Notes from Trigonometry

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Preface

During Fall 2001 I taught trigonometry for the first time. As a supplement to the class lectures I would prepare a one or two page handout for each lecture.

During Winter 2002 I taught trigonometry again and took these handouts and expanded them into four or five page sets of notes. This collection of notes came together to form this book.

These notes mainly grew out of a desire to cover topics not usually covered in trigonometry, such as the Pythagorean theorem (Lecture 2), proof by contradiction (Lecture 16), limits (Lecture 18) and proof by induction (Lecture 23). As well as giving a geometric basis for the relationships of trigonometry.

Since these notes grew as a supplement to a textbook, the majority of the problems in the supplemental problems (of which there are several for nearly every lecture) are more challenging and less routine than would normally come from a textbook of trigonometry. I will say that every problem does have an answer. Perhaps someday I will go through and add an appendix with the solutions to the problems.

These notes may be freely used and distributed. I only ask that if you find these notes useful that you send suggestions on how to improve them, ideas for interesting trigonometry problems or point out errors in the text. I can be contacted at the following e-mail address.

`butler@math.byu.edu`

I would like to thank the Brigham Young University's mathematics department for allowing me the chance to teach the trigonometry class and not dragging me over hot coals for my exuberant copying of lecture notes. I would also like to acknowledge the influence of James Cannon. Many of the beautiful proofs and ideas grew out of material that I learned from him.

These notes were typeset using LaTeX and the images were prepared in Geometer's Sketchpad.

Lecture 1

The usefulness of mathematics

In this lecture we will discuss the aim of an education in mathematics, namely to help develop your thinking abilities. We will also outline several broad approaches to help in developing problem solving skills.

1.1 What can I learn from math?

To begin consider the following taken from Abraham Lincoln's *Short Autobiography* (here Lincoln is referring to himself in the third person).

> He studied and nearly mastered the six books of Euclid since he was a member of congress.
>
> He began a course of rigid mental discipline with the intent to improve his faculties, especially his powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow-lawyers, half a dozen in a room, filled the air with interminable snoring.

"Euclid" refers to the book *The Elements* which was written by the Greek mathematician Euclid and was the standard textbook of geometry for over two thousand years. Now it is unlikely that Abraham Lincoln ever had any intention of becoming a of mathematician. So this raises the question of why he would spend so much time studying the subject. The answer I believe can be stated as follows: Mathematics is bodybuilding for your mind.

Now just as you don't walk into a gym and start throwing all the weights onto a single bar, neither would you sit down and expect to solve difficult problems.

Your ability to solve problems must be developed, and one of the many ways to develop your your problem solving ability is to do mathematics.

Now let me carry this analogy with bodybuilding a little further. When I played football in high school I would spend just as much time in the weight room as any member of the team. But I never developed huge biceps, a flat stomach or any of a number of features that many of my teammates seemed to gain with ease. Some people have bodies that respond to training and bulk up right away, and then some bodies do not respond to training as quickly.

You will probably notice the same thing when it comes to doing mathematics. Some people pick up the subject quickly and fly through it, while others struggle to understand the basics. It is this latter group that I would like to address. Don't give up. You have the ability to understand and enjoy math inside of you, be patient, do your exercises and practice thinking through problems. Your ability to do mathematics will come, it will just take time.

1.2 Problem solving techniques

There are a number of books written on the subject of mathematical problem solving. One of the best, and most famous, is *How to Solve It* by George Polya. The following basic outline is adopted from his ideas. Essentially there are four steps involved in solving a problem.

UNDERSTANDING THE PROBLEM—Before beginning to solve any problem you must *understand* what it is that you are trying to solve. Look at the problem. There are two parts, what you are given and what you are trying to show. Clearly identify these parts. What are you given? What are you trying to show? Is it reasonable that there is a connection between the two?

DEVISING A PLAN—Once we understand the problem that we are trying to solve we need to find a way to connect what we are given to what we are trying to show, we need a plan. Mathematicians are not very original and often use the same ideas over and over, so look for similar problems, i.e. problems with the same conclusion or the same given information. Try solving a simpler version of the problem, or break the problem into smaller (simpler) parts. Work through an example. Is there other information that would help in solving the problem? Can you get that information from what you have? Are you using all of the given information?

CARRYING OUT THE PLAN—Once you have a plan, carry it out. *Check each step*. Can you see clearly that the step is correct?

LOOKING BACK—With the problem finished look at the solution. Is there a way to check your answer? Is your answer reasonable? For example, if you are finding the height of a mountain and you get 24,356 miles you might be suspicious. Can you see your solution at a glance? Can you give a different proof?

You should review this process several times. When you feel like you have run into a wall on a problem come back and start working through the questions. Often times it is just a matter of *understanding* the problem that prevents its solution.

1.3 The ultimate in problem solving

There is one method of problem solving that is so powerful, so universal, so simple that it will always work.

Try something. If it doesn't
work then try something else.
But never give up.

While this might seem too easy, it is actually a very powerful problem solving method. Often times our first attempt to solve a problem will fail. The secret is to keep trying. Along the same lines the following idea is helpful to keep in mind.

The road to wisdom? Well it's
plain and simple to express:
err and err and err again
but less and less and less.

1.4 Take a break

Let us return one more time to the bodybuilding analogy. You do not decide to go into the gym one morning and come out looking like a Greek sculpture in the afternoon. The body needs time to heal and grow. By the same token, your mind also needs time to relax and grow.

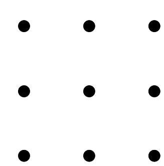
In solving mathematical problems you might sometimes feel like you are pushing against a brick wall. Your mind will be tired and you don't want to think anymore. In this situation one of the most helpful things to do is to *walk away from the problem for some time*. Now this does not need to mean physically walk away, just stop working on it and let your mind go on to something else, and then come back to the problem later.

When you return the problem will often be easier. There are two reasons for this. First, you have a fresh perspective and you might notice something about the problem that you had not before. Second, your subconscious mind will often keep working on the problem and have found a missing step while you were doing something else. At any rate, you will have relieved a bit of stress and will feel better.

There is a catch to this. In order for this to be as effective as possible you have to truly desire to find a solution. If you don't care your mind will stop working on it. Be passionate about your studies and learn to look forward to the joy and challenge of solving problems.

1.5 Supplemental problems

1. Without lifting your pencil, connect the nine dots shown below only using four line segments. *Hint*: there is a solution, don't add any constraints not given by the problem.



2. You have written ten letters to ten friends. After writing them you put them into pre-addressed envelopes but forgot to make sure that the right letter went with the right envelope. What is the probability that *exactly* nine of the letters will get to the correct friend?

3. You have a 1000 piece puzzle that is completely disassembled. Let us call a "move" anytime we connect two blocks of the puzzle (the blocks might be single pieces or consist of several pieces already joined). What is the minimum number of moves to connect the puzzle? What is the maximum number of moves to connect the puzzle?

Lecture 2

Geometric foundations

In this lecture we will introduce some of the basic notation and ideas to be used in studying triangles. Our main result will be to show that the sum of the angles in a triangle is $180°$.

2.1 What's special about triangles?

The word trigonometry comes from two root words. The first is *trigonon* which means "triangle" and the second is *metria* which means "measure." So literally trigonometry is the study of measuring triangles. Examples of things that we can measure in a triangle are the lengths of the sides, the angles (which we will talk about soon), the area of the triangle and so forth.

So this class is devoted to studying triangles. But there aren't similar classes dedicated to studying four-sided objects or five-sided objects or etc.... So what distinguishes the triangle?

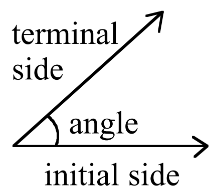
Let us perform a thought experiment. Imagine that you made a triangle and a square out of sticks and that the corners were joined by a peg of some sort through a hole, so essentially the corners were single points. Now grab one side of each shape and lift it up. What happened? The triangle stayed the same and didn't change its shape, on the other hand the square quickly lost its "squareness" and turned into a different shape.

So triangles are *rigid*, that is they are not easily moved into a different shape or position. It is this property that makes triangles important.

Returning to our experiment, we can make the square rigid by adding in an extra side. This will break the square up into a collection of triangles, each of which are rigid, and so the entire square will now become rigid. We will often work with squares and other polygons (many sided objects) by breaking them up into a collection of triangles.

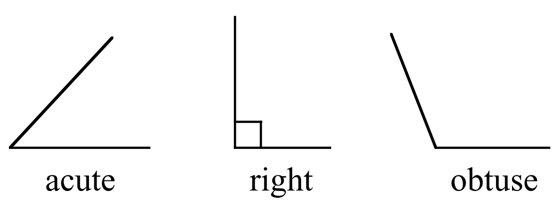
2.2 Some definitions on angles

An angle is when two rays (think of a ray as "half" of a line) have their end point in common. The two rays make up the "sides" of the angle, called the initial and terminal side. A picture of an angle is shown below.



Most of the time when we will talk about angles we will be referring to the measure of the angle. The measure of the angle is a number associated with the angle that tells us how "close" the rays come to each other, another way to think of the number is a measure of the amount of rotation to get from one side to the other.

There are several ways to measure angles as we shall see later on. The most prevalent is the system of degrees ('°'). In degrees we split up a full revolution into 360 parts of equal size, each part being one degree. An angle with measure $180°$ looks like a straight line and is called a *linear angle*. An angle with $90°$ forms a *right angle* (it is the angle found in the corners of a square and so we will use a square box to denote angles with a measure of $90°$). Acute angles are angles that have measure less than $90°$ and obtuse angles are angles that have measure between $90°$ and $180°$. Examples of some of these are shown below.



Some angles are associated in pairs. For example, two angles that have their measures adding to $180°$ are called *supplementary angles* or linear pairs. Two angles that have their measure adding to $90°$ are called *complementary angles*. Two angles that have the exact same measure are called *congruent angles*.

Example 1 What are the supplement and the complement of $32°$?

Solution Since supplementary angles need to add to $180°$ the supplementary angle is $180° - 32° = 148°$. Similarly, since complementary angles need to add to $90°$ the complementary angle is $90° - 32° = 58°$.

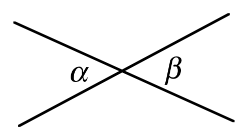
2.3 Symbols in mathematics

When we work with objects in mathematics it is convenient to give them names. These names are arbitrary and can be chosen to best suit the situation or mood. For example if we are in a romantic mood we could use '$\heartsuit$', or any number of other symbols.

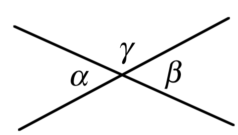
Traditionally in mathematics we use letters from the Greek alphabet to denote angles. This is because the Greeks were the first to study geometry. The Greek alphabet is shown below (do not worry about memorizing this).

α	–	alpha	ι	–	iota	ρ	–	rho
β	–	beta	κ	–	kappa	σ	–	sigma
γ	–	gamma	λ	–	lambda	τ	–	tau
δ	–	delta	μ	–	mu	υ	–	upsilon
ϵ	–	epsilon	ν	–	nu	ϕ	–	phi
ζ	–	zeta	ξ	–	xi	χ	–	chi
η	–	eta	o	–	omicron	ψ	–	psi
θ	–	theta	π	–	pi	ω	–	omega

Example 2 In the diagram below show that the angles α and β are congruent. This is known as the vertical angle theorem.



Solution First let us begin by marking a third angle, γ, such as shown in the figure below.



Now the angles α and γ form a straight line and so are supplementary, it follows that $\alpha = 180° - \gamma$. Similarly, β and γ also form a straight line and so again we have that $\beta = 180° - \gamma$. So we have

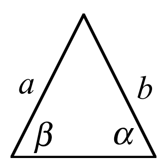
$$\alpha = 180° - \gamma = \beta,$$

which shows that the angles are congruent.

One last note on notation. Throughout the book we will tend to use capital letters $(A, B, C, \ldots)$ to represent points and lowercase letters $(a, b, c, \ldots)$ to represent line segments or length of line segments. While it a goal to be consistent it is not always convenient, however it should be clear from the context what we are referring to whenever our notation varies.

2.4 Isoceles triangles

A special group of triangles are the isoceles triangles. The root *iso* means "same" and isoceles triangles are triangles that have at least two sides of equal length. A useful fact from geometry is that if two sides of the triangle have equal length then the corresponding angles (i.e. the angles *opposite* the sides) are congruent. In the picture below it means that if $a = b$ then $\alpha = \beta$.



The geometrical proof goes like this. Pick up and "turn over" the triangle and put it back down on top of the old triangle keeping the vertex where the two sides of equal length come together at the same point. The triangle that is turned over will exactly match the original triangle and so in particular the angles (which have now traded places) must also exactly match, i.e. they are congruent.

A similar process will show that if two angles in a triangle are congruent then the sides opposite the two angles have the same length. Combining these two fact means that in a triangle having equal sides is the same as having equal angles.

One special type of isoceles triangle is the *equilateral* triangle which has all three of the sides of equal length. Applying the above argument twice shows that all the angles of such a triangle are congruent.

2.5 Right triangles

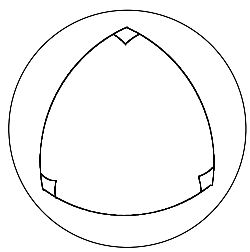
In studying triangles the most important triangles will be the *right triangles*. Right triangles, as the name implies, are triangles with a right angle. Triangles can be places into two large categories. Namely, right and oblique. Oblique triangles are triangles that do not have a right angle.

So useful are the right triangles that we will study oblique triangles through combinations of right triangles.

2.6 Angle sum in triangles

It would be useful to know if there was a relation that existed between the angles in a triangle. For example, do they sum up to a certain value? Many of us have been raised on the mantra, "180° in a triangle, ohmmm," but is this always true? The answer is, sort of.

To see why this is not always true, imagine that you have a globe, or any sphere, in front of you. At the North Pole draw two line segments down to the equator and join these line segments along the equator. The resulting triangle will have an angle sum of more than 180°. An example of what this would look like is shown below. (Keep in mind on the sphere these lines are straight.)

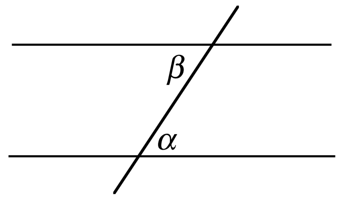


Now that we have ruined our faith in the sum of the angles in triangles, let us restore it. The fact that the triangle added up to more than 180° relied on us using a globe, or sphere, to draw our triangle on. The sphere behaves differently than a piece of paper. The study of behavior of geometric objects on a sphere is called *spherical geometry*. The study of geometric objects on a piece of paper is called *planar geometry* or *Euclidean geometry*.

The major difference between the two is that in spherical geometry there are no *parallel lines* (i.e. lines which do not intersect) while in Euclidean geometry given a line and a point not on the line there is one unique parallel line going through the point. There are other geometries that are studied that have infinitely many parallel lines going through a point, these are called *hyperbolic geometries*. In our class we will *always* assume that we are in Euclidean geometry.

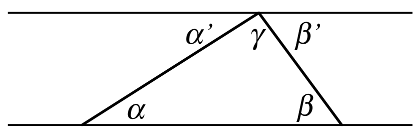
One consequence of there being one and only one parallel line through a given point to another line is that the opposite interior angles formed by a line that goes through both parallel lines are congruent. Pictorially, this means that the angles α and β are congruent in the picture at the top of the next page.

Using the ideas of opposite interior angles we can now easily verify that the angles in any triangle in the plane must always add to 180°. To see this, start with any triangle and form two parallel lines, one that goes through one side of the triangle and the other that runs through the third vertex, such as shown on the next page.



We have that $\alpha = \alpha'$ and $\beta = \beta'$ since these are pairs of opposite interior angles of parallel lines. Notice now that the angles α', β' and γ form a linear angle and so in particular we have,

$$\alpha + \beta + \gamma = \alpha' + \beta' + \gamma = 180°.$$

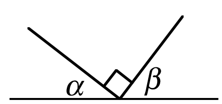


Example 3 Find the measure of the angles of an equilateral triangle.

Solution We noted earlier that the all of the angles of an equilateral triangle are congruent. Further, they all add up to 180° and so each of the angles must be one-third of 180°, or 60°.

2.7 Supplemental problems

1. True/False. In the diagram below the angles α and β are complementary. Justify your answer.



2. Give a quick sketch of how to prove that if two angles of a triangle are congruent then the sides opposite the angles have the same length.

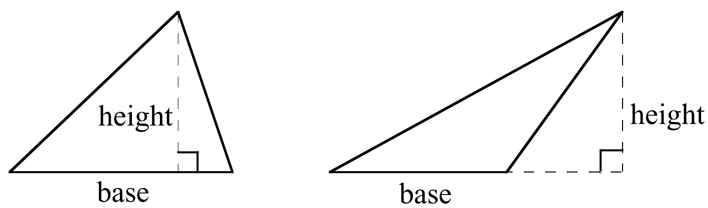
3. One approach to solving problems is proof by superposition. This is done by proving a special case, then using the special case to prove the general case. Using proof by superposition show that the area of a triangle is

$$\frac{1}{2}(\text{base})(\text{height}).$$

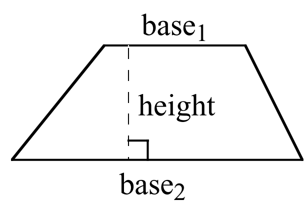
This should be done in the following manner:

(i) Show the formula is true for right triangles. *Hint*: a right triangle is half of another familiar shape.

(ii) There are now two general cases. (What are they? *Hint*: examples of each case are shown below, how would you describe the difference between them?) For each case break up the triangle in terms of right triangles and use the results from part (i) to show that they also have the same formula for area.



4. A trapezoid is a four sided object with two sides parallel to each other. An example of a trapezoid is shown below.

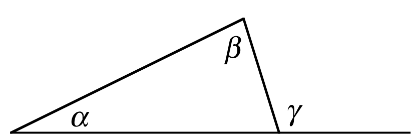


Show that the area of a trapezoid is given by the following formula.

$$\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$$

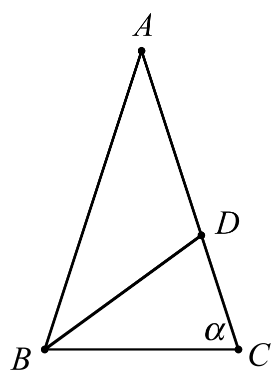
Hint: Break the shape up into two triangles.

5. In the diagram below prove that the angles α, β and γ satisfy $\alpha + \beta = \gamma$. This is known as the exterior angle theorem.

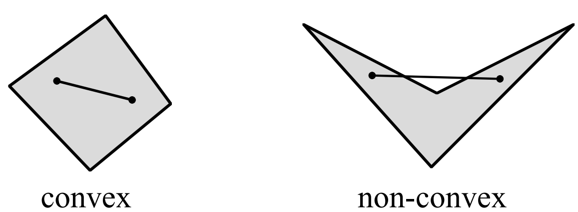


6. True/False. A triangle can have two obtuse angles. Justify your answer. (Remember that we are in Euclidean geometry.)

7. In the diagram below find the measure of the angle α given that $AB = AC$ and $AD = BD = BC$ ($AB = AC$ means the segment connecting the points A and B has the same length as the segment connecting the points A and C, similarly for the other expression). *Hint*: use all of the information and the relationships that you can to label as many angles as possible in terms of α and then get a relationship that α must satisfy.



8. A convex polygon is a polygon where the line segment connecting any two points on the inside of the polygon will lie completely inside the polygon. All triangles are convex. Examples of a convex and non-convex quadrilateral are shown below.



Show that the angle sum of a quadrilateral is 360°.

Show that the angle sum of a convex polygon with n sides is $(n-2)180°$.

Lecture 3

The Pythagorean theorem

In this lecture we will introduce the Pythagorean theorem and give three proofs for the theorem as well as some applications.

3.1 The Pythagorean theorem

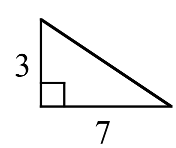
The Pythagorean theorem is named after the Greek philosopher Pythagorus, though it was known well before his time in different parts of the world such as the Middle East and China. The Pythagorean theorem is correctly stated in the following way.

> Given a right triangle with sides of length a, b and c (c being the longest side, which is also called the hypotenuse) then $a^2 + b^2 = c^2$.

In this theorem, as with every theorem, it is important that we say what our assumptions are. The values a, b and c are not just arbitrary but are associated with a definite object. So in particular if you say that the Pythagorean theorem is $a^2 + b^2 = c^2$ then you are only partially right. Mathematics is precise.

Before we give a proof of the Pythagorean theorem let us consider an example of its application.

> **Example 1** Use the Pythagorean theorem to find the missing side of the right triangle shown below.



Solution In this triangle we are given the lengths of the "legs" (i.e. the sides joining the right angle) and we are missing the hypotenuse, or c. And so in particular we have that

$$3^2 + 7^2 = c^2 \quad \text{or} \quad c^2 = 58 \quad \text{or} \quad c = \sqrt{58} \approx 7.616$$

Note in the example that there are two values given for the missing side. The value $\sqrt{58}$ is the *exact* value for the missing side. In other words it is an expression that refers to the unique number satisfying the relationship. The other number, 7.616, is an approximation to the answer (the '$\approx$' sign is used to indicate an approximation). Calculators are wonderful at finding approximations but bad at finding exact values. Make sure when answering the questions that your answer is in the requested form.

Also, when dealing with expressions that involve square roots there is a temptation to simplify along the following lines, $\sqrt{a^2 + b^2} = a + b$. This seems reasonable, just taking the square root of each term, but it is not correct. Erase any thought of doing this from your mind.

This does not work because there are several operations going on in this relationship. There are terms being squared, terms being added and terms having the square root taken. Rules of algebra dictate which operations must be done first, for example one rule says that if you are taking a square root of terms being added together you first must add then take the square root. Most of the rules of algebra are intuitive and so do not worry too much about memorizing them.

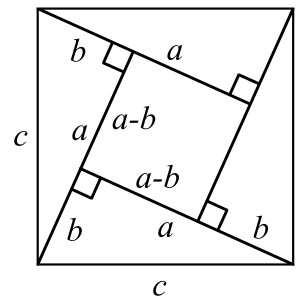
3.2 The Pythagorean theorem and dissection

There are literally hundreds of proofs for the Pythagorean theorem. We will not try to go through them all but there are books that contain collections of proofs of the Pythagorean theorem.

Our first method of proof will be based on the principle of *dissection*. In dissection we calculate a value in two different ways. Since the value doesn't change based on the way that we calculate it the two methods to calculate will be equal. These two calculations being equal will give birth to relationships, which if done correctly will be what we are after.

For our proof by dissection we first need something to calculate. So starting with a right triangle we will make four copies and place them as shown on the next page. The result will be a large square formed of four triangles and a small square (you should verify that the resultant shape is a square before proceeding).

The value that we will calculate is the area of the figure. First we can compute the area in terms of the large square. Since the large square has sides of length c the area of the large square is c^2.



The second way we will calculate area is in terms of the pieces making up the large square. The small square has sides of length $(a-b)$ and so its area is $(a-b)^2$. Each of the triangles has area $(1/2)ab$ and there are four of them.

Putting all of this together we get the following.

$$c^2 = (a-b)^2 + 4 \cdot \frac{1}{2}ab = (a^2 - 2ab + b^2) + 2ab = a^2 + b^2$$

3.3 Scaling

Imagine that you made a sketch on paper made out of rubber and then stretched or squished the paper in a nice uniform manner. The sketch that you made would get larger or smaller, but would always appear essentially the same.

This process of stretching or shrinking is *scaling*. Mathematically, scaling is when you multiply all distances by a positive number, say k. When $k > 1$ then we are stretching distances and everything is getting larger. When $k < 1$ then we are shrinking distances and everything is getting smaller.

What effect does scaling have on the size of objects?

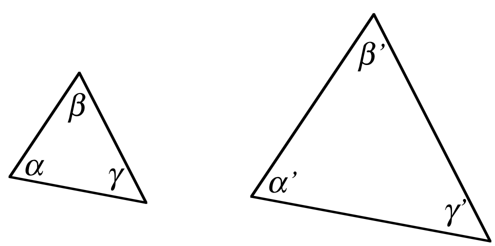
Lengths: The effect of scaling on paths is to multiply the total length by a factor of k. This is easily seen when the path is a straight line, but it is also true for paths that are not straight since all paths can be approximated by straight line segments.

Areas: The effect of scaling on areas is to multiply the total area by a factor of k^2. This is easily seen for rectangles and any other shape can be approximated by rectangles.

Volumes: The effect of scaling on volumes is to multiply the total volume by a factor of k^3. This is easily seen for cubes and any other shape can be approximated by cubes.

> **Example 2** You are boxing up your leftover fruitcake from the holidays and you find that the box you are using will only fit half of the fruitcake. You go grab a box that has double the dimensions of your current box in every direction. Will the fruitcake exactly fit in the new box?
>
> *Solution* The new box is a scaled version of the previous box with a scaling factor of 2. Since the important aspect in this question is the volume of the box, then looking at how the volume changes we see that the volume increases by a factor of 2^3 or 8. In particular the fruitcake will not fit exactly but only occupy one fourth of the box. You will have to wait three more years to acquire enough fruitcake to fill up the new box.

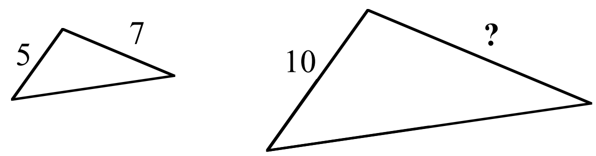
Scaling plays an important role in trigonometry, though often behind the scenes. This is because of the relationship between scaling and similar triangles. Two triangles are similar if the corresponding angle measurements of the two triangles match up. In other words, in the picture below we have that the two triangles are similar if $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$.



Essentially, similar triangles are triangles that look like each other, but are different sizes. Or in other words, *similar triangles are scaled versions of each other*.

The reason this is important is because it is often hard to work with full size representations of triangles. For example, suppose that we were trying to measure the distance to a star using a triangle. Such a triangle could never fit inside a classroom, nevertheless we draw a picture and find a solution. How do we know that our solution is valid? Because of scaling. Scaling says that the triangle that is light years across behaves the same way as a similar triangle that we draw on our paper.

> **Example 3** Given that the two triangles shown on the top of the next page are similar find the length of the indicated side.
>
> *Solution* Since the two triangles are similar they are scaled versions of each other. If we could figure out the scaling factor, then we would



only need to multiply the length of 7 by our scaling factor to get our final answer.

To figure out the scaling factor, we note that the side of length 5 became a side of length 10. In order to achieve this we had to scale by a factor of 2. So in particular, the length of the indicated side is 14.

3.4 The Pythagorean theorem and scaling

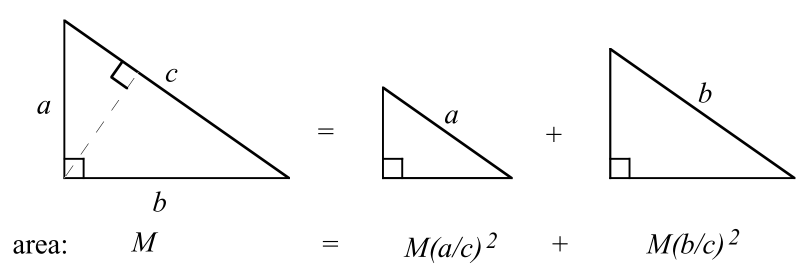
To use scaling to prove the Pythagorean theorem we must first produce some similar triangles. This is done by cutting our right triangle up into two smaller right triangles, which are similar as shown below. So in essence we now have three right triangles all similar to one another, or in other words they are scaled versions of each other. Further, these triangles will have hypotenuses of length a, b and c.

To get from a hypotenuse of length c to a hypotenuse of length a we would scale by a factor of (a/c). Similarly, to get from a hypotenuse of length c to a hypotenuse of length b we would scale by a factor of (b/c).

In particular, if the triangle with the hypotenuse of c has area M then the triangle with the hypotenuse of a will have area $M(a/c)^2$. This is because of the effect that scaling has on areas. Similarly, the triangle with a hypotenuse of b will have area $M(b/c)^2$.

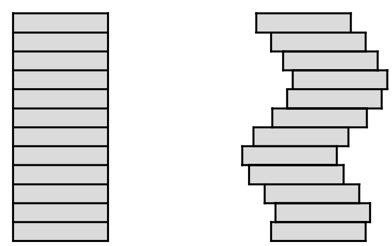
But these two smaller triangles exactly make up the large triangle. In particular, the area of the large triangle can be found by adding the areas of the two smaller triangles. So we have,

$$M = M\left(\frac{a}{c}\right)^2 + M\left(\frac{b}{c}\right)^2 \qquad \text{which simplifies to} \qquad c^2 = a^2 + b^2.$$



3.5 Cavalieri's principle

Imagine that you had a huge stack of books in front of you piled up straight and square. Now you push some books to the left and some books to the right to make a new shape. While the new shape may look different it is still made up of the same books and so you have the same area as what you started with. Pictorially, an example is shown below.



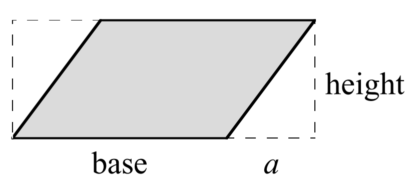
This is the spirit of Cavalieri's principle. That is if you take a shape and then shift portions of it left or right, but never change any of the widths, then the total area does not change. While a proof of this principle is outside the scope of this class, we will verify it for a special case.

Example 4 Verify Cavalieri's principle for parallelograms.

Solution Parallelograms are rectangles which have been "tilted" over. The area of the original shape, the rectangle, is the base times the height. So to verify Cavalieri's principle we need to show that the area of the parallelogram is also the base times the height.

Now looking at the picture below we can make the parallelogram part of a rectangle with right triangles to fill in the gaps. In particular the area of the parallelogram is the area of the rectangle minus the area of the two triangles or in other words,

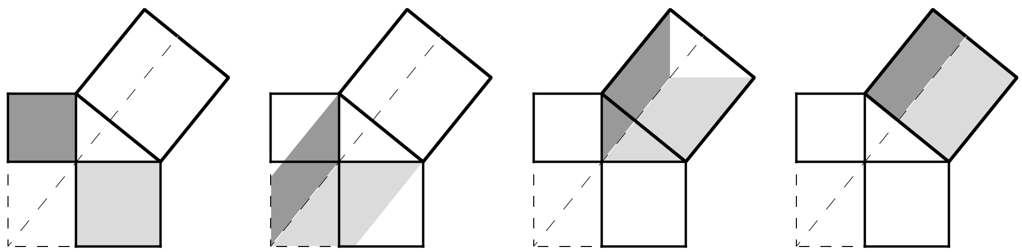
$$\text{area} = (\text{base} + a)(\text{height}) - 2 \cdot \frac{1}{2}a(\text{height}) = (\text{base})(\text{height}).$$



3.6 The Pythagorean theorem and Cavalieri's principle

Imagine starting with a right triangle and then constructing squares off of each side of the triangle. The area of the squares would be a^2, b^2 and c^2. So we can prove the Pythagorean theorem by showing that the squares on the legs of the right triangle will *exactly* fill up the square on the hypotenuse of the right triangle.

The process is shown below. Keep your eye on the area during the steps.



The first step won't change area because we shifted the squares to parallelograms and using Cavalieri's principle the areas are the same as the squares we started with. The second step won't change area because we moved the parallelograms and area does not depend on where something is positioned. On the final step we again use Cavalieri's principle to show that the area is not changed by shifting from the parallelograms to the rectangles.

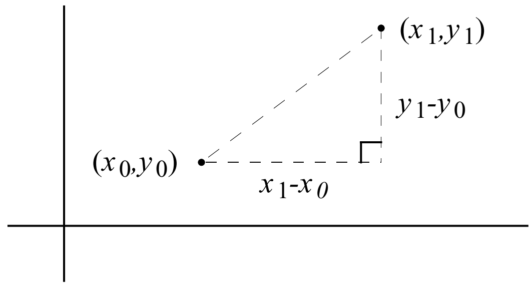
3.7 The beginning of measurement

The Pythagorean theorem is important because it marked the beginning of the measurement of distances.

For example, suppose that we wanted to find the distance between two points in the plane, call the points (x_0, y_0) and (x_1, y_1). The way we will think about distance is as the length of the shortest path that connects the two points. In the plane this shortest path is the straight line segment between the two points.

In order to use the Pythagorean theorem we need to introduce a right triangle into the picture. We will do this in a very natural way as is shown on the next page.

The lengths of the legs of the triangle are found by looking at what they represent. The length on the bottom represents how much we have changed our x value, which is $x_1 - x_0$. The length on the side represents how much we have changed our y value, which is $y_1 - y_0$. With two sides of our right triangle we can



find the third, which is our distance, by the Pythagorean theorem. So we have,

$$\text{distance} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

Example 5 Find the distance between the point $(1.3, 4.2)$ and the point $(5.7, -6.5)$. Round the answer to two decimal places.

Solution Using the formula just given for distance we have

$$\text{distance} = \sqrt{(1.3 - 5.7)^2 + (4.2 - (-6.5))^2} \approx 11.57$$

Example 6 Geometrically a circle is defined as the collection of all points that are a given distance, called the radius, away from a central point. Use the distance formula to show that the point (x, y) is on a circle of radius r centered at (h, k) if and only if

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is the algebraic definition of a circle.

Solution The point (x, y) is on the circle if and only if it is distance r away from the center point (h, k). So according to the distance formula a point (x, y) on the circle must satisfy,

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

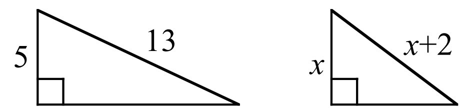
Squaring the left and right hand sides of the formula we get

$$(x - h)^2 + (y - k)^2 = r^2.$$

One important circle that we will encounter throughout these notes is the unit circle. This circle is the circle with radius 1 and centered at the origin. From the previous example we know that the unit circle can be described algebraically by $x^2 + y^2 = 1$.

