Optimal Arbitrage under Constraints

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Submitted in partial fulfillment of the

Requirements for the degree

of Doctor of Philosophy

in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2013

ABSTRACT

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In this thesis, we investigate the existence of relative arbitrage opportunities in a Markovian model of a financial market, which consists of a bond and stocks, whose prices evolve like Itô processes. We consider markets where investors are constrained to choose from among a restricted set of investment strategies. We show that the upper hedging price of (i.e. the minimum amount of wealth needed to superreplicate) a given contingent claim in a constrained market can be expressed as the supremum of the fair price of the given contingent claim under certain unconstrained auxiliary Markovian markets. Under suitable assumptions, we further characterize the upper hedging price as viscosity solution to certain variational inequalities. We, then, use this viscosity solution characterization to study how the imposition of stricter constraints on the market affect the upper hedging price. In particular, if relative arbitrage opportunities exist with respect to a given strategy, we study how stricter constraints can make such arbitrage opportunities disappear.

Contents

Acknowledgments

I would like to take this opportunity to thank my advisor, Prof. Ioannis Karatzas, for the knowledge, guidance and motivation that I received from him throughout my PhD years. He has been an outstanding researcher in his area; our numerous discussions during research meetings as well as his lectures in class have always greatly inspired me. His enthusiasm for the subject is infectious.

Prof. Karatzas introduced me to the problem that I have addressed in this thesis. I received further insight into the problem and, broadly speaking, into the subject of Mathematical Finance and Stochastic Portfolio Theory, during our research meetings at Intech or in Columbia University. These meetings were participated by Johannes Ruf, Radka Pickova, Tomoyuki Ichiba, Adrian Banner, Vasilios Papathanikolos, Daniel Fernholz, Robert Fernholz and Philip Whitman. I would like to thank each of them for the numerous insights that I received by listening to them. Special thanks is due to Johannes Ruf and Philip Whitman for sharing insightful perspectives regarding the problem. I am indebted to Robert Fernholz, whose fundamental research in Stochastic Portfolio Theory, first published in 2002, has inspired numerous scholars and motivated my thesis.

I am grateful to Professors Marcel Nutz, Victor de la Pena, Philip Protter and Johannes Ruf for agreeing to serve on my dissertation committee and give numerous suggestions.

I am deeply indebted to my friends in New York. Their company and support has been vital in taking me through all these years.

I express my deepest gratitude to Ivy Ng, for supporting me in all possible ways over the last few years.

No thanks are enough for my parents and sister. They have always been by my side. I dedicate this thesis to them.

To my parents and sister

Chapter 1

Introduction

We place ourselves in a market with stocks and a bond. The market participants are rational investors who prefer more wealth to less. An interesting problem would be to find out whether it is possible to have guaranteed profit starting with zero initial wealth, loosely referred to as "arbitrage", probably in allusion to the judicious investments that have to be made to achieve it; the French word "arbitrage" derives from the Latin word "arbitrari" which means "to give judgement". If arbitrage is possible, it would be interesting to find out the optimum way to achieve it, if an optimum exists, where the criterion for optimization depends on the investor's preferences. In this thesis, the criterion for optimization would be the amount of wealth generated by the investment.

Consider a self-financing investment strategy π which starts with \$1 and dynamically trades in stocks and bond. Suppose that there exists another self-financing strategy ρ which also starts with \$1 and dynamically trades in stocks and bond, but gives higher pay-off than π. We will call ρ a "relative arbitrage" opportunity with respect to π. If π invests only in the bond, then ρ is a classical arbitrage opportunity. Investing in π cannot be considered a judicious decision in this case, unless market constraints restrict investment in ρ, and it would be of interest to investors to be aware of the existence of such "relative arbitrage" opportunities. Market constraints could arise, for example, in the form of prohibition of short-selling of stocks or prohibition of borrowing more than a certain amount of money and such, one or more of which an investment in ρ might necessitate.

Suppose that agent A wants to have the same pay-off as that of π at a given time point T, but suppose that market constraints do not allow him to invest in ρ. Suppose also that agent B is allowed to invest in ρ. In that case B can offer to pay A the same pay-off as that of π at time T in exchange for \$1 at the beginning and make guaranteed profit by investing in ρ.

More generally, suppose agent A wants to guarantee a certain pay-off at time T. For example, agent A could be an airlines company who wants to have an arrangement so that it does not have to pay more than \$D (say) per gallon of jet fuel at the end of one year. It might then want to buy a call option on jet fuel with strike price of \$D per gallon, which would guarantee, as pay-off, the positive part of the difference between the price of jet fuel per gallon at the end of one year and \$D. Suppose now that there exists a self-financing strategy by which agent A can replicate or super-replicate the claim by trading in stocks and the bond, starting with an initial wealth w and all the while obeying the constraints imposed on him. The minimum initial wealth for which such a strategy exists is called the upper hedging price ($\mathrm{UHP}_{\mathrm{A}}$) of the claim for agent A under his given set of constraints. Agent A, if judicious, would definitely not be willing to pay any more than $\mathrm{UHP}_{\mathrm{A}}$ for an agreement which would pay him the claim at time T. Consider another agent B, who operates under a different set of constraints and for whom the upper hedging price for the claim under his set of constraints is UHP_B which is less than UHP_A. Then, agents A and B could enter into an agreement by which A pays UHP_A to B at the beginning and B pays the claim to A at the terminal time T, thus resulting in guaranteed profit for B.

The problem of determining whether arbitrage possibilities exist in the market, with the generality with which it has been posed in the first paragraph, is difficult to solve. We will instead, in this thesis, fix an investment strategy π and determine whether there exists an investment strategy which satisfies a given set of market constraints and is a relative arbitrage opportunity with respect to π. We will also study how with increasingly stringent constraints, relative arbitrage opportunities with respect to π first cease to exist in the constrained market and then how the upper hedging price in the constrained market blows up. As a by-product we will also get the fair price of a claim, if it exists, under a given set of constraints. This will enable investors to understand how arbitrage opportunities might arise out of different constraints being imposed on different players in the market. At the same time, it will give us a better understanding of certain aspects of an investment strategy which make relative arbitrage with respect to it possible. Performance of fund managers are often measured in relative terms, which makes the study of relative arbitrage all the more important. Performance of mutual funds is compared to a relevant index such as the S&P 500 index or to other mutual funds in their sector. Performance of hedge fund managers are generally measured in absolute terms, i.e. relative to a strategy which invests only in bonds.

Suppose that all the agents in the market have to obey the same set of market constraints. Consider a contingent claim which is exactly replicable under such constraints, and suppose that there does not exist any relative arbitrage opportunity with respect to this replicating strategy. Then the initial wealth of the replicatng strategy is the “fair price” of the claim under the given set of constraints, i.e. a price at which neither the seller nor the buyer makes a risk free profit. Given a set of constraints, a claim may not be replicable. In that case, the concept of fair price will not exist, and we will have to resort to the concept of upper hedging price. In fact for a particular set of constraints, a claim may not even be super-replicable starting with any finite initial wealth.

Most of the classical work in mathematical finance has been directed towards finding the fair price of contingent claims which can be replicated by trading dynamically in stocks and the bond. They model the stock prices as semimartingale processes and impose mathematical assumptions on the market which essentially rule out “arbitrage” and at the same time are equivalent to the existence of an equivalent probability measure under which the discounted stock prices (the ratio of the stock price and the bond price) are local martingales. This equivalence is the content of the *First Fundamental Theorem of Asset Pricing.* If the replicating strategy satisfies some suitable square-integrability condition, then the initial wealth of the replicating strategy turns out to be the expectation of the discounted contingent claim under the equivalent probability measure. If all investment strategies are constrained to satisfy the afore-mentioned square-integrability condition, then this is also the fair price of the contingent claim. Such an approach dates back to the fundamental and seminal work of J.M. Harrison and H.R. Pliska from the late 1970s, which was first published as Harrison and Pliska (1981). There they analysed a finite sample space and assumed that the market satisfies the condition of *No Arbitrage (NA).* Harrison and Kreps (1979) and Kreps (1981) introduced the condition of *No Free Lunch (NFL)* for infinite probability spaces. Though this was a significant improvement, NFL was not amenable to intuition. Delbaen and Schachermayer (1994) and Delbaen and Schachermayer (1998) introduced the condition of *No Free Lunch with Vanishing Risk (NFLVR),* which is much more intuitive and economically justifiable. We refer the reader to Delbaen and Schachermayer (2006) for a thorough introduction to NA, NFLVR and other notions of arbitrage.

