A, B] \rightarrow \mathbb{C}$. Suppose that the function f is continuous on C . Then it follows from (4) that

$$(6) \quad \left| \int_C f(z) dz \right| = \left| \int_A^B f(\zeta(t)) \zeta'(t) dt \right| \leq \int_A^B |f(\zeta(t))| |\zeta'(t)| dt.$$

Suppose that $|f(z)| \leq M$ on C , where M is a real constant, then we have

$$(7) \quad \left| \int_C f(z) dz \right| \leq M \int_A^B |\zeta'(t)| dt.$$

Let us investigate the integral on the right hand side of (7) more closely. If $\zeta(t) = x(t) + iy(t)$, then

$$|\zeta'(t)| = \left| \frac{dx}{dt} + i \frac{dy}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt},$$

where $s(t)$ is the length of the arc C between $\zeta(A)$ and $\zeta(t)$. It follows that the integral

$$\int_A^B |\zeta'(t)| dt$$

is the length of the arc C . We have proved the following result.

THEOREM 4B. *Suppose that a function f is continuous on a contour C . Then*

$$\left| \int_C f(z) dz \right| \leq ML,$$

where L is the length of the contour C and where M is a real constant such that $|f(z)| \leq M$ on C .

REMARK. We usually write

$$\int_C f(z) dz = \int_A^B f(\zeta(t)) |\zeta'(t)| dt.$$

In this notation, we have

$$L = \int_C |dz|,$$

and (6) can be represented by

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z)| |dz|.$$

4.4. Equivalent Curves

EXAMPLE 4.4.1. Consider the arc C' given by the function $\zeta : [0, 1] \rightarrow \mathbb{C}$ where $\zeta(t) = (1 + i)t^2$. Consider also the arc C'' given by the function $\xi : [0, \pi/2] \rightarrow \mathbb{C}$ where $\xi(\tau) = (1 + i)\sin \tau$. Clearly $\zeta(0) = \xi(0)$, so that the two arcs have the same initial point. Also, $\zeta(1) = \xi(\pi/2)$, so that the two arcs have the same terminal point. Furthermore, $\zeta([0, 1]) = \xi([0, \pi/2])$, so that the two arcs have the same trace. Suppose that the function f is continuous on C' and C'' . On the one hand, we have

$$\int_{C'} f(z) dz = \int_0^1 f(\zeta(t)) \zeta'(t) dt = \int_0^1 f((1 + i)t^2) 2(1 + i)t dt.$$

On the other hand, we have

$$\int_{C''} f(z) dz = \int_0^{\pi/2} f(\xi(\tau)) \xi'(\tau) d\tau = \int_0^{\pi/2} f((1 + i)\sin \tau) (1 + i) \cos \tau d\tau.$$

If we perform a formal change of variables

$$t^2 = \sin \tau \quad \text{and} \quad 2t \, dt = \cos \tau \, d\tau,$$

then we see in fact that

$$\int_{C'} f(z) \, dz = \int_{C''} f(z) \, dz.$$

This is not surprising, considering that basically the two arcs “are the same”.

DEFINITION. Two curves C' and C'' are said to be equivalent if they have the same trace and if

$$\int_{C'} f(z) \, dz = \int_{C''} f(z) \, dz$$

holds for all functions f which are continuous in a region containing this trace.

REMARKS. (1) It can be shown that two simple arcs are equivalent if they have the same initial points, the same terminal points and the same trace, with the convention that in the case of closed arcs, the arcs must be followed in the same direction.

(2) In fact, the definition

$$\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz + \dots + \int_{C_k} f(z) \, dz$$

of a contour integral in terms of integrals over arcs as discussed earlier is made in the same spirit.

(3) The practical importance of these considerations is that when we consider integrals over a simple contour, we may choose the most convenient parameterization of the given contour.

EXAMPLE 4.4.2. Consider the integral

$$\int_C \frac{dz}{z},$$

where C is a contour that avoids the origin. Suppose first of all that C is an arc $\zeta : [A, B] \rightarrow \mathbb{C}$. Then

$$\int_C \frac{dz}{z} = \int_A^B \frac{\zeta'(t)}{\zeta(t)} \, dt = \text{var}(\log z, C).$$

Here the variation function $\text{var}(\log z, C)$ is interpreted in the following way: We choose a branch of the logarithmic function at the initial point $z_1 = \zeta(A)$ of the arc C and then let $\log z$ vary continuously as z follows C to the terminal point $z_2 = \zeta(B)$ of the arc C . In other words, the function $\log \zeta(t)$ must be continuous in $[A, B]$. Then we calculate $\log \zeta(B) - \log \zeta(A)$. This result is then extended to contours by addition. Let us interpret this geometrically. Note that $\log z = \log |z| + i \arg z$. Since $\log |z|$ is single valued, its variation around a closed contour C is 0. In this case, we have

$$\text{var}(\log z, C) = \text{var}(i \arg z, C),$$

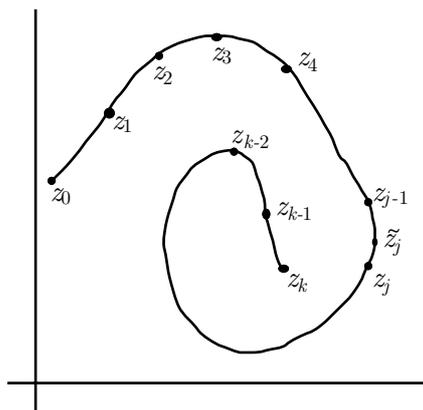
and this gives a value $2\pi i$ every time the closed contour winds round the origin.

4.5. Riemann Sums

The following brief discussion of complex integrals in terms of Riemann sums will further demonstrate the independence of the integral from the parameterization of the arcs in question. The discussion is heuristic, as we only want to illustrate ideas.

Suppose that C is an arc, with initial point z_0 and terminal point z_k . Suppose further that we divide the arc C into subarcs C_1, \dots, C_k in the following way. The points z_0, z_1, \dots, z_k are points on the arc C , and they occur in the given order as we follow the arc C from z_0 to z_k . For every $j = 1, \dots, k$, the subarc C_j is then the part of C between z_{j-1} and z_j , with initial point z_{j-1} and terminal point z_j .

For every $j = 1, \dots, k$, we write $\Delta z_j = z_j - z_{j-1}$, and we let \tilde{z}_j denote a point on the subarc C_j .



As in real variables, we can then construct the Riemann sum

$$S = \sum_{j=1}^k f(\tilde{z}_j) \Delta z_j.$$

We now consider subdivisions of the arc C which are made finer and finer by subdivision into more and more subarcs. The precise requirement will be

$$k \rightarrow \infty \quad \text{and} \quad \max_{1 \leq j \leq k} |\Delta z_j| \rightarrow 0.$$

When the subdivision becomes arbitrarily fine, the Riemann sum S has a unique limit, independent of the manner of subdivision. This limit is the integral

$$\int_C f(z) dz.$$

PROBLEMS FOR CHAPTER 4

- Consider the integral $\int_C z^n dz$, where $n \in \mathbb{Z}$ and C is a closed contour on the complex plane.
 - Suppose that $n \geq 0$. Use Theorem 4A to explain why the integral is equal to zero.
 - Suppose that $n < -1$, and that the contour C does not pass through the origin $z = 0$. Use Theorem 4A to explain why the integral is equal to zero.
 - What is the value of the integral if $n = -1$ and C is the unit circle $\{z : |z| = 1\}$, followed in the positive (anticlockwise) direction?
- Sketch each of the arcs $z = 2 + it$, $z = e^{-\pi it}$, $z = e^{4\pi it}$ and $z = 1 + it + t^2$ for $t \in [0, 1]$.
 - Using Theorem 4A if appropriate, integrate each of the functions $f(z) = 4z^3$, $f(z) = \bar{z}$ and $f(z) = 1/z$ over each of the arcs in part (a).

3. Suppose that a function $f(z)$ satisfies $f'(z) = 0$ throughout a domain $D \subseteq \mathbb{C}$. Use Theorem 4A to prove that $f(z)$ is constant in D .
4. Suppose that C_1 is the semicircle from 1 to -1 through i , followed in the positive (anticlockwise) direction. Suppose also that C_2 is the semicircle from 1 to -1 through $-i$, followed in the negative (clockwise) direction. Show that

$$\int_{C_1} z^3 dz = \int_{C_2} z^3 dz \quad \text{and} \quad \int_{C_1} \bar{z} dz \neq \int_{C_2} \bar{z} dz.$$

Use Theorem 4A to comment on the two results.

5. Suppose that $\alpha = a + ib$, where $a, b \in \mathbb{R}$ are fixed. By integrating the function $e^{\alpha t}$ over an interval $[0, T]$ and equating real parts, show that

$$(a^2 + b^2) \int_0^T e^{at} \cos bt dt = e^{aT} (a \cos bT + b \sin bT) - a.$$

6. Suppose that $f(t) = f_1(t) + if_2(t)$ and $g(t) = g_1(t) + ig_2(t)$ are differentiable complex valued functions of a real variable t . Deduce the formulas

$$(f + g)'(t) = f'(t) + g'(t) \quad \text{and} \quad (fg)'(t) = f(t)g'(t) + f'(t)g(t)$$

from known results of these types for real valued functions $f_1(t)$, $f_2(t)$, $g_1(t)$ and $g_2(t)$.

7. Consider an arc $z(t) = x(t) + iy(t)$, where $t \in [A, B]$. Use

$$\frac{dz}{dt} = \frac{dx}{dt} + i \frac{dy}{dt}$$

to interpret $z'(t)$ as the complex representation of a vector tangent to the arc at any point where $z'(t)$ is non-zero.

8. Suppose that $\zeta(t) = t^2$ for $t \in [-1, 0]$ and $\zeta(t) = it^2$ for $t \in [0, 1]$. Show that the curve $\zeta : [-1, 1] \rightarrow \mathbb{C}$ is an arc (although its trace has a corner).
9. Consider the curve $\zeta : [-1, 1] \rightarrow \mathbb{C}$, given by $\zeta(t) = -t$ for $t \in [-1, 0]$ and $\zeta(t) = t + it^3 \sin(1/t)$ for $t \in (0, 1]$.
- Show that $\zeta : [-1, 1] \rightarrow \mathbb{C}$ is an arc.
 - Determine all points of self-intersection of this arc; in other words, find all points $z \in \mathbb{C}$ such that there exist $t_1, t_2 \in [-1, 1]$ satisfying $t_1 \neq t_2$ and $z = \zeta(t_1) = \zeta(t_2)$.

10. Suppose that $C = \{z : |z| = 1\}$ is the unit circle, followed in the positive (anticlockwise) direction. Evaluate each of the following integrals:

a) $\int_C \frac{dz}{z}$

b) $\int_C \frac{dz}{|z|}$

c) $\int_C \frac{|dz|}{z}$

d) $\int_C \frac{dz}{z^2}$

e) $\int_C \frac{dz}{|z^2|}$

f) $\int_C \frac{|dz|}{z^2}$

11. Suppose that $C = \{z : |z| = 1\}$ is the unit circle, followed in the positive (anticlockwise) direction.
 a) Use Theorem 4B to show that

$$\left| \int_C \frac{dz}{4+3z} \right| \leq 2\pi.$$

- b) By dividing the circle C into its left half and its right half and applying Theorem 4B to each half, establish the better bound

$$\left| \int_C \frac{dz}{4+3z} \right| \leq \frac{6\pi}{5}.$$

12. Consider the circle $C = \{z : |z - 1| = 1/2\}$, followed in the positive (anticlockwise) direction with initial point $z = 1/2$. Evaluate the integral

$$\int_C \frac{dz}{(z^2 - 1)^{1/2}},$$

given that the integrand is equal to the derivative of the function $\log(z + (z^2 - 1)^{1/2})$.

13. Writing $f = u + iv$ and $z = x + iy$, computation suggests the identity

$$\int_C f(z) dz = \int_C (u(x, y) dx - v(x, y) dy) + i \int_C (u(x, y) dy + v(x, y) dx).$$

Suppose now that the arc C is given by $\zeta(t) = \xi(t) + i\eta(t)$ for $t \in [A, B]$. Show from first definition that the identity holds.

14. Suppose that the arc C_1 is given by $z = \zeta_1(t)$ for $t \in [A, B]$, and that the arc C_2 is given by $z = \zeta_2(\tau)$ for $\tau \in [\alpha, \beta]$. Suppose further that there is a differentiable function $\phi : [A, B] \rightarrow [\alpha, \beta]$ such that $\phi(A) = \alpha$, $\phi(B) = \beta$. Show that the two arcs C_1 and C_2 are equivalent.
15. Suppose that C denotes the ellipse $x = a \cos t$ and $y = b \sin t$, where $a, b \in \mathbb{R}$ are positive and fixed, and $t \in [0, 2\pi]$, so that C is followed in the positive (anticlockwise) direction.

- a) By referring to Example 4.4.2 if necessary, explain why $\frac{1}{2\pi i} \int_C \frac{dz}{z} = 1$.

- b) Hence show that $\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{1}{ab}$.

INTRODUCTION TO COMPLEX ANALYSIS

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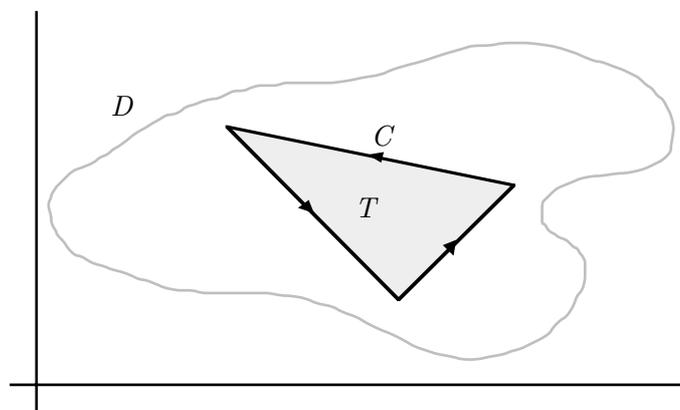
Chapter 5

CAUCHY'S INTEGRAL THEOREM

5.1. A Restricted Case

Cauchy's integral theorem states that in a simply connected domain, the integral of an analytic function over a closed contour is zero. The proof of this general result is rather involved. Here we first study a special case of the theorem in order to develop the basic properties of analytic functions.

THEOREM 5A. *Suppose that a function f is analytic in a domain D . Suppose further that the closed triangular region T lies in D , and that C denotes the boundary of T in the positive (anticlockwise) direction.*



Then

$$\int_C f(z) dz = 0.$$

We shall give two proofs of this result, usually known as Cauchy’s integral theorem for a triangular path. The first of these proofs, given next, is based on an additional assumption that the derivative $f'(z)$ is continuous in D .

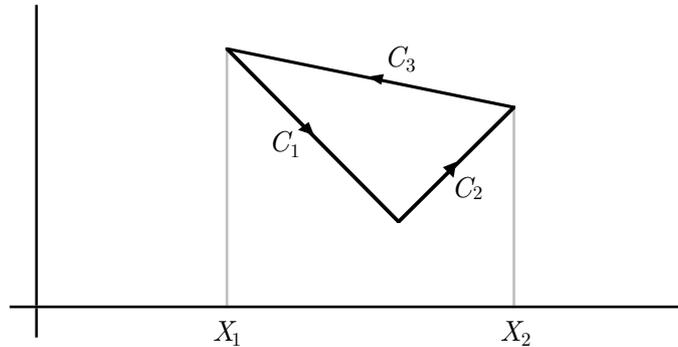
PROOF OF THEOREM 5A. Write $f(z) = u(x, y) + iv(x, y)$, where u and v are real valued. Since f' exists and is continuous, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold, and the four partial derivatives are continuous. On the other hand, we can write

$$\int_C f(z) dz = \int_C (u + iv)(dx + idy) = \int_C (u dx - v dy) + i \int_C (v dx + u dy).$$

Suppose that $C = C_1 \cup C_2 \cup C_3$, a union of the three straight directed edges.



Consider the integral

$$\int_C u dx.$$

We can write

$$\int_C u dx = \int_{C_1} u dx + \int_{C_2} u dx + \int_{C_3} u dx.$$

For each of the three integrals on the right hand side, y can be represented as a linear function of x , unless the edge is vertical, in which case the integral vanishes. Suppose that the projection of the triangle T on the x -axis is the line segment $X_1 \leq x \leq X_2$. Suppose also that the vertical line with abscissa x intersects the triangle in $h_1(x)$ and $h_2(x)$, where $h_1(x) \leq h_2(x)$ (in the diagram, $h_1(x)$ describes C_1 and C_2 , while $h_2(x)$ describes C_3). Then

$$\begin{aligned} \int_C u dx &= \int_{X_1}^{X_2} u(x, h_1(x)) dx + \int_{X_2}^{X_1} u(x, h_2(x)) dx = - \int_{X_1}^{X_2} (u(x, h_2(x)) - u(x, h_1(x))) dx \\ &= - \int_{X_1}^{X_2} \left(\int_{h_1(x)}^{h_2(x)} \frac{\partial u}{\partial y}(x, y) dy \right) dx = - \int_T \frac{\partial u}{\partial y} dx dy. \end{aligned}$$

Note that the third equality above follows from the continuity of $\partial u/\partial y$. Similarly

$$\int_C v dy = \int_T \frac{\partial v}{\partial x} dx dy.$$

Hence

$$\int_C (u \, dx - v \, dy) = - \int_T \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \, dx \, dy = 0.$$

We can also show that

$$\int_C v \, dx = - \int_T \frac{\partial v}{\partial y} \, dx \, dy \quad \text{and} \quad \int_C u \, dy = \int_T \frac{\partial u}{\partial x} \, dx \, dy,$$

so that

$$\int_C (v \, dx + u \, dy) = \int_T \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dx \, dy = 0.$$

The result follows. \circ

5.2. Analytic Functions in a Star Domain

In this section, we shall use Theorem 5A to establish the existence of an indefinite integral and the Cauchy integral theorem for analytic functions in a certain class of domains.

DEFINITION. A domain $D \subseteq \mathbb{C}$ is called a star domain if there exists a point $z_0 \in D$ such that for every point $z \in D$, the line segment joining z and z_0 also lies in D . In this case, the point z_0 is called a star centre of the domain D .

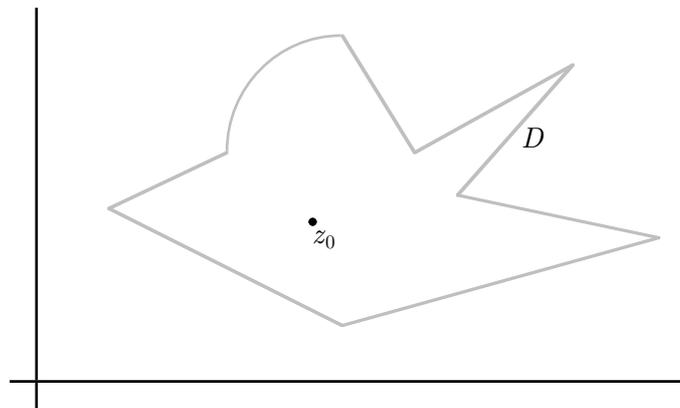
EXAMPLE 5.2.1. The disc $\{z : |z| < 1\}$ is a star domain. Every point in this domain is a star centre.

EXAMPLE 5.2.2. The complex plane \mathbb{C} is a star domain. Again, every point in this domain is a star centre.

EXAMPLE 5.2.3. The complex plane \mathbb{C} with the non-negative real axis $\{x + iy : x \geq 0, y = 0\}$ deleted is a star domain. Every point on the remaining part of the real axis is a star centre.

EXAMPLE 5.2.4. The set $\{x + iy : |xy| < 1\}$ is a star domain. The point 0 is the only star centre.

EXAMPLE 5.2.5. The interior of the set shown below is a star domain, with a star centre z_0 as shown.

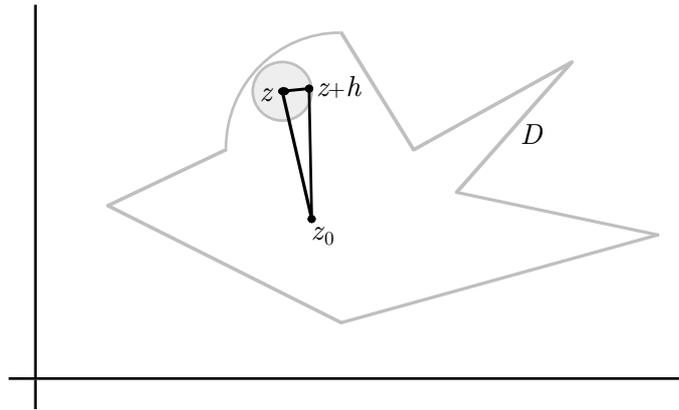


THEOREM 5B. Suppose that a function f is analytic in a star domain D . Then there exists a function F , analytic in D and such that $F'(z) = f(z)$ for every $z \in D$.

PROOF. Suppose that $z_0 \in D$ is a star centre. For every $z \in D$, define

$$(1) \quad F(z) = \int_{[z_0, z]} f(\zeta) \, d\zeta,$$

where, for every $z_1, z_2 \in D$, $[z_1, z_2]$ denotes the directed line segment from z_1 to z_2 . Since $z \in D$, there exists an ϵ -neighbourhood of z which is contained in D . Furthermore, for every $h \in \mathbb{C}$ satisfying $|h| < \epsilon$, the point $z+h$ lies in this ϵ -neighbourhood of z . It follows that the closed triangular region with vertices z_0, z and $z+h$ lies in D .



By Theorem 5A, we have

$$\int_{[z_0, z]} f(\zeta) \, d\zeta + \int_{[z, z+h]} f(\zeta) \, d\zeta + \int_{[z+h, z_0]} f(\zeta) \, d\zeta = 0.$$

In other words,

$$\int_{[z_0, z+h]} f(\zeta) \, d\zeta - \int_{[z_0, z]} f(\zeta) \, d\zeta = \int_{[z, z+h]} f(\zeta) \, d\zeta.$$

It follows from (1) that

$$F(z+h) - F(z) = \int_{[z, z+h]} f(\zeta) \, d\zeta.$$

If $h \neq 0$, then

$$(2) \quad \frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{[z, z+h]} (f(\zeta) - f(z)) \, d\zeta.$$

Since the function f is continuous at z , it follows that given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. This means that if $|h| < \delta$, then $|f(\zeta) - f(z)| < \epsilon$ holds for every $\zeta \in [z, z+h]$. Theorem 4B now gives

$$(3) \quad \left| \int_{[z, z+h]} (f(\zeta) - f(z)) \, d\zeta \right| \leq \epsilon|h|.$$

Combining (2) and (3), we have

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \epsilon.$$

This gives

$$\lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

and completes the proof of the theorem. \circ

If we examine our proof carefully, then it is not difficult to see that we have in fact established the following result.

THEOREM 5C. *Suppose that a function f is continuous in a star domain D . Suppose further that*

$$\int_C f(z) dz = 0$$

for every closed triangular contour C lying in D . Then there exists a function F , analytic in D and such that $F'(z) = f(z)$ for every $z \in D$.

We can also deduce the Cauchy integral theorem for a star domain.

THEOREM 5D. *Suppose that a function f is analytic in a star domain D . Suppose further that C is a closed contour lying in D . Then*

$$\int_C f(z) dz = 0.$$

PROOF. By Theorem 5B, there exists a function F , analytic in D and such that $F'(z) = f(z)$ for every $z \in D$. The result now follows from Remark (1) immediately after Theorem 4A. \circ

EXAMPLE 5.2.6. Consider the contour integral

$$\int_{|z|=3} \frac{e^z + \sin z}{z^2 - 16} dz,$$

where the contour of integration is the circle centred at 0 and with radius 3, followed in the positive (anticlockwise) direction. Note that the function in question is analytic in the disc $D = \{z : |z| < 4\}$, clearly a star domain. It follows from Theorem 5D that the integral is 0.

EXAMPLE 5.2.7. Suppose that $0 < r < R$. Consider the contour integral

$$\int_{|z|=r} \frac{R+z}{(R-z)z} dz,$$

where the contour of integration is the circle centred at 0 and with radius r , followed in the positive (anticlockwise) direction. For every $z \in \mathbb{C}$, note that using partial fractions, we have

$$\frac{R+z}{(R-z)z} = \frac{1}{z} + \frac{2}{R-z}.$$

It follows that

$$\int_{|z|=r} \frac{R+z}{(R-z)z} dz = \int_{|z|=r} \frac{1}{z} dz + \int_{|z|=r} \frac{2}{R-z} dz.$$

Next, note that the function

$$\frac{2}{R-z}$$

is analytic in the star domain $D = \{z : |z| < R\}$. It follows from Theorem 5D that the last integral is 0, so that

$$(4) \quad \int_{|z|=r} \frac{R+z}{(R-z)z} dz = \int_{|z|=r} \frac{1}{z} dz = 2\pi i,$$

in view of Example 4.4.2. On the other hand, the contour can be described by $z = re^{i\theta}$, where $\theta \in [0, 2\pi]$. This formal substitution leads to the expression $dz = ire^{i\theta} d\theta = iz d\theta$ and

$$\int_{|z|=r} \frac{R+z}{(R-z)z} dz = \int_0^{2\pi} \frac{R+re^{i\theta}}{R-re^{i\theta}} i d\theta.$$

Next, note that

$$\frac{R+re^{i\theta}}{R-re^{i\theta}} = \frac{(R+re^{i\theta})(R-re^{-i\theta})}{(R-re^{i\theta})(R-re^{-i\theta})} = \frac{R^2-r^2+2iRr\sin\theta}{R^2-2Rr\cos\theta+r^2},$$

so that

$$(5) \quad \int_{|z|=r} \frac{R+z}{(R-z)z} dz = \int_0^{2\pi} \frac{R^2-r^2+2iRr\sin\theta}{R^2-2Rr\cos\theta+r^2} i d\theta.$$

Combining (4) and (5) and equating imaginary parts, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos\theta+r^2} d\theta = 1.$$

5.3. Nested Triangles

In this section, we shall give a second proof of Theorem 5A, without the additional assumption that the derivative $f'(z)$ is continuous in D . This proof is based on the following well-known result in real analysis: Suppose that

$$a_1 \leq a_2 \leq a_3 \leq \dots \quad \text{and} \quad b_1 \geq b_2 \geq b_3 \geq \dots$$

Suppose further that $a_k \leq b_k$ for every $k \in \mathbb{N}$, and that $b_k - a_k \rightarrow 0$ as $k \rightarrow \infty$. Then there exists a unique number $\ell \in \mathbb{R}$ such that $a_k \rightarrow \ell$ and $b_k \rightarrow \ell$ as $k \rightarrow \infty$. This is a special case of the Cantor intersection theorem. In other words, if the intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \dots$$

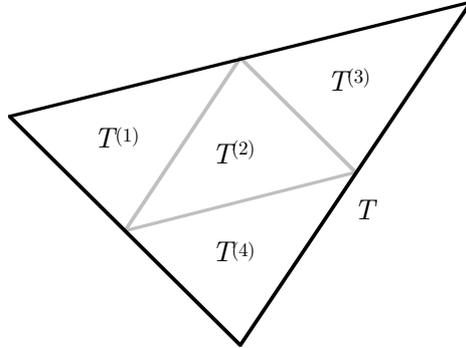
are nested, so that each contains all subsequent ones, and if their lengths decrease to 0, then the intervals collapse to a unique point.

We shall now prove Theorem 5A by the method of bisection.

Suppose that a function f is analytic in a domain D . Suppose further that the closed triangular region T lies in D , and that C denotes the boundary of T in the positive (anticlockwise) direction. Write

$$I(T) = \int_C f(z) dz.$$

We now divide T into four triangular regions by joining the midpoints of the three sides of T as shown in the diagram.



Suppose that the four triangular regions so obtained are denoted by $T^{(j)}$, where $j = 1, 2, 3, 4$, with boundaries $C^{(j)}$ in the positive (anticlockwise) direction. Then since integrals over the common sides cancel each other, we have

$$I(T) = I(T^{(1)}) + I(T^{(2)}) + I(T^{(3)}) + I(T^{(4)}),$$

where for $j = 1, 2, 3, 4$,

$$I(T^{(j)}) = \int_{C^{(j)}} f(z) dz.$$

Since the maximum is never less than the average, at least one of these four triangular regions $T^{(j)}$ must satisfy

$$(6) \quad |I(T^{(j)})| \geq \frac{1}{4}|I(T)|.$$

We denote this triangular region by T_1 , with the convention that if more than one of the four triangular regions $T^{(j)}$ satisfies (6), then we choose one under some fixed rule. This process can now be repeated indefinitely, so that we obtain a sequence of nested triangles

$$T = T_0 \supseteq T_1 \supseteq T_2 \supseteq T_3 \supseteq \dots \supseteq T_k \supseteq \dots$$

with the property

$$|I(T_k)| \geq \frac{1}{4}|I(T_{k-1})|,$$

so that

$$(7) \quad |I(T_k)| \geq 4^{-k}|I(T)|.$$

Note now that the sequence of nested triangular regions must collapse to a point $z^* \in D$. Suppose now that $\epsilon > 0$ is chosen. Since D is open and the function f is analytic at z^* , there exists a δ -neighbourhood $\{z : |z - z^*| < \delta\}$ of z^* , contained in D and such that

$$(8) \quad \left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon$$

whenever $|z - z^*| < \delta$. Furthermore, we can choose k so large that

$$(9) \quad T_k \subset \{z : |z - z^*| < \delta\}.$$

Note that since

$$\int_{C_k} dz = 0 \quad \text{and} \quad \int_{C_k} z dz = 0,$$

we have

$$I(T_k) = \int_{C_k} f(z) dz = \int_{C_k} (f(z) - f(z^*) - (z - z^*)f'(z^*)) dz.$$

In view of (8) and (9), we have

$$|f(z) - f(z^*) - (z - z^*)f'(z^*)| \leq \epsilon |z - z^*| \leq \epsilon d_k,$$

where d_k denotes the diameter of T_k . It follows from Theorem 4B that

$$(10) \quad |I(T_k)| \leq \epsilon d_k L_k,$$

where L_k denotes the perimeter of T_k . Observe now that

$$(11) \quad d_k = 2^{-k}d \quad \text{and} \quad L_k = 2^{-k}L,$$

where d and L denote respectively the diameter and perimeter of T . Combining (7), (10) and (11), we obtain

$$|I(T)| \leq \epsilon dL.$$

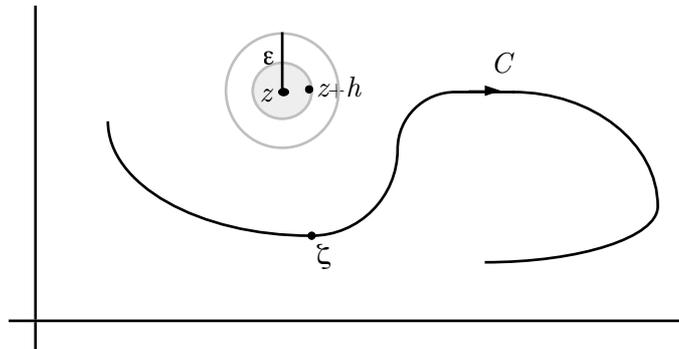
Since $\epsilon > 0$ is arbitrary, we must have $I(T) = 0$. This completes the proof of Theorem 5A.

5.4. Further Examples

EXAMPLE 5.4.1. Suppose that C is any contour. For any $z \in \mathbb{C}$ not lying on C , consider the integral

$$I(z) = \int_C \frac{d\zeta}{\zeta - z}.$$

We shall show that the function $I(z)$ is continuous at z . Since $z \notin C$, there exists $\epsilon > 0$ such that the ϵ -neighbourhood of z does not meet C . Suppose that $h \in \mathbb{C}$ satisfies $|h| < \epsilon/2$.



Then

$$I(z+h) - I(z) = \int_C \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) d\zeta = h \int_C \frac{d\zeta}{(\zeta - z - h)(\zeta - z)}.$$

Note next that for any $\zeta \in C$, we have

$$|\zeta - z| > \epsilon \quad \text{and} \quad |\zeta - z - h| > \frac{\epsilon}{2},$$

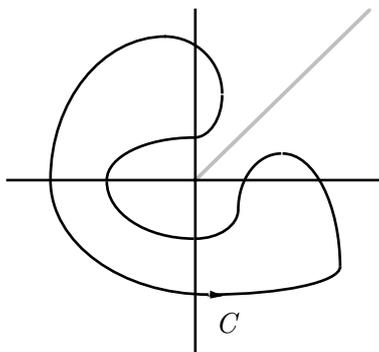
and so it follows from Theorem 4B that

$$|I(z+h) - I(z)| \leq \frac{2L|h|}{\epsilon^2},$$

where L is the length of C . This clearly tends to 0 as $h \rightarrow 0$.

The final example in this chapter exhibits the possibility of defining a continuous logarithm.

EXAMPLE 5.4.2. Consider the domain obtained by deleting from \mathbb{C} the origin 0 and a half-line starting from 0. This is a star domain in which the function $1/z$ has a continuous derivative. Suppose that C is a closed contour that does not meet this half-line.



Then

$$\int_C \frac{d\zeta}{\zeta} = 0.$$

Furthermore, the integral

$$\int_{z_0}^z \frac{d\zeta}{\zeta}$$

is independent of the path joining z_0 to z in this domain, and can therefore be used to define a continuous logarithm.

PROBLEMS FOR CHAPTER 5

1. Give an example to show that the conclusion of Theorem 5D may not hold if D is not a star domain.
2. Suppose that $R > 0$ is fixed. By integrating the function $(R-z)^{-1}$ over the circle $C = \{z : |z| = r\}$, where $0 < r < R$, and referring to Example 5.2.7, show that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R \cos \theta}{R^2 - 2Rr \cos \theta + r^2} d\theta = \frac{r}{R^2 - r^2}.$$

3. a) Suppose that C is the rectangle with vertices at $\pm b$ and $\pm b + ia$, where $a, b > 0$. Explain why

$$\int_C e^{-z^2} dz = 0.$$

- b) Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where C_1, C_2, C_3, C_4 represent the four edges of C followed in the positive (anticlockwise) direction, with initial point $z = -b$. Show that

$$\left| \int_{C_2} e^{-z^2} dz \right| \leq e^{-b^2} \int_0^a e^{y^2} dy \quad \text{and} \quad \left| \int_{C_4} e^{-z^2} dz \right| \leq e^{-b^2} \int_0^a e^{y^2} dy.$$

- c) Explain why

$$\int_{-b}^b e^{-(x+ia)^2} dx - \int_{-b}^b e^{-x^2} dx \rightarrow 0 \quad \text{as } b \rightarrow \infty.$$

Deduce that the integral

$$\int_{-\infty}^{\infty} e^{-(x+ia)^2} dx$$

is independent of the choice of $a > 0$.

4. Suppose that a function $f(z)$ is analytic in $\{z : |z| < R\}$ and continuous in $\{z : |z| \leq R\}$, where $R > 0$ is fixed. Suppose further that C denotes the circle $\{z : |z| = R\}$.

- a) Suppose that $r < R$. Explain why

$$\int_C f(z) dz = \int_0^{2\pi} f(Re^{i\theta}) Re^{i\theta} i d\theta - \int_0^{2\pi} f(re^{i\theta}) re^{i\theta} i d\theta.$$

- b) The function $f(z)z$ is continuous in $\{z : |z| \leq R\}$, and so uniformly continuous in $\{z : |z| \leq R\}$. This implies that given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(Re^{i\theta})Re^{i\theta} - f(re^{i\theta})re^{i\theta}| < \epsilon$ whenever $R - \delta < r < R$. Use this to show that

$$\left| \int_C f(z) dz \right| < 2\pi\epsilon.$$

- c) Explain why it follows that

$$\int_C f(z) dz = 0.$$

- d) Explain also why this result does not follow directly from Theorem 5D.

5. Suppose that a function $f(z)$ is continuous on a closed contour C . Suppose further that $f(z)$ can be uniformly approximated with arbitrary precision by a polynomial; in other words, given any $\epsilon > 0$, there exists a polynomial $P(z)$ such that $|f(z) - P(z)| < \epsilon$ for every $z \in C$. Prove that

$$\int_C f(z) dz = 0.$$

6. Suppose that a function $f(z)$ is analytic in $\{z : |z| \leq 1\}$. By considering a suitable integral over the unit circle $\{z : |z| = 1\}$, show that

$$\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| \geq 1.$$

7. Suppose that C is a closed contour, and that D is a domain not containing any point of C . By noting Examples 4.4.2 and 5.4.1, show that the integral

$$n(C, z_0) = \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0}$$

is independent of the choice of $z_0 \in D$.

[REMARK: The value $n(C, z_0)$ is called the winding number of the contour C round the point z_0 , and measures the number of times the contour winds round the point z_0 .]

8. Suppose that C is a contour $z = r(\theta)e^{i\theta}$ for $\theta \in [0, 2\pi]$, where $r(\theta) > 0$ for every $\theta \in [0, 2\pi]$. Suppose further that $r(0) = r(2\pi)$, so that C is a closed contour. Let D be the domain containing the origin $z = 0$ and with boundary C .
- Show that D is a star domain with the origin $z = 0$ as a star centre.
 - Suppose that $z_0 \notin D \cup C$. Explain why the half line $L = \{\lambda z_0 : \lambda \in [1, \infty)\}$ satisfies $L \cap C = \emptyset$. Show also that $\mathbb{C} \setminus L$ is a star domain with star centre $z = 0$.
 - Explain why

$$\frac{1}{2\pi i} \int_C \frac{dz}{z - z_0} = \begin{cases} 0 & \text{if } z_0 \notin D \cup C, \\ 1 & \text{if } z_0 \in D. \end{cases}$$

[HINT: For the case $z_0 \in D$, refer to Problem 7 if necessary.]

- d) Suppose that $P(z)$ is a polynomial with no roots on the contour C . By referring to Problem 1 in Chapter 3 if necessary, show that the number of roots of $P(z)$ in D is given by

$$\frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)} dz.$$

9. Suppose that $P(z)$ is a polynomial of degree k and with distinct roots z_1, \dots, z_k . Suppose further that C is a closed contour which does not contain any of these roots. By referring to Problem 7 if necessary, show that

$$\frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)} dz = n(C, z_1) + \dots + n(C, z_k).$$

10. Suppose that two star domains D_1 and D_2 both have the point z_0 as star centre. Show that $D_1 \cap D_2$ and $D_1 \cup D_2$ are both star domains with star centre z_0 .

INTRODUCTION TO COMPLEX ANALYSIS

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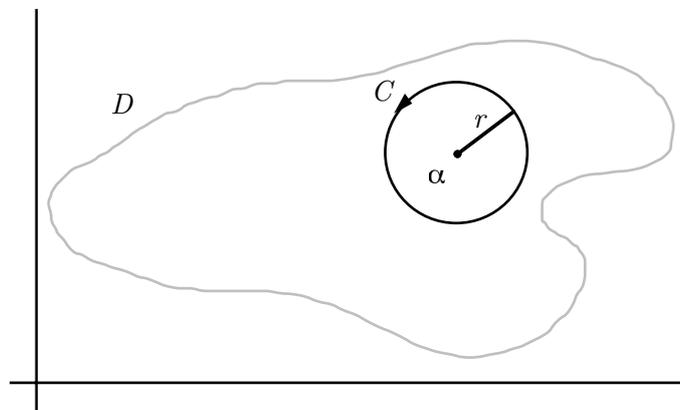
Chapter 6

CAUCHY'S INTEGRAL FORMULA

6.1. Introduction

In this chapter, we study a remarkable formula due to Cauchy and which shows that the values of an analytic function at the interior points of a disc are determined by the values of the function on the boundary of the disc.

THEOREM 6A. *Suppose that a function f is analytic in a domain D . Suppose further that the closed disc $\{z : |z - \alpha| \leq r\}$ is contained in D , and that C denotes the circle $\{z : |z - \alpha| = r\}$ followed in the positive (anticlockwise) direction.*

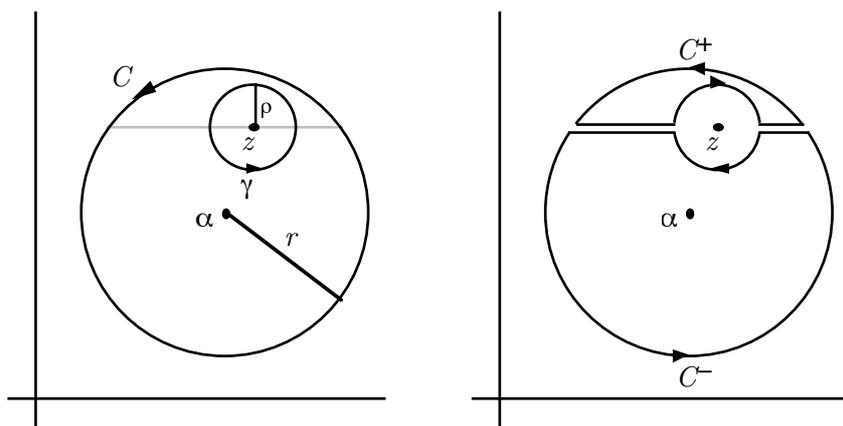


Then for every $z \in D$ satisfying $|z - \alpha| < r$, we have

$$(1) \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

REMARK. Theorem 6A is a special case of Cauchy's integral formula in a simply connected domain which we shall study in Chapter 9.