3.8 Supplemental problems

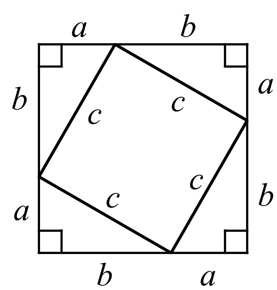
1. Using the Pythagorean theorem find the length of the missing side of the triangles shown below.



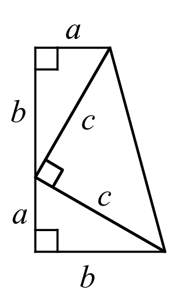
2. You have tied a balloon onto a 43 foot string anchored to the ground. At noon on a windy day you notice that the shadow of the balloon is 17 feet from where the string is anchored. How high up is the balloon at this time?

3. A Pythagorean triple is a combination of three numbers, (a, b, c), such that $a^2 + b^2 = c^2$, i.e. they form the sides of a right triangle. Show that for any choice of m and n that $(m^2 - n^2, 2mn, m^2 + n^2)$ is a Pythagorean triple.

4. Give another proof by dissection of the Pythagorean theorem using the figure shown below.



5. Give yet another proof by dissection of the Pythagorean theorem using the figure shown below. *Hint*: the area of a trapezoid is $(1/2)(\text{base}_1 + \text{base}_2)(\text{height})$.

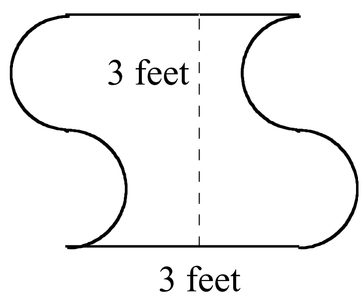


6. True/False. Two triangles are similar if they have two pairs of angles which are congruent. Justify your answer. (Recall that two triangles are similar if and only if *all* of their angles are congruent.)

7. Suppose that you grow a garden in your back yard the shape of a square and you plan to double your production by increasing the size of the garden. If you want to still keep a square shape, by what factor should you increase the length of the sides?

8. In our proof of the Pythagorean theorem using scaling verify that the two triangles that we got from cutting our right triangle in half are similar to the original triangle.

9. What is the area of the object below? What principle are you using?



10. Use Cavalieri's principle to show that the area of a triangle is one half of the base times the height.

11. When Greek historians first traveled to see the great pyramids of Egypt they ran across a difficult problem, how to measure the height of the pyramids. Fortunately, the historians were familiar with triangles and scaling. So they measured the length of the shadow of the pyramid at the same time that they measured the shadow of a pole of known height and then used scaling to calculate the height of the pyramid.

    Suppose that they used a ten foot pole that cast an eight foot shadow at the same time the pyramid cast a shadow that was 384 feet in length from the center of the pyramid. Using this information estimate the height of the pyramid.

Lecture 4

Angle measurement

In this lecture we will look at the two popular systems of angle measurement, degrees and radians.

4.1 The wonderful world of π

The number π (pronounced like "pie") is among the most important numbers in mathematics. It arises in a wide array of mathematical applications, such as statistics, mechanics, probability, and so forth. Mathematically, π is defined as follows.

$$\pi = \frac{\text{circumference of a circle}}{\text{diameter of a circle}} \approx 3.14159265\ldots$$

Since any two circles are scaled versions of each other it does not matter what circle is used to find an estimate for π.

> **Example 1** Use the following scripture from the King James Version of the Bible to estimate π.
>
> And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about. – 1 Kings 7:23
>
> *Solution* The verse describes a round font with a diameter of approximately 10 cubits and a circumference of approximately 30 cubits. Using the definition of π we get.
>
> $$\pi \approx \frac{30}{10} = 3$$

One thing to note about the preceding example is that it is not exact. This by no means implies that the Bible is incorrect (though some use this scripture to argue so), it most likely means there was some rounding error along the way.

4.2 Circumference and area of a circle

From the definition of π we can solve for the circumference of a circle. From which we get the following,

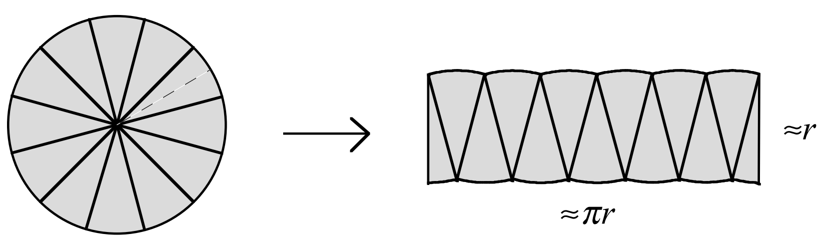
$$\begin{aligned} \text{circumference} &= \pi \cdot (\text{diameter}) \\ &= 2\pi r \qquad \text{(where } r \text{ is the radius of the circle).} \end{aligned}$$

The diameter of a circle is how wide the circle is at its widest point. The radius of the circle is the distance from the center of the circle to the edge. Thus the diameter which is all the way across is twice the radius which is half-way across.

One of the great observations of the Greeks was connecting the number π which came from the circumference of the circle to the area of the circle. The idea connecting them runs along the following lines. Take a circle and slice it into a large number of pie shaped wedges. Then take these pie shaped wedges and rearrange them to form a shape that looks like a rectangle with dimensions of half the circumference and the radius. As the number of pie shaped wedges increases the shape looks more and more like the rectangle, and so the circle has the same area as the rectangle, so we have,

$$\text{area} = \left(\frac{1}{2}(2\pi r)\right) r = \pi r^2.$$

Pictorially, this is seen below.



4.3 Gradians and degrees

The way we measure angles is somewhat arbitrary and today there are two major systems of angle measurement, degrees and radians, and one minor system of angle measurement, gradians.

Gradians are similar to degrees but instead of splitting up a circle into 360 parts we break it up into 400 parts. Gradians are not very widely used and this will be our only mention of them. Even though it is not a widely used system most calculators will have a 'drg' button which will convert to and from degrees, radians and gradians.

The most common way to measure angles in the real world (i.e. for surveying and such) is the system of degrees which we have already encountered. Degrees splits a full revolution into 360 parts each part being called 1°. The choice of 360 dates back thousand of years to the Babylonians, who most likely chose 360 based on the number of days in a year, but there are other possibilities as well.

While degrees is based on breaking up a circle into 360 parts, we will actually allow any number to be a degree measure when we are working with angles in the standard position in the plane. Recall that an angle is composed of two rays that come together at a point, an angle is in standard position when one of the sides of the angle, the initial side, is the positive x axis (i.e. to the right of the origin).

A positive number for degree measurement means that to get the second side of the angle, the terminal side, we move in a *counter*-clockwise direction. A negative number indicates that we move in a clockwise direction.

When an angle is greater than 360° (or similarly less than -360°) then this represents an angle that has come "full-circle" or in other words it wraps once and possibly several times around the origin. With this in mind, we will call two angles *co-terminal* if they end up facing the same direction. That is they differ only by a multiple of 360° (in other words a multiple of a revolution). An example of two angles which are co-terminal are 45° and 405°. A useful fact is that any angle can be made co-terminal with an angle between 0° and 360° by adding or subtracting multiples of 360°.

Example 2 Find an angle between 0° and 360° that is co-terminal with the angle 6739°.

Solution One way we can go about this is to keep subtracting off 360° until we get to a number that is between 0° and 360°. But with numbers like this such a process could take quite a while to accomplish. Instead, consider the following,

$$\frac{6739^\circ}{360^\circ} \approx 18.7194\ldots.$$

Since 360° represents one revolution by dividing our angle through by 360° the resulting number is how many revolutions our angle makes. So in particular our angle makes 18 revolutions plus a little more. So to find our angle that we want we can subtract off 18 revolutions and the result will be an angle between 0° and 360°. So our final answer is,

$$6739^\circ - 18 \cdot 360^\circ = 259^\circ.$$

4.4 Minutes and seconds

It took mathematics a long time to adopt our current decimal system. For thousands of years the best way to represent a fraction of a number was with fractions (and sometimes curiously so). But they needed to be able to measure just a fraction of an angle. To accommodate this they adopted the system of minutes and seconds.

One minute (denoted by $'$) corresponds to 1/60 of a degree. One second (denoted by $''$) correspond to 1/60 of a minute, or 1/3600 of a degree. This is analogous to our system of time measurement where we think of a degree representing one hour.

This system of degrees and minutes allowed for accurate measurement. For example, $1''$ is to $360°$ as 1 second is to 15 days. As another example, if we let the equator of the earth correspond to $360°$ then one second would correspond to about 101 feet.

Most commonly the system of minutes and seconds is used today in cartography, or map making. For example, BYU is located at approximately $40°14'44''$ north latitude and $111°38'44''$ west longitude.

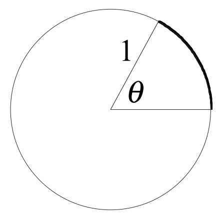
The system of minutes and seconds is also sometimes used on woodworking machines, but for the most part it is not commonly used. Most handhold scientific calculators are also equipped to convert between the decimal system and $D°M'S''$. For these two reasons we will not spend time mastering this system.

Example 3 Convert $51.1265°$ to $D°M'S''$ form.

Solution It's easy to see that we will have $51°$, it is the minutes and seconds that will pose the greatest challenge to us. Since there are $60'$ in one degree, to convert $.1265°$ into minutes we multiply by 60. So we get that $.1265° = 7.59'$. So we have $7'$. Now we have $.59'$ to convert to seconds. Since there are $60''$ in one minute, to convert $.59'$ into seconds we multiply by 60. So we get that $.59' = 35.4''$. Combining this altogether we have $51.625° = 51°7'35.4''$.

4.5 Radian measurement

For theoretical applications the most common system of angle measurement is radians (sometimes denoted by *rads* and sometimes denoted by nothing at all). Radian angle measurement can be related to the edge of the unit circle (recall the unit circle is a circle with radius 1). In radian measurement we measure an angle in standard position by measuring the distance traveled along the edge of the unit circle to where the second part of the angle intercepts the unit circle. Similarly as with degrees a positive angle means you travel counter-clockwise and a negative angle means you travel clockwise. Pictorially, the measure of the angle θ is given by the length of the arc shown below.



The circumference of the unit circle is 2π and so a full revolution corresponds to an angle measure of 2π in radians, half of a revolution corresponds to an angle measure of π radians and so on. Two angles, measured in radians, will be coterminal if they differ by a multiple of 2π.

4.6 Converting between radians and degrees

We have two ways to measure angles, so in particular we have two ways to measure a full revolution. In degrees a full revolution corresponds to $360°$ while in radians a full revolution corresponds to 2π *rads*. So we have that $360° = 2\pi$ *rads*. This can be rearranged to give the following useful (though not quite correct) relationship,

$$\frac{180°}{\pi\ rads} = 1 = \frac{\pi\ rads}{180°}.$$

This gives a way to convert from radians to degrees or from degrees to radians.

Example 3 Convert $240°$ to radians and $\frac{3\pi}{8}$ *rads* to degrees.

Solution For the first conversion we want to cancel the degrees and be left in radians, so we multiply through by $(\pi/180°)$. Doing so we get the following,

$$240° = 240° \cdot \frac{\pi}{180°} = \frac{4}{3}\pi.$$

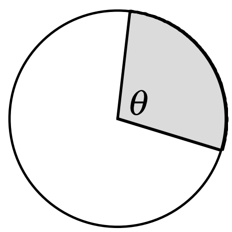
For the second conversion we want to cancel the radians and be left in degrees, so we multiply through by $(180°/\pi)$. Doing so we get the following,

$$\frac{3\pi}{8} = \frac{3\pi}{8} \cdot \frac{180°}{\pi} = 67.5°.$$

4.7 Wonderful world of radians

If we can use degrees to measure any angle then why would we need any other way to measure an angle? Put differently, what is useful about radians? The short, and correct, answer is that when using angles measured in radians a lot of equations simplify. This explains its popular use in theoretical applications.

As an example of this simplification, consider the formula for finding the area of a pie shaped wedge of a circle. In other words, given the angle θ find the area of the shaded portion below.



To find this area we will use proportions. That is the proportion the area of the pie shaped wedge compared to the total area is the same as the proportion of the angle θ compared to a full revolution. We will do this proportion twice, once for each system of angle measurement.

Using degrees we get:

$$\frac{\text{area}}{\pi r^2} = \frac{\theta}{360} \qquad \text{which simplifies to} \qquad \text{area} = \frac{\theta \pi r^2}{360}.$$

Using radians we get:

$$\frac{\text{area}}{\pi r^2} = \frac{\theta}{2\pi} \qquad \text{which simplifies to} \qquad \text{area} = \frac{\theta r^2}{2}.$$

Between these two equations, the one in radians is *much* easier to work with.

4.8 Supplemental problems

1. Your local park has recently installed a new circular duck pond that takes you 85 paces to walk around. How many paces is it across the pond from one edge to the other at the widest point? Round your answer to the nearest pace.

2. Suppose that you were to wrap a piece of string around the equator of the Earth. How much additional string would you need if you wanted the string

one foot off the Earth all the way around? (Assume that the equator is a perfect circle.) *Hint*: you have all of the information you need to answer the question.

3. Suppose that a new system of angle measurement has just been announced called *percentees*. In this system one revolution is broken up into 100 parts, (so 100 *percentees* make up one revolution). Using this new system of angle measurement, complete the following.

    (a) Convert $976°$ to *percentees*.

    (b) Convert -86.7 *percentees* to radians.

    (c) Find an angle between 800 and 900 *percentees* that is co-terminal with -327 *percentees*.

4. True/False. The length of an arc of a circle (i.e. a portion of the edge of the circle) with radius r and a central angle of θ (where θ is measured in radians) has a length of θr. Justify your answer. *Hint*: use proportions.

5. Suppose that you are on a new fad diet called the "Area Diet," wherein you can eat anything you want as long as your total daily intake does not exceed a certain total area. Your angle loving friend has brought over a pizza to share with you and wants to know at what central angle to cut your slice. If the pizza has a radius of 8 inches and you have 24 square inches allotted to the meal, then what angle (to the nearest degree) should you have your friend cut the pizza?

6. My sister suffers from *crustophobia* a condition in which she can eat everything on a pizza expect for the outer edge by the crust. After the family has sat down and eaten a circular pizza with a radius of 8 inches I bet my sister that she has eaten more than 40 in^2. We measure the length of her leftover crust and find it to be 9 inches. Who wins the bet and why? (Assume that the part that she did not eat, i.e. the edge, has zero area.)

Lecture 5

Trigonometry with right triangles

In this lecture we will define the trigonometric functions in terms of right triangles and explore some of the basic relationships that these functions satisfy.

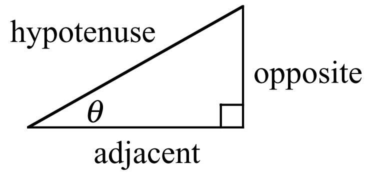
5.1 The trigonometric functions

Suppose we take any triangle and take a ratio of two of its sides, if we were to look at any similar triangle and the corresponding ratio we would always get the same value. This is because similar triangles are scaled versions of each other and in scaling we would multiply both the top and bottom terms of the ratio by the same amount, so the scaling factor will cancel itself out. Mathematicians would describe this ratio as an *invariant* under scaling, that is the ratio is something that does not change with scaling.

Since these values do not change we can give specific names to these ratios. In particular, we will give names to the ratios of the sides of a right triangle, and these will be the trigonometric functions.

For any acute angle θ we can construct a right triangle with one of the angles being θ. In this triangle (as with every triangle) there are three sides. We will call these the adjacent (denoted by *adj* which is the leg of the right triangle that forms one side of the angle θ), the opposite (denoted by *opp* which is the leg of the right triangle that is opposite the angle θ, i.e. does not form a part of the angle) and the hypotenuse (denoted by *hyp* which is the longest side of the right triangle. Pictorially, these are located as shown below.

The six trigonometric functions are sine (sin), cosine (cos), tangent (tan), cotangent (cot), secant (sec) and cosecant (csc). They are defined in terms of ratios in

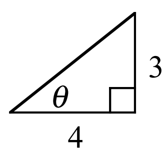


the following way,

$$\begin{aligned}\sin(\theta) &= \frac{opp}{hyp}, \qquad \cos(\theta) = \frac{adj}{hyp}, \qquad \tan(\theta) = \frac{opp}{adj},\\ \csc(\theta) &= \frac{hyp}{opp}, \qquad \sec(\theta) = \frac{hyp}{adj}, \qquad \cot(\theta) = \frac{adj}{opp}.\end{aligned}$$

To help remember these you can use the acronym SOHCAHTOA (pronounced "sew-ka-toe-a") to get the relationships for the sine, cosine and tangent function (i.e. the Sine is the Opposite over the Hypotenuse, the Cosine is the Adjacent over the Hypotenuse and the Tangent is the Opposite over the Adjacent).

Example 1 Using the right triangle shown below find the six trigonometric functions for the angle θ.



Solution First, we can use the Pythagorean theorem to find the length of the hypotenuse. Since we have that the adjacent side has length 4 the opposite side has length 3 then the hypotenuse has length $\sqrt{3^2+4^2} = 5$. Using the defining ratios we get,

$$\begin{aligned}\sin(\theta) &= \frac{3}{5}, \qquad \cos(\theta) = \frac{4}{5}, \qquad \tan(\theta) = \frac{3}{4},\\ \csc(\theta) &= \frac{5}{3}, \qquad \sec(\theta) = \frac{5}{4}, \qquad \cot(\theta) = \frac{4}{3}.\end{aligned}$$

The trigonometric functions take an angle and return a value. But there is more then one way to measure an angle, and 1° is not the same angle as 1 *rad*. So when you are working a problem using a calculator make sure that your calculator is in the correct angle mode. For example, if you are working a problem that involves degrees, make sure your calculator is set in degrees and not radians. Otherwise, you are likely to get a wrong answer.

5.2 Using the trigonometric functions

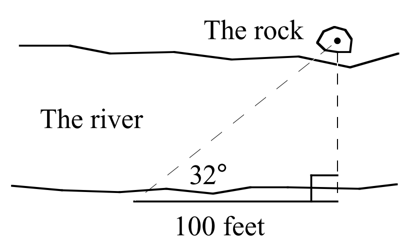
Now we know how given a right triangle to find the trigonometric functions associated with the acute angles of the triangle. We also know that every right triangle with those acute angles will have the same ratios, or in other words the same values for the trigonometric functions.

If we knew these ratios and the length of one side of the right triangle then we could find the lengths of the other sides of the right triangle. But this can only be done if we know what the ratios are.

So how do we get these ratios? Historically, the ratios were found by careful calculations (some of which we will do shortly) for a large number of angles. These ratios were then compiled and published in big books, so that they could be looked up. Over the course of the last twenty years such books have become obsolete. This is because of the invention and widespread use of scientific calculators. These calculators can compute quickly and accurately the trigonometric functions, not to mention being easier to carry around.

Now we have our way to get the ratios, and so if we know the length of one side of a right triangle we can find the length of the other two sides. This is shown in the following example.

Example 2 One day you stroll down to the river and take a walk along the river bank. At one point in time you notice a rock directly across from you. After walking 100 feet downstream you now have to turn an angle of 32° with the river to be looking directly at the rock. How wide is the river? (A badly drawn picture is shown below to help visualize the situation.)



Solution From the picture we see that this boils down to finding the length of a side of a right triangle. The angle that we know about is 32° and we know that the length of the side adjacent to the angle is 100 feet. We want to know about the length of the side that is opposite to the angle. Looking at our choices for the trigonometric functions we see that the tangent function relates all three of these, i.e. the angle, the adjacent side and the opposite side. So we have

$$\tan(32^\circ) = \frac{opp}{100\,\text{ft}} \qquad \text{or} \qquad opp = \tan(32^\circ)100\,\text{ft} \approx 62.5\,\text{ft},$$

so the river is approximately sixty two and a half feet across.

An important step in the above example was determining which trigonometric function to use. Since most calculators only have the sine, cosine and tangent functions available we will usually select from one of these three. To know which one we look at our triangle.

Specifically, we will usually have two sides involved, one side that we already know the length of and the other that we are trying to find the length of. We see how these sides relate to the angle that we know. In the example they were the adjacent and opposite sides and so we used the tangent. If they had been the adjacent and hypotenuse sides we would have used the cosine. Finally, if they had been the opposite and hypotenuse sides we would have used the sine.

5.3 Basic Identities

From the definitions of the trigonometric functions we can find relationships between them. For example, if we flip the sine function over we get the cosecant functions. This leads to the *reciprocal identities*, namely,

$$\csc(\theta) = \frac{1}{\sin(\theta)}, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \cot(\theta) = \frac{1}{\tan(\theta)}.$$

This explains why calculators do not have buttons for all of the trigonometric functions. Namely, we can get the cosecant, secant and cotangent functions by taking the reciprocal of the sine, cosine and tangent functions respectively.

Another way that we can relate the trigonometric functions is as ratios of trigonometric functions. These are known as the *quotient identities*. The two most important are,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)},$$

though others also exist.

5.4 The Pythagorean identities

The Pythagorean identities follow from the Pythagorean theorem. Namely we have,

$$\cos^2(\theta) + \sin^2(\theta) = \left(\frac{adj}{hyp}\right)^2 + \left(\frac{opp}{hyp}\right)^2 = \frac{(adj)^2 + (opp)^2}{(hyp)^2} = 1.$$

This leads to other variations. If we divide both sides by $\cos^2(\theta)$ and then use the reciprocal and quotient identities we get,

$$\frac{\cos^2(\theta)+\sin^2(\theta)}{\cos^2(\theta)}=\frac{1}{\cos^2(\theta)} \qquad \text{or} \qquad 1+\tan^2(\theta)=\sec^2(\theta).$$

Similarly, if we divide both sides of the equation by $\sin^2(\theta)$ we get,

$$\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2(\theta)}=\frac{1}{\sin^2(\theta)} \qquad \text{or} \qquad \cot^2(\theta)+1=\csc^2(\theta).$$

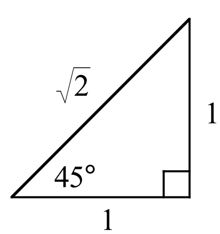
These three equations form the Pythagorean identities and are useful in simplifying expressions.

5.5 Trigonometric functions with some familiar triangles

Up to this point we have done much talking about the trigonometric functions, but we have yet to find any exact values of the trigonometric functions.

We can use some simple triangles to compute the exact values for the angles 30°, 45° and 60°. Later on we will see how to get more exact values from these.

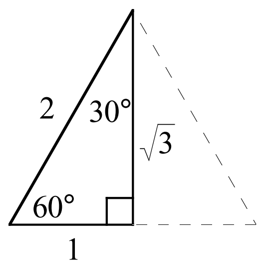
For 45° (or $\pi/4$ *rads*) we start by constructing an isoceles right triangle, since it is isoceles the two acute angles are congruent and so they have to be 45° (in order for the angles to add to 180°). If we let the lengths of the legs of the right triangle be 1 then by the Pythagorean theorem the hypotenuse will have length $\sqrt{2}$. The triangle is illustrated below.



From this triangle we get the following,

$$\sin(45^\circ)=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}, \qquad \cos(45^\circ)=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}, \qquad \tan(45^\circ)=\frac{1}{1}=1.$$

For 30° and 60° (or $\pi/6$ *rads* and $\pi/3$ *rads*) we start by constructing an equilateral triangle with sides of length 2 and then by cutting it in half we will get a right triangle with angles of 30° and 60°. The hypotenuse will be of length 2, and one of the legs is half the length of the side of the equilateral triangle, so has length 1. Using the Pythagorean theorem we find that the length of the third side of the right triangle is $\sqrt{3}$. The triangle is illustrated below.



Using this triangle we can find the exact values of the trigonometric functions for 30°,

$$\sin(30^\circ) = \frac{1}{2}, \qquad \cos(30^\circ) = \frac{\sqrt{3}}{2}, \qquad \tan(30^\circ) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

and for 60°,

$$\sin(60^\circ) = \frac{\sqrt{3}}{2}, \qquad \cos(60^\circ) = \frac{1}{2}, \qquad \tan(60^\circ) = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

5.6 A word of warning

Sometimes it is tempting to simplify the expression $\sin(a+b)$ as $\sin(a) + \sin(b)$. But this does not work. Erase any thought from your mind of simplifying in this manner.

As an example of why it does not work, consider the following.

$$\sin(30^\circ + 30^\circ) = \sin(60^\circ) = \sqrt{3}/2$$
$$\sin(30^\circ) + \sin(30^\circ) = 1/2 + 1/2 = 1$$

These do not match. Later on we will find the correct formula for simplifying such expressions.

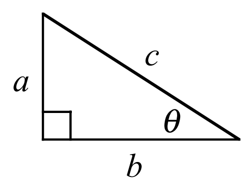
5.7 Supplemental problems

1. Find the values of the trigonometric functions given

$$\cos(\theta) = \frac{x^2 - 1}{x^2 + 1} \qquad \text{(where } x > 1\text{).}$$

2. Use the right triangle shown below to prove the following relationships for the acute angle θ.

$$\cos(\theta) = \sin(90° - \theta), \quad \cot(\theta) = \tan(90° - \theta), \quad \csc(\theta) = \sec(90° - \theta).$$

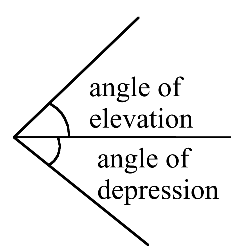


These are known as the complementary angle identities. Note that this is where the "co" comes from in cosine, cotangent and cosecant, i.e. they are the sine, tangent and secant of the complimentary angle.

3. Write $\sin(\theta)$, $\cos(\theta)$, $\sec(\theta)$ and $\csc(\theta)$ as ratios of trigonometric functions.

4. One day you happen to find yourself walking alongside an iron rod and you notice that by turning 81° to your left you see a building that was as it were floating in the air. After walking another 500 feet you now notice that the building is directly to your left. How far away is the building when you looked the second time?

5. In describing physical problems that involve height, that is length up and down, we often will use the terms angle of elevation and angle of depression. The angle of elevation is the angle that you have to look up from the horizontal, the angle of depression is how much you look down. These are illustrated below.

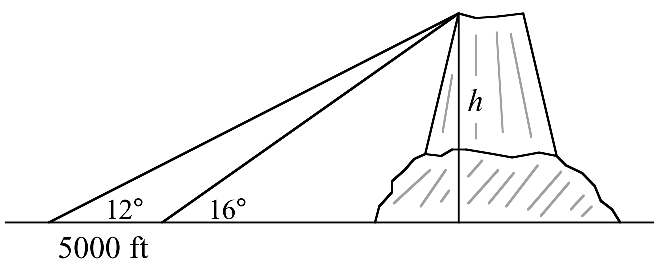


With this in mind, answer the following question.

As you continue to walk along the iron rod you notice in front of you a tree which is white above all that is white. To see the top of the tree you have to look up at an angle of elevation of $25°$. If you reach the bottom of the tree after walking another 100 feet, how tall is the tree? (Assume your eye level is five feet above the ground.)

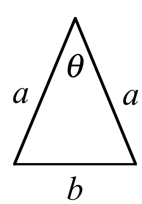
6. One day you and a vertically-challenged friend find yourself in possession of the ring of power heading toward Mount Doom. At first you have to look up at an angle of $12°$ to see the top of Mount Doom. After walking another 5000 feet you now have to look up at an angle of $16°$ to see the top of Mount Doom. How tall is Mount Doom? Find the answer without using the law of sines.

   A badly drawn picture is shown below. Round your answer to the nearest 100 feet.



Note: drawing is not to scale

7. In the diagram below express b in terms of a and θ. *Hint*: try breaking up the triangle into right triangles and use the information that you have.



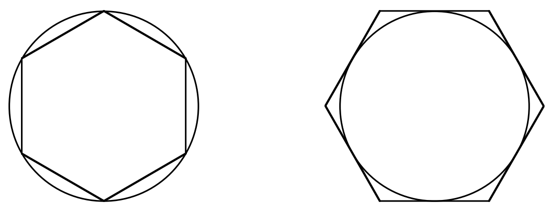
8. A regular polygon is a polygon with all sides of equal length and all angles of equal measure. Using this complete the following.

   (a) For a regular polygon with n sides inscribed in the unit circle (an example is shown below on the left for $n = 6$) express the *total length* of the sides of the polygon in terms of n and trigonometric functions.

   (b) For a regular polygon with n sides with a unit circle inscribed in the polygon (an example is shown below on the right for $n = 6$) express the *total length* of the sides of the polygon in terms of n and trigonometric functions.

(c) There exists a unique number always bigger than the answer to part (a) and always smaller than the answer to part (b). What is the number and why?

(d) Using a calculator compute the values for parts (a) and (b) when n is 180. Round your answers to six decimal places. Does your answer to part (c) lie in between these two values?



9. (a) Let the circumference of a polygon denote the sum of the lengths of all the sides. Given an n-sided regular polygon with a circumference of 1 (i.e. each side of the polygon has length $(1/n)$), find the area of the polygon in terms of n and trigonometric functions. *Hint*: the area of the triangle shown in problem 7 is $(b^2 \cot(\theta/2))/4$ (you do not have to prove this, but it is within your reach).

   (b) As n get large the value of the answer to part (a) gets closer and closer to a number. What is the number and why? *Hint*: as n gets large what does the polygon look like?

Lecture 6

Trigonometry with circles

In this lecture we will generalize the trigonometric functions so that we can use any angle. Along the way we will explore some interesting properties of symmetry.

6.1 The unit circle in its glory

Right triangles are wonderful for exploring the trigonometric functions, but they have a very serious limitation. Namely, we can only put acute angles in right triangles (that is angles between 0° and 90°). But there are many, many angles that are not acute.

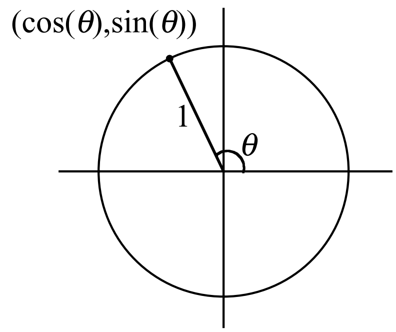
To be able to work with the trigonometric functions of any angle we will define the trigonometric functions by using the unit circle (recall that a unit circle is a circle with a radius of 1). We will first define the cosine and sine functions in terms of the unit circle. Then we will define the rest of the trigonometric functions as combinations of sine and cosine.

So for any angle begin by constructing the angle in standard position, that is the first part of the angles will be the positive x axis. The second part of the angle will intersect the unit circle at some point and we will define the cosine of the angle to be the x coordinate of the point and the sine of the angle to be the y coordinate of the point. This is illustrated at the top of the next page.

With the sine and cosine functions defined we will get the other four trigonometric functions by using the identities we found last time,

$$\sec(\theta) = \frac{1}{\cos(\theta)}, \quad \csc(\theta) = \frac{1}{\sin(\theta)}, \quad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.$$

Note that the values of the trigonometric functions depend only on where the terminal side of the angle intersects the unit circle. If two angles intersect the unit circle at the same point then they will have the same values for all of the

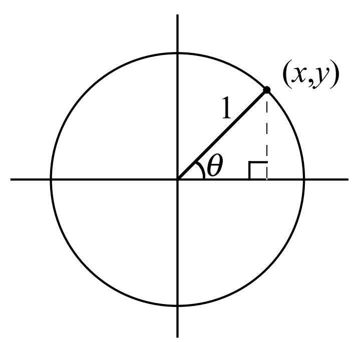


trigonometric functions. In particular, two angles that are co-terminal will have the same values for all of the trigonometric functions.

So we have that every angle is associated with a point on the unit circle, and every point on the unit circle is associated with an angle (infinitely many of them actually because every angle has infinitely many co-terminal angles).

6.2 Different, but not that different

We have now defined the trigonometric functions for acute angles in two ways, namely as ratios in a right triangle and as points on the unit circle. In order for the subject to make sense, these two definitions should agree. So consider the picture below for any acute angle θ.



We have formed a right triangle by dropping down a line from the point of intersection to the x axis. Since this circle is a unit circle the length of the hypotenuse is 1 (i.e. the hypotenuse is the length of the radius). The adjacent side of the triangle is the change in the left/right direction which in our case is the value x and the opposite side of our triangle is the change in the up/down direction which in our case is the value of y. So finding the sine and cosine from ratios we have,

$$\cos(\theta) = \frac{x}{1} = x, \qquad \sin(\theta) = \frac{y}{1} = y.$$

Which agrees with finding the sine and cosine as a point on the unit circle. So the two ways of defining the trigonometric functions agree for acute angles. Of course, with the unit circle we can now find the trigonometric functions for any angle.

6.3 The quadrants of our lives

When we worked with right triangles the trigonometric functions were *always* positive. But now we have a wider range of angles, and one of the big differences that confronts us is that the sine and cosine functions can return positive or negative values depending on where the angle is.

To help keep track of the signs we will split the plane up into four parts, this is done naturally by using the x axis and the y axis. We will call each one of these four parts a quadrant and unimaginatively give them the names I, II, III and IV where we start in the upper right and go in a counterclockwise direction.

Now recall that the sine function corresponds to the y values and so where the y values are positive the sine function is positive and where the y values are negative the sine function is negative. In a similar way the cosine function is related to the x values. Once we know the signs of the sine and cosine function we can then find the sign for the tangent function (recall that the tangent function is a ratio of the sine and cosine functions). Going through and marking the signs of the various functions we get the following chart.

sin + cos - tan - II	sin + cos + tan + I
III sin - cos - tan +	IV sin - cos + tan -

6.4 Using reference angles

Our work with right triangles has not been in vain. For one thing, many real world problems can be described in terms of right triangles and so it is good to get an intuitive understanding of the relationships of right triangles. But more importantly, right triangles are useful for evaluating the trigonometric functions for acute angles. This is helpful because we can use information about acute angles to find the value of the trigonometric functions for any angles.

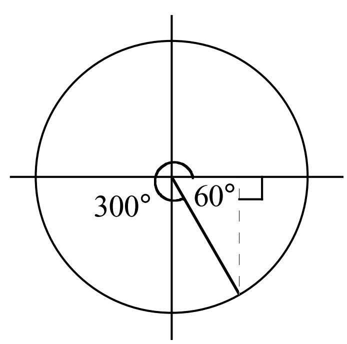
This is done by using *reference angles*, every angle has a reference angle (an acute angle between $0°$ and $90°$). To find the reference angle draw the angle in standard position. From the point where the angle intersects the unit circle drop a line straight down (or up) to the x axis. This forms a right triangle. The acute angle of this right triangle located at the origin is the reference angle.