On the other hand, Stochastic Portfolio Theory (SPT), is not predicated upon the absence of arbitrage in the market. Instead of imposing conditions on the market to make it free of arbitrage, it tries to characterize market conditions which make arbitrage possible. It was introduced by Robert Fernholz in the seminal paper Fernholz and Shay (1982) and built upon in the monograph Fernholz (2002). The interested reader can also see the survey paper Fernholz and Karatzas (2009). SPT studies the construction of portfolios with controlled behavior and in particular those which can take advantage of certain market conditions in order to create arbitrage opportunities. Fernholz and Karatzas (2010) model the stock price processes as Itô processes and characterize the existence of relative arbitrage with respect to the market portfolio by the smallest non-negative solution of a parabolic partial differential equation. In the case that relative arbitrage exists with respect to the market portfolio, it also determines the investment strategy which super-replicates the market portfolio starting with the minimum wealth required for such super-replication.

Fernholz and Karatzas (2009) show that the existence of the equivalent martingale measure is not essential for the calculation of the fair price of a contingent claim. They model the stock prices as Itô processes. Under the assumption that there exists a square-integrable market price of risk, they define a stochastic discount factor process. Note here that the assumption of the existence of a square-integrable market price of risk, implicitly imposes the condition that

NFLVR fails if and only if NA fails; see Proposition 3.2 of Karatzas and Kardaras (2007). They also assume that the market is "complete" and show that the fair price of a contingent claim is the expectation of the contingent claim weighted by the stochastic discount factor at the terminal time. If the fair price of the wealth generated at time T by an investment strategy is less than the initial wealth, then there exists relative arbitrage with respect to this strategy. For a Markovian market, and under certain fairly general conditions, Fernholz and Karatzas (2010) determine the optimal portfolio to make arbitrage profit if there exists relative arbitrage with respect to the "market portfolio". Ruf (2011) does not assume market completeness, but shows, under certain conditions, that a contingent claim which depends only on the final stock price, can be replicable by an investment strategy. In this case, the fair price of the claim is again the expectation of the contingent claim weighted by the stochastic discount factor. When an equivalent martingale measure exists, the stochastic discount factor becomes its Radon-Nikodým derivative with respect to the original probability measure, and we get back to the situation of classical Mathematical Finance.

Cvitanić and Karatzas (1993) determine the upper hedging price, i.e. the minimum wealth needed to super-replicate a given contingent claim, when market constraints restrict investment in certain investment strategies. They show that the upper hedging price of the contingent claim can be expressed as the supremum of the fair price of the contingent claim in certain unconstrained auxilliary markets, in which the contingent claim is replicable.

In this thesis, we will not impose conditions in the market which will rule out arbitrage opportunities. Instead, we will consider a market where relative arbitrage opportunities with respect to a given investment strategy might exist.

Given an investment strategy we will consider different sets of constraints and determine how stringent we need to be so that none of the investment strategies satisfying the constraints present relative arbitrage opportunity with respect to the given strategy. Note here, that in a market, where all the investors have the same constraints imposed on them, assumption of no arbitrage has the economic justification that if there is arbitrage, investors will immediately take advantage of it and arbitrage will disappear. However, arbitage opportunities that arise out of different constraints being imposed on different investors, continue to exist.

An investor may be subject to constraints on his choice of investment strategies because of several reasons. We will discuss some commonly occurring constraints. However, our work in the thesis will address only a specific type of constraints, viz. those in which the vector of proportion of wealth invested in the stocks is constrained to lie in a given closed convex set at all times and the wealth process is positive at all times. We will leave other types of constraints for future research.

One of the most important market constraints is that of prohibition of short selling. Short selling comes with a huge downside potential. As an InvestorGuide article puts it, “most short sellers put a limit to how much they are willing to lose, but as stock prices rise they can become vulnerable to a short ’squeeze’, in which long investors buy stocks and demand delivery. As short sellers buy to cover their losses, the price continues to rise, triggering more short sellers to cover their losses, etc”. Hedge funds are allowed to engage in short selling, but they are only open to particular types of investors specified by regulators. These investors are typically institutions, such as pension funds, university endowments and foundations, or high net worth individuals. Mutual funds, on the other hand, are open to the general public and are not allowed to engage in any risky investments. Most mutual funds are not allowed to short sell unless they satisfy several conditions. Short selling is difficult also for individual investors. Thus this is an example of a constraint which some investors have to obey while others do not. This same prohibition of short selling will apply to all investors if the restriction is market-wide. For example, many emerging markets do not allow short selling out of the fear that it would disrupt orderly markets by causing panic selling, high volatility, and market crashes. Prohibition of short selling in a stock can be fit into our framework through constraining the proportion of wealth invested in the stock to be non-negative at all times.

Another interesting constraint is the uptick rule, which was eliminated in 2007 but has been up for consideration again. As per the rule, every short sale transaction is required to be entered at a price that is higher than the price of the previous trade. This rule was in effect to prevent traders known as "pool operators" from driving down a stock price through heavy short selling, then buying the shares for a large profit. Such a constraint does not fit into our framework.

Leverage constraints are those in which the extent of the total short position of an investor is limited to a fixed fraction of his wealth. They can be fit into our framework by requiring that the proportion of wealth invested in shorting is greater than a given constant. Liquidity constraints, which limit the investments in the stock market to a fixed proportion of the total wealth and require the rest to be maintained as capital reserve, are also amenable to our framework. The decline of Bear Stearns and Lehman Brothers have proved the importance of such constraints.

Some fund managers restrict their investments to stocks which belong to their field of expertise. This can be seen as a constraint which prohibits trading in certain stocks.

1.0.1 Preview

We start by reviewing some background material in Chapter 2. In Section 2.1 we present the market model for the stock prices and the bond. In Section 2.2 we specialize this to the Markovian market model, i.e. one, in which the dynamics of the stock prices and the bond price depend only on time and the current stock prices. In Section 2.3 we discuss the market price of risk and the associated stochastic discount factor. Section 2.4 introduces the definition of strategies and portfolios. For us, a strategy is a specification of the proportion of wealth to be invested in each stock. A portfolio is a strategy which invests the entire wealth in the stock market. Section 2.5 presents the concept of strong and weak relative arbitrage with respect to strategies. In Section 2.6 we discuss the notion of upper hedging price process for European contingent claims. These contingent claims need not be the terminal wealth generated by strategies. However, in complete markets, any strictly positive contingent claim can be replicated by a strategy, i.e., can be expressed as the terminal wealth generated by a strategy, and in this case we compute the fair price of the contingent claim. For Markovian markets and when the replicating strategy is Markovian, this fair price is a function of time and the current stock price and can be characterized as being the smallest non-negative solution to a certain partial differential equation. This section is a restatement of some of the results from Fernholz and Karatzas (2010).