PROOF OF THEOREM 6A. Suppose that γ is a circle of radius ρ and centred at z , followed in the positive direction. Suppose further that ρ is sufficiently small so that γ lies in the interior of C . Note that a horizontal line through the point z intersects C at two points and intersects γ at two points and gives rise to two line segments inside C and outside γ .



The part of the two circles above this line and the two line segments give rise to a simple closed contour C^+ followed in the positive direction and which can be shown to lie in a star domain lying in D but not containing z , so that

$$\int_{C^+} \frac{f(\zeta)}{\zeta - z} d\zeta = 0,$$

in view of Theorem 5D. Similarly, the part of the two circles below this line and the two line segments give rise to a simple closed contour C^- followed in the positive direction and which again can be shown to lie in a star domain lying in D but not containing z , so that

$$\int_{C^-} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

It is easily seen that

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta - \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C^+} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{C^-} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

so that

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We can write

$$(2) \quad \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \int_\gamma \frac{d\zeta}{\zeta - z} + \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta.$$

The first integral on the right hand side of (2) is studied in a similar way as in Example 4.2.2, and we have

$$(3) \quad \int_{\gamma} \frac{d\zeta}{\zeta - z} = 2\pi i.$$

On the other hand, note that f is continuous at z . It follows that given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. If we choose ρ so that $\rho < \delta$, then

$$\left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \leq \frac{\epsilon}{\rho}$$

for every $\zeta \in \gamma$, and so it follows from Theorem 4B that

$$(4) \quad \left| \int_{\gamma} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq 2\pi\epsilon.$$

Combining (2)–(4), we obtain

$$\left| \int_C \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi i f(z) \right| \leq 2\pi\epsilon.$$

The result follows immediately, since ϵ is arbitrary. \circ

6.2. Derivatives

An important consequence of Cauchy's integral formula is that we can show that an analytic function possesses derivatives of all orders. This is a rather remarkable result, and much nicer than in real analysis. We shall establish this result by a number of steps.

THEOREM 6B. *Suppose that a function f is analytic in a domain D . Then the derivative f' is analytic in D .*

REMARK. Recall that in our first proof of Theorem 5A, we use the additional assumption that f' is continuous in D . In view of Theorem 6B, it appears that this extra assumption is superfluous. However, our proof of Theorem 6B below will depend on Theorem 6A, whose proof uses Theorem 5D, which follows somewhat from Theorem 5A. Hence we cannot reasonably use our first proof of Theorem 5A without running into the danger of a “circular argument” of deducing two results from each other and possibly establishing neither. Note, however, that our second proof of Theorem 5A in Section 5.3 saves us from this dubious distinction.

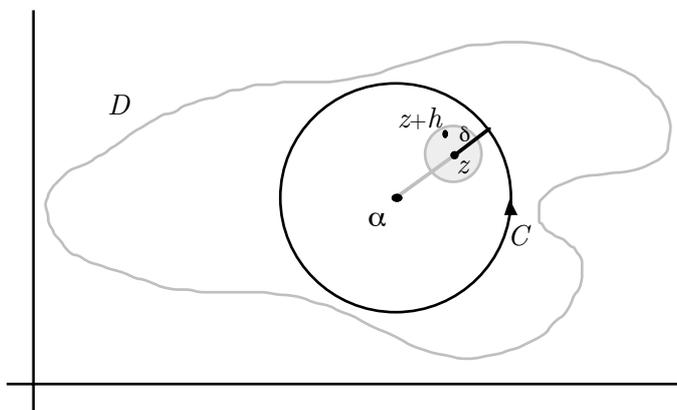
THEOREM 6C. *Suppose that a function f is analytic in a domain D . Then the derivative $f^{(n)}$ exists for every $n \in \mathbb{N}$ and is analytic in D .*

THEOREM 6D. *Suppose that a function f is analytic in a domain D . Then, in the notation of Theorem 6A, we have*

$$(5) \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for every $n \in \mathbb{N}$.

PROOF OF THEOREM 6B. Note that every $z \in D$ is contained inside some circle C with α as centre, and so (1) is valid. Suppose that $h \in \mathbb{C}$ has sufficiently small modulus so that $z + h$, as well as z , lies inside the circle C .



Then by (1), we have

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{1}{2\pi i h} \int_C \left(\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z} \right) d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta + \frac{h}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)} d\zeta. \end{aligned}$$

Since z is inside the circle C , the number

$$\delta = \min_{\zeta \in C} |\zeta - z| > 0.$$

If $|h| < \delta/2$, then for every $\zeta \in C$, we have

$$|\zeta - z - h| \geq |\zeta - z| - |h| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

On the other hand, the circle C is a closed and bounded set. It follows that there exists some real constant M such that $|f(\zeta)| \leq M$ for every $\zeta \in C$. Recall that the circle C has radius r . It now follows from Theorem 4B that

$$\left| \frac{h}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)} d\zeta \right| \leq \frac{|h|}{2\pi} \frac{2M}{\delta^3} 2\pi r = \frac{2Mr|h|}{\delta^3} \rightarrow 0$$

as $h \rightarrow 0$. This establishes the existence of f' in D and (5) for $n = 1$. We now repeat the argument, starting with (5) with $n = 1$. This establishes the existence of f'' in D , and so the analyticity of f' in D . \circ

PROOF OF THEOREM 6C. Suppose that $f^{(n)}$ is analytic in D . Applying Theorem 6B to the function $f^{(n)}$, we conclude that $f^{(n+1)}$ is analytic in D . The conclusion now follows by induction. \circ

PROOF OF THEOREM 6D. It follows from Theorem 6C that $f^{(n)}$ is analytic in D . Applying Theorem 6A to the function $f^{(n)}$, we have

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_C \frac{f^{(n)}(\zeta)}{\zeta - z} d\zeta.$$

Integrating this by parts n times gives (5). \circ

6.3. Further Consequences

THEOREM 6E. (CAUCHY'S ESTIMATE) *Suppose that a function f is analytic in a domain D . Suppose further that the closed disc $\{z : |z - \alpha| \leq r\}$ is contained in D , and that there exists a positive constant M such that $|f(z)| \leq M$ in this disc. Then*

$$|f^{(n)}(\alpha)| \leq \frac{n!M}{r^n}.$$

PROOF. It follows from Theorem 6D that

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - \alpha)^{n+1}} d\zeta.$$

Note now that for every $\zeta \in C$, we have $|\zeta - \alpha| = r$, so that

$$\left| \frac{f(\zeta)}{(\zeta - \alpha)^{n+1}} \right| \leq \frac{M}{r^{n+1}}.$$

It now follows from Theorem 4B that

$$|f^{(n)}(\alpha)| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}$$

as required. \circ

THEOREM 6F. (LIOUVILLE'S THEOREM) *An entire and non-constant function f cannot be bounded in \mathbb{C} .*

PROOF. Suppose on the contrary that f is bounded. Then there exists a positive constant M such that $|f(z)| \leq M$ for every $z \in \mathbb{C}$. For every $\alpha \in \mathbb{C}$, Cauchy's estimate for $n = 1$ gives

$$|f'(\alpha)| \leq \frac{M}{r}.$$

This inequality is valid for every $r > 0$ since the closed disc $\{z : |z - \alpha| \leq r\}$ is clearly contained in \mathbb{C} . It follows that we must have $f'(\alpha) = 0$ for every $\alpha \in \mathbb{C}$. Hence it follows from Theorem 4A that

$$f(z) - f(0) = \int_0^z f'(\zeta) d\zeta = 0,$$

so that f is constant, a contradiction. \circ

THEOREM 6G. (MORERA'S THEOREM) *Suppose that f is continuous in a domain D . Suppose further that*

$$\int_C f(z) dz = 0$$

holds for every closed triangular contour C which together with its interior lies in D . Then f is analytic in D .

PROOF. Suppose that $z \in D$. Then there exists an ϵ -neighbourhood D_z of z lying entirely in D . Clearly D_z is a star domain. It follows from Theorem 5C that there exists a function F , analytic in D_z and such that $F' = f$ in D_z . By Theorem 6B, f is analytic in D_z , and so analytic at z . Since $z \in D$ is arbitrary, the result follows immediately. \circ

EXAMPLE 6.3.1. We can now prove the Fundamental theorem of algebra, that every non-constant polynomial $P(z)$ has at least one root. It is easily checked that every such non-constant polynomial satisfies $|P(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$. Hence the function $1/P(z)$ is bounded outside some circle $\{z : |z| = r\}$. Suppose that $P(z)$ does not vanish. Then $1/P(z)$ is an entire function. Hence it is continuous and so bounded in the closed set $\{z : |z| \leq r\}$. It follows that it is bounded in \mathbb{C} . By Liouville's theorem, it must be constant, a contradiction.

PROBLEMS FOR CHAPTER 6

- Suppose that a function $f = u + iv$ is analytic in a region D . Show that all partial derivatives of u and v are continuous in D . Show also that uv is harmonic in D .
- Suppose that $f(z)$ is an entire function. Suppose further that there exist $M \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $|f(z)| \leq M|z|^m$ whenever $|z|$ is large.
 - Use Cauchy's estimate to show that $f^{(n)}(0) = 0$ for every integer $n > m$.
 - Deduce that $f(z)$ is a polynomial of degree at most m .
- Suppose that a function $f(z)$ is analytic in the closed disc $\{z : |z| \leq R\}$, where $R > 0$ is fixed.
 - Prove Gauss's mean value theorem, that

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) d\theta.$$

- Prove also that for every $n \in \mathbb{N} \cup \{0\}$, we have

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi R^n} \int_0^{2\pi} |f(Re^{i\theta})| d\theta.$$

- Suppose that L is the length of the image of the circle $\{z : |z| = R\}$ under f , so that

$$L = \int_C |f'(z)| |dz| = R \int_0^{2\pi} |f'(Re^{i\theta})| d\theta.$$

Show that $L \geq 2\pi R |f'(0)|$.

- By first expressing the integral in polar coordinates, show that

$$\int_{\{z:|z|\leq R\}} f(x+iy) dx dy = \pi R^2 f(0).$$

- Suppose that $f(z)$ is an entire function. Suppose further that there exists $M \in \mathbb{R}$ such that $\Re f(z) \leq M$ for every $z \in \mathbb{C}$. By applying Liouville's theorem to the function $e^{f(z)}$, show that $f(z)$ is constant.
- Suppose that $f(z)$ is an entire function, and that $g(R) \rightarrow 0$ as $R \rightarrow \infty$. Suppose further that for all large R , the inequality $|f(z)| \leq Rg(R)$ holds whenever $|z| = R$. By proceeding along the lines of the proof of Liouville's theorem, show that $f(z)$ is constant.
- Suppose that a function $f(\zeta)$ is continuous on a contour C . Show that the function

$$F(z) = \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{satisfies} \quad F'(z) = \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

for every $z \notin C$, so that $F(z)$ is analytic off the contour C .

- Suppose that a function $f(\zeta)$ is continuous on a contour C . Consider again the function $F(z)$ in Problem 6. Use Morera's theorem to show that $F(z)$ is analytic off the contour C .

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 7

TAYLOR SERIES, UNIQUENESS AND THE MAXIMUM PRINCIPLE

7.1. Remarks on Series

The purpose of this chapter is to show that every analytic function can be represented by a Taylor series, and to use the Taylor series to study further properties of such functions.

Here we do not propose to have a systematic study of series. Such a study is postponed until Chapter 16. In this section, we shall make a brief review of standard terminology.

We are concerned with power series of the form

$$(1) \quad a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots,$$

where $a_0, a_1, a_2, \dots \in \mathbb{C}$ and $z_0 \in \mathbb{C}$ are fixed, and where z belongs to some region in the complex plane \mathbb{C} . For every $N \in \mathbb{N}$, the N -th partial sum of the series (1) is defined by

$$(2) \quad s_N(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots + a_N(z - z_0)^N.$$

We are interested in the sequence $s_N(z)$ of partial sums.

Suppose that a function $f(z)$ and a sequence of functions $s_N(z)$ are defined in a region G in the complex plane \mathbb{C} . The sequence $s_N(z)$ is said to converge uniformly to $f(z)$ in G if, given any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon)$ such that

$$|s_N(z) - f(z)| < \epsilon$$

for every $N > N_0$ and every $z \in G$. Note here that the notion of uniformity implies the independence of N_0 from the choice of z in G . We also write

$$f(z) = \lim_{N \rightarrow \infty} s_N(z)$$

uniformly in G .

7.2. Taylor Series

In particular, if $s_N(z)$ is given by (2), then we write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

uniformly in G .

THEOREM 7A. (TAYLOR'S THEOREM) *Suppose that a function f is analytic in the domain $\{z : |z - z_0| < R\}$, where $z_0 \in \mathbb{C}$ and $R > 0$ are fixed. Suppose further that $0 \leq r < R$. Then*

$$(3) \quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

uniformly in $\{z : |z - z_0| \leq r\}$.

DEFINITION. The series (3) is called the Taylor series of the function f at z_0 .

Theorem 7A follows easily from the following special case.

THEOREM 7B. *Suppose that a function g is analytic in the domain $\{z : |z| < R\}$, where $R > 0$ is fixed. Suppose further that $0 \leq r < R$. Then*

$$(4) \quad g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$$

uniformly in $\{z : |z| \leq r\}$.

We shall show that Theorem 7A follows from Theorem 7B. Suppose that $|z - z_0| < R$. If we write $\zeta = z - z_0$, then $|\zeta| < R$. We now use the substitution $f(z) = g(\zeta)$. Suppose that f is analytic in the region $\{z : |z - z_0| < R\}$. Then clearly g is analytic in the region $\{\zeta : |\zeta| < R\}$. It then follows from Theorem 7B that

$$f(z) = g(\zeta) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \zeta^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

uniformly in $\{\zeta : |\zeta| \leq r\}$, and so uniformly in $\{z : |z - z_0| \leq r\}$. It remains to prove Theorem 7B.

PROOF OF THEOREM 7B. Let the real number ρ be chosen to satisfy $r < \rho < R$, and let C denote the circle $\{\zeta : |\zeta| = \rho\}$, followed in the positive (anticlockwise) direction. By Theorem 6A, we have

$$g(z) = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta - z} d\zeta$$

for every z satisfying $|z| \leq r$. Since (see Remark (1) below)

$$(5) \quad \frac{1}{\zeta - z} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \frac{z^2}{\zeta^3} + \dots + \frac{z^{n-1}}{\zeta^n} + \frac{z^n}{\zeta^n} \frac{1}{\zeta - z},$$

we have

$$(6) \quad g(z) = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta} d\zeta + \dots + \frac{z^{n-1}}{2\pi i} \int_C \frac{g(\zeta)}{\zeta^n} d\zeta + z^n g_n(z),$$

where

$$(7) \quad g_n(z) = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta^n(\zeta - z)} d\zeta.$$

By Theorems 6A and 6D, we have

$$(8) \quad \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta^{k+1}} d\zeta = \frac{g^{(k)}(0)}{k!}$$

for every $k = 0, \dots, n-1$, so that (6) becomes

$$(9) \quad g(z) = g(0) + g'(0)z + \dots + \frac{g^{(n-1)}(0)}{(n-1)!} z^{n-1} + z^n g_n(z).$$

To complete the proof of Theorem 7B, it suffices to show that

$$|z^n g_n(z)| \rightarrow 0$$

uniformly in $\{z : |z| \leq r\}$ as $n \rightarrow \infty$. Clearly, the circle $C = \{\zeta : |\zeta| = \rho\}$ is closed and bounded, and the function g is continuous on C . It follows that there exists a positive real constant M such that $|g(\zeta)| \leq M$ for every $\zeta \in C$. Hence for every $\zeta \in C$ and every $|z| \leq r$, we have

$$\left| \frac{g(\zeta)}{\zeta^n(\zeta - z)} \right| \leq \frac{M}{\rho^n(\rho - r)}.$$

It follows from Theorem 4B that

$$\left| \int_C \frac{g(\zeta)}{\zeta^n(\zeta - z)} d\zeta \right| \leq \frac{M}{\rho^n(\rho - r)} 2\pi\rho,$$

and so

$$|z^n g_n(z)| \leq \frac{r^n}{2\pi} \frac{M}{\rho^n(\rho - r)} 2\pi\rho = \frac{M\rho}{\rho - r} \left(\frac{r}{\rho}\right)^n.$$

Since $r < \rho$, the right hand side clearly converges to 0 as $n \rightarrow \infty$ independently of the choice of z in the set $\{z : |z| \leq r\}$. \circ

REMARKS. (1) To derive the identity (5), note that if $w \neq 1$, then the identity

$$\frac{1}{1-w} = 1 + w + w^2 + w^3 + \dots + w^{n-1} + \frac{w^n}{1-w}$$

is easily verified. Now substitute $w = z/\zeta$ and then divide by ζ to obtain (5).

(2) It is easily seen that the function $g_n(z)$ in (7) is analytic in the domain $\{z : |z| < R\}$. If $z \neq 0$, then the analyticity follows immediately from (9). To show that $g_n(z)$ is analytic at 0, note that the

function $g(\zeta)/\zeta^n$ is continuous on the circle C , and the result follows from Problem 6 in Chapter 6. Note also that (7) and (8) for $k = n$ give

$$g_n(0) = \frac{g^{(n)}(0)}{n!}.$$

These observations, combined with (9), immediately lead to the following finite version of Taylor's theorem.

THEOREM 7C. *Under the hypotheses of Theorem 7A, we have*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + f_n(z)(z - z_0)^n,$$

where $f_n(z)$ is analytic in the domain $\{z : |z - z_0| < R\}$ and

$$f_n(z_0) = \frac{f^{(n)}(z_0)}{n!}.$$

A very good reason for studying Taylor series is their polynomial-like behaviour, although great care needs to be exercised. An example is the following result.

THEOREM 7D. *Suppose that a function f is analytic in the domain $\{z : |z - z_0| < R\}$, where $z_0 \in \mathbb{C}$ and $R > 0$ are fixed. Then the series obtained through term-by-term differentiation of the Taylor series (3) of $f(z)$ converges uniformly to $f'(z)$ in any closed disc $\{z : |z - z_0| \leq r\}$, where $r < R$. Furthermore, the differentiated series is the Taylor series of $f'(z)$.*

PROOF. Since f is analytic in the domain $\{z : |z - z_0| < R\}$, it follows from Theorem 6B that f' is also analytic in $\{z : |z - z_0| < R\}$. By Theorem 7A, the function f' has its Taylor series

$$f'(z) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(z_0)}{n!}(z - z_0)^n = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}n(z - z_0)^{n-1},$$

the same series as obtained through term-by-term differentiation of the Taylor series of f . The uniform convergence of this series to $f'(z)$ in $\{z : |z - z_0| \leq r\}$ follows from Theorem 7A. \circ

EXAMPLE 7.2.1. The function $g(z) = e^z$ is entire. Also, for every $n \in \mathbb{N}$, we have $g^{(n)}(z) = e^z$, so that $g^{(n)}(0) = 1$. It now follows from Theorem 7B that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

uniformly in $\{z : |z| \leq r\}$ for every $r > 0$.

EXAMPLE 7.2.2. The function $g(z) = \sin z$ is entire. Also, for every $n \in \mathbb{N}$, we have

$$g^{(n)}(z) = \begin{cases} \cos z & \text{if } n = 1, 5, 9, \dots, \\ -\sin z & \text{if } n = 2, 6, 10, \dots, \\ -\cos z & \text{if } n = 3, 7, 11, \dots, \\ \sin z & \text{if } n = 4, 8, 12, \dots, \end{cases}$$

so that

$$g^{(n)}(0) = \begin{cases} 1 & \text{if } n = 1, 5, 9, \dots, \\ 0 & \text{if } n = 2, 6, 10, \dots, \\ -1 & \text{if } n = 3, 7, 11, \dots, \\ 0 & \text{if } n = 4, 8, 12, \dots \end{cases}$$

It now follows from Theorem 7B that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

uniformly in $\{z : |z| \leq r\}$ for every $r > 0$. Applying Theorem 7D and differentiating term-by-term, we obtain

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

uniformly in $\{z : |z| \leq r\}$ for every $r > 0$.

EXAMPLE 7.2.3. Consider the function

$$h(z) = \frac{\sin z}{z}.$$

From Example 7.2.2 and Theorem 7C, we have

$$\sin z = z + z^2 g_2(z),$$

where the function $g_2(z)$ is entire. It follows that for $z \neq 0$, we have

$$\frac{\sin z}{z} = 1 + z g_2(z),$$

and so $h(z) \rightarrow 1$ as $z \rightarrow 0$. Note that the function $h(z)$ is analytic at any $z \neq 0$. If we define $h(0) = 1$, then the function h is continuous at 0. We say that h has a removable singularity at 0.

EXAMPLE 7.2.4. Consider the function

$$k(z) = \frac{1 - \cos z}{z^2}.$$

From Example 7.2.2 and Theorem 7C, we have

$$\cos z = 1 - \frac{z^2}{2} + z^3 g_3(z),$$

where the function $g_3(z)$ is entire. It follows that for $z \neq 0$, we have

$$\frac{1 - \cos z}{z^2} = \frac{1}{2} - z g_3(z),$$

and so $k(z) \rightarrow 1/2$ as $z \rightarrow 0$. Note that the function $k(z)$ is analytic at any $z \neq 0$. If we define $k(0) = 1/2$, then the function k is continuous at 0.

7.3. Uniqueness

Recall the Cauchy integral formula as given by Theorem 6A. To determine the value of an analytic function at interior points of a disc, we need the values of the function on the boundary of the disc.

On the other hand, if we know the values of $f(z_0), f'(z_0), f''(z_0), \dots$ of an analytic function f at a point z_0 in a domain D , then the Taylor series determines $f(z)$ in some disc $\{z : |z - z_0| < R\}$ centred at z_0 . It follows that if $f(z)$ is known in some infinitely differentiable short arc in the disc $\{z : |z - z_0| < R\}$,

then $f(z)$ is uniquely determined in the disc $\{z : |z - z_0| < R\}$, since the derivatives of $f(z)$ can be calculated by differentiation of $f(z)$ on this arc.

The purpose of this section is to extend this second observation to show that an analytic function f is uniquely determined in a domain D , and not just a disc centred at z_0 , by the values of $f^{(n)}(z_0)$ at some z_0 in D . We shall also show that an analytic function f is determined in a domain D by the values on a short continuous curve C in D .

THEOREM 7E. *Suppose that two functions f and g are analytic in a domain D . Suppose further that $z_0 \in D$, and that $f(z) = g(z)$ in some ϵ -neighbourhood of z_0 . Then $f(z) = g(z)$ for every $z \in D$.*

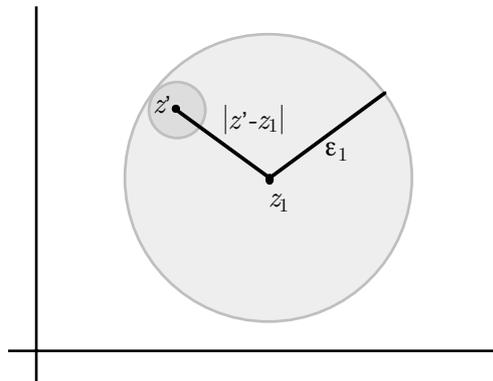
REMARK. The proof is based on the following argument: A domain D is open and connected, and therefore cannot be written as a disjoint union of two non-empty open sets.

PROOF OF THEOREM 7E. For every $z \in D$, write $h(z) = f(z) - g(z)$. Then the function h is analytic in D . Let

$$S_1 = \{z_1 \in D : h(z) = 0 \text{ in some neighbourhood of } z_1\} \quad \text{and} \quad S_2 = D \setminus S_1.$$

To prove the theorem, it clearly suffices to show that $S_1 = D$.

Suppose that $z_1 \in S_1$. Then there exists $\epsilon_1 > 0$ such that $h(z) = 0$ in $\{z : |z - z_1| < \epsilon_1\}$. Suppose now that $|z' - z_1| < \epsilon_1$.



Then clearly

$$\{z : |z - z'| < \epsilon_1 - |z' - z_1|\} \subseteq \{z : |z - z_1| < \epsilon_1\},$$

and so it follows that $h(z) = 0$ in the neighbourhood $\{z : |z - z'| < \epsilon_1 - |z' - z_1|\}$ of z' . This shows that $z' \in S_1$ whenever $|z' - z_1| < \epsilon_1$. It follows that $\{z : |z - z_1| < \epsilon_1\} \subseteq S_1$, so that S_1 is open.

Suppose next that $z_2 \in S_2$. Since $z_2 \in D$, there exists $R > 0$ such that the disc $\{z : |z - z_2| < R\}$ is contained in D and so it follows from Theorem 7A that the Taylor series expansion

$$h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(z_2)}{n!} (z - z_2)^n$$

is valid in the disc $\{z : |z - z_2| \leq r\}$ for every $r < R$. Since h is not identically zero in this latter disc, there exists a smallest n such that $h^{(n)}(z_2) \neq 0$. By Theorem 7C, we can then write

$$h(z) = h_n(z)(z - z_2)^n,$$

where $h_n(z)$ is analytic in the disc $\{z : |z - z_2| < R\}$ and

$$h_n(z) \rightarrow \frac{h^{(n)}(z_2)}{n!} \neq 0$$

as $z \rightarrow z_2$. It follows from continuity of h_n that there exists $\epsilon_2 > 0$ such that $h(z) \neq 0$ in the punctured disc $\{z : 0 < |z - z_2| < \epsilon_2\}$, and so $\{z : |z - z_2| < \epsilon_2\} \subseteq S_2$. Hence S_2 is open.

Clearly $S_1 \neq \emptyset$, since $z_0 \in S_1$. Suppose, on the contrary, that $S_1 \neq D$. Then the two open sets S_1 and S_2 are both non-empty. Clearly

$$S_1 \cup S_2 = D \quad \text{and} \quad S_1 \cap S_2 = \emptyset.$$

In view of our earlier remark, this is absurd. Hence we must have $S_2 = \emptyset$, and so $S_1 = D$. \circ

In fact, a slight elaboration of the ideas in part of the above proof gives the following two results.

THEOREM 7F. *Suppose that a function f is analytic in the domain D . Suppose further that $z_0 \in D$, and that $f(z_0) = 0$. Then either $f(z)$ is identically zero in D or else there exists $n \in \mathbb{N}$ such that*

$$f(z) = (z - z_0)^n g(z),$$

where the function g is analytic in D , and

$$g(z_0) = \frac{f^{(n)}(z_0)}{n!} \neq 0.$$

DEFINITION. If the latter conclusion of Theorem 7F holds, then we say that the function has a zero of order n at z_0 . Furthermore, if $n = 1$, then we say that the function f has a simple zero at z_0 .

THEOREM 7G. *Suppose that a function f is analytic in the domain D . Suppose further that $f(z)$ is not identically zero in D . Then for every $z_0 \in D$ such that $f(z_0) = 0$, there exists $\epsilon > 0$ such that $f(z) \neq 0$ for every $0 < |z - z_0| < \epsilon$. In other words, the zeros of f are isolated.*

PROOF OF THEOREM 7F. Clearly there exists $R > 0$ such that the disc $\{z : |z - z_0| < R\}$ is contained in D . Suppose that $f^{(n)}(z_0) = 0$ for every $n \in \mathbb{N}$. Then it follows from Theorem 7A that $f(z)$ is identically zero in this disc. Let the function g be identically zero in D . Then $f(z) = g(z)$ in some neighbourhood of z_0 . It now follows from Theorem 7E that $f(z) = 0$ for every $z \in D$. Suppose next that $f^{(k)}(z_0) \neq 0$ for some $k \in \mathbb{N}$. Then there exists a smallest $k \in \mathbb{N}$ such that $f^{(k)}(z_0) \neq 0$. Denote this value of k by n . By Theorem 7C, we can then write

$$f(z) = f_n(z)(z - z_0)^n,$$

where $f_n(z)$ is analytic in the disc $\{z : |z - z_0| < R\}$ and

$$f_n(z) \rightarrow \frac{f^{(n)}(z_0)}{n!} \neq 0$$

as $z \rightarrow z_0$. We now define $g(z) = f_n(z)$ in this disc, and by $g(z) = f(z)(z - z_0)^{-n}$ in the remainder of D to complete the proof. \circ

PROOF OF THEOREM 7G. It follows from Theorem 7F that there exists $n \in \mathbb{N}$ and an analytic function g in D such that $f(z) = (z - z_0)^n g(z)$, where $g(z_0) \neq 0$. It follows from the continuity of g that there exists $\epsilon > 0$ such that $g(z) \neq 0$ if $|z - z_0| < \epsilon$. The result follows immediately. \circ

EXAMPLE 7.3.1. Suppose that two functions f and g are analytic in a domain D and not identically zero in D . Suppose further that $z_0 \in D$, and that $f(z_0) = g(z_0) = 0$. Then by Theorem 7F, there exist $m, n \in \mathbb{N}$ and functions F and G analytic in D and satisfying $F(z_0) \neq 0$ and $G(z_0) \neq 0$ such that

$$f(z) = (z - z_0)^m F(z) \quad \text{and} \quad g(z) = (z - z_0)^n G(z).$$

Then

$$\frac{f(z)}{g(z)} = (z - z_0)^k \frac{F(z)}{G(z)} \quad \text{and} \quad \frac{f'(z)}{g'(z)} = (z - z_0)^k \frac{mF(z) + (z - z_0)F'(z)}{nG(z) + (z - z_0)G'(z)},$$

where $k = m - n$. Consider now the special case $m = n = 1$, so that $k = 0$. We have

$$\lim_{z \rightarrow z_0} \frac{f'(z)}{g'(z)} = \lim_{z \rightarrow z_0} \frac{F(z) + (z - z_0)F'(z)}{G(z) + (z - z_0)G'(z)} = \lim_{z \rightarrow z_0} \frac{F(z)}{G(z)} = \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)}.$$

This is l'Hopital's rule.

We complete this section by proving the following result, which shows that an analytic function is determined in a domain D by the values on a short continuous curve in D .

THEOREM 7H. *Suppose that a function f is analytic in the domain D . Suppose further that z_n is a sequence of distinct points having a limit $z_0 \in D$, and that $f(z_n) = g(z_n)$ for every $n \in \mathbb{N}$. Then $f(z) = g(z)$ for every $z \in D$.*

PROOF. For every $z \in D$, write $h(z) = f(z) - g(z)$. Then the function h is analytic in D . Furthermore, it follows from continuity that $h(z_0) = 0$. Since $h(z_n) = 0$ for every $n \in \mathbb{N}$ and $z_n \rightarrow z_0$ as $n \rightarrow \infty$, the zero z_0 of the function h is not isolated. It follows from Theorem 7G that $h(z)$ is identically zero in D . \circ

7.4. The Maximum Principle

Let us return to Cauchy's integral formula, as given by Theorem 6A. If we take z to be the centre α of the circle C , then (1) in Chapter 6 gives

$$(10) \quad f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \alpha} d\zeta,$$

so that

$$(11) \quad |f(\alpha)| \leq \max_{\zeta \in C} |f(\zeta)|,$$

in view of Theorem 4B. In other words, the modulus of an analytic function at a point in a domain never exceeds the maximum modulus of the function on the boundary of any disc centred at that point and contained in the domain.

In this section, we shall establish the following stronger result.

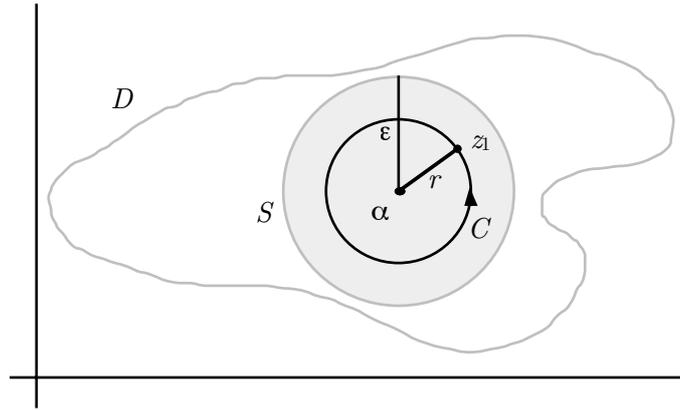
THEOREM 7J. (MAXIMUM PRINCIPLE) *Suppose that a function f is analytic in a domain D . Then $|f(z)|$ cannot have a maximum anywhere in D unless $f(z)$ is constant in D .*

PROOF. Suppose on the contrary that there exists $\alpha \in D$ such that

$$(12) \quad |f(\alpha)| \geq |f(z)|$$

for every $z \in D$. Since D is open, there exists an ϵ -neighbourhood S of α which is contained in D . If $|f(z)| = |f(\alpha)|$ for every $z \in S$, then it follows from Example 3.3.5 that $f(z)$ is constant in S , and so constant in D by Theorem 7E. We may therefore assume that there exists $z_1 \in S$ such that

$$(13) \quad |f(z_1)| < |f(\alpha)|.$$



Let $|z_1 - \alpha| = r$. Clearly $r < \epsilon$. If we denote by C the circle in the positive (anticlockwise) direction centred at α and with radius r , then (10) holds. Furthermore, writing $\zeta = \alpha + re^{it}$, we have

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{it}) dt,$$

so that

$$(14) \quad |f(\alpha)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + re^{it})| dt.$$

On the other hand, it clearly follows from (12) that

$$(15) \quad |f(\alpha)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + re^{it})| dt.$$

However, note that $z_1 = \alpha + re^{it_1}$ for some $t_1 \in [0, 2\pi]$. It follows from continuity that there exists an interval $I \subseteq [0, 2\pi]$ for which $|f(\alpha + re^{it})| < |f(\alpha)|$ for every $t \in I$. It follows that equality cannot hold in (15), so that

$$(16) \quad |f(\alpha)| > \frac{1}{2\pi} \int_0^{2\pi} |f(\alpha + re^{it})| dt.$$

Note now that (14) and (16) contradict each other, and this concludes the proof of the theorem. \circ

The following alternative form of the Maximum principle is perhaps more useful.

THEOREM 7K. *Suppose that a function f is analytic in a bounded domain D , and continuous in the closed region \overline{D} . Then $|f(z)|$ attains its maximum on the boundary of D .*

PROOF. It is well known from real analysis that $|f(z)|$ assumes its maximum somewhere in the closed bounded region \overline{D} . By Theorem 7J, this maximum cannot be attained in D , and so must be attained on the boundary of D . \circ

PROBLEMS FOR CHAPTER 7

- Obtain the Taylor series for the function $(1 - z)^{-1}$ at $z = 0$. Deduce from this the Taylor series for the function $(1 - z)^{-2}$ at $z = 0$. In what open discs centred at $z = 0$ are these series valid?
- Suppose that a function $f(z)$ is analytic in the disc $\{z : |z| < R\}$, where $R > 0$ is fixed, and has Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

a) Show that

$$\int_0^z f(\zeta) d\zeta = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$$

uniformly in $\{z : |z| \leq r\}$ for every $r < R$.

- b) Denote the integral by $F(z)$. Explain why the series in (a) is the Taylor series for $F(z)$.
- Deduce from the Taylor series for $(1 - z)^{-1}$ at $z = 0$ in Problem 1 the Taylor series for the function $\log(1 - z)$ at $z = 0$, where $\log 1 = 0$. In what open discs centred at $z = 0$ is this series valid?
 - Suppose that $\alpha \in \mathbb{C}$ is fixed. By interpreting the function $(1 + z)^\alpha$ as $e^{\alpha \log(1+z)}$, with $\log 1 = 0$, show that

$$(1 + z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha - 1)}{2!} z^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} z^3 + \dots$$

uniformly in $\{z : |z| \leq r\}$ for every $r < 1$.

- Show that the function $f(z)$, defined by $f(0) = 1$ and $f(z) = z^{-1} \sin z$ when $z \neq 0$, is entire.
 - Obtain the Taylor series for $f(z)$ at $z = 0$.
 - Obtain the Taylor series for the integral $\int_0^z f(\zeta) d\zeta$ at $z = 0$.
- Suppose that $P(z)$ is a polynomial of degree at most 3. Using partial fractions if necessary, find a_0 , a_1 and a_2 such that

$$\frac{P(z)}{(z^2 + 1)(z - 1)(z - 2)} = a_0 + a_1 z + a_2 z^2 + \dots$$

valid whenever $|z| < 1$.

- Suppose that a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges uniformly to an analytic function $f(z)$ in the disc $D = \{z : |z| \leq R\}$, where $R > 0$ is fixed. For every $N \in \mathbb{N}$ and $z \in D$, let

$$s_N(z) = \sum_{n=0}^N a_n z^n.$$

Then given any $\epsilon > 0$, there exists N_0 such that $|f(z) - s_N(z)| < \epsilon$ for every $N > N_0$ and $z \in D$.

a) Show that for every $k \in \mathbb{N} \cup \{0\}$ and every $N > N_0$,

$$\left| \int_C (f(z) - S_N(z)) z^{-k-1} dz \right| \leq \frac{2\pi\epsilon}{R^k},$$

where $C = \{z : |z| = R\}$, followed in the positive (anticlockwise) direction.

b) Now let $N > k$. Show that

$$\left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}} dz - a_k \right| \leq \frac{\epsilon}{R^k}.$$

c) Deduce that $a_k = f^{(k)}(0)/k!$ for every $k \in \mathbb{N} \cup \{0\}$.

[REMARK: This shows that if a power series converges uniformly to an analytic function $f(z)$ in D , then it is the Taylor series for $f(z)$.]

8. Suppose that two functions $f(z)$ and $g(z)$ are analytic in the disc $D = \{z : |z| < R\}$, where $R > 0$ is fixed, with Taylor series

$$f(z) = a_0 + a_1z + a_2z^2 + \dots \quad \text{and} \quad g(z) = b_0 + b_1z + b_2z^2 + \dots$$

respectively.

a) Without worrying about convergence problems, multiply the two series together to obtain another power series $c_0 + c_1z + c_2z^2 + \dots$. Check that $c_n = a_0b_n + a_1b_{n-1} + \dots + a_nb_0$ for every $n \in \mathbb{N} \cup \{0\}$.

b) Suppose that $f(z)g(z)$ has a Taylor series. Show that the coefficient of the term z^n in the Taylor series is given by the value at $z = 0$ of the function

$$\frac{1}{n!} \frac{d^n}{dz^n} (f(z)g(z)).$$

c) Explain why the power series in (a) is the Taylor series for $f(z)g(z)$.

9. Suppose that two functions $f(z)$ and $g(z)$ are analytic in a bounded region D and continuous in \overline{D} . Suppose further that $f(z) = g(z)$ for every z on the boundary of D . Show from the Maximum principle that $f(z) = g(z)$ for every $z \in D$.

10. Suppose that a function $f(z)$ is analytic in a closed bounded region D . Suppose further that $f(z) \neq 0$ for any $z \in D$. Show that $|f(z)|$ assumes its minimum value on the boundary of D .

11. Using l'Hopital's rule, or otherwise, evaluate each of the following limits:

a) $\lim_{z \rightarrow \pi} \frac{\sin z}{\pi - z}$

b) $\lim_{z \rightarrow i} \frac{e^{\pi z} + 1}{z^2 + 1}$

12. Suppose that $f(z)$ is analytic in the disc $D = \{z : |z| \leq R\}$, where $R > 0$ is fixed. Suppose further that $f(0) = 0$ and $|f(z)| \leq M$ whenever $|z| = R$.

a) Explain why $f(z) = zg(z)$ for some function $g(z)$ analytic in D .

b) By applying the Maximum principle on the function $g(z)$, prove Schwarz's lemma, that

$$|f(z)| < \frac{M}{R}|z|$$

whenever $0 < |z| < R$, unless $f(z) = cz$ for some constant $c \in \mathbb{C}$.

13. Suppose that C is a contour of length L . Suppose further that a function $f(z)$ is continuous on C , and $|f(z)| \leq M$ for every $z \in C$. Show that, unless $|f(z)| = M$ for every $z \in C$, we have

$$\left| \int_C f(z) dz \right| < ML.$$

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 8

ISOLATED SINGULARITIES AND LAURENT SERIES

8.1. Removable Singularities

Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Observe that it is not necessary for f to be defined at the point z_0 . We say that the function f has an isolated singularity at z_0 . Our purpose is to show that there are only three possible ways in which $f(z)$ can behave in a punctured neighbourhood of z_0 . To illustrate the first of these, let us first consider the following examples.