Reference angles are useful because the value of the trigonometric functions for the reference angle will match the value of the trigonometric functions for the angle except possibly for the sign.

To determine the sign of the angle we note what quadrant the angle lies in and then use the chart about the signs of the trigonometric functions in the quadrants to fix any sign problems.

Example 1 Find the exact values for the sine, cosine and tangent of $2820°$.

Solution First we will simplify matters and find a co-terminal angle to $2820°$ between $0°$ and $360°$. Using the process from a previous lecture we get that a co-terminal angle is $300°$. This is in quadrant IV, and so the sine function will be negative, the cosine function will be positive and the tangent function will be negative. Drawing the triangle in we get the picture as shown below.



In particular we have that the reference angle is $60°$. We already have that $\sin(60°) = \sqrt{3}/2$, $\cos(60°) = 1/2$ and $\tan(60°) = \sqrt{3}$. Using the information about the value of the reference angle and the fact that our angle is in the fourth quadrant we have,

$$\sin(2820°) = -\frac{\sqrt{3}}{2}, \quad \cos(2820°) = \frac{1}{2}, \quad \tan(2820°) = -\sqrt{3}.$$

6.5 The Pythagorean identities

The identities that we developed last time still carry through to the unit circle. In particular, the Pythagorean identities still hold. This follows from noting that the algebraic definition of the unit circle is $x^2 + y^2 = 1$. Then recall that the cosine function is the x value of a point on the unit circle and the sine function is the y value of a point on the unit circle. So we get the following,

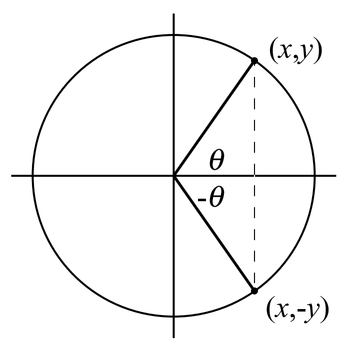
$$\cos^2(\theta) + \sin^2(\theta) = x^2 + y^2 = 1.$$

6.6 A man, a plan, a canal: Panama!

A useful tool of mathematics is symmetry. Symmetry deals with how an object is similar to itself. A verbal example of symmetry is a palindrome which is the same either forwards or backwards.

The unit circle is highly symmetric (i.e. you can fold it in half any number of ways and the two halves will overlap). We can use this to our advantage to get some important relationships that the trigonometric functions satisfy.

As a first example, consider the two angles θ and $-\theta$. These two angles will lie straight across the x axis from each other (i.e. if the point (x, y) on the unit circle is associated with the angle θ then the point $(x, -y)$ is associated with the angle $-\theta$). Pictorially, this is seen below.



The point that corresponds to the angle θ is $(\cos(\theta), \sin(\theta))$ and the point that corresponds to the angle $-\theta$ is the point $(\cos(-\theta), \sin(-\theta))$. Using these relationships and symmetry we have,

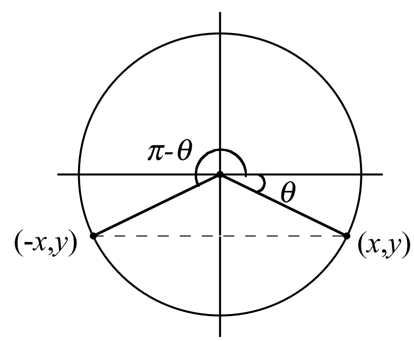
$$\cos(-\theta) = \cos(\theta), \qquad \sin(-\theta) = -\sin(\theta).$$

We will often find relationships for the sine and cosine functions and then use the reciprocal and quotient identities to extend these relationships to the other trigonometric identities. In this case we have the following for the tangent function.

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

Similarly we can show $\cot(-\theta) = -\cot(\theta)$, $\csc(-\theta) = -\csc(\theta)$ and $\sec(-\theta) = \sec(\theta)$. All of these together form the *even/odd'er identities* which we shall discuss in more detail later on.

As a second example of using symmetry, consider the two angles θ and $\pi - \theta$ (or if you prefer to work in degrees, θ and $180° - \theta$). In this case the symmetry is around the y axis (i.e. if the point (x, y) is associated with the angle θ then the point $(-x, y)$ is associated with the angle $\pi - \theta$). Pictorially, this is seen below.



The point that is associated with the angle θ is $(\cos(\theta), \sin(\theta))$ and the point associated with the angle $\pi - \theta$ is $(\cos(\pi - \theta), \sin(\pi - \theta))$. Using this and the relationship from symmetry we get,

$$\cos(\pi - \theta) = -\cos(\theta), \qquad \sin(\pi - \theta) = \sin(\theta).$$

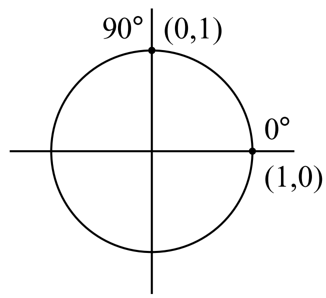
As with before we can use the information about the sine and cosine functions to show that $\tan(\pi - \theta) = -\tan(\theta)$, $\cot(\pi - \theta) = -\cot(\theta)$, $\csc(\pi - \theta) = \csc(\theta)$ and $\sec(\pi - \theta) = -\sec(\theta)$.

One of the most important ways we use symmetry is finding additional values for angles that satisfy a given relationship. For example, suppose we have that $\sin(32°) = a$ for some value a and we want to find another angle θ that also has $\sin(\theta) = a$. From the second type of symmetry we know that if we go across horizontally that we do not change the value for the sine function, and in particular that $\sin(180° - 32°) = \sin(32°) = a$. So a second angle would be 148°.

In general, by using symmetry around the x and y axis we can find additional values for angles for the cosine and sine functions respectively. For the cosine this amounts to the angle $-\theta$ and for the sine this will amount to the angle $180° - \theta$, and then any multiple of 360° added to these.

6.7 More exact values of the trigonometric functions

We have already been able to get the exact values of the trigonometric functions for the angles 30°, 45° and 60°. We will now expand our list to include 0° and 90° (or 0 and $\pi/2$ *rads*). This will be done by using the unit circle and examining the points as indicated below.



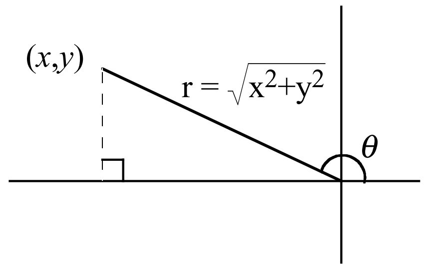
The point that corresponds to 0° is located at $(1, 0)$ and so $\cos(0°) = 1$ and $\sin(0°) = 0$. Similarly the point that correspond to 90° is located at $(0, 1)$ and so $\cos(90°) = 0$ and $\sin(90°) = 1$. Using the values of sine and cosine we have that $\tan(0°) = 0/1 = 0$ and that $\tan(90°) = 1/0$ which is undefined. Combining this with what we did last time we have the chart shown below.

Angle	0° or 0	30° or $\pi/6$	45° or $\pi/4$	60° or $\pi/3$	90° or $\pi/2$
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan(\theta)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef.

Using these values for the trigonometric functions and reference angles we can now find the exact value for a large number of angles. These are shown at the end of this lecture.

6.8 Extending to the whole plane

We can extend the trigonometric functions beyond the unit circle and indeed to every point in the plane except the origin. So for any (x, y) except the origin consider the picture shown below.



Here $r = \sqrt{x^2 + y^2}$ is the distance to the origin and will always be positive. Then we can define the trigonometric functions in terms of x, y and r as follows,

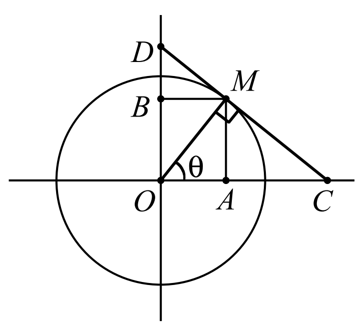
$$\sin(\theta) = \frac{y}{r}, \qquad \cos(\theta) = \frac{x}{r}, \qquad \tan(\theta) = \frac{y}{x},$$
$$\csc(\theta) = \frac{r}{y}, \qquad \sec(\theta) = \frac{r}{x}, \qquad \cot(\theta) = \frac{x}{y}.$$

This works by taking any point in the plane (x, y) and associating it with (or scaling it to) a point on the unit circle, namely the point $(x/r, y/r)$, which is associated with the same angle θ. We then let the trigonometric functions for the point (x, y) to be defined as the trigonometric functions for the point on the unit circle $(x/r, y/r)$.

In particular, as with the unit circle, we can associate every point in the plane except the origin with an angle. The idea of taking a point and scaling it to a point on the unit circle will play an important role later on.

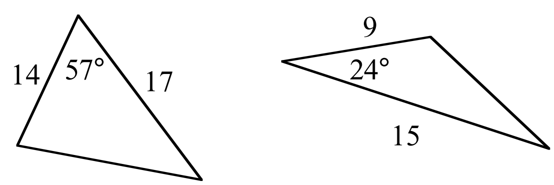
6.9 Supplemental problems

1. Given that the circle shown below is the unit circle match each of the six trigonometric functions for the angle θ to one of the following lengths, OA, OB, OC, OD, MC and MD. *Hint*: find the angle formed by going from O to D to C in terms of θ.

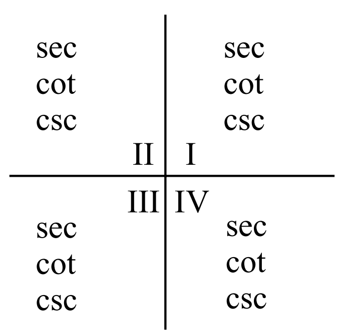


2. Show that if we know the lengths of two sides (call them a and b) and the angle in between those two sides (call it γ) of a triangle then the area is given by
$$\text{area} = \frac{1}{2}ab\sin(\gamma)$$
Does this formula also apply when γ is obtuse? *Hint*: we already know that the area of a triangle is (1/2)(base)(height) so let the side of length a be the base and then find the height.

3. Using the result from the previous problem, find the area of the two triangles shown below.



4. Fill in the chart below by indicating whether the function is positive or negative in each quadrant.



5. What symmetry exists between the angle θ and $\theta + \pi$ (or $\theta + 180°$ if working in degrees)? *Hint*: try putting in a couple of values for θ and see how the two angles compare.

   Using symmetry, write $\sin(\pi + \theta)$ in terms of $\cos(\theta)$ and/or $\sin(\theta)$. Repeat for $\cos(\pi + \theta)$.

   Find a relationship between $\tan(\pi + \theta)$ and $\tan(\theta)$.

6. What symmetry exists between the angle θ and $(\pi/2) - \theta$ (or $90° - \theta$ if working in degrees)? *Hint*: try putting in a couple of values for θ and see how the two angles compare.

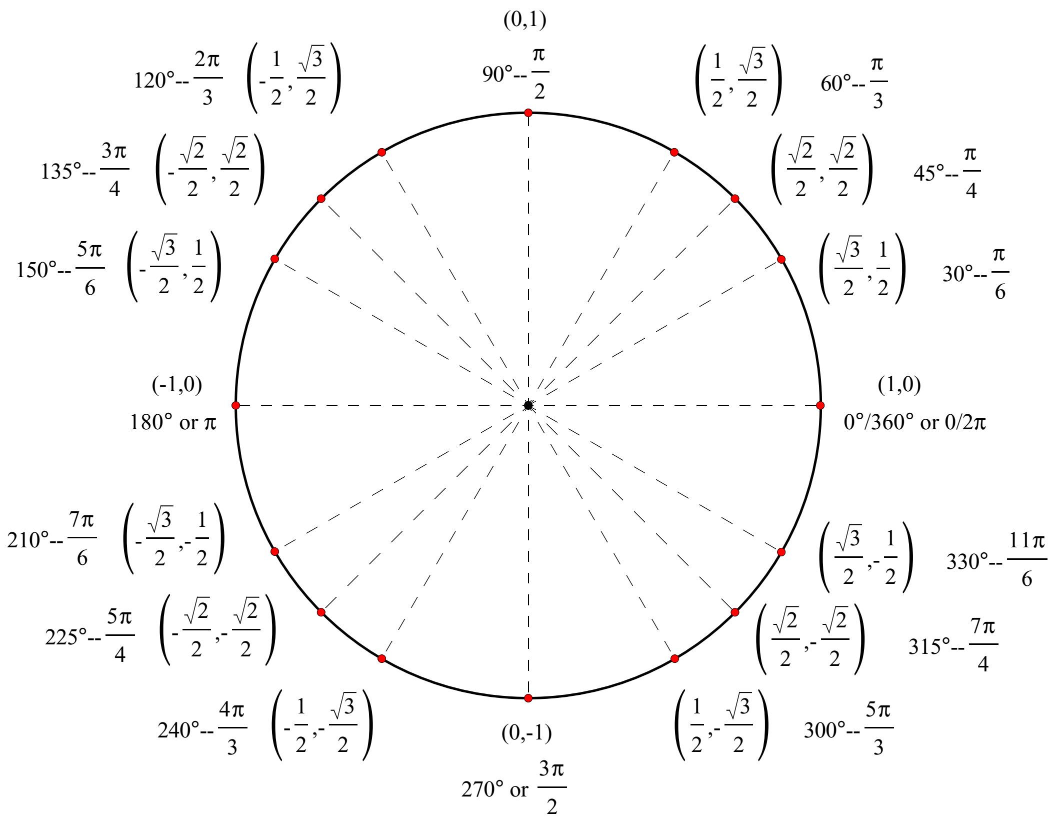
Using symmetry, write $\sin((\pi/2)-\theta)$ in terms of $\cos(\theta)$ and/or $\sin(\theta)$. Repeat for $\cos((\pi/2)-\theta)$.

7. Fill in the chart below with exact values. Here β refers to the reference angle of θ.

θ	Quadrant	β	$\cos(\beta)$	$\sin(\beta)$	$\cos(\theta)$	$\sin(\theta)$
$585°$						
$-(25\pi/6)$						

8. Verify that if x and y are not both 0 then the point $(x/r, y/r)$ is on the unit circle, where $r = \sqrt{x^2+y^2}$. Also using the definition of the sine and cosine function on the unit circle and the reciprocal and quotient identities show that we have the following.

$$\begin{aligned} \sin(\theta) &= \frac{y}{r}, & \cos(\theta) &= \frac{x}{r}, & \tan(\theta) &= \frac{y}{x}, \\ \csc(\theta) &= \frac{r}{y}, & \sec(\theta) &= \frac{r}{x}, & \cot(\theta) &= \frac{x}{y}. \end{aligned}$$



The Unit Circle

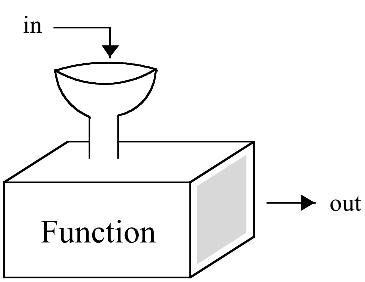
Lecture 7

Graphing the trigonometric functions

In this lecture we will explore what functions are and develop a way to graphically represent a function.

7.1 What is a function?

We have been taking a lot of time to discuss the trigonometric functions, now let us step back a second and see what a function is. Pictorially, we can think of a function as a machine such as is drawn below.



In our picture of a function there are two openings, one is for what we put into our function, the other is for what comes out of our function. The machine in the middle, the function, is a rule that assigns to every input a *unique* output. (It is this uniqueness that determines whether or not the rule, or the machine, is truly a function.)

The domain of the function deals with asking the question, what can we put into our function? We will be looking mostly at the sine and cosine functions. For these functions our input will be angles, and from the last lecture we know that for *any* angle we can get a value for the sine and cosine functions. So for these two functions the domain is everything. (Mathematically we would denote this by $(-\infty, \infty)$, this does not represent a point but an interval.)

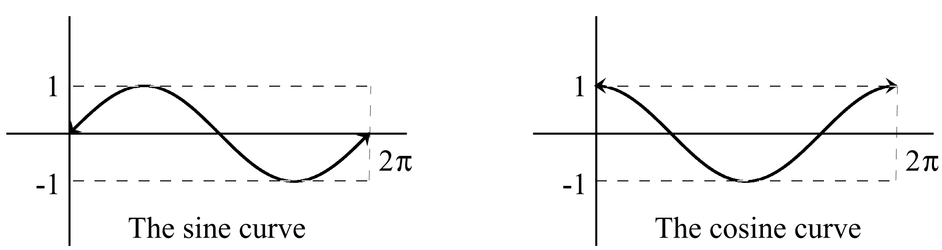
The range of the function deals with asking the question, what comes out of our function? For the trigonometric functions our outputs are numbers and in particular since the sine and cosine functions are based on the unit circle their values can only be values that are achieved on the unit circle. So the sine and cosine functions have as their range the numbers between -1 and 1, including the values of -1 and 1. (Mathematically we would denote this by $[-1, 1]$, again this represents an interval.)

7.2 Graphically representing a function

When trying to understand the behavior of a function it is often convenient to be able to look at a representation of the function so that we can see its behavior at a glance. We will do this by constructing a graph (or pretty picture).

To construct a graph of a function recall that a function works in pairs of numbers, the input and the corresponding output. We will associate these pairs of numbers with points in the plane by associating the input with the x coordinate and the corresponding output with the y coordinate.

As an example from last time we showed that $\sin(0) = 0$ and so one point that is on the curve of the sine function is the point $(0, 0)$. If we were to go through and find the values of the sine function for a large number of angles and then plot these as points we would start to see a curve emerge, what we will call the sine curve. A similar process will create the cosine curve. These are shown below



When we deal with graphing the trigonometric functions we will *always* work in radians. This is not because we cannot graph in degrees, but rather there are some deeper hidden reasons as to why we choose radians. We will catch a glimpse of these reasons later on.

7.3 Over and over and over again

If we were to graph the sine and cosine curves correctly we would have to put in a lot of values. However, we can save ourselves some work by making an observation. We know that if two angles are co-terminal they will have the same values for the trigonometric functions, for example $\sin(x + 2\pi) = \sin(x)$. In particular the trigonometric functions are repeating.

To make a graph of the trigonometric function we only need to determine what it looks like on an interval that contains a complete revolution. Once we have that we just copy it over and over to get the complete graph for the function.

Functions that have this property are called periodic and the minimum amount of time it takes to repeat is the period. The sine and cosine functions are 2π periodic while the tangent function is π periodic.

7.4 Even and odd functions

The graphs of some functions exhibit symmetry. There are two special types of symmetry that we will encounter when graphing functions.

The first type of symmetry is around the y axis. Imagine graphing the function then folding it in half along the y axis. If the two halves exactly match up then it is symmetrical around the y axis. Such a function is called an *even* function and satisfies the relationship $f(-x) = f(x)$. Examples of even trigonometric functions are the cosine and secant functions.

The second type of symmetry is around the origin. Imagine graphing the function then rotating the graph a half revolution around the origin. If it looks the same as before then it is symmetrical around the origin. Such a function is called an *odd* function and satisfies the relationship $f(-x) = -f(x)$. Examples of odd trigonometric functions are the sine, cosecant, tangent and cotangent functions.

Example 1 Determine whether the following function is even, odd or neither.

$$f(x) = \sin^2(x) - \cos(x)$$

Solution To determine if a function is even, odd or neither we will look at how the function $f(-x)$ relates to the function $f(x)$. If they are the same then the function is even, if they are the negative of each other then the function is odd, if neither of these are true then the function is neither. So using the even/odd'er identities we get the following.

$$\begin{aligned} f(-x) &= \sin^2(-x) - \cos(-x) = (\sin(-x))^2 - \cos(-x) \\ &= (-\sin(x))^2 - \cos(x) = \sin^2(x) - \cos(x) \\ &= f(x) \end{aligned}$$

So the function is even.

7.5 Manipulating the sine curve

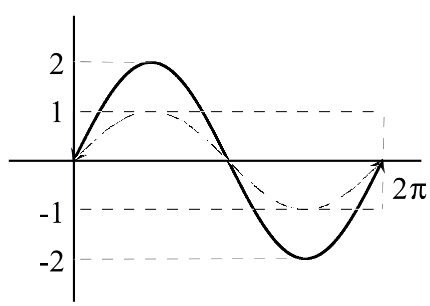
Once we have gotten a graph for the sine function (or any function for that matter) we may want to manipulate the graph. There are four basic manipulations that we can make to any graph: move it left or right, move it up or down, stretch it horizontally, and stretch it vertically. These are done by adding constants into our basic function, i.e.

$$y = a\sin(bx - c) + d.$$

We will first look at these constants individually and then as a whole.

- The value a. This is outside the function and so deals with the output (i.e. the y values). This constant will change the *amplitude* of the graph, or how tall the graph is. The amplitude is half the distance from the top of the curve to the bottom of the curve.

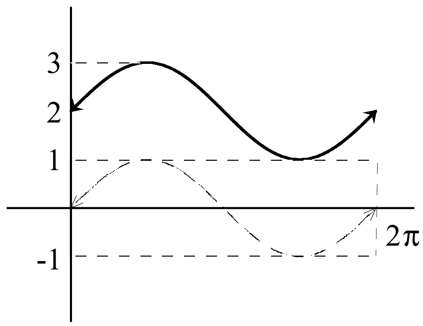
  The new amplitude will be the value $|a|$, that is the absolute value of a. For example the original curve $y = \sin(x)$ has amplitude 1 while the curve $y = 2\sin(x)$ will have amplitude 2, as seen below.



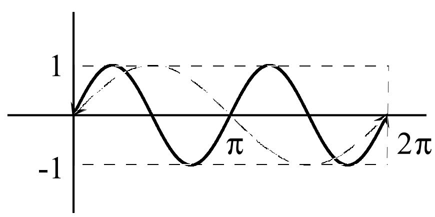
  If the value of a is negative then in addition to changing the amplitude it will also flip the curve over.

- The value d. This again is outside and so will effect the y values of the graph. This constant will vertically shift the graph up and down (depending on if d is positive or negative).

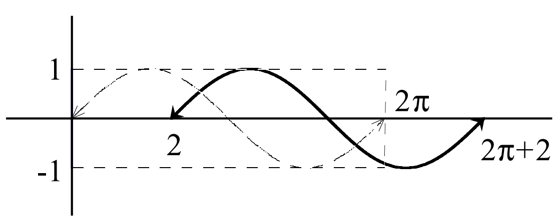
  For an example the curve $y = \sin(x) + 2$ is 2 above the curve $y = \sin(x)$, as seen below.



- The value b. This is inside the function and so effects the input or domain (i.e. the x values). This constant will stretch or shrink the graph horizontally. However, it will not change the period directly. For example the function $y = \sin(2x)$ does not have period 2. The period can be found by the following rule, if $y = \sin(bx)$ then the period is $(2\pi/b)$ (i.e. the original period divided by the constant b). So in particular the function $y = \sin(2x)$ will have period $(2\pi/2) = \pi$. The functions $y = \sin(x)$ and $y = \sin(2x)$ are shown below.



- The constant c. This is on the inside and deals with moving the function horizontally left/right. For example the curve $y = \sin(x - 2)$ is the graph $y = \sin(x)$ shifted horizontally to the right 2 units, as seen below.



This constant is not independent of the others. In particular, it depends on the value of b. Let us now turn to exploring how and why this is so.

7.6 The wild and crazy inside terms

In graphing functions changing the inside terms seems to do things that are counter-intuitive. As an example consider the function $y = a\sin(bx - c) + d$. This function will have period $2\pi/b$ and a horizontal shift of c/b. Not what we would expect.

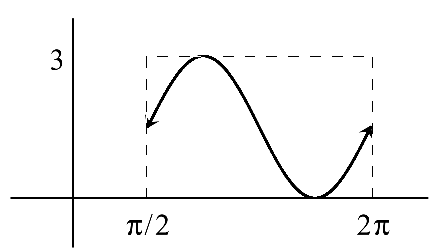
To see where these strange values arise recall that one period of the sine curve corresponds to one revolution around the circle. So one period begins at 0 and ends at 2π. If we are interested in exploring one period of our modified curve we would do it by finding when the inside expression is 0 (this is the start of the period) and when it is 2π (this is the end of the period). In particular we have the following,

$$\begin{array}{rccl} \textit{start} & bx - c = 0 & \text{or} & x = (c/b) \\ \textit{end} & bx - c = 2\pi & \text{or} & x = (2\pi/b) + (c/b). \end{array}$$

Note that the start of the period is now at the value c/b, this is why our horizontal shift is c/b. The difference between the start and the end represents the period, that is how long it takes to repeat, and so the period will be $2\pi/b$.

Example 2 Given that the graph shown below is one period of the sine curve find the amplitude, vertical shift, period and horizontal shift. Using these values write an equation for the curve in the form,

$$y = a\sin(bx - c) + d$$



Solution From the graph the height between the lowest and highest values is 3, and so the amplitude is half that or 3/2. The vertical shift (keeping in mind that the sine curve should start at the origin) is

3/2. The period is the length from the beginning to the end and so is $2\pi - \pi/2 = 3\pi/2$. Finally, the horizontal shift is $\pi/2$.

With these values in hand we can now start finding a, b c and d. The amplitude is a and so $a = 3/2$. The vertical shift is d and so $d = 3/2$. The period is $2\pi/b$ so $2\pi/b = 3\pi/2$ or $b = 4/3$. The horizontal shift is c/b so $(c = b\pi/2 = 2\pi/3)$. Putting these together we have,

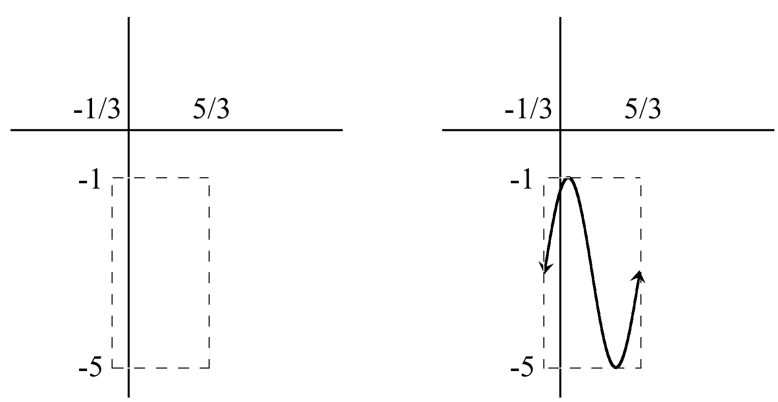
$$y = \frac{3}{2}\sin\left(\frac{4}{3}x - \frac{2\pi}{3}\right) + \frac{3}{2}.$$

In this example we had a specific period of the sine curve given to us. What if we were given the whole sine curve and were asked to find an expression of the form $y = a\sin(bx - c) + d$, which period should we use? The correct answer is any of them. You can choose any full period to determine your constants. Note that the constants will depend upon which period you choose but they will all correspond to the same curve.

Example 3 Given the following function find the amplitude, vertical shift, period and horizontal shift. Then use these values to graph one period of the function.

$$y = 2\sin\left(\pi x + \frac{\pi}{3}\right) - 3$$

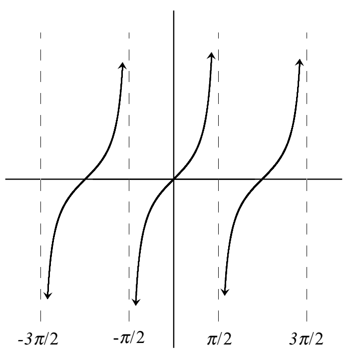
Answer: From the equation we can read off the amplitude, which is $a = 2$ and the vertical shift which is $d = -3$. To find the period we take the value $b = \pi$ and divide it into 2π which gives a period of $(2\pi/\pi) = 2$. To find the horizontal shift we take the value of $c = -\pi/3$ and divide it by $b = \pi$ to get a horizontal shift of $(-\pi/3)/\pi = -1/3$.



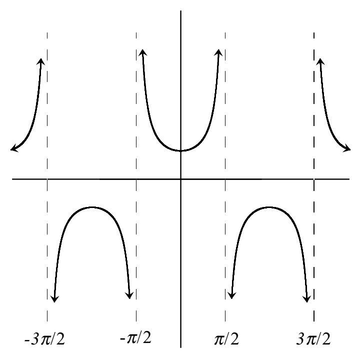
> To graph the function we first use the vertical and horizontal shift to find where the curve starts. We can then use the information about the amplitude and the period to draw a box that will tightly contain one period of the curve. The box for our problem is shown above on the left. With the box in place we then draw in one period of the sine curve, exactly filling the box, to get our required graph. This is shown above on the right.

7.7 Graphs of the other trigonometric functions

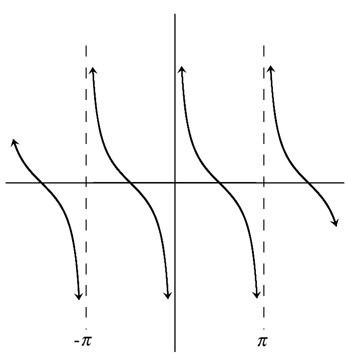
We can repeat a similar discussion for the other trigonometric functions but we will abstain from doing this. For reference we will include the graphs of the trigonometric functions to help gain an intuitive feel for what these functions do.



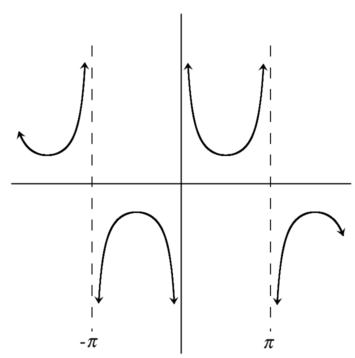
The tangent curve



The secant curve



The cotangent curve



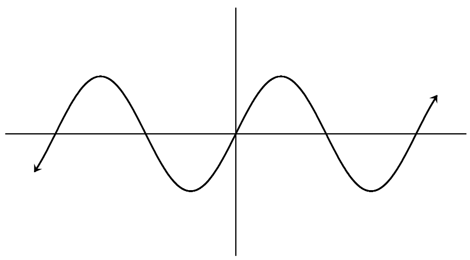
The cosecant curve

These functions have either the sine or cosine in the denominator. In particular, the sine and cosine will periodically take on the value of zero. When this occurs the function is trying to divide by zero. This causes the vertical asymptotes seen in these graphs.

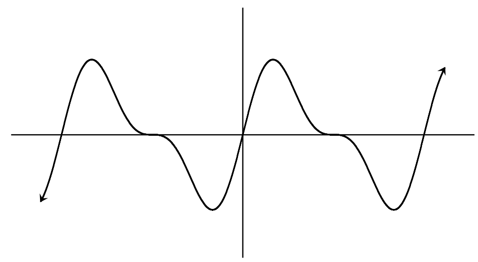
7.8 Why these functions are useful

The sine and cosine functions turn out to be incredibly useful for one very important reason, they repeat in a regular pattern (i.e. they are periodic). There are a vast array of things that repeat periodically, the rising and setting of the sun, the motion of a spring up and down, the tides of the ocean and so forth and so forth.

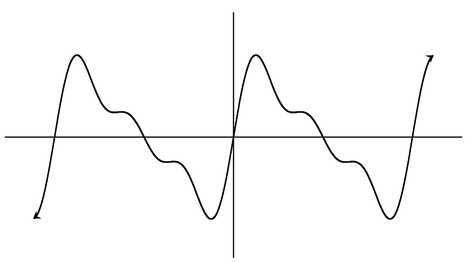
It turns out that all periodic behavior can be studied through combinations of the sine and cosine functions. This is a branch of mathematics known as Fourier analysis. We will not go into it at this time as it requires a substantial Calculus background, instead we will look at some pictures to see what can happen as we combine more and more trigonometric functions.



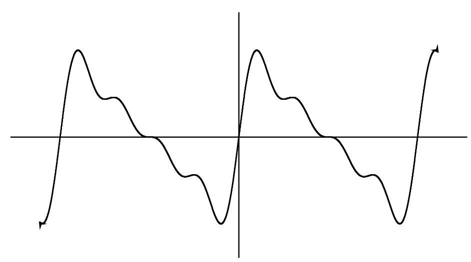
$2\sin(x)$



$2\sin(x)+\sin(2x)$



$2\sin(x)+\sin(2x)+(2/3)\sin(3x)$



$2\sin(x)+\sin(2x)+(2/3)\sin(3x)+(1/2)\sin(4x)$

7.9 Supplemental problems

1. Determine whether the following functions are even, odd, or neither.

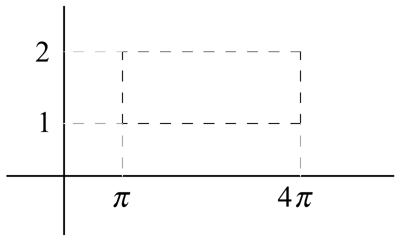
   (a) $\sin(x)\cos(x)$

   (b) $\sin(x)+\cos(x)$

   (c) $\sin(\cos(x))$

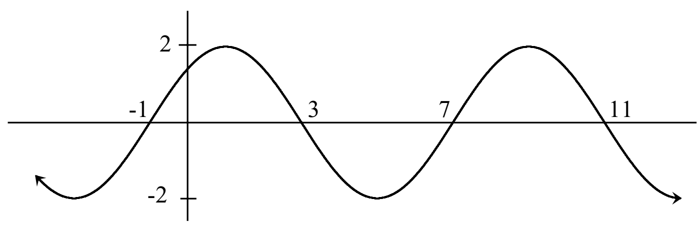
2. There is one and only one function that is both even and odd, what function is it? Justify your answer.

3. In the box below draw in one period of the sine curve so that it will completely fill the box. Then fill in the indicated values below and write the function in the form $y = a\sin(bx - c) + d$.



Amplitude: *Period:*
Horizontal shift: *Vertical shift:*

4. One day you find yourself ruling a country and two of your royal advisors are in an argument and come to you hoping that you can settle the issue. They present to you the following graph.