In Chapter 3, we approach the problem of finding the upper hedging price of a contingent claim under market constraints. The vector of proportions of wealth invested in each stock is constrained to lie in a given closed convex set at each time point. Our approach here is similar to that of Cvitanić and Karatzas (1993). They showed that the upper hedging price of the contingent claim under market constraints can be evaluated as the supremum of the fair price in certain unconstrained auxiliary markets. These auxiliary markets are constructed using dual processes, which essentially act like Lagrange multipliers. In Section 3.1 we present some of the tools from convex analysis that we use in the exposition, and also derive some useful results. We consider more general constraint sets than those in Cvitanić and Karatzas (1993). For example, market constraints which prohibit short selling of stocks and investments in the bond market do not satisfy the conditions imposed by Cvitanić and Karatzas (1993), but satisfy our conditions. In Section 3.2, we introduce the class of dual processes mentioned above and use them to construct auxiliary processes which, in turn, define auxiliary discount factors. The upper hedging price of the contingent claim under the given constraints, is then shown to be the supremum, over the auxiliary processes, of the expectation of the contingent claim discounted by the auxiliary discount factors. Under slightly more restrictive conditions on the market model and the constraint set, which allow the interpretation of auxiliary markets, this result has been proved in Cvitanić and Karatzas (1993). Our contribution and the main result of this chapter is that in the special case of Markovian markets, the dual processes can be taken to be Markovian. However, for this result, we have to impose some conditions on the contingent claims. We suppose that the contingent claim Y is the terminal wealth generated by the Markovian stratgey π starting with initial wealth \$w, and that $\pi(\cdot, \cdot)$ is locally bounded as a function of stock prices. It is clear, that Y will be a multiple of the wealth generated by π starting with \$1 at time t, the multiplicative factor being the wealth generated by π up to time t starting with initial wealth \$w. We define the arbitrage coefficient of π at time t under a given set of constraints to be the upper hedging price at time t of the terminal wealth generated by π starting with \$1 at time t. The value of the upper hedging price process of Y at time t, will then be a multiple of the arbitrage coefficient of π at time t, the multiplicative factor being the wealth generated by π up to time t starting with initial wealth \$w. When the market and the strategy π are Markovian, the arbitrage coefficient of π at a time t will be a function of time and the current stock price. Relative arbitrage with respect to π exists over the time horizon $[t, T]$, if and only if the value of the arbitrage coefficient of π at time t is less than 1.

In Chapter 4, we put ourselves in the Markovian market framework and restrict attention to contingent claims which can be expressed as being the terminal wealth generated by Markovian trading strategies; but shift the focus from the contingent claims and their upper hedging price to the replicating strategies and their arbitrage coefficient. In Section 4.1, we present dynamic programming principles followed by the upper hedging price process and the arbitrage coefficient process. In Section 4.2, we characterize the arbitrage coefficient of a strategy in terms of the viscosity subsolution and supersolution of certain partial differential equations. The result, proved in Section 3.2, that the dual processes can be taken to be Markovian is essential over here. Our viscosity solutions approach follows the exposition of Pham (2009) and of Soner and Touzi (2003). Soner and Touzi (2003) develop a similar viscosity solution characterization for contingent claims which are functions of the final stock price, under more restrictive assumptions on the market model. Sections 4.3, 4.4 and 4.5 contain the technical proofs of the results presented in Section 4.2. Sections 4.6 and 4.7 contain some auxiliary technical results.

In Section 5.1, we present some comparison results for these viscosity solutions and discuss their relevance in the discussion of relative arbitrage. Section

5.2 contains a very important part of the thesis, where we use the viscosity solutions characterization to study how the constraints affect the value of the arbitrage coefficient and thus govern the existence or absence of relative arbitrage. Convex polyhedral constraint sets are particularly amenable to our analysis and most of our examples involve such constraint sets. In Section 5.3, we discuss polyhedral convex sets. Section 5.4 contains some examples.

1.1 Notation

Vectors and matrices

We will write $\mathbf{1}$ to denote the vector with all elements equal to 1, $\mathbf{0}$ to denote the vector with all elements equal to 0, and e_i to denote the vector with all elements 0 except for the i-th element being 1. The dimensions of the respective vectors will be clear from the context. Given a set A of vectors in $\mathbb{R}^k$, we will denote the vector space generated by A as $\text{vect}(A)$. Given $x, y \in \mathbb{R}^k$, we will write $x \geq y$ to mean $x_i \geq y_i,\ i = 1, 2, \ldots, k$. The ordered elements of x will be denoted by $x_{(i)}, i = 1, 2, \ldots, k$ with $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(k)}$. We will denote by $\text{diag}(x)$ a $k \times k$ diagonal matrix with x as its diagonal. For any matrix $\sigma = ((\sigma_{ij}))$, we will denote its i-th row and j-th column by σ_{i*} and σ_{*j} respectively.

We will denote the Euclidean norm by $\|\cdot\|$. For any $k \in \mathbb{N}, \delta > 0$ and $x \in \mathbb{R}^k$, $B^k(x, \delta)$ will denote the open ball of radius δ around x, i.e. $B^k(x, \delta) = \left\{y \in \mathbb{R}^k : \|y - x\| < \delta\right\}$. We denote by $\mathcal{S}(n)$, the set of all symmetric $n \times n$ matrices. Given matrices $X, Y \in \mathcal{S}(n)$, we say $X \geq Y$ if $X - Y$ is nonnegative definite and $X > Y$ if $X - Y$ is positive definite. For a square matrix A, we will denote by $\text{tr}(A)$ the trace of the matrix A.

Functions

Given any function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}^k$, we will write $f(x)$ to denote the vector $(f(x_1), \ldots, f(x_k))$. For example, we will write $\log(x)$ to denote $(\log(x_1), \ldots, \log(x_k))$. For a vector field $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, we will denote $\nabla_x g(x, y) := \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_m}\right)(x, y)$. The function sgn: $\mathbb{R} \to \{-1, 1\}$ is defined as

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{1.1}$$

Sets

The closure of a set A will be denoted by $\bar{A}$, its boundary by ∂A, its interior by A^o and its relative interior by $\mathrm{ri}(A)$. Given a set $C = [a, b] \times B^k(x, \delta) \subset \mathbb{R}_+ \times \mathbb{R}^k$, we will denote its parabolic boundary by $\partial_p C = \big((a, b] \times \partial B^k(x, \delta)\big) \cup \big(\{b\} \times B^k(x, \delta)\big)$.

Probability spaces

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we will drop the argument ω from the random variable $X(\omega)$, and denote it by X. If the random variable X is a function of another random variable Y, then we will denote the function again by X; the meaning will be clear from the context.

Chapter 2

Market Model and Pricing of Contingent Claims

2.1 General market model

We place ourselves in a model $\mathcal{M}$ for a financial market consisting of a money market

$$\mathrm{d}B(t) = B(t)\mathfrak{r}(t)\mathrm{d}t, \quad B(0) = 1, \tag{2.1}$$

and of n stocks with capitalizations $X_i(\cdot) > 0$ that satisfy

$$\begin{gathered} dX_i(t) = X_i(t)\left(\mathfrak{b}_i(t)\mathrm{d}t + \sum_{\nu=1}^{d} \mathfrak{s}_{i\nu}(t)dW_\nu(t)\right), \\ X_i(0) = x_i > 0, \; i = 1, 2, \ldots, n. \end{gathered} \tag{2.2}$$

These are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are driven by the d-dimensional Brownian motion $W(\cdot) = (W_1(\cdot), \ldots, W_d(\cdot))'$ with $d \geq n$. The filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$, which represents the flow of information in the market, is assumed to be right-continuous with $\mathcal{F}(0) = \{0, \Omega\}$, mod. $\mathbb{P}$. We shall assume that the vector-valued process $\mathcal{X}(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'$ of capitalizations, the non-negative interest rate process $\mathfrak{r}(\cdot)$, the vector valued process $\mathfrak{b}(\cdot) = (\mathfrak{b}_1(\cdot), \ldots, \mathfrak{b}_n(\cdot))'$

of *mean rates of return* for the various stocks, and the $(n \times d)-$matrix-valued process $\mathfrak{s}(\cdot) = (\mathfrak{s}_{i\nu}(\cdot))_{1\le i\le n, 1\le \nu\le d}$ of stock-price *volatilities* , are all $\mathbb{F}$-progressively measurable. Finally, we denote by

$$\mathfrak{a}_{ij}(t) = \sum_{\nu=1}^{d} \mathfrak{s}_{i\nu}(t)\mathfrak{s}_{j\nu}(t) = (\mathfrak{s}(t)\mathfrak{s}'(t)))_{ij},$$

the (i, j)-th element of the non-negative definite matrix-valued *covariance process* $\mathfrak{a}(\cdot) = (\mathfrak{a}_{ij}(\cdot))_{1\le i,j\le n}$ of the stocks in the market.