EXAMPLE 8.1.1. The function

$$f(z) = \frac{\sin z}{z}$$

is analytic in the punctured disc $\{z : 0 < |z| < R\}$. However, the quotient is not defined at $z = 0$. However, note that the function $\sin z$ is entire. By Theorem 7C, we can write

$$\sin z = z + z^3 g(z),$$

where g is an entire function. It follows that for $z \neq 0$, we have

$$f(z) = \frac{\sin z}{z} = 1 + z^2 g(z).$$

Note that the function $1 + z^2 g(z)$ is entire. It follows that if we make the further definition $f(0) = 1$, then f is now analytic at $z = 0$, and we have removed the isolated singularity.

EXAMPLE 8.1.2. Suppose that a function f is analytic in a domain D , and that $z_0 \in D$. We define the function g in D by writing

$$(1) \quad g(z_0) = f'(z_0),$$

and writing

$$(2) \quad g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

if $z \neq z_0$. It is easily seen from Theorem 7C that g is analytic in D . However, note that the function g , defined by (2), is analytic in the domain $D \setminus \{z_0\}$. It also has an isolated singularity at z_0 , which is removed by the definition (1).

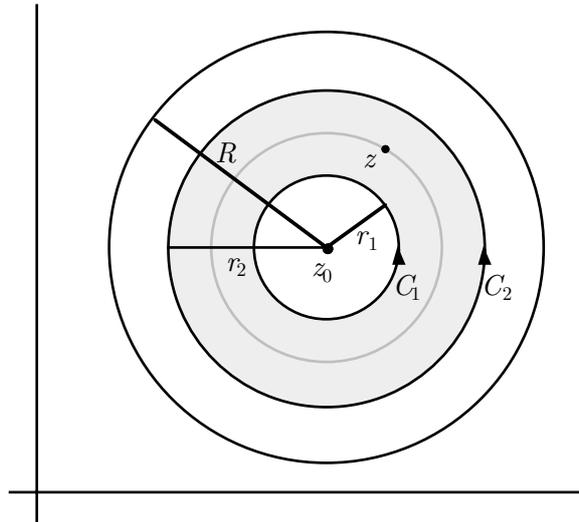
DEFINITION. Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that by assigning a suitable value for $f(z_0)$, the function f can be made to be analytic in the disc $\{z : |z - z_0| < R\}$. Then we say that f has a removable singularity at z_0 .

THEOREM 8A. (RIEMANN'S THEOREM ON REMOVABLE SINGULARITIES) *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that*

$$(3) \quad \lim_{z \rightarrow z_0} (z - z_0)f(z) = 0.$$

Then f has a removable singularity at z_0 .

PROOF. Suppose that z is a point in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Then $0 < |z - z_0| < R$. Let r_1 and r_2 satisfy $0 < r_1 < |z - z_0| < r_2 < R$, and let C_1 and C_2 denote two circles in the positive (anticlockwise) direction, centred at z_0 , and of radius r_1 and r_2 respectively.



The function g , defined by $g(z) = f'(z)$ and for $\zeta \neq z$ by

$$(4) \quad g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z},$$

is clearly analytic in the punctured disk $\{\zeta : 0 < |\zeta - z_0| < R\}$. Then it can be shown, as in the proof of Theorem 6A, that

$$\int_{C_1} g(\zeta) d\zeta = \int_{C_2} g(\zeta) d\zeta.$$

Combining this with (4), we have

$$(5) \quad \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \int_{C_1} \frac{d\zeta}{\zeta - z} = \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \int_{C_2} \frac{d\zeta}{\zeta - z}.$$

Note now that the function

$$\frac{1}{\zeta - z}$$

is analytic in the star domain $\{\zeta : |\zeta - z_0| < |z - z_0|\}$ which contains the contour C_1 . It follows that

$$(6) \quad \int_{C_1} \frac{d\zeta}{\zeta - z} = 0.$$

On the other hand, by Cauchy's integral formula as given by Theorem 6A, we have

$$(7) \quad \int_{C_2} \frac{d\zeta}{\zeta - z} = 2\pi i.$$

Furthermore, in view of the condition (3), we have, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|(\zeta - z_0)f(\zeta)| < \epsilon$ whenever $|\zeta - z_0| < \delta$. Without loss of generality, we may assume that

$$(8) \quad \delta < \frac{1}{2}|z - z_0|.$$

If we now take $r_1 = \delta$, then

$$\left| \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta \right| = \left| \int_{C_1} \frac{(\zeta - z_0)f(\zeta)}{(\zeta - z_0)(\zeta - z)} d\zeta \right| \leq \frac{\epsilon}{\delta(|z - z_0| - \delta)} 2\pi\delta = \frac{2\pi\epsilon}{|z - z_0| - \delta} \leq \frac{4\pi\epsilon}{|z - z_0|},$$

in view of Theorem 4B and (8). Since $\epsilon > 0$ is arbitrary, we conclude that

$$(9) \quad \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Combining (5)–(7) and (9), we obtain

$$(10) \quad f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Note now that (10) holds for every z in the punctured disc $\{z : 0 < |z - z_0| < r_2\}$. Note also that the integral on the right hand side of (10) represents an analytic function in the disc $\{z : |z - z_0| < r_2\}$ (see the proof of Theorem 6B). It follows that if we define

$$f(z_0) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z_0} d\zeta,$$

then the function f is analytic in the disc $\{z : |z - z_0| < r_2\}$. \circ

REMARKS. (1) Note that condition (3) will be satisfied if $f(z)$ is continuous at z_0 , or if $|f(z)|$ is bounded.

(2) Since an analytic function is continuous, it follows that removable singularities at z_0 can be overcome by defining

$$f(z_0) = \lim_{z \rightarrow z_0} f(z).$$

8.2. Poles

DEFINITION. Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that

$$(11) \quad f(z) = \frac{g(z)}{(z - z_0)^n},$$

where $n \in \mathbb{N}$ and the function g is analytic in some neighbourhood of z_0 , with $g(z_0) \neq 0$. Then we say that f has a pole of order n at z_0 . Furthermore, if $n = 1$, then we say that f has a simple pole at z_0 .

THEOREM 8B. *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Then f has a pole at z_0 if and only if*

$$(12) \quad \lim_{z \rightarrow z_0} |f(z)| = \infty;$$

in other words, given any $E > 0$, there exists $\delta > 0$ such that $|f(z)| > E$ whenever $0 < |z - z_0| < \delta$.

PROOF. Note first of all that (12) follows immediately from (11), since $g(z_0) \neq 0$. Suppose now that (12) holds. Then $f(z) \neq 0$ in some punctured disc $\{z : 0 < |z - z_0| < r\}$, where $r \leq R$. It follows that the function

$$F(z) = \frac{1}{f(z)}$$

is analytic in $\{z : 0 < |z - z_0| < r\}$, and has an isolated singularity at z_0 . On the other hand, it follows from (12) that $F(z) \rightarrow 0$ as $z \rightarrow z_0$. Hence by Theorem 8A, F has a removable singularity at z_0 . If we define $F(z_0) = 0$, then F is now analytic in the disc $\{z : |z - z_0| < r\}$. Clearly $F(z)$ is not identically zero in $\{z : |z - z_0| < r\}$. It follows from Theorem 7F that there exists $n \in \mathbb{N}$ such that

$$F(z) = (z - z_0)^n h(z),$$

where the function h is analytic in $\{z : |z - z_0| < r\}$, with $h(z_0) \neq 0$. Hence

$$g(z) = \frac{1}{h(z)}$$

is analytic in some neighbourhood of z_0 , and (11) holds. Clearly $g(z_0) \neq 0$. \circ

REMARK. Note that a function f has a pole of order n at z_0 if and only if the function $1/f$ has a zero of order n at z_0 .

8.3. Essential Singularities

DEFINITION. Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Suppose further that the isolated singularity at z_0 is neither removable nor a pole. Then we say that f has an essential singularity at z_0 .

EXAMPLE 8.3.1. The function $e^{1/z}$ is analytic at every $z \neq 0$. It has an isolated singularity at $z = 0$. Let us restrict z to be real numbers, and consider $e^{1/x}$, where $x > 0$. Clearly

$$\lim_{x \rightarrow 0^+} e^{1/x} = \lim_{y \rightarrow +\infty} e^y = \infty,$$

so that the singularity is not removable. On the other hand, for every $n \in \mathbb{N}$,

$$\lim_{x \rightarrow 0^+} x^n e^{1/x} = \lim_{y \rightarrow +\infty} \frac{e^y}{y^n} = \infty,$$

so that the singularity is not a pole of order n . Hence $e^{1/z}$ has an essential singularity at $z = 0$.

To illustrate the wild behaviour of an analytic function near an essential singularity, we mention Picard's theorem that such a function assumes all values except possibly one in any neighbourhood of an essential singularity. The following result is somewhat weaker, and shows that such a function comes arbitrarily close to any given complex number in any neighbourhood of an essential singularity.

THEOREM 8C. (CASORATI-WEIERSTRASS) *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an essential singularity at z_0 . Then given any $w \in \mathbb{C}$ and any real numbers $\epsilon > 0$ and $\delta > 0$, there exists z in the punctured disc satisfying*

$$0 < |z - z_0| < \delta \quad \text{and} \quad |f(z) - w| < \epsilon.$$

PROOF. Suppose on the contrary that the conclusion does not hold. Then there exist $w \in \mathbb{C}$ and real numbers $\epsilon > 0$ and $\delta > 0$ such that $|f(z) - w| \geq \epsilon$ whenever $0 < |z - z_0| < \delta$. It follows that the function

$$g(z) = \frac{1}{f(z) - w}$$

is analytic and bounded in the punctured disc $\{z : 0 < |z - z_0| < \delta\}$, with an isolated singularity at z_0 which is removable, in view of Theorem 8A. It follows that by defining $g(z_0)$ appropriately, the function g is analytic in the disc $\{z : |z - z_0| < \delta\}$. On the other hand, the function g is clearly not identically zero in $\{z : |z - z_0| < \delta\}$. Furthermore, note that

$$f(z) = w + \frac{1}{g(z)}.$$

If $g(z_0) \neq 0$, then f is analytic at z_0 . If $g(z_0) = 0$, then f has a pole at z_0 . In either case, the conclusion contradicts the assumption that f has an essential singularity at z_0 , and this completes the proof. \square

8.4. Isolated Singularities at Infinity

The behaviour of a function $f(z)$ at $z = \infty$ can be studied via the behaviour of the function $f(1/\zeta)$ at $\zeta = 0$. A punctured neighbourhood $\{\zeta : 0 < |\zeta| < R^{-1}\}$ of 0 then plays the same role as the "punctured" neighbourhood $\{z : R < |z| < \infty\}$ of ∞ .

Suppose now that a function $f(z)$ is analytic in the domain $\{z : R < |z| < \infty\}$. Then by using $z = 1/\zeta$ and considering $\zeta = 0$, we see that the function $f(z)$ has an isolated singularity at $z = \infty$. This may be a removable singularity, a pole or an essential singularity.

Corresponding to Theorem 8A, suppose that $|f(z)/z| \rightarrow 0$ as $|z| \rightarrow \infty$. Then the singularity is removable by defining $f(\infty)$ suitably to make $f(z)$ continuous at $z = \infty$. In other words, we need to define

$$f(\infty) = \lim_{\zeta \rightarrow 0} f\left(\frac{1}{\zeta}\right).$$

In the special case that $f(\infty) = 0$, then we say that f has a zero at $z = \infty$. Furthermore, if f is not identically zero, then, corresponding to Theorem 7F, there exists $n \in \mathbb{N}$ such that

$$f(z) = \frac{h(z)}{z^n},$$

where $h(z)$ is analytic in $\{z : R < |z| < \infty\}$, and $h(\infty) \neq 0$. In this case, we say that f has a zero of order n at $z = \infty$.

Corresponding to Theorem 8B, suppose that $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$. Then f has a pole at $z = \infty$, and there exists $n \in \mathbb{N}$ such that

$$f(z) = z^n h(z),$$

where $h(z)$ is analytic in $\{z : R < |z| < \infty\}$, and $h(\infty) \neq 0$. In this case, we say that f has a pole of order n at $z = \infty$.

Corresponding to Theorem 8C, suppose that the isolated singularity at $z = \infty$ is neither removable nor a pole. Then it is an essential singularity. In this case, given any $w \in \mathbb{C}$ and any real numbers $\epsilon > 0$ and $N > 0$, there exists z in the domain $\{z : R < |z| < \infty\}$ satisfying

$$|z| > N \quad \text{and} \quad |f(z) - w| < \epsilon.$$

In other words, the function $f(z)$ comes arbitrarily close to any given complex number in any neighbourhood of $z = \infty$.

8.5. Further Examples

EXAMPLE 8.5.1. The function

$$f(z) = \frac{e^z - 1}{z(z-1)}$$

is analytic at every $z \in \mathbb{C}$ except for isolated singularities at $z = 0, 1$. At $z = 1$, it has a simple pole; note that we can write

$$f(z) = \frac{g(z)}{z-1} \quad \text{with} \quad g(z) = \frac{e^z - 1}{z},$$

and $g(1) \neq 0$. At $z = 0$, it has a removable singularity, since

$$\lim_{z \rightarrow 0} \frac{e^z - 1}{z(z-1)} = \lim_{z \rightarrow 0} \frac{e^z}{2z-1} = -1$$

by l'Hopital's rule. It follows that if we define $f(0) = -1$, then f is analytic at $z = 0$. The function $f(z)$ also has an isolated singularity at $z = \infty$. To study the isolated singularity at $z = \infty$, note first of all that

$$\lim_{|z| \rightarrow \infty} \frac{e^z - 1}{z(z-1)}$$

does not exist. To see this, note that

$$\lim_{x \rightarrow +\infty} \frac{e^x - 1}{x(x-1)} = +\infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{e^x - 1}{x(x-1)} = 0.$$

Hence the singularity is not removable. Suppose next that $n \in \mathbb{N}$ is given and fixed. Then

$$h(z) = \frac{f(z)}{z^n} = \frac{e^z - 1}{z^{n+1}(z - 1)}$$

is not analytic at $z = \infty$, since

$$\lim_{|z| \rightarrow \infty} \frac{e^z - 1}{z^{n+1}(z - 1)}$$

does not exist. To see this, note that

$$\lim_{x \rightarrow +\infty} \frac{e^x - 1}{x^{n+1}(x - 1)} = +\infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{e^x - 1}{x^{n+1}(x - 1)} = 0.$$

Hence the singularity is not a pole. It follows that $f(z)$ has an essential singularity at $z = \infty$.

EXAMPLE 8.5.2. The function

$$f(z) = \frac{(z^2 - 4)(z - 1)^4}{(\sin \pi z)^4}$$

is analytic at every $z \in \mathbb{C}$ except for isolated singularities at $z = 0, \pm 1, \pm 2, \dots$, where the denominator vanishes. Note also that the numerator vanishes at $z = 1, \pm 2$. Note that the function $\sin \pi z$ has simple zeros at $z = 0, \pm 1, \pm 2, \dots$. It follows that f has poles of order 4 at $z = 0, -1, \pm 3, \pm 4, \pm 5, \dots$. Next, note that the function $(z^2 - 4)(z - 1)^4$ has simple zeros at $z = \pm 2$. It follows that f has poles of order 3 at $z = \pm 2$. To study the isolated singularity at $z = 1$, note that by Theorem 7C, we have

$$\sin \pi z = -\pi(z - 1) + g(z)(z - 1)^2,$$

where g is entire. It follows that

$$\lim_{z \rightarrow 1} \frac{(z^2 - 4)(z - 1)^4}{(\sin \pi z)^4} = \lim_{z \rightarrow 1} \frac{z^2 - 4}{(\pi - g(z)(z - 1))^4} = -\frac{3}{\pi^4},$$

and so f has a removable singularity at $z = 1$. Finally, the singularity at $z = \infty$ is not isolated, since there does not exist any $R > 0$ such that the function $f(z)$ is analytic in the domain $\{z : R < |z| < \infty\}$.

8.6. Laurent Series

EXAMPLE 8.6.1. Suppose that the function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with a pole of order m at z_0 . Then

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where the function g is analytic in $\{z : |z - z_0| < R\}$, with $g(z_0) \neq 0$. By Theorem 7C, we have

$$g(z) = g(z_0) + g'(z_0)(z - z_0) + \frac{g''(z_0)}{2!}(z - z_0)^2 + \dots + \frac{g^{(m-1)}(z_0)}{(m-1)!}(z - z_0)^{m-1} + g_m(z)(z - z_0)^m,$$

where $g_m(z)$ is analytic in the disc $\{z : |z - z_0| < R\}$. It follows that

$$f(z) = \frac{g(z_0)}{(z - z_0)^m} + \frac{g'(z_0)}{(z - z_0)^{m-1}} + \frac{g''(z_0)}{2!(z - z_0)^{m-2}} + \dots + \frac{g^{(m-1)}(z_0)}{(m-1)!(z - z_0)} + g_m(z).$$

The expression

$$\frac{g(z_0)}{(z-z_0)^m} + \frac{g'(z_0)}{(z-z_0)^{m-1}} + \frac{g''(z_0)}{2!(z-z_0)^{m-2}} + \cdots + \frac{g^{(m-1)}(z_0)}{(m-1)!(z-z_0)}$$

is called the principal part of f at z_0 . If we use Theorem 7A instead, then we can show that

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n$$

for suitable choices of the coefficients a_n .

Our main task in this section is to generalize this example. The first step in this direction can be summarized by the following result.

THEOREM 8D. *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an isolated singularity at z_0 . Then there exist unique functions f_1 and f_2 such that*

- (a) $f(z) = f_1(z) + f_2(z)$ in $\{z : 0 < |z - z_0| < R\}$,
- (b) f_1 is analytic in \mathbb{C} except possibly at z_0 ,
- (c) $f_1(z) \rightarrow 0$ as $|z| \rightarrow \infty$, and
- (d) f_2 is analytic in the disc $\{z : |z - z_0| < R\}$.

PROOF. We begin the proof in the same way as for Theorem 8A. Suppose that z is a point in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Let r_1 and r_2 satisfy $0 < r_1 < |z - z_0| < r_2 < R$, and let C_1 and C_2 denote two circles in the positive (anticlockwise) direction, centred at z_0 , and of radius r_1 and r_2 respectively. On combining (5)–(7), we obtain

$$(13) \quad f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Write

$$(14) \quad f_1(z) = -\frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{and} \quad f_2(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Part (a) follows immediately. For part (d), note that the second integral in (14) represents an analytic function in the disc $\{z : |z - z_0| < r_2\}$ (as in the proof of Theorems 6B and 8A). For part (b), note that the first integral in (14) represents an analytic function in the annulus $\{z : |z - z_0| > r_1\}$ (similar to the proof of Theorem 6B). Note next that $f_2(z)$ and $f(z)$ are independent of the choice of r_1 , so that it follows from (a) that $f_1(z)$ is also independent of the choice of r_1 . Similarly, $f_2(z)$ is independent of the choice of r_2 . It is easy to see that

$$\lim_{|z| \rightarrow \infty} \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Part (c) follows immediately. To show that the functions f_1 and f_2 are unique, suppose that g_1 and g_2 are functions having the same properties as f_1 and f_2 respectively. Then

$$f_1(z) - g_1(z) = g_2(z) - f_2(z)$$

in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Let

$$F(z) = \begin{cases} g_2(z) - f_2(z) & \text{if } |z - z_0| < R, \\ f_1(z) - g_1(z) & \text{if } |z - z_0| > 0. \end{cases}$$

Then F is entire. On the other hand, it follows from part (c) that $F(z) \rightarrow 0$ as $|z| \rightarrow \infty$. Hence F is bounded. It follows from Liouville's theorem that F is constant in \mathbb{C} , and so we must have $F(z) = 0$ for every $z \in \mathbb{C}$. This completes the proof. \circ

We can now state our generalization of Example 8.6.1.

THEOREM 8E. *Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an isolated singularity at z_0 . For every $n \in \mathbb{Z}$, let*

$$(15) \quad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a circle in the positive (anticlockwise) direction centred at z_0 and of radius r , where $0 < r < R$. Then the series

$$(16) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is convergent in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Furthermore, this convergence is uniform in any annulus $\{z : r_1 < |z - z_0| < r_2\}$, where $0 < r_1 < r_2 < R$.

REMARK. To say that the series converges uniformly to $f(z)$ in the annulus $\{z : r_1 < |z - z_0| < r_2\}$, we mean given any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon, r_1, r_2)$, independent of the choice of z , such that

$$\left| f(z) - \sum_{n=-N_1}^{N_2} a_n (z - z_0)^n \right| < \epsilon$$

for every z in the annulus $\{z : r_1 < |z - z_0| < r_2\}$ whenever $N_1 > N_0$ and $N_2 > N_0$.

DEFINITION. The series (16) is called the Laurent series for the function f at z_0 .

PROOF OF THEOREM 8E. The first step in our proof is to show that if the series in (16) converges to $f(z)$ uniformly on the circle C centred at z_0 and of radius r , where $0 < r < R$, then the coefficients a_n are given by (15). Suppose that $n \in \mathbb{Z}$ is chosen and fixed. For any $\epsilon > 0$, we can choose N_1 and N_2 so large that $-N_1 \leq n \leq N_2$ and

$$\left| f(z) - \sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right| < \epsilon$$

for every $z \in C$. Then it follows from Theorem 4B that

$$(17) \quad \left| \frac{1}{2\pi i} \int_C \left(f(z) - \sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right) \frac{dz}{(z - z_0)^{n+1}} \right| \leq \frac{\epsilon}{r^n}.$$

Since

$$\frac{1}{2\pi i} \int_C (z - z_0)^k dz = \begin{cases} 1 & \text{if } k = -1, \\ 0 & \text{if } k \neq -1, \end{cases}$$

we have

$$\frac{1}{2\pi i} \int_C \left(\sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right) \frac{dz}{(z - z_0)^{n+1}} = a_n,$$

so that (17) can be simplified to

$$\left| \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz - a_n \right| \leq \frac{\epsilon}{r^n}.$$

Since $\epsilon > 0$ is arbitrary, (15) follows immediately. It now remains to show that $f(z)$ can be represented in the form (16) in the punctured disc $\{z : 0 < |z - z_0| < R\}$, and that the convergence is uniform in any annulus $\{z : r_1 < |z - z_0| < r_2\}$, where $0 < r_1 < r_2 < R$. Suppose that $0 < r_1 < r_2 < R$. Following Theorem 8D, we can write

$$(18) \quad f(z) = f_1(z) + f_2(z),$$

where $f_1(z)$ and $f_2(z)$ are uniquely determined and satisfy conditions (b)–(d) of Theorem 8D. Since f_2 is analytic in the disc $\{z : |z - z_0| < R\}$, it follows from Theorem 7A that the Taylor series

$$(19) \quad f_2(z) = \sum_{n=0}^{\infty} A_n(z - z_0)^n$$

converges in the disc $\{z : |z - z_0| < R\}$, uniformly in the closed disc $\{z : |z - z_0| \leq r_2\}$. To study $f_1(z)$, write

$$w = \frac{1}{z - z_0} \quad \text{or} \quad z = \frac{1}{w} + z_0.$$

Then

$$f_1(z) = f_1\left(\frac{1}{w} + z_0\right)$$

is an entire function of w , and so it follows from Theorem 7A that the Taylor series

$$(20) \quad f_1\left(\frac{1}{w} + z_0\right) = \sum_{m=1}^{\infty} B_m w^m$$

converges in \mathbb{C} , uniformly in the closed disc $\{w : |w| \leq 1/r_1\}$. Note that the constant term B_0 in the Taylor series is missing, since B_0 corresponds to the value of the function at $w = 0$, or $z = \infty$, and this is 0 in view of condition (c) in Theorem 8D. However, (20) is equivalent to saying that the series

$$(21) \quad f_1(z) = \sum_{m=1}^{\infty} B_m(z - z_0)^{-m}$$

converges in $\mathbb{C} \setminus \{0\}$, uniformly in $\{z : |z - z_0| \geq r_1\}$. The result now follows on combining (18), (19) and (21). \circ

DEFINITION. The series

$$f_1(z) = \sum_{n=-\infty}^{-1} a_n(z - z_0)^n,$$

where a_n is given by (15), is called the principal part of the function f at z_0 .

The next result highlights the relationship between the principal part of a function and the nature of the isolated singularity.

THEOREM 8F. Suppose that a function f is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an isolated singularity at z_0 . Suppose further that the Laurent coefficients a_n are given by (15).

- The function f either is analytic or has a removable singularity at z_0 if and only if $a_n = 0$ for every $n < 0$.
- The function f has a pole at z_0 if and only if a positive but finite number of coefficients a_n with $n < 0$ are non-zero.
- The function f has an essential singularity at z_0 if and only if an infinite number of coefficients a_n with $n < 0$ are non-zero.

PROOF. Note first of all that if f has a removable singularity at z_0 , then f can be made analytic at z_0 by a suitable choice of $f(z_0)$. Part (a) now follows on observing that an analytic function has a Taylor series, and that a Laurent series with no principal part is a Taylor series. To prove part (b), note first of all that if a positive but finite number of coefficients a_n with $n < 0$ are non-zero, then there exists $m > 0$ such that $a_{-m} \neq 0$ but $a_n = 0$ for every $n < -m$. In this case, we have

$$f(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n,$$

so that

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $m \in \mathbb{N}$ and the function g is analytic in some neighbourhood of z_0 , with $g(z_0) = a_{-m} \neq 0$. This shows that f has a pole of order m at z_0 . The converse is given in Example 8.6.1. Part (b) follows. Part (c) follows immediately from (a) and (b). \circ

EXAMPLE 8.6.2. The observation that a Laurent series is unique enables us to use different methods to find the coefficients apart from the formula (15). Consider, for example, the function $e^{1/z}$. Using the substitution $z = 1/w$ on the Taylor series

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!},$$

we obtain the Laurent series

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = \dots + \frac{1}{3!z^3} + \frac{1}{2!z^2} + \frac{1}{z} + 1.$$

We conclude this chapter by making a remark on various equivalent definitions of analyticity in a domain D . The reader is advised to check the following theorem very carefully.

THEOREM 8G. *For any function f and any domain D , the following statements are equivalent:*

- (a) $f(z)$ is analytic in D .
- (b) $f(z)$ has continuous derivatives of all orders in D .
- (c) $f'(z)$ exists and is continuous in D .
- (d) $f'(z)$ exists in D .
- (e) $f'(z)$ exists in D except possibly at a finite number of points in D , and $f(z)$ is continuous at these exceptional points.
- (f) $f(z)$ can be represented uniformly by its Taylor series in the neighbourhood of every point in D .

PROBLEMS FOR CHAPTER 8

1. For each of the functions below, classify all the singular points in \mathbb{C} :

- | | | |
|---------------------------------|------------------------------|-------------------------------------|
| a) $f(z) = e^z$ | b) $f(z) = \frac{\cos z}{z}$ | c) $f(z) = \frac{z^2 + 1}{z^2 - 1}$ |
| d) $f(z) = \frac{z^4}{z^3 + z}$ | e) $f(z) = \frac{z}{\cos z}$ | |

2. Show that the principal parts of the function $f(z) = 8z^3(z + 1)^{-1}(z - 1)^{-2}$ at $z = -1$ and $z = 1$ are respectively $-2(z + 1)^{-1}$ and $4(z - 1)^{-2} + 10(z - 1)^{-1}$.

3. For each of the functions below, find the principal part at the given points:

a) $f(z) = \frac{e^z}{z^4}$ at the point $z = 0$

b) $f(z) = \frac{z^6}{(1-z)^3}$ at the point $z = 1$

c) $f(z) = \frac{\sin z}{(z-2\pi)^2}$ at the point $z = 2\pi$

4. Expand the function $(z-1)/(z+1)$ in powers of $1/z$.

5. For each of the functions below, use partial fractions if appropriate and find the principal part at each of its singular points in \mathbb{C} :

a) $f(z) = \frac{12}{z^2(z^2+4)}$

b) $f(z) = \frac{z^4+1}{z(z^2+1)^2}$

c) $f(z) = \frac{48z^6}{(z-1)^2(z-2)}$

d) $f(z) = \frac{z^9+1}{(z-1)^3(z^2+4)^2}$

6. Suppose that $f(z) = b_{-m}z^{-m} + b_{-m+1}z^{-m+1} + \dots + b_0 + b_1z + \dots + b_kz^k$, where $m, k \in \mathbb{N}$. Suppose further that $f(z)$ has Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

at the point $z = 0$. Show by direct calculation that $a_n = b_n$ whenever $-m \leq n \leq k$ and $a_n = 0$ otherwise.

7. a) Consider the function $f(z) = e^{1/z}$. Note that for every $k \in \mathbb{Z}$, the coefficient for the term z^k in the Laurent series of $f(z)$ at $z = 0$ is given by

$$a_k = \frac{1}{2\pi i} \int_C \frac{e^{1/\zeta}}{\zeta^{k+1}} d\zeta,$$

where C is the circle $\{z : |z| = 1\}$ followed in the positive (anticlockwise) direction. Show that

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\cos \theta} \cos(\sin \theta + k\theta) d\theta.$$

b) Find the Laurent series for the function $f(z) = e^{1/z}$ at $z = 0$ without using part (a).

c) Deduce that for every $n \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{\pi} \int_0^{\pi} e^{\cos \theta} \cos(\sin \theta - n\theta) d\theta = \frac{1}{n!}.$$

INTRODUCTION TO COMPLEX ANALYSIS

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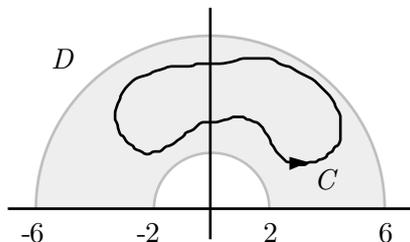
Chapter 9

CAUCHY'S INTEGRAL THEOREM REVISITED

9.1. Simply Connected Domains

Cauchy's integral theorem states that in a simply connected domain, the integral of an analytic function over a closed contour is zero. In Chapter 5, we have studied a special case of the theorem in order to develop the basic properties of analytic functions. More precisely, we have studied the special case when the domain in question is a star domain. For most purposes in pure and applied mathematics, this special case is adequate. However, by introducing further line segments, one may be able to extend the results to curves which may not lie in star domains.

EXAMPLE 9.1.1. It is easy to see that the domain $D = \{z : 2 < |z| < 6 \text{ and } \Im z > 0\}$ is the part of the annulus $\{z : 2 < |z| < 6\}$ in the upper half plane. It is not difficult to check that D is not a star domain.



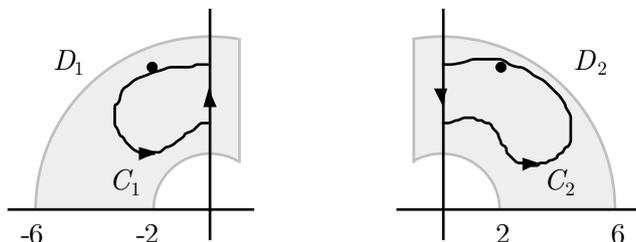
Let

$$D_1 = \{z : 2 < |z| < 6 \text{ and } \Im z > 0 \text{ and } \Re z < 1\}$$

and

$$D_2 = \{z : 2 < |z| < 6 \text{ and } \Im z > 0 \text{ and } \Re z > -1\}.$$

It is not difficult to see that $D = D_1 \cup D_2$, and that D_1 and D_2 are star domains with star centres $-2 + 5i$ and $2 + 5i$ respectively.



If C is a simple closed contour lying in D , then by introducing line segments along the imaginary axis, it is not difficult to see that there is a simple closed contour C_1 in D_1 and a simple closed contour C_2 in D_2 such that for any function f analytic in D , we have

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz.$$

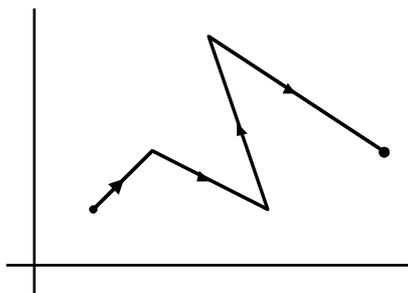
Note now that we can apply Theorem 5D to the two integrals on the right hand side.

However, it is of great theoretical interest to formulate results that are less restricting. Here we introduce the idea of a simply connected domain. This can be done in a number of ways. Here we use the Jordan curve theorem for simple closed polygons.

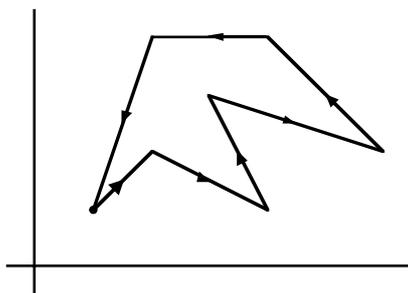
DEFINITION. By a polygonal curve, we mean a curve $\zeta : [A, B] \rightarrow \mathbb{C}$ which is continuous and piecewise linear. In other words, there exists a dissection

$$A = A_1 < B_1 = A_2 < B_2 = \dots = A_k < B_k = B$$

such that for every $j = 1, \dots, k$, the edge $\zeta : [A_j, B_j] \rightarrow \mathbb{C}$ is of the form $\zeta(t) = \alpha_j t + \beta_j$, where $\alpha_j, \beta_j \in \mathbb{C}$ and we assume further that $\alpha_j \neq 0$.



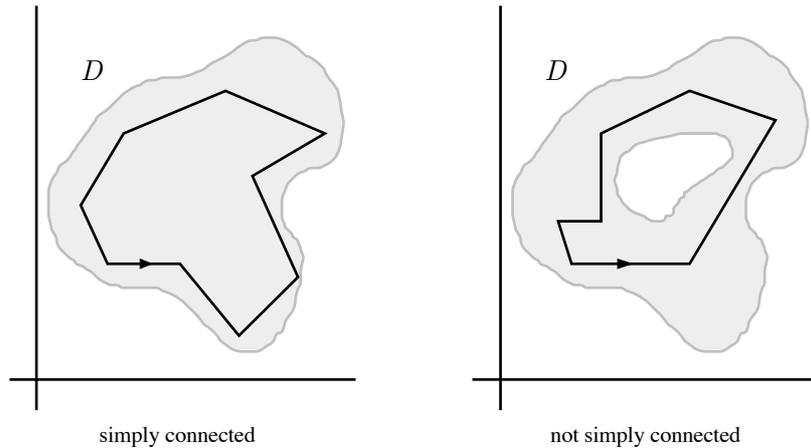
DEFINITION. By a simple closed polygon, we mean a polygonal curve that is closed and does not intersect itself; in other words, if $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, with the one exception that $\zeta(A) = \zeta(B)$.



It is not hard to prove the Jordan curve theorem for a simple closed polygon, that such a polygon divides the plane \mathbb{C} into two domains, the bounded one interior to the polygon and the unbounded one exterior to the polygon. Here we shall not give the proof.

DEFINITION. A domain D is said to be simply connected if the interior of every simple closed polygon in D is contained in D .

REMARK. Recall that a domain is an open connected set. A simply connected domain is one which is free of holes or cuts in its interior.



9.2. Cauchy's Integral Theorem

In this section, we indicate the proof of the following generalization of Theorem 5B.

THEOREM 9A. Suppose that a function f is analytic in a simply connected domain D . Then there exists a function F , analytic in D and such that $F'(z) = f(z)$ for every $z \in D$.

In view of Remark (1) immediately after Theorem 4A, Theorem 9A immediately leads to the following generalization of Theorem 5D.

THEOREM 9B. Suppose that a function f is analytic in a simply connected domain D . Suppose further that C is a closed contour lying in D . Then

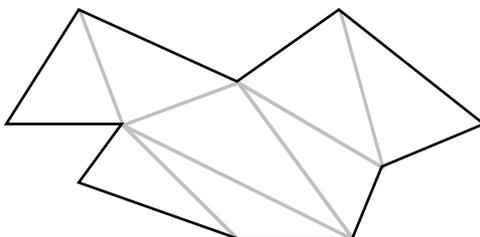
$$\int_C f(z) dz = 0.$$

EXAMPLE 9.2.1. The punctured plane $D = \{z : z \neq 0\}$ is not simply connected. Although the function $f(z) = 1/z$ is analytic in D ,

$$\int_C \frac{dz}{z} = 2\pi i \neq 0$$

if C is the positive (anticlockwise) oriented unit circle centred at 0. This also shows that there is no single valued branch of $\log z$ in D , and confirms that the condition that D is simply connected in Theorems 9A and 9B is essential.

The proof of Theorem 9A depends on a process known sometimes as “triangulation”. It can be shown by induction that a simple closed polygon of $k + 2$ sides can be decomposed by diagonals into a set of k triangles.



We shall first prove the following generalization of Theorem 5A.

THEOREM 9C. *Suppose that a function f is analytic in a simply connected domain D . Suppose further that C is a simple closed polygon in D . Then*

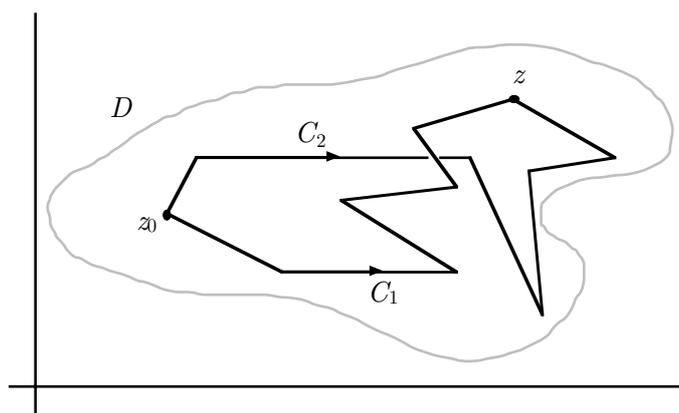
$$\int_C f(z) dz = 0.$$

PROOF. We may assume, without loss of generality, that C is in the positive (anticlockwise) direction. Suppose that the triangulation process gives rise to triangles T_1, \dots, T_k , with boundaries C_1, \dots, C_k in the positive (anticlockwise) direction. Then it is easy to see that

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \dots + \int_{C_k} f(z) dz.$$

The result follows on applying Theorem 5A to each of the integrals on the right hand side. \circ

We now sketch a proof of Theorem 9A. Suppose that $z_0 \in D$ is fixed. For every $z \in D$, let C_1 and C_2 denote polygonal curves from z_0 to z that lie entirely in D .



We shall first of all indicate that

$$(1) \quad \int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

Let C be the closed polygonal curve obtained by C_1 followed by $-C_2$. To show (1), it suffices to show that

$$(2) \quad \int_C f(z) dz = 0.$$

It can be shown by induction that the closed polygonal curve C consists of a number of line segments followed in opposite directions and a number of simple closed polygonal curves. (2) now follows in view of Theorem 9C. Note that (1) shows that the integral is independent of the polygonal curve chosen. We can therefore define

$$F(z) = \int_{z_0}^z f(\zeta) d\zeta,$$

where the integral is taken along any polygonal curve from z_0 to z . It follows that if $|h|$ is sufficiently small, then the segment $[z, z+h]$ lies entirely in D , and that

$$F(z+h) - F(z) = \int_{[z, z+h]} f(\zeta) d\zeta.$$

The proof of Theorem 9A is now completed in the same way as in the proof of Theorem 5B.

9.3. Cauchy's Integral Formula

Suppose that f is analytic in a simply connected domain D . Suppose further that C is a closed contour in D , and that the point z does not lie on C . If $z \in D$, then the function

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

is analytic in D , apart from a removable singularity at $\zeta = z$. Furthermore, this singularity is removed by defining $g(z) = f'(z)$. It now follows from Theorem 9B that

$$\int_C \frac{f(\zeta) - f(z)}{\zeta - z} dz = 0, \quad \text{and so} \quad \int_C \frac{f(\zeta)}{\zeta - z} dz = f(z) \int_C \frac{dz}{\zeta - z}.$$

Here

$$n(C, z) = \frac{1}{2\pi i} \int_C \frac{dz}{\zeta - z} = \frac{\text{var}(i \arg(\zeta - z), C)}{2\pi i}$$

is the winding number, and counts the number of times the contour C winds round the point z in the positive (anticlockwise) direction. Hence

$$(3) \quad \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} dz = n(C, z) f(z).$$

Note also that if $z \notin D$, then both sides of (3) are 0. It follows that (3) holds whenever $z \in C$.

9.4. Analytic Logarithm

Suppose that f is analytic and non-zero in a simply connected domain D . Let $z_0 \in D$ be fixed. For every $z \in D$, we can define

$$(4) \quad g(z) = \int_{z_0}^z \frac{f'(\zeta)}{f(\zeta)} d\zeta + \text{Log } f(z_0),$$

where the integral is over any contour in D from z_0 to z , and is independent of the choice of the contour, in view of Theorem 9B. Differentiating (4) with respect to z , we obtain $g'(z)f(z) = f'(z)$. It follows that

$$\frac{d}{dz} (e^{-g(z)} f(z)) = e^{-g(z)} f'(z) - g'(z) e^{-g(z)} f(z) = 0,$$

so that $e^{-g(z)}f(z)$ is constant in D . Since $e^{-g(z_0)}f(z_0) = 1$, it follows that $e^{-g(z)}f(z) = 1$ for every $z \in D$. In other words, $f(z) = e^{g(z)}$ for every $z \in D$, so that f has an analytic logarithm in D .