One of them then says, "May your reign be long and prosperous and may you confirm that this is a graph of the function $y = 2\sin((\pi/4)x + (\pi/4))$."

After which the other says, "May your reign be long and prosperous and may you confirm that this is a graph of the function $y = -2\sin((\pi/4)x - (3\pi/4))$."

Which one is right? Justify your answer.

5. Given that $y = a\sin(bx - c) + d$, where a, b, c and d are known values, what are the largest and smallest values that y can achieve? Justify your answer. (Assume that a is positive.)

Lecture 8

Inverse trigonometric functions

In this lecture we will explore how given an output of a trigonometric function to find the angle associated with it. This will be done through developing the inverse trigonometric functions.

8.1 Going backwards

Over the last few lectures we have been examining the trigonometric functions. These functions will take in an angle and return a number. Sometimes we might want to go backwards, that is we have a number and we want to find an angle that corresponds to it.

Example 1 Find the acute angle θ such that,

$$\sin(\theta) = \frac{1}{2}.$$

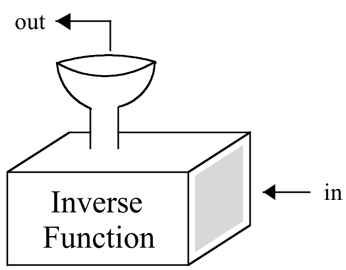
Solution From a previous lecture we constructed the table below,

Angle	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan(\theta)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef.

Looking at the line for the sine function we see that the sine function will return a value of $1/2$ for the angle $30°$ or $\pi/6$ *rads*.

8.2 What inverse functions are

An inverse function reverses the direction of our original function. Imagine that the inverse function is our function machine with the directions reversed. Pictorially, our inverse function machine would look like the following.



In the first example we put the value 1/2 into our inverse function and we got back an angle of 30°.

Mathematically, we will say that the inverse function (which we denote as f^{-1}) satisfies the following,

$$f^{-1}(y) = x \qquad \text{implies} \qquad f(x) = y.$$

In particular, the following statement will be true whenever it makes sense,

$$f(f^{-1}(y)) = y.$$

However, we shall soon see that it is not always true that

$$f^{-1}(f(x)) = x.$$

8.3 Problems taking the inverse functions

The reason that it is not always true that $f^{-1}(f(x)) = x$ is because there can be multiple values of x that map to the same value of y. Consider our first example. We found that one angle that mapped to the value of 1/2 under the sine function was 30°, but there are other angles. For instance, 150°, 390°, −210° and so forth all map to the value of 1/2 under the sine function.

This presents a problem because in order to be a function every input must have a *unique* output. So if we were to create an inverse trigonometric function we would need to find a way that for every input we assigned a unique angle (something not automatic with a periodic function, such as the trigonometric functions are).

8.4 Defining the inverse trigonometric functions

To overcome the problem of having multiple angles mapping to the same value we will restrict our domain before finding the inverse. Visually this is represented as throwing away most of our curve.

The way we throw out chunks of our curve is somewhat arbitrary. The only real requirement is that the domain that we are left with will hit all the value of our range *uniquely*. It is in this setting that the inverse trigonometric functions can be created.

The three most popular trigonometric functions are the sine, cosine and tangent. We will create the trigonometric functions for these by first limiting our domain and then swapping the corresponding inputs and outputs (an inverse function swaps the role of inputs and outputs and so the domains and ranges also swap).

To denote the inverse functions we will add *arc* in front of the function name. For example the arcsine function (denoted by 'arcsin') is the inverse function of the sine function and so forth.

Combining all this, we get the following table when the inverse functions return angles in radians.

Function	Inverse of	Domain	Range
arcsin	sin	$[-1, 1]$	$[-\pi/2, \pi/2]$
arccos	cos	$[-1, 1]$	$[0, \pi]$
arctan	tan	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$

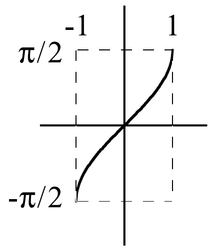
If our inverse functions are returning angles in degrees then the table would be filled out in the following way.

Function	Inverse of	Domain	Range
arcsin	sin	$[-1, 1]$	$[-90°, 90°]$
arccos	cos	$[-1, 1]$	$[0°, 180°]$
arctan	tan	$(-\infty, \infty)$	$(-90°, 90°)$

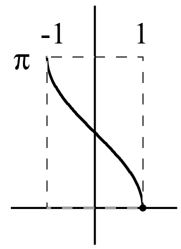
Graphs of these functions are shown on the next page. Graphically, inverse functions are the original function (domain restricted) which have been flipped around the line $y = x$.

Another notation to use for the inverse trigonometric functions is with an exponent of '-1', i.e. $\arcsin(x) = \sin^{-1}(x)$.

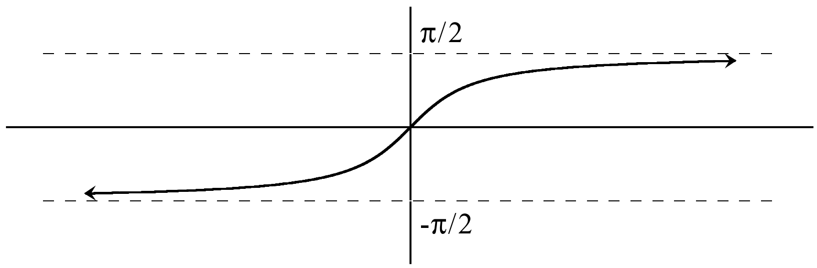
This notation is perfectly acceptable, however be careful. There is the grand temptation to interpret $\sin^{-1}(x)$ as $(1/\sin(x)) = \csc(x)$. But the arcsine and cosecant functions are *very* different. This can be seen by comparing their graphs. This goes for all of the trigonometric functions.



The arcsin curve



The arccos curve



The arctan curve

8.5 So in answer to our quandary

So by the nature of the inverse trigonometric functions we have that as long as y is in the range of the original function that the following equations hold,

$$\sin(\arcsin(y)) = y, \qquad \cos(\arccos(y)) = y, \qquad \tan(\arctan(y)) = y.$$

If we compose these functions in the reverse order then the relationship will hold only when x lies in the restricted domain. That is,

$$\begin{aligned}
\arcsin(\sin(x)) &= x \begin{cases} \text{when } x \text{ is between } -\pi/2 \text{ and } \pi/2 \textit{ rads} \\ \text{or when } x \text{ is between } -90^\circ \text{ and } 90^\circ, \end{cases} \\
\arccos(\cos(x)) &= x \begin{cases} \text{when } x \text{ is between } 0 \text{ and } \pi \textit{ rads} \\ \text{or when } x \text{ is between } 0^\circ \text{ and } 180^\circ, \end{cases} \\
\arctan(\tan(x)) &= x \begin{cases} \text{when } x \text{ is between } -\pi/2 \text{ and } \pi/2 \textit{ rads} \\ \text{or when } x \text{ is between } -90^\circ \text{ and } 90^\circ. \end{cases}
\end{aligned}$$

8.6 The other inverse trigonometric functions

So far we have only talked about the inverse trigonometric functions for sine cosine and tangent,but we have left out the inverse functions for the cosecant, secant and cotangent. Looking at your calculators you will notice that the calculator has left these out as well. So the problem arises, how do we evaluate these inverse trigonometric functions. We do it with a twist. Consider the following.

$$\begin{array}{cl} \text{Suppose that} & \operatorname{arccsc}(y) = x, \\ \text{then it follows that} & y = \csc(x), \\ \text{which implies} & 1/y = \sin(x) \\ \text{and so we have} & \arcsin(1/y) = x = \operatorname{arccsc}(y). \end{array}$$

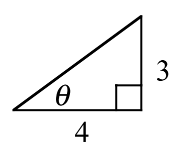
We can repeat the procedure for the other two functions and get,

$$\operatorname{arcsec}(y) = \arccos\left(\frac{1}{y}\right), \qquad \operatorname{arccot}(y) = \arctan\left(\frac{1}{y}\right).$$

8.7 Using the inverse trigonometric functions

Now that we have gone through the work of describing these functions we will use them in some applications.

Example 2 Find the acute angle θ in the triangle below.



Solution This is a right triangle and so we can represent the trigonometric functions as ratios. In particular, we know the sides opposite and adjacent the angle θ. The trigonometric function that relates these two sides is the tangent function and so we have,

$$\tan(\theta) = \frac{3}{4} \qquad \text{or} \qquad \theta = \arctan\left(\frac{3}{4}\right) \approx 36.87^\circ \text{ or } .6435 \textit{ rads.}$$

Note in this example that we had numbers in the triangle, but the same process would have worked if there had been variables on the side to express θ as a function of the variables.

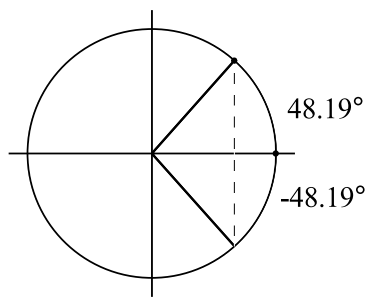
Example 3 Find *all* angles θ such that

$$\cos(\theta) = \frac{2}{3}$$

Solution Using the arccos function we get the following,

$$\theta = \arccos\left(\frac{2}{3}\right) \approx 48.19^\circ \text{ or } .8411 \; \textit{rads}.$$

This produces only one angle. We can add full revolutions to get more, but that will still only be half of the desired angles. In general there will be two angles with the same cosine (or same sine) value in a revolution. Since we are working with the cosine we can find the second angle by dropping a line straight down as shown in the picture below (recall that the x values correspond to the cosine function so by dropping straight down we keep the same cosine value).



So in addition to 48.19° we also have -48.19° or 311.81° and so our final answer is,

$$48.19^\circ \text{ plus multiples of } 360^\circ \text{ and } 311.81^\circ \text{ plus multiples of } 360^\circ.$$

In radians our final answer would be,

$$.8411 \text{ plus multiples of } 2\pi \text{ and } 5.4421 \text{ plus multiples of } 2\pi.$$

For our last example we will show how to simplify a particular type of trigonometric expression into an algebraic expression, that is, an expression involving no trigonometric functions. The type of trigonometric expression we will explore is a trigonometric function composed with an inverse trigonometric function. First an example and then we will summarize the steps that we took in simplifying.

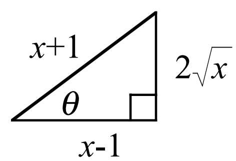
Example 4 Simplify the following to an algebraic expression.

$$\cot\left(\arccos\left(\frac{x-1}{x+1}\right)\right)$$

Solution First note that the output of the arccosine function is an angle and so in particular the term inside represents an angle. Since it is an angle, let us give it an angle name, say θ. So we have,

$$\theta = \arccos\left(\frac{x-1}{x+1}\right) \qquad \text{and so} \qquad \cos(\theta) = \frac{x-1}{x+1}.$$

From this last expression we can construct a right triangle where we know the length of two of the sides and use the Pythagorean theorem to solve for the third side. This process will produce the triangle shown below.



With this triangle in place we can then use the triangle to get the value for $\cot(\theta)$. Putting it all together we have,

$$\cot\left(\arccos\left(\frac{x-1}{x+1}\right)\right) = \frac{x-1}{2\sqrt{x}}.$$

The basic steps in this process are,

1. Set the inside term to an angle θ.
2. Use the properties of the inverse trigonometric functions to rewrite the expression as a trigonometric function of θ.
3. Using the expression from the previous step draw a right triangle and use the Pythagorean theorem to find the length of the missing side.
4. Using the right triangle from the previous step evaluate the outside trigonometric function.

8.8 Supplemental problems

1. True/False. The arctangent function is even. Justify your answer.
2. Show that $\arccos(x) = 90^\circ - \arcsin(x)$ if the arcsine function is returning angles in degrees or $\arccos(x) = (\pi/2) - \arcsin(x)$ if the arcsine function is returning angles in radians. *Hint*: Use the identity $\cos(90^\circ - \theta) = \sin(\theta)$.

Lecture 9

Working with trigonometric identities

In this lecture we will expand upon our trigonometric skills by learning how to manipulate and verify trigonometric identities.

9.1 What the equal sign means

In mathematics we often will use the '=' sign with two different meanings in mind. Namely, it is used to denote identities and conditional relationships.

An *identity* represents a relationship that is *always* true. We have seen several examples of this. For instance the Pythagorean identity, $\cos^2(\theta) + \sin^2(\theta) = 1$ is true for every value of θ and so is an identity.

A *conditional relationship* represents an equation that is *sometimes (possibly never)* true. We have also seen examples of this. For instance in the last lecture we found that the relationship $\cos(\theta) = 2/3$ is satisfied for some but not all θ.

So the '=' sign gets a lot of usage and you need to be careful to see whether it is being used to represent an identity or a conditional relationship. (Some mathematical zealots will use the '$\equiv$' sign to denote an identity, we shall not adopt this practice here.)

For now we will focus on identities and save looking at conditional relationships for later. The most important part of working with identities is being able to manipulate them, bend them to your will so to speak. To learn how to do this we will look at a variety of techniques from algebra.

9.2 Adding fractions

An important skill to have is the ability to add fractions correctly. To add fractions we first work to get a common denominator, and then add the numerators. The process is shown below.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

Note in particular that we do not add fractions by adding their numerators and denominators, that is $(a/b)+(c/d)\neq(a+c)/(b+d)$. This is a common mistake, almost any example shows that this does not work, one example is $(1/2)+(1/2)\neq(1/2)=(1+1)/(2+2)$.

Example 1 Simplify the following expression.

$$\frac{\sin(\theta)}{1+\cos(\theta)}+\frac{\cos(\theta)}{\sin(\theta)}$$

Solution First we will get a common denominator so we can add and then simplify whatever we have left.

$$\begin{aligned}
\frac{\sin(\theta)}{1+\cos(\theta)}+\frac{\cos(\theta)}{\sin(\theta)} &= \frac{\sin(\theta)\sin(\theta)}{(1+\cos(\theta))\sin(\theta)}+\frac{(1+\cos(\theta))\cos(\theta)}{(1+\cos(\theta))\sin(\theta)}\\
&= \frac{\sin(\theta)\sin(\theta)+\cos(\theta)(1+\cos(\theta))}{(1+\cos(\theta))\sin(\theta)}\\
&= \frac{\sin^2(\theta)+\cos^2(\theta)+\cos(\theta)}{(1+\cos(\theta))\sin(\theta)}\\
&= \frac{(1+\cos(\theta))}{(1+\cos(\theta))\sin(\theta)}=\frac{1}{\sin(\theta)}=\csc(\theta)
\end{aligned}$$

In this example in addition to adding we also used the Pythagorean identity and a reciprocal identity. Often throughout the process of simplifying expressions and verifying identities we will repeatedly use many of the identities that we have found up to this point.

Another thing to note is that in this process we cancelled terms in our numerator and denominator. This must be done with exceeding care and can only be done when both the term on the top and the term on the bottom multiply everything else. It is NOT true that $(a+b)/a=1+b$ or that $(ab+cb)/(b+d)=a+c$.

As a good practice you should always double check your cancellation. If you are in doubt as to whether or not you can cancel then you probably can't.

9.3 The conju-what? The conjugate

One very useful algebraic trick to use in simplifying some expressions is the *conjugate*. The conjugate basically means change the sign in the middle. So for example the conjugate of $1+\cos(\theta)$ is $1-\cos(\theta)$ (i.e. we changed the sign in the middle). This is useful because when multiplying conjugates the "cross terms" cancel, that is,

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2.$$

Use of the conjugate is particularly helpful in getting terms that have expressions like $1 \pm \cos(x)$ or $1 \pm \sin(x)$ in the denominator out of fractional form. This is because of the Pythagorean identities. For example,

$$(1-\cos(x))(1+\cos(x)) = 1 - \cos^2(x) = \sin^2(x).$$

Example 2 Rewrite the following expression so that it is not in fractional form.

$$\frac{1}{1+\sin(x)}$$

Solution We will start with the expression and multiply through both the top and the bottom by the conjugate. (We need to multiply both the top and the bottom so that the total value of the expression does not change.) Doing this, we get the following.

$$\begin{aligned}
\frac{1}{1+\sin(x)} &= \frac{1(1-\sin(x))}{(1+\sin(x))(1-\sin(x))} = \frac{1-\sin(x)}{1-\sin^2(x)} \\
&= \frac{1-\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} - \frac{\sin(x)}{\cos(x)}\frac{1}{\cos(x)} \\
&= \sec^2(x) - \tan(x)\sec(x)
\end{aligned}$$

Note that in this example we broke up the fraction into two pieces. This is no problem when we break up addition in the numerator, i.e. $(a+b)/c = (a/c)+(b/c)$, but does not work for terms in the denominator.

9.4 Dealing with square roots

Sometimes when dealing with expressions we will need to work with square roots. When doing so there are some important things to remember. The first is that the square root does not break up over addition, i.e. $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$, but does break up over multiplication, i.e. $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

The second is that the expression $\sqrt{x^2}$ does not always equal x, but rather $\sqrt{x^2} = |x|$. In other words, if you square a number and then take the square root you will be left with the absolute value of what you started with. You can drop the absolute value sign when you are certain that the value will be positive.

Example 3 Simplify so as to remove the square root in the following expression.

$$\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

Solution Note that in the denominator inside the square root that we have $1+\cos(\theta)$. This is a wonderful expression to use conjugates with. So starting by multiplying through by the conjugates, we will get the following.

$$\begin{aligned}
\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} &= \sqrt{\frac{(1-\cos(\theta))(1-\cos(\theta))}{(1+\cos(\theta))(1-\cos(\theta))}} = \sqrt{\frac{(1-\cos(\theta))^2}{1-\cos^2(\theta)}} \\
&= \sqrt{\frac{(1-\cos(\theta))^2}{\sin^2(\theta)}} = \frac{|1-\cos(\theta)|}{|\sin(\theta)|} = \frac{1-\cos(\theta)}{|\sin(\theta)|}
\end{aligned}$$

In the last step we can drop the absolute value sign on the term $1-\cos(\theta)$ because it will *always* be nonnegative, or in other words bigger than or equal to zero. But we cannot drop the absolute value sign on the term $\sin(\theta)$ because it can sometimes be negative.

9.5 Verifying trigonometric identities

Up to this point we have not been verifying identities but just putting tools in place to simplify expressions. Verifying an identity requires simplifying one expression to another expression.

When verifying identities the following guidelines are helpful to keep in mind.

- Work with one side at a time and manipulate it to the other side. The most straightforward way to do this is to simplify one side to the other directly, but we can always transform both sides to a common expression if we see no direct way to connect the two.

It is important not to mix the sides together. This is because we are trying to prove that the two sides are equal, but in order to mix sides you have to assume that they are equal. This is known as circular logic and should be avoided at all costs.

Similarly, you should not square both sides. Essentially, treat the two sides as completely separate *until* you have shown that they are equal.

- Humans are designed to make things less complicated (think of it as the second law of thermodynamics). So when picking what side to start with work with the most complicated side and simplify to the other side.
- Look for common terms that can be factored out and cancelled, look for fractions that can be added, ways to use the conjugate, ways to simplify square roots (but never introduce square roots), ways to use rules of algebra such as FOILing (i.e. $(a+b)(c+d) = ac + ad + bc + bd$), etc....
- When you are stuck try putting everything in terms of sines and cosines (this is possible because of the reciprocal and quotient identities). Sometimes expressions are easier to work with in this form.

Example 4 Verify the following identity.

$$\tan(\theta) + \cot(\theta) = \sec(\theta)\csc(\theta)$$

Solution Looking at this we seem to have no direct connection between the two sides. Let us start with the left hand side and put everything in terms of sine and cosine and see if something marvelous happens.

$$\begin{aligned}
\tan(\theta) + \cot(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} + \frac{\cos(\theta)}{\sin(\theta)} \\
&= \frac{\sin(\theta)\sin(\theta)}{\cos(\theta)\sin(\theta)} + \frac{\cos(\theta)\cos(\theta)}{\cos(\theta)\sin(\theta)} \\
&= \frac{\sin^2(\theta) + \cos^2(\theta)}{\cos(\theta)\sin(\theta)} = \frac{1}{\cos(\theta)\sin(\theta)} \\
&= \frac{1}{\cos(\theta)}\frac{1}{\sin(\theta)} = \sec(\theta)\csc(\theta)
\end{aligned}$$

Example 5 Verify the following identity.

$$\frac{\cos(\theta)\cot(\theta)}{1 - \sin(\theta)} - 1 = \csc(\theta)$$

Solution Between the left and the right the left looks more complicated. So we will start with the left and try to get the right.

$$\begin{aligned}
\frac{\cos(\theta)\cot(\theta)}{1-\sin(\theta)} - 1 &= \frac{\cos(\theta)(\cos(\theta)/\sin(\theta))}{1-\sin(\theta)} - 1 \\
&= \frac{\cos^2(\theta)}{\sin(\theta)(1-\sin(\theta)} - 1 \\
&= \left(\frac{\cos^2(\theta)}{\sin(\theta)(1-\sin(\theta))}\right)\left(\frac{1+\sin(\theta)}{1+\sin(\theta)}\right) - 1 \\
&= \frac{\cos^2(\theta)(1+\sin(\theta))}{\sin(\theta)(1-\sin^2(\theta))} - 1 \\
&= \frac{\cos^2(\theta)(1+\sin(\theta))}{\sin(\theta)\cos^2(\theta)} - 1 \\
&= \frac{1+\sin(\theta)}{\sin(\theta)} - 1 \\
&= \frac{1}{\sin(\theta)} + \frac{\sin(\theta)}{\sin(\theta)} - 1 \\
&= \frac{1}{\sin(\theta)} = \csc(\theta)
\end{aligned}$$

9.6 Supplemental problems

1. When verifying an identity you should not square both sides. To see why this does not work show that,

$$(\cos(\theta) - \sin(\theta))^2 = (\sin(\theta) - \cos(\theta))^2,$$

even though,

$$\cos(\theta) - \sin(\theta) \neq \sin(\theta) - \cos(\theta).$$

2. Simplify the following expression so as to remove the square root.

$$\sqrt{\frac{\sec(x) + \tan(x)}{\sec(x) - \tan(x)}}$$

Lecture 10

Solving conditional relationships

In this lecture we will work on solving conditional relationships.

10.1 Conditional relationships

A conditional relationship is an equation that is sometimes (possibly never) true. The important thing we will do with conditional relationships is solve for the angle(s) that makes the statement true.

The main technique that we will develop is taking our relationship and simplifying it down to the point where we have one (or several) equations where we have a trigonometric function being equal to a number and then using methods we have learned previously to actually solve for the angle(s).

10.2 Combine and conquer

If your expression has several terms of the same type combine them together. (Our goal as always is to make the equation as simple as possible in hopes that we can get a handle on it.)

Example 1 Solve for the acute angle that satisfies the following conditional relationship.

$$3\sin(x) = \sin(x) + 1$$

Solution Both sides of the conditional relationship have a $\sin(x)$ and so let us combine and see what we get.

$$3\sin(x) - \sin(x) = 1 \quad \text{or} \quad 2\sin(x) = 1 \quad \text{or} \quad \sin(x) = \frac{1}{2}$$

With this in hand we can now look up on a table and find that the acute angle that gives a value of $1/2$ with the sine function is $30°$ or $\pi/6$.

Sometimes we will have terms that we cannot combine together, but if we *group* all the terms on one side we can factor a common expression out. This is useful because if there are several terms multiplying together that give a value of zero then one of the terms must be zero. This is by no means surprising, but by many means useful.

Example 2 Solve for all the angles between $0°$ and $360°$ (or between 0 and 2π radians) such that the following conditional relationship is satisfied.

$$2\sin(x)\cos(x) = \sqrt{3}\cos(x)$$

Solution Since we cannot combine these terms let us group them all together and see what we can factor out. In particular we have,

$$2\sin(x)\cos(x) - \sqrt{3}\cos(x) = 0 \quad \text{or} \quad \cos(x)(2\sin(x) - \sqrt{3}) = 0.$$

We now have two terms multiplying together that give a value of 0 so our only solutions will be when one of the terms is 0. We can now break this up into two smaller problems. Namely, when does each term equal 0.

When does $\cos(x) = 0$? We can look up one value on a table and get $90°$ (or $\pi/2$). Since we are dealing with the cosine function on the unit circle we would go straight down and get our other solution of $270°$ (or $3\pi/2$).

When does $2\sin(x) - \sqrt{3} = 0$? First we can rearrange this equation and see that this is the same as asking when $\sin(x) = \sqrt{3}/2$. We can look up on a table and see that this will happen at $60°$ (or $\pi/3$). Since we are now dealing with the sine function on the unit circle we would go straight across and get our other solution as $120°$ (or $2\pi/3$).

Combining these two we will have that our solution is,

$$60°, 90°, 120°, 270° \qquad \text{or in radians} \qquad \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$$

One important thing to note in the previous example is that we did not begin by dividing both sides by $\cos(x)$. This seems like a quite logical and consistent thing to do, but there is a subtle reason that we cannot. When solving conditional relationships, we are looking at x over all possibilities and trying to determine which ones satisfy the relationship. When we divide both sides of the equation by $\cos(x)$ then when $\cos(x)$ is 0 we are dividing by 0 which is bad mathematics. So we have the following rule in solving conditional relationships: *You cannot divide to cancel terms if the term is ever zero.*

10.3 Use the identities

Sometimes grouping similar terms together and factoring will not cut the mustard. Then we start getting more sophisticated. We will sometimes need to turn to the identities to help us. The identities can be used in several ways, primary is their ability to simplify complex expressions that might be on one side of the equation. In particular they can be used in combining terms.

Example 3 Solve for all the angles for which the following conditional relationship is satisfied.

$$\sin(x) = \cos(x)$$

Solution In this example we cannot combine the terms since they are not similar and if we grouped the terms on one side, we would not be able to factor out any common expression. So after staring at the equation for some time we come up with a plan. Namely, all we have here is the sine and cosine function, and the tangent function is the sine over the cosine. So let us divide both sides by the term $\cos(x)$ (here it will be alright to divide because we are not cancelling terms). So we have the following,

$$\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)} \qquad \text{or} \qquad \tan(x) = 1.$$

Now we have gotten to our ideal situation, a function being equal to a number. Looking up on our chart we see that the tangent function is 1 when our angle is $45°$ (or $\pi/4$). The tangent function is nice because it is periodic with period of $180°$ or π and so our final solution is,

$$45° + k180° \text{ for } k = 0, \pm1, \pm2, \ldots \text{ or}$$
$$\pi/4 + k\pi \text{ for } k = 0, \pm1, \pm2, \ldots.$$

10.4 'The' square root

When simplifying some expressions we will take the square root. Now the word "the" in "the square root" is misleading. For any number that is not 0 there will always be *two* square roots. One has a positive value and the other has a negative value. This is often denoted by '$\pm$'.

So for example, if you have the expression $\tan^2(x) = 3$ and you take the square roots to simplify then you would get $\tan(x) = \pm\sqrt{3}$ which breaks up into the two equations $\tan(x) = \sqrt{3}$ and $\tan(x) = -\sqrt{3}$.

10.5 Squaring both sides

Sometimes we will have a relationship that we are unable to simplify using what we have learned to this point. One thing to try is squaring both sides of the formula. This can help because the Pythagorean identities involves trigonometric functions that are squared, and so while there is no easy way to replace $\sin(\theta)$ in terms of $\cos(\theta)$ there is an easy way to replace $\sin^2(\theta)$ in terms of $\cos(\theta)$. Before squaring it might be necessary to move some terms around to get the best results.

Unfortunately, this method has a large drawback in that in addition to producing correct solutions it can also produce "false" solutions. In other words you will get answers from this process that appear to be a solution but are actually not. So always check your answers after solving using this method. (In general it is a good idea to check your answers when solving any conditional relationship.)

Example 4 Find all of the angles between $0°$ and $360°$ (or between 0 and 2π radians) that satisfy the following conditional relationship.

$$\sin(x) - 1 = \cos(x)$$

Solution We seem extraordinarily stuck. And so let us try our new method of squaring both sides, which gives,

$$(\sin(x) - 1)^2 = (\cos(x))^2 \quad \text{or} \quad \sin^2(x) - 2\sin(x) + 1 = \cos^2(x).$$

Now looking at this we want to use the Pythagorean identities in a way that will simplify this expression. Since there is a sine term that is not squared it makes more sense to replace the squared cosine term then the squared sine term. So we get the following,

$$\begin{aligned} \sin^2(x) - 2\sin(x) + 1 &= 1 - \sin^2(x) \\ \text{or} \quad 2\sin^2(x) - 2\sin(x) &= 0 \\ \text{or} \quad 2\sin(x)(\sin(x) - 1) &= 0. \end{aligned}$$

This breaks down into two simpler equations that we can solve. Namely, when does $\sin(x) = 0$ and when does $\sin(x) = 1$.

We have that $\sin(x) = 0$ at the left-most and right-most points of the unit circle which are at $0°$ and $180°$ (or 0 and π radians). Also, $\sin(x) = 1$ at the top-most point of the unit circle which is at $90°$ (or $\pi/2$ radians). So now we have our list of answers but more appropriately we should call these "possible" answers. So we will now check them.

Test the angle	Which gives	Include?
$0°$ (0 *rads*)	$\sin(0) - 1 = \cos(0)$ or -1=1	No
$90°$ ($\pi/2$ *rads*)	$\sin(90°) - 1 = \cos(90°)$ or 0=0	Yes
$180°$ (π *rads*)	$\sin(180°) - 1 = \cos(180°)$ or -1=-1	Yes

So we had a false solution hanging around. Our final answer will thus be $90°$ and $180°$ (or $\pi/2$ and π radians).

10.6 Expanding the inside terms

So far we have been solving conditional relationships for terms which only have simple variables, such as x, inside. We can expand our methods so that we can have more interesting expressions inside, such as $(4x - 80°)$.

The process is best seen with an example. Afterwards, we will summarize the steps that we used.

Example 5 Find all of the angles x, between $0°$ and $360°$ that satisfy the following conditional relationship.

$$\sin(5x) = \frac{1}{2}$$

Solution First let us look at how the inside expression varies. Since we know that x goes from $0°$ to $360°$ then we have that $5x$ goes from $0°$ to $1800°$ (i.e. multiply the values by 5).

For a moment let us set $\theta = 5x$. Now let us consider the related problem of finding all of the angles θ, between $0°$ to $1800°$ that satisfy $\sin(\theta) = 1/2$. This is a much easier problem than the one that we started with. We already know that one such value for θ is $30°$. Since we are using the sine function to get our second value we would go straight across horizontally and get a second angle of $150°$. Once we have nailed down these two angles then we keep adding multiples of $360°$ until we get out of the range for θ. So we have that,

$$\theta = 30°, 150°, 390°, 510°, 750°, 870°, 1110°, 1230°, 1470°, 1590°.$$

Now we recall that $\theta = 5x$. So we can replace θ by $5x$ in the above expression and to solve for x we divide through by 5. So our final answer is,

$$x = 6^\circ, 30^\circ, 78^\circ, 102^\circ, 150^\circ, 174^\circ, 222^\circ, 246^\circ, 294^\circ, 318^\circ.$$

The basic steps in this process are,

1. Give the inside expression a name such as θ or whatever else is convenient. From the range given for our variable, find the corresponding range for θ.

2. Solve the related problem of when the conditional relationship is satisfied with θ inside over the range given in the previous step.

3. Replace θ by the inside expression that you started with and solve for the variable.

10.7 Supplemental problems

1. Solve for the unique acute angle θ that satisfies the following,

$$\theta = \arccos\left(\frac{1}{2}\sin(\theta)\right).$$

2. Find all the solutions between 90° and 270° to

$$\sin(5x - 80^\circ) = \frac{\sqrt{3}}{2}.$$

3. Find all the solutions between 0 and 2π (or if you prefer to work in degrees between 0° and 360°) to the following conditional relationship,

$$2\sin(2x)\cos(3x) = \cos(3x).$$

Lecture 11

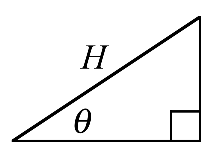
The sum and difference formulas

In this lecture we will learn how to work with terms such as $\sin(x+y)$. Along the way we will learn the useful tool of projection.

11.1 Projection

A proper discussion of projection must wait until later. For now we will use a very simple and straightforward version. Namely, given a hypotenuse of a right triangle and an acute angle we will find expressions for the lengths of the legs of the right triangle.

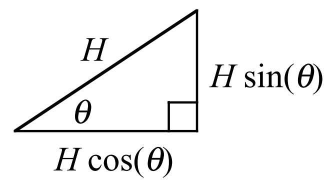
To find our formulas for projection consider the picture below where we know the length of the hypotenuse (which we will call H) and the acute angle θ.



Using the definition of trigonometric functions as ratios of right triangles we can find the length of the missing sides. So we have,

$$\sin(\theta) = \frac{opp}{H} \qquad \text{or} \qquad opp = H\sin(\theta),$$
$$\cos(\theta) = \frac{adj}{H} \qquad \text{or} \qquad adj = H\cos(\theta).$$

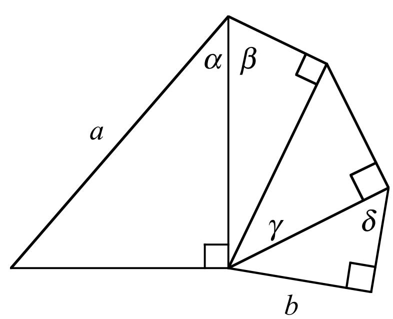
So knowing the length of the hypotenuse and an acute angle we can then fill in the lengths of the other sides of the triangle, as is shown at the top of the next page.



To see why we use the name projection, imagine standing directly over the triangle with a bright flashlight. If we point our flashlight straight down the hypotenuse will cast a shadow (i.e. project an image of itself) onto one of the legs, and the length of that shadow is the length of the leg.

By itself projection may not seem useful, but we can use projection over and over and over and....

Example 1 In the picture below find the length of the side b in terms of a and a combination of sines and/or cosines.