Consider the following assumption:

Assumption 2.1.1. *For every $T \in (0, \infty)$*

$$\int_0^T \left(\mathfrak{r}(t) + \sum_{i=1}^{n} |\mathfrak{b}_i(t)| + \sum_{i=1}^{n} \mathfrak{a}_{ii}(t) \right) \mathrm{d}t < \infty, \qquad a.s. \tag{2.3}$$

If Assumption 2.1.1 holds, then with

$$\gamma_i(t) = \mathfrak{b}_i(t) - \frac{1}{2}\mathfrak{a}_{ii}(t),$$

we can define

$$Y_i(t) = Y_i(0) + \int_0^t \gamma_i(s)\mathrm{d}s + \sum_{\nu=1}^{d} \int_0^t \mathfrak{s}_{i\nu}(s)dW_\nu(s)\mathrm{d}s, \quad i = 1, \ldots, n. \tag{2.4}$$

By Itô's lemma, we can see that

$$X_i(t) = e^{Y_i(t)},\ 0 \le t < \infty, \tag{2.5}$$

has the dynamics of (2.2). Progressive measurability of $\mathfrak{b}(\cdot)$ and $\mathfrak{s}(\cdot)$ and the integrability condition (2.3) guarantee that $Y_i(\cdot)$ is a continuous $\mathcal{F}(t)$-adapted process, and hence $X_i(\cdot)$, defined in (2.5), is a strictly positive continuous adapted process for each $i = 1, 2, \ldots, n$.

In this setting, the Brownian motion $W(\cdot)$ need not be adapted to the filtration $\mathbb{F}$. However, it is adapted to the $\mathbb{P}$-augmentation $\mathbb{G} = \{\mathcal{G}(t)\}_{0\le t<\infty}$ of the filtration $\mathbb{F}$, provided that $d = n$ and $\mathfrak{s}$ is invertible.

2.2 Markovian market model

As a special case of the market model specified in the previous section, sometimes we will consider a Markovian market model where the interest rate process and the mean rates of return and volatility coefficients of the stock price processes $X_i(\cdot)$ depend only on time and on the current market configuration i.e. $\mathfrak{b}(t) = b(t, \mathcal{X}(t))$, $\mathfrak{s}(t) = \sigma(t, \mathcal{X}(t))$ and $\mathfrak{r}(t) = r(t, \mathcal{X}(t))$, and hence $\mathfrak{a}_{ij}(t) = a_{ij}(t, \mathcal{X}(t))$, where $r(\cdot,\cdot), b_i(\cdot,\cdot), \sigma_{ij}(\cdot,\cdot)$ and $a_{ij}(\cdot,\cdot)$ are Borel measurable functions from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}$ for $1 \le i \le n, 1 \le j \le d$. That is, the stock price processes satisfy the stochastic differential equation,

$$\begin{aligned} dX_i(t) &= X_i(t)\left(b_i(t,\mathcal{X}(t))dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t,\mathcal{X}(t))dW_\nu(t)\right), \\ X_i(0) &= x_i > 0,\ i = 1, 2, \ldots, n. \end{aligned} \tag{2.6}$$

We will assume that

Assumption 2.2.1. *There exists a unique-in-distribution weak solution to the SDE* (2.6).[1]

Remark 2.2.1. From the proof of Theorem 1.1 in Athreya et al. (2002)(pg 31), we see that sufficient conditions for the existence of weak solution to (2.6) is that the functions $x_i b_i(t,x)$ and $x_i\sigma_{ij}(t,x)$ are continuous functions of x and $|x_i b_i(t,x)| \le$

[1] As pointed out by Ioannis Karatzas in a personal communication (2012), Assumption 2.2.1 may not be needed for our work. If there are multiple solutions to the SDE (2.6), we might be able to use the techniques for selecting Markov solutions as in Krylov (1973) and Chapter 12 of Stroock and Varadhan (1979). See also page 15 of Ruf (2011). This remains to be worked upon.

$C(1 + |x|)$ for all $x \in [0, \infty)^d, i = 1, 2, \ldots, n$. The proof has been given in the time homogeneous case but it works also for the time dependent case. Stroock and Varadhan (1979) gives uniqueness of solutions up until the first hitting time of $\partial([0, \infty)^d)$. Now if we assume that $x_i b_i(t, x)$, $x_i \sigma_{ij}(t, x)$ are locally Lipschitz and satisfy the linear growth condition in the space variable, then by Theorem 9.4.1 in Friedman (2006), we can conclude that X_i takes values in $(0, \infty)^n$ for all $t \geq 0$, and hence uniqueness in distribution holds for the weak solution of (2.6). The other required conditions involving the degeneracy of the diffusion matrix on the boundary of the positive orthant and the Fichera drift are trivially satisfied. Being locally Lipschitz implies continuity and boundedness on compact subsets of $\mathbb{R}^d$. Thus, all we need to assume for the existence of a unique-in-distribution weak solution is that $x_i b_i(t, x)$ and $x_i \sigma_{ij}(t, x)$ are locally Lipschitz and satisfy the linear growth condition in the space variable for $i = 1, 2, \ldots, n$.

Remark 2.2.2. Remark 5.4.31 in Karatzas and Shreve (1991) suggests that if $x_i x_j a_{ij}(t, x)$ is twice continuously differentiable in the state variable, or less restrictively if $x_i \sigma_{ij}(t, x)$ is locally Lipschitz in the state variable, and if $x_i b_i(t, x)$ is continuously differentiable in the state variable, then there exists at most one solution. Having assumed so, we need only linear growth of $x_i b_i(t, x)$ in the space variable in order to guarantee existence of a solution. Thus another sufficient condition for the existence of a unique solution is that $x_i \sigma_{ij}(t, x)$ is locally Lipschitz in x, and $x_i b_i(t, x)$ is continuously differentiable in x and satisfies the linear growth condition in x. The linear growth condition on $x_i \sigma_{ij}(t, x)$ in x as mentioned in Remark 2.2.1 is replaced here by the existence of a continuous space derivative of $x_i b_i(t, x)$.

Also, X_i takes values in $(0, \infty)$ iff its reciprocal $\Xi_i = 1/X_i$ takes values in $(0, \infty)$. Therefore, conditions which guarantee non-explosive solutions for the SDE for Ξ (which can be written down using Itô's lemma), guarantee that X_i takes values in $(0, \infty)$.