PROBLEMS FOR CHAPTER 9

1. Suppose that T is a triangle, followed in the positive (anticlockwise) direction. Suppose further that for any $a \in \mathbb{C} \setminus T$, we write

$$n(T, a) = \frac{1}{2\pi i} \int_T \frac{dz}{z - a}.$$

- a) Show that $n(T, a) = 0$ for every $a \in \mathbb{C}$ outside T .
 b) Suppose now that $a \in \mathbb{C}$ is inside T . By relating T to a suitably small circular path centred at the point a , show that $n(T, a) = 1$.
2. Suppose that C is simple closed polygon, followed in the positive (anticlockwise) direction. Suppose further that for any $a \in \mathbb{C} \setminus C$, we write

$$n(C, a) = \frac{1}{2\pi i} \int_C \frac{dz}{z - a}.$$

- a) Show that $n(C, a) = 0$ for every $a \in \mathbb{C}$ outside C .
 b) Suppose now that $a \in \mathbb{C}$ is inside C . Apply the “triangulation” process to C and show that $n(C, a) = 1$ if the point a does not lie on the boundary of any of the triangles that arise from the process.
 c) Suppose now that $a \in \mathbb{C}$ is inside C and lies on the boundary of some of the triangles that arise from the “triangulation” process. Explain why we also have $n(C, a) = 1$.
3. Suppose that $D \subseteq \mathbb{C}$ is a domain. For every contour C lying in D and for every $a \notin D$, write

$$n(C, a) = \frac{1}{2\pi i} \int_C \frac{dz}{z - a}.$$

- a) Suppose that D is simply connected. Explain why $n(C, a) = 0$.
 b) Suppose that D is not simply connected. Show that there exists a contour C lying in D and a point $a \notin D$ such that $n(C, a) \neq 0$.
4. Deduce from the conclusion of Problem 3 that every star domain is simply connected.
5. Suppose that $D \subseteq \mathbb{C}$ is a domain. Suppose further that every function which is analytic and non-zero in D has an analytic logarithm in D , so that in particular, for every $a \notin D$, there exists a function $g(z)$ analytic in D and such that $z - a = e^{g(z)}$ for every $z \in D$. Use the conclusion of Problem 3 to show that D is simply connected.
6. Suppose D is a bounded domain. For any $\epsilon > 0$, let D_ϵ denote the larger domain containing D and every point in \mathbb{C} whose distance from D is less than ϵ . Give an example of a simply connected domain D such that D_ϵ is not simply connected for any $\epsilon > 0$.
7. Suppose that $f(z)$ is an entire function with a finite number of zeros. Show that there exist a polynomial $P(z)$ and an entire function $g(z)$ such that $f(z) = P(z)e^{g(z)}$.
8. Suppose that $f(z)$ is analytic and non-zero in the disc $\{z : |z| \leq R\}$, where $R > 0$ is fixed. Prove the following special case of Jensen’s formula, that

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta.$$

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Chapter 10

RESIDUE THEORY

10.1. Cauchy's Residue Theorem

If we extend Cauchy's integral theorem to functions having isolated singularities, then the integral is in general not equal to zero. Instead, each singularity contributes a term called the residue. Our principal aim in this section is to show that this residue depends only on the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of the function near the singularity z_0 , since all the other powers of $z - z_0$ has single valued integrals and so integrate to zero.

DEFINITION. By a simple closed contour or Jordan contour, we mean a contour $\zeta : [A, B] \rightarrow \mathbb{C}$ such that $\zeta(t_1) \neq \zeta(t_2)$ whenever $t_1 \neq t_2$, with the one exception $\zeta(A) = \zeta(B)$.

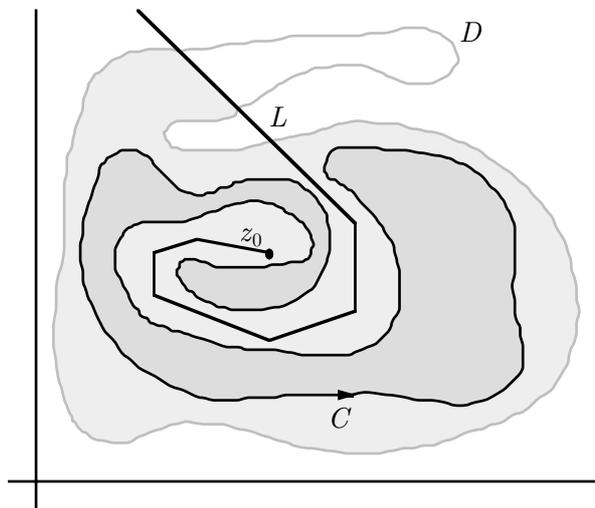
THEOREM 10A. *Suppose that a function f is analytic in a simply connected domain D , except for an isolated singularity at z_0 , and that*

$$f_1(z) = \sum_{n=-\infty}^{-1} a_n(z - z_0)^n$$

is the principal part of f at z_0 . Suppose further that C is a Jordan contour in D followed in the positive (anticlockwise) direction and not passing through z_0 . Then

$$\frac{1}{2\pi i} \int_C f_1(z) dz = \begin{cases} a_{-1} & \text{if } z_0 \text{ lies inside } C, \\ 0 & \text{if } z_0 \text{ lies outside } C. \end{cases}$$

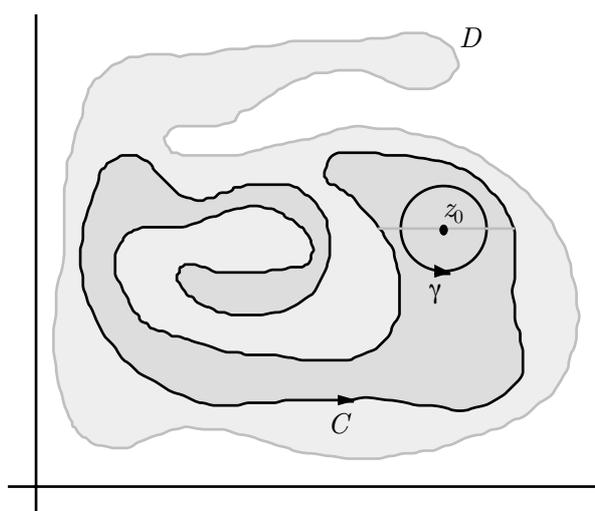
PROOF. Suppose first of all that z_0 is outside C . Then z_0 is in the exterior domain of C which also contains the point at ∞ . It follows that z_0 can be joined to the point at ∞ by a simple polygonal curve L , as shown in the picture below.



The Jordan contour C is clearly contained in the simply connected domain obtained when L is deleted from the complex plane. In fact, it is contained in a simply connected domain which is a subset of $D \setminus L$, as shown by the shaded part in the picture above. Clearly f is analytic in this simply connected domain, so it follows from Theorem 9B that

$$\int_C f(z) dz = 0.$$

Suppose next that z_0 is inside C . Then there exists $r > 0$ such that the closed disc $\{z : |z - z_0| \leq r\}$ is inside C . Let γ denote the boundary of this disc, followed in the positive (anticlockwise) direction.

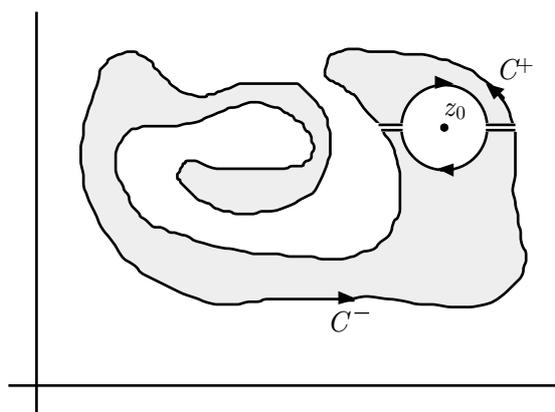


We now draw a horizontal line through the point z_0 . Following this line to the left from z_0 , it first intersects γ and then C (for the first time). Draw a line segment joining these two intersection points. Similarly, following this line to the right from z_0 , it first intersects γ and then C (for the first time). Again draw a line segment joining these two intersection points. Note that these two line segments are inside C and outside γ . We now divide C into two parts by cutting it at the two intersection points

mentioned. It can be shown that one part of this, together with the part of γ above the horizontal line and the two line segments, gives rise to a simple closed contour C^+ followed in the positive direction and which can be shown to lie in a simply connected domain lying in D but not containing z_0 . Clearly f is analytic in this simply connected domain, so that

$$\int_{C^+} f(z) dz = 0,$$

in view of Theorem 9B.



Similarly, the other part of C , together with the part of γ below the horizontal line and the two line segments, gives rise to a simple closed contour C^- followed in the positive direction and which again can be shown to lie in a simply connected domain lying in D but not containing z_0 . Clearly f is analytic in this simply connected domain, so that

$$\int_{C^-} f(z) dz = 0.$$

It is easily seen that

$$\int_C f(z) dz - \int_\gamma f(z) dz = \int_{C^+} f(z) dz + \int_{C^-} f(z) dz,$$

so that

$$\int_C f(z) dz = \int_\gamma f(z) dz.$$

By Theorem 8E, we have

$$\int_\gamma f(z) dz = 2\pi i a_{-1}.$$

It follows that

$$\int_C f(z) dz = 2\pi i a_{-1}.$$

Finally, note that $f_2(z) = f(z) - f_1(z)$ is analytic in D , so that

$$\int_C f_2(z) dz = 0,$$

whence

$$\int_C f(z) dz = \int_C f_1(z) dz.$$

The result follows. \circ

DEFINITION. The value a_{-1} in Theorem 10A is called the residue of the function f at z_0 , and denoted by $\text{res}(f, z_0)$.

We are now in a position to state and prove a simple version of Cauchy's residue theorem.

THEOREM 10B. Suppose that the function f is analytic in a simply connected domain D , except for isolated singularities at z_1, \dots, z_k . Suppose further that C is a Jordan contour in D followed in the positive (anticlockwise) direction and not passing through z_1, \dots, z_k . Then

$$\frac{1}{2\pi i} \int_C f(z) dz = \sum_{\substack{j=1 \\ z_j \text{ inside } C}}^k \text{res}(f, z_j).$$

PROOF. For every $j = 1, \dots, k$, let $f_j(z)$ denote the principal part of $f(z)$ at z_j . By Theorem 8D, f_j is analytic in \mathbb{C} except at z_j . It follows that the function

$$g(z) = f(z) - \sum_{j=1}^k f_j(z)$$

is analytic in D , so that

$$\int_C g(z) dz = 0$$

by Theorem 9B, and so

$$\int_C f(z) dz = \sum_{j=1}^k \int_C f_j(z) dz.$$

The result now follows from Theorem 10A. \circ

10.2. Finding the Residue

In order to use Theorem 10B to evaluate the integral

$$\int_C f(z) dz,$$

we need a technique to evaluate the residues at the isolated singularities.

Suppose that $f(z)$ has a removable singularity at z_0 . Then $f(z)$ has a Taylor series expansion which is valid in a neighbourhood of z_0 . The residue is clearly 0.

Suppose that $f(z)$ has a simple pole at z_0 . Then we can write

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z),$$

where $g(z)$ is analytic at z_0 , so that $(z - z_0)g(z) \rightarrow 0$ as $z \rightarrow z_0$. It follows that the residue is given by

$$a_{-1} = \lim_{z \rightarrow z_0} (z - z_0)f(z).$$

Suppose that $f(z)$ has a pole of order m at z_0 . Then we can write

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{z - z_0} + g(z),$$

where $g(z)$ is analytic at z_0 , so that

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \dots + a_{-1}(z - z_0)^{m-1} + (z - z_0)^m g(z)$$

is analytic at z_0 . Differentiating $m - 1$ times gives

$$\frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z)) = a_{-1}(m - 1)! + \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m g(z)).$$

Since $g(z)$ is analytic at z_0 , we have

$$\lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m g(z)) = 0.$$

It follows that the residue is given by

$$a_{-1} = \frac{1}{(m - 1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}}((z - z_0)^m f(z)).$$

DEFINITION. A function is said to be meromorphic in a domain D if it is analytic in D except for poles.

EXAMPLE 10.2.1. The function

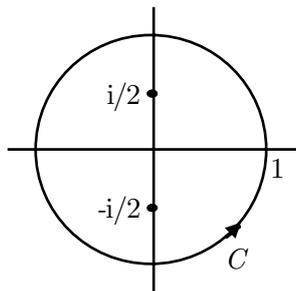
$$f(z) = \frac{e^{2iz}}{1 + 4z^2}$$

has simple poles at $z = \pm i/2$, with residues

$$\operatorname{res}\left(f, \frac{i}{2}\right) = \lim_{z \rightarrow i/2} \left(z - \frac{i}{2}\right) f(z) = \lim_{z \rightarrow i/2} \frac{e^{2iz}}{4(z + i/2)} = \frac{e^{-1}}{4i}$$

and

$$\operatorname{res}\left(f, -\frac{i}{2}\right) = \lim_{z \rightarrow -i/2} \left(z + \frac{i}{2}\right) f(z) = \lim_{z \rightarrow -i/2} \frac{e^{2iz}}{4(z - i/2)} = -\frac{e}{4i}.$$



It follows from Cauchy's residue theorem that if $C = \{z : |z| = 1\}$ is the circle with centre 0 and radius 1, followed in the positive (anticlockwise) direction, then

$$\int_C \frac{e^{2iz}}{1+4z^2} dz = 2\pi i \left(\frac{e^{-1}}{4i} - \frac{e}{4i} \right) = \frac{\pi}{2} \left(\frac{1}{e} - e \right).$$

EXAMPLE 10.2.2. The function

$$f(z) = \frac{e^z}{z^4}$$

has a pole of order 4 at $z = 0$, with residue

$$\text{res}(f, 0) = \frac{1}{3!} \lim_{z \rightarrow 0} \frac{d^3}{dz^3} (z^4 f(z)) = \frac{1}{3!} \lim_{z \rightarrow 0} \frac{d^3}{dz^3} e^z = \frac{1}{6}.$$

It follows from Cauchy's residue theorem that if C is any Jordan contour with 0 inside and followed in the positive (anticlockwise) direction, then

$$\int_C \frac{e^z}{z^4} dz = 2\pi i \left(\frac{1}{6} \right) = \frac{\pi i}{3}.$$

EXAMPLE 10.2.3. Suppose that a function f is analytic in a simply connected domain D , and that $z_0 \in D$. Suppose further that C is a Jordan contour in D , followed in the positive (anticlockwise) direction and with z_0 inside. If $f(z_0) \neq 0$, then the function

$$F(z) = \frac{f(z)}{z - z_0}$$

has a simple pole at z_0 , with residue

$$\lim_{z \rightarrow z_0} (z - z_0)F(z) = f(z_0).$$

Applying Cauchy's residue theorem, we obtain Cauchy's integral formula

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = f(z_0).$$

If $f(z_0) = 0$, then $F(z)$ has a removable singularity at z_0 . The same result follows instead from Cauchy's integral theorem.

10.3. Principle of the Argument

In this section, we shall show that the residue theorem, when applied suitably, can be used to find the number of zeros of an analytic function, as well as the number of zeros minus the number of poles of a meromorphic function.

The main idea underpinning our discussion can be summarized by the following two results.

THEOREM 10C. *Suppose that a function f is analytic in a neighbourhood of z_0 . Suppose further that f has a zero of order m at z_0 . Then the function f'/f is analytic in a punctured neighbourhood of z_0 , with a simple pole at z_0 with residue m .*

THEOREM 10D. Suppose that a function f is analytic in a punctured neighbourhood of z_0 . Suppose further that f has a pole of order m at z_0 . Then the function f'/f is analytic in a punctured neighbourhood of z_0 , with a simple pole at z_0 with residue $-m$.

PROOF OF THEOREM 10C. We can write $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1}g(z) + (z - z_0)^m g'(z)}{(z - z_0)^m g(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Since $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$, the function $g'(z)/g(z)$ is analytic in a neighbourhood of z_0 . The result follows. \circ

PROOF OF THEOREM 10D. We can write $f(z) = (z - z_0)^{-m} g(z)$, where $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{-m(z - z_0)^{-m-1}g(z) + (z - z_0)^{-m} g'(z)}{(z - z_0)^{-m} g(z)} = \frac{-m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Since $g(z)$ is analytic in a neighbourhood of z_0 and $g(z_0) \neq 0$, the function $g'(z)/g(z)$ is analytic in a neighbourhood of z_0 . The result follows. \circ

The main result in this section is the Principle of the argument, as stated below.

THEOREM 10E. Suppose that a function f is meromorphic in a simply connected domain D . Suppose further that C is a Jordan curve in D , followed in the positive (anticlockwise) direction, and that f has no zeros or poles on C . If N denotes the number of zeros of f in the interior of C , counted with multiplicities, and if P denotes the number of poles of f in the interior of C , counted with multiplicities, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P.$$

PROOF. Note that by Theorems 10C and 10D, the poles of the function f'/f are precisely at the zeros and poles of f . Furthermore, a zero of f of order m gives rise to a residue m for f'/f , so that the residues of f'/f arising from the zeros of f are equal to the number of zeros of f counted with multiplicities, and this number is N . On the other hand, a pole of f of order m gives rise to a residue $-m$ for f'/f , so that the residues of f'/f arising from the poles of f are equal to minus the number of poles of f counted with multiplicities, and this number is P . It follows that the sum of the residues is equal to $N - P$. The result now follows from Theorem 10B applied to the function f'/f . \circ

REMARKS. (1) Note that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \text{var}(\log f(z), C) = \frac{1}{2\pi i} \text{var}(i \arg f(z), C) = \frac{1}{2\pi} \text{var}(\arg f(z), C).$$

It follows that the conclusion of Theorem 10E can be expressed in the form

$$N - P = \frac{1}{2\pi} \text{var}(\arg f(z), C),$$

in terms of the variation of the argument of $f(z)$ along the Jordan curve C .

(2) Note also that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f(C)} \frac{dw}{w} = n(f(C), 0).$$

(3) Theorem 10E can be generalized in the following way. Suppose that a function f is meromorphic in a simply connected domain D , and that all its zeros and poles in D are simple. Suppose further that C is a Jordan curve in D , followed in the positive (anticlockwise) direction, and that f has no zeros or poles on C . If a_1, \dots, a_N denote the zeros of f in the interior of C , and if b_1, \dots, b_P denote the poles of f in the interior of C , then

$$(1) \quad \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} g(z) dz = \sum_{j=1}^N g(a_j) - \sum_{k=1}^P g(b_k)$$

for every function g analytic in D . To see this, simply note that any simple zero or simple pole z_0 of f , where $g(z_0) \neq 0$, gives rise to a simple pole of $(f'/f)g$ with residue

$$\lim_{z \rightarrow z_0} (z - z_0) \frac{f'(z)}{f(z)} g(z) = g(z_0) \lim_{z \rightarrow z_0} (z - z_0) \frac{f'(z)}{f(z)} = \begin{cases} g(z_0) & \text{if } z_0 \text{ is a simple zero of } f, \\ -g(z_0) & \text{if } z_0 \text{ is a simple pole of } f; \end{cases}$$

on the other hand, if $g(z_0) = 0$, then $(f'/f)g$ has a removable singularity at z_0 . In fact, (1) remains valid if the zeros and poles of f are of higher order, provided that all zeros and poles are counted with multiplicities. Note also that the choice $g(z) = 1$ in D gives Theorem 10E again. A particular useful choice of f is given by the entire function $f(z) = \sin \pi z$, with simple zeros at every $n \in \mathbb{Z}$. Since

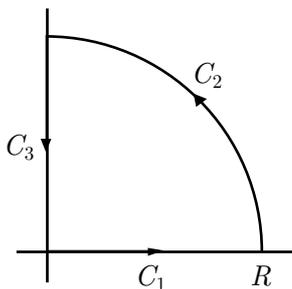
$$\frac{f'(z)}{f(z)} = \frac{\pi \cos \pi z}{\sin \pi z} = \pi \cot \pi z,$$

it follows from (1) that

$$(2) \quad \frac{1}{2\pi i} \int_C g(z) \pi \cot \pi z dz = \sum_{n \text{ inside } C} g(n)$$

for every function g analytic in D . This may be used to obtain a variety of infinite series expansions. See Chapter 16.

EXAMPLE 10.3.1. To find the number of zeros of the function $f(z) = z^4 + z^3 - 2z^2 + 2z + 4$ in the first quadrant of the complex plane, we use the Jordan curve $C = C_1 \cup C_2 \cup C_3$, where $C_1 = [0, R]$ is the straight line segment along the real axis from 0 to R , C_2 is the circular path $\zeta : [0, \pi/2] \rightarrow \mathbb{C}$, given by $\zeta(t) = Re^{it}$, and $C_3 = [iR, 0]$ is the straight line segment along the imaginary axis from iR to 0. Here R is taken to be a large positive real number.



On C_1 , we have $z = x > 0$, so that

$$f(z) = f(x) = x^4 + x^3 - 2x^2 + 2x + 4 \geq \begin{cases} x^4 + x^3 + 4 & \text{if } 0 \leq x \leq 1 \\ 2x + 4 & \text{if } x \geq 1 \end{cases}$$

is clearly positive, so that $\text{var}(\arg f(z), C_1) = 0$. Next, note that

$$f(z) = z^4 \left(1 + \frac{z^3 - 2z^2 + 2z + 4}{z^4} \right).$$

On C_2 , we have $|z| = R$, so that

$$\left| \frac{z^3 - 2z^2 + 2z + 4}{z^4} \right| \leq \frac{R^3 + 2R^2 + 2R + 4}{R^4} < \frac{2R^3}{R^4} = \frac{2}{R}$$

whenever $R > 8$, say. It follows that on C_2 when R is large enough, we have $f(z) = R^4 e^{4it}(1+w)$, where $|w| < 2/R$, so that $\text{var}(\arg f(z), C_2) = 2\pi + \epsilon_1$, where $\epsilon_1 \rightarrow 0$ as $R \rightarrow \infty$. Finally, on C_3 , we have $z = iy$, where $y > 0$, so that

$$f(z) = f(iy) = (y^4 + 2y^2 + 4) + i(2y - y^3) = (y^2 + 1)^2 + 3 + i(2y - y^3).$$

Note that $\Re f(iy) > 0$, so that $f(iy)$ is in the first or fourth quadrant of the complex plane. In fact, when $R > 0$ is large, $f(iR)$ is much nearer the real axis than the imaginary axis, while $f(0) = 4$ is on the positive real axis. It follows that $\text{var}(\arg f(z), C_3) = \epsilon_2$, where $\epsilon_2 \rightarrow 0$ as $R \rightarrow \infty$. We now conclude that $\text{var}(\arg f(z), C) = 2\pi + \epsilon_1 + \epsilon_2$, where $\epsilon_1, \epsilon_2 \rightarrow 0$ as $R \rightarrow \infty$. On the other hand, C is a closed contour, so that $\text{var}(\arg f(z), C)$ must be an integer multiple of 2π . It follows that $\text{var}(\arg f(z), C) = 2\pi$. Note now that the function f has no poles in the first quadrant. It follows from the Argument principle that f has exactly one zero inside the contour C for all large R . Hence f has exactly one zero in the first quadrant of the complex plane.

To find the number of zeros in a region, the following result provides an opportunity to either bypass the Argument principle or at least enable one to apply the Argument principle to a simpler function. Needless to say, the proof is based on an application of the Argument principle.

THEOREM 10F. (ROUCHÉ'S THEOREM) *Suppose that functions f and g are analytic in a simply connected domain D , and that C is a Jordan contour in D . Suppose further that $|f(z)| > |g(z)|$ on C . Then f and $f + g$ have the same number of zeros inside C .*

We shall give two proofs of this result. The first is the one given in most texts.

FIRST PROOF OF THEOREM 10F. Consider the function

$$F(z) = \frac{f(z) + g(z)}{f(z)}.$$

The condition $|f(z)| > |g(z)|$ on C ensures that both f and $f + g$ have no zeros on C . On the other hand, note that

$$(3) \quad |F(z) - 1| = \left| \frac{g(z)}{f(z)} \right| < 1$$

for every $z \in C$. By Remark (2) after Theorem 10E, we have

$$\frac{1}{2\pi i} \int_C \frac{F'(z)}{F(z)} dz = \frac{1}{2\pi i} \int_{F(C)} \frac{dw}{w} = n(F(C), 0).$$

In view of (3), the closed contour $F(C)$ is contained in the open disc $\{w : |w - 1| < 1\}$ with centre 1 and radius 1. This disc does not contain the point 0, so that $n(F(C), 0) = 0$. Hence

$$\frac{1}{2\pi i} \int_C \frac{F'(z)}{F(z)} dz = 0.$$

It follows from the Argument principle that the function F has the same number of zeros and poles inside C . Note now that the poles of F are precisely the zeros of f , and the zeros of F are precisely the zeros of $f + g$. \circ

SECOND PROOF OF THEOREM 10F. For every $\tau \in [0, 1]$, let

$$N(\tau) = \frac{1}{2\pi i} \int_C \frac{f'(z) + \tau g'(z)}{f(z) + \tau g(z)} dz.$$

The condition $|f(z)| > |g(z)|$ on C ensures that

$$|f(z) + \tau g(z)| \geq |f(z)| - \tau |g(z)| \geq |f(z)| - |g(z)| > 0$$

on C , so that $f + \tau g$ does not have any zeros (or poles) on C . In fact, there is a positive lower bound for $|f(z) + \tau g(z)|$ on C independent of τ . It follows easily from this that $N(\tau)$ is continuous in $[0, 1]$. By the Argument principle, $N(\tau)$ is an integer for every $\tau \in [0, 1]$. Hence $N(\tau)$ must be constant in $[0, 1]$. In particular, we must have $N(0) = N(1)$. Clearly, $N(0)$ is the number of zeros of f inside C , and $N(1)$ is the number of zeros of $f + g$ inside C . \circ

EXAMPLE 10.3.2. To determine the number of solutions of $e^z = 2z + 1$ with $|z| < 1$, we write

$$f(z) = -2z \quad \text{and} \quad g(z) = e^z - 1,$$

so that $f(z) + g(z) = e^z - 2z - 1$. We therefore need to find the number of zeros of $f + g$ inside the unit circle $C = \{z : |z| = 1\}$. Clearly, f has precisely one zero, at $z = 0$, inside C . On the other hand, note that

$$e^z - 1 = \int_{[0, z]} e^\zeta d\zeta = \int_0^1 e^{zt} z dt.$$

If $z \in C$, then $|e^{zt}| \leq e^t$, and so

$$|g(z)| = |e^z - 1| \leq \int_0^1 |e^{zt} z| dt \leq \int_0^1 e^t dt = e - 1.$$

Since $|f(z)| = 2$ whenever $z \in C$, it follows that $|f(z)| > |g(z)|$ on C . By Rouché's theorem, $f + g$ has precisely one zero inside C .

PROBLEMS FOR CHAPTER 10

- Write down the Taylor series for e^w about the origin $w = 0$.
 - Using the substitution $w = 1/z^2$ in (a), find the Laurent series for the function e^{1/z^2} about the origin $z = 0$.
 - Find the residue of the function e^{1/z^2} at the origin $z = 0$.
 - What type of singularity does the function e^{1/z^2} have at the origin $z = 0$?
- Suppose that $f(z) = g(z)/h(z)$, where the functions $g(z)$ and $h(z)$ are analytic at z_0 . Suppose further that $g(z_0) \neq 0$ and $h(z)$ has a simple zero at z_0 . Use l'Hopital's rule to show that

$$\text{res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{g(z)}{h'(z)} = \frac{g(z_0)}{h'(z_0)}.$$

3. For each of the functions $f(z)$ given below, find all the singularities in \mathbb{C} , find the residues at these singularities, and evaluate the integrals

$$\int_{C'} f(z) dz \quad \text{and} \quad \int_{C''} f(z) dz,$$

where C' and C'' are circular paths centred at the origin $z = 0$, of radius $1/2$ and 2 respectively, followed in the positive (anticlockwise) direction:

$$\text{a) } f(z) = \frac{1}{z(z-1)} \qquad \text{b) } f(z) = \frac{z}{z^4+1} \qquad \text{c) } f(z) = \frac{z^3+2}{(z^4-1)(z+1)}$$

4. Suppose that C is a circular path centred at the origin $z = 0$, of radius 1 , followed in the positive (anticlockwise) direction. Show each of the following:

$$\text{a) } \int_C \frac{e^{\pi z}}{4z^2+1} dz = \pi i; \qquad \text{b) } \int_C \frac{e^z}{z^3} dz = \pi i.$$

5. Find the number of zeros of $f(z) = z^4 + z^3 + 5z^2 + 2z + 4$ in the first quadrant of the complex plane. Find also the number of zeros of the function in the fourth quadrant.

6. Consider the equation $2z^5 + 8z - 1 = 0$.

- Writing $f(z) = 2z^5$ and $g(z) = 8z - 1$, use Rouché's theorem to show that all the roots of this equation lie in the open disc $\{z : |z| < 2\}$.
- Writing $f(z) = 8z - 1$ and $g(z) = 2z^5$, use Rouché's theorem to show that this equation has exactly one root in the open disc $\{z : |z| < 1\}$.
- How many roots does this equation have in the open annulus $\{z : 1 < |z| < 2\}$? Justify your assertion.

7. Show that the equation $z^6 + 4z^2 = 1$ has exactly two roots in the open disc $\{z : |z| < 1\}$.

[HINT: Use Rouché's theorem. You will need to make a good choice for $f(z)$ and $g(z)$. Do not give up if your first guess does not work.]

INTRODUCTION TO COMPLEX ANALYSIS

W W L CHEN

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Chapter 11

EVALUATION OF DEFINITE INTEGRALS

11.1. Introduction

The calculus of residues often provides an efficient method for evaluating certain real and complex integrals. This is particularly important when it is not possible to find indefinite integrals explicitly. Even in cases when ordinary methods of calculus can be applied, the use of residues often proves to be a labour saving device.

Naturally, the calculus of residues gives rise to complex integrals, and this suggests that we may be at a disadvantage if we want to evaluate real integrals. In practice, this is seldom the case, since a complex integral is equivalent to two real integrals.

However, there are limitations to this approach. The integrand must be closely associated with some analytic function. We usually want to integrate some elementary functions, and these can be extended to the complex domain. Also, the techniques of complex integration applies to closed curves while a real integral is over an interval. It follows that we need a device to reduce our problem to one which concerns integration over closed curves. There are a number of ways to achieve this, depending on circumstances. The technique is best learned by studying typical examples, and complete mastery does not guarantee success.

11.2. Rational Functions over the Unit Circle

We shall be concerned mainly with integrals of the type

$$\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta,$$

where $f(x, y)$ is a real valued rational function in the real variables x and y .

If we use the substitution

$$z = e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad dz = ie^{i\theta} d\theta,$$

then

$$\frac{1}{z} = \cos \theta - i \sin \theta \quad \text{and} \quad d\theta = -i \frac{dz}{z}.$$

We can therefore write

$$\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad \text{and} \quad \sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right),$$

so that

$$\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta = -i \int_C f \left(\frac{1}{2} \left(z + \frac{1}{z} \right), \frac{1}{2i} \left(z - \frac{1}{z} \right) \right) \frac{dz}{z},$$

where C is the unit circle $\{z : |z| = 1\}$, followed in the positive (anticlockwise) direction.

EXAMPLE 11.2.1. Suppose that the real number $a > 1$ is fixed. Consider the integral

$$\int_0^\pi \frac{d\theta}{a + \cos \theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{a + \cos \theta}.$$

Using the substitution $z = e^{i\theta} = \cos \theta + i \sin \theta$, we have

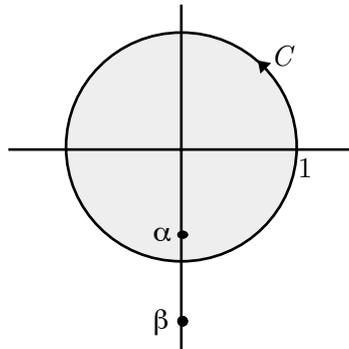
$$\int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = -i \int_C \frac{dz}{z \left(a + \frac{1}{2} \left(z + \frac{1}{z} \right) \right)} = -i \int_C \frac{2dz}{z^2 + 2az + 1},$$

where C is the unit circle $\{z : |z| = 1\}$, followed in the positive (anticlockwise) direction. It follows that

$$\int_0^\pi \frac{d\theta}{a + \cos \theta} = -i \int_C \frac{dz}{z^2 + 2az + 1}.$$

If we factorize the denominator $z^2 + 2az + 1$, we obtain roots

$$\alpha = -a + \sqrt{a^2 - 1} \quad \text{and} \quad \beta = -a - \sqrt{a^2 - 1}.$$



Clearly $|\beta| > 1$. Since $\alpha\beta = 1$, it follows that $|\alpha| < 1$. Hence the function

$$\frac{1}{z^2 + 2az + 1}$$

is analytic in some simply connected domain containing the unit circle C , except for a simple pole at $z = \alpha$ inside C , with residue

$$\operatorname{res}\left(\frac{1}{z^2 + 2az + 1}, \alpha\right) = \lim_{z \rightarrow \alpha} (z - \alpha) \frac{1}{z^2 + 2az + 1} = \lim_{z \rightarrow \alpha} \frac{1}{z - \beta} = \frac{1}{\alpha - \beta} = \frac{1}{2\sqrt{a^2 - 1}}.$$

It follows from Cauchy's residue theorem that

$$\int_C \frac{dz}{z^2 + 2az + 1} = 2\pi i \operatorname{res}\left(\frac{1}{z^2 + 2az + 1}, \alpha\right) = \frac{2\pi i}{2\sqrt{a^2 - 1}},$$

and so

$$\int_0^\pi \frac{d\theta}{a + \cos \theta} = \frac{\pi}{\sqrt{a^2 - 1}}.$$

EXAMPLE 11.2.2. Now let $w \in \mathbb{C} \setminus [-1, 1]$, and consider the integral

$$F(w) = \int_0^\pi \frac{d\theta}{w + \cos \theta}.$$

Note that we have excluded the closed interval $[-1, 1]$ to ensure that the denominator of the integrand does not vanish. One can show that $F(w)$ exists in the domain $\mathbb{C} \setminus [-1, 1]$, so that $F(w)$ is analytic there. We know from Example 11.2.1 that

$$(1) \quad F(w) = \frac{\pi}{(w^2 - 1)^{1/2}}$$

on the real axis to the right of the point $w = 1$. It follows from Theorem 7H that (1) holds for every $w \in \mathbb{C} \setminus [-1, 1]$. Note, however, that the branch of the square root must be chosen so that it is positive for $w > 1$.

11.3. Rational Functions over the Real Line

We shall be concerned mainly with integrals of the type

$$(2) \quad \int_{-\infty}^{\infty} f(x) dx,$$

where $f(x)$ is a real valued rational function in the real variable x . Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the real line, so that the integral (2) is convergent.

Consider first of all the integral

$$\int_{-R}^R f(x) dx,$$

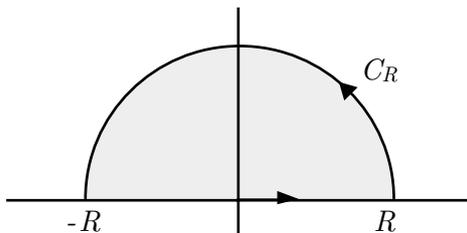
where $R > 0$. We then extend the definition of the rational function f to the complex domain, and consider also the integral

$$\int_{C_R} f(z) dz,$$

where C_R is the semicircular arc given by $z = Re^{it}$, where $t \in [0, \pi]$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $[-R, R]$ denotes the line segment from $-R$ to R .



By Cauchy's residue theorem, we have

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(f, z_i),$$

where the summation is taken over all the poles of f inside the Jordan contour C . It is easily shown that

$$\int_{C_R} f(z) dz \rightarrow 0$$

as $R \rightarrow \infty$, so that

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{\Im z_i > 0} \text{res}(f, z_i),$$

where the summation is taken over all the poles of f in the upper half plane.

EXAMPLE 11.3.1. Suppose that the real number $a > 0$ is fixed. Consider the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^3} dx.$$

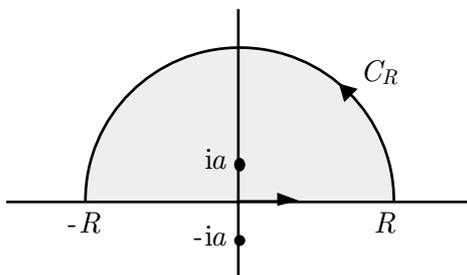
To evaluate this integral, note that the rational function

$$f(z) = \frac{z^2}{(z^2 + a^2)^3}$$

has poles of order 3 at $z = \pm ia$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > a$.



By Cauchy's residue theorem, we have

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i \operatorname{res}(f, ia).$$

Since

$$\begin{aligned} \operatorname{res}(f, ia) &= \frac{1}{2} \lim_{z \rightarrow ia} \frac{d^2}{dz^2} \left((z - ia)^3 \frac{z^2}{(z^2 + a^2)^3} \right) = \frac{1}{2} \lim_{z \rightarrow ia} \left(\frac{2}{(z + ia)^3} - \frac{12z}{(z + ia)^4} + \frac{12z^2}{(z + ia)^5} \right) \\ &= \frac{1}{2} \left(\frac{2}{(2ia)^3} - \frac{12ia}{(2ia)^4} - \frac{12a^2}{(2ia)^5} \right) = -\frac{i}{16a^3}, \end{aligned}$$

it follows that

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = \frac{\pi}{8a^3}.$$

Note now that

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{R^2}{(R^2 - a^2)^3} \pi R \rightarrow 0$$

as $R \rightarrow \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^3} dx = \int_{-\infty}^{\infty} f(x) dx = \frac{\pi}{8a^3}.$$

EXAMPLE 11.3.2. Consider the integral

$$\int_{-\infty}^{\infty} \frac{x^2 + 3}{x^4 + 5x^2 + 4} dx.$$

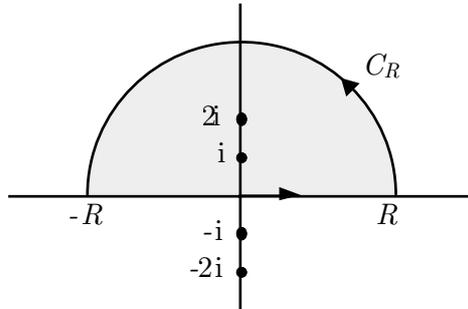
To evaluate this integral, note that the rational function

$$f(z) = \frac{z^2 + 3}{z^4 + 5z^2 + 4} = \frac{z^2 + 3}{(z^2 + 1)(z^2 + 4)}$$

has simple poles at $z = \pm i$ and $z = \pm 2i$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > 2$.



By Cauchy's residue theorem, we have

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i (\operatorname{res}(f, i) + \operatorname{res}(f, 2i)).$$

By Problem 2 in Chapter 10, we have

$$\operatorname{res}(f, i) = \lim_{z \rightarrow i} \frac{z^2 + 3}{4z^3 + 10z} = \frac{1}{3i} \quad \text{and} \quad \operatorname{res}(f, 2i) = \lim_{z \rightarrow 2i} \frac{z^2 + 3}{4z^3 + 10z} = \frac{1}{12i}.$$

It follows that

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = \frac{5\pi}{6}.$$

Note now that

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{R^2 + 3}{R^4 - 5R^2 - 4} \pi R \rightarrow 0$$

as $R \rightarrow \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{x^2 + 3}{x^4 + 5x^2 + 4} dx = \int_{-\infty}^{\infty} f(x) dx = \frac{5\pi}{6}.$$

11.4. Rational and Trigonometric Functions over the Real Line

We shall be concerned mainly with integrals of the type

$$(3) \quad \int_{-\infty}^{\infty} f(x)e^{ix} dx,$$

where $f(x)$ is a real valued rational function in the real variable x . Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the real line, so that the integral (3) is convergent. Note that the real and imaginary parts of the integral (3) are respectively

$$\int_{-\infty}^{\infty} f(x) \cos x dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \sin x dx.$$

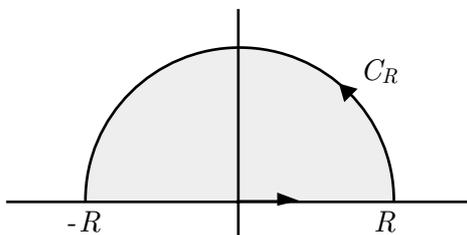
Consider first of all the integral

$$\int_{-R}^R f(x)e^{ix} dx,$$

where $R > 0$. We consider also the integral

$$\int_{C_R} f(z)e^{iz} dz,$$

where C_R is the semicircular arc given by $z = Re^{it}$, where $t \in [0, \pi]$.



Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $[-R, R]$ denotes the line segment from $-R$ to R . By Cauchy's residue theorem, we have

$$(4) \quad \int_{-R}^R f(x)e^{ix} dx + \int_{C_R} f(z)e^{iz} dz = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(f(z)e^{iz}, z_i),$$

where the summation is taken over all the poles of $f(z)e^{iz}$ inside the Jordan contour C .

To study the second integral in (4), we prove the following estimate.

THEOREM 11A. (JORDAN'S LEMMA) *Suppose that $R > 0$. Suppose further that C_R is the semicircular arc given by $z = Re^{it}$, where $t \in [0, \pi]$. Then*

$$(5) \quad \int_{C_R} |e^{iz}| |dz| < \pi.$$

PROOF. Note that

$$(6) \quad \begin{aligned} \int_{C_R} |e^{iz}| |dz| &= \int_0^\pi |e^{iRe^{it}}| |iRe^{it}| dt = R \int_0^\pi |e^{iR(\cos t + i \sin t)}| dt \\ &= R \int_0^\pi e^{-R \sin t} dt = 2R \int_0^{\pi/2} e^{-R \sin t} dt. \end{aligned}$$

Since

$$\sin t \geq \frac{2}{\pi} t \quad \text{whenever } 0 \leq t \leq \frac{\pi}{2},$$

it follows that

$$(7) \quad \int_0^{\pi/2} e^{-R \sin t} dt \leq \int_0^{\pi/2} e^{-2Rt/\pi} dt = \frac{\pi}{2R} (1 - e^{-R}) < \frac{\pi}{2R}.$$

The inequality (5) follows on combining (6) and (7). \circ

It follows easily from Theorem 11A that

$$\int_{C_R} f(z)e^{iz} dz \rightarrow 0$$

as $R \rightarrow \infty$, so that

$$\int_{-\infty}^{\infty} f(x)e^{ix} dx = 2\pi i \sum_{\Im z_i > 0} \text{res}(f(z)e^{iz}, z_i),$$

where the summation is taken over all the poles of $f(z)e^{iz}$ in the upper half plane.

REMARK. In view of Jordan's lemma, we may consider integrals of the form (3) where the degree of the denominator of the rational function f exceeds the degree of the numerator of f by only 1. Note, however, that the argument in this case only establishes the existence of the integral (3) as

$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x)e^{ix} dx.$$

EXAMPLE 11.4.1. Suppose that the real number $a > 0$ is fixed. Consider the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx.$$

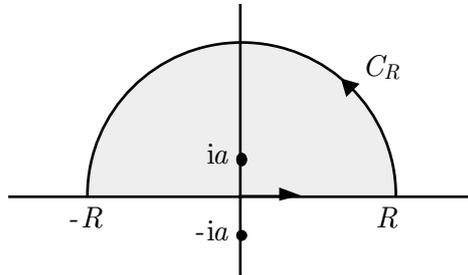
To evaluate this integral, note that the function

$$F(z) = \frac{e^{iz}}{z^2 + a^2}$$

has simple poles at $z = \pm ia$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > a$.



By Cauchy's residue theorem, we have

$$\int_{-R}^R F(x) dx + \int_{C_R} F(z) dz = 2\pi i \operatorname{res}(F, ia).$$

Since

$$\operatorname{res}(F, ia) = \lim_{z \rightarrow ia} \left((z - ia) \frac{e^{iz}}{z^2 + a^2} \right) = \lim_{z \rightarrow ia} \frac{e^{iz}}{z + ia} = \frac{e^{-a}}{2ia},$$

it follows that

$$\int_{-R}^R F(x) dx + \int_{C_R} F(z) dz = \frac{\pi e^{-a}}{a}.$$

Note now that

$$\left| \int_{C_R} F(z) dz \right| \leq \frac{1}{R^2 - a^2} \int_{C_R} |e^{iz}| |dz| < \frac{\pi}{R^2 - a^2} \rightarrow 0$$

as $R \rightarrow \infty$. Hence

$$\int_{-\infty}^{\infty} F(x) dx = \frac{\pi e^{-a}}{a},$$

so that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \Re \int_{-\infty}^{\infty} F(x) dx = \frac{\pi e^{-a}}{a}.$$

EXAMPLE 11.4.2. Suppose that the real numbers $a > 0$ and $b > 0$ are fixed and different. Consider the integral

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)} dx.$$

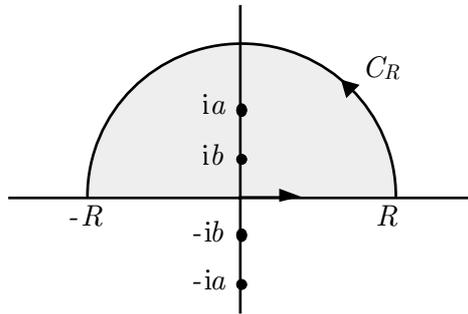
To evaluate this integral, note that the function

$$F(z) = \frac{z^3 e^{iz}}{(z^2 + a^2)(z^2 + b^2)}$$

has simple poles at $z = \pm ia$ and $z = \pm ib$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > \max\{a, b\}$.



By Cauchy's residue theorem, we have

$$\int_{-R}^R F(x) dx + \int_{C_R} F(z) dz = 2\pi i (\text{res}(F, ia) + \text{res}(F, ib)).$$

Since

$$\text{res}(F, ia) = \lim_{z \rightarrow ia} \left((z - ia) \frac{z^3 e^{iz}}{(z^2 + a^2)(z^2 + b^2)} \right) = \lim_{z \rightarrow ia} \frac{z^3 e^{iz}}{(z + ia)(z^2 + b^2)} = \frac{a^2 e^{-a}}{2(a^2 - b^2)}$$

and

$$\text{res}(F, ib) = \lim_{z \rightarrow ib} \left((z - ib) \frac{z^3 e^{iz}}{(z^2 + a^2)(z^2 + b^2)} \right) = \lim_{z \rightarrow ib} \frac{z^3 e^{iz}}{(z^2 + a^2)(z + ib)} = \frac{b^2 e^{-b}}{2(b^2 - a^2)},$$

it follows that

$$\int_{-R}^R F(x) dx + \int_{C_R} F(z) dz = \pi i \left(\frac{a^2 e^{-a}}{a^2 - b^2} + \frac{b^2 e^{-b}}{b^2 - a^2} \right).$$

Note now that

$$\left| \int_{C_R} F(z) dz \right| \leq \frac{R^3}{(R^2 - a^2)(R^2 - b^2)} \int_{C_R} |e^{iz}| |dz| < \frac{\pi R^3}{(R^2 - a^2)(R^2 - b^2)} \rightarrow 0$$

as $R \rightarrow \infty$. Hence

$$\int_{-\infty}^{\infty} F(x) dx = \frac{\pi i (a^2 e^{-a} - b^2 e^{-b})}{a^2 - b^2},$$

so that

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)} dx = \Im \int_{-\infty}^{\infty} F(x) dx = \frac{\pi (a^2 e^{-a} - b^2 e^{-b})}{a^2 - b^2}.$$

EXAMPLE 11.4.3. Consider the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

To evaluate this integral, note that the function

$$F(z) = \frac{e^{iz}}{z}$$

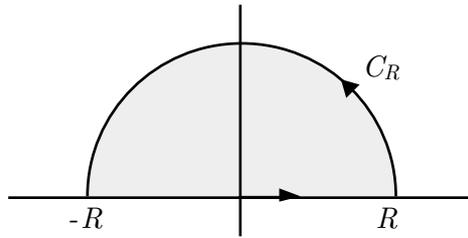
has a simple pole at $z = 0$. We consider instead the function

$$G(z) = \frac{e^{iz} - 1}{z}$$

which has a removable singularity at $z = 0$. Consider now the Jordan contour

$$C = [-R, R] \cup C_R,$$

where $R > 0$.



By Cauchy's integral theorem, we have

$$\int_{-R}^R G(x) dx + \int_{C_R} G(z) dz = 0,$$

so that

$$\int_{-R}^R G(x) dx = \int_{C_R} \frac{dz}{z} - \int_{C_R} \frac{e^{iz}}{z} dz.$$

Note that

$$\int_{C_R} \frac{dz}{z} = \pi i \quad \text{and} \quad \left| \int_{C_R} \frac{e^{iz}}{z} dz \right| \leq \frac{1}{R} \int_{C_R} |e^{iz}| |dz| < \frac{\pi}{R}.$$

Hence

$$\left| \int_{-R}^R G(x) dx - \pi i \right| < \frac{\pi}{R}.$$

Since

$$\int_{-R}^R \frac{\sin x}{x} dx = \Im \int_{-R}^R G(x) dx,$$

it follows that

$$\left| \int_{-R}^R \frac{\sin x}{x} dx - \pi \right| < \frac{\pi}{R},$$

so that letting $R \rightarrow \infty$, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

We shall return to this example later.

Note that the previous two examples do not fit the discussion at the beginning of this section, since the degrees of the denominators of the rational functions in question do not exceed the degrees of the numerators by at least 2. In fact, we have a non-trivial convergence problem for the integral

$$\int_{-\infty}^{\infty} f(x) \sin x dx.$$

The argument formally establishes the existence of this integral as

$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x) \sin x dx,$$

and not as

$$\lim_{\substack{X_1 \rightarrow \infty \\ X_2 \rightarrow \infty}} \int_{-X_1}^{X_2} f(x) \sin x dx.$$

However, it turns out that this does not cause any difficulties, since the functions $f(x) \sin x$ in question turn out to be even functions of x , so that for $X_1, X_2 > 0$, we have

$$\int_{-X_1}^{X_2} f(x) \sin x dx = \int_0^{X_1} f(x) \sin x dx + \int_0^{X_2} f(x) \sin x dx.$$

Let us examine the problem more carefully. Consider the integral

$$\int_{-\infty}^{\infty} f(x)e^{ix} dx,$$

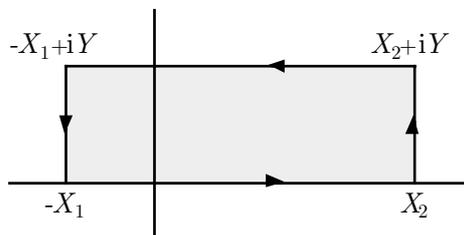
where $f(x)$ is a real valued rational function in the real variable x . Suppose now that the degree of the denominator of f exceeds the degree of the numerator of f by exactly 1, and that f has no poles on the real line. To establish the existence of the integral, we need to study the integral

$$\int_{-X_1}^{X_2} f(x)e^{ix} dx,$$

where $-X_1 < 0 < X_2$, and consider the limit as $X_1 \rightarrow \infty$ and $X_2 \rightarrow \infty$. Clearly we cannot use the semicircular arc. We shall use instead a rectangular contour

$$C = [-X_1, X_2] \cup [X_2, X_2 + iY] \cup [X_2 + iY, -X_1 + iY] \cup [-X_1 + iY, -X_1],$$

where $Y > 0$. Here $[Z_1, Z_2]$, where $Z_1, Z_2 \in \mathbb{C}$, denotes the line segment from Z_1 to Z_2 .



By Cauchy's residue theorem, we have

$$(8) \quad \int_{-X_1}^{X_2} f(x)e^{ix} dx + \int_{[X_2, X_2+iY]} f(z)e^{iz} dz + \int_{[X_2+iY, -X_1+iY]} f(z)e^{iz} dz + \int_{[-X_1+iY, -X_1]} f(z)e^{iz} dz \\ = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(f(z)e^{iz}, z_i),$$

where the summation is taken over all the poles of $f(z)e^{iz}$ inside the rectangular contour C . When X_1, X_2 and Y are large, then all the poles of the function $f(z)e^{iz}$ in the upper half plane are inside the contour C .

Under our hypotheses, the function $zf(z)$ is bounded. Suppose that $|zf(z)| \leq M$ for every $z \in \mathbb{C}$.

Note first of all that

$$\int_{[X_2, X_2+iY]} f(z)e^{iz} dz = i \int_0^Y f(X_2 + iy)e^{i(X_2+iy)} dy.$$

Since

$$|f(X_2 + iy)| \leq \frac{M}{|X_2 + iy|} \leq \frac{M}{X_2},$$

we have

$$(9) \quad \left| \int_{[X_2, X_2+iY]} f(z)e^{iz} dz \right| \leq \frac{M}{X_2} \int_0^Y e^{-y} dy = \frac{M}{X_2} (1 - e^{-Y}) < \frac{M}{X_2}.$$

Similarly,

$$(10) \quad \left| \int_{[-X_1+iY, -X_1]} f(z)e^{iz} dz \right| < \frac{M}{X_1}.$$

Next, note that

$$\int_{[X_2+iY, -X_1+iY]} f(z)e^{iz} dz = - \int_{-X_1}^{X_2} f(x + iY)e^{i(x+iY)} dx.$$

Since

$$|f(x + iY)| \leq \frac{M}{|x + iY|} \leq \frac{M}{Y},$$

we have

$$(11) \quad \left| \int_{[X_2+iY, -X_1+iY]} f(z)e^{iz} dz \right| \leq \frac{M}{Y} \int_{-X_1}^{X_2} e^{-Y} dx = \frac{Me^{-Y}}{Y} (X_1 + X_2).$$

Combining (8)–(11), we conclude that for sufficiently large X_1, X_2 and Y , we have

$$(12) \quad \left| \int_{-X_1}^{X_2} f(x)e^{ix} dx - 2\pi i \sum_{\Im z_i > 0} \text{res}(f(z)e^{iz}, z_i) \right| < \frac{M}{X_1} + \frac{M}{X_2} + \frac{Me^{-Y}}{Y} (X_1 + X_2).$$

Note that the left hand side of (12) is independent of Y . For fixed X_1 and X_2 , we have

$$\frac{Me^{-Y}}{Y} (X_1 + X_2) \rightarrow 0$$

as $Y \rightarrow \infty$. It follows that

$$\left| \int_{-X_1}^{X_2} f(x)e^{ix} dx - 2\pi i \sum_{\Im z_i > 0} \text{res}(f(z)e^{iz}, z_i) \right| \leq \frac{M}{X_1} + \frac{M}{X_2}.$$

Letting $X_1 \rightarrow \infty$ and $X_2 \rightarrow \infty$, we conclude that

$$\int_{-\infty}^{\infty} f(x)e^{ix} dx = 2\pi i \sum_{\Im z_i > 0} \text{res}(f(z)e^{iz}, z_i).$$

11.5. Bending Round a Singularity

We shall first indicate the ideas by two examples.

EXAMPLE 11.5.1. Recall Example 11.4.3, and consider again the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

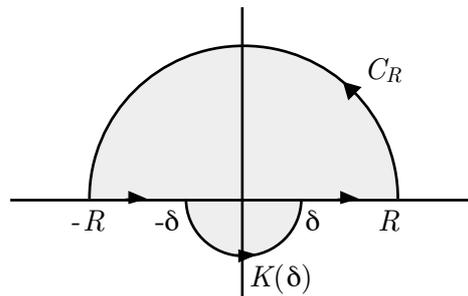
If we use the function

$$F(z) = \frac{e^{iz}}{z}$$

to evaluate this integral, then the Jordan contour $C = [-R, R] \cup C_R$ discussed earlier is unsuitable, since the singular point $z = 0$ is on the contour. Let us consider instead the Jordan contour

$$C = [-R, -\delta] \cup K(\delta) \cup [\delta, R] \cup C_R,$$

where $R > \delta > 0$, and where $K(\delta)$ denotes the semicircular arc $z = \delta e^{it}$, where $t \in [\pi, 2\pi]$.



By Cauchy's residue theorem, we have

$$\int_{-R}^{-\delta} F(x) dx + \int_{K(\delta)} F(z) dz + \int_{\delta}^R F(x) dx + \int_{C_R} F(z) dz = 2\pi i \text{res}(F, 0).$$

Note that the function $F(z)$ is analytic in \mathbb{C} apart from a simple pole at $z = 0$ with residue 1, so that $\text{res}(F, 0) = 1$. It follows that

$$F(z) = \frac{1}{z} + G(z),$$

where $G(z)$ is entire. Furthermore, it is easy to show that

$$\int_{K(\delta)} \frac{dz}{z} = \pi i.$$

Hence

$$\int_{-R}^{-\delta} F(x) dx + \int_{\delta}^R F(x) dx + \int_{K(\delta)} G(z) dz + \int_{C_R} F(z) dz = \pi i.$$

Since $G(z)$ is entire, there exists $M > 0$ such that $|G(z)| < M$ whenever $|z| \leq 1$, so that for every $\delta < 1$, we have

$$\left| \int_{K(\delta)} G(z) dz \right| \leq M\pi\delta.$$

On the other hand, a simple application of Jordan's lemma gives

$$\left| \int_{C_R} F(z) dz \right| < \frac{\pi}{R}.$$

It follows that if $\delta < 1$, then

$$\left| \int_{-R}^{-\delta} F(x) dx + \int_{\delta}^R F(x) dx - \pi i \right| < M\pi\delta + \frac{\pi}{R}.$$

Letting $\delta \rightarrow 0$ and $R \rightarrow \infty$, we obtain

$$\int_{-\infty}^{\infty} F(x) dx = \pi i.$$

Taking imaginary parts gives

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

EXAMPLE 11.5.2. Suppose that the real number $a > 0$ is fixed. Consider the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx.$$

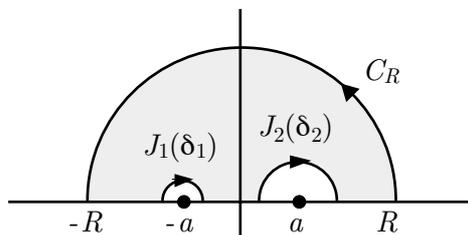
To evaluate this integral, note that the function

$$F(z) = \frac{e^{iz}}{a^2 - z^2}$$

has simple poles at $z = \pm a$. Consider now the Jordan contour

$$C = [-R, -a - \delta_1] \cup J_1(\delta_1) \cup [-a + \delta_1, a - \delta_2] \cup J_2(\delta_2) \cup [a + \delta_2, R] \cup C_R,$$

where $R > 2a$ and $0 < \delta_1, \delta_2 < a$.



Here $-J_1(\delta_1)$ denotes the semicircular arc $z = -a + \delta_1 e^{it}$, where $t \in [0, \pi]$, and $-J_2(\delta_2)$ denotes the semicircular arc $z = a + \delta_2 e^{it}$, where $t \in [0, \pi]$. By Cauchy's integral theorem, we have

$$\begin{aligned} & \int_{-R}^{-a-\delta_1} F(x) dx - \int_{-J_1(\delta_1)} F(z) dz + \int_{-a+\delta_1}^{a-\delta_2} F(x) dx \\ & - \int_{-J_2(\delta_2)} F(z) dz + \int_{a+\delta_2}^R F(x) dx + \int_{C_R} F(z) dz = 0. \end{aligned}$$

Note that the function $F(z)$ is analytic in \mathbb{C} apart from simple pole at $z = \pm a$ with residues

$$\operatorname{res}(F, -a) = \lim_{z \rightarrow -a} (z+a)F(z) = \frac{e^{-ia}}{2a} \quad \text{and} \quad \operatorname{res}(F, a) = \lim_{z \rightarrow a} (z-a)F(z) = -\frac{e^{ia}}{2a}.$$

It follows that

$$F(z) = \frac{e^{-ia}}{2a(z+a)} + G_1(z) = -\frac{e^{ia}}{2a(z-a)} + G_2(z),$$

where $G_1(z)$ is analytic in $\{z : |z+a| \leq a\}$ and $G_2(z)$ is analytic in $\{z : |z-a| \leq a\}$. Furthermore, it is easy to show that

$$\int_{-J_1(\delta_1)} \frac{e^{-ia}}{2a(z+a)} dz = \frac{\pi i e^{-ia}}{2a} \quad \text{and} \quad \int_{-J_2(\delta_2)} \frac{e^{ia}}{2a(z-a)} dz = \frac{\pi i e^{ia}}{2a}.$$

Hence

$$\begin{aligned} & \int_{-R}^{-a-\delta_1} F(x) dx + \int_{-a+\delta_1}^{a-\delta_2} F(x) dx + \int_{a+\delta_2}^R F(x) dx \\ & - \int_{-J_1(\delta_1)} G_1(z) dz - \int_{-J_2(\delta_2)} G_2(z) dz + \int_{C_R} F(z) dz \\ & = \frac{\pi i e^{-ia}}{2a} - \frac{\pi i e^{ia}}{2a} = \frac{\pi i}{2a} (e^{-ia} - e^{ia}). \end{aligned}$$

Since $G_1(z)$ is analytic in $\{z : |z+a| \leq a\}$, there exists $M_1 > 0$ such that $|G_1(z)| < M_1$ whenever $|z+a| \leq a$, so that for every $\delta_1 < a$, we have

$$\left| \int_{-J_1(\delta_1)} G_1(z) dz \right| \leq M_1 \pi \delta_1.$$

Since $G_2(z)$ is analytic in $\{z : |z-a| \leq a\}$, there exists $M_2 > 0$ such that $|G_2(z)| < M_2$ whenever $|z-a| \leq a$, so that for every $\delta_2 < a$, we have

$$\left| \int_{-J_2(\delta_2)} G_2(z) dz \right| \leq M_2 \pi \delta_2.$$

On the other hand, a simple application of Jordan's lemma gives

$$\left| \int_{C_R} F(z) dz \right| < \frac{\pi}{R^2 - a^2}.$$

It follows that if $0 < \delta_1, \delta_2 < a$ and $R > 2a$, then

$$\begin{aligned} & \left| \int_{-R}^{-a-\delta_1} F(x) dx + \int_{-a+\delta_1}^{a-\delta_2} F(x) dx + \int_{a+\delta_2}^R F(x) dx - \frac{\pi i}{2a} (e^{-ia} - e^{ia}) \right| \\ & < M_1 \pi \delta_1 + M_2 \pi \delta_2 + \frac{\pi}{R^2 - a^2}. \end{aligned}$$

Letting $\delta_1, \delta_2 \rightarrow 0$ and $R \rightarrow \infty$, we obtain

$$\int_{-\infty}^{\infty} F(x) dx = \frac{\pi i}{2a} (e^{-ia} - e^{ia}).$$

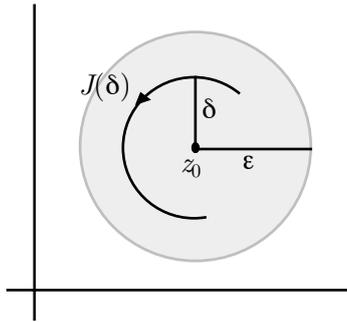
Taking real parts gives

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx = \frac{\pi \sin a}{a}.$$

REMARK. Note that in Example 11.5.1, we have bent round the singularity in question by using a contour with the singularity in the interior, whereas in Example 11.5.2, we have bent round the singularities in question by using a contour with the singularities in the exterior.

We have used the following general result.

THEOREM 11B. Suppose that a function $F(z)$ is analytic in an ϵ -neighbourhood of z_0 , apart from a simple pole at z_0 with residue a_{-1} . Suppose further that $0 \leq t_1 < t_2 \leq 2\pi$. For every positive $\delta < \epsilon$, let $J(\delta)$ denote a circular arc of the form $z = z_0 + \delta e^{it}$, where $t \in [t_1, t_2]$.



Then

$$\lim_{\delta \rightarrow 0} \int_{J(\delta)} F(z) dz = ia_{-1}(t_2 - t_1).$$

PROOF. We can write

$$F(z) = \frac{a_{-1}}{z - z_0} + G(z),$$

where $G(z)$ is analytic in the closed disc $\{z : |z - z_0| \leq \delta\}$. Then

$$\int_{J(\delta)} F(z) dz = \int_{J(\delta)} \frac{a_{-1}}{z - z_0} dz + \int_{J(\delta)} G(z) dz.$$

It is easy to check that

$$\int_{J(\delta)} \frac{a_{-1}}{z - z_0} dz = ia_{-1}(t_2 - t_1).$$

On the other hand, since $G(z)$ is analytic in the closed disc $\{z : |z - z_0| \leq \delta\}$, it is bounded in this disc, and so there exists $M > 0$ such that $|G(z)| \leq M$ whenever $|z - z_0| \leq \delta$, whence

$$\left| \int_{J(\delta)} G(z) dz \right| \leq M \int_{J(\delta)} |dz| \leq 2\pi M \delta \rightarrow 0$$

as $\delta \rightarrow 0$. The result follows. \circ

11.6. Integrands with Branch Points

Consider an integral of the type

$$(13) \quad \int_0^{\infty} x^{\alpha} f(x) dx,$$

where $f(x)$ is a real valued rational function in the real variable x . Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the positive real axis and at most a simple pole at the origin. We shall also assume that $0 < \alpha < 1$.

The problem here is that the function $z^{\alpha} f(z)$ is not single valued. However, this is precisely the circumstance that makes it possible to find the integral. The simplest technique is to first of all make the substitution $x = u^2$, and note that

$$(14) \quad \int_0^{\infty} x^{\alpha} f(x) dx = 2 \int_0^{\infty} u^{2\alpha+1} f(u^2) du.$$

We now consider the function

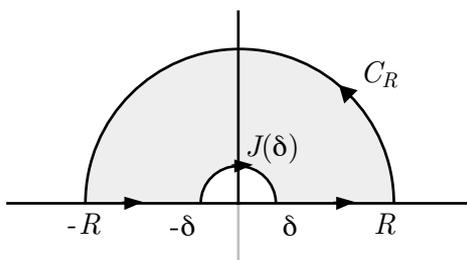
$$F(z) = z^{2\alpha+1} f(z^2).$$

For the function $z^{2\alpha}$, by choosing the branch so that the argument of $z^{2\alpha}$ lies between $-\pi\alpha$ and $3\pi\alpha$, it is easy to see that this is well defined and analytic in the region obtained from \mathbb{C} by deleting the origin and the negative imaginary axis. It follows that as long as a Jordan contour avoids this cut, then we can use Cauchy's residue theorem on $F(z)$.

We shall consider the Jordan contour

$$C = [-R, -\delta] \cup J(\delta) \cup [\delta, R] \cup C_R,$$

where $R > \delta > 0$, $-J(\delta)$ denotes the semicircular arc of the form $z = \delta e^{it}$, where $t \in [0, \pi]$, and C_R denotes the semicircular arc of the form $z = R e^{it}$, where $t \in [0, \pi]$.



By Cauchy's residue theorem, we have

$$\int_{-R}^{-\delta} F(z) dz + \int_{J(\delta)} F(z) dz + \int_{\delta}^R F(z) dz + \int_{C_R} F(z) dz = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(F, z_i),$$

where the summation is taken over all the poles of F inside the Jordan contour C . It is easily shown that

$$\int_{J(\delta)} F(z) dz \rightarrow 0 \quad \text{and} \quad \int_{C_R} F(z) dz \rightarrow 0$$

as $\delta \rightarrow 0$ and $R \rightarrow \infty$ respectively, so that

$$\int_{-\infty}^{\infty} F(z) dz = 2\pi i \sum_{\Im z_i > 0} \text{res}(F, z_i),$$

where the summation is taken over all the poles of F in the upper half plane. On the other hand, note that $(-z)^{2\alpha} = e^{2\pi i\alpha} z^{2\alpha}$, and so

$$\int_{-\infty}^{\infty} z^{2\alpha+1} f(z^2) dz = \int_{-\infty}^{\infty} (z^{2\alpha+1} + (-z)^{2\alpha+1}) f(z^2) dz = (1 - e^{2\pi i\alpha}) \int_0^{\infty} u^{2\alpha+1} f(u^2) du.$$

Since $e^{2\pi i\alpha} \neq 1$, this gives us a way of calculating the integral on the right hand side of (14).

EXAMPLE 11.6.1. Suppose that the real number $\alpha \in (0, 1)$ is fixed. Consider the integral

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx.$$

The substitution $x = u^2$ gives

$$(15) \quad \int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx = 2 \int_0^{\infty} \frac{u^{2\alpha-1}}{u^2+1} du.$$

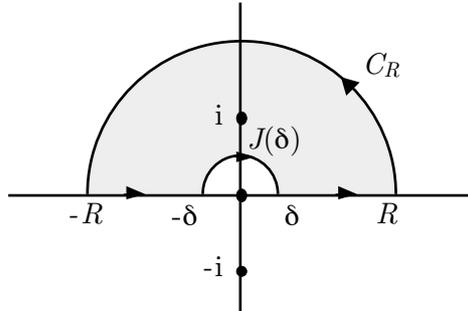
To evaluate this integral, note that the function

$$F(z) = \frac{z^{2\alpha-1}}{z^2+1}$$

has a singularity at $z = 0$ and simple poles at $z = \pm i$. Consider now the Jordan contour

$$C = [-R, -\delta] \cup J(\delta) \cup [\delta, R] \cup C_R,$$

where $R > 1 > \delta > 0$.



By Cauchy's residue theorem, we have

$$\int_{-R}^{-\delta} F(z) dz + \int_{J(\delta)} F(z) dz + \int_{\delta}^R F(z) dz + \int_{C_R} F(z) dz = 2\pi i \operatorname{res}(F, i).$$

Since

$$\operatorname{res}(F, i) = \lim_{z \rightarrow i} (z - i)F(z) = -\frac{1}{2}e^{\pi i\alpha},$$

it follows that

$$\int_{-R}^{-\delta} F(z) dz + \int_{J(\delta)} F(z) dz + \int_{\delta}^R F(z) dz + \int_{C_R} F(z) dz = -\pi i e^{\pi i\alpha}.$$

Note now that

$$\left| \int_{J(\delta)} F(z) dz \right| \leq \frac{\delta^{2\alpha+1}}{\delta^2 - \delta^4} \pi \delta \rightarrow 0 \quad \text{and} \quad \left| \int_{C_R} F(z) dz \right| \leq \frac{R^{2\alpha+1}}{R^4 - R^2} \pi R \rightarrow 0$$

as $\delta \rightarrow 0$ and $R \rightarrow \infty$ respectively. Hence

$$(1 - e^{2\pi i\alpha}) \int_0^\infty \frac{u^{2\alpha+1}}{u^2 + u^4} du = \int_{-\infty}^\infty \frac{z^{2\alpha+1}}{z^2 + z^4} dz = -\pi i e^{\pi i\alpha},$$

so that

$$\int_0^\infty \frac{u^{2\alpha+1}}{u^2 + u^4} du = -\frac{\pi i e^{\pi i\alpha}}{1 - e^{2\pi i\alpha}} = -\frac{\pi i}{e^{-\pi i\alpha} - e^{\pi i\alpha}} = \frac{\pi}{2 \sin \pi\alpha}.$$

It now follows from (15) that

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin \pi\alpha}.$$

In fact, the integral (13) can be studied without the substitution $x = u^2$. However, the contour that we use will not be a Jordan contour. We shall consider the function

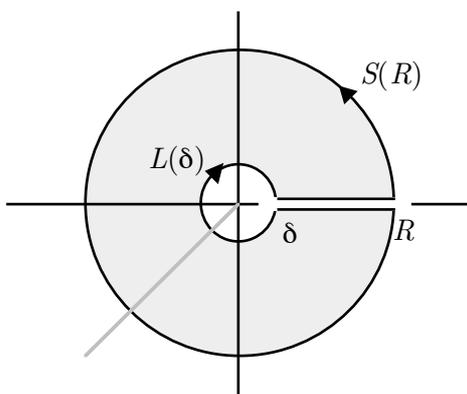
$$F(z) = z^\alpha f(z),$$

and choose a branch of z^α so that the argument lies between 0 and $2\pi\alpha$.

We now consider the contour

$$C = [\delta, R] \cup S(R) \cup [R, \delta] \cup L(\delta),$$

where $R > \delta > 0$, $S(R)$ denotes the circle of the form $z = Re^{it}$, where $t \in [0, 2\pi]$, and $L(\delta)$ denotes the circle of the form $z = \delta e^{it}$, where $t \in [0, 2\pi]$.



Clearly C is not a Jordan contour. However, there clearly exists $t_0 \in (0, 2\pi)$ such that the line segment joining δe^{it_0} and Re^{it_0} does not pass through any singularities of $f(z)$ in $\{z : |z| \leq R\}$. Now let

$$C_1 = [\delta, R] \cup S_1(R) \cup [Re^{it_0}, \delta e^{it_0}] \cup L_1(\delta),$$

where $S_1(R)$ denotes the circular arc of the form $z = Re^{it}$, where $t \in [0, t_0]$, and $L_1(\delta)$ denotes the circular arc of the form $z = \delta e^{it}$, where $t \in [0, t_0]$. Also let

$$C_2 = [R, \delta] \cup L_2(\delta) \cup [\delta e^{it_0}, Re^{it_0}] \cup S_2(R),$$

where $S_2(R)$ denotes the circular arc of the form $z = Re^{it}$, where $t \in [t_0, 2\pi]$, and $L_2(\delta)$ denotes the circular arc of the form $z = \delta e^{it}$, where $t \in [t_0, 2\pi]$.

It is easy to see that

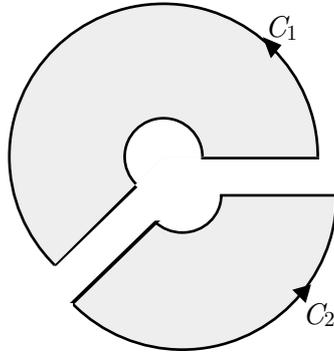
$$(16) \quad \int_C F(z) dz = \int_{C_1} F(z) dz + \int_{C_2} F(z) dz.$$

Clearly there exists $\epsilon_0 > 0$ such that C_1 is a Jordan contour in the simply connected domain

$$D_1 = \{z \neq 0 : -\epsilon_0 < \arg z < t_0 + \epsilon_0\}$$

and C_2 is a Jordan contour in the simply connected domain

$$D_2 = \{z \neq 0 : t_0 - \epsilon_0 < \arg z < 2\pi + \epsilon_0\}.$$



Applying Cauchy's residue theorem, we obtain

$$\int_{C_1} F(z) dz = 2\pi i \sum_{z_i \text{ inside } C_1} \text{res}(F, z_i) \quad \text{and} \quad \int_{C_2} F(z) dz = 2\pi i \sum_{z_i \text{ inside } C_2} \text{res}(F, z_i),$$

so that

$$(17) \quad \int_C F(z) dz = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(F, z_i).$$

Note that

$$(18) \quad \int_{C_1} F(z) dz = \int_{[\delta, R]} F(z) dz + \int_{S_1(R)} F(z) dz + \int_{[Re^{it_0}, \delta e^{it_0}]} F(z) dz + \int_{L_1(\delta)} F(z) dz,$$

and

$$(19) \quad \int_{C_2} F(z) dz = \int_{[R, \delta]} F(z) dz + \int_{L_2(\delta)} F(z) dz + \int_{[\delta e^{it_0}, Re^{it_0}]} F(z) dz + \int_{S_2(R)} F(z) dz.$$

When we evaluate the integrals in (18), we need $0 \leq \arg z \leq t_0$. Hence

$$(20) \quad \int_{[\delta, R]} F(z) dz = \int_{\delta}^R F(x) dx.$$

When we evaluate the integrals in (19), we need $t_0 \leq \arg z \leq 2\pi$. Hence

$$(21) \quad \int_{[R, \delta]} F(z) dz = \int_{[R, \delta]} z^\alpha f(z) dz = - \int_{\delta}^R x^\alpha e^{2\pi i \alpha} f(x) dx = -e^{2\pi i \alpha} \int_{\delta}^R F(x) dx.$$

Also

$$(22) \quad \int_{[Re^{it_0}, \delta e^{it_0}]} F(z) dz + \int_{[\delta e^{it_0}, Re^{it_0}]} F(z) dz = 0.$$

It follows from (16), (18)–(22) that

$$(23) \quad \begin{aligned} \int_C F(z) dz &= \int_{[\delta, R]} F(z) dz + \int_{[R, \delta]} F(z) dz + \int_{S_1(R)} F(z) dz + \int_{S_2(R)} F(z) dz \\ &\quad + \int_{L_1(\delta)} F(z) dz + \int_{L_2(\delta)} F(z) dz \\ &= (1 - e^{2\pi i \alpha}) \int_{\delta}^R F(x) dx + \int_{S(R)} F(z) dz + \int_{L(\delta)} F(z) dz. \end{aligned}$$

It is easily shown that

$$\int_{L(\delta)} F(z) dz \rightarrow 0 \quad \text{and} \quad \int_{S(R)} F(z) dz \rightarrow 0$$

as $\delta \rightarrow 0$ and $R \rightarrow \infty$ respectively, so it follows from (17) and (23) that

$$(1 - e^{2\pi i \alpha}) \int_0^{\infty} F(x) dx = 2\pi i \sum_{z_i} \text{res}(f, z_i),$$

where the summation is taken over all the poles of F in $\mathbb{C} \setminus \{0\}$.

EXAMPLE 11.6.2. Suppose that the real number $\alpha \in (0, 1)$ is fixed. Consider again the integral

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx.$$

To evaluate this integral, note that the function

$$F(z) = \frac{z^{\alpha-1}}{1+z} = \frac{z^{\alpha}}{z(1+z)},$$

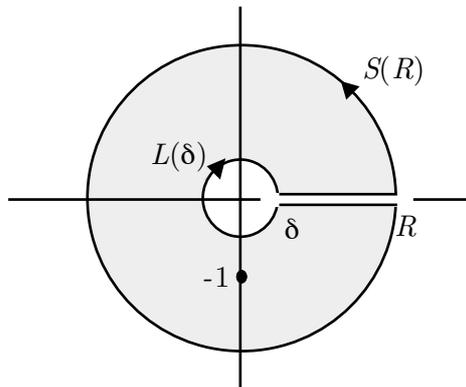
has a simple pole at $z = -1$, with residue

$$\text{res}(F, -1) = \lim_{z \rightarrow -1} (z+1)F(z) = \lim_{z \rightarrow -1} z^{\alpha-1} = e^{\pi i(\alpha-1)} = -e^{\pi i \alpha}.$$

Consider now the contour

$$C = [\delta, R] \cup S(R) \cup [R, \delta] \cup L(\delta),$$

where $R > 1 > \delta > 0$.



By Cauchy's residue theorem and our earlier observation, we have

$$(24) \quad \int_C F(z) dz = (1 - e^{2\pi i \alpha}) \int_{\delta}^R F(x) dx + \int_{S(R)} F(z) dz + \int_{L(\delta)} F(z) dz = -2\pi i e^{\pi i \alpha}.$$

Note now that

$$\left| \int_{L(\delta)} F(z) dz \right| \leq \frac{\delta^{\alpha-1}}{1-\delta} 2\pi\delta \rightarrow 0 \quad \text{and} \quad \left| \int_{S(R)} F(z) dz \right| \leq \frac{R^{\alpha-1}}{R-1} 2\pi R \rightarrow 0$$

as $\delta \rightarrow 0$ and $R \rightarrow \infty$ respectively, so it follows from (24) that

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx = -\frac{2\pi i e^{\pi i \alpha}}{1 - e^{2\pi i \alpha}} = \frac{\pi}{\sin \pi \alpha}.$$

We next turn our attention to integrals of the types

$$(25) \quad \int_0^{\infty} f(x) \log x dx \quad \text{and} \quad \int_0^{\infty} f(x) \log^2 x dx,$$

where $f(x)$ is a real valued rational function in the real variable x . Here we shall assume that the degree of the denominator of f exceeds the degree of the numerator of f by at least 2, and that f has no poles on the non-negative real axis, so that the integrals (25) are convergent. We shall also assume that f is an even function; in other words, $f(-x) = f(x)$ for every $x \in \mathbb{R} \setminus \{0\}$.

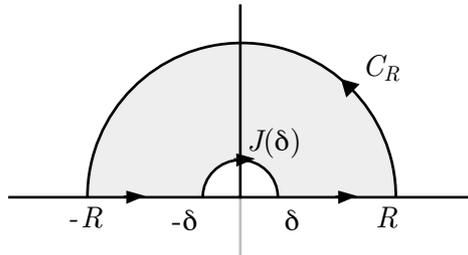
We shall consider the function

$$F(z) = f(z) \log^2 z$$

and the Jordan contour

$$C = [\delta, R] \cup C_R \cup [-R, -\delta] \cup J(\delta),$$

where $R > \delta > 0$. Here C_R denotes the semicircular arc $z = Re^{it}$, where $t \in [0, \pi]$, and $-J(\delta)$ denotes the semicircular arc $z = \delta e^{it}$, where $t \in [0, \pi]$.