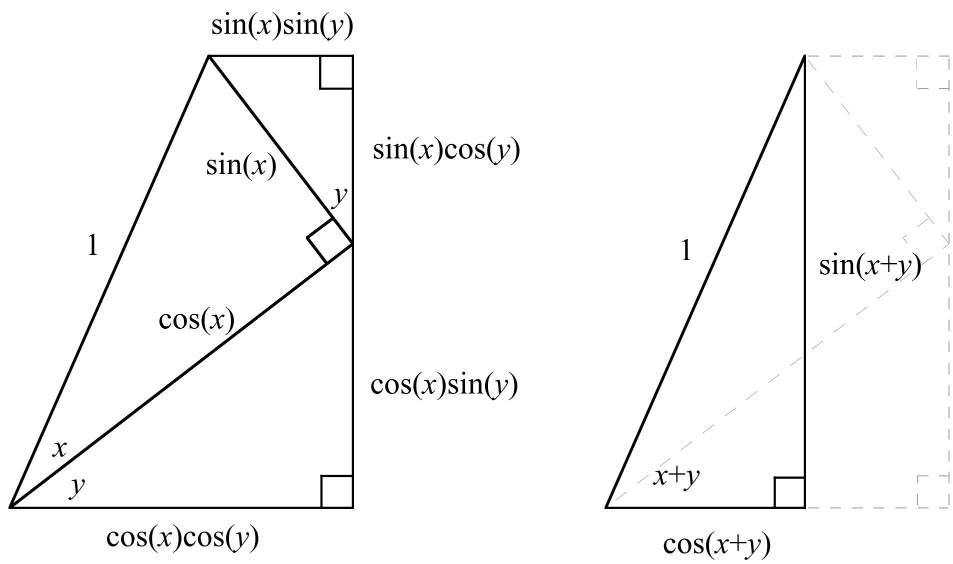
Solution We start with the length a which is a hypotenuse and then we keep projecting until we reach the side with length b. Doing so we will end up with,

$$b = a\cos(\alpha)\sin(\beta)\cos(\gamma)\sin(\delta).$$

11.2 Sum formulas for sine and cosine

With our new tool of projection in hand we will derive the sum formulas for the sine and cosine functions, that is we will find formulas for $\sin(x+y)$ and $\cos(x+y)$. To do this we will start with right triangles with angles x and y and then combine them together to form an angle $x+y$. Recall that scaling triangles will not change the value of the ratios and so we will scale our triangles so that they will fit together and so that the length of the longest side is 1. We will throw in one more triangle to flush out the picture.

Now we will use projection on this shape and fill in the lengths of all of the sides. At the same time we will construct another right triangle in the diagram with an angle of $x + y$ and a hypotenuse of 1 and use projection on that triangle.



We can use these two diagrams to compute the same lengths in two different ways. Doing so, we get the following formulas,

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y), \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y). \end{aligned}$$

Example 2 Use the sum formula for the cosine function to find the exact value for $\cos(75°)$.

Solution Since $75° = 45° + 30°$ we have,

$$\begin{aligned} \cos(75°) &= \cos(45° + 30°) \\ &= \cos(45°)\cos(30°) - \sin(45°)\sin(30°) \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}. \end{aligned}$$

11.3 Difference formulas for sine and cosine

To get the difference formulas for the sine and cosine we can either go through another process or we can modify what we already have. It is usually easier to modify what we already have.

So to get our difference formulas we will use our sum formulas and the even/odd'er identities (i.e. $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$). Combining these we get the following,

$$\begin{aligned}\sin(x-y) &= \sin(x+(-y)) = \sin(x)\cos(-y) + \cos(x)\sin(-y) \\ &= \sin(x)\cos(y) - \cos(x)\sin(y),\end{aligned}$$

$$\begin{aligned}\cos(x-y) &= \cos(x+(-y)) = \cos(x)\cos(-y) - \sin(x)\sin(-y) \\ &= \cos(x)\cos(y) + \sin(x)\sin(y).\end{aligned}$$

We can use the difference formulas in the same way that we can use the sum formulas.

11.4 Sum and difference formulas for tangent

To find the sum and difference formulas for the tangent function we can combine the results of the sum and difference formulas for the sine and cosine function (recall that the tangent function is the sine function over the cosine function). Putting these together, we get the following.

$$\begin{aligned}\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)} \\ &= \frac{\left(\frac{\sin(x)\cos(y)}{\cos(x)\cos(y)}\right) + \left(\frac{\cos(x)\sin(y)}{\cos(x)\cos(y)}\right)}{\left(\frac{\cos(x)\cos(y)}{\cos(x)\cos(y)}\right) - \left(\frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}\right)} = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\end{aligned}$$

We can get our difference formula in the same way, but let us instead modify what we already have. Using the sum function for the tangent function along with the fact that tangent is an odd function we get,

$$\begin{aligned}\tan(x-y) &= \tan(x+(-y)) = \frac{\tan(x)+\tan(-y)}{1-\tan(x)\tan(-y)} \\ &= \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}.\end{aligned}$$

Example 3 Verify the following,

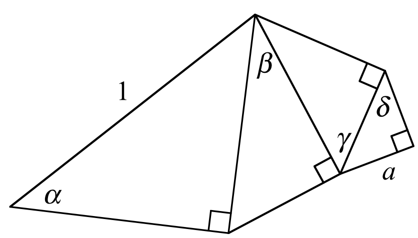
$$\tan\left(\theta+\frac{\pi}{4}\right)\cdot\tan\left(\theta-\frac{\pi}{4}\right) = -1$$

Solution We use the sum and difference formulas on the right hand side then substitute in 1 wherever we have $\tan(\pi/4)$.

$$\begin{aligned}\tan\left(\theta+\frac{\pi}{4}\right)\cdot\tan\left(\theta-\frac{\pi}{4}\right) &= \left(\frac{\tan(\theta)+1}{1-\tan(\theta)}\right)\cdot\left(\frac{\tan(\theta)-1}{1+\tan(\theta)}\right)\\ &= \frac{\tan(\theta)-1}{1-\tan(\theta)} = -1\end{aligned}$$

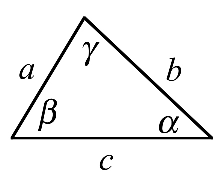
11.5 Supplemental problems

1. In the figure below express the length a in terms of a product of sines and/or cosines of various angles.



2. In the triangle below show that,

$$c = a\cos(\beta) + b\cos(\alpha)$$



3. Use difference formulas to prove the following,

$$\begin{aligned}\cos(90^\circ-\theta) &= \sin(\theta),\\ \sin(90^\circ-\theta) &= \cos(\theta).\end{aligned}$$

4. Can you use the difference formula for the tangent function to prove that $\tan(90^\circ-\theta)=\cot(\theta)$? Explain.

5. Prove
$$\cot(x+y) = \frac{\cot(x)\cot(y) - 1}{\cot(x) + \cot(y)}$$
Hint: you might want to use a technique similar to what we used for the tangent function.

6. Show that,
$$\cos(x+y)\cos(2y) + \sin(x+y)\sin(2y) = \cos(x+y)\cos(2x) + \sin(x+y)\sin(2x).$$

7. Using the sum and difference formulas derive the Pythagorean identity, namely show $\cos^2(x) + \sin^2(x) = 1$.

8. Given that $\tan(a) = 1/5$, $\tan(b) = 1/8$, $\tan(c) = 2$ and $\tan(d) = 1$, which is larger, $\tan(a+b)$ or $\tan(c-d)$? Justify your answer *without* using a calculator.

Lecture 12

Heron's formula

In this lecture we will develop another way to find the area of a triangle. Namely Heron's formula, which gives the area of a triangle given the length of all three sides.

12.1 The area of triangles

At this point we have two formulas for finding the area of triangles, namely (1/2)(base)(height) and $(1/2)ab\sin(\gamma)$. Let us now produce a third.

Recall that triangles are rigid. One consequence of this is that there is at most only one triangle with three sides of given lengths. Since there is only one triangle, there will only be one area associated with a triangle with the given lengths of the sides. So it seems reasonable that knowing only the lengths of the sides of a triangle that we should be able to find the area of a triangle.

It turns out that finding the area with the length of sides is possible and is done by Heron's formula. This formula has been known for quite some time, Heron himself lived thousands of years ago, and as is the way of mathematics a large number of proofs have emerged for it. We will present a geometrical proof that involves some trigonometry. Do not worry about reproducing the proof, rather look for the ideas and connections that are used.

12.2 The plan

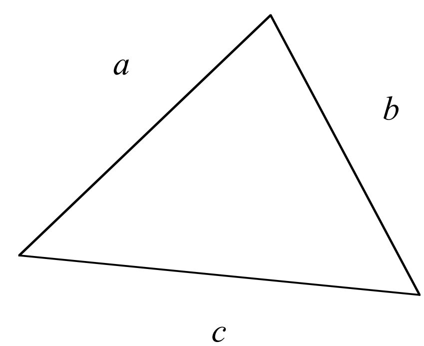
With any problem that we have we need to start with a plan. To do this we look at our goal, which is to find the area of a triangle, and think about how to get there. We already know some ways to find the area of a triangle, so let us try to use them. We know for instance that the area is half of the base times the height. Unfortunately, we don't have a way to directly apply this relationship yet.

But what if we were to break the triangle up? Aha. Let us break up the triangle into a collection of smaller triangles that we can easily figure out the areas for. Right triangles would be best if we can find a way to do it, since with right triangles we have easy ways to compute area. Then we could just add the areas together and get it all back in terms of the original lengths of the sides.

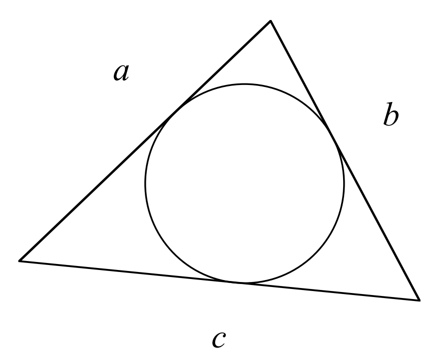
So our plan is to break up a triangle into smaller (hopefully right) triangles. Then add up the areas of these triangles and put everything back in terms of the lengths of the sides.

12.3 Breaking up is easy to do

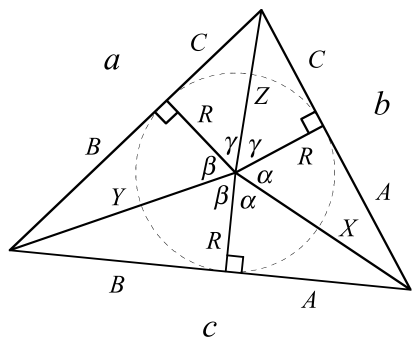
Start with any triangle, for example, such as the one shown below.



A fun and useful fact from geometry is that in any such triangle we can inscribe a circle (i.e. put a circle inside the triangle so that it just touches the edges). This will produce a picture such as is shown below.



Now from the center of this circle we will draw lines to each vertex of the triangle and to each side where the circle just touches the side. At the points where the side and the circle just touch this will form a right angle and so we will have right triangles. In fact we will have a total of six right triangles. To help us work with these triangles we label everything we can. Doing this we get the picture below.



12.4 The little ones

With our picture in place it is quite quick to add up the area of all these little triangles. Since they are all right triangles we can use some simple formulas and we will get that the total area is,

$$\text{area} = 2 \cdot \frac{1}{2}AR + 2 \cdot \frac{1}{2}BR + 2 \cdot \frac{1}{2}CR = (A + B + C)R.$$

We would be done except that we do not know what A, B, C and R are. What we do know is the lengths of the sides of the big triangle, i.e. a, b and c. We need to find a way to rewrite A, B, C and R as expressions of a, b and c.

12.5 Rewriting our terms

From the triangle we get the relationships

$$A + C = b, \qquad A + B = c, \qquad B + C = a.$$

Now if this were an algebra class we would take some time to take these three equations and solve for A, B and C in terms of a, b and c. But we are here to learn about trigonometry and so we will jump to the end and get the following,

$$A = \frac{1}{2}(-a + b + c), \qquad B = \frac{1}{2}(a - b + c), \qquad C = \frac{1}{2}(a + b - c).$$

If you feel uneasy about this last step, please double check that the calculations are correct.

We have three down and one term left to go. Solving for R is by far the most interesting step in this proof and we will have to build up to it. First, from the right triangles that we formed we can find the values for the trigonometric functions for the angles formed at the center, i.e. for α, β and γ. Doing so we get,

$$\sin(\alpha) = \frac{A}{X}, \qquad \sin(\beta) = \frac{B}{Y}, \qquad \sin(\gamma) = \frac{C}{Z},$$

$$\cos(\alpha) = \frac{R}{X}, \qquad \cos(\beta) = \frac{R}{Y}, \qquad \cos(\gamma) = \frac{R}{Z}.$$

Now we will pull the rabbit out of the hat, and do it using trigonometry. So using the sum formulas from last time and substituting in the values we just found at the appropriate time we will get,

$$\begin{aligned} 0 &= \sin(180^\circ) = \sin(\alpha+\beta+\gamma) = \sin((\alpha+\beta)+\gamma) \\ &= \sin(\alpha+\beta)\cos(\gamma) + \sin(\gamma)\cos(\alpha+\beta) \\ &= \sin(\alpha)\cos(\beta)\cos(\gamma) + \cos(\alpha)\sin(\beta)\cos(\gamma) \\ &\qquad + \cos(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\sin(\beta)\sin(\gamma) \\ &= \frac{A}{X}\frac{R}{Y}\frac{R}{Z} + \frac{R}{X}\frac{B}{Y}\frac{R}{Z} + \frac{R}{X}\frac{R}{Y}\frac{C}{Z} - \frac{A}{X}\frac{B}{Y}\frac{C}{Z} \\ &= \frac{1}{XYZ}(R^2(A+B+C) - ABC). \end{aligned}$$

From this it follows that,

$$0 = R^2(A+B+C) - ABC \quad \text{or} \quad R^2 = \frac{ABC}{A+B+C} \quad \text{or} \quad R = \sqrt{\frac{ABC}{A+B+C}}.$$

12.6 All together

Before we finish up Heron's formula we need to add some small details. First note that if we add up the outside edge of the big triangle in two ways we get,

$$a+b+c = 2A+2B+2C \quad \text{or} \quad A+B+C = \frac{1}{2}(a+b+c) = s,$$

(i.e. we will define this new term s as half the value of the total distance around the triangle (or half the "circumference" of the triangle)).

With this new term defined we then get that,

$$\begin{array}{rcccccc} A &=& \frac{1}{2}(-a+b+c) &=& \frac{1}{2}(a+b+c)-a &=& s-a, \\ B &=& \frac{1}{2}(a-b+c) &=& \frac{1}{2}(a+b+c)-b &=& s-b, \\ C &=& \frac{1}{2}(a+b-c) &=& \frac{1}{2}(a+b+c)-c &=& s-c. \end{array}$$

Let us now with one deft stroke finish this business.

$$\begin{aligned} \text{area} &= (A+B+C)R = (A+B+C)\sqrt{\frac{ABC}{A+B+C}} \\ &= \sqrt{(A+B+C)ABC} = \sqrt{s(s-a)(s-b)(s-c)}, \end{aligned}$$

where $s = (a+b+c)/2$.

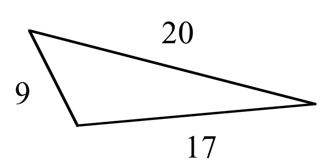
12.7 Heron's formula, properly stated

We have now derived Heron's formula. Mathematically, it would be presented as follows.

Given a triangle with sides of length a, b and c, then the area enclosed by the triangle is given by,

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{1}{2}(a+b+c).$$

Example 1 Find the area of the triangle below.



Solution Since we know the lengths of the sides of this triangle we can use Heron's formula to find the total area. First, solving for s we have $s = (9+17+20)/2 = 23$. And so we get,

$$\text{area} = \sqrt{23(23-9)(23-17)(23-20)} = \sqrt{5796} \approx 76.13.$$

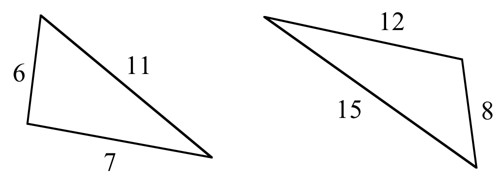
12.8 Supplemental problems

1. Express $\cos(x + y + z)$ in terms of $\cos(x)$, $\cos(y)$, $\cos(z)$, $\sin(x)$, $\sin(y)$ and $\sin(z)$. *Hint*: use a technique similar to what we did for $\sin(x + y + z)$.

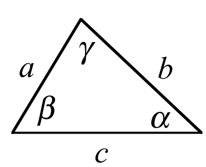
2. Express $\cos(3x)$ in terms involving only $\cos(x)$. *Hint*: $3x = x + x + x$.

3. Show that $x = \cos(20°)$ is one solution of the relationship $8x^3 - 6x - 1 = 0$. *Hint*: put the value $x = 20°$ into your answer to the previous problem and simplify.

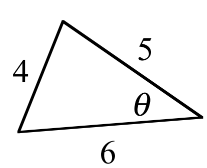
4. Use Heron's formula to find the area of the triangles shown below. Round your answers to two decimal places.



5. When we have more then one way to compute the same value we can often combine the two together to derive useful information. For example, using formulas for area show that the angle γ in the picture below satisfies the relationship,
$$\sin(\gamma) = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ab},$$
where $s = (a + b + c)/2$.



Use this to find the angle γ in the picture below.



Lecture 13

Double angle identity and such

In this lecture we will explore applications of the sum formulas. In particular we will derive a number of new identities, namely the double angle identity, power reduction identity and the half angle identity.

13.1 Double angle identities

After mathematicians find one relationship they then will go and examine all of the consequences that come from it. So now let us take our sum and difference formula and examine some of the consequences that come from them.

One of the easiest consequences of the sum and difference formulas are the double angle identities. As the name implies a double angle is twice the original angle, which can be found by adding the angle to itself. And so we get the following,

$$\begin{aligned} \sin(2x) &= \sin(x+x) \\ &= \sin(x)\cos(x) + \cos(x)\sin(x) \\ &= 2\sin(x)\cos(x) \end{aligned}$$

$$\begin{aligned} \cos(2x) &= \cos(x+x) \\ &= \cos(x)\cos(x) - \sin(x)\sin(x) \\ &= \cos^2(x) - \sin^2(x) \end{aligned}$$

Example 1 Given that $\sin(\theta) = 3/5$ and that $\cos(\theta) = 4/5$ find $\sin(2\theta)$ and $\cos(2\theta)$.

Solution Since we know the sine and cosine values of the angle we can apply the double angle formulas from above and get,

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta) = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25},\\ \cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}.\end{aligned}$$

An amazing thing about these identities is how much information we can get without actually knowing what the angle θ is.

Starting with the double angle identity for the cosine function we can use the Pythagorean identity to rewrite it in different ways. Namely, we can have the following,

$$\begin{aligned}\cos(2x) &= \cos^2(x)-\sin^2(x)\\ &= \cos^2(x)-(1-\cos^2(x))\\ &= 2\cos^2(x)-1,\end{aligned}$$

$$\begin{aligned}\cos(2x) &= \cos^2(x)-\sin^2(x)\\ &= (1-\sin^2(x))-\sin^2(x)\\ &= 1-2\sin^2(x).\end{aligned}$$

13.2 Power reduction identities

Starting with these last two forms for $\cos(2x)$ we can manipulate and solve for the terms $\cos^2(x)$ and $\sin^2(x)$.

$$\begin{array}{lcl}\cos(2x) = 2\cos^2(x)-1 & \text{so} & \cos^2(x) = \dfrac{1+\cos(2x)}{2}\\ \cos(2x) = 1-2\sin^2(x) & \text{so} & \sin^2(x) = \dfrac{1-\cos(2x)}{2}\end{array}$$

These are called the power reduction identities since we start with the term on the left hand side with a square power and the terms on the right side do not have square power terms in them (i.e. we reduced the highest power term by one).

We can use these formulas multiple times (sometimes in conjunction with other identities) to reduce expressions with powers higher than degree two.

Example 2 Rewrite $\sin^4(x)$ to an expression that does not have any terms with a power greater then one or two different trigonometric functions multiplied together.

Solution We can use power reduction again and again, doing so we will get,

$$\begin{aligned}
\sin^4(x) &= \sin^2(x)\sin^2(x) = \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right) \\
&= \frac{1-2\cos(2x)+\cos^2(2x)}{4} \\
&= \frac{1-2\cos(2x)+((1+\cos(4x))/2)}{4} \\
&= \frac{3-4\cos(2x)+\cos(4x)}{8}.
\end{aligned}$$

13.3 Half angle identities

Starting with the power reduction identities we can simultaneously take the square root of both sides and replace all of the x terms with $x/2$. Doing so we will get the following (remember that when taking square roots there are two possibilities and so we need to add the "$\pm$" sign).

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}} \qquad \text{and} \qquad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

These equations allow us to find the value of the sine and cosine of half the angle if we already know the value of the cosine function of the original angle. The '$\pm$' sign is handled by determining in which quadrant the angle $x/2$ lies, and then using the appropriate signs for the functions.

Example 3 Find the exact value of $\sin(\theta/2)$ and $\cos(\theta/2)$ given that $\cos(\theta) = -1/8$ and that $3\pi < \theta < 7\pi/2$.

Solution Before we start throwing out our formulas, we need to first determine where the angle $\theta/2$ lies. We already know the range for θ and so starting with this relationship and dividing through by 2 we get $3\pi/2 < \theta/2 < 7\pi/4$, and so the angle $\theta/2$ lies in the fourth quadrant. So we know that the $\sin(\theta/2)$ will be negative and that the $\cos(\theta/2)$ will be positive. Now we can proceed, and we get the following,

$$\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{1-(-1/8)}{2}} = -\frac{3}{4}, \\
\cos\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1+(-1/8)}{2}} = \frac{\sqrt{7}}{4}.
\end{aligned}$$

To find a half angle identity for the tangent function we can do a similar procedure, or we can use a combination of the double angle and power reduction identities. If we do the later we get,

$$\begin{aligned}
\tan\left(\frac{x}{2}\right) &= \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sin^2(x/2)}{\cos(x/2)\sin(x/2)} = \frac{(1-\cos(x))/2}{\sin(x)/2} = \frac{1-\cos(x)}{\sin(x)}, \text{ or} \\
\tan\left(\frac{x}{2}\right) &= \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sin(x/2)\cos(x/2)}{\cos^2(x/2)} = \frac{\sin(x)/2}{(1+\cos(x))/2} = \frac{\sin(x)}{1+\cos(x)}.
\end{aligned}$$

13.4 Supplemental problems

1. In this problem we will compute the exact value of $\sin(18°)$. To do this complete the following steps.

   (i) Show that if $\alpha = 18°$ then $\sin(2\alpha) = \cos(3\alpha)$.

   (ii) Write the expression for (i) in terms of $\cos(\alpha)$'s and $\sin(\alpha)$'s. All the terms have a common factor that can be cancelled. Write everything that is left in terms of $\sin(\alpha)$'s.

   (iii) Moving everything over to one side you should now have a quadratic in terms of sines. That is you should have an equation of the form $a\sin^2(\alpha) + b\sin(\alpha) + c = 0$ for some values a, b and c. Now use the quadratic equation to solve for $\sin(\alpha)$, namely you will have,

   $$\sin(\alpha) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   One of the values is negative, but $\sin(18°) > 0$ and so we can throw that out, and thus we get our final answer.

   Hint: you can check your answer by computing $\sin(18°)$ and your final answer with a calculator and checking that they are equal.

2. Express $\sin(4x)$ in terms of $\sin(s)$'s and $\cos(x)$'s. *Hint*: try using the double angle identities.

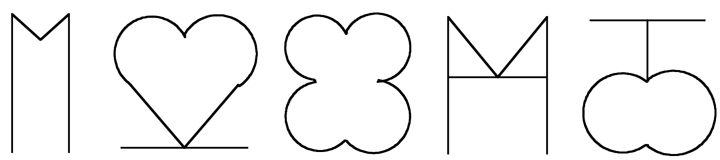
3. Find the exact value of the following.

   $$\sin\left(\frac{\pi}{9}\right)\cos\left(\frac{\pi}{36}\right) - \cos\left(\frac{\pi}{9}\right)\sin\left(\frac{\pi}{36}\right)$$

   Hint: this takes more then one step.

4. In mathematics there are two important skills. The one most people think of is the ability to answer questions. But just as important, if not more so, is the ability to ask the right questions. In mathematics the way that we look for questions to ask is to look for patterns, things that seem to follow a behavior. Once a pattern is recognized often a good question can be asked.

   (a) As a warmup exercise look at the symbols below. Identify what the next symbol in the sequence should look like. To do it you have to identify the pattern.

   (b) Now we will look for a pattern involving trigonometry. Starting with $\cos(\pi/4) = \sqrt{2}/2$ use the half angle identities to find $\cos(\pi/8)$, $\cos(\pi/16)$ and $\cos(\pi/32)$. *Hint*: all these angles are in the first quadrant so we always take the '+' version of the formula for the half-angle identity, also simplify the expression at each stage as much as possible before moving to the next stage.

   (c) Look at the answers for part (b). There should be a pattern emerging. Looking at that pattern what would you expect the following to be,

   $$\cos\left(\frac{\pi}{2^{(n+1)}}\right) = \quad ?$$

   Note that for $n = 1$ we have that this is $\cos(\pi/4) = \sqrt{2}/2$. (You do not need to actually prove that it actually looks like what you say, we will save that for another day.)

5. Verify the following for $-1 \le x \le 1$,

   $$2\arccos\left(\sqrt{\frac{x+1}{2}}\right) = \arccos(x).$$

   (This is a non-trivial problem. You might want to consider the double angle identity for cosine, namely $\cos(2\theta) = 2\cos^2(\theta) - 1$.)

6. Verify the double angle formula for cotangent, namely,

   $$\cot(2x) = \frac{1}{2}\cot(x) - \frac{1}{2}\tan(x).$$

7. (a) We have the double angle identity for sine and cosine. Find the double angle identity for tangent. In other words, express

$$\tan(2x) = (\text{stuff involving } \tan(x)).$$

   (b) Given that $\tan(x) = 2/5$ and $\sin(x) < 0$ find the exact value for $\tan(2x)$.

8. Given $\cos(\theta) = -7/32$ and $5\pi/2 < \theta < 3\pi$ find the exact value for $\sin(\theta/4)$. *Hint*: as an intermediate step you may want to find the exact value for $\cos(\theta/2)$.

9. Show that,

$$\frac{\cos(x) + \sin(x)}{\cos(x) - \sin(x)} = \sec(2x) + \tan(2x).$$

10. Show that,

$$\sec^2(\theta) = 2 - \frac{2}{1 + \sec(2\theta)}.$$

11. Rewrite the following as an algebraic expression.

$$\tan\left(\frac{1}{2}\arctan(x)\right)$$

Lecture 14

Product to sum and vice versa

In this lecture we will continue examining consequences of the sum and difference formulas. In particular we will derive the sum to product, product to sum and identity with no name.

14.1 Product to sum identities

The name of this identity tells what it does. Namely we will take a product (i.e. two trigonometric functions multiplying together) and rewrite it as a sum (i.e. two trigonometric functions adding together). This is possible because the sum and difference formulas for the cosine function (and similarly for the sine) look amazingly like each other except for the sign in the middle. In particular, when we combine them together we get cancellation.

$$\begin{aligned}\cos(x+y)+\cos(x-y) &= (\cos(x)\cos(y)-\sin(x)\sin(y))+ \\ &\quad\ (\cos(x)\cos(y)+\sin(x)\sin(y)) \\ &= 2\cos(x)\cos(y)\end{aligned}$$

If we take this last equation and divide both sides by 2 we get,

$$\cos(x)\cos(y)=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$$

In a similar fashion we can get the other product to sum identities, namely,

$$\begin{aligned}\sin(x)\sin(y) &= \frac{1}{2}[\cos(x-y)-\cos(x+y)], \\ \sin(x)\cos(y) &= \frac{1}{2}[\sin(x+y)+\sin(x-y)], \\ \cos(x)\sin(y) &= \frac{1}{2}[\sin(x+y)-\sin(x-y)].\end{aligned}$$

Example 1 Find the exact value of $\sin(52.5°)\cos(7.5°)$.

Solution We do not have the exact values for either of these angles and it might take us some time to find them. So let us simplify by using the product to sum identities and see what happens.

$$\begin{aligned}\sin(52.5°)\cos(7.5°) &= \frac{1}{2}[\sin(52.5° + 7.5°) + \sin(52.5° - 7.5°)]\\ &= \frac{1}{2}[\sin(60°) + \sin(45°)]\\ &= \frac{\sqrt{3} + \sqrt{2}}{4}\end{aligned}$$

Example 2 Write the expression $4\cos(3x)\cos(5x)$ as a sum of two trigonometric functions.

Solution By a straightforward application of product to sum we get,

$$\begin{aligned}4\cos(3x)\cos(5x) &= 4\frac{1}{2}[\cos(3x + 5x) + \cos(3x - 5x)]\\ &= 2\cos(8x) + 2\cos(2x)\end{aligned}$$

14.2 Sum to product identities

These identities do the opposite of the product to sum. Now we will start with a sum and rewrite the expression as a product.

To derive these identities we will start with the product to sum identities and use substitution. First, note that the variable names are completely arbitrary and we could use any names for our variables that we choose. So let us choose to use new names, say u and v. Then we know from the product to sum identity that,

$$\cos(u)\cos(v) = \tfrac{1}{2}[\cos(u + v) + \cos(u - v)] \quad \text{or}$$
$$\cos(u + v) + \cos(u - v) = 2\cos(u)\cos(v).$$

And now for any arbitrary x and y let $u = (x + y)/2$ and $v = (x - y)/2$. The purpose of this is not immediately apparent, but notice the following,

$$\begin{aligned}u + v &= \frac{x + y}{2} + \frac{x - y}{2} = \frac{2x}{2} = x,\\ u - v &= \frac{x + y}{2} - \frac{x - y}{2} = \frac{2y}{2} = y.\end{aligned}$$

If we substitute in these values of u and v into our equation we have,

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right).$$

We can repeat a similar procedure with the other product to sum identities and get the rest of the sum to product identities. Namely, these are,

$$\begin{aligned}\cos(x)-\cos(y) &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),\\ \sin(x)+\sin(y) &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),\\ \sin(x)-\sin(y) &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).\end{aligned}$$

Example 3 Find all of the zeroes between $0°$ and $360°$ to the equation

$$\sin(3x)-\sin(x)=0.$$

Solution First we will use the sum to product identity to rewrite the problem. In particular this is the same as solving,

$$2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)=2\cos(2x)\sin(x)=0.$$

So our problem breaks into two parts, when does $\cos(2x)=0$ and when does $\sin(x)=0$. Since x goes between $0°$ and $360°$, $2x$ will go between $0°$ and $720°$ and cosine is zero at the top and bottom of the unit circle, and so we get,

$$2x=90°,270°,450°,630° \quad \text{or} \quad x=45°,135°,225°,315°.$$

The function $\sin(x)$ is 0 at the left and right ends of the unit circle and so that will contribute solutions of $0°$ and $180°$. So our final answer is,

$$x=0°,45°,135°,180°,225°,315°.$$

14.3 The identity with no name

We will explore one last identity that unfortunately does not have a common name, but one which is still useful and we will need soon.

Before we can get to the identity we need to build up some ideas necessary for its implementation. Let us start with any pair of numbers a and b where at least one (and usually both) are not zero and consider the point in the plane,

$$\left(\frac{a}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}}\right).$$

Such a point satisfies the relationship $x^2 + y^2 = 1$ (this can easily be verified). But the points that satisfy $x^2+y^2 = 1$ are not just any ordinary points, these are points on the unit circle. When we were defining the trigonometric functions we said that every angle is associated with a point on the unit circle and that every point on the unit circle is associated with an angle (infinitely many angles actually).

In particular, there is a unique angle between $0°$ and $360°$, call it θ, such that,

$$\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}.$$

With this idea in place we now have enough to derive the identity with no name.

$$\begin{aligned} a\sin(x) + b\cos(x) &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin(x) + \frac{b}{\sqrt{a^2+b^2}}\cos(x)\right) \\ &= \sqrt{a^2+b^2}(\cos(\theta)\sin(x) + \sin(\theta)\cos(x)) \\ &= \sqrt{a^2+b^2}\sin(x+\theta) \end{aligned}$$

All that remains is to determine the angle θ. The actual angle can only lie in one of two places, namely either in quadrants I and II or quadrants III and IV. So we will break up our solution for the angle θ into two possibilities. These are given as follows,

$$\theta = \begin{cases} \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) & \text{if } b \geq 0 \\ 360° - \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) & \text{if } b < 0 \end{cases}$$

Example 4 Use the identity with no name to rewrite the following expression as a single sine function. Round the answer for θ to two decimal places.

$$12\sin(3\heartsuit) - 5\cos(3\heartsuit)$$

Solution From the identity with no name we know that the expression can be rewritten in the form,

$$\sqrt{12^2+5^2}\sin(3\heartsuit + \theta) = 13\sin(3\heartsuit + \theta).$$

All that remains is to find the angle θ. In this expression we have that $b = -5 < 0$ and so we will use the second method to find θ and we will get,

$$\theta = 360° - \arccos\left(\frac{12}{13}\right) \approx 337.38°,$$

so we can rewrite the expression as,

$$13\sin(3\heartsuit + 337.38°).$$

14.4 Supplemental problems

1. Express $\cos^6(x)$ as a sum of cosines added together, that is your final answer can have no cosine terms multiplied together. *Hint*: write $\cos^6(x)$ as a product of $\cos^2(x)$'s, use power reduction on each term and then FOIL and touch up what's left with various identities.

2. Find all solutions between 0 and 2π of the equation $\cos(4x) + \cos(2x) = 0$.

3. Find the exact value of $\sin(75°) + \sin(15°)$ by using a sum to product relationship.

4. Rewrite $\sin(x) + \sin(2x) + \sin(3x) + \sin(4x)$ as a *product* involving trigonometric functions. Your final answer should not have any sums or differences. *Hint*: there are several ways to start, but any way that you start should involve first grouping the terms into two parts, work on each part, factor out what is common and then keep going.

5. Write $4\sin(x)\sin(2x)\cos(3x)$ as a sum of trigonometric functions, your final answer should not have any trigonometric functions multiplying together.