Consider the functions

$$\widetilde{b}(t,x) := b\left(t,e^{x}\right),\ \widetilde{\sigma}(t,x) := \sigma\left(t,e^{x}\right),\ \widetilde{a}(t,x) = a\left(t,e^{x}\right),\quad x \in \mathbb{R}^n, \tag{2.7}$$

$$\widetilde{\gamma}_i(t,x) = \widetilde{b}_i(t,x) - \frac{1}{2}\widetilde{a}_{ii}(t,x),\ i = 1,2,\ldots,n,\quad x \in \mathbb{R}^n, \tag{2.8}$$

where for $\mathbb{R}^n \ni x = (x_1,\ldots,x_n)$ we write $e^x = (e^{x_1},\ldots,e^{x_n})$. Suppose $\mathcal{Y}(\cdot) = (Y_1(\cdot),Y_2(\cdot),\ldots,Y_n(\cdot))$ is a solution to the SDE

$$\begin{aligned} dY_i(t) &= \widetilde{\gamma}_i(t,\mathcal{Y}(t))dt + \sum_{\nu=1}^{d} \widetilde{\sigma}_{i\nu}(t,\mathcal{Y}(t))dW_\nu(t), \\ Y_i(0) &= \ln x_i,\ i = 1,2,\ldots,n. \end{aligned} \tag{2.9}$$

An application of Itô's lemma shows that

$$X_i(t) := e^{Y_i(t)},\ i = 1,2,\ldots,n, \tag{2.10}$$

is a solution to (2.6). Now, for any solution $\mathcal{X}(\cdot)$ of (2.6), let ζ denote the first time that $\mathcal{X}(\cdot)$ hits $\partial[0,\infty)^n$. By an application of Itô's lemma, we see that up until the stopping time ζ, $Y_i(t) := \ln X_i(t), i = 1,2,\ldots,n$, solves the SDE (2.9). Therefore, conditions which imply uniqueness-in-distribution of solutions of (2.9), also imply uniqueness-in-distribution of solutions of (2.6) up until the first hitting time of $\partial[0,\infty)^n$.

Remark 2.2.3. It follows from Remark 2.2.1, that sufficient condition for the existence of a unique-in-distribution weak solution of (2.9) is that $b_i(t,x), \sigma_{ij}(t,x)$ are locally Lipschitz and satisfy the linear growth condition in the space variable. This condition is much less stringent than requiring that $x_i b_i(t,x), x_i\sigma_{ij}(t,x)$ are locally Lipschitz and satisfy the linear growth condition in the space variable.

2.3 Market price of risk and strict local martingales

We shall assume from now on that

Assumption 2.3.1. *There exists a market price of risk* $\Theta : [0, \infty) \times \Omega \to \mathbb{R}^d$*, an* $\mathbb{F}$*-progressively measurable process that satisfies*

$$\mathfrak{s}(t,\omega)\Theta(t,\omega) = \mathfrak{b}(t,\omega) - \mathfrak{r}(t,\omega)\mathbf{1}, \forall (t,\omega) \in [0,\infty) \times \Omega; \tag{2.11}$$

$$\textit{and, } \mathbb{P}\left(\int_0^T \|\Theta(t,\omega)\|^2 \mathrm{d}t < \infty, \forall T \in (0,\infty)\right) = 1. \tag{2.12}$$

The existence of market-price-of-risk process $\Theta(\cdot)$ allows us to introduce an associated exponential local martingale

$$Z^{\Theta}(t) := \exp\left\{-\int_0^t \Theta'(s)\mathrm{d}W(s) - \frac{1}{2}\|\Theta(s)\|^2 \mathrm{d}s\right\}, \quad 0 \le t < \infty. \tag{2.13}$$

This process is also a supermartingale; it is a martingale, if and only if $\mathbb{E}(Z^{\Theta}(T)) = 1$ holds for all $T \in (0, \infty)$. For the purpose of this work it is important to allow such exponential processes to be strict local martingales; that is, not to exclude the possibility $\mathbb{E}(Z^{\Theta}(T)) < 1$ for some $T \in (0, \infty)$.

The quantity $Z^{\Theta}(t)/B(t)$ will arise repeatedly in our calculations. To alleviate notation we will denote

$$H^{0,\Theta}(t) := Z^{\Theta}(t)/B(t). \tag{2.14}$$

For $0 \le t \le s \le T$, we will denote

$$\widetilde{H}^{0,\Theta}(t,s) := H^{0,\Theta}(s)/H^{0,\Theta}(t). \tag{2.15}$$

The role of the superscript 0 in $H^{0,\Theta}$ will become clear after we introduce a class of dual processes in Section 3.2.

2.4 Strategies and portfolios

Consider now a small investor who decides at each time t, which proportion $\mathfrak{p}_i(t)$ of current wealth to invest in the i-th stock, $i = 1, 2, \ldots, n$; the proportion $1 - \sum_{i=1}^n \mathfrak{p}_i(t) =: \mathfrak{p}_0(t)$ gets invested in the money market. We define an investment strategy on the bounded interval $I \subset [0, \infty)$ to be such a $\mathbb{G}$-progressively measurable process $\mathfrak{p} : I \times \Omega \to \mathbb{R}^n$ such that

$$\int_I (|\mathfrak{p}'(t,\omega)\mathfrak{b}(t,\omega)| + \mathfrak{p}'(t,\omega)\mathfrak{a}(t,\omega)\mathfrak{p}(t,\omega))\mathrm{d}t < \infty, \text{ for } \mathbb{P} - \text{a.e. } \omega \in \Omega. \tag{2.16}$$

We denote by $V^{v,\mathfrak{p}}(s,t)$ the wealth generated by an investment strategy $\mathfrak{p}$ in the time interval $[s,t]$ starting with capital v at time s. Also note that $V^{v,\mathfrak{p}}(s,t) = vV^{1,\mathfrak{p}}(s,t)$. Thus for any fixed s, the wealth process $V^{v,\mathfrak{p}}(s,\cdot)$ satisfies the initial condition $V^{v,\mathfrak{p}}(s,s) = v$ and for $t > s$, the dynamics

$$\begin{aligned} \mathrm{d}V^{v,\mathfrak{p}}(s,t) &= \sum_{i=1}^n \mathfrak{p}_i(t) V^{v,\mathfrak{p}}(s,t) \frac{\mathrm{d}X_i(t)}{X_i(t)} + \mathfrak{p}_0(t) V^{v,\mathfrak{p}}(s,t)\mathfrak{r}(t)\mathrm{d}t \\ &= V^{v,\mathfrak{p}}(s,t)\Bigg([\mathfrak{p}'(t)\mathfrak{b}(t) + \mathfrak{p}_0(t) r(t)]\mathrm{d}t + \mathfrak{p}'(t)\mathfrak{s}(t)\mathrm{d}W(t) \Bigg). \end{aligned} \tag{2.17}$$

The condition (2.16) implies that the wealth process $V^{v,\mathfrak{p}}(\cdot,\cdot)$ is strictly positive.

Let K be a non-empty closed convex subset of $\mathbb{R}^n$. Suppose that the market is so constrained that the investor can only choose strategies $\mathfrak{p}(\cdot)$ that satisfy

$$\mathfrak{p}(t) \in K \text{ for Lebesgue a.e. } t \in I \text{ a.s.}. \tag{2.18}$$

We will denote such a market model by $\mathcal{M}(K)$. Strategies which satisfy (2.18) will be called *admissible* on the time interval I and the constraint set K; their collection will be denoted $\mathcal{H}(I,K)$.

A strategy $\mathfrak{p}(\cdot) \in \mathcal{H}(I,K)$ with $\sum_{i=1}^n \mathfrak{p}_i(s,\omega) = 1$ for all $(s,\omega) \in I \times \Omega$ will be called a *portfolio* on I with constraint set K. A portfolio never invests in the money market, and never borrows from it. Note that a portfolio can be seen as a strategy

$\mathfrak{p}(\cdot)$ constrained so that $\mathfrak{p}(s) \in K_0, s \in I$, where K_0 is the closed convex set

$$K_0 := \left\{ p \in \mathbb{R}^n \middle| \sum_{i=1}^{n} p_i = 1 \right\}. \tag{2.19}$$

A strategy which prohibits short-selling, i.e., for which $\mathfrak{p}_i(\cdot) \geq 0, i = 1, 2, \ldots, n$, is called a long-only strategy. A long-only portfolio of particular interest is the *market portfolio*; this invests in all stocks in proportion to their relative weights

$$\mu_i(t) := \frac{X_i(t)}{X(t)}, i = 1, 2, \ldots, n, \text{ where } X(t) := X_1(t) + \ldots + X_n(t). \tag{2.20}$$

If the strategy depends only on time and the current market configuration, we will call it a *Markovian strategy.*

2.5 Relative arbitrage

The following notion was introduced in Fernholz (2002): given $0 \leq t < T < \infty$ and any two investment strategies $\mathfrak{p}(\cdot)$ and $\mathfrak{q}(\cdot)$ in $\mathcal{H}([t, T], \mathbb{R}^n)$, we call $\mathfrak{p}(\cdot)$ an *arbitrage relative to* $\mathfrak{q}(\cdot)$ over $[t, T]$, if

$$\begin{aligned} \mathbb{P}(V^{1,\mathfrak{p}}(t, T) \geq V^{1,\mathfrak{q}}(t, T)) = 1, \\ \text{and } \mathbb{P}(V^{1,\mathfrak{p}}(t, T) > V^{1,\mathfrak{q}}(t, T)) > 0. \end{aligned} \tag{2.21}$$

We call such relative arbitrage *strong*, if

$$\mathbb{P}\left(V^{1,\mathfrak{p}}(t, T) > V^{1,\mathfrak{q}}(t, T)\right) = 1.$$

Consider the situation where the following assumption holds.