By Cauchy's residue theorem, we have

$$\int_{[\delta, R]} F(z) dz + \int_{C_R} F(z) dz + \int_{[-R, -\delta]} F(z) dz + \int_{J(\delta)} F(z) dz = 2\pi i \sum_{z_i \text{ inside } C} \text{res}(F, z_i).$$

If we impose the restriction $-\pi/2 \leq \arg z < 3\pi/2$, then

$$\int_{[\delta, R]} F(z) dz = \int_{\delta}^R f(x) \log^2 x dx,$$

and

$$\begin{aligned} \int_{[-R, -\delta]} F(z) dz &= \int_{\delta}^R (\log x + i\pi)^2 f(-x) dx = \int_{\delta}^R (\log x + i\pi)^2 f(x) dx \\ &= \int_{\delta}^R f(x) \log^2 x dx + 2\pi i \int_{\delta}^R f(x) \log x dx - \pi^2 \int_{\delta}^R f(x) dx. \end{aligned}$$

It is easily shown that

$$\int_{J(\delta)} F(z) dz \rightarrow 0 \quad \text{and} \quad \int_{C_R} F(z) dz \rightarrow 0$$

as $\delta \rightarrow 0$ and $R \rightarrow \infty$ respectively. It follows that

$$(26) \quad 2 \int_0^{\infty} f(x) \log^2 x dx + 2\pi i \int_0^{\infty} f(x) \log x dx - \pi^2 \int_0^{\infty} f(x) dx = 2\pi i \sum_{\text{Im} z_i > 0} \text{res}(F, z_i),$$

where the summation is taken over all the poles of F in the upper half plane. The integrals (25) can then be found on equating real and imaginary parts.

EXAMPLE 11.6.3. Consider the integrals

$$\int_0^{\infty} \frac{\log x}{1+x^2} dx \quad \text{and} \quad \int_0^{\infty} \frac{\log^2 x}{1+x^2} dx.$$

To evaluate these integrals, note that the function

$$F(z) = \frac{\log^2 z}{1+z^2}$$

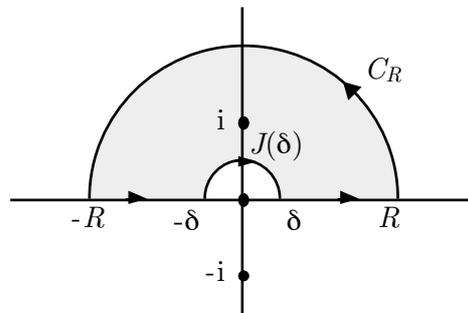
has simple poles at $z = \pm i$. In particular,

$$\text{res}(F, i) = \lim_{z \rightarrow i} (z - i)F(z) = \lim_{z \rightarrow i} \frac{\log^2 z}{z + i} = -\frac{\pi^2}{8i}.$$

Consider now the Jordan contour

$$C = [\delta, R] \cup C_R \cup [-R, -\delta] \cup J(\delta),$$

where $R > 1 > \delta > 0$.



By Cauchy's residue theorem, we have

$$\int_{[\delta, R]} F(z) dz + \int_{C_R} F(z) dz + \int_{[-R, -\delta]} F(z) dz + \int_{J(\delta)} F(z) dz = 2\pi i \text{res}(F, i) = -\frac{\pi^3}{4}.$$

On $J(\delta)$, we have $z = \delta e^{it}$, so that

$$\left| \frac{\log^2 z}{1+z^2} \right| \leq \frac{|\log \delta + it|^2}{1-\delta^2} \leq \frac{\log^2 \delta + \pi^2}{1-\delta^2}.$$

Hence

$$\left| \int_{J(\delta)} F(z) dz \right| \leq \frac{\log^2 \delta + \pi^2}{1-\delta^2} \pi \delta \rightarrow 0$$

as $\delta \rightarrow 0$. On C_R , we have $z = R e^{it}$, so that

$$\left| \frac{\log^2 z}{1+z^2} \right| \leq \frac{|\log R + it|^2}{R^2-1} \leq \frac{\log^2 R + \pi^2}{R^2-1}.$$

Hence

$$\left| \int_{C_R} F(z) dz \right| \leq \frac{\log^2 R + \pi^2}{R^2-1} \pi R \rightarrow 0$$

as $R \rightarrow \infty$. It follows from (26) that

$$2 \int_0^\infty \frac{\log^2 x}{1+x^2} dx + 2\pi i \int_0^\infty \frac{\log x}{1+x^2} dx - \pi^2 \int_0^\infty \frac{dx}{1+x^2} = -\frac{\pi^3}{4}.$$

It is well known that

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Hence

$$2 \int_0^\infty \frac{\log^2 x}{1+x^2} dx + 2\pi i \int_0^\infty \frac{\log x}{1+x^2} dx = \frac{\pi^3}{4}.$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty \frac{\log^2 x}{1+x^2} dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\log x}{1+x^2} dx = 0.$$

We conclude this chapter by considering an example which does not easily fall into any general discussion. Some of the calculation is unpleasant, and we shall omit some details.

EXAMPLE 11.6.4. We wish to evaluate the integral

$$\int_0^\pi \log \sin x dx.$$

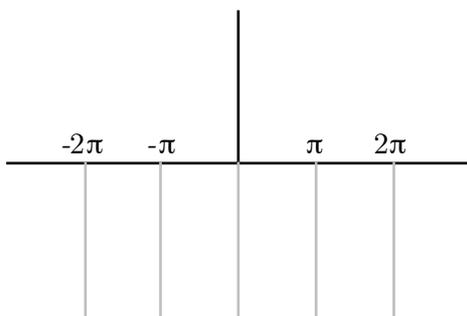
To do this, consider the function

$$(27) \quad 1 - e^{2iz} = e^{iz}(e^{-iz} - e^{iz}) = -2ie^{iz} \sin z.$$

Note that if $z = x + iy$, where $x, y \in \mathbb{R}$, then

$$1 - e^{2iz} = 1 - e^{-2y}(\cos 2x + i \sin 2x),$$

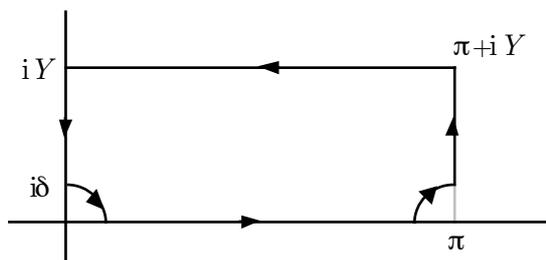
so that the function is real and non-positive if and only if $x = n\pi$ and $y \leq 0$, where $n \in \mathbb{Z}$. Consider the simply connected domain D obtained from \mathbb{C} by deleting all half lines of the form $\{z = n\pi + iy : y \leq 0\}$, where $n \in \mathbb{Z}$.



In this domain, the principal branch of $\log(1 - e^{2iz})$, with imaginary part between $-\pi$ and π , is single valued and analytic. Consider the Jordan contour

$$C = [\delta, \pi - \delta] \cup T_1(\delta) \cup [\pi + i\delta, \pi + iY] \cup [\pi + iY, iY] \cup [iY, i\delta] \cup T_2(\delta),$$

where $\delta > 0$ is small and $Y > 0$ is large. Here $-T_1(\delta)$ denotes the circular arc $z = \pi + \delta e^{it}$, where $t \in [\pi/2, \pi]$, and $-T_2(\delta)$ denotes the circular arc $z = \delta e^{it}$, where $t \in [0, \pi/2]$.



By Cauchy's integral theorem, we have

$$\int_{[\delta, \pi - \delta]} \log(1 - e^{2iz}) dz + \int_{T_1(\delta)} \log(1 - e^{2iz}) dz + \int_{[\pi + i\delta, \pi + iY]} \log(1 - e^{2iz}) dz + \int_{[\pi + iY, iY]} \log(1 - e^{2iz}) dz + \int_{[iY, i\delta]} \log(1 - e^{2iz}) dz + \int_{T_2(\delta)} \log(1 - e^{2iz}) dz = 0.$$

Using the periodicity of the integrand, we have

$$\int_{[\pi + i\delta, \pi + iY]} \log(1 - e^{2iz}) dz + \int_{[iY, i\delta]} \log(1 - e^{2iz}) dz = 0.$$

Furthermore, it can be shown that

$$\int_{T_1(\delta)} \log(1 - e^{2iz}) dz \rightarrow 0 \quad \text{and} \quad \int_{T_2(\delta)} \log(1 - e^{2iz}) dz \rightarrow 0$$

as $\delta \rightarrow 0$, and that

$$\int_{[\pi + iY, iY]} \log(1 - e^{2iz}) dz \rightarrow 0$$

as $Y \rightarrow \infty$. It follows that

$$(28) \quad \int_0^\pi \log(1 - e^{2ix}) dx = 0.$$

Next, consider the function $\log(-2ie^{ix} \sin x)$. If we choose $\log e^{ix} = ix$, then the imaginary part lies between 0 and π . To obtain the principal branch of the logarithm with imaginary part between $-\pi$ and π , we must choose $\log(-i) = -i\pi/2$. Hence

$$\log(-2ie^{ix} \sin x) = \log 2 - \frac{i\pi}{2} + ix + \log \sin x,$$

so that

$$(29) \quad \int_0^\pi \log(-2ie^{ix} \sin x) dx = \pi \log 2 - \frac{i\pi^2}{2} + \frac{i\pi^2}{2} + \int_0^\pi \log \sin x dx.$$

Combining (27)–(29), we conclude that

$$\int_0^\pi \log \sin x dx = -\pi \log 2.$$

PROBLEMS FOR CHAPTER 11

1. Show each of the following:

$$\begin{array}{ll} \text{a) } \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} = \frac{2\pi}{\sqrt{3}} & \text{b) } \int_0^{2\pi} \frac{d\theta}{1 - 2a \cos \theta + a^2} = \frac{2\pi}{1 - a^2}, \text{ where } a \in \mathbb{C} \text{ and } |a| < 1 \\ \text{c) } \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi & \text{d) } \int_0^{\infty} \frac{x^2}{(1 + x^2)^2} dx = \frac{\pi}{4} \quad \text{e) } \int_0^{\infty} \frac{dx}{1 + x^6} = \frac{\pi}{3} \\ \text{f) } \int_{-\infty}^{\infty} \frac{\cos x}{(1 + x^2)^2} dx = \frac{\pi}{e} & \text{g) } \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-a}, \text{ where } a \in \mathbb{R} \text{ and } a > 0 \\ \text{h) } \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right), \text{ where } a, b \in \mathbb{R} \text{ and } a > b > 0 \end{array}$$

2. Suppose that $a \in \mathbb{R}$ and $0 < a < 1$. By integrating the function

$$\frac{e^{az}}{e^z + 1}$$

around a rectangle with vertices at $\pm R$ and $\pm R + 2\pi i$, show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = \frac{\pi}{\sin \pi a}.$$

3. Show each of the following:

$$\begin{array}{ll} \text{a) } \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx = \pi & \text{b) } \int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)} dx = \pi \\ \text{c) } \int_0^{\infty} \frac{\log x}{x^4 + 1} dx = -\frac{\pi^2 \sqrt{2}}{16} & \text{d) } \int_0^{\infty} \frac{\log x}{x^2 - 1} dx = \frac{\pi^2}{4} \end{array}$$

INTRODUCTION TO COMPLEX ANALYSIS

W W L CHEN

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Chapter 12

HARMONIC FUNCTIONS AND CONFORMAL MAPPINGS

12.1. A Local Property of Analytic Functions

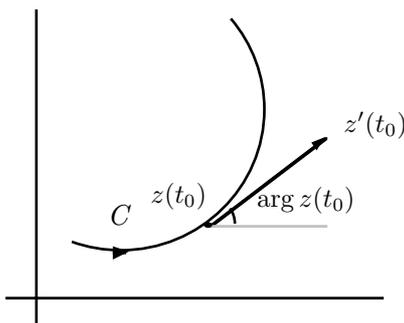
Consider an arc C given by $z(t)$, where $t \in [A, B]$. For every $t \in [A, B]$, we can write

$$z(t) = x(t) + iy(t),$$

where $x(t), y(t) \in \mathbb{R}$. Then the vector

$$z'(t) = x'(t) + iy'(t)$$

has slope dy/dx , which is also the slope of the arc C . It follows that if $z'(t_0) \neq 0$, then the vector $z'(t_0)$ is tangent to the arc at the point $z_0 = z(t_0)$, and $\arg z'(t_0)$ is the angle this directed tangent makes with the positive x -axis.



Suppose now that C lies in a domain D , and that a function $f(z)$ is analytic in D . Consider the arc $f(C)$ given by $w(t) = f(z(t))$, where $t \in [A, B]$. By the Chain rule,

$$w'(t) = f'(z(t))z'(t).$$

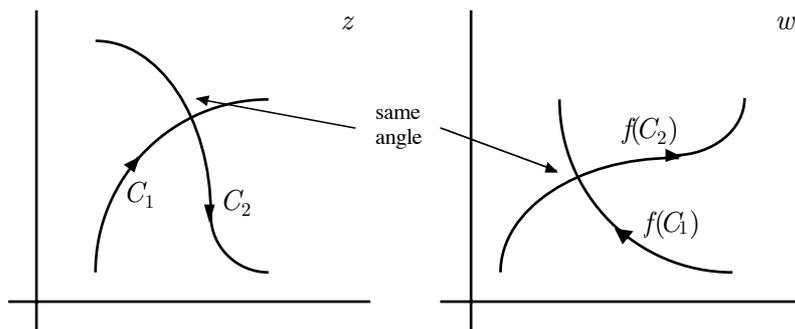
Suppose now that $z'(t_0) \neq 0$ and $f'(z_0) \neq 0$, where $z_0 = z(t_0)$. Then $w'(t_0) \neq 0$, and

$$\arg w'(t_0) = \arg f'(z_0) + \arg z'(t_0).$$

We can interpret this geometrically in the following way: The angle between the directed tangent to C at $z_0 = z(t_0)$ and the directed tangent to $f(C)$ at $f(z_0)$ is $\arg f'(z_0)$.

In other words, under the mapping f , the directed tangent to any arc through z_0 is rotated by an angle $\arg f'(z_0)$, independent of the choice of the arc through z_0 . This also means that if two arcs C_1 and C_2 intersect at z_0 at an angle, then the two arcs $f(C_1)$ and $f(C_2)$ intersect at $f(z_0)$ at the same angle.

DEFINITION. An analytic function f is said to be conformal at z_0 if the following condition is satisfied: If two arcs C_1 and C_2 meet at z_0 , then the angle from $f(C_1)$ to $f(C_2)$ at $f(z_0)$ is the same as the angle from C_1 to C_2 at z_0 .



We have in fact proved the following result.

THEOREM 12A. Suppose that a function f is analytic in a domain D , and that $z_0 \in D$. Suppose further that $f'(z_0) \neq 0$. Then f is conformal at z_0 .

REMARK. Conformality is considered a local property of analytic functions. Note also that

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|.$$

This shows that $|f'(z_0)|$ is a local scaling factor of the function f at z_0 , and is independent of the direction of z from z_0 .

To say that an analytic function is conformal usually means that it is locally one-to-one. In particular, we have the following result.

THEOREM 12B. Suppose that a function f is analytic and one-to-one in a domain D . Then f is conformal at every point in D .

This follows immediately from the result below and Theorem 12A.

THEOREM 12C. Suppose that a non-constant function f is analytic in a domain D , and that $z_0 \in D$. Suppose further that $f'(z_0) = 0$. Then f cannot be one-to-one in any disc containing z_0 .

PROOF. Write $w_0 = f(z_0)$. Since $f(z)$ is not identically constant, the function $g(z) = f(z) - w_0$ is not identically zero. If $f'(z_0) = 0$, then $g(z)$ has a zero of finite order at least 2 at z_0 . Since the zeros of an analytic function are isolated, we can choose $r > 0$ so small that both $g(z)$ and $f'(z)$ have no zeros in the punctured disc $\{z : 0 < |z - z_0| \leq r\}$. Then

$$m = \min_{z \in C} |g(z)| > 0,$$

where $C = \{z : |z - z_0| = r\}$ denotes the boundary of the disc. Let $w \in \mathbb{C}$ satisfy $0 < |w - w_0| < m$. Then

$$|w_0 - w| < |g(z)|$$

on C . It follows from Rouché's theorem that the functions $g(z)$ and $g(z) + (w_0 - w)$ have the same number of zeros inside C . Hence

$$g(z) + (w_0 - w) = f(z) - w$$

has at least two zeros inside C . Clearly none of these zeros can be z_0 . Since $f'(z) \neq 0$ inside the punctured disc, it follows that these zeros must be simple, and so distinct. \circ

EXAMPLE 12.1.1. The exponential function $f(z) = e^z$ has non-zero derivative at every $z \in \mathbb{C}$, and is therefore conformal at every $z \in \mathbb{C}$. Note, however, that $f : \mathbb{C} \rightarrow \mathbb{C}$ is not one-to-one. On the other hand, the function is one-to-one if we restrict its domain of definition to any strip of the form

$$\{z = x + iy : a \leq y \leq b\},$$

where $0 < b - a < 2\pi$. We therefore say that the exponential function is locally one-to-one.

12.2. Laplace's Equation

Recall that a continuous function $\phi(x, y)$ that satisfies Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in a domain $D \subseteq \mathbb{C}$ is said to be harmonic in D . In Theorem 3E, we have shown that such a function can be written as the real part of an analytic function f in D .

Our main task in this section is to show that a harmonic function can be carried from one domain to another by analytic functions. More precisely, we prove the following result.

THEOREM 12D. *Suppose that $D, D' \subseteq \mathbb{C}$ are domains, and that $f : D \rightarrow D'$ is a one-to-one and onto analytic function. Suppose further that for every $z \in D$, we write $w = f(z)$, where $z = x + iy$ and $w = u + iv$, with $x, y, u, v \in \mathbb{R}$. Then for every function $\phi(x, y)$ harmonic in D , the function $\psi(u, v)$, defined by*

$$(1) \quad \psi(u, v) = \phi(x(u, v), y(u, v)),$$

is harmonic in D' .

Theorem 12D is particularly useful in applications which involve the solution of the Dirichlet problem concerning the question of finding a harmonic function in a domain D which takes specified values on the boundary of D . Once we solve this problem for a particular domain, we can use Theorem 12D to find solutions on all domains which can be obtained from D by a one-to-one and onto analytic function, so

long as the boundary values correspond. We can therefore select the domain which makes the problem simplest.

Note that (1) can be written in the form

$$\psi(w) = \phi(f^{-1}(w)).$$

Our first task is therefore to establish the analyticity of the inverse function $f^{-1} : D' \rightarrow D$.

THEOREM 12E. *Suppose that $D, D' \subseteq \mathbb{C}$ are domains, and that $f : D \rightarrow D'$ is a one-to-one and onto analytic function. Then the inverse function $f^{-1} : D' \rightarrow D$ is analytic in D' . Suppose further that for every $z \in D$, we write $w = f(z)$, where $z = x + iy$ and $w = u + iv$, with $x, y, u, v \in \mathbb{R}$. Then*

$$(2) \quad \frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} \quad \text{and} \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}.$$

PROOF. To prove the first assertion, it suffices to prove that for every $w_0 = f(z_0) \in D'$, the limit

$$\lim_{w \rightarrow w_0} \frac{z - z_0}{w - w_0}$$

exists. To establish this, note first of all that by Theorem 12C, we have

$$\lim_{z \rightarrow z_0} \frac{w - w_0}{z - z_0} \neq 0.$$

Since f is continuous in D , we clearly have $w \rightarrow w_0$ as $z \rightarrow z_0$. Since f is one-to-one in D , we clearly have $w \neq w_0$ when $z \neq z_0$. It follows that

$$\lim_{w \rightarrow w_0} \frac{z - z_0}{w - w_0} = \left(\lim_{z \rightarrow z_0} \frac{w - w_0}{z - z_0} \right)^{-1}.$$

To complete the proof of the theorem, note that (2) are the Cauchy-Riemann equations of the analytic function f^{-1} . \circ

PROOF OF THEOREM 12D. Note that

$$\frac{\partial \psi}{\partial u} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial x}{\partial u} \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial \phi}{\partial y},$$

so that

$$(3) \quad \frac{\partial^2 \psi}{\partial u^2} = \frac{\partial}{\partial u} \left(\frac{\partial x}{\partial u} \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial \phi}{\partial y} \right) = \frac{\partial^2 x}{\partial u^2} \frac{\partial \phi}{\partial x} + \frac{\partial x}{\partial u} \frac{\partial}{\partial u} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial^2 y}{\partial u^2} \frac{\partial \phi}{\partial y} + \frac{\partial y}{\partial u} \frac{\partial}{\partial u} \left(\frac{\partial \phi}{\partial y} \right).$$

On the other hand,

$$(4) \quad \frac{\partial}{\partial u} \left(\frac{\partial \phi}{\partial x} \right) = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) = \frac{\partial x}{\partial u} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial y}{\partial u} \frac{\partial^2 \phi}{\partial x \partial y}.$$

Similarly,

$$(5) \quad \frac{\partial}{\partial u} \left(\frac{\partial \phi}{\partial y} \right) = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) + \frac{\partial y}{\partial u} \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) = \frac{\partial x}{\partial u} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial y}{\partial u} \frac{\partial^2 \phi}{\partial y^2}.$$

Combining (3)–(5), we obtain

$$(6) \quad \frac{\partial^2 \psi}{\partial u^2} = \frac{\partial^2 x}{\partial u^2} \frac{\partial \phi}{\partial x} + \frac{\partial^2 y}{\partial u^2} \frac{\partial \phi}{\partial y} + \left(\frac{\partial x}{\partial u} \right)^2 \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \frac{\partial^2 \phi}{\partial x \partial y} + \left(\frac{\partial y}{\partial u} \right)^2 \frac{\partial^2 \phi}{\partial y^2}.$$

A similar argument gives

$$(7) \quad \frac{\partial^2 \psi}{\partial v^2} = \frac{\partial^2 x}{\partial v^2} \frac{\partial \phi}{\partial x} + \frac{\partial^2 y}{\partial v^2} \frac{\partial \phi}{\partial y} + \left(\frac{\partial x}{\partial v}\right)^2 \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} \frac{\partial^2 \phi}{\partial x \partial y} + \left(\frac{\partial y}{\partial v}\right)^2 \frac{\partial^2 \phi}{\partial y^2}.$$

Adding (6) and (7), we have

$$(8) \quad \frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = \left(\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2}\right) \frac{\partial \phi}{\partial x} + \left(\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2}\right) \frac{\partial \phi}{\partial y} + 2 \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial x}{\partial v} \frac{\partial y}{\partial v}\right) \frac{\partial^2 \phi}{\partial x \partial y} \\ + \left(\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\right) \frac{\partial^2 \phi}{\partial x^2} + \left(\left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\right) \frac{\partial^2 \phi}{\partial y^2}.$$

Suppose now that $\phi(x, y)$ harmonic in D . Then

$$(9) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in D . On the other hand, the Cauchy-Riemann equations (2) give

$$(10) \quad \frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = 0 \quad \text{and} \quad \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} = 0,$$

as well as

$$(11) \quad \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} = 0 \quad \text{and} \quad \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 = \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2$$

in D' . Combining (8)–(11), it is easily seen that

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 0$$

in D' , so that $\psi(u, v)$ is harmonic in D' . \circ

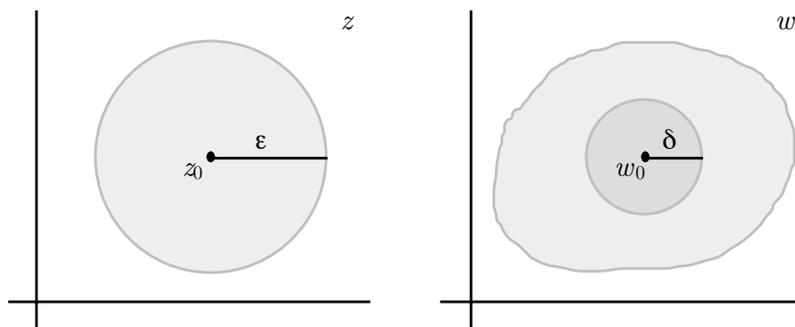
12.3. Global Properties of Analytic Functions

We begin by studying the following result which can be considered both local and global. It can be proved by means of Rouché's theorem in the same spirit as the proof of Theorem 12C.

THEOREM 12F. (OPEN MAPPING THEOREM) *Suppose that a non-constant function f is analytic in a domain D . Then $f(S)$ is open for any open set $S \subseteq D$. More specifically, suppose that $z_0 \in D$, and that $w_0 = f(z_0)$. Then for all sufficiently small $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\{w : |w - w_0| < \delta\} \subseteq f(\{z : |z - z_0| < \epsilon\}).$$

In other words, w_0 is an interior point of $f(D)$.



PROOF. Since $f(z)$ is not identically constant, the function $g(z) = f(z) - w_0$ is not identically zero, and has a zero of finite order at z_0 . Since the zeros of an analytic function are isolated, we can choose $r < \epsilon$ so small that $g(z)$ has no zeros in the punctured disc $\{z : 0 < |z - z_0| \leq r\}$. Then

$$\delta = \min_{z \in C} |g(z)| > 0,$$

where $C = \{z : |z - z_0| = r\}$ denotes the boundary of the disc. Let $w \in \mathbb{C}$ satisfy $|w - w_0| < \delta$. Then

$$|w_0 - w| < |g(z)|$$

on C . It follows from Rouché's theorem that the functions $g(z)$ and $g(z) + (w_0 - w)$ have the same number of zeros inside C . Hence

$$g(z) + (w_0 - w) = f(z) - w$$

has a solution inside C , so that $w \in f(\{z : |z - z_0| < r\})$. It follows that

$$\{w : |w - w_0| < \delta\} \subseteq f(\{z : |z - z_0| < r\}).$$

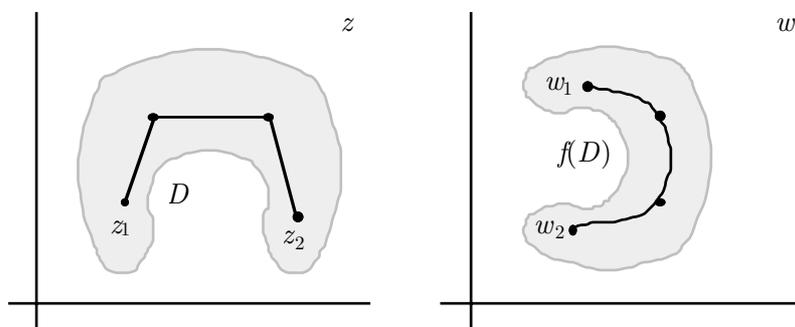
The result follows. \circ

Let us examine Theorem 12D again. We have assumed that both D and D' are domains. The following result allows us to somewhat relax our assumptions.

THEOREM 12G. *Suppose that a non-constant function f is analytic in a domain D . Then $f(D)$ is a domain.*

REMARK. Recall that an open set S is connected if every two points in S can be joined by the union of a finite number of line segments lying in S . An easy theorem in real analysis states that any contour can be approximated arbitrarily well by the union of a finite number of line segments. It follows that S is connected if every two points in S can be joined by a contour lying in S .

PROOF OF THEOREM 12G. To show that $f(D)$ is a domain, we need to show that it is open and connected. In view of Theorem 12F, it remains to show that $f(D)$ is connected. Suppose that $w_1, w_2 \in f(D)$. Then there exist $z_1, z_2 \in D$ such that $f(z_1) = w_1$ and $f(z_2) = w_2$. Since D is connected, z_1 and z_2 can be joined by the union of a finite number of line segments lying in S . The image of each line segment under f is an arc in $f(D)$, since f is differentiable in D .



It follows that w_1 and w_2 can be joined by a contour lying in $f(D)$. \circ

We conclude this section by stating the following result.

THEOREM 12H. (RIEMANN MAPPING THEOREM) *Suppose that D is a simply connected domain in \mathbb{C} which is different from \mathbb{C} . Then there exists a one-to-one and onto analytic function of the type $f : D \rightarrow \mathcal{U}$, where $\mathcal{U} = \{w : |w| < 1\}$ denotes the unit open disc.*

REMARKS. (1) If we prescribe a point in D and a direction through this point, then there is a unique function of the type described which maps this point and direction to the origin and the positive x -axis respectively.

(2) The proof can be split into three steps. Let $z_0 \in D$ be fixed. One begins by showing that the collection \mathcal{S} of one-to-one analytic functions of the type $f : D \rightarrow \mathcal{U}$ and satisfying the conditions $f(z_0) = 0$ and $f'(z_0) > 0$ is non-empty. One then shows that there is an extremal member in \mathcal{S} with greatest $f'(z_0)$. Finally, one shows that if a member in \mathcal{S} is not an onto function, then it cannot be this extremal member. It follows that the extremal member satisfies the requirements of the theorem.

(3) Unfortunately, Theorem 12H is a purely existence theorem, and so cannot be used in conjunction with Theorem 12D. In Chapters 13–14, we shall study some techniques which may enable us to construct such a function.

PROBLEMS FOR CHAPTER 12

1. Let C_1 and C_2 be two straight lines that meet at the origin at an angle ϕ . Consider the function $f(z) = z^3$.
 - a) At what angle do the two lines $f(C_1)$ and $f(C_2)$ meet?
 - b) Comment on the solution in (a).
2. Discuss angles at the origin under the mapping $f(z) = z^\alpha$, where $0 < \alpha < 1$.
3. Use the Open mapping theorem to prove the Maximum principle.
4. Explain why the conclusion of the Riemann mapping theorem cannot hold when $D = \mathbb{C}$.

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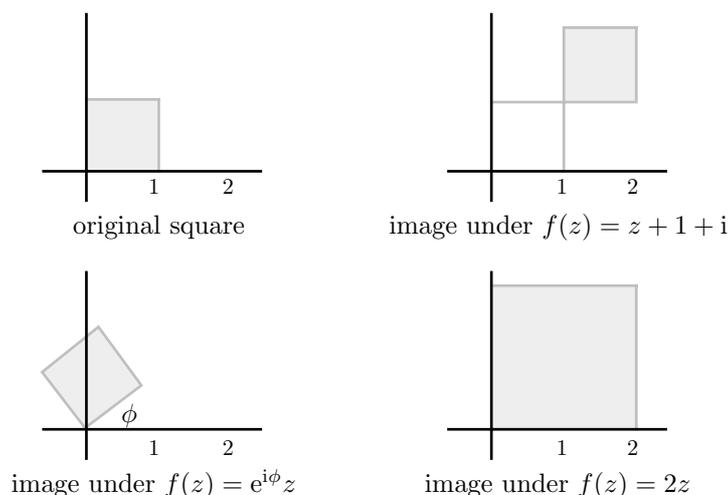
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Chapter 13

MÖBIUS TRANSFORMATIONS

13.1. Linear Functions

EXAMPLE 13.1.1. Consider the square $\{z = x + iy \in \mathbb{C} : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. The pictures below show the images of this square under the functions $f(z) = z + 1 + i$, $f(z) = e^{i\phi}z$ and $f(z) = 2z$. Note that the image of the square in each case is also a square.



The function $f(z) = z + 1 + i$ is an example of a function of the type $f(z) = z + c$, where $c \in \mathbb{C}$ is fixed. This function describes a translation on the complex plane \mathbb{C} , where every point is shifted by a vector corresponding to the complex number c . The function $f(z) = e^{i\phi}z$, where $\phi \in \mathbb{R}$ is fixed, describes a rotation on the complex plane \mathbb{C} , where every point is rotated in the anticlockwise direction by an angle ϕ about the origin. The function $f(z) = 2z$ is an example of a function of the type $f(z) = \rho z$, where

$\rho \in \mathbb{R}$ is positive and fixed. This function describes a magnification on the complex plane \mathbb{C} , where the distance between points is magnified by a factor ρ , noting that $|\rho z_1 - \rho z_2| = \rho|z_1 - z_2|$ for every $z_1, z_2 \in \mathbb{C}$.

It is easily seen that if we take the domain and codomain of each of the above functions to be the complex plane \mathbb{C} , then $f : \mathbb{C} \rightarrow \mathbb{C}$ is both one-to-one and onto. Furthermore, any geometric object in \mathbb{C} has an image under f which is similar to itself.

DEFINITION. A linear function is a function $f : \mathbb{C} \rightarrow \mathbb{C}$ of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ are fixed, and $a \neq 0$.

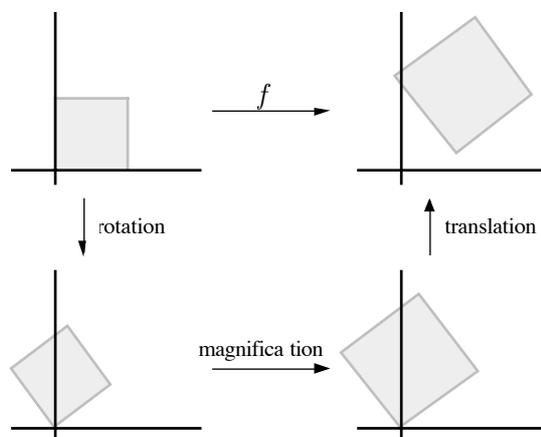
EXAMPLE 13.1.2. Let us return to the three examples earlier. For the function $f(z) = z + c$, we have $a = 1$ and $b = c$. For the function $f(z) = e^{i\phi}z$, we have $a = e^{i\phi}$ and $b = 0$. For the function $f(z) = \rho z$, we have $a = \rho$ and $b = 0$.

THEOREM 13A. Any linear function $f : \mathbb{C} \rightarrow \mathbb{C}$ is the composition of a rotation, a magnification and a translation. Furthermore, it is one-to-one and onto.

PROOF. Suppose that $f(z) = az + b$ for every $z \in \mathbb{C}$. Write $a = \rho e^{i\phi}$, where $\rho, \phi \in \mathbb{R}$ and $\rho > 0$. Then $f = f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = e^{i\phi}z \quad \text{and} \quad f_2(z) = \rho z \quad \text{and} \quad f_3(z) = z + b.$$

We have the picture below:



The last assertion follows from the observation that composition of functions preserves the one-to-one and onto properties. ○

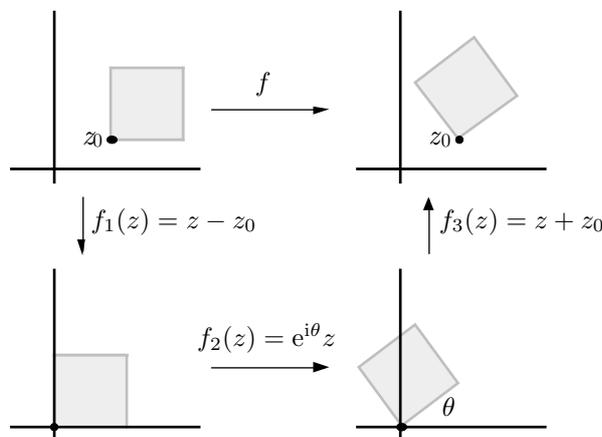
THEOREM 13B. The composition of any two linear functions is also a linear function.

PROOF. Suppose that $f_1(z) = a_1z + b_1$ and $f_2(z) = a_2z + b_2$, where $a_1, b_1, a_2, b_2 \in \mathbb{C}$ and $a_1, a_2 \neq 0$. Then $(f_2 \circ f_1)(z) = a_2(a_1z + b_1) + b_2 = a_1a_2z + (a_2b_1 + b_2)$. Clearly $a_1a_2, a_2b_1 + b_2 \in \mathbb{C}$ and $a_1a_2 \neq 0$. ○

EXAMPLE 13.1.3. Suppose that $z_0 \in \mathbb{C}$ is fixed. Consider the linear function $f : \mathbb{C} \rightarrow \mathbb{C}$ which rotates the complex plane \mathbb{C} in the anticlockwise direction by an angle θ about the point z_0 . We may adopt the following strategy: Translate the point z_0 to the origin, then rotate in the anticlockwise direction by an angle θ about the origin, and then translate the origin back to the point z_0 . Then $f = f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = z - z_0 \quad \text{and} \quad f_2(z) = e^{i\theta}z \quad \text{and} \quad f_3(z) = z + z_0.$$

We have the picture below:



Hence

$$f(z) = e^{i\theta}(z - z_0) + z_0 = e^{i\theta}z + z_0(1 - e^{i\theta}).$$

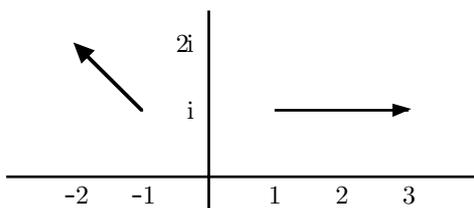
Alternatively, we may adopt the following strategy: Rotate in the anticlockwise direction by an angle θ about the origin, and then translate the image of z_0 under this rotation back to z_0 . Then $f = g_2 \circ g_1$, where

$$g_1(z) = e^{i\theta}z \quad \text{and} \quad g_2(z) = z + (z_0 - e^{i\theta}z_0).$$

Hence

$$f(z) = e^{i\theta}z + (z_0 - e^{i\theta}z_0) = e^{i\theta}z + z_0(1 - e^{i\theta}).$$

EXAMPLE 13.1.4. Consider the linear function $f : \mathbb{C} \rightarrow \mathbb{C}$ which maps the horizontal arrow shown to the other arrow shown.



We may adopt the following strategy: Translate the tip of the arrow from $3 + i$ to the origin, magnify the arrow by a factor $1/\sqrt{2}$, rotate it about its tip (now at the origin) in the anticlockwise direction by an angle $3\pi/4$, and finally translate its tip from the origin to the point $-2 + 2i$. Then $f = f_4 \circ f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = z - (3 + i) \quad \text{and} \quad f_2(z) = \frac{1}{\sqrt{2}}z \quad \text{and} \quad f_3(z) = e^{3\pi i/4}z \quad \text{and} \quad f_4(z) = z + (-2 + 2i).$$

Hence

$$f(z) = \frac{e^{3\pi i/4}}{\sqrt{2}}(z - 3 - i) - 2 + 2i = \left(-\frac{1}{2} + \frac{i}{2}\right)(z - 3 - i) - 2 + 2i = \left(-\frac{1}{2} + \frac{i}{2}\right)z + i.$$

13.2. The Inversion Function

Consider the inversion function

$$w = f(z) = \frac{1}{z}.$$

This function can be considered a function of the type $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, where $\overline{\mathbb{C}}$ denotes the extended complex plane, so that $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We write formally $f(0) = \infty$ and $f(\infty) = 0$.

Let us first study some geometric properties of this function. For our purposes, the point at ∞ is considered to belong to every line on the extended complex plane.

REMARKS. (1) A line passing through the origin contains all points of the form $z = re^{i\theta}$, where $\theta \in \mathbb{R}$ is fixed and $r \in \mathbb{R}$. The images of these points under the inversion function are of the form

$$w = \frac{1}{z} = \frac{1}{r}e^{-i\theta}.$$

They form a line through the origin. Note that the point at ∞ and the origin change roles under the inversion function.

(2) A line not passing through the origin consists of all points of the form $z = x + iy$, where $x, y \in \mathbb{R}$ and $Ax + By = C$, where $A, B, C \in \mathbb{R}$ are fixed and $C \neq 0$. The images of these points under the inversion function are of the form $w = u + iv$, where $u, v \in \mathbb{R}$ and $w = 1/z$. It is easy to see that

$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2},$$

so that

$$(1) \quad x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = -\frac{v}{u^2 + v^2}.$$

It follows that

$$\frac{Au}{u^2 + v^2} - \frac{Bv}{u^2 + v^2} = C.$$

This can be rewritten in the form

$$u^2 + v^2 - \frac{A}{C}u + \frac{B}{C}v = 0,$$

the equation of a circle passing through the origin.

(3) Note now that the inverse of the inversion function is the inversion function itself. It follows from the previous observation that a circle passing through the origin becomes a line not passing through the origin under the inversion function.

(4) A circle not passing through the origin consists of all points of the form $z = x + iy$, where $x, y \in \mathbb{R}$ and $x^2 + y^2 + Ax + By = C$, where $A, B, C \in \mathbb{R}$ are fixed and $C \neq 0$. The images of these points under the inversion function are of the form $w = u + iv$, where $u, v \in \mathbb{R}$ and $w = 1/z$. In view of (1), we have

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{Au}{u^2 + v^2} - \frac{Bv}{u^2 + v^2} = C.$$

This can be rewritten in the form

$$u^2 + v^2 - \frac{A}{C}u + \frac{B}{C}v = \frac{1}{C},$$

the equation of a circle not passing through the origin.

We now state a result which includes these four remarks.

THEOREM 13C. *The inversion function $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by $f(z) = 1/z$ for every non-zero $z \in \mathbb{C}$, and $f(0) = \infty$ and $f(\infty) = 0$, is one-to-one and onto. On the other hand, its inverse function is itself. Furthermore, the image under this function of a line or a circle in $\overline{\mathbb{C}}$ is also a line or a circle in $\overline{\mathbb{C}}$.*

REMARKS. (1) We have in fact shown the following: Under the inversion function, the image of a line through the origin is a line through the origin, the image of a line not through the origin is a circle through the origin, the image of a circle through the origin is a line not through the origin, and the image of a circle not through the origin is a circle not through the origin.

(2) If we think of a line as a circle of infinite radius, then we can think of circles and lines as belonging to the “same” class. The inversion function therefore maps members of this class to members of this class.