6. Using the identity with no name rewrite the expression below in terms of a single sine function. Round your value for θ to two decimal places.

$$-3\sin(2x) + 4\cos(2x)$$

7. Given that $y = 3\sin(x) - \sqrt{7}\cos(x)$ find the largest value that y can achieve. Also find a value for x where y achieves this maximum value (round your answer to two decimal places). *Hint*: if we have $y = a\sin(\heartsuit)$ then the largest value y can achieve is a and it does so when $\heartsuit = 90°$.

Lecture 15

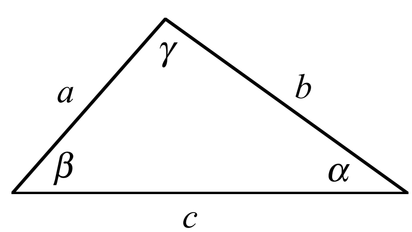
Law of sines and cosines

In this lecture we will introduce the law of sines and cosines which will allow us to explore oblique triangles.

15.1 Our day of liberty

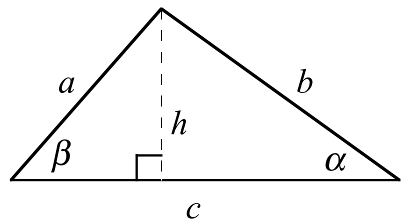
We can now free ourselves from using only right triangles and be able to work with all sorts of triangles. We will do it by introducing the law of sines and the law of cosines. Our derivation of these laws will be through use of right triangles, but these laws will let us put the right triangles in the background once proved.

For our notation in the lecture we will let a, b and c will represent the length of the sides of a triangle while the quantities α, β and γ will represent the measure of the corresponding angles. Namely, they will match up according to the picture below.



15.2 The law of sines

For this law, start with any arbitrary triangle and from one of the vertexes draw a line straight down to the base. This will split the triangle up into two smaller right triangles, such as is shown below,



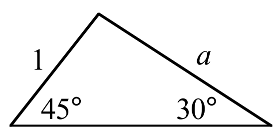
We can calculate the value of h in two different ways (once for each right triangle that it is attached to). This is done by using projection and we get that,

$$a\sin(\beta) = h = b\sin(\alpha) \qquad \text{or} \qquad \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)}.$$

We can repeat this whole process by drawing a line from one of the other vertexes down to its corresponding base. Combining that result with what we already have we end up with the following.

$$\textbf{Law of sines} \qquad \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

Example 1 In the triangle below find the length of the side a.

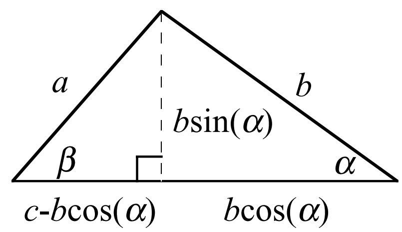


Solution We hope to use the law of sines, but before we start we need to make sure that we *can* use the law of sines. To be able to use the law of sines there needs to be two sides and two angles (with the angles opposite the sides) involved. Further, we need to know at least three of these four (that way we can actually solve for the fourth). So examining this triangle we see that we do know two angles and one side and want to know the fourth side. So we *can* use the law of sines and doing so we get the following,

$$\frac{a}{\sin(45°)} = \frac{1}{\sin(30°)} \qquad \text{so} \qquad a = \frac{\sin(45°)}{\sin(30°)} = \sqrt{2}.$$

15.3 The law of cosines

Again we start with our arbitrary triangle and again we draw a line from the vertex down to the base. We now have a triangle similar as to what we had before. So



we use projection on the small triangle on the right hand side to label all of the lengths of the triangle, doing so we will get the picture above.

Since we have right triangles we can use the Pythagorean theorem on these triangles. In particular, we can use the Pythagorean theorem on the small triangle on the left hand side and we get the following.

$$\begin{aligned} a^2 &= (b\sin(\alpha))^2 + (c - b\cos(\alpha))^2 \\ &= b^2\sin^2(\alpha) + c^2 - 2bc\cos(\alpha) + b^2\cos^2(\alpha) \\ &= b^2 + c^2 - 2bc\cos(\alpha) \end{aligned}$$

We can repeat this procedure by dropping down the other vertexes to the other side and we get some various forms of the same relationship, all of these are various forms of the law of cosines.

$$\textbf{Law of cosines} \quad \begin{cases} a^2 = b^2 + c^2 - 2bc\cos(\alpha) \\ b^2 = a^2 + c^2 - 2ac\cos(\beta) \\ c^2 = a^2 + b^2 - 2ab\cos(\gamma) \end{cases}$$

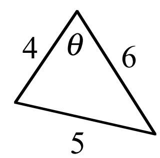
If we are in a right triangle where $\gamma = 90°$ then we would have $\cos(\gamma) = 0$. In this situation the law of cosines simplifies to give $a^2 + b^2 = c^2$, or in other words the Pythagorean theorem. So the law of cosines can be thought of as a generalization of the Pythagorean theorem.

We can also rearrange the terms involved in the law of cosines to solve for the cosine of the angle. In particular this will allow us to solve for the angle of a triangle given the lengths of all of the sides of the triangle. Doing this we get the following equations (also considered forms of the law of cosines).

$$\textbf{Law of cosines} \quad \begin{cases} \cos(\alpha) = (b^2 + c^2 - a^2)/(2bc) \\ \cos(\beta) = (a^2 + c^2 - b^2)/(2ac) \\ \cos(\gamma) = (a^2 + b^2 - c^2)/(2ab) \end{cases}$$

Example 2 In the triangle on the top of the next page find the angle θ (round the answer to two decimal places).

Solution We hope to use the law of cosines. Looking at the law of cosines there are four variables, namely the length of all of the sides



and one angle. In order to use the law of cosines we need to know at least three of the four. Since we know the length of all of the sides we are okay to proceed with using the law of cosines. Doing so we get the following,

$$\cos(\theta) = \frac{4^2 + 6^2 - 5^2}{2(4)(6)} = \frac{9}{16} \quad \text{so} \quad \theta = \arccos\left(\frac{9}{16}\right) \approx 55.77°.$$

15.4 The triangle inequality

From the law of cosines we can derive a very important mathematical rule. First, recall that the cosine function has its range of values between -1 and 1 and in particular $-\cos(\gamma) \leq 1$. With this in mind, consider the following.

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \leq a^2 + 2ab + b^2 = (a+b)^2$$

Or in particular we can take the square roots of both sides and get the following.

Triangle inequality $\qquad c \leq a + b$

This also has the alternate forms $a \leq b + c$ and $b \leq a + c$.

In words, the triangle inequality says the following, *the direct route is the shortest.* If you want to move from one point on a triangle to another then going on the segment that connects the two points will always have you travel a distance that is less than or equal to going along the other two segments.

One of the most useful properties of the triangle inequality is to test whether or not you have a triangle. If you add up the two shortest sides of a triangle and it is less than the longest side, then it is no triangle at all.

Notice that in the triangle inequality we have "less than or equal to," what would happen if we had equality? This would form a strange looking "triangle," namely, the triangle would not look like a triangle but rather a line segment. Sometimes there is concern over whether this truly is a triangle. At any rate it is good to think of it as an "extreme" example of a triangle. (Often times by studying extreme examples, i.e. worst case scenarios, we can get an idea of some behavior of an object.)

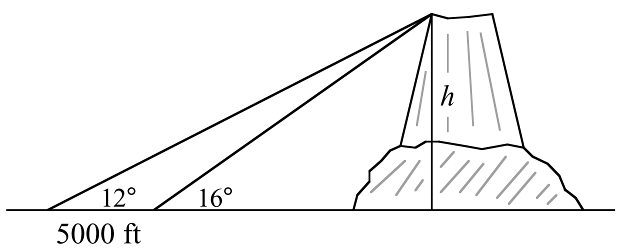
The triangle inequality is used extensively in mathematics. Particularly in calculus and any branch of mathematics that has to deal with measurement of space.

15.5 Supplemental problems

1. Verify the law of sines and the law of cosines if the triangle has an obtuse angle.

2. One day you and a vertically-challenged friend find yourself in possession of the ring of power heading toward Mount Doom. At first you have to look up at an angle of 12° to see the top of Mount Doom. After walking another 5000 feet you now have to look up at an angle of 16° to see the top of Mount Doom. How tall is Mount Doom?

   A badly drawn picture is shown below. Round your answer to the nearest 100 feet.

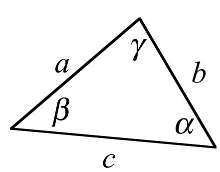


Note: drawing is not to scale

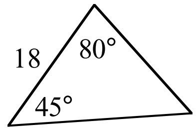
3. Your friend calls you up and tells you that he is planning to buy a triangular piece of land with sides of length 4000 feet, 2000 feet, and 6500 feet. What advice would you give to your friend?

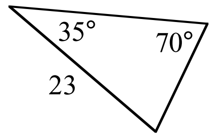
4. For our last formula for the area of a triangle, show that the area of the triangle shown below is given by,

$$\text{area} = \frac{a^2 \sin(\beta)\sin(\gamma)}{2\sin(\alpha)}.$$



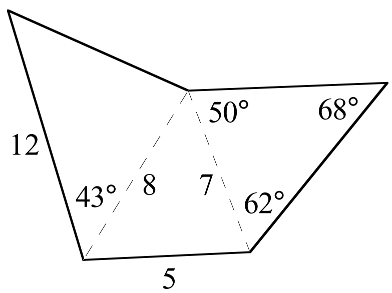
Hint: we already have shown that the area of a triangle is $\frac{1}{2}ab\sin(\gamma)$, so using the law of sines try to express b in terms of a.





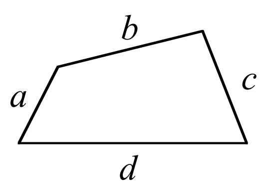
5. Using the formula from the previous question find the area of the triangles shown above. Round your answers to two decimal places.

6. Using only the information shown in the picture below find the total area. Round your answer to two decimal places.



Hint: this has been broken up into three triangles and we have three formulas to find area that use information about the length and the angles.

7. Using the triangle inequality show that $d \leq a + b + c$ in the figure below. *Hint*: to use the triangle inequality you will need a triangle.



8. The law of sines and cosines are well known, but there is also a lesser known law, called the law of tangents.

$$\textbf{Law of tangents} \qquad \frac{\tan(\alpha + \beta/2)}{\tan(\alpha - \beta/2)} = \frac{a+b}{a-b}$$

Verify the law of tangents formula.

Lecture 16

Bubbles and contradiction

In this lecture we will look at an application of what we have learned to show, among other things, why bees make hives with hexagon shapes.

16.1 A back door approach to proving

Up to this point we have seen a large amount of mathematics done in a very direct manner. We start with what we are given, a triangle for example, and then develop the relationship we are interested in.

However, in trying to show that a statement is true it is sometimes easier to take a back door approach to the problem. To do this we use a type of proof known as proof by contradiction.

To see how proof by contradiction works imagine that you are shown two envelopes marked A and B. In one of the envelopes there is a prize and in the other there is nothing. You are then told that the envelope marked B does not have the prize. What do you do?

Your first reaction is to say take the envelope marked A. That is the right action, but let us think carefully why. We know that only one of these envelopes has the prize, if we knew which one had the prize we would be done. But if we also knew which one does not have the prize then we can eliminate that as a possibility and narrow it down to an easier choice, in this case a choice of one.

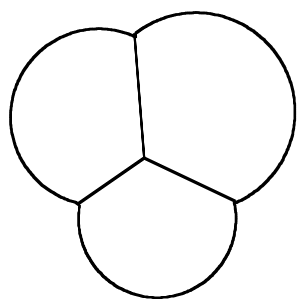
In mathematics we try to prove that some relationship is true. But, for any relationship there are only two possibilities, either it is true or not true. The indirect manner of proof, or proof by contradiction, will be to eliminate the possibility that it is not true, and thus show that it must be true.

We will examine a proof by contradiction. The actual result of what we talk about in this lecture is not important, what is important is to see how we can tie ideas together and prove a non-trivial fact.

16.2 Bubbles

Imagine that you are blowing bubbles. What shape will the bubbles be in? Most likely every bubble that you have ever seen came in only one shape, and that is round or spherical. Why do bubbles always come in round shapes? Here is the interesting property, bubbles form minimum surfaces. That is if you take all of the shapes that enclose the same amount of space (or volume) the bubble will take the least amount of surface area to do so.

Bubbles will form a minimum surface because the soap particles that make up the bubble are attracted to one another. The particles thus pull together as much as possible and in the process make the surface area as small as possible. This is why they form spheres. But another interesting question to look at is what happens when two or more bubbles connect. If you blow a bunch of bubbles at one time they seem to come together at angles of 120°. For example, shown below is what three bubbles coming together might look like (seen from directly overhead).



This is an interesting property and it would be interesting to try and prove it. But an ability to do so is slightly out of our grasp as this time. Instead, we shall prove a simpler problem.

16.3 A simpler problem

In mathematics if we cannot prove something then we turn to the next best thing, which is to prove a simpler version of what we are trying to do. In this case instead of looking at the three dimensional problem we will examine an analogous problem in two dimensions.

Imagine that you have two pieces of transparent plastic which are connected by a series of small rods at various points. Now imagine dipping this into a big vat of bubble mix and then pulling it out. There would now be a collection of soap films connecting the various rods.

From above these soap films would look like a network of lines that connected the rods. The idea that the soap film forms a minimum surface connecting the rods will translate into a collection of lines that connect the points where the rods are located and has total length as small as possible.

We will call any collection of line segments that connects a collection of points in the plane (connected in the sense that you can go from any point to any other point by traveling along these line segments) a *network*. A network that has the shortest possible total length of the line segments we will call a *soapy network*.

Previously we claimed that when bubbles meet that they do so in angles of $120°$. This should translate over into soapy networks, and so we make the following claim.

> **Claim:** In every soapy network whenever three (or more) lines come together they will always form angles of $120°$.

One way we could prove this claim is to examine *every* soapy network and look at *every* time three or more lines came together in that network and then prove that *every* angle involved is $120°$. However, this would literally take forever since there are infinitely many possibilities to check. So a direct proof would be an undesirable approach to proving this.

Since a direct approach seems to fail us let us use our new indirect approach and use proof by contradiction. So to use proof by contradiction we need to eliminate the possibility that it is not true. Let us look at what would make this statement not true, and call it our anti-claim.

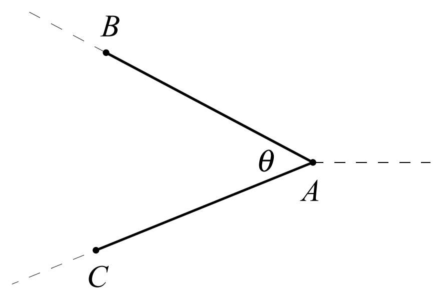
> **Anti-claim:** There is at least one soapy network and at least one place in that soapy network where three (or more) lines come together and do not form angles all of $120°$.

So to show our claim is true, we will show this our anti-claim is false.

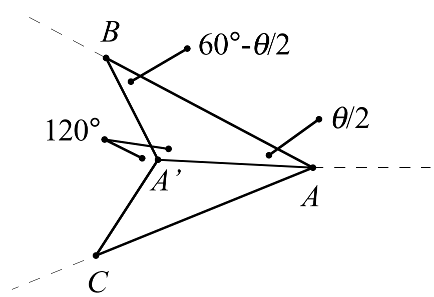
16.4 A meeting of lines

So suppose that we have a soapy network connecting a series of points and there is a time when three or more lines meet and do not form angles all of $120°$.

Now since the angles are not all $120°$ then there is at least one of the angles that is less than $120°$. If this were not the case then all of the angles would have to add up to more than $360°$, but this is impossible. We will denote the point where the lines come together by A and we will let θ denote the angle that is less than $120°$. This angle is formed by two line segments and we can mark off a small length on both line segments and get the points B and C (so B and C are both the same distance away from A, we will call that distance d). So our picture is like the one shown below. (The dotted lines represent that this is only a part of our network and that it extends for some time in various directions.)



Now with our picture we can draw a line that will bisect the angle θ (i.e. the line will cut θ in half). This line will also bisect the line segment connecting the points B and C. If we let the point A' vary along this angle bisector from the point A to the midpoint of the segment connecting the points B and C then the angle that is formed from going from B to A' to C (denoted by $\angle BA'C$) will vary smoothly from θ to 180°. In particular, there exists some point between A and the midpoint of B and C where the measure of $\angle BA'C$ is 120°, it is this point that we will mark A'. Now our picture is as shown below.



In this picture we have now formed two triangles which we will denote by $\triangle BA'A$ and $\triangle CA'A$. These two triangles are *congruent* (or in other words you can take one and put it exactly on top of the other). In particular, we can solve for the angles of the triangles in term of θ. We have,

$$\angle AA'B = \angle AA'C = 120^\circ, \qquad \angle A'AB = \angle A'AC = \theta/2,$$
$$\angle ABA' = \angle ACA' = 60^\circ - \theta/2.$$

With the angles of the triangle in place we can now use the law of sines and get the following. (Here $\overline{AB}$ refers to the distance of the line segment connecting the points A and B and similarly for the other terms.)

$$\frac{\overline{AB}}{\sin(120^\circ)} = \frac{\overline{A'B}}{\sin(\theta/2)} = \frac{\overline{A'A}}{\sin(60^\circ - (\theta/2))}$$

We can now solve for the length of the sides of the triangles. Using the difference formulas, some exact values for the sine and cosine functions and that $\overline{AB} = d$ we get the following.

$$\begin{aligned}
\overline{A'A} &= \frac{d\sin\left(60^\circ - (\theta/2)\right)}{\sin(120^\circ)} = d\cos\left(\frac{\theta}{2}\right) - \frac{d}{\sqrt{3}}\sin\left(\frac{\theta}{2}\right) \\
\overline{A'B} = \overline{A'C} &= \frac{d\sin\left(\theta/2\right)}{\sin(120^\circ)} = \frac{2d}{\sqrt{3}}\sin\left(\frac{\theta}{2}\right)
\end{aligned}$$

Now consider the following:

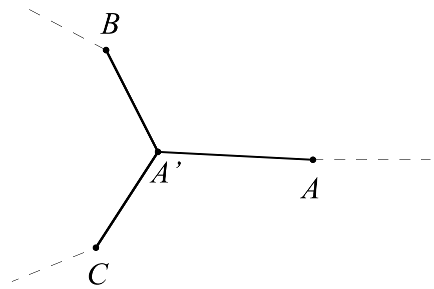
$$\begin{aligned}
\overline{A'B} + \overline{A'C} + \overline{A'A} &= \frac{2d}{\sqrt{3}}\sin\left(\frac{\theta}{2}\right) + \frac{2d}{\sqrt{3}}\sin\left(\frac{\theta}{2}\right) \\
&\quad + \left(d\cos\left(\frac{\theta}{2}\right) - \frac{d}{\sqrt{3}}\sin\left(\frac{\theta}{2}\right)\right) \\
&= d\sqrt{3}\sin\left(\frac{\theta}{2}\right) + d\cos\left(\frac{\theta}{2}\right) \\
&= 2d\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\theta}{2}\right) + \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\right) \\
&= 2d\left(\cos(30^\circ)\sin\left(\frac{\theta}{2}\right) + \sin(30^\circ)\cos\left(\frac{\theta}{2}\right)\right) \\
&= 2d\sin\left(30^\circ + \frac{\theta}{2}\right) \\
&< 2d \\
&= \overline{AB} + \overline{AC}
\end{aligned}$$

Since $0^\circ < \theta < 120^\circ$ then $30^\circ < 30^\circ + \theta/2 < 90^\circ$, it follows that $\sin(30^\circ + \theta/2) < 1$.

Now this seems like a lot of hard work. But we are done. We just need to look at what we have shown.

In the last step we showed that the total length of the segments $A'B$, $A'C$ and $A'A$ is less than the total length of the segments AB and AC. In particular, if we take out segments AB and AC and put in segments $A'B$, $A'C$ and $A'A$ then our picture will look like the following.

We can still connect all of the points together and so the result will still be a network of the points that we started with. But the total length of the lines in this



network is now shorter than it was before. But this is *impossible* since we started with the shortest total length possible for our network. So our assumption that we started with must be false. Recall that our assumption was,

> **Anti-claim:** There is at least one soapy network and at least one place in that soapy network where three (or more) lines come together and do not form angles all of 120°.

In particular, since this is false then the following must be true,

> **Claim:** In every soapy network whenever three (or more) lines come together they will always form angles of 120°.

And we are done with our proof.

16.5 Bees and their mathematical ways

Note that this says that if we are trying to use as little material as possible in breaking up the plane into compartments then we would want shapes that have angles of all 120°. It turns out that the hexagon is a shape that can do this. (A hexagon is a six sided polygon)

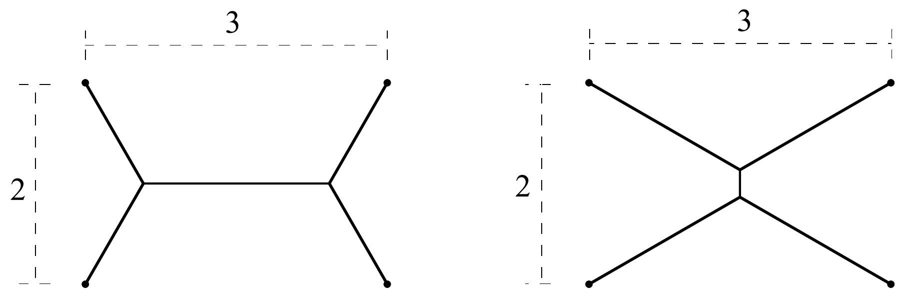
Bees seemed to have known this for millions of years and their hives are broken up into hexagonal compartments. Perhaps we underestimate the mathematical abilities of bees.

16.6 Supplemental problems

1. Show that in a soapy network we can *never* have four or more lines come together.

2. We have shown that in a soapy network that whenever three lines come together that they do so at angles of 120°. We have *not* shown that if

whenever three lines come together they do so at angles of 120° then the network is soapy, this turns out to be false. To see this find the total length of the line segments shown in the two networks below (assume that the angles formed are all 120°, round your answers to two decimal places).



Lecture 17

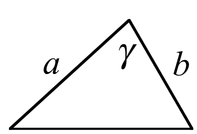
Solving triangles

In this lecture we will see how to take some information about a triangle and use it to fill in missing information.

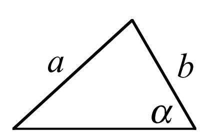
17.1 Solving triangles

To solve a triangle means to take some given information about the triangle and find the measure of the angles and the lengths of the sides of the triangle. In solving a triangle we will need to be given at least three pieces of information. These are usually some combination of the length of the sides and the measure of the angles, denoted by S and A respectively.

Note that it is important in what *order* we are given the information. For example if we have a side then an angle then a side (which we shall denote by SAS) then we will have different information then if we have a side then a side then an angle (which we shall denote by SSA). Examples of each are shown below.



SAS

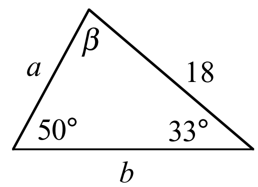


SSA

17.2 Two angles and a side

The AAS or ASA or SAA cases are all handled in the same manner. This is because if we know two angles of a triangle then we can automatically get the third at no cost (recall that all of the angles add up to 180°). We can then use the law of sines to solve for the lengths of the sides.

Example 1 Fill in the missing information (i.e. solve the triangle) shown below.



Solution First let us find the angle β. Since the sum of the angles in the triangle is 180° we have,

$$\beta = 180^\circ - 50^\circ - 33^\circ = 97^\circ.$$

So by the law of sines we have,

$$\frac{18}{\sin(50^\circ)} = \frac{a}{\sin(33^\circ)} = \frac{b}{\sin(97^\circ)}.$$

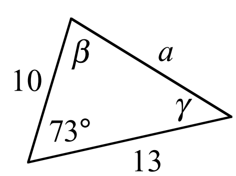
We can now solve for the values a and b and get,

$$a = \frac{18\sin(33^\circ)}{\sin(50^\circ)} \approx 12.8, \qquad b = \frac{18\sin(97^\circ)}{\sin(50^\circ)} \approx 23.32.$$

17.3 Two sides and an included angle

This is known as the *SAS* case, triangles of this variety can be solved by first using the law of cosines to find the third side and then using either the law of cosines or the law of sines to find the missing angles.

Example 2 Fill in the missing information (i.e. solve the triangle) shown below.



Solution First we can use the law of cosines to solve for a,

$$a^2 = 10^2 + 13^2 - 2(10)(13)\cos(73^\circ) \approx 192.98 \quad \text{so} \quad a \approx 13.89.$$

Now to solve for one of the angles let us use the law of cosines.

$$\cos(\beta) = \frac{(13.89)^2 + 10^2 - 13^2}{2(13.89)(10)} \approx .4462 \quad \text{so} \quad \beta \approx 63.5^\circ.$$

Finally, to solve for the third angle we can simply use the fact that all of the angles add to 180° and get,

$$\gamma = 180^\circ - 73^\circ - 63.5^\circ = 43.5^\circ.$$

17.4 The scalene inequality

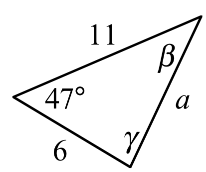
In our last example we used the law of cosines to solve for one of the missing angles. This was not the only way that we could have solved for this angle, we could have instead used the law of sines.

But the law of sines has a hidden catch. This is because the sine function has some ambiguity built into it. Namely, between 0° and 180° (the range of angles that can be used to make a triangle) there are almost always two angles that give the same sine value (i.e. the angle θ and the angle $180^\circ - \theta$). [One thing to note is that the cosine function does *not* have this ambiguity and so the law of cosines will never lead you astray when solving for an angle.]

To combat such ambiguity the following fact, known as the *scalene inequality*, is useful. *Given two sides of a triangle the longer side is opposite the bigger angle.*

A useful consequence of this fact is that since a triangle cannot have two obtuse angles, then given two sides of a triangle the angle opposite the shorter side must be acute. This will help us correct any ambiguity that we might come across.

Example 3 Find the length of the missing sides and then use the law of sines to find the missing angles of the triangle shown below.



Solution Using the law of cosines we have,

$$a^2 = 6^2 + 11^2 - 2(6)(11)\cos(47^\circ) \approx 66.98 \quad \text{so} \quad a \approx 8.18.$$

Now the law of sines states,

$$\frac{8.18}{\sin(47^\circ)} = \frac{6}{\sin(\beta)} = \frac{11}{\sin(\gamma)},$$

so we have that,

$$\sin(\beta) = \frac{6\sin(47^\circ)}{8.18} \approx .5362 \text{ and } \sin(\gamma) = \frac{11\sin(47^\circ)}{8.18} \approx .983.$$

Now if we were to plug these into our calculator we would have that $\beta \approx 32.42^\circ$ and $\gamma \approx 79.42^\circ$. If we add up all of these angles we see that the sum of angles in our triangle is,

$$47^\circ + 32.42^\circ + 79.42^\circ = 158.84^\circ.$$

Something is wrong, namely, we have run into the ambiguity of using the law of sines. What we should state is that after using our calculator that we have,

$$\beta \approx 32.42^\circ \text{ or } 147.58^\circ \qquad \text{and} \qquad \gamma \approx 79.42^\circ \text{ or } 100.58^\circ$$

Now the question is which is which. The scalene inequality comes to our aid. Since the side that is opposite the angle β is shorter than the side that is opposite the angle γ it must be that β is acute. So we have that $\beta = 32.42^\circ$. Since we already know another angle, namely 47°, we can easily verify that γ must be 100.58°.

One of the useful applications of the scalene inequality is to check your work after solving a triangle. If the longest side is not opposite the biggest angle or if the shortest side is not opposite the smallest angle then somewhere along the lines there was an error in solving the triangle.

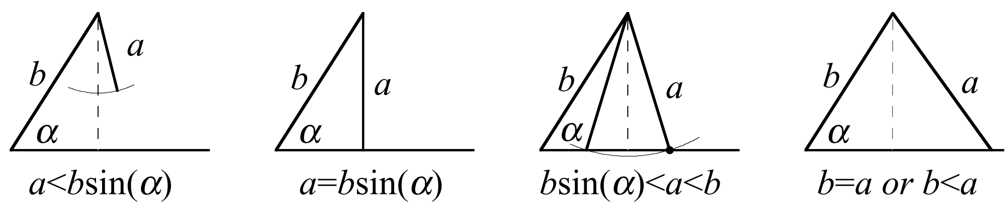
17.5 Three sides

A SSS triangle is a triangle where we know the length of all the sides. We use the law of cosines to find one of the missing angles and then use the law of sines and/or law of cosines to find the other angles.

17.6 Two sides and a not included angle

This is called the SSA case (rarely is this referred to as angle-side-side). This is by far the most interesting possibility. All the other triangles up to this point have one and only one solution, but this has the possibility of no solutions, one solution, or even two solutions.

So suppose that we have an acute angle α and the lengths of two sides a and b (where the side of length a is opposite the angle α). Then start by drawing the angle α with one side of length b and then the other side extending an unknown length. To complete the triangle imagine now putting the length of side a onto the other end of the length of side b. We then have several possibilities, and these are shown below. (Note: $b\sin(\alpha)$ is the distance represented by the dotted line in the pictures.)

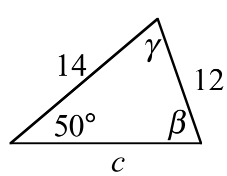


This is summarized in the following table.

Sides satisfy	# of solns.	Notes
$a < b\sin(\alpha)$	0	Side a isn't long enough to make a triangle.
$a = b\sin(\alpha)$	1	Angle β is a right angle.
$b\sin(\alpha) < a < b$	2	Use both the obtuse and acute angle β.
$b \leq a$	1	Angle β is an acute angle.

When we are actually solving a triangle of this type we will use the law of sines.

Example 4 Fill in the missing information of the triangle shown below. If more then one solution exists, provide both solutions.



Solution This is a SSA triangle and so we must first determine how many solutions there are. In the previous notation we have that $a = 12$, $b = 14$ and $\alpha = 50°$. So we have that $b\sin(\alpha) \approx 10.72$, and in particular we have that $b\sin(\alpha) < a < b$. So there are two solutions. From the law of sines we get.

$$\frac{12}{\sin(50^\circ)} = \frac{14}{\sin(\beta)} \quad \text{or} \quad \sin(\beta) = \frac{14\sin(50^\circ)}{12} \approx .8937,$$

so $\beta \approx 63.34^\circ$ or 116.66°.

For $\beta = 63.34^\circ$ we get that $\gamma = 180^\circ - 50^\circ - 63.34^\circ = 66.66^\circ$. Then using law of sines we get

$$\frac{c}{\sin(66.66^\circ)} = \frac{12}{\sin(50^\circ)} \qquad \text{or} \qquad c = \frac{12\sin(66.66^\circ)}{\sin(50^\circ)} \approx 14.38.$$

For $\beta = 116.66^\circ$ we get that $\gamma = 180^\circ - 50^\circ - 116.66^\circ = 13.34^\circ$. Then using law of sines we get,

$$\frac{c}{\sin(13.34^\circ)} = \frac{12}{\sin(50^\circ)} \qquad \text{or} \qquad c = \frac{12\sin(13.34^\circ)}{\sin(50^\circ)} \approx 3.62.$$

Thus our two solutions to our triangle are:

$$\begin{array}{lll} \beta = 63.34^\circ, & \gamma = 66.66^\circ, & c = 14.38; \\ \beta = 116.66^\circ, & \gamma = 13.34^\circ, & c = 3.62. \end{array}$$

17.7 Surveying

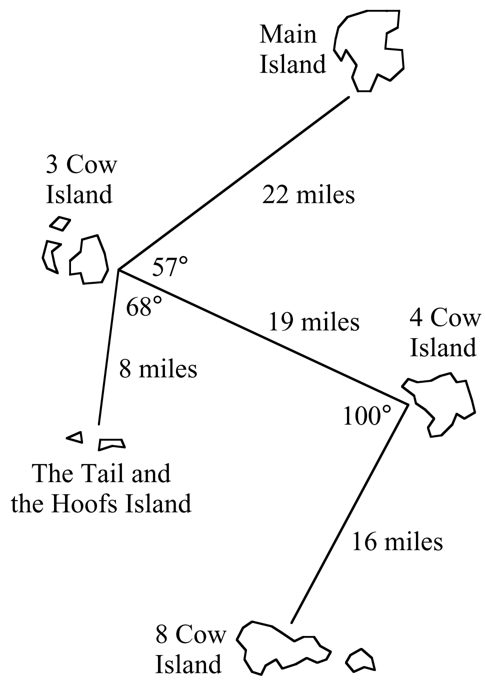
In surveying there are only two things that can be measured directly, namely angles and distances. We can then take this information about angles and distances to get unavailable information about other angles and distances.

So essentially surveying boils down to two elements; collecting accurate information about angles and distances, and using that information to solve triangles. We have now mastered one of these skills.

Example 5 Johnny Lingo has been taking his new bride, Mahana, on a tour of the surrounding islands. He has kept meticulous notes about the trip as shown on the top of the next page. He is now planning to return from eight cow island to the main island on a direct route. How far is the trip? Round your answer to one decimal place.

Solution This can be solved by repeated application of the law of cosines. First we can use the law of cosines to solve for the distance between the Main Island and 4 Cow Island. Namely, we have,

$$\text{distance} = \sqrt{22^2 + 19^2 - 2(19)(22)\cos(57^\circ)} \approx 19.74 \text{ miles}.$$



With this distance we can also solve for the angle formed by going from 3 Cow Island to 4 Cow Island to the Main Island. Namely, we have,

$$\text{angle} = \arccos\left(\frac{(19.74)^2 + 19^2 - 22^2}{2(19.74)(19)}\right) \approx 69.18°.$$

We now can find our distance by solving the SAS triangle which consists of going from the Main Island to 4 Cow Island then to 8 Cow Island. This has a length of 19.74, an angle of 169.18° and a length of 16 miles. So we have,

$$\begin{aligned} \text{Length of trip} &= \sqrt{(19.74)^2 + 16^2 - 2(19.74)(16)\cos(169.18°)} \\ &\approx 35.6 \text{ miles.} \end{aligned}$$

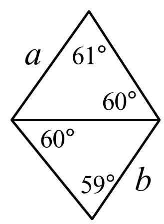
Note that we did not need to use the information about the Tail and the Hoofs Island to answer the question.