Assumption 2.5.1. *All eigenvalues of the covariance matrix process $a(\cdot)$ are uniformly bounded away from infinity; that is*

$$\xi' \mathfrak{a}(t) \xi = \xi' \mathfrak{s}(t) \mathfrak{s}'(t) \xi \leq K \|\xi\|^2, \ \ \forall t \in [0, \infty) \text{ and } \xi \in \mathbb{R}^n. \tag{2.22}$$

We shall refer to Assumption 2.5.1 as the uniform boundedness condition on the covariance structure of $\mathcal{M}$.

Proposition 6.1 and (6.12) in Fernholz and Karatzas (2009) then asserts the following:

Proposition 2.5.1. *Suppose Assumption 2.3.1 and Assumption 2.5.1 hold in our market model $\mathcal{M}$. Let $\mathfrak{p}(\cdot)$ and $\mathfrak{q}(\cdot)$ be two portfolios which are uniformly bounded in (t,ω). Suppose that for some real number $T > 0$, $\mathfrak{p}(\cdot)$ is an arbitrage relative to $\mathfrak{q}(\cdot)$ on the time horizon $[0,T]$. Then, the process $H^{0,\Theta}(t)V^{v,\mathfrak{q}}(t), 0 \leq t \leq T$ is a strict local martingale and a strict supermartingale, namely $\mathbb{E}(H^{0,\Theta}(T)V^{v,\mathfrak{q}}(T)) < v$.*

The process $Z^{\Theta}(\cdot)$ of (2.13) is also a strict local martingale on $[0,T]$, i.e. $\mathbb{E}(Z^{\Theta}(T)) < 1$.

2.6 Hedging European contingent claims

We will now broach the issue of hedging strictly positive European contingent claims in our market model $\mathcal{M}(K)$ and over a time horizon $[0,T]$ with a real number $T > 0$. Consider a contingent claim which is an $\mathcal{F}(T)$-measurable random variable $Y : \Omega \to (0,\infty)$ with

$$0 < y := \mathbb{E}\left(YH^{0,\Theta}(T)\right) < \infty.$$

From the point of view of the seller of the contingent claim, this random amount represents a liability that has to be covered with the right amount of initial funds $u^Y(0,T,K)$ at time $t = 0$ and the right trading strategy during the interval $[0,T]$, so that at the end of the time-horizon (time $t = T$) the initial funds have grown enough to cover the liability without risk. Suppose now the seller of the contingent claim can at time t pass on the liability of the contingent claim to another person, who in exchange should want the right amount of money $u^Y(t,T,K)$ from the seller, so that with that amount and the right trading strategy during the interval $[t,T]$, the fund $u^Y(t,T,K)$ should have grown enough to cover the liability without risk. Thus we are interested in the so-called *upper hedging price*

$$u^Y(t,T,K) := \inf\{w > 0 | \exists \mathfrak{p}(\cdot) \in \mathcal{H}([t,T],K) \text{ s.t. } V^{w,\mathfrak{p}}(t,T) \geq Y, \text{ a.s.}\} \tag{2.23}$$

the smallest amount of capital needed at time t that makes such riskless hedging possible in the constrained market. From equation (2.13), by an application of Itô's formula, we have,

$$\mathrm{d}H^{0,\Theta}(s) = -H^{0,\Theta}(s)\Big(\mathfrak{r}(s)\mathrm{d}s + \Theta'(s)\mathrm{d}W(s).\Big) \tag{2.24}$$

It follows that for $\mathfrak{p}(\cdot) \in \mathcal{H}([t,T],K)$ and any $w > 0$ and any $t \leq s \leq T$,

$$\mathrm{d}(H^{0,\Theta}(s)V^{w,\mathfrak{p}}(t,s)) = H^{0,\Theta}(s)V^{w,\mathfrak{p}}(t,s)\Big(\mathfrak{p}'(s)\mathfrak{s}(s) - \Theta'(s)\Big)\mathrm{d}W(s), \tag{2.25}$$

so that $H^{0,\Theta}(\cdot)V^{w,\mathfrak{p}}(t,\cdot)$ is a non-negative local martingale and a supermartingale on $[t,T]$. If the set on the right-hand side of (2.23) is not empty, then for any $w > 0$ in this set and for any $\mathfrak{p}(\cdot) \in \mathcal{H}([t,T],K)$, it follows from $V^{w,\mathfrak{p}}(t,T) = wV^{1,\mathfrak{p}}(t,T)$ that,

$$\begin{aligned} &\mathbb{E}\left[H^{0,\Theta}(T)V^{w,\mathfrak{p}}(t,T)\middle|\mathcal{F}_t\right] - H^{0,\Theta}(t)w \leq 0 \\ \text{i.e. } &w \geq \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)V^{w,\mathfrak{p}}(t,T)\middle|\mathcal{F}_t\right] \geq \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)Y\middle|\mathcal{F}_t\right] \end{aligned} \tag{2.26}$$

and because $w > 0$ is arbitrary we deduce that

$$u^Y(t,T,K) \geq \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)Y\middle|\mathcal{F}_t\right]. \tag{2.27}$$

This inequality holds trivially if the set on the right-hand side of (2.23) is empty, since then we have $u^Y(t,T,K) = \infty$. *Note* here that the market price of risk Θ need not be unique, but $u^Y(t,T,K) \geq \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)Y\middle|\mathcal{F}_t\right]$ holds for *all* progressively measurable processes $\Theta(\cdot)$ that satisfy (2.11).

2.6.1 Complete markets

Now consider the non-negative martingale

$$M^{\Theta}(t,s) := \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)Y \middle| \mathcal{F}_s\right],\ t \le s \le T.$$

Suppose that we can impose the following structural assumption on the filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$, the "flow of information" in the market.