13.3. A Generalization

If we extend any linear function discussed in §13.1 to a function of the type $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ by writing $f(\infty) = \infty$, then it is easy to see that this extended function $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is also one-to-one and onto, and that its inverse function $f^{-1} : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is also a linear function.

Note also that the class of all circles and lines in $\overline{\mathbb{C}}$ is carried to itself by all linear functions as well as the inversion function. We now try to generalize these two types of functions.

DEFINITION. A Möbius transformation, or a bilinear transformation, is a rational function $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ of the form

$$(2) \quad T(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$. We write formally

$$(3) \quad T\left(-\frac{d}{c}\right) = \infty \quad \text{and} \quad T(\infty) = \frac{a}{c}.$$

REMARKS. (1) Note that if $ad - bc = 0$, then

$$T'(z) = \frac{ad - bc}{(cz + d)^2} = 0,$$

so that $T(z)$ is constant. Hence the requirement $ad - bc \neq 0$ is essential.

(2) To justify (3), note that the function $T(z)$ has a simple pole at $z = -d/c$, and that

$$\lim_{|z| \rightarrow \infty} T(z) = \frac{a}{c}.$$

(3) Since

$$T'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0$$

for every $z \in \mathbb{C}$ satisfying $z \neq -d/c$, it follows that a Möbius transformation is conformal at every point in \mathbb{C} where it is analytic.

(4) The case $c = 0$ and $d = 1$ reduces to $T(z) = az + b$, a linear function.

(5) The case $a = d = 0$ and $b = c = 1$ reduces to $T(z) = 1/z$, the inversion function.

(6) If $c \neq 0$, then it is easy to check that

$$(4) \quad \frac{az + b}{cz + d} = \frac{a}{c} + \left(b - \frac{ad}{c}\right) \frac{1}{cz + d}.$$

(7) Writing $w = T(z)$, then (2) can be written in the form $cwz - az + dw - b = 0$, and this is linear in both z and w . This is the reason for calling such a function a bilinear transformation.

The following result is a generalization of Theorems 13A and 13C.

THEOREM 13D. *Suppose that $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a Möbius transformation. Then*

- (a) *T is the composition of a sequence of translations, magnifications, rotations and inversions;*
- (b) *$T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is one-to-one and onto;*
- (c) *the inverse function $T^{-1} : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is also a Möbius transformation;*
- (d) *T maps the class of circles and lines in $\overline{\mathbb{C}}$ to itself; and*
- (e) *for every Möbius transformation $S : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, $S \circ T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is also a Möbius transformation.*

PROOF. (a) Suppose that

$$T(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. If $c = 0$, then we must have $ad \neq 0$, so that

$$T(z) = \frac{az + b}{d} = \frac{a}{d}z + \frac{b}{d}.$$

In this case, T is a linear function, and the result follows from Theorem 13A. On the other hand, if $c \neq 0$, then we use the identity (4). We can write $T = T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = cz + d \quad \text{and} \quad T_2(z) = \frac{1}{z} \quad \text{and} \quad T_3(z) = \left(b - \frac{ad}{c}\right)z + \frac{a}{c}.$$

It is easy to check that T_1 and T_3 are linear functions, while T_2 is the inversion function. The result now follows from Theorem 13A.

(b) and (d) follow from (a) on noting that translations, magnifications, rotations and inversions all have the properties in question, and that composition of functions preserves these properties.

(c) and (e) are left as exercises. \circ

EXAMPLE 13.3.1. Suppose that $a \in \mathbb{C}$ is fixed and $|a| < 1$. Consider the Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by

$$T(z) = \frac{a - z}{1 - \bar{a}z},$$

where $\bar{a} \in \mathbb{C}$ denotes the complex conjugate of a . Note that

$$|T(z)|^2 = \frac{|a - z|^2}{|1 - \bar{a}z|^2} = \frac{|a|^2 - 2\Re(\bar{a}z) + |z|^2}{1 - 2\Re(\bar{a}z) + |a|^2|z|^2}.$$

It is easy to see that if $|z| = 1$, then $|T(z)| = 1$. It follows from Theorem 13D(d) that the image under T of the unit circle $\{z : |z| = 1\}$ is the unit circle itself. On the other hand, the inequality $|T(z)| < 1$ is equivalent to the inequality

$$|a|^2 + |z|^2 < 1 + |a|^2|z|^2,$$

which is equivalent to the inequality

$$(1 - |a|^2)(1 - |z|^2) > 0,$$

which is equivalent to the inequality $|z| < 1$, in view of the assumption $|a| < 1$. It now follows from this observation and Theorem 13D(b) that the interior $\{z : |z| < 1\}$ of the unit circle must be mapped onto itself by T .

DEFINITION. A fixed point $z \in \overline{\mathbb{C}}$ of a Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a solution of the equation $T(z) = z$.

THEOREM 13E. A Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ has at most two distinct fixed points in $\overline{\mathbb{C}}$ unless $T(z) = z$ identically.

PROOF. Suppose that

$$T(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. If $c = 0$, then $ad \neq 0$, so that T is a linear function. In this case, the equation $T(z) = z$ becomes $az + b = dz$. If $T(z)$ is not identically equal to z , then $a \neq d$ or $b \neq 0$, so that this equation has at most one solution in \mathbb{C} . Suppose next that $c \neq 0$. Then clearly ∞ is not a fixed point. The equation $T(z) = z$ is now a quadratic equation, and so has at most two distinct roots in \mathbb{C} . \circ

It follows from Theorem 13E that a Möbius transformation must be the identity function if it has three fixed points. Suppose now that S and T are Möbius transformations, and that there exist distinct $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ such that $S(z_j) = T(z_j)$ for $j = 1, 2, 3$. By Theorem 13D(c)(e), the composition $S^{-1} \circ T$ is also a Möbius transformation. Clearly $(S^{-1} \circ T)(z_j) = z_j$ for $j = 1, 2, 3$, so that $S^{-1} \circ T$ has three fixed points, and so must be the identity function. In other words, $(S^{-1} \circ T)(z) = z$, and so $S(z) = T(z)$, for every $z \in \overline{\mathbb{C}}$. We summarize this observation below.

THEOREM 13F. Suppose that two Möbius transformations $S : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ and $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ are equal at three distinct points in $\overline{\mathbb{C}}$. Then $S(z) = T(z)$ for every $z \in \overline{\mathbb{C}}$.

Next, we show that three points determine uniquely a Möbius transformation.

THEOREM 13G. Suppose that $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ are distinct, and that $w_1, w_2, w_3 \in \overline{\mathbb{C}}$ are also distinct. Then there exists a unique Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ such that $T(z_j) = w_j$ for $j = 1, 2, 3$.

PROOF. To establish the existence of such a function, note that $T_1 : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by

$$T_1(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)},$$

is a Möbius transformation, with $T_1(z_1) = 0$, $T_1(z_2) = 1$ and $T_1(z_3) = \infty$. Similarly, $T_2 : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by

$$T_2(w) = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)},$$

is a Möbius transformation, with $T_2(w_1) = 0$, $T_2(w_2) = 1$ and $T_2(w_3) = \infty$. Clearly $T = T_2^{-1} \circ T_1$ is a Möbius transformation such that $T(z_j) = w_j$ for $j = 1, 2, 3$. The uniqueness follows from Theorem 13F. \circ

EXAMPLE 13.3.2. To find a Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ such that $T(0) = 2$, $T(1) = 3$ and $T(6) = 4$, note that $T_1 : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by

$$T_1(z) = \frac{(z - 0)(1 - 6)}{(z - 6)(1 - 0)} = \frac{-5z}{z - 6},$$

is a Möbius transformation, with $T_1(0) = 0$, $T_1(1) = 1$ and $T_1(6) = \infty$. Similarly, $T_2 : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by

$$T_2(w) = \frac{(w - 2)(3 - 4)}{(w - 4)(3 - 2)} = \frac{-w + 2}{w - 4},$$

is a Möbius transformation, with $T_2(2) = 0$, $T_2(3) = 1$ and $T_2(4) = \infty$. We now have to calculate $T = T_2^{-1} \circ T_1$. Note that

$$T_2^{-1}(z) = \frac{4z + 2}{z + 1},$$

so that

$$T(z) = \frac{4\left(\frac{-5z}{z-6}\right) + 2}{\left(\frac{-5z}{z-6}\right) + 1} = \frac{-20z + 2(z-6)}{-5z + (z-6)} = \frac{-18z - 12}{-4z - 6} = \frac{9z + 6}{2z + 3}.$$

13.4. Finding Particular Möbius Transformations

Recall that a Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by

$$T(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$, maps the class of circles and lines in $\overline{\mathbb{C}}$ to itself. Suppose that a circle or line contains the pole $z = -d/c$ of T , then its image under T is unbounded, and is therefore a line rather than a circle. Suppose, on the other hand, that a circle or line does not contain the pole $z = -d/c$ of T , then its image under T cannot contain the point at ∞ , and is therefore a circle rather than a line.

Note next that a circle or line splits the extended complex plane into two domains. Here we adopt the convention that a line contains the point at ∞ , whereas a half plane not including its boundary line does not contain the point at ∞ . Since $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is one-to-one and onto, and since an analytic function

maps domains to domains, it follows that the image of any domain arising from a circle or line must be mapped onto a domain arising from the image of this circle or line under T .

REMARK. Strictly speaking, the function

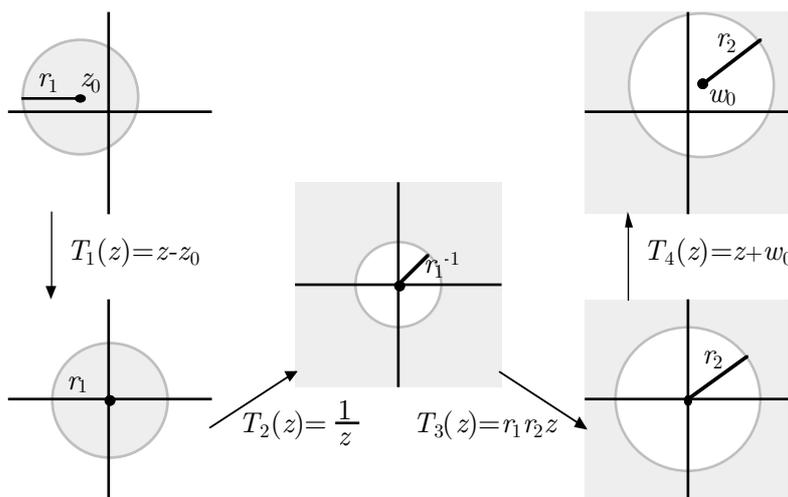
$$T(z) = \frac{az + b}{cz + d}$$

is one-to-one, onto and analytic if we take the domain of T to be $\mathbb{C} \setminus \{-d/c\}$ and the codomain of T to be $\mathbb{C} \setminus \{a/c\}$.

EXAMPLE 13.4.1. Suppose that $z_0, w_0 \in \mathbb{C}$ and $r_1, r_2 > 0$ are fixed, and that we are required to find a Möbius transformation $T : \mathbb{C} \rightarrow \mathbb{C}$ which maps the disc $\{z : |z - z_0| < r_1\}$ to the annulus $\{w : |w - w_0| > r_2\}$. This can be achieved by taking $T = T_4 \circ T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = z - z_0 \quad \text{and} \quad T_2(z) = \frac{1}{z} \quad \text{and} \quad T_3(z) = r_1 r_2 z \quad \text{and} \quad T_4(z) = z + w_0.$$

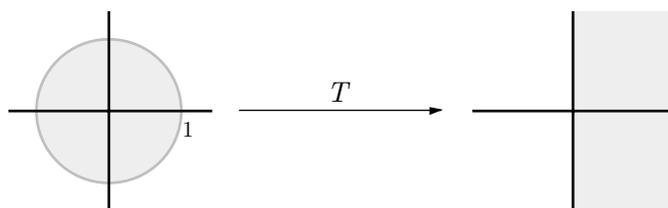
We have the picture below:



Note that T_1 is a translation which takes the centre of the disc $\{z : |z - z_0| < r_1\}$ to the origin. Then the inversion T_2 turns a disc into an annulus. We now apply a magnification T_3 and then use the translation T_4 to position the disc so that its centre is at w_0 . It is easy to see that

$$T(z) = \frac{r_1 r_2}{z - z_0} + w_0 = \frac{w_0 z + (r_1 r_2 - w_0 z_0)}{z - z_0}.$$

EXAMPLE 13.4.2. Suppose that we are required to find a Möbius transformation T which maps the unit disc $\{z : |z| < 1\}$ to the right half plane $\{w : \Re w > 0\}$.



Our first step is to find a Möbius transformation $S : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ which maps the unit circle $\{z : |z| = 1\}$ to the imaginary axis $\{w : \Re w = 0\}$. For

$$S(z) = \frac{az + b}{cz + d}$$

to map the unit circle to a line, the unit circle must contain the pole $z = -d/c$ of S . Suppose that we choose the point $z = 1$ to be this pole. In this case, we may take, for example, $c = 1$ and $d = -1$. Next, some point on the unit circle must have image 0 under S . Suppose that we choose $z = -1$ to be this point. In this case, we may take, for example, $a = 1$ and $b = 1$. Note that $ad - bc \neq 0$, and these choices give

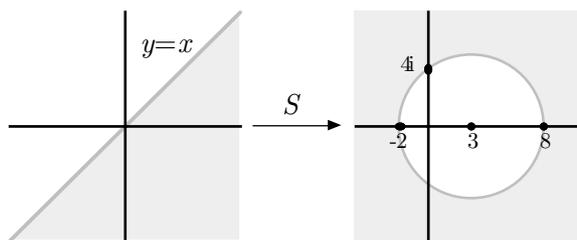
$$S(z) = \frac{z + 1}{z - 1}.$$

Note that $S(1) = \infty$ and $S(-1) = 0$. We also know that the image of the unit circle under S is a line through the origin, but at this point, we do not know whether this line is the imaginary axis. To check what this line is, we use a third point on the unit circle, the point $z = i$, say. It is easy to check that $S(i) = -i$, on the imaginary axis. We therefore conclude that S maps the unit circle to the imaginary axis. It follows that S maps the unit disc $\{z : |z| < 1\}$ to one of the half planes arising from the imaginary axis, but at this point, we do not know whether it is $\{w : \Re w < 0\}$ or $\{w : \Re w > 0\}$. To check which half plane this is, we can use the point $z = 0$. It is easy to check that $S(0) = -1$. Unfortunately, this is in $\{w : \Re w < 0\}$ instead of $\{w : \Re w > 0\}$. This little problem can be eradicated by rotating $S(z)$ about the origin by an angle π . In other words, the Möbius transformation

$$T(z) = e^{i\pi} S(z) = -S(z) = \frac{-z - 1}{z - 1}$$

satisfies our requirements.

EXAMPLE 13.4.3. Suppose that we are required to find a Möbius transformation S which maps the half plane $\{z = x + iy : y < x\}$ to the annulus $\{w : |w - 3| > 5\}$.



Our first step is to find a Möbius transformation $S : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ which maps the line $\{z = x + iy : y = x\}$ to the circle $\{w : |w - 3| = 5\}$. Consider first of all the transformation

$$S_1(z) = \sqrt{2}e^{-i\pi/4}z = (1 - i)z$$

(the magnification here by $\sqrt{2}$ serves only to simplify the arithmetic), where we attempt to map the line $\{z = x + iy : y = x\}$ to the real axis $\{z : \Im z = 0\}$. Next, we shall find a Möbius transformation S_2 which maps the real axis $\{z : \Im z = 0\}$ to the circle $\{w : |w - 3| = 5\}$. To do this, we shall use the Möbius transformation S_2 which maps the points $0, 1, \infty$, say, on the real axis to the points $8, 4i, -2$, say, on the circle. From the proof of Theorem 13G, the inverse Möbius transformation S_2^{-1} is given by

$$z = S_2^{-1}(w) = \frac{(w - 8)(4i + 2)}{(w + 2)(4i - 8)} = \frac{w - 8}{2i(w + 2)}.$$

Simple calculation gives

$$w = S_2(z) = \frac{4z - 8i}{-2z - i},$$

and so the Möbius transformation $S = S_2 \circ S_1$, given by

$$S(z) = (S_2 \circ S_1)(z) = \frac{4(1-i)z - 8i}{-2(1-i)z - i},$$

maps the line $\{z = x + iy : y = x\}$ to the circle $\{w : |w - 3| = 5\}$. It follows that S maps the half plane $\{z = x + iy : y < x\}$ to the disc $\{w : |w - 3| < 5\}$ or the annulus $\{w : |w - 3| > 5\}$. To check which this is, we can use the point $z = 1$. It is easy to check that $S(1) = -4 + 4i$. This is in the annulus $\{w : |w - 3| > 5\}$. It follows that

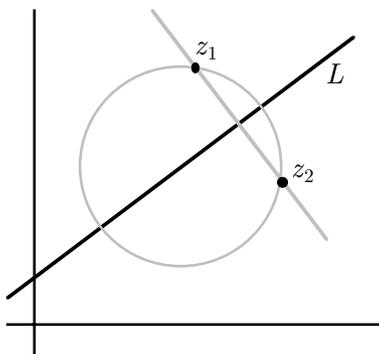
$$S(z) = \frac{4(1-i)z - 8i}{-2(1-i)z - i}$$

satisfies our requirements.

13.5. Symmetry and Möbius Transformations

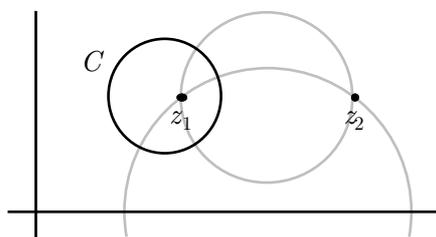
DEFINITION. We say that two points $z_1, z_2 \in \mathbb{C}$ are symmetric with respect to a line L if L is the perpendicular bisector of the line segment joining z_1 and z_2 .

Suppose that $z_1, z_2 \in \mathbb{C}$ are symmetric with respect to a line L . Then it is easy to see that every circle or line passing through both z_1 and z_2 intersects L at right angles.



Using this observation, we make the following definition.

DEFINITION. We say that two points $z_1, z_2 \in \mathbb{C}$ are symmetric with respect to a circle C if every circle or line passing through both z_1 and z_2 intersects C at right angles.



REMARKS. (1) Consider the circle $C = \{z : |z - z_0| = r\}$ with centre z_0 and radius r . Then it can be shown that two points z_1 inside C and z_2 outside C are symmetric with respect to C if and only if there exist $\rho, \theta \in \mathbb{R}$ satisfying $0 < \rho < r$ and such that

$$z_1 = z_0 + \rho e^{i\theta} \quad \text{and} \quad z_2 = z_0 + \frac{r^2}{\rho} e^{i\theta};$$

in other words, if and only if $(z_1 - z_0)(\overline{z_2 - z_0}) = r^2$.

(2) We also say that the centre of a circle C and the point at ∞ are symmetric with respect to the circle C .

(3) Note that a line can be interpreted as a circle of infinite radius. It follows that our definition covers symmetry with respect to both lines and circles.

THEOREM 13H. (SYMMETRY PRINCIPLE) *Suppose that $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a Möbius transformation. Suppose further that C is a circle or line in $\overline{\mathbb{C}}$. Then two points $z_1, z_2 \in \overline{\mathbb{C}}$ are symmetric with respect to C if and only if $T(z_1)$ and $T(z_2)$ are symmetric with respect to $T(C)$.*

PROOF. Note that T maps the class of lines and circles in $\overline{\mathbb{C}}$ to itself. Note also that T is conformal at all points where it is analytic, and so preserves orthogonality. \circ

EXAMPLE 13.5.1. Let us return to Example 13.3.1. Suppose that $a \in \mathbb{C}$ is fixed and $|a| < 1$. Suppose also that $\lambda \in \mathbb{C}$ is fixed and $|\lambda| = 1$. Then the Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, given by

$$T(z) = \lambda \frac{z - a}{az - 1},$$

maps the unit disc $D = \{z : |z| < 1\}$ onto itself. Note here that we have introduced an extra rotation λ about the origin. We shall now attempt to show that any Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ which maps the unit disc D onto itself must be of this form. Clearly T maps the unit circle $C = \{z : |z| = 1\}$ onto itself. Next, let $a \in \mathbb{C}$ be the unique point satisfying $T(a) = 0$. Then $|a| < 1$. Suppose now that a and a^* are symmetric with respect to the unit circle C . Then by the Symmetry principle, $T(a)$ and $T(a^*)$ are symmetric with respect to the circle $T(C) = C$. Since $T(a) = 0$, we must have $T(a^*) = \infty$. It follows that $T(z)$ must have a zero at $z = a$ and a pole at $z = a^*$. Note now that $a\overline{a^*} = 1$, so that $a^* = 1/\overline{a}$. Hence

$$T(z) = \lambda \frac{z - a}{az - 1}$$

for some $\lambda \in \mathbb{C}$. Recall now that $|T(z)| = 1$ whenever $|z| = 1$. In particular, we require

$$|T(1)| = \left| \lambda \frac{1 - a}{a - 1} \right| = 1.$$

It follows that $|\lambda| = 1$.

PROBLEMS FOR CHAPTER 13

1. Find a Möbius transformation that takes the points $0, 2, -2$ to the points $-2, 0, 2$ respectively.
2. Show that the Möbius transformation $w = \frac{z - i}{z + i}$ maps the upper half plane $\{z : \Im z > 0\}$ onto the disc $\{w : |w| < 1\}$.

3. Suppose that C is a given circle or line, and that C' is also a given circle or line. Does there exist a Möbius transformation that maps C onto C' ? If so, is this Möbius transformation unique? Justify your assertions.
4. The cross ratio of four distinct points $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$ is defined by

$$X(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

and by the obvious limit if one of the points is ∞ . Show that the cross ratio is invariant under Möbius transformation; in other words, for every Möbius transformation $T : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, we have

$$X(T(z_1), T(z_2), T(z_3), T(z_4)) = X(z_1, z_2, z_3, z_4).$$

[HINT: Note that every Möbius transformation is a composition of translations, rotations, magnifications and inversions.]

5. Use the invariance of the cross ratio to find a Möbius transformation that takes the points $0, 1, \infty$ to the points $-i, 1, i$ respectively.
[HINT: Suppose that the Möbius transformation takes z to w .]
6. Show that a Möbius transformation $w = f(z)$ maps the upper half plane $\{z : \Im z > 0\}$ onto the disc $\{w : |w| < 1\}$ if and only if it is of the form

$$w = \lambda \frac{z - a}{z - \bar{a}},$$

where $a, \lambda \in \mathbb{C}$ satisfy $|\lambda| = 1$ and $\Im a > 0$.

7. a) Construct a one-parameter family of Möbius transformations that map the real axis onto the unit circle by mapping the points $0, \lambda, \infty$ to the points $-i, 1, i$ respectively, where λ is a non-zero real parameter.
b) What point of the upper half plane gets mapped to the centre of the circle?
c) For what values of λ is the upper half plane $\{z : \Im z > 0\}$ mapped onto the disc $\{w : |w| < 1\}$?
Onto the annulus $\{w : |w| > 1\}$?
d) Taking note of Problem 6, comment whether the family includes all Möbius transformations that map the real axis onto the unit circle.
8. Find all Möbius transformations that map the disc $\{z : |z - 1| < 2\}$ onto the upper half plane $\{w : \Im w > 0\}$ and takes $z = 1$ to $w = i$.
9. Show that if either of the transformations $w = a + \frac{bz}{1 - cz}$ and $w = c + \frac{bz}{1 - az}$ maps the unit disc onto the unit disc, then they both do.
10. Find a transformation that maps $A = \{z = x + iy : |z| < 1 \text{ and } y > 0\}$, the upper half of the unit disc, onto the first quadrant.

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 14

SCHWARZ-CHRISTOFFEL TRANSFORMATIONS

14.1. Introduction

Recall that a function $f(z)$ is conformal at every point where it is analytic and has non-zero derivative. In this chapter, we shall study the situation at points where $f(z)$ is not conformal.

Suppose that $x_0 \in \mathbb{R}$ is fixed. Consider a function $f(z)$ with derivative

$$f'(z) = (z - x_0)^\alpha,$$

where $-1 < \alpha < 1$. Here we have chosen the branch of the argument so that

$$-\frac{\pi}{2} < \arg(z - x_0) \leq \frac{3\pi}{2},$$

introducing a branch cut along the axis $\{x_0 + iy : y \leq 0\}$. We shall study the image of the real axis under this mapping f .

Suppose first of all that z lies on the real axis and $z > x_0$. Then $f(z)$ is conformal at such a point z , since $f'(z) \neq 0$. Note also that

$$\arg f'(z) = \alpha \arg(z - x_0) = 0$$

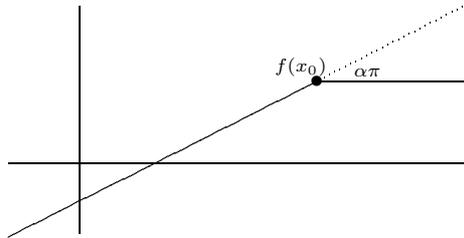
for all such points z , ignoring multiples of 2π . Since the tangent at every point of the half line (x_0, ∞) has slope 0, it follows that the tangent at every point of the image curve $f((x_0, \infty))$ has slope $\arg f'(z) = 0$. Hence $f((x_0, \infty))$ is a half line parallel to the real axis and has left hand end point $f(x_0)$.

Suppose next that z lies on the real axis and $z < x_0$. Again $f(z)$ is conformal at such a point z , since $f'(z) \neq 0$. Note also that

$$\arg f'(z) = \alpha \arg(z - x_0) = \alpha\pi$$

for all such points z , again ignoring multiples of 2π . It follows easily that $f((-\infty, x_0))$ is a half line making an angle $\alpha\pi$ with the horizontal axis.

Summarizing the above, we have the following diagram which describes the image of the real axis under f .



14.2. A Generalization

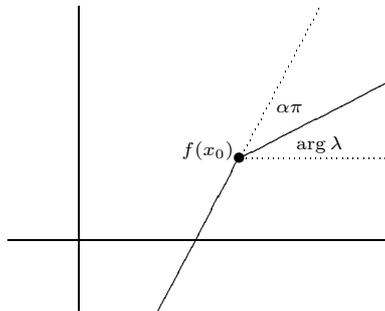
Again, suppose that $x_0 \in \mathbb{R}$ is fixed. Consider a function $f(z)$ with derivative

$$f'(z) = \lambda(z - x_0)^\alpha,$$

where $\lambda \in \mathbb{C}$ is non-zero and $-1 < \alpha < 1$. Then

$$\arg f'(z) = \arg \lambda + \alpha \arg(z - x_0).$$

In other words, there is an extra rotation by $\arg \lambda$ from the case in the previous section. This leads to the following diagram which describes the image of the real axis under f .



Suppose now that $x_1, \dots, x_k \in \mathbb{R}$ are fixed, and that $x_1 < \dots < x_k$. Consider a function $f(z)$ with derivative

(1)
$$f'(z) = \lambda(z - x_1)^{\alpha_1} \dots (z - x_k)^{\alpha_k},$$

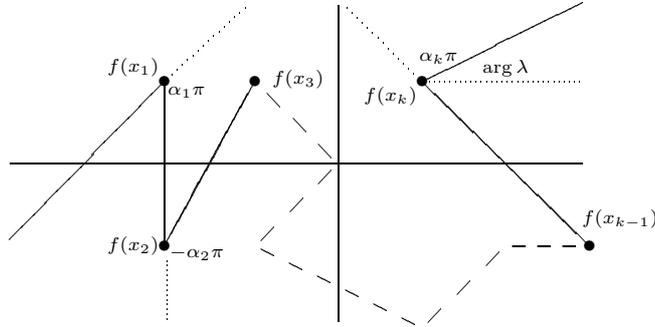
where $\lambda \in \mathbb{C}$ is non-zero and $-1 < \alpha_1, \dots, \alpha_k < 1$. Then

$$\arg f'(z) = \arg \lambda + \alpha_1 \arg(z - x_1) + \dots + \alpha_k \arg(z - x_k).$$

It is easy to see that if z is on the real axis, then

$$\arg f'(z) = \begin{cases} \arg \lambda & \text{if } z > x_k, \\ \arg \lambda + \alpha_k \pi & \text{if } x_{k-1} < z < x_k, \\ \vdots & \\ \arg \lambda + \alpha_2 \pi + \dots + \alpha_k \pi & \text{if } x_1 < z < x_2, \\ \arg \lambda + \alpha_1 \pi + \dots + \alpha_k \pi & \text{if } z < x_1. \end{cases}$$

This leads to the following diagram which describes the image of the real axis under f .



Suppose now that a function $f(z)$ satisfies (1). Then it is analytic on the complex plane \mathbb{C} with a few branch cuts at x_1, \dots, x_k . More precisely, it is analytic in the domain

$$\mathbb{C} \setminus (\{x_1 + iy : y \leq 0\} \cup \dots \cup \{x_k + iy : y \leq 0\}).$$

It follows that for any $z \in \mathcal{H}$, where \mathcal{H} denotes the upper half plane, we can write

$$(2) \quad f(z) = \int_{[z_0, z]} f'(\zeta) d\zeta + B = \lambda \int_{[z_0, z]} (\zeta - x_1)^{\alpha_1} \dots (\zeta - x_k)^{\alpha_k} d\zeta + B.$$

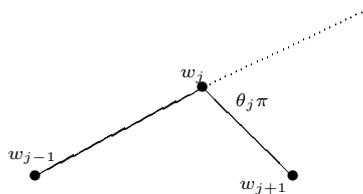
Here, z_0 is a suitably chosen point in \mathcal{H} or its boundary. Also, for every $z \in \mathcal{H}$, $[z_0, z]$ denotes the straight line segment from z_0 to z .

DEFINITION. A function $f(z)$ of the form (2) is called a Schwarz-Christoffel transformation.

14.3. Polygons

Note that the function (2) maps the real axis onto a polygonal path. We now wish to construct a one-to-one analytic function that maps the upper half plane \mathcal{H} onto the interior of a given polygon P . The idea is to tailor a Schwarz-Christoffel transformation to achieve this.

Suppose that the vertices of the polygon P are given by w_1, \dots, w_k in the anticlockwise direction. Let us follow the edges of the polygon P . At vertex w_j , suppose that we make a right turn of angle $\theta_j\pi$, where $-1 < \theta_j < 1$, with the convention that $\theta_j < 0$ denotes a left turn.



Since P is a polygon and its vertices are given in the anticlockwise direction, we must have

$$\theta_1\pi + \dots + \theta_k\pi = -2\pi.$$

It is an elementary fact in geometry that if we know the vertices w_1, \dots, w_{k-1} and angles $\theta_1\pi, \dots, \theta_{k-1}\pi$ of the polygon P , then the last vertex w_k and angle $\theta_k\pi$ are uniquely determined. The idea is therefore

to find real numbers $x_1 < \dots < x_{k-1}$ to act as preimages of the vertices w_1, \dots, w_{k-1} , and to assume that $x = \infty$ is the preimage of the vertex w_k .

Suppose that $x_1 < \dots < x_{k-1}$. Clearly the function

$$g(z) = \int_{[z_0, z]} (\zeta - x_1)^{\theta_1} \dots (\zeta - x_{k-1})^{\theta_{k-1}} d\zeta$$

maps the real line onto some polygon Q of k sides. However, the polygon Q may not be the polygon P , but at least it has the required right hand turn angles $\theta_1, \dots, \theta_{k-1}$ at the vertices $g(x_1), \dots, g(x_{k-1})$. We can adjust the lengths of the sides of the polygon Q by choosing x_1, \dots, x_{k-1} carefully, so that Q is similar to the polygon P . Once this is achieved, we can then map the polygon Q to the polygon P by a linear transformation.

We state, without proof, the following important result.

THEOREM 14A. *Suppose that P is a polygon with vertices w_1, \dots, w_k in the anticlockwise direction, with corresponding right turns of angles $\theta_1\pi, \dots, \theta_k\pi$ respectively, where $-1 < \theta_1, \dots, \theta_k < 1$. Then there exists a function of the form*

$$f(z) = A \int_{[z_0, z]} (\zeta - x_1)^{\theta_1} \dots (\zeta - x_{k-1})^{\theta_{k-1}} d\zeta + B,$$

where $A, B \in \mathbb{C}$, that maps the upper half plane \mathcal{H} one-to-one and conformally onto the interior of P , with

$$f(x_1) = w_1, \quad \dots, \quad f(x_{k-1}) = w_{k-1}, \quad f(\infty) = w_k.$$

REMARKS. (1) Note that we do not even need to have very precise information on w_k and θ_k .

(2) Certain infinite regions can sometimes be thought of as infinite polygons. In this case, it is sometimes convenient to take w_k as the point at infinity, as we need no information on the angle θ_k when we use Theorem 14A.

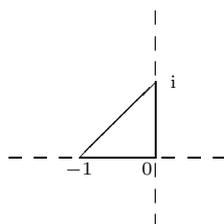
(3) It can be shown that a Schwarz-Christoffel transformation can be uniquely determined by three points, as is the case for Möbius transformations. This can be interpreted as three degrees of freedom in our construction of the transformation. One of these is used by taking $f(\infty) = w_k$. We can therefore afford to choose x_1 and x_2 freely, subject to the restriction that $-\infty < x_1 < x_2 < \infty$.

(4) Occasionally, we may choose extra points apart from x_1 and x_2 due to symmetry properties of the polygon P . We shall illustrate this point in Examples 14.4.3–14.4.5 below.

(5) Note that the integrals involved may be impossible to calculate in practice. Numerical techniques are often used. However, we shall not discuss these here.

14.4. Examples

EXAMPLE 14.4.1. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane \mathcal{H} to the inside of the triangle with vertices at -1 , 0 and i . The boundary of the triangle is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = i, \quad w_2 = -1, \quad w_3 = 0,$$

so that

$$\theta_1 = -3/4, \quad \theta_2 = -3/4, \quad \theta_3 = -1/2.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0, z]} (\zeta - x_1)^{-3/4} (\zeta - x_2)^{-3/4} d\zeta + B.$$

We may choose $x_1 = -1$ and $x_2 = 1$, and obtain, using $z_0 = 0$,

$$f(z) = A \int_{[0, z]} (\zeta + 1)^{-3/4} (\zeta - 1)^{-3/4} d\zeta + B = A \int_{[0, z]} (\zeta^2 - 1)^{-3/4} d\zeta + B.$$

We need $f(-1) = i$ and $f(1) = -1$. It follows that

$$A \int_0^{-1} (\zeta^2 - 1)^{-3/4} d\zeta + B = i \quad \text{and} \quad A \int_0^1 (\zeta^2 - 1)^{-3/4} d\zeta + B = -1.$$

Writing

$$\kappa = \int_0^1 (\zeta^2 - 1)^{-3/4} d\zeta,$$

we have

$$-A\kappa + B = i \quad \text{and} \quad A\kappa + B = -1,$$

so that

$$A = \frac{-1-i}{2\kappa} \quad \text{and} \quad B = \frac{i-1}{2}.$$

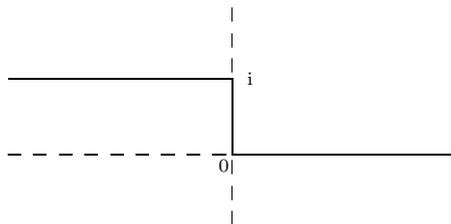
Hence

$$f(z) = \frac{-1-i}{2\kappa} \int_{[0, z]} (\zeta^2 - 1)^{-3/4} d\zeta + \frac{i-1}{2}.$$

EXAMPLE 14.4.2. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane \mathcal{H} to the set

$$P = \{z = x + iy : x > 0 \text{ and } y > 0\} \cup \{z = x + iy : x \leq 0 \text{ and } y > 1\}.$$

The boundary of P is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = i, \quad w_2 = 0, \quad w_3 = \infty,$$

so that

$$\theta_1 = 1/2 \quad \text{and} \quad \theta_2 = -1/2.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0, z]} (\zeta - x_1)^{1/2} (\zeta - x_2)^{-1/2} d\zeta + B'.$$

We may choose $x_1 = -1$ and $x_2 = 1$, and obtain

$$\begin{aligned} f(z) &= A \int_{[z_0, z]} (\zeta + 1)^{1/2} (\zeta - 1)^{-1/2} d\zeta + B' = A \int_{[z_0, z]} \left(\frac{\zeta + 1}{\zeta - 1} \right)^{1/2} d\zeta + B' \\ &= A \left((z^2 - 1)^{1/2} + \log(z + (z^2 - 1)^{1/2}) \right) + B. \end{aligned}$$

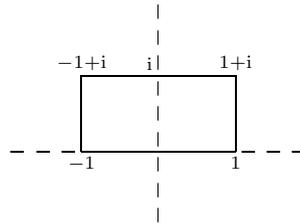
We shall omit some of the painful analysis, and claim that we can choose a branch of the function which is analytic in the upper half plane \mathcal{H} . We need $f(-1) = i$ and $f(1) = -1$. It follows that by choosing a suitable branch of the logarithm, we have

$$A \log(-1) + B = i \quad \text{and} \quad A \log 1 + B = 0,$$

so that $A = 1/\pi$ and $B = 0$. Hence

$$f(z) = \frac{1}{\pi} \left((z^2 - 1)^{1/2} + \log(z + (z^2 - 1)^{1/2}) \right).$$

EXAMPLE 14.4.3. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane \mathcal{H} to the inside of the rectangle with vertices at ± 1 and $\pm 1 + i$. The boundary of the rectangle is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = -1 + i, \quad w_2 = -1, \quad w_3 = 1, \quad w_4 = 1 + i, \quad w_5 = i$$

(here we have used an extra point w_5 in order to create some symmetry; see Remark (4) in the previous section), so that

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = -1/2 \quad \text{and} \quad \theta_5 = 0.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0, z]} (\zeta - x_1)^{-1/2} (\zeta - x_2)^{-1/2} (\zeta - x_3)^{-1/2} (\zeta - x_4)^{-1/2} d\zeta + B.$$

We shall choose

$$x_1 = -\alpha, \quad x_2 = -1, \quad x_3 = 1, \quad x_4 = \alpha,$$

where $\alpha > 1$ will be determined later. Note that we are attempting to benefit from the symmetry here. With such a choice, we obtain, using $z_0 = 0$,

$$\begin{aligned} f(z) &= A \int_{[0,z]} (\zeta + \alpha)^{-1/2} (\zeta + 1)^{-1/2} (\zeta - 1)^{-1/2} (\zeta - \alpha)^{-1/2} d\zeta + B \\ &= A \int_{[0,z]} (\zeta^2 - 1)^{-1/2} (\zeta^2 - \alpha^2)^{-1/2} d\zeta + B = A \int_{[0,z]} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} + B. \end{aligned}$$

We need

$$f(-\alpha) = -1 + i, \quad f(-1) = -1, \quad f(1) = 1, \quad f(\alpha) = 1 + i.$$

It follows that

$$(3) \quad A \int_0^{-\alpha} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} + B = -1 + i,$$

$$(4) \quad A \int_0^{-1} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} + B = -1,$$

$$(5) \quad A \int_0^1 \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} + B = 1,$$

$$(6) \quad A \int_0^{\alpha} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} + B = 1 + i.$$

Subtracting (4) from (3) and subtracting (5) from (6), we obtain respectively

$$A \int_{-1}^{-\alpha} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} = i \quad \text{and} \quad A \int_1^{\alpha} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} = i,$$

which are in fact the same equation (note that symmetry is at work here). Multiplying the denominator by i , we obtain

$$(7) \quad A \int_1^{\alpha} \frac{d\zeta}{\sqrt{(\zeta^2 - 1)(\alpha^2 - \zeta^2)}} = 1.$$

On the other hand, if $B = 0$, then (4) and (5) are the same, and can be represented by

$$(8) \quad A \int_0^1 \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} = 1.$$

It follows that our choice of α should be made so that

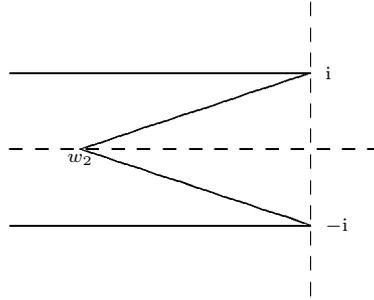
$$\int_0^1 \frac{d\zeta}{\sqrt{(1 - \zeta^2)(\alpha^2 - \zeta^2)}} = \int_1^{\alpha} \frac{d\zeta}{\sqrt{(\zeta^2 - 1)(\alpha^2 - \zeta^2)}}.$$

We can then take A to be the reciprocal of the common value of these two integrals.

EXAMPLE 14.4.4. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane \mathcal{H} to the domain

$$P = \mathbb{C} \setminus \{z = x \pm i : x \leq 0\}.$$

The boundary of the set P is described by the solid edges in the picture below when the point w_2 is taken to infinity along the negative real axis.