17.8 Supplemental problems

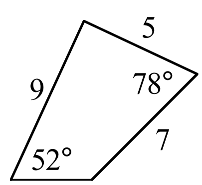
1. We have been able to solve triangles given SSS, AAS, ASA, SAA, SAS and SSA. Is it possible to solve a triangle given AAA, that is given only the angles of the triangle is it possible to fill in all of the missing values? Justify your answer.

2. There is other information in a triangle then just the measure of the angles and the lengths of the sides. For example, there is the area of the triangle. So given that a triangle has angles of 82° and 39° and an area of 60 square units, solve the triangle.

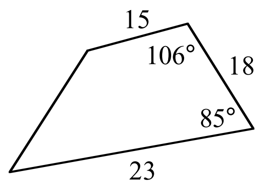
3. In the picture below which length is larger a or b? Justify your answer. *Hint*: use the scalene inequality to compare both a and b to a common length.

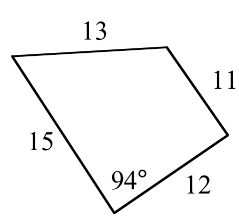


4. Once you have mastered solving triangles you can then solve any shape (i.e. find the missing angles and lengths) by breaking the shape up into triangles, solving each of those and then putting the triangles back together. Using this idea, solve the quadrilateral shown below, if more than one solution exists then provide both solutions. Round the values for the lengths and the angles to two decimal places. *Hint*: there may be more than one solution.

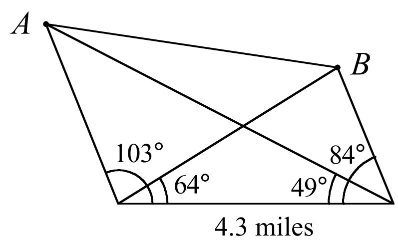


5. Find the area of the quadrilaterals shown below. Round your answers to two decimal places. *Hint*: we already have lots of ways to find the area of triangles so try breaking up the shape into two triangles.



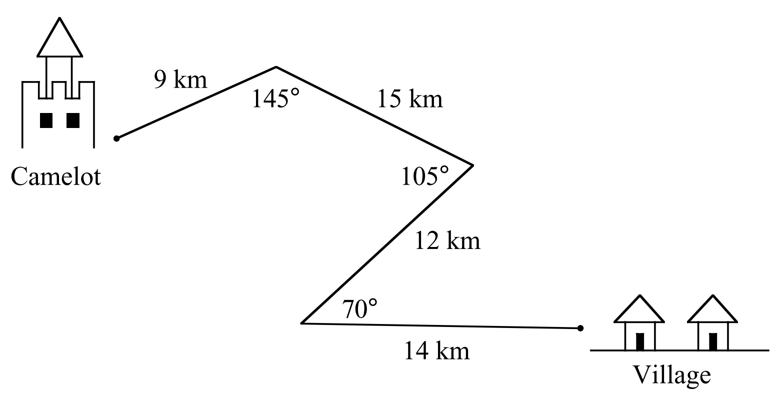


6. Surveying can be used to find distances which are impossible to measure directly. As an example, let points A and B denote the top of two mountains, we can then measure the angles that are formed by A, B and two points that are a known distance apart and get the picture shown below. Using the given information find the distance between A and B (something which is almost impossible to measure directly). Round your answer to one decimal place.



7. Before King Arthur sets out in his quest for the Holy Grail he first wants to determine the airspeed velocity of an unladen African swallow. To do this he sends a group of trained African swallows away to a village and at noon on a certain day they are released, at the same time ye olde royal hour glass in Camelot starts keeping track of time. The fastest swallow makes the trip from the village to the castle in 93 minutes. However, King Arthur forgot to get an estimate on the distance as a swallow flies between the village and Camelot. He turns to you as the royal trigonometrist to determine the straight line distance from the village to Camelot and gives you the information shown below.

   Determine the straight line distance and then use this information to determine the airspeed velocity of an unladen swallow measured in kilometers per hour. Round your answers to the nearest whole number.



Lecture 18

Introduction to limits

In this lecture we will introduce limits, which form the foundation for calculus and other advanced studies in mathematics.

18.1 One, two, infinity...

The problem of how to deal with infinity has plagued mathematicians for thousands of years. Paradoxes dealing with infinity go all the way back to the Greeks and a philosopher named Zeno. He set forth several paradoxes one of which is along the lines of the following.

> The great Achilles was to race a tortoise in a 100 meter race. Since Achilles was 10 times faster than the tortoise (this is a very fast tortoise) the tortoise was given a 10 meter head start. Now clearly in such a race Achilles should win, but before he can win he must get to where the tortoise started. By the time Achilles gets to that point the tortoise has since moved and now Achilles must get to where the Tortoise is now at, but again by the time he gets there the tortoise has again moved. Indeed, before Achilles can pass the tortoise he must "catch up" to him infinitely often. How then can Achilles win?

The paradox can be easily resolved with the idea that it is possible for infinitely many things to be added together and get a finite amount (not at all an obvious fact). For example, by the time Achilles has caught up to where the tortoise started the tortoise has moved 1 meter. By the time that Achilles catches up a second time the tortoise has moved another .1 meters. By the time that Achilles catches up a third time the tortoise has moved another .01 meters. And so forth and so forth.

If we look at the total distance traveled by Achilles as he is catching up we see that we will have traveled,

$$10,\ 11,\ 11.1,\ 11.11,\ 11.111,\ 11.1111, \ldots$$

This is a sequence of numbers that is converging (i.e. getting closer and closer) to the number $11.11\ldots = 100/9$. So when Achilles will have run 100/9 meters he will then be tied with the tortoise and from there he will easily win the race.

18.2 Limits

Often we cannot evaluate something directly, for example, no one has ever actually added infinitely many numbers together as in the example with Zeno's paradox, but we can sometimes say what should happen with great confidence.

We do this by examining what happens to our values as we get closer and closer to what it is that we are trying to evaluate, this is called a *limiting process*. If as we get closer and closer to what we are trying to evaluate our values are heading towards a certain number, then we call that number the *limit*.

When we looked at the sequence 10, 11, 11.1, 11.11 we were doing a limiting process and we saw that these values were heading towards the number $11.11\ldots$ which is the limit of this limiting process.

It is important to remember that the limit is not necessarily what happens, only what we expect to happen based on what is happening nearby.

A large area of mathematics known as *analysis* is built up around the idea of limits. For example, calculus is essentially the study of two important kinds of limits (called the derivative and the integral).

18.3 The squeezing principle

We can sometimes find the value of a limit by bounding it above and below in the limiting process by two other limiting processes which have matching limits at the place that we are interested in. Since our limiting process is in between these two, as they come together our process has to be squeezed to the same limit as the other two processes. This is called the *squeezing principle*.

To picture this, imagine a large metal press with two plates that are coming together and in between the two plates there is a ball bouncing. Now we can only say that as the plates are coming together that the ball is somewhere in between the two. But where is the ball when they have come together? It can only be in one place, squished in between the two plates. This is what is happening with the squeezing principle.

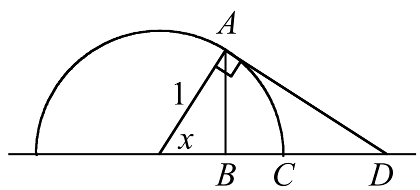
18.4 A trigonometry limit

With the idea of the squeezing principle in hand we can now evaluate an important limit. Specifically we are going to determine what happens to the values of the function,

$$\frac{\sin(x)}{x},$$

as the value of x (measured in radians) approaches 0.

Consider the diagram shown below.



From the diagram we have that $\overline{AB} \leq \overline{AC} \leq \overline{AD}$. Further, we can find the values of these lengths in terms of x. To find $\overline{AB}$ we can use the right triangle at the points A, B and the origin which has a hypotenuse of length 1 and an acute angle x to get that $\overline{AB} = \sin(x)$. Similarly, we can get $\overline{AD} = \tan(x)$. To find the length $\overline{AC}$ we note that it is an arc of a circle with radius 1 and central angle x (with x measured in radians), so from geometry we have that $\overline{AC} = x$.

It follows that for x between 0 and $\pi/2$ radians that,

$$\sin(x) \leq x \leq \tan(x).$$

In a fraction if you make a denominator smaller than the total value gets larger and if you make the denominator larger the total value gets smaller. In particular using the relationship we just found we have,

$$\cos(x) = \frac{\sin(x)}{\tan(x)} \leq \frac{\sin(x)}{x} \leq \frac{\sin(x)}{\sin(x)} = 1.$$

We have now been able to put the function $\sin(x)/x$ in between the two functions 1 and $\cos(x)$ both of which go to 1 as x gets closer and closer to 0. Therefore we can apply the squeezing principle and conclude that the function $\sin(x)/x$ will also go to 1 as x goes to 0. Using mathematical notation we would say this in the following way,

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Example 1 Using the relationship,

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1,$$

for x between 0 and $\pi/2$ radians show that,

$$\frac{\sin(2x)}{2} \leq \frac{\sin^2(x)}{x} \leq \sin(x),$$

is also satisfied for x between 0 and $\pi/2$ radians. From this, find what happens to the values of $\sin^2(x)/x$ as x approaches 0.

Solution First note that $\sin^2(x)/x = \sin(x)(\sin(x)/x)$. So using the given relationship we have,

$$\frac{\sin(2x)}{2} = \sin(x)\cos(x) \leq \sin(x)\frac{\sin(x)}{x} \leq \sin(x) \cdot 1 = \sin(x)$$

We have now been able to put the term $\sin^2(x)/x$ in between two functions. Looking at these functions they both go to the value of 0 as x goes to 0. By the squeezing principle we have that the function $\sin^2(x)/x$ will approach the value of 0 as x goes to 0.

18.5 Supplemental problems

1. Two trains start out ten miles apart on the same track and head toward each other, each going at five miles per hour. Between the two trains is a mathematical superfly who travels at a speed of ten miles per hour. The fly started on one train and is flying back and forth between the two trains. The fly is super in the sense that it can instantaneously turn around and start flying the other direction when it reaches one of the trains. Before the two trains collide the superfly will have made *infinitely* many trips back and forth between the two trains. How far will the fly have traveled? *Hint*: there is a very, very easy way to get the answer and a very, very hard way to get the answer; use the easy way.

2. (a) Show that for x between 0 and $\pi/2$ radians that,

$$1 \leq \frac{\tan(x)}{x} \leq \sec(x)$$

Hint: try to find a way to use what we already know about the expression $\sin(x)/x$ between 0 and $\pi/2$ radians.

(b) Check the answer to part (i) numerically by filling in the table below (round your values to five decimal places, make sure that your calculator is set in radian mode).

x	1	$\tan(x)/x$	$\sec(x)$
1			
0.1			
0.01			

(c) Using the information above and the squeezing principle find the value for the following.

$$\lim_{x\to 0} \frac{\tan(x)}{x}$$

3. (a) Show that for x between 0 and $\frac{\pi}{2}$ radians that

$$2\cos^2(x) \leq \frac{\sin(2x)}{x} \leq 2\cos(x).$$

Hint: try using the double angle identity.

(b) Using the relationship given in part (a) find

$$\lim_{x\to 0} \frac{\sin(2x)}{x}.$$

4. (a) Show that for x between 0 and $\frac{\pi}{2}$ radians that

$$0 \leq \frac{1-\cos(x)}{x} \leq \frac{\sin(x)}{1+\cos(x)}.$$

Hint: multiply the middle term by the conjugate of $1 - \cos(x)$ and simplify what is left using the relationships we already know.

(b) Using the relationship given in part (a) find

$$\lim_{x\to 0} \frac{1-\cos(x)}{x}.$$

5. Show that,

$$\lim_{h\to 0} \left(\frac{\sin(x+h) - \sin(x)}{h} \right) = \cos(x),$$

and that,

$$\lim_{h\to 0} \left(\frac{\cos(x+h) - \cos(x)}{h} \right) = -\sin(x).$$

6. We found that the limit of $\sin(x)/x$ as x approached 0 in radians is 1. Show that the limit of $\sin(x)/x$ as x approaches 0 in degrees is $\pi/180$.

Lecture 19

Viète's formula

In this lecture we will use the idea of limits to develop a representation for the function $\sin(x)/x$ and in particular we will derive Viète's formula.

19.1 A remarkable formula

Repeated use of the double angle formula for the sine function gives us the following relationship.

$$\begin{aligned}
\sin(x) &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
&= 4\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right) \\
&= 8\sin\left(\frac{x}{8}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right) \\
&= \cdots \\
&= 2^n\sin\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cdots\cos\left(\frac{x}{2^n}\right)
\end{aligned}$$

If we divide both sides of this equation by x we will have the following.

$$\begin{aligned}
\frac{\sin(x)}{x} &= \frac{2^n\sin(x/2^n)}{x}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cdots\cos\left(\frac{x}{2^n}\right) \\
&= \frac{\sin(x/2^n)}{(x/2^n)}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cdots\cos\left(\frac{x}{2^n}\right)
\end{aligned}$$

Now as n gets large (or in other words as n goes to infinity) the first term on the right goes to the value of 1. To see this let $u = (x/2^n)$, then as n gets large the value of u goes to zero, since the first term can be simply written as $\sin(u)/u$, so it follows that the first term is going to 1. At the same time as n gets large we just keep adding more and more cosine terms on. In particular, if we let n go to infinity then we have the following.

$$\frac{\sin(x)}{x} = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cdots$$

Where the '$\cdots$' mean we keep multiplying cosine terms forever.

19.2 Viète's formula

This last formula is valid for any value of x and so in particular it is valid for the value $x = \pi/2$. If we plug that value into both sides we will get the following.

$$\frac{\sin(\pi/2)}{(\pi/2)} = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cdots$$

This last expression can be greatly simplified and using a result from an earlier supplementary problem we get the following relationship which is known as Viète's formula.

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

This is one of the first formulas which gave a way to determine the numerical value of π as as infinite product. However, while it is completely accurate in calculating the value of π it is also extremely slow and so has no practical application.

Lecture 20

Introduction to vectors

In this lecture we will introduce vectors and how to combine and manipulate them.

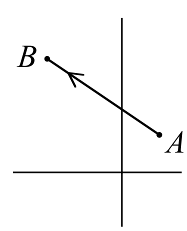
20.1 The wonderful world of vectors

In physics there is sometimes a need to describe objects that have both a direction and a magnitude. Two examples of this are force and velocity (the magnitude of velocity is what we commonly call speed). To describe such objects we will use *vectors*. Think of a vector as an object that has both a direction and a magnitude.

Vectors become a very convenient way to describe physical problems and relationships, and are widely used both in physics as well as in mathematics.

20.2 Working with vectors geometrically

We can represent a vector in the plane by connecting two points with a line segment. In the picture below we will connect points A and B with a vector which we shall denote by $\overrightarrow{AB}$.

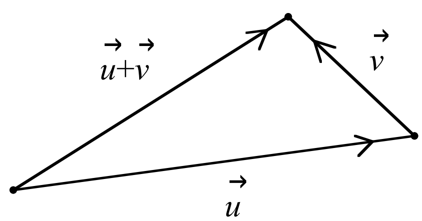


Remember that direction is also important in a vector, so the vector that goes from A to B is different from the vector that goes from B to A. To help signify direction we will introduce arrows into our pictures.

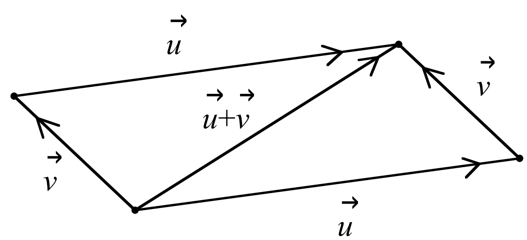
The point that our vector starts at (A in our picture) is called the initial point and the point that the vector ends at is called the terminal point (B in our picture). We will also adopt the names of the tail and the head respectively.

Two important operations that can be done with vectors are addition and scaling.

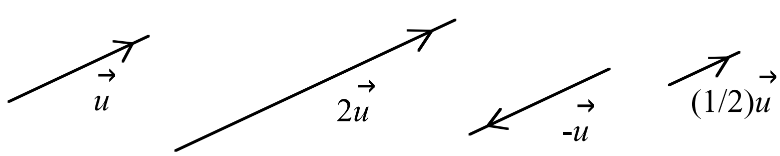
To add vectors we chain them together. For example suppose we want to find the vector $\vec{u}+\vec{v}$ (i.e. $\vec{u}$ and $\vec{v}$ are vectors that we are going to add together), then put the tail of $\vec{v}$ onto the head of $\vec{u}$. Then the vector $\vec{u}+\vec{v}$ will start at the tail of $\vec{u}$ and end at the head of $\vec{v}$. This is shown below.



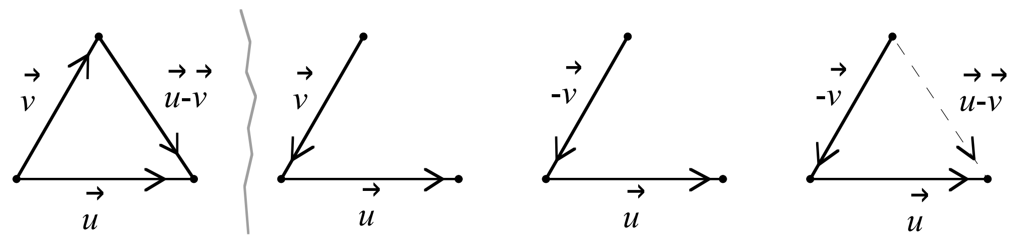
It doesn't matter what order we add vectors in and so we could also have put the tail of $\vec{v}$ onto the head of $\vec{u}$. If we represented both ways of adding vectors in the same picture we would see a parallelogram emerge (i.e. the opposite sides go in the same direction and so are parallel). Hence the rule for adding vectors is sometimes called the parallelogram rule. Vectors turn out to be a good way of describing relationships of parallelograms.



The other operation that we can perform with a vector is scaling. Scaling deals with changing the length of the vector and is done by multiplying a vector by a constant. So the vector $2\vec{u}$ is a vector that is twice as long as $\vec{u}$. When the constant is negative the vector will *reverse* the direction as well as change length. Examples of this are shown below.



We can combine these operations of scaling and addition to describe how to do subtraction of vectors. If we want to find the vector $\vec{u} - \vec{v}$ first start by putting the tails of the vectors $\vec{u}$ and $\vec{v}$ together. Then the vector $\vec{u} - \vec{v}$ can be found by going from the head of $\vec{v}$ to the head of $\vec{u}$. This is shown below, also shown below is the process which shows why this works.



In this process we first started by drawing in $\vec{u}$ and $\vec{v}$ with their tails together. Multiplying $\vec{v}$ by -1 changed the direction which we represented by reversing the direction of the arrow of $\vec{v}$. Then we added the two vectors $-\vec{v}$ and $\vec{u}$ to get the vector $\vec{u} - \vec{v}$ which is exactly where we described it.

20.3 Working with vectors algebraically

It is nice to have a geometric picture in mind when we are working with vectors. However, it is often unrealistic to work with vectors in a purely geometric setting. This is because we are inherently imperfect artists and can only get at best approximations to the vectors which approximations get progressively worse as the difficulty of the problem increases.

So often times we will choose to work with vectors in a form that allows for more precision in manipulating them. In particular, we will do it in a way that is algebraically convenient.

Recall that a vector is something that represents a distance and a magnitude. One way to incorporate this information is to break up the vector in pieces (called components) that describe how much the vector moves in relationship to each direction.

Returning to the picture we had at the beginning suppose that A was located at the point $(1, 1)$ and that B was located at the point $(-2, 2)$. Then we could say that our vector $\overrightarrow{AB}$ changed the x value by -3 and changed the y value by 1. We will write this as $\overrightarrow{AB} = \langle -3, 1\rangle$ (we use the '$\langle$' and '$\rangle$' to help distinguish vectors from points, remember that a vector is not a point but rather a *displacement*).

In general, if we have a vector going from the initial point $A = (x_0, y_0)$ to the terminal point $B = (x_1, y_1)$ then we have,

$$\overrightarrow{AB} = \langle x_1 - x_0, y_1 - y_0 \rangle.$$

One great advantage of working with vectors algebraically is the ease of addition and scaling of vectors. For these operations we will work in each component, and so we have,

$$\begin{array}{lrcl} \text{Addition:} & \langle a, b\rangle + \langle c, d\rangle & = & \langle a+c, b+d\rangle. \\ \text{Scaling:} & k\langle a, b\rangle & = & \langle ka, kb\rangle. \end{array}$$

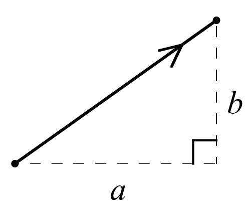
Many of the same rules of arithmetic that we have grown up with still apply when working with vectors. These rules include,

$$\begin{aligned} \vec{u} + \vec{v} &= \vec{v} + \vec{u}, \\ (\vec{u} + \vec{v}) + \vec{w} &= \vec{v} + (\vec{u} + \vec{w}), \\ \vec{u} + \vec{0} &= \vec{u} \quad \text{where } \vec{0} = \langle 0, 0\rangle, \\ (c+d)\vec{u} &= c\vec{u} + d\vec{u}, \\ c(\vec{u} + \vec{v}) &= c\vec{u} + c\vec{v}. \end{aligned}$$

20.4 Finding the magnitude of a vector

We started by saying that a vector has both a magnitude and a direction. So let us look at how to find the magnitude. If we were dealing with vectors geometrically then we would find the magnitude by pulling out a measuring stick and finding the length of the line segment representing the vector.

Algebraically, it is not much different. The magnitude is the length of the vector between the tail and the head. If we have a vector $\langle a, b\rangle$ then we can find the magnitude by the Pythagorean theorem (see the picture below).



We will denote the magnitude of a vector $\vec{v}$ by $\|\vec{v}\|$ and in particular we have,

$$\|\vec{v}\| = \sqrt{a^2 + b^2}.$$

Scaling and magnitude have a nice relationship, namely that if we scale a vector by a value of c then we multiply the magnitude by $|c|$ (magnitude is *always* a non-negative number). The reason that this works is shown below.

$$\|c\vec{v}\| = \|\langle ca, cb\rangle\| = \sqrt{(ca)^2 + (cb)^2} = \sqrt{c^2(a^2 + b^2)} = |c|\sqrt{a^2 + b^2} = |c|\|\vec{v}\|$$

20.5 Working with direction

With a way to find magnitude we now turn to direction. This is trickier to get our finger on. What exactly is a direction? A good way to think about direction is as a *unit vector*. A unit vector is a vector with length one and so all of the important information about the vector is contained in the direction.

A useful fact is that every vector, besides the zero vector (i.e. $\langle 0, 0\rangle$), can be represented in a *unique* way as a positive scalar times a unit vector. Namely, we have the following.

$$\vec{u} = \|\vec{u}\| \left(\frac{1}{\|\vec{u}\|}\vec{u} \right)$$

The important thing to note is that $(1/\|\vec{u}\|)\vec{u}$ is a unit vector. This follows from the argument just given about multiplying the magnitude of the vector by the same amount as you scale the vector.

Example 1 Find a unit vector in the same direction as

$$\langle 2, -5\rangle$$

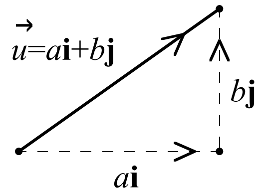
Solution Proceeding with the idea just given we will divide this vector by its magnitude and get a unit vector pointing in the same direction. So we will get the following vector.

$$\frac{1}{\|\langle 2, -5\rangle\|}\langle 2, -5\rangle = \frac{1}{\sqrt{2^2 + (-5)^2}}\langle 2, -5\rangle = \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle$$

There are two very important unit vectors that have been given names, these are called the standard unit vectors. They are $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. These are useful in giving another way to represent vectors in component form. Namely, we have the following,

$$\langle a, b\rangle = \langle a, 0\rangle + \langle 0, b\rangle = a\langle 1, 0\rangle + b\langle 0, 1\rangle = a\mathbf{i} + b\mathbf{j}.$$

When you see a vector $\vec{u}$ in the form $a\mathbf{i} + b\mathbf{j}$, think of a as how much the vector is moving in the x direction and b as how much the vector is moving in the y direction. This is shown in the picture below.



20.6 Another way to think of direction

If we have a vector $\langle a, b\rangle$ that is a unit vector then we also have that $a^2 + b^2 = \|\langle a, b\rangle\|^2 = 1$. So we can think of (a, b) as a point on the unit circle.

Now we can see trigonometry working its way back into the area of vectors. Recall that every angle is associated with a point on the unit circle and that every point on the unit circle is associated with an angle. In particular, there is a *unique* angle between $0°$ and $360°$ such that $(a, b) = (\cos(\theta), \sin(\theta))$. Where θ is measured in standard position (i.e. $0°$ is the positive x axis).

20.7 Between magnitude-direction and component form

So we have another way to describe a vector, namely in a very pure sense as a magnitude and a direction where the direction is given in relation to some fixed direction. So suppose that we are now given a vector, $\vec{v}$, described as a magnitude and direction and we want to put the vector into component form.

First, let us handle direction. We are given an angle θ where we will assume that θ is measured in standard position. By our discussion above we have that the angle θ corresponds to a unit vector, and in particular it corresponds to the unit vector $\langle\cos(\theta), \sin(\theta)\rangle$.

Now we have our unit vector, but it may not be the right length. So we scale it by the magnitude. So suppose that our magnitude of the vector is k then by scaling the vector that we just found by k we get the component representation of the vector, namely,

$$\vec{v} = \langle k\cos(\theta), k\sin(\theta)\rangle = (k\cos(\theta))\mathbf{i} + (k\sin(\theta))\mathbf{j}.$$

Now let us go the other way. Suppose that we have a vector that is in component form and we want to describe it as a magnitude and direction. We have already seen how to find magnitude so let us focus on the direction.

Imagine that the vector has its tail at the origin so that the head of the vector will be at some point in the plane (a, b) (which by the way are the same a and b as we use to represent our vector in component form, i.e. the same a and b as $\langle a, b\rangle$).

Now we recall from when we first learned about the trigonometric functions that for every point in the plane, besides the origin, there corresponds an angle θ and further if our point is (a, b) we have,

$$\tan(\theta) = \frac{b}{a}.$$

Ideally we could just take the arctangent and be done, but there is a catch. The arctangent function only returns values in a 180° range, namely between −90° and 90°. It is possible that the angle that we want is actually outside of this range, and so we might need to compensate. In particular, we get the following.

$$\theta = \begin{cases} \arctan(b/a) & \text{if } a \geq 0 \\ \arctan(b/a) + 180^\circ & \text{if } a < 0 \end{cases}$$

Example 2 Convert $\langle 3, 7\rangle$ into magnitude-direction form.

Solution First we find the magnitude.

$$\|\langle 3, 7\rangle\| = \sqrt{3^2 + 7^2} = \sqrt{58}$$

Looking at our vector we have that $b = 7 \geq 0$ and so we can find the direction in the following way.

$$\theta = \arctan\left(\frac{7}{3}\right) \approx 66.8^\circ$$

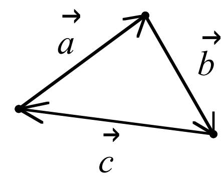
So our vector has magnitude $\sqrt{58}$ and is pointing in a direction of approximately 66.8°.

20.8 Applications to physics

Many problems in physics reduce down to dealing with vectors. In particular, an important rule is that if an object is at rest then the sum of the forces acting on the object have to sum to zero. If we are using vectors to represent the forces, then this idea translates into the sum of the vectors being zero.

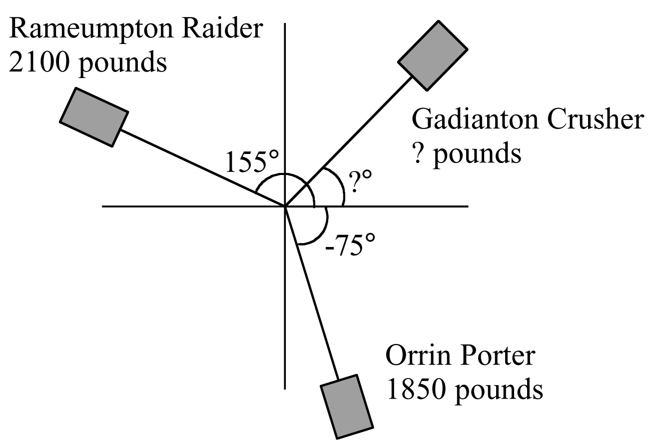
20.9 Supplemental problems

1. Find the terminal point of the vector $3\mathbf{i} - 2\mathbf{j}$ if the vectors initial point is $(5, 1)$.



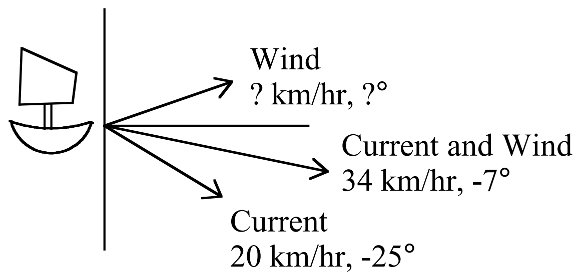
2. Find the exact value for the vector $\vec{a} + \vec{b} + \vec{c}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are as shown in the picture above. Justify your answer.

3. You have recently decided to enter the Mormon Truck Circuit (MTC) with your new truck the "Gadianton Crusher." Your first event is a three way tug of war in which three cars are all chained to a central point and try to pull the other two off. The first truck, the "Orrin Porter," likes to position himself at an angle of $-75°$ and pull with a force of 1850 pounds. The second truck, the "Rameumpton Raider," likes to position himself at an angle of $155°$ and pull with a force of 2100 pounds. At what angle and with what force should you position the Gadianton Crusher so that the central point where the cars are chained does not move? Round your answers to the nearest whole number. *Hint*: for the central point not to move the forces acting on it need to add up to 0, a badly drawn picture is shown below.



4. One day you find yourself sailing to the promised land on the "Bountiful or bust." For the last few days you have been traveling along a current in the ocean. The current has been carrying the boat at a speed of 20 kilometers per hour in a direction of $-25°$ (see the picture below). Suddenly a great wind arises and the boat is now traveling at a speed of 34 kilometers per hour in a direction of $-7°$ (i.e. the combined effects of the current and the wind cause the boat to travel 34 kilometers per hour in a direction of $-7°$).

If the boat got off the current then at what speed and what direction would the boat travel? In other words, at what direction and what speed would the boat be traveling with only the wind? Round your answers to one decimal place. A badly drawn picture is shown below. *Hint*: to find the effect of the wind "subtract" the current from the combination of the current and wind.



Lecture 21

The dot product and its applications

In this lecture we will explore a new way of "combining" vectors together, namely the dot product. One important application of the dot product will be to find angles between vectors.

21.1 A new way to combine vectors

In the last lecture we introduced vectors and saw how to manipulate and combine vectors through scaling and addition. This time we will look at a new way of combining them together, but now the result will not be a vector but rather a number that we will call the *dot product*.

Specifically, if we have the vectors $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$ then the dot product (which we will unimaginatively denote by a '$\cdot$') is given by,

$$\vec{u} \cdot \vec{v} = ac + bd.$$

That is we multiply the x components and the y components and we add up the results.

The dot product obeys some nice properties. For example,

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u}, \\ \vec{u} \cdot \vec{0} &= 0, \\ \vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}. \end{aligned}$$

Another important relationship emerges when we take the dot product of a vector, $\vec{u}$, with itself. We get

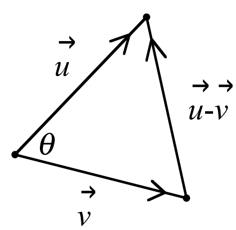
$$\vec{u} \cdot \vec{u} = aa + bb = a^2 + b^2 = \|\vec{u}\|^2.$$

In other words the dot product can be used to find the magnitude of vectors. Namely, we have $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$.

21.2 The dot product and the law of cosines

Since we can relate the dot product to length we can now relate the dot product to our various rules that we have learned for triangles involving length. One rule in particular provides an amazing application of the dot product, namely the law of cosines.

Starting with two non-zero vectors, $\vec{u}$ and $\vec{v}$, we can put the tails together and we almost get a triangle. The third side of the triangle we can find by the vector $\vec{u}-\vec{v}$. Finally, let the angle θ denote the angle between the vectors $\vec{u}$ and $\vec{v}$ so that our picture looks like the one shown below.



The three sides of this triangle have length $\|\vec{u}\|$, $\|\vec{v}\|$ and $\|\vec{u}-\vec{v}\|$, so we can now apply the law of cosines to this triangle and we will get the following.

$$\begin{aligned} \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos(\theta) &= \|\vec{u}-\vec{v}\|^2 \\ &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \end{aligned}$$

After cancelling we get,

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta).$$

This provides an alternative way of finding the dot product and gives rise to the greatest uses of the dot product. For example, we can rearrange this last relationship to solve for θ.

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \quad \text{or} \quad \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right).$$

So the dot product can provide a method to determine the angle between vectors. (The angle between vectors is the angle that is formed by putting the tails of the vectors together.)

Example 1 Find the dot product of two vectors the first of which has a magnitude of 12 and a direction of 53° and the second of which has a magnitude of 7 and a direction of 87° (where the angles are measured in standard position).

Solution We do not have the component form of these vectors and so we cannot directly apply the definition of the dot product. We could of course find the component form, but let us see if there is not a better way.

Notice that we can find the angle between the two vectors by taking the difference of their angles. In particular, the angle between these two vectors is $87° - 53° = 34°$. We already have the magnitudes of these vectors and so we can apply our new relationship for dot product and get the following.

$$\vec{u} \cdot \vec{v} = (12)(7)\cos(34°) \approx 69.63$$

Example 2 Find the angle between the vectors $\vec{u} = \langle 3, -8 \rangle$ and $\vec{v} = \langle -4, -2 \rangle$.