Assumption 2.6.1. *Every local martingale of the filtration $\mathbb{F}$ can be represented as a stochastic integral, with respect to the driving Brownian motion $W(\cdot)$, of some $\mathbb{G}$-progressively measurable integrand.*

Assumption 2.6.2. *We have $d = n$ and $\mathfrak{s}(s)$ is invertible, $\forall\ s\ \in [t,T]$.*

Under Assumption 2.6.2 there is a unique marke price of risk, viz.

$$\Theta(\cdot) = \mathfrak{s}^{-1}(\cdot)\Big(\mathfrak{b}(\cdot) - \mathfrak{r}(\cdot)\mathbf{1}\Big).$$

Under Assumption 2.6.1, one can represent the non-negative martingale $M^{\Theta}(t,s)$ as a stochastic integral

$$M^{\Theta}(t,s) = M^{\Theta}(t,t) + \int_t^s \psi'(u)\mathrm{d}W(u), \quad t \le s \le T$$

for some $\mathbb{G}$-progressively measurable and a.s. square-integrable process $\psi : [t,T] \times \Omega \to \mathbb{R}^d$. With

$$w_t^Y := M^{\Theta}(t,t) = \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)Y \middle| \mathcal{F}_t\right]$$

$$V^*(t,\cdot) := M^{\Theta}(t,\cdot)/\Big(w_t^Y \widetilde{H}^{0,\Theta}(t,\cdot)\Big)$$

$$\mathfrak{p}^Y(\cdot) := \frac{\mathfrak{a}^{-1}(\cdot)\mathfrak{s}(\cdot)}{\widetilde{H}^{0,\Theta}(t,\cdot)w_t^Y V^*(t,\cdot)}\left[\psi(\cdot) + M^{\Theta}(t,\cdot)\Theta(\cdot)\right] \tag{2.28}$$

we have

$$w_t^Y V^{\mathfrak{p}^Y}(t,T) = Y \ \text{ a.s.}$$

Therefore, the investment strategy $\mathfrak{p}^Y$ is in $\mathcal{H}([t,T],\mathbb{R}^n)$ and replicates the contingent claim Y. This implies that $M^{\Theta}(t,t)$ belongs to the set on the right-hand side of (2.23), and so

$$M^{\Theta}(t,t) \geq u^Y(t,T,\mathbb{R}^n).$$

But we have already proved in (2.27) the reverse inequality, which gives us the *Black-Scholes-type formula*

$$u^Y(t,T,\mathbb{R}^n) = \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)Y \middle| \mathcal{F}_t\right] \tag{2.29}$$

for the unconstrained upper hedging price of (2.23), under Assumptions (2.6.1) and (2.6.2).

The above thus shows that, for any $\mathcal{F}_T$-measurable strictly positive contingent claim Y with $\mathbb{E}\left(YH^{0,\Theta}(T)\right) < \infty$, and for any $t \in [0,T]$, there exists a strategy $\mathfrak{p} \in \mathcal{H}([t,T],\mathbb{R}^n)$ such that

$$Y = w_t^Y V^{\mathfrak{p}}(t,T) \text{ a.s.}$$

where $w_t^Y = \mathbb{E}\left[\widetilde{H}^{0,\Theta}(t,T)Y \middle| \mathcal{F}_t\right]$. Therefore, given any contingent claim $Y > 0$, we can assume without loss of generality that it is the terminal wealth generated by a strategy $\mathfrak{p}$ starting with some wealth $w > 0$.

The process $\left\{N^{0,\Theta}(t)\right\}_{0\leq t\leq T}$, defined by

$$N^{0,\Theta}(t) = \widetilde{H}^{0,\Theta}(0,t)u^Y(t,T,\mathbb{R}^n) = \mathbb{E}\left[\widetilde{H}^{0,\Theta}(0,T)Y \middle| \mathcal{F}_t\right]$$

is a martingale. Define,

$$\mathfrak{z}_{\mathbb{R}^n}(t) = \mathbb{E}[\widetilde{H}^{0,\Theta}(t,T)V^{\mathfrak{p}}(t,T) \big| \mathcal{F}_t].$$

In the following chapters we will consider situations where the investment strategy is constrained to take values in a given closed convex set K. The subscript $\mathbb{R}^n$ in

$\mathfrak{z}_{\mathbb{R}^n}$ alludes to the unconstrained situation under consideration.
Note that, by (2.29)

$$u^Y(t,T,\mathbb{R}^n) = wV^{\mathfrak{p}}(0,t)\mathfrak{z}_{\mathbb{R}^n}(t). \tag{2.30}$$

In order to replicate the contingent claim $Y = wV^{\mathfrak{p}}(0,T)$, we can start with wealth $wV^{\mathfrak{p}}(0,t)$ at time t and use the investment strategy $\mathfrak{p}$ in the interval $[t,T]$. Instead, from (2.30) and the definition of upper hedging price as in (2.23), we see that we can super-replicate Y by starting even with wealth $wV^{\mathfrak{p}}(0,t)\mathfrak{z}_{\mathbb{R}^n}(t)$, and following the investment strategy $\mathfrak{p}^Y$ as defined in (2.28). $wV^{\mathfrak{p}}(0,t)\mathfrak{z}_{\mathbb{R}^n}(t)$ is also the minimum amount of wealth needed at time t to super-replicate Y. Therefore,

$$0 < \mathfrak{z}_{\mathbb{R}^n}(t) \leq 1,\ 0 \leq t \leq T.$$

Thus, $\mathfrak{z}_{\mathbb{R}^n}(t)$ has the interpretation of being the minimum amount of wealth needed at time t to be able to super-replicate, by time T, the terminal wealth generated by the investment strategy $\mathfrak{p}$ starting with 1\$ at time t. When $\mathfrak{z}_{\mathbb{R}^n}(t) < 1$, (2.30) implies that Y can be super-replicated using the strategy $\mathfrak{p}^Y$ starting with less wealth at time t, than the investment strategy $\mathfrak{p}$ would necessitate. In other words, $\mathfrak{z}_{\mathbb{R}^n}(t) < 1$ implies there exists strong arbitrage relative to $\mathfrak{p}$ on the time horizon $[t,T]$.
When the market and the strategy are both Markovian, i.e., the stock price processes satisfy the stochastic differential equation (2.6) and the strategy $\mathfrak{p}(\cdot)$ is of the form $\mathfrak{p}(\cdot) = \pi(\cdot, \mathcal{X}(\cdot))$ then $\mathfrak{z}_{\mathbb{R}^n}(t)$ is of the form

$$\mathfrak{z}_{\mathbb{R}^n}(t) = z_{\mathbb{R}^n}(t, \mathcal{X}(t)) := \mathbb{E}[\widetilde{H}^{0,\Theta}(t,T)V^{\mathfrak{p}}(t,T)\big|\mathcal{X}(t)], \tag{2.31}$$

for a suitable function $z_{\mathbb{R}^n}(\cdot,\cdot)$ on $(0,T) \times (0,\infty)^n$.

Assumption 2.6.3. *$z_{\mathbb{R}^n}(\cdot,\cdot)$ is locally $C^{1,2}$, the market is Markovian and π is Markovian.*

Under this assumption, an application of Itô's lemma and the martingale property of $N^{0,\Theta}(\cdot)$ show that for each $(t,x) \in [0,T] \times \mathbb{R}^n_+$ there exists a neighborhood $\mathcal{U}_{t,x}$ of

(t, x) in which $z_{\mathbb{R}^n}(\cdot, \cdot)$ satisfies the partial differential equation

$$\begin{aligned} &\frac{\partial w}{\partial t}(t,x) + \sum x_i \frac{\partial w}{\partial x_i}(t,x)\left(a_{i*}(t,x)\pi(t,x) + r(t,x)\right) + \\ &\qquad + \frac{1}{2}\sum x_i x_j a_{ij}(t,x)\frac{\partial^2 w}{\partial x_i \partial x_j}(t,x) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^n_+, \qquad (2.32) \\ &w(T,x) = 1, \quad x \in \mathbb{R}^n_+. \end{aligned}$$

It is easy to see that $w(\cdot, \cdot) \equiv 1$ is a trivial solution to (2.32). In fact, (2.32) can have multiple solutions. The same argument as in Theorem 1 in Fernholz and Karatzas (2010) and Proposition 2 in Ruf (2011), shows that the function $z_{\mathbb{R}^n}(\cdot, \cdot)$ is the smallest nonnegative function of class $C^{1,2}$ on $[0, T] \times \mathbb{R}^n_+$ that satisfies (2.32). Thus, as discussed in Section 9.2 in Fernholz and Karatzas (2010), under Assumption 2.6.3, the existence of strong arbitrage relative to π over the time interval $[0, T]$ is equivalent to failure of uniqueness on the part of the Cauchy problem (2.32) over the strip $[0, T] \times \mathbb{R}^n_+$.