Let us write, in our notation,

$$w_1 = i, \quad w_2 = \infty, \quad w_3 = -i, \quad w_4 = \infty$$

(note again the symmetry; see Remark (4) in the previous section), so that

$$\theta_1 = 1, \quad \theta_2 = -1, \quad \theta_3 = 1.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0, z]} (\zeta - x_1)(\zeta - x_2)^{-1}(\zeta - x_3) d\zeta + B'.$$

We shall choose

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1,$$

and note that we are attempting to benefit from the symmetry here. We obtain

$$\begin{aligned} f(z) &= A \int_{[z_0, z]} (\zeta + 1)\zeta^{-1}(\zeta - 1) d\zeta + B' = A \int_{[z_0, z]} (\zeta^2 - 1)\zeta^{-1} d\zeta + B' \\ &= A \int_{[z_0, z]} \left(\zeta - \frac{1}{\zeta} \right) d\zeta + B' = A \left(\frac{z^2}{2} - \log z \right) + B. \end{aligned}$$

We need

$$f(-1) = i, \quad f(0) = -\infty, \quad f(1) = -i.$$

It follows that by choosing a suitable branch of the logarithm, we have

$$A \left(\frac{1}{2} - i\pi \right) + B = i \quad \text{and} \quad A \left(\frac{1}{2} - 0 \right) + B = -i,$$

so that $A = -2/\pi$ and $B = 1/\pi - i$. Hence

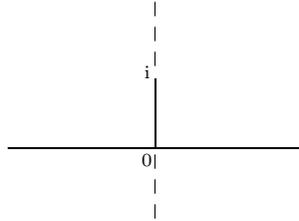
$$f(z) = -\frac{2}{\pi} \left(\frac{z^2}{2} - \log z \right) + \left(\frac{1}{\pi} - i \right).$$

Note that $|f(z)| \rightarrow \infty$ as $z \rightarrow 0$.

EXAMPLE 14.4.5. We wish to find a Schwarz-Christoffel transformation that maps the upper half plane \mathcal{H} to the domain

$$P = \mathcal{H} \setminus \{z = yi : y \leq 1\}.$$

The boundary of the set P is described by the solid edges in the picture below.



Let us write, in our notation,

$$w_1 = 0, \quad w_2 = i, \quad w_3 = 0, \quad w_4 = \infty$$

(note again the symmetry as well as the use of the point 0 twice), so that

$$\theta_1 = -1/2, \quad \theta_2 = 1, \quad \theta_3 = -1/2.$$

Following Theorem 14A, we consider a function of the form

$$f(z) = A \int_{[z_0, z]} (\zeta - x_1)^{-1/2} (\zeta - x_2) (\zeta - x_3)^{-1/2} d\zeta + B'.$$

We shall choose

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1,$$

and note again that we are attempting to benefit from the symmetry here. We obtain

$$f(z) = A \int_{[z_0, z]} (\zeta + 1)^{-1/2} \zeta (\zeta - 1)^{-1/2} d\zeta + B' = A \int_{[z_0, z]} (\zeta^2 - 1)^{-1/2} \zeta d\zeta + B' = A(z^2 - 1)^{1/2} + B.$$

We need

$$f(-1) = 0, \quad f(0) = i, \quad f(1) = 0.$$

It follows that by choosing a suitable branch of the function which is positive for large positive z , we have

$$Ai + B = i \quad \text{and} \quad B = 0,$$

so that $A = 1$ and $B = 0$. Hence

$$f(z) = (z^2 - 1)^{1/2}.$$

PROBLEMS FOR CHAPTER 14

1. Use these notes and without reproducing proofs, find a transformation that maps the unit disc $D = \{z : |z| < 1\}$ onto the domain $D' = \mathcal{H} \setminus \{z = yi : y \leq 1\}$, where \mathcal{H} denotes the upper half plane.

2. For each of the sets A below, find a Schwarz-Christoffel transformation that maps the upper half plane \mathcal{H} onto the set A :

- a) A is an open triangular region with vertices ± 1 and $i\sqrt{3}$.
- b) A is the region above the polygonal path

$$\{z = x + i : x \leq 0\} \cup \{z = x + (1 - x)i : 0 \leq x \leq 1\} \cup \{z = x : x \geq 1\}.$$

- c) $A = \{z = x + iy : y > 0 \text{ or } |x| < 1\}$.

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 15

LAPLACE'S EQUATION REVISITED

15.1. Use of Möbius Transformations

Recall that Laplace's equation involves finding a harmonic function in a given region and which satisfies given boundary conditions. In this chapter, we shall illustrate very briefly the use of transformations to simplify this problem. Note, however, that we are not discussing the general problem of the solution of Laplace's equation; that is a topic in partial differential equations. Here we shall satisfy ourselves on how to use a few simple cases of Laplace's equation to obtain solutions in more complicated situations. We first discuss an example which uses Möbius transformations.

EXAMPLE 15.5.1. Consider the lens region formed by the intersection of the two discs

$$\{z \in \mathbb{C} : |z + 1|^2 < 2\} \cap \{z \in \mathbb{C} : |z - 1|^2 < 2\}.$$

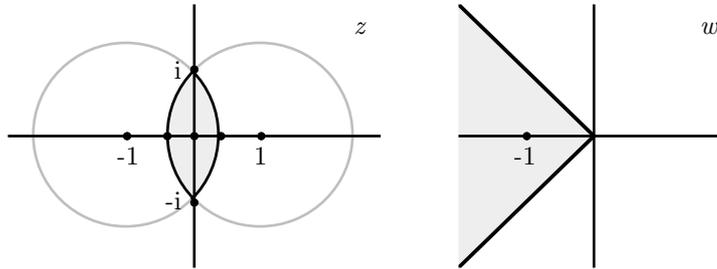
Here the two discs both have radius $\sqrt{2}$ and are centred at $z = -1$ and $z = 1$ respectively (see the picture on the next page). Suppose that we are required to find a harmonic function ϕ in this region with $\phi = 1$ on the right hand boundary and $\phi = 0$ on the left hand boundary. Note that both the right hand and left hand boundaries are parts of circles and intersect at $z = \pm i$. If we use a Möbius transformation with pole at $z = i$, then both boundaries are transformed into straight lines. Let us try the transformation

$$w = f(z) = \frac{z + i}{z - i}.$$

Then $f(i) = \infty$ and $f(-i) = 0$. By considering, for example, $f(\sqrt{2} - 1)$ and $f(1 - \sqrt{2})$, it is not difficult to show that the right hand and left hand boundaries are transformed into the half lines $\arg w = 3\pi/4$ and $\arg w = 5\pi/4$ respectively. Note also that $f(0) = -1$. It follows that the lens region is transformed into the region

$$\left\{ w \in \mathbb{C} : \frac{3\pi}{4} < \arg w < \frac{5\pi}{4} \right\}.$$

We summarize the above discussion in the pictures below:



It is easy to check that the function

$$\psi(w) = \frac{2}{\pi} \left(\frac{5\pi}{4} - \arg w \right)$$

is harmonic in this region and satisfies $\psi(w) = 1$ when $\arg w = 3\pi/4$ and $\psi(w) = 0$ when $\arg w = 5\pi/4$. It follows that our required harmonic function is given by

$$\phi(z) = \frac{2}{\pi} \left(\frac{5\pi}{4} - \arg \left(\frac{z+i}{z-i} \right) \right).$$

15.2. Use of Schwarz-Christoffel Transformations

We now discuss three examples which use Schwarz-Christoffel transformations.

EXAMPLE 15.2.1. We wish to find a non-constant harmonic function in the region above the polygonal path given in Example 14.4.2, with boundary condition $\phi = 0$ on the polygonal path. Here $\phi = \text{const}$ can be interpreted as lines of flow on a river over a step on the river bed. Recall that the Schwarz-Christoffel transformation

$$f(z) = \frac{1}{\pi} \left((z^2 - 1)^{1/2} + \log(z + (z^2 - 1)^{1/2}) \right)$$

maps the upper half plane onto the region in question. We now need to find a non-constant harmonic function ψ on the upper half plane with boundary condition $\psi = 0$ on the real line. For example, the function

$$\psi(z) = \Im z$$

satisfies the requirements. We now need to invert the function $f(z)$ to obtain a harmonic function

$$\phi(w) = \Im(f^{-1}(w))$$

in the original region.

EXAMPLE 15.2.2. We wish to find a harmonic function in the slit plane given in Example 14.4.4, with boundary conditions $\phi = 1$ on the upper slit and $\phi = -1$ on the lower slit. Here $\phi = \text{const}$ can be interpreted as equipotential lines in a region around two semi-infinite conducting plates with opposite charges. Recall that the Schwarz-Christoffel transformation

$$f(z) = -\frac{2}{\pi} \left(\frac{z^2}{2} - \log z \right) + \left(\frac{1}{\pi} - i \right)$$

maps the upper half plane onto the region in question. Furthermore, it maps the negative and positive real axis onto the upper and lower slits respectively. We now need to find a harmonic function ψ on the upper half plane with boundary conditions $\psi = 1$ on the negative real axis and $\psi = -1$ on the positive real axis. For example, the function

$$\psi(z) = \frac{2}{\pi} \arg z - 1$$

satisfies the requirements (here we take the principal value of the argument). We now need to invert the function $f(z)$ to obtain a harmonic function

$$\phi(w) = \frac{2}{\pi} \arg(f^{-1}(w)) - 1$$

in the original region.

EXAMPLE 15.2.3. We wish to find a non-constant harmonic function in the slit upper half plane given in Example 14.4.5, with boundary condition $\phi = 0$ on the slit and the real axis. Here $\phi = \text{const}$ can be interpreted as lines of flow past an obstacle. Recall that the Schwarz-Christoffel transformation

$$f(z) = (z^2 - 1)^{1/2}$$

maps the upper half plane onto the region in question. We now need to find a harmonic function ψ on the upper half plane with boundary conditions $\psi = 0$ on the real line. For example, the function

$$\psi(z) = \Im z$$

satisfies the requirements. Note now that

$$f^{-1}(w) = (w^2 + 1)^{1/2}.$$

We therefore obtain the harmonic function

$$\phi(w) = \Im((w^2 + 1)^{1/2})$$

in the original region. Here we choose a branch of the square root that is positive for large positive w .

INTRODUCTION TO COMPLEX ANALYSIS

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Chapter 16

UNIFORM CONVERGENCE

16.1. Uniform Convergence of Sequences

Recall that if a sequence a_n of complex numbers converges to a , then, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a| < \epsilon$ whenever $n > N$.

We can extend this to pointwise convergence in a region $D \subseteq \mathbb{C}$ in a natural way. A sequence of complex valued functions $a_n(z)$ defined on D converges pointwise to a function $a(z)$ defined on D if, given any $\epsilon > 0$ and any $z \in D$, there exists $N \in \mathbb{R}$ such that $|a_n(z) - a(z)| < \epsilon$ whenever $n > N$. Here the value of N may depend on the choice of $z \in D$. Indeed, for any fixed $z \in D$, we simply consider the convergence of the sequence $a_n(z)$ of complex numbers to the complex number $a(z)$. The region D does not play any essential part in the argument apart from providing the complex numbers z in question.

In this chapter, we introduce the idea of uniformity to the question of convergence. Put simply, uniformity transfers the dependence of N on z to dependence of N only on the region D containing the complex numbers z in question. More precisely, we have the following definition.

DEFINITION. Suppose that $D \subseteq \mathbb{C}$ is a region. We say that a sequence of complex valued functions $a_n(z)$ converges uniformly in D to a function $a(z)$, denoted by $a_n(z) \rightarrow a(z)$ as $n \rightarrow \infty$ uniformly in D , if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a(z)| < \epsilon$ whenever $n > N$.

REMARK. Note that N no longer depends on the choice of $z \in D$. Note also that a precise definition can be given by requiring $N \in \mathbb{R}$ to satisfy

$$\sup_{z \in D} |a_n(z) - a(z)| < \epsilon$$

whenever $n > N$.

EXAMPLE 16.1.1. Consider the sequence

$$a_n(z) = \frac{z}{n}$$

in the region $D = \{z : |z| < 1\}$. Note first of all that for every fixed $z \in D$, we have $a_n(z) \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, given any $\epsilon > 0$, we have, for every $z \in D$, that

$$|a_n(z) - 0| = \frac{|z|}{n} < \frac{1}{n} < \epsilon$$

whenever $n > 1/\epsilon$. Hence $a_n(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly in D . Now consider the same sequence in the region $D = \mathbb{C}$. Note that

$$|a_n(z) - 0| < \epsilon \quad \text{if and only if} \quad n > \frac{|z|}{\epsilon}.$$

It is therefore impossible to find a suitable N independent of the choice of $z \in \mathbb{C}$. Hence $a_n(z)$ converges to 0, but not uniformly, in \mathbb{C} .

16.2. Consequences of Uniform Convergence

In this section, we show that uniform convergence carries a number of properties of the sequence over to the limit function. The following three results concern respectively continuity, integrability and differentiability.

THEOREM 16A. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that $a_n(z) \rightarrow a(z)$ as $n \rightarrow \infty$ uniformly in D . Then $a(z)$ is continuous in D .*

PROOF. Suppose that $z_0 \in D$ is fixed. For every $z \in D$, we have

$$a(z) - a(z_0) = a(z) - a_n(z) + a_n(z) - a_n(z_0) + a_n(z_0) - a(z_0),$$

so that

$$(1) \quad |a(z) - a(z_0)| \leq |a_n(z) - a(z)| + |a_n(z) - a_n(z_0)| + |a_n(z_0) - a(z_0)|.$$

Given any $\epsilon > 0$, there exists N (independent of the choice of $z \in D$) such that

$$(2) \quad |a_n(z) - a(z)| < \frac{\epsilon}{3} \quad \text{and} \quad |a_n(z_0) - a(z_0)| < \frac{\epsilon}{3}$$

whenever $n > N$. We now choose any $n > N$ and consider the function $a_n(z)$. Clearly this function is continuous at z_0 . Hence given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$(3) \quad |a_n(z) - a_n(z_0)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |z - z_0| < \delta.$$

Combining (1)–(3), we conclude that $|a(z) - a(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$, so that $a(z)$ is continuous at z_0 . Since $z_0 \in D$ is arbitrary, the result follows. \circ

EXAMPLE 16.2.1. Consider the sequence $a_n(z) = z^n$ on the real interval $[0, 1]$. Each function $a_n(z)$ is clearly continuous in $[0, 1]$. Also $a_n(z) \rightarrow 0$ as $n \rightarrow \infty$ if $z \in [0, 1)$ and $a_n(1) \rightarrow 1$ as $n \rightarrow \infty$, so that the limit function is not continuous in $[0, 1]$. In view of Theorem 16A, it is clear that this discontinuity is caused by the lack of uniform convergence of $a_n(z)$ in $[0, 1]$.

THEOREM 16B. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that $a_n(z) \rightarrow a(z)$ as $n \rightarrow \infty$ uniformly in D . Then for any contour C lying in D , we have*

$$\lim_{n \rightarrow \infty} \int_C a_n(z) dz = \int_C a(z) dz.$$

PROOF. Note first of all that the integrals exist, since integrability over C is a consequence of continuity in D . Suppose now that the contour C has length L . Given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a(z)| < \epsilon/L$ whenever $n > N$. Then

$$\left| \int_C a_n(z) dz - \int_C a(z) dz \right| \leq L \sup_{z \in C} |a_n(z) - a(z)| \leq \epsilon$$

whenever $n > N$. \circ

THEOREM 16C. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is analytic in a disc $D = \{z : |z - z_0| < R\}$. Suppose further that $a_n(z) \rightarrow a(z)$ as $n \rightarrow \infty$ uniformly in $D_r = \{z : |z - z_0| \leq r\}$ for every $r \in [0, R)$. Then $a(z)$ is analytic in D , and $a'_n(z) \rightarrow a'(z)$ as $n \rightarrow \infty$ uniformly in D_r for every $r \in [0, R)$.*

PROOF. Suppose that T is any triangular path in D . We now choose $r \in [0, R)$ so that $T \subseteq D_r$. Then

$$\int_T a(z) dz = \lim_{n \rightarrow \infty} \int_T a_n(z) dz = 0.$$

Here the second equality follows from Cauchy's integral theorem, while the first equality follows from Theorem 16B, in view of uniform convergence in D_r . The assertion that $a(z)$ is analytic in D now follows from Morera's theorem (Theorem 6G). Suppose next that $r \in [0, R)$ is fixed. We now choose $\rho = (r + R)/2$, so that $r < \rho < R$, and let C_ρ denote the circle $\{\zeta : |\zeta - z_0| = \rho\}$, followed in the positive (anticlockwise) direction (the reader is advised to draw a picture). For every $z \in D_r$, we have, by Cauchy's integral formula, that

$$a'_n(z) - a'(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{a_n(\zeta) - a(\zeta)}{(\zeta - z)^2} d\zeta.$$

Note that for every $\zeta \in C_\rho$, we have $|\zeta - z| \geq \rho - r$. Also, in view of the uniform convergence of the sequence $a_n(z)$ in D_ρ , we have, given any $\epsilon > 0$, there exists N such that for every $z \in D_\rho$,

$$|a_n(z) - a(z)| < \frac{(\rho - r)^2 \epsilon}{\rho}$$

whenever $n > N$. It follows that for every $z \in D_r$, we have

$$|a'_n(z) - a'(z)| < \rho \sup_{\zeta \in C_\rho} \left| \frac{a_n(\zeta) - a(\zeta)}{(\zeta - z)^2} \right| \leq \epsilon$$

whenever $n > N$. Hence $a'_n(z) \rightarrow a'(z)$ as $n \rightarrow \infty$ uniformly in D_r . \circ

Note that Theorem 16C is restricted to discs. However, as far as application is concerned, this is not a serious restriction. For any point z in an arbitrary domain $D \subseteq \mathbb{C}$, we can always find an open disc D' such that $z \in D' \subseteq D$, and so we can apply Theorem 16C to the disc D' . We immediately have the following result.

THEOREM 16D. *Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is analytic in a domain $D \subseteq \mathbb{C}$. Suppose further that $a_n(z) \rightarrow a(z)$ as $n \rightarrow \infty$ uniformly in D . Then $a(z)$ is analytic in D . Furthermore, for every $z \in D$ and every $k \in \mathbb{N}$, we have $a_n^{(k)}(z) \rightarrow a^{(k)}(z)$ as $n \rightarrow \infty$.*

16.3. Cauchy Sequences

Suppose that a sequence of complex numbers a_n converges to a . Then given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a| < \epsilon/2$ whenever $n > N$. It follows that

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \epsilon$$

whenever $m, n > N$.

DEFINITION. We say that a sequence of complex numbers a_n is a Cauchy sequence if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a_m| < \epsilon$ whenever $m, n > N$.

In the last section of this chapter, we shall prove the following result.

THEOREM 16E. (GENERAL PRINCIPLE OF CONVERGENCE) *A sequence of complex numbers a_n is convergent if and only if it is a Cauchy sequence. In other words, a sequence a_n of complex numbers is convergent if and only if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a_m| < \epsilon$ whenever $m, n > N$.*

DEFINITION. Suppose that $D \subseteq \mathbb{C}$ is a region. We say that a sequence of complex valued functions $a_n(z)$ is a uniform Cauchy sequence in D , if, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a_m(z)| < \epsilon$ whenever $m, n > N$.

We have the following important result.

THEOREM 16F. (GENERAL PRINCIPLE OF UNIFORM CONVERGENCE) *Suppose that $D \subseteq \mathbb{C}$ is a region. A sequence of complex valued functions $a_n(z)$ converges uniformly in D if and only if it is a uniform Cauchy sequence in D .*

PROOF. It is simple to show that uniform convergence implies uniform Cauchy. To prove the converse, note that for every fixed $z \in D$, the sequence of complex numbers $a_n(z)$ is a Cauchy sequence. It follows from Theorem 16E that $a_n(z)$ converges to $a(z)$, say. Since $a_n(z)$ is a uniform Cauchy sequence in D , it follows that, given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that for every $z \in D$, $|a_n(z) - a_m(z)| < \epsilon$ whenever $m, n > N$. Letting $m \rightarrow \infty$, we conclude that $|a_n(z) - a(z)| \leq \epsilon$ whenever $n > N$. \circ

16.4. Uniform Convergence of Series

Recall that the convergence of a series depends on the convergence of the sequence of partial sums.

DEFINITION. Suppose that $D \subseteq \mathbb{C}$ is a region. We say that a series of complex valued functions

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly in D if the sequence of partial sums

$$s_N(z) = \sum_{n=1}^N a_n(z)$$

converges uniformly in D .

We immediately have the following analogues of Theorems 16A, 16B, 16D, 16E and 16F. They can be established by applying the earlier results to the sequence of partial sums.

THEOREM 16G. Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that the series

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly to a function $s(z)$ in D . Then $s(z)$ is continuous in D .

THEOREM 16H. Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is continuous in a region $D \subseteq \mathbb{C}$. Suppose further that the series

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly to a function $s(z)$ in D . Then for any contour C lying in D , we have

$$\sum_{n=1}^{\infty} \int_C a_n(z) dz = \int_C s(z) dz.$$

In other words, we can interchange the order of summation and integration.

THEOREM 16J. Suppose that for every $n \in \mathbb{N}$, the function $a_n(z)$ is analytic in a domain $D \subseteq \mathbb{C}$. Suppose further that the series

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly to a function $s(z)$ in D . Then $s(z)$ is analytic in D . Furthermore, for every $z \in D$ and every $k \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} a_n^{(k)}(z) = s^{(k)}(z).$$

In other words, we can interchange the order of summation and differentiation.

THEOREM 16K. (GENERAL PRINCIPLE OF CONVERGENCE) A series

$$\sum_{n=1}^{\infty} a_n$$

of complex numbers converges if and only if, given any $\epsilon > 0$, there exists $N_0 \in \mathbb{R}$ such that

$$\left| \sum_{n=N_1+1}^{N_2} a_n \right| < \epsilon$$

whenever $N_2 > N_1 > N_0$.

THEOREM 16L. (GENERAL PRINCIPLE OF UNIFORM CONVERGENCE) *Suppose that $D \subseteq \mathbb{C}$ is a region. A series*

$$\sum_{n=1}^{\infty} a_n(z)$$

of complex valued functions converges uniformly in D if and only if, given any $\epsilon > 0$, there exists $N_0 \in \mathbb{R}$ such that for every $z \in D$,

$$\left| \sum_{n=N_1+1}^{N_2} a_n(z) \right| < \epsilon$$

whenever $N_2 > N_1 > N_0$.

We can also establish the following uniform versions of the Comparison test and the Ratio test.

THEOREM 16M. (WEIERSTRASS M-TEST) *Suppose that $D \subseteq \mathbb{C}$ is a region. Suppose further that $a_n(z)$ is a sequence of complex valued functions such that $|a_n(z)| \leq M_n$ for every $z \in D$, where the real series*

$$\sum_{n=1}^{\infty} M_n$$

of non-negative terms is convergent. Then the series

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly (and absolutely) in D .

PROOF. Using the Triangle inequality, we have

$$\left| \sum_{n=N_1+1}^{N_2} a_n(z) \right| \leq \sum_{n=N_1+1}^{N_2} |a_n(z)| \leq \sum_{n=N_1+1}^{N_2} M_n.$$

Given any $\epsilon > 0$, it follows from Theorem 16K that there exists N_0 such that

$$\sum_{n=N_1+1}^{N_2} M_n < \epsilon$$

whenever $N_2 > N_1 > N_0$. It follows that for every $z \in D$,

$$\left| \sum_{n=N_1+1}^{N_2} a_n(z) \right| < \epsilon$$

whenever $N_2 > N_1 > N_0$. The result now follows from Theorem 16L. \circ

THEOREM 16N. (RATIO TEST) Suppose that $D \subseteq \mathbb{C}$ is a region. Suppose further that $a_n(z)$ is a sequence of complex valued functions such that $a_1(z)$ is bounded in D , and

$$(4) \quad \left| \frac{a_{n+1}(z)}{a_n(z)} \right| \leq R < 1$$

for every $z \in D$, where R is constant. Then the series

$$\sum_{n=1}^{\infty} a_n(z)$$

converges uniformly (and absolutely) in D .

PROOF. Note that the condition (4) implies $|a_n(z)| \leq R^{n-1}|a_1(z)|$ for every $n \in \mathbb{N}$. On the other hand, there exists $M \in \mathbb{R}$ such that $|a_1(z)| \leq M$ for every $z \in D$. It follows that for every $z \in D$ and every $n \in \mathbb{N}$, we have $|a_n(z)| \leq MR^{n-1}$. The result now follows from the Weierstrass M -test, noting that the geometric series

$$\sum_{n=1}^{\infty} MR^{n-1}$$

converges. \circ

EXAMPLE 16.4.1. The series

$$(5) \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges absolutely for every z satisfying $\Re z > 1$. To see this, note that writing $z = x + iy$, where $x, y \in \mathbb{R}$, we have

$$\frac{1}{n^z} = \frac{1}{n^{x+iy}} = \frac{1}{n^x} n^{-iy} = \frac{1}{n^x} e^{-iy \log n} = \frac{1}{n^x} (\cos(y \log n) - i \sin(y \log n)),$$

so that

$$\left| \frac{1}{n^z} \right| = \frac{1}{n^x}.$$

Since $x > 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^x}$$

of non-negative terms is convergent. It follows from the Comparison test that the series (5) converges absolutely. Suppose now that $\delta > 0$ is fixed. Consider the region $D = \{z : \Re z > 1 + \delta\}$. Then for every $z \in D$, we have

$$\left| \frac{1}{n^z} \right| = \frac{1}{n^x} < \frac{1}{n^{1+\delta}}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$$

of non-negative terms is convergent. It follows from the Weierstrass M -test that the series (5) converges uniformly in D . We comment here that the series (5) is called the Riemann zeta function, and is crucial in the study of the distribution of prime numbers. Indeed, the study of this function has led to much of the development in complex analysis.

EXAMPLE 16.4.2. In Chapter 10, we discussed the function $\pi \cot \pi z$, and showed that it has simple poles at the (real) integers with residue 1. Here we shall make a more detailed study. Consider the function

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Let us first of all study this function in the region $D_R = \{z : |z| < R\}$, where $R > 0$ is fixed. Let $N \in \mathbb{N}$ satisfy $N > 2R$, and write $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{z} + \sum_{n=1}^N \frac{2z}{z^2 - n^2} \quad \text{and} \quad f_2(z) = \sum_{n=N+1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Clearly the function $f_1(z)$ is analytic in D_R , with the exception of simple poles at the (real) integers in D_R . Consider next the function $f_2(z)$ in D_R . For every $z \in D_R$ and every $n > N > 2R$, we have

$$\left| \frac{2z}{z^2 - n^2} \right| \leq \frac{2R}{n^2 - R^2} = \frac{1}{n^2} \frac{2R}{1 - (R/n)^2} < \frac{8R}{3n^2}.$$

It follows from the Weierstrass M -test that the series for $f_2(z)$ converges uniformly in D_R , and is analytic in D_R in view of Theorem 16J. Hence $f(z)$ is analytic in D_R , with the exception of simple poles at the (real) integers in D_R . It follows that $f(z)$ is meromorphic in \mathbb{C} , with simple poles at the (real) integers. It is easy to check that all these simple poles have residue 1. Note also that we can write

$$f(z) = z \sum_{n \in \mathbb{Z}} \frac{1}{z^2 - n^2}.$$

We shall show that $f(z) = \pi \cot \pi z$. For convenience, we shall change notation, and show that

$$(6) \quad \sum_{n \in \mathbb{Z}} \frac{1}{a^2 - n^2} = \frac{\pi \cot \pi a}{a}$$

whenever $a \notin \mathbb{Z}$. Consider the function

$$g(z) = \frac{\pi \cot \pi z}{a^2 - z^2}.$$

Since the function $\pi \cot \pi z$ has simple poles at every $n \in \mathbb{Z}$ with residue 1, and since $a \notin \mathbb{Z}$, it follows that $g(z)$ has simple poles at every $n \in \mathbb{Z}$ and at $z = \pm a$, with residues

$$\operatorname{res}(g, n) = \frac{1}{a^2 - n^2} \quad \text{and} \quad \operatorname{res}(g, \pm a) = -\frac{\pi \cot \pi a}{2a}.$$

For every $N \in \mathbb{N}$, let C_N denote the boundary of the rectangular domain

$$\left\{ z = x + iy : |x| < N + \frac{1}{2} \text{ and } |y| < N \right\},$$

followed in the positive (anticlockwise) direction. If $N > |a|$, then we have

$$\frac{1}{2\pi i} \int_{C_N} \frac{\pi \cot \pi z}{a^2 - z^2} dz = \sum_{-N \leq n \leq N} \frac{1}{a^2 - n^2} - \frac{\pi \cot \pi a}{a}.$$

Clearly (6) will follow if we show that the integral on the left hand side converges to 0 as $N \rightarrow \infty$. It can be shown that $|\cot \pi z| \leq \coth \pi$ for every $z \in C_N$. Hence for every $N > |a|$, we have

$$\left| \int_{C_N} \frac{\pi \cot \pi z}{a^2 - z^2} dz \right| \leq (8N + 2) \sup_{z \in C_N} \left| \frac{\pi \cot \pi z}{a^2 - z^2} \right| \leq \frac{(8N + 2)\pi \coth \pi}{N^2 - |a|^2} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

16.5. Application to Power Series

Let $z, \alpha \in \mathbb{C}$. In this section, we shall study series of the type

$$(7) \quad \sum_{n=0}^{\infty} a_n (z - \alpha)^n \quad (a_0, a_1, a_2, \dots \in \mathbb{C}),$$

known commonly as power series.

THEOREM 16P. *Suppose that the series given by (7) converges for a particular value $z = z_0$. Then, for every $r < |z_0 - \alpha|$, the series converges uniformly (and absolutely) in the disc $D_r = \{z : |z - \alpha| \leq r\}$.*

PROOF. Suppose that

$$\sum_{n=0}^{\infty} a_n (z_0 - \alpha)^n$$

converges. Then $a_n (z_0 - \alpha)^n \rightarrow 0$ as $n \rightarrow \infty$, and so there exists $M \in \mathbb{R}$ such that $|a_n (z_0 - \alpha)^n| \leq M$ for every $n \in \mathbb{N} \cup \{0\}$. For every $z \in D_r$, we have

$$|a_n (z - \alpha)^n| \leq M \left| \frac{z - \alpha}{z_0 - \alpha} \right|^n \leq M \left| \frac{r}{z_0 - \alpha} \right|^n$$

for every $n \in \mathbb{N} \cup \{0\}$. The result now follows from the Weierstrass M -test, noting that the geometric series

$$\sum_{n=0}^{\infty} M \left| \frac{r}{z_0 - \alpha} \right|^n$$

converges. \circ

THEOREM 16Q. (CONVERGENCE THEOREM FOR POWER SERIES) *For the power series given by (7), exactly one of the following holds:*

- The series converges absolutely for every $z \in \mathbb{C}$.*
- There exists a positive real number R such that the series converges absolutely for every $z \in \mathbb{C}$ satisfying $|z - \alpha| < R$ and diverges for every $z \in \mathbb{C}$ satisfying $|z - \alpha| > R$.*
- The series diverges for every $z \neq \alpha$.*

SKETCH OF PROOF. In the notation of Theorem 16P, consider

$$S = \{r \geq 0 : (7) \text{ converges absolutely in } D_r\}.$$

Then S contains the number 0. In view of Theorem 16P, S must be an interval with lower end-point 0, so that $S = [0, \infty)$, $S = \{0\}$ or there exists some positive number R such that $S = [0, R)$ or $S = [0, R]$. The first two possibilities correspond to (a) and (c) respectively, while the last possibility corresponds to (b). \circ

DEFINITION. The number R in Theorem 16Q is called the radius of convergence of the series (7). We also say that $R = 0$ if case (c) occurs, and that $R = \infty$ if case (a) occurs.

We now show that differentiation of a power series can be carried out term by term, and that the series so obtained converges to the derivative.

THEOREM 16R. *Suppose that the power series given by (7) has radius of convergence $R > 0$. Then it represents an analytic function $f(z)$ in the open disc $D = \{z : |z - \alpha| < R\}$. Furthermore, the derivatives of $f(z)$ can be obtained by differentiating the series term by term.*

PROOF. For every $r < R$, it follows from Theorem 16P that the series converges uniformly in the disc $D_r = \{z : |z - \alpha| < r\}$. It now follows from Theorem 16J that the series converges to an analytic function $f(z)$ in D_r , and the derivatives of $f(z)$ can be obtained by differentiating the series term by term. Since the above holds for any $r < R$, the result follows. \circ

EXAMPLE 16.5.1. Suppose that $f(t)$ is a complex valued function continuous (and so bounded) on the closed real interval $[0, 1]$. Consider the function

$$F(z) = \int_0^1 e^{-zt} f(t) dt.$$

For any fixed $z \in \mathbb{C}$, we have the power series (here t is the variable)

$$(8) \quad e^{-zt} = \sum_{n=0}^{\infty} \frac{(-zt)^n}{n!},$$

with infinite radius of convergence. It follows from Theorem 16P that the series (8) converges uniformly in $[0, 1]$, and so can be multiplied by the bounded function $f(t)$ and integrated term by term, in view of Theorem 16H. Hence

$$(9) \quad F(z) = \sum_{n=0}^{\infty} \int_0^1 \frac{(-zt)^n}{n!} f(t) dt = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \int_0^1 t^n f(t) dt.$$

Furthermore, if $|f(t)| \leq M$, where M is a fixed positive number, then

$$\left| \int_0^1 t^n f(t) dt \right| \leq M \int_0^1 t^n dt = \frac{M}{n+1}.$$

Suppose now that $R > 0$ is fixed. If $|z| < R$, then

$$\left| \frac{(-z)^n}{n!} \int_0^1 t^n f(t) dt \right| \leq \frac{MR^n}{(n+1)!}.$$

Note that the series

$$\sum_{n=0}^{\infty} \frac{MR^n}{(n+1)!}$$

converges, so it follows from the Weierstrass M -test that the series in (9) converges uniformly in the disc $\{z : |z| < R\}$. By Theorem 16J, the function $F(z)$ is analytic in $\{z : |z| < R\}$. Since $R > 0$ is arbitrary, it follows that $F(z)$ is entire.

16.6. Cauchy Sequences

In this section, we shall prove Theorem 16E. Clearly a convergent sequence of complex numbers is Cauchy. It remains to show that a Cauchy sequence of complex numbers is convergent.

The proof of this result usually involves the Bolzano-Weierstrass theorem which states that every bounded sequence of complex numbers has a convergent subsequence. Here, we shall give a proof without using the Bolzano-Weierstrass theorem.

Assume, first of all, that the sequence a_n is real. Since a_n is a Cauchy sequence, it follows that there exists an increasing sequence of natural numbers

$$N_1 < N_2 < \dots < N_p < \dots$$

such that

$$|a_n - a_m| < \frac{1}{2^p}$$

whenever $n, m \geq N_p$ (we simply take $\epsilon = 2^{-p}$ for every $p \in \mathbb{N}$). In particular, we have

$$|a_{N_{p+1}} - a_{N_p}| < \frac{1}{2^p}$$

for every $p \in \mathbb{N}$. For every $p \in \mathbb{N}$, let

$$b_p = a_{N_p} - \frac{1}{2^{p-1}}.$$

Then

$$b_{p+1} - b_p = a_{N_{p+1}} - a_{N_p} + \frac{1}{2^p} \geq \frac{1}{2^p} - |a_{N_{p+1}} - a_{N_p}| > 0,$$

so that the sequence b_p is increasing. Note next that

$$|b_p| = \left| a_{N_p} - \frac{1}{2^{p-1}} \right| \leq |a_{N_p} - a_{N_1}| + |a_{N_1}| + \frac{1}{2^{p-1}} \leq \frac{1}{2} + |a_{N_1}| + \frac{1}{2^{p-1}},$$

so that the sequence b_p is bounded. Hence the sequence b_p converges to L , say, as $p \rightarrow \infty$.

We now show that $a_n \rightarrow L$ as $n \rightarrow \infty$. Given any $\epsilon > 0$, we now choose $p \in \mathbb{N}$ so large that

$$\frac{1}{2^p} < \frac{\epsilon}{4} \quad \text{and} \quad |b_p - L| < \frac{\epsilon}{4}.$$

Suppose that $n \geq N_p$. Then

$$|a_n - L| \leq |a_n - a_{N_p}| + |a_{N_p} - b_p| + |b_p - L| < \frac{1}{2^p} + \frac{1}{2^{p-1}} + \frac{\epsilon}{4} < \epsilon$$

as required.

Suppose now that the sequence a_n is complex valued. Then we can write $a_n = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$. If a_n is a Cauchy sequence, then it is easy to see that the real sequences x_n and y_n are real Cauchy sequences. It follows that both x_n and y_n converge, and so a_n converges.

PROBLEMS FOR CHAPTER 16

- Suppose that $a_n(z) \rightarrow a(z)$ and $b_n(z) \rightarrow b(z)$ as $n \rightarrow \infty$ uniformly in a region D .
 - Show that $a_n(z) + b_n(z) \rightarrow a(z) + b(z)$ as $n \rightarrow \infty$ uniformly in D .
 - Suppose that $f(z)$ is bounded in D . Show that $a_n(z)f(z) \rightarrow a(z)f(z)$ as $n \rightarrow \infty$ uniformly in D .
 - Write $f(z) = 1/z$ and $a_n(z) = 1/n$. Find a region D such that $a_n(z)$ converges uniformly in D but $a_n(z)f(z)$ does not converge uniformly in D .

- For each of the following power series, find a number R such that the series converges for $|z| < R$ and diverges for $|z| > R$:

- $\sum_{n=0}^{\infty} 2^n z^n$
- $\sum_{n=1}^{\infty} n^2 z^n$
- $\sum_{n=1}^{\infty} \frac{2^n z^{2n}}{n^2 + n}$
- $\sum_{n=0}^{\infty} \frac{3^n z^n}{4^n + 5^n}$

- Show that each of the following represents an entire function:

- $\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{1/2}}$
- $\sum_{n=1}^{\infty} \frac{z^n}{2^{n^2}}$
- $\sum_{n=1}^{\infty} \frac{1}{2^n n^z}$

- Show that each of the following functions is meromorphic in \mathbb{C} , and find the residues at the poles:

- $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)}$
- $\sum_{n=1}^{\infty} \frac{1}{(z+n)^2}$

- Show that for every $z \notin \mathbb{Z}$, we have $\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2} = \left(\frac{\pi}{\sin \pi z}\right)^2$.

- Show that except at the poles, we have $\sum_{n=-\infty}^{\infty} \frac{z}{n^2 + z^2} = \frac{\pi}{\tanh \pi z}$.

- By writing the series as $1/z$ plus a sum over all natural numbers, evaluate $\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}$.

- By letting $z \rightarrow 0$, show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

- Consider the exponential series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which converges for every $z \in \mathbb{C}$. Suppose further that $e(z)$ is the sum of the series.

- Show that the series converges uniformly in the disc $D_R = \{z : |z| < R\}$ for every real number $R > 0$.
- Suppose that D is a bounded region in \mathbb{C} . Explain why the series converges uniformly in D .
- Show that for every $z \in \mathbb{C}$ satisfying $|z| = R$, we have

$$\left| \sum_{n=N+1}^M \frac{z^n}{n!} \right| \geq \frac{R^M}{M!} - R^M \left(\frac{1}{R} + \frac{1}{R^2} + \dots + \frac{1}{R^{M-N-1}} \right) \geq R^M \left(\frac{1}{M!} - \frac{1}{R-1} \right).$$

- d) Use (c) to show that the series does not converge uniformly in \mathbb{C} .
- e) Explain carefully why $e(z)$ is an entire function in \mathbb{C} .
[REMARK: In view of the unfavourable conclusion of (d), you should take extra care here.]
- f) Show that $e'(z) = e(z)$ for every $z \in \mathbb{C}$ and $e(0) = 1$.
- g) Let $g(z) = e(-z)e(z)$. Show that $g'(z) = 0$ for every $z \in \mathbb{C}$, and deduce that $e(-z)e(z) = 1$ for every $z \in \mathbb{C}$.
- h) Suppose that $a \in \mathbb{C}$ is fixed. By studying the function $g_a(z) = e(-z)e(z+a)$, show that $e(z+a) = e(z)e(a)$ for every $z \in \mathbb{C}$.
8. This question makes use of the function $e(z)$ discussed in Problem 7. Suppose that for every $z \in \mathbb{C}$, we write

$$c(z) = \frac{e(iz) + e(-iz)}{2} \quad \text{and} \quad s(z) = \frac{e(iz) - e(-iz)}{2i}.$$

- a) By using the Taylor series for $e(iz)$ and $e(-iz)$, find the Taylor series for $c(z)$ and $s(z)$.
- b) Show that $c'(z) = -s(z)$ and $s'(z) = c(z)$ for every $z \in \mathbb{C}$.
- c) By studying the function $h(z) = c^2(z) + s^2(z)$, show that $c^2(z) + s^2(z) = 1$ for every $z \in \mathbb{C}$.