Solution This is a straightforward application of the dot product.

$$\begin{aligned} \theta &= \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) \\ &= \arccos\left(\frac{(3)(-4) + (-8)(-2)}{\sqrt{(3)^2 + (-8)^2}\sqrt{(-4)^2 + (-2)^2}}\right) \\ &= \arccos\left(\frac{2}{\sqrt{365}}\right) \approx 83.99° \end{aligned}$$

21.3 Orthogonal

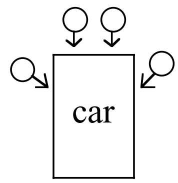
Two vectors are perpendicular to one another if they meet at an angle of 90°. In particular if $\vec{u}$ and $\vec{v}$ are perpendicular we have the following,

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(90°) = 0.$$

So we can use the dot product to test if two vectors are perpendicular. In general, we will say that two vectors whose dot product is zero are *orthogonal*. So two vectors that are perpendicular to one another are said to be orthogonal. By this convention we will say that $\vec{0}$ (i.e. the zero vector) is orthogonal to every vector.

21.4 Projection

Imagine that you have an old car that dies at random, inconvenient intervals (for a lot of students this is not too hard to imagine). Now imagine that you had several people pushing your car as shown in the picture below.

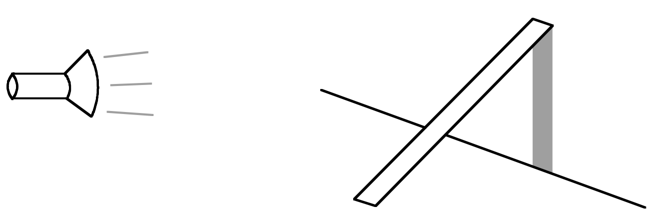


Now here is a question, are the people that push on the sides of the car as effective as those who are pushing on the back of the car?

The answer is, no. Ultimately the goal is to move the car forward, but the people who are pushing on the sides are wasting energy because they are not pushing exactly in the direction that we want the car to go. So we might want to ask the question, how much of their force is going to moving the car forward?

To answer this question we can break up their force (the vector) into two pieces, one piece that points in the direction that we want the car to go and the other piece pointing orthogonal to the way we want the car to go. The part pointing in the direction of the way we want the car to go is the *effective* force, i.e. the amount of force someone pushing from behind would have to give to produce the same result. The part pointing in the orthogonal direction is the wasted force, the totally ineffectual portion of their force.

This process of breaking up a vector into pieces is *projection*. For another example of projection, imagine that there is a stick leaning against a wall in a dark room. Now take a flashlight and shine it in the direction of the stick with the flashlight parallel to the ground. This will cause a shadow to fall behind the stick, that shadow is the *projection* of the stick onto the wall. This is shown below.

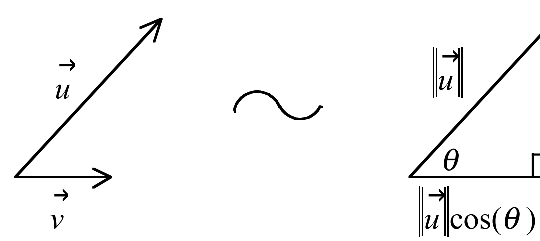


21.5 Projection with vectors

In terms of vectors projection deals with finding how much of one vector (say $\vec{u}$) points in the same direction as another vector (say $\vec{v}$).

The answer to this is a vector and we will find it by determining the direction and the magnitude of the vector. The direction is simple because it needs to point in the same direction as $\vec{v}$ and so to find the direction we will need to find a unit vector that points in the same direction as $\vec{v}$, from last time we know that such a vector is $(1/\|\vec{v}\|)\vec{v}$.

For magnitude, start by imagining that we put the two vectors with their tails together and then drop a line straight down from the head of $\vec{u}$ to a line that is extended from $\vec{v}$ to form a right triangle. Finally, if we let θ be the angle between our two vectors then by projection we can find the magnitude of our vector. Namely we have that the magnitude of the projection should be $\|\vec{u}\|\cos(\theta)$.



With our direction and magnitude we can now solve for the projection of the vector $\vec{u}$ onto the vector $\vec{v}$ (which we shall denote by $\mathrm{proj}_{\vec{v}}(\vec{u})$).

$$\mathrm{proj}_{\vec{v}}(\vec{u}) = (\|\vec{u}\|\cos(\theta))\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = \left(\frac{\|\vec{u}\|\|\vec{v}\|\cos(\theta)}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

21.6 The perpendicular part

We talked about breaking our vector up into two parts one that points in the same direction and another that points in the orthogonal, or perpendicular, direction.

We can use projection to find the part of the vector $\vec{u}$ that lies in the same direction as $\vec{v}$. Once we have this, to find the orthogonal part we subtract off the projection, i.e. the vector $\vec{u} - \mathrm{proj}_{\vec{v}}(\vec{u})$. This resulting vector will be orthogonal to $\vec{v}$. This is verified in the following way.

$$(\vec{u} - \mathrm{proj}_{\vec{v}}(\vec{u}))\cdot\vec{v} = (\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v})\cdot\vec{v} = \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} = 0$$

Thus given any vector, $\vec{u}$, and any other non-zero vector $\vec{v}$ we can break $\vec{u}$ into two parts, one in the same direction as $\vec{v}$ and one orthogonal to $\vec{v}$. This is summarized below.

$$\begin{aligned}\text{Part of } \vec{u} \text{ parallel to } \vec{v} &= \text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} \\ \text{Part of } \vec{u} \text{ perpendicular to } \vec{v} &= \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\end{aligned}$$

Example 3 Write the vector $\langle 3, -4\rangle$ in two parts, one of which is parallel to the vector $\langle 2, 3\rangle$ and one of which is perpendicular to the vector $\langle 2, 3\rangle$.

Solution First we will find the parallel part, this is simply projection and so by using the projection formula we get the following vector,

$$\left(\frac{\langle 3,-4\rangle\cdot\langle 2,3\rangle}{\langle 2,3\rangle\cdot\langle 2,3\rangle}\right)\langle 2,3\rangle = \frac{-6}{13}\langle 2,3\rangle = \left\langle \frac{-12}{13}, \frac{-18}{13}\right\rangle.$$

With the parallel part in hand we now subtract it from the original vector to get the perpendicular part, namely we will have,

$$\langle 3,-4\rangle - \left\langle \frac{-12}{13}, \frac{-18}{13}\right\rangle = \left\langle \frac{51}{13}, \frac{-34}{13}\right\rangle.$$

21.7 Supplemental problems

1. If $\vec{u} = \langle \cos(\theta), \sin(\theta)\rangle$ and $\vec{v} = \langle \cos(\phi), \sin(\phi)\rangle$ then find and simplify $\vec{u}\cdot\vec{v}$. (Simplify means to reduce it to a single term.)

2. True/False. For any two values a and b the vectors $\langle a, b\rangle$ and $\langle b, -a\rangle$ are orthogonal. Justify your answer.

3. Given that the vector $\vec{u}$ has a magnitude of 9 and is positioned at an angle of 19° (here all the angles are measured in standard position) and that the vector $\vec{v}$ is positioned at an angle of 73°, find the magnitude of $\vec{v}$ if $\vec{u}\cdot\vec{v} = 34$. Round your answer to two decimal places.

4. Given that the vectors $\vec{u}$ and $\vec{v}$ are orthogonal show that they satisfy,
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

5. Given that the vectors $\vec{u}$ and $\vec{v}$ satisfy,
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2,$$
show that $\vec{u}$ and $\vec{v}$ are orthogonal.

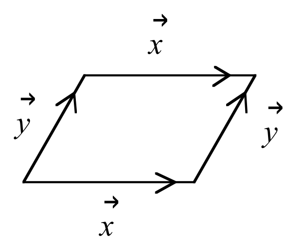
6. Give any two vectors $\vec{u}$ and $\vec{v}$, prove the following.

$$\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = 4(\vec{u}\cdot\vec{v})$$

7. (a) Prove the following for any vectors $\vec{x}$ and $\vec{y}$.

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$$

   (b) Indicate where $\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$ are in the parallelogram below.



   (c) This result is known as the law of parallelograms. If our parallelogram is a rectangle what famous result do we get?

8. Projection is a function that takes a vector and returns another vector. Show that the projection function satisfies the following,

   (a) $\text{proj}_{\vec{v}}(\vec{u}+\vec{w}) = \text{proj}_{\vec{v}}(\vec{u}) + \text{proj}_{\vec{v}}(\vec{w})$,

   (b) $\text{proj}_{\vec{v}}(a\vec{u}) = a\text{proj}_{\vec{v}}(\vec{u})$.

   [Note: any function that satisfies these two properties are *linear*. Linear functions form the backbone for much of mathematics.]

Lecture 22

Introduction to complex numbers

In this lecture we will introduce complex numbers, a number system that includes i, the imaginary number.

22.1 You want me to do what?

Several hundred years ago mathematicians were stuck. They needed a number that when squared would become negative. Unfortunately, they were coming up short, since if you square a positive number you get a positive number and if you square a negative number you also get a positive number.

So in the spirit of good mathematics, when they didn't have a number to do what they wanted, they made a new number, called i, and said that i was the number such that when you squared it you got -1. That is $i^2 = -1$.

Note that we have that $i^2 = -1$, $i^3 = i^2 i = -i$ and $i^4 = i^2 i^2 = (-1)(-1) = 1$ and so on. In fact any power of i is equal to one of i, -1, $-i$, 1, to determine which one you only need to figure out the remainder of the number when dividing by 4. This is because $i^4 = 1$ and so we can break off a lot of i^4 terms and they will all become 1's which do not change the value.

Example 1 Simplify the expression i^{2002}.

Answer: We can rewrite 2002 as $4 \cdot 500 + 2$. In this form it is now easy to see what happens.

$$i^{2002} = i^{4\cdot 500+2} = i^{4\cdot 500} i^2 = -1$$

22.2 Complex numbers

With the introduction of this new number, mathematicians were in a sense able to get a complete number system. Gauss, considered one of the greatest mathematicians of all time, was the first person to show that if we include i (and all that follows) that we can find the roots of any polynomial.

With i in hand we can now describe the complex numbers. A complex number is a number of the form $a + bi$ where a and b are real numbers (the numbers that you grew up with and that you love so well). The values a and b denote the "real" and "imaginary" parts of the number respectively.

We will say that two complex numbers are the same if and only if they have the same real parts and the same imaginary parts.

22.3 Working with complex numbers

Just as with real numbers we can add, subtract, multiply and divide complex numbers.

To add and subtract numbers we add and subtract their real and imaginary parts. This is shown in the following.

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\ (a+bi)-(c+di) &= (a-c)+(b-d)i\end{aligned}$$

To multiply we FOIL the terms and use the fact that $i^2 = -1$ to simplify the result.

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac-bd)+(ad+bc)i$$

Division represents a problem because of the complex number that is in the denominator. To handle this we need to find a way to change the complex number to a real number. To our rescue comes the conjugate. The conjugate of a complex number is the complex number with the sign of the imaginary part changed. So for example, if we denote conjugation by putting a bar over the number then we have that $\overline{a+bi} = a - bi$. Using conjugates we get the following.

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{(ac+bd)+(-ad+bc)i}{(c^2+d^2)+(-cd+cd)i} = \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i$$

Example 2 Simplify the following complex number.

$$\frac{8+15i}{7-i} - \frac{27}{5+5i}$$

Solution This is a matter of careful manipulation. Applying the rules we have just learned we will get the following.

$$\begin{aligned}
\frac{8+15i}{7-i} - \frac{27}{5+5i} &= \frac{(8+15i)}{(7-i)} \cdot \frac{(7+i)}{(7+i)} - \frac{27}{5+5i} \cdot \frac{(5-5i)}{(5-5i)} \\
&= \frac{56+8i+105i-15}{7^2+1^2} - \frac{135-135i}{5^2+5^2} \\
&= \frac{-94+248i}{50} = -1.88+4.96i
\end{aligned}$$

22.4 Working with numbers geometrically

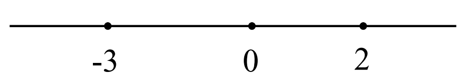
When we want to draw a picture that represents the real numbers we start by drawing a line and then say that the points on the line correspond to the real numbers and the real numbers correspond to points on the line.

We would like an analogous system for the complex numbers. But for complex numbers we are working not with one parameter, but two (i.e. the real and imaginary parts of the number). So geometrically instead of a line we will use a plane where every point corresponds to a number and every number corresponds to a point. This plane is called the complex plane.

In particular, we will associate the x coordinate of the point with the real part of the complex number and the y coordinate of the point with the imaginary part of the complex number. So for example the complex number $2-\sqrt{7}i$ corresponds to the point $(2,-\sqrt{7})$ in the plane.

22.5 Absolute value

It would be nice to have an absolute value function for the imaginary numbers much like we do for the real numbers. First, let us consider what the absolute value function does for real numbers. To start consider the line shown below (i.e. this line is drawn to represent the real numbers).



We know that $|2|=2$ and $|-3|=3$ but what does that mean. If we look at the picture we can see that the distance from 0 to 2 is 2 and that the distance from -3 to 0 is 3. In particular, the absolute value function measures *distance* away from zero.

For working with absolute value in the complex numbers (which is sometimes referred to as modulus) we will adopt this same idea. Namely, we will have,

$$|a + bi| = \textit{distance from } a + bi \textit{ to } 0.$$

To find this distance think of the geometrical picture. We are going to find the distance from the point (a, b) to the point $(0, 0)$. So by the Pythagorean theorem we get,

$$|a + bi| = \sqrt{a^2 + b^2}.$$

This absolute value function measures the distance from the point to the origin. We can also use it to measure the distance between any two arbitrary complex numbers. Namely, if we have the complex numbers z_1 and z_2 then we will denote the distance between them by $|z_1 - z_2|$. (In terms of the picture think of the "$-z_2$" as putting the origin at z_2 and then it becomes just a matter of finding the distance from a number to the origin.)

Example 3 Find the distance between $2 - 5i$ and $-1 - i$.

Solution This is a straightforward application of applying the distance formula and so we will get.

$$\text{distance} = |(2 - 5i) - (-1 - i)| = |3 - 4i| = \sqrt{(3)^2 + (-4)^2} = 5$$

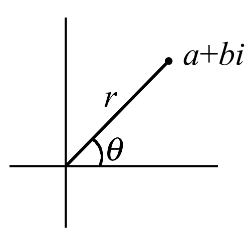
22.6 Trigonometric representation of complex numbers

As with vectors we can describe complex numbers in more than one way. The trigonometric form of a complex number is given in the following way,

$$z = r(\cos(\theta) + i\sin(\theta)),$$

(some books will use $\text{cis}(\theta)$ for a shorthand way of saying $\cos(\theta) + i\sin(\theta)$).

In terms of the graphical representation r and θ are related to a complex number $a + bi$ as shown below.



In particular r represents the distance to zero (i.e. the origin). So given a complex number $z = a + bi$ we have,

$$r = |z| = \sqrt{a^2 + b^2}.$$

To find the angle θ we use the fact that every point except the origin is associated with an angle in the plane. In particular the point corresponding to $a + bi$ will relate to the angle θ in the following way.

$$\tan(\theta) = \frac{b}{a}.$$

To find θ we would take the arctangent, but we come across the problem of not having enough values in the range of the arctangent function. To fix this we will find θ in one of two ways depending on where $a + bi$ is in the complex plane. Namely, we will have the following.

$$\theta = \begin{cases} \arctan(b/a) & \text{if } a \geq 0 \\ \arctan(b/a) + 180^\circ & \text{if } a < 0 \end{cases}$$

If we wanted our angle in radians we would replace 180° by π.

One thing to note is that θ is not unique. In particular if we add any multiple of a full revolution to θ then we will still get the same complex number ($\theta + 360^\circ$ or $\theta + 2\pi$ for example). We will use this to our advantage later.

22.7 Working with numbers in trigonometric form

In trigonometric form there is no easy way to add or subtract numbers, but there is a beautiful way to multiply and divide. Suppose we have $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$ then for multiplication we get the following.

$$\begin{aligned} z_1 z_2 &= (r_1(\cos(\theta_1) + i\sin(\theta_1))) \cdot (r_2(\cos(\theta_2) + i\sin(\theta_2))) \\ &= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) \\ &\qquad + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))] \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

That is, to multiply the numbers we multiply the radiuses to get the new radius and add the angles to get the new angle.

Similarly, it can be shown for division that we get.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

22.8 Supplemental problems

1. Show that $z\overline{z} = |z|^2$. *Hint*: recall that $\overline{z}$ refers to the conjugate of z, one way to start is to write $z = a + bi$ and then work through the calculations.

2. What is the distance between $4 - i$ and $-2 + 3i$?

3. Describe the set of points, z, in the complex plane that satisfy $|z - (2 - i)| \leq 3$. A picture is an acceptable answer. *Hint*: what does the absolute value of a difference of complex numbers refer to? What is the corresponding geometrical interpretation?

Lecture 23

De Moivre's formula and induction

In this lecture we will learn about mathematical induction. In particular, we will prove De Moivre's formula which will allow us to easily find powers and roots of complex numbers.

23.1 You too can learn to climb a ladder

Mathematical induction is a powerful tool that allows us to prove an infinite number of cases of a problem very quickly. The process of proving an infinite number of cases sounds daunting but it boils down to just two steps. Namely, proving the first case and then proving that if one case is true then it must be that the next case is also true.

These steps of mathematical induction are analogous to climbing up a ladder. To climb any ladder you first have to get on the ladder and then you have to move from rung to rung to get up. So if you can understand how to climb a ladder, you can understand mathematical induction. Conversely, if you have not yet learned to climb a ladder then studying mathematical induction will help teach you how.

23.2 Before we begin our ladder climbing

Before climbing up a ladder we would place it against a wall or some other solid surface, that is we would prepare. Similarly we have preparations when we want to use mathematical induction. Before we begin we must *understand* what we are trying to prove.

The best way to learn mathematics is to do mathematics and so as we are learning about induction we will do an induction problem. So using induction we will prove De Moivre's formula.

De Moivre's formula is stated in the following way,

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta) \qquad \text{for } n = 1, 2, 3, \ldots.$$

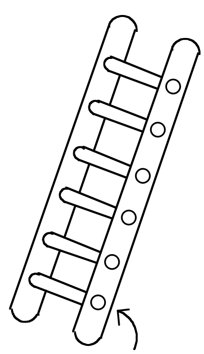
This is an amazing formula. It basically gives an *easy* way to find powers of a particular type of complex number. For example, if De Moivre's formula is true (which we do not yet know, but will soon) then we have the following,

$$(\cos(1^\circ) + i\sin(1^\circ))^{360} = \cos(360^\circ) + i\sin(360^\circ) = 1.$$

Could you imagine having to actually take that term and multiply it 360 times, this is going to be such a cool formula. But first we have to prove it.

23.3 The first step: the first step

Whether you have been climbing ladders professionally for years, or are new to the competitive world of ladder climbing, everyone has to start climbing a ladder in the same way. They first have to get on the ladder.



The first step

The idea of starting to climb the ladder is analogous in mathematical induction to proving the first case. After all, we are trying to show that infinitely many cases are true and so we should at least show the first one is true.

Showing the first case is true is done in a variety of ways depending upon the problem. We are trying to show De Moivre's formula for $n = 1, 2, 3, \ldots$ and so our first case will be when $n = 1$. So to prove the first case we put $n = 1$ into both sides of De Moivre's formula and verify that they are equal.

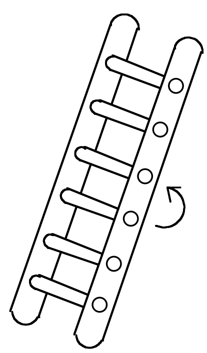
$$(\cos(\theta) + i\sin(\theta))^1 = \cos(1 \cdot \theta) + i\sin(1 \cdot \theta)$$

A quick glance and we can see that this is trivially true.

At this point we emit a loud "Wahoo!" to express our joy. We have one case down, now infinitely many to go.

23.4 The second step: rinse, lather, repeat

In climbing a ladder you will soon learn that as you go from rung to rung you do the same thing over and over. For example the process of moving from the first to the second rung is no different then moving from the 121st to the 122nd rung. So when reviewing manuals on the subject of ladder climbing you will see that they only give a general method on how to move one step at a time. For example, to get from one rung to the next you would first move one foot up and then the other (but not both at the same time).



Moving on up

For mathematical induction we want to duplicate the process of moving up the ladder. So think of the rungs of the ladder as the individual cases that we are trying to prove. We want to show that if some case is true (i.e. we are on some rung of the ladder) then the next case will also be true (i.e. we can get to the next rung of the ladder).

To do this we will start by assuming that the kth case is true and then prove that the $(k+1)$st case must also be true. In terms of our problem of proving De Moivre's formula, we will assume that,

$$(\cos(\theta) + i\sin(\theta))^k = \cos(k\theta) + i\sin(k\theta),$$

and then prove that,

$$(\cos(\theta) + i\sin(\theta))^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta).$$

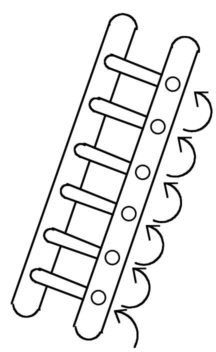
This is the most challenging part of an induction proof. The key element is being able to find a way to use what we know about the kth case to prove something about the $(k+1)$st case. Shown below is the argument for this step for De Moivre's formula. Look carefully for how we use information about the kth case to simplify.

$$\begin{aligned}(\cos(\theta)+i\sin(\theta))^{k+1} &= (\cos(\theta)+i\sin(\theta))^{k}(\cos(\theta)+i\sin(\theta))^{1}\\ &= (\cos(k\theta)+i\sin(k\theta))(\cos(\theta)+i\sin(\theta))\\ &= (\cos(k\theta)\cos(\theta)-\sin(k\theta)\sin(\theta))\\ &\quad +i(\cos(k\theta)\sin(\theta)+\sin(k\theta)\cos(\theta))\\ &= \cos((k+1)\theta)+i\sin((k+1)\theta)\end{aligned}$$

We can now emit as many loud "Wahoo"'s as our heart has room to contain. We are done, let us see why.

23.5 Enjoying the view

Starting with any ladder we know how to get on the first step. Once we are on the first step we can get to the second step. Once we are on the second step we can get to the third step. And so on and so on until we run out of steps.



To infinity and beyond...

Mathematical induction follows the same way. We know that the first case is true (this is by the first step). We also know that since the first case is true that the second case must also be true (this is by the second step). We also know that since the second case is now true that the third case must also be true (this is again by the second step). And so on and so on we apply the second step until we've shown that *every* case is true. In particular we have now shown,

$$(\cos(\theta)+i\sin(\theta))^{n}=\cos(n\theta)+i\sin(n\theta) \qquad \text{for } n=1,2,3,\ldots.$$

23.6 Applying De Moivre's formula

Now that we have De Moivre's formula we should explore some of the uses and consequences of it. Note that De Moivre's formula deals with powers of certain types of complex numbers, namely $\cos(\theta) + i\sin(\theta)$. These are the same kind of terms that we came across when we were dealing with trigonometric representation of complex numbers. So De Moivre's formula can be used to find powers of all kinds of complex numbers.

Example 1 Find the exact value for $(1+i)^{10}$.

Solution First we will convert $1+i$ into its trigonometric form and then use De Moivre's formula to simplify it. Doing this we will get the following.

$$\begin{aligned}
(1+i)^{10} &= \left(\sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]\right)^{10} \\
&= \left(\sqrt{2}\left[\cos(45^\circ) + i\sin(45^\circ)\right]\right)^{10} \\
&= \left(\sqrt{2}\right)^{10}(\cos(45^\circ) + i\sin(45^\circ))^{10} \\
&= (\sqrt{2})^{10}(\cos(450^\circ) + i\sin(450^\circ)) \\
&= 32(0+i) = 32i
\end{aligned}$$

Another thing to notice about De Moivre's formula is that it deals with sines and cosines of multiples of an angle. In particular we can use De Moivre's formula to find equations for sine and cosine of multiples of an angle in terms of sine and cosine of the angle.

Example 2 Use De Moivre's formula to find equations for $\cos(2\theta)$ and $\sin(2\theta)$ in terms of $\cos(\theta)$'s and $\sin(\theta)$'s.

Solution Putting $n = 2$ into De Moivre's formula we get the following.

$$\begin{aligned}
\cos(2\theta) + i\sin(2\theta) &= (\cos(\theta) + i\sin(\theta))^2 \\
&= (\cos^2(\theta) - \sin^2(\theta)) + i(2\sin(\theta)\cos(\theta))
\end{aligned}$$

If we take this equation and set the real parts and the imaginary parts equal to each other we then get the following two equations.

$$\begin{aligned}
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
\sin(2\theta) &= 2\sin(\theta)\cos(\theta)
\end{aligned}$$

23.7 Finding roots

One of the most useful applications of De Moivre's formula is finding nth roots of complex numbers. A number u is an nth root of z if $u^n = z$. If we write z in trigonometric form, i.e. $z = r(\cos(\theta) + i\sin(\theta))$, then one of the nth roots of z is given by the following.

$$\sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right) \qquad \text{(an } n\text{th root of } z\text{)}$$

One question to ask is if whether there are any more nth roots of z. For example there are two square roots and so we would expect that in general that there would be a total of n, nth roots of z. De Moivre's formula will also give us these as well.

Recall that in representing a number in trigonometric form that the angle θ is not unique but that we can add any multiple of $360°$ (or 2π *rads*) to the angle and get other representations for z. Each one of these representations will correspond to a root and by considering all of the possibilities we will get all of the roots.

Specifically, if we have $z = r(\cos(\theta) + i\sin(\theta))$ then there are n nth roots of z and they are given by the following.

$$\sqrt[n]{r}\left(\cos\left(\frac{\theta + k\cdot 360°}{n}\right) + i\sin\left(\frac{\theta + k\cdot 360°}{n}\right)\right) \qquad \text{for } k = 0, 1, \ldots, n-1$$

Note that if we put in $k = n$ then this will correspond to the same root as when $k = 0$. This is why we only need to consider k for n different values. (The roots will just keep repeating every time we go through a cycle of n values.)

The nth roots of z will be evenly spaced, $1/n$th of a revolution, around a circle of radius $\sqrt[n]{r}$.

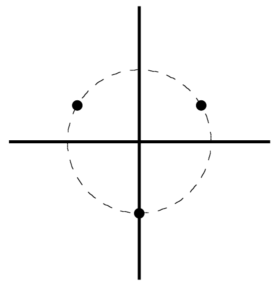
Example 3 Find the cube roots of i.

Solution First note that $i = 1(\cos(90°) + i\sin(90°))$. So in particular by applying our formula for roots above (with $n = 3$) we will get the following.

$$\sqrt[3]{i} = \sqrt[3]{1}\left(\cos\left(\frac{90° + k\cdot 360°}{3}\right) + i\sin\left(\frac{90° + k\cdot 360°}{3}\right)\right)$$

Plugging the values of $k = 0, 1, 2$ we get the following.

$$\begin{aligned} k = 0: &\qquad \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ k = 1: &\qquad -\frac{\sqrt{3}}{2} + \frac{1}{2}i \\ k = 2: &\qquad -i \end{aligned}$$



Graphically, these roots are around the unit circle as shown above.

23.8 Supplemental problems

1. Using De Moivre's formula find equations for $\cos(3x)$ and $\sin(3x)$ in terms of $\cos(x)$'s and $\sin(x)$'s.

2. In an earlier homework assignment we found the following pattern:

$$\cos\left(\frac{\pi}{2^{(n+1)}}\right) = \frac{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}{2}, \qquad \text{with } n \text{ square roots in total.}$$

   Using mathematical induction prove this relationship is true for $n = 1, 2, \ldots$. *Hint*: first verify it is true for the first case and then use the half angle formula for cosine to show that if it is true for one case then it is also true for the next case.

Appendix A

Collection of equations

Over the course of these lectures we have encountered a large number of relationships. Collected here are some of the most useful formulas, though by no means is this all there is.

Reciprocal identities

$$\csc(\theta) = \frac{1}{\sin(\theta)}, \qquad \sec(\theta) = \frac{1}{\cos(\theta)}, \qquad \cot(\theta) = \frac{1}{\tan(\theta)},$$

$$\sin(\theta) = \frac{1}{\csc(\theta)}, \qquad \cos(\theta) = \frac{1}{\sec(\theta)}, \qquad \tan(\theta) = \frac{1}{\cot(\theta)}.$$

Quotient identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.$$

Pythagorean identities

$$\cos^2(\theta) + \sin^2(\theta) = 1, \qquad 1 + \tan^2(\theta) = \sec^2(\theta), \qquad \cot^2(\theta) + 1 = \csc^2(\theta).$$

Complementary angle identities

$$\cos(90^\circ - \theta) = \sin(\theta), \qquad \csc(90^\circ - \theta) = \sec(\theta), \qquad \cot(90^\circ - \theta) = \tan(\theta),$$

$$\sin(90^\circ - \theta) = \cos(\theta), \qquad \sec(90^\circ - \theta) = \csc(\theta), \qquad \tan(90^\circ - \theta) = \cot(\theta).$$

Even/odd'er identities

$$\cos(-\theta) = \cos(\theta), \qquad \sin(-\theta) = -\sin(\theta), \qquad \tan(-\theta) = -\tan(\theta).$$

$$\sec(-\theta) = \sec(\theta), \qquad \csc(-\theta) = -\csc(\theta), \qquad \cot(-\theta) = -\cot(\theta).$$

Sum and difference formulas

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y)+\cos(x)\sin(y),\\
\sin(x-y) &= \sin(x)\cos(y)-\cos(x)\sin(y),\\
\cos(x+y) &= \cos(x)\cos(y)-\sin(x)\sin(y),\\
\cos(x-y) &= \cos(x)\cos(y)+\sin(x)\sin(y),
\end{aligned}$$

$$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}, \qquad \tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}.$$

Double angle identities

$$\sin(2x)=2\sin(x)\cos(x),$$

$$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1=1-2\sin^2(x).$$

Power reduction identities

$$\cos^2(x)=\frac{1+\cos(2x)}{2}, \qquad \sin^2(x)=\frac{1-\cos(2x)}{2}.$$

Half-angle identities

$$\cos\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1+\cos(x)}{2}}, \qquad \sin\left(\frac{x}{2}\right)=\pm\sqrt{\frac{1-\cos(x)}{2}},$$

$$\tan\left(\frac{x}{2}\right)=\frac{1-\cos(x)}{\sin(x)}=\frac{\sin(x)}{1+\cos(x)}.$$

Product to sum identities

$$\begin{aligned}
\cos(x)\cos(y) &= \frac{1}{2}(\cos(x+y)+\cos(x-y)),\\
\sin(x)\sin(y) &= \frac{1}{2}(\cos(x-y)-\cos(x+y)),\\
\sin(x)\cos(y) &= \frac{1}{2}(\sin(x+y)+\sin(x-y)),\\
\cos(x)\sin(y) &= \frac{1}{2}[\sin(x+y)-\sin(x-y)].
\end{aligned}$$

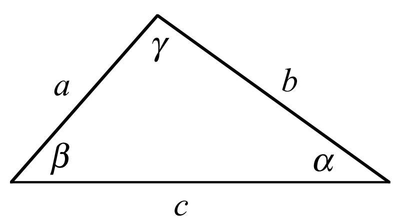
Sum to product identities

$$\begin{aligned}
\cos(x)+\cos(y) &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),\\
\cos(x)-\cos(y) &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),\\
\sin(x)+\sin(y) &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),\\
\sin(x)-\sin(y) &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).
\end{aligned}$$

The identity with no name

$$a\sin(x)+b\cos(x)=(\sqrt{a^2+b^2})\sin(x+\theta)$$

$$\text{where } \theta = \begin{cases} \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) & \text{if } b \geq 0 \\ 360^\circ - \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) & \text{if } b < 0 \end{cases}$$



Law of sines

$$\frac{a}{\sin(\alpha)}=\frac{b}{\sin(\beta)}=\frac{c}{\sin(\gamma)}$$

Law of cosines

$$\begin{array}{lcl}
a^2=b^2+c^2-2bc\cos(\alpha) & \text{or} & \cos(\alpha)=(b^2+c^2-a^2)/(2bc),\\
b^2=a^2+c^2-2ac\cos(\beta) & \text{or} & \cos(\beta)=(a^2+c^2-b^2)/(2ac),\\
c^2=a^2+b^2-2ab\cos(\gamma) & \text{or} & \cos(\gamma)=(a^2+b^2-c^2)/(2ab).
\end{array}$$

Area formulas for triangles

$$\frac{1}{2}(\text{base})(\text{height}), \qquad \text{or} \qquad \frac{1}{2}ab\sin(\gamma), \qquad \text{or} \qquad \frac{a^2\sin(\beta)\sin(\gamma)}{2\sin(\alpha)},$$

$$\text{or} \qquad \sqrt{a(s-a)(s-b)(s-c)} \qquad \text{where } s=\frac{a+b+c}{2}.$$

Limit relationships

$$\text{For } 0 < x < \frac{\pi}{2} \text{ we have } \cos(x) \leq \frac{\sin(x)}{x} \leq 1,$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

Dot product

$$\begin{gathered} \langle a, b\rangle \cdot \langle c, d\rangle = ac + bd, \quad \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}, \\ \vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}, \\ \vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos(\theta), \quad \vec{u}\cdot\vec{u} = \|\vec{u}\|^2, \end{gathered}$$

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Complex numbers in trigonometric form

$$a + bi = r[\cos(\theta) + i\sin(\theta)], \text{ where } r = |a+bi| = \sqrt{a^2+b^2}$$

$$\text{and } \theta = \begin{cases} \arctan(b/a) & \text{if } a \geq 0 \\ \arctan(b/a) + 180^\circ & \text{if } a < 0 \end{cases}.$$

De Moivre's formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta),$$

$$(r[\cos(\theta) + i\sin(\theta)])^{(1/n)} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + k\cdot 360^\circ}{n}\right) + i\sin\left(\frac{\theta + k\cdot 360^\circ}{n}\right)\right)$$

$$\text{for } k = 0, 1, \ldots, n-1.$$