Chapter 3

Upper Hedging Price under Constraints: Stochastic Representation

We saw in (2.27), that for any constraint set $K \subseteq \mathbb{R}^n$, $u^Y(t,T,K) \geq \mathbb{E}[\widetilde{H}^{0,\Theta}(t,T)Y|\mathcal{F}_t]$. If $K = \mathbb{R}^n$, the reverse inequality holds under Assumptions (2.6.1) and (2.6.2) or under conditions presented in Theorems 4.1 and 4.2 in Ruf (2012), thus giving the Black-Scholes type formula $u^Y(t,T,\mathbb{R}^n) = \mathbb{E}[\widetilde{H}^{0,\Theta}(t,T)Y|\mathcal{F}_t]$. When $K \subsetneq \mathbb{R}^n$, Cvitanić and Karatzas (1993) showed that the upper hedging price $u^Y(t,T,K)$ can be represented as the essential supremum of the unconstrained upper hedging prices (i.e., with $K = \mathbb{R}^n$) in certain auxiliary markets. These auxiliary markets are constructed using certain dual processes, which play a vital role similar to Lagrange multipliers.

In the special case of Markovian markets, we show that the auxiliary markets can be taken to be Markovian, for which we will only need to consider dual processes which depend on time and the current stock price. We also observe that the dual processes can be taken to be uniformly bounded. The last observation allows us to choose a broader class of constraints than that in Cvitanić and Karatzas (1993). This enlargement of the class of constraints will prove useful in studying how the possibility for relative arbitrage disappears as the constraints become stricter and stricter.

3.1 Constraint set K

For a given closed, convex subset $K \neq \emptyset$ of $\mathbb{R}^n$, let us define $\zeta_K : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$\zeta_K(\kappa) \triangleq \sup_{\pi \in K}(-\pi'\kappa), \ \kappa \in \mathbb{R}^n. \tag{3.1}$$

This is the support function of the convex set $-K$. It is a closed (i.e., lower semi-continuous), proper (i.e., not identically $+\infty$) convex function, which is finite on its effective domain

$$\widetilde{K} \triangleq \{\kappa \in \mathbb{R}^n; \zeta_K(\kappa) < \infty\}. \tag{3.2}$$

The effective domain, $\widetilde{K}$, is a convex cone, called the *barrier cone* of $-K$. In particular,

$$0 \in \widetilde{K} \text{ and } \zeta_K(0) = 0.$$

Henceforth, we will drop the subscript K from ζ_K. We will use the subscript only when dealing with several constraint sets, in order to distinguish the respective ζ's. The function ζ is positively homogeneous,

$$\zeta(\alpha\kappa) = \alpha\zeta(\kappa), \ \forall \kappa \in \mathbb{R}^n, \alpha \geq 0,$$

and subadditive,

$$\zeta(\kappa_1 + \kappa_2) \leq \zeta(\kappa_1) + \zeta(\kappa_2), \ \forall \kappa_1, \kappa_2 \in \mathbb{R}^n.$$

According to Rockafellar (1970), Theorem 13.1, pg 112,

$$\pi \in K \Leftrightarrow \zeta(\kappa) + \pi'\kappa \geq 0, \ \ \forall \kappa \in \widetilde{K}.$$

Lemma 3.1.1. *If K is a closed convex cone, then $\zeta(\kappa) \leq 0$ for all $\kappa \in \widetilde{K}$.*

Proof. Suppose there exists $\widetilde{\pi} \in K$, $\kappa \in \widetilde{K}$ such that $-\widetilde{\pi}'\kappa = \delta > 0$. Since $\alpha\widetilde{\pi} \in K$ for all $\alpha > 0$, hence $\sup_{\pi \in K}(-\pi'\kappa) = \infty$, contradicting the fact that $\kappa \in \widetilde{K}$. Therefore, $-\pi'\kappa \leq 0$ for all $\kappa \in \widetilde{K}$, $\pi \in K$, which proves our claim. □

We will assume that

Assumption 3.1.1. *The function $\zeta(\cdot)$ of (3.1) is locally bounded on $\widetilde{K}$.*

Assumption 3.1.1 holds in particular when ζ is a continuous function on $\widetilde{K}$. Theorem 10.2 in Rockafellar (1970) guarantees that ζ is a continuous function on $\widetilde{K}$, if $\widetilde{K}$ is locally simplical; see also Remark 5.1 in Cvitanić and Karatzas (1993).

Unlike Cvitanić and Karatzas (1993), we do not assume that ζ is bounded from below on $\mathbb{R}^n$, i.e.

$$\zeta(\kappa) \geq \zeta_0, \ \forall \kappa \in \mathbb{R}^n, \text{ for some } \zeta_0 \in \mathbb{R}. \tag{3.3}$$

We will see in Lemma 3.1.2 that ζ is bounded from below on $\mathbb{R}^n$ if and only if the constraint set K contains points arbitrarily close to 0. Corollary 3.1.2 shows that in that case ζ will take only non-negative values on $\widetilde{K}$. This observation will make some computations immediate, whenever K contains points arbitrarily close to 0.

Lemma 3.1.2. *The function $\zeta(\cdot)$ of (3.1) is bounded from below on $\mathbb{R}^n$ iff $K \cap \bar{B}(0, \delta) \neq \emptyset$ for any $\delta > 0$.*

Proof. Suppose ζ is bounded from below by ζ_0. Since $\zeta(0) = 0$, without loss of generality we can assume that $\zeta_0 \leq 0$. Suppose also that there exists $\delta > 0$ such that

$$K \cap \bar{B}(0, \delta) = \emptyset.$$

Then by Corollary 11.4.2 in Rockafellar (1970), we get that there exists a hyperplane separating K and $\bar{B}(0,\delta)$ strongly. Thus there exists a vector b and a real number β such that

$$b^T x > \beta, \ \forall x \in K$$

and

$$b^T x < \beta, \ \forall x \in \bar{B}(0,\delta).$$

Since $0 \in \bar{B}(0,\delta)$, hence $\beta > 0$. Let

$$\kappa = \frac{1}{\beta}(\zeta_0 - 1) < 0, \quad \nu = -\kappa b.$$

Then for any $\pi \in K$, we have

$$-\pi'\nu = \kappa\pi' b < \kappa\beta = \zeta_0 - 1.$$

Hence

$$\zeta(\nu) = \sup_{\pi\in K}(-\pi'\nu) < \zeta_0 - 1,$$

which contradicts the assumption that ζ is bounded from below by ζ_0. Hence

$$K \cap \bar{B}(0,\delta) \neq \emptyset \text{ for all } \delta > 0.$$

To prove sufficiency, suppose that

$$K \cap \bar{B}(0,\delta) \neq \emptyset \text{ for all } \delta > 0.$$

Then we will show that there exists $\pi \in K$ such that $\alpha\pi \in K$ for all $\alpha \in [0,1]$. Take any $\delta > 0$. Choose $\pi_0 \in K \cap \bar{B}(0,\delta)$, such a choice being possible by assumption. For each integer $n \geq 1$, choose

$$\pi_n \in K \cap \bar{B}(0,\delta/n).$$

Since K is convex,

$$\alpha\pi_0 + (1-\alpha)\pi_n \in K.$$

Now,

$$\lim_{n\to\infty} \alpha\pi_0 + (1-\alpha)\pi_n = \alpha\pi_0,$$

and since K is closed, $\alpha\pi_0 \in K$. Since α was chosen arbitrarily between $[0,1]$, this holds for all $\alpha \in [0,1]$. Now take any $\nu \in \widetilde{K}$. Suppose $\pi_0^T\nu = \epsilon_0$. If $\epsilon_0 > 0$, then

$$\{\alpha\pi_0^T\nu : \alpha \in [0,1]\} = [0, \epsilon_0].$$

Therefore, we have, $\{-\alpha\pi_0^T\nu : \alpha \in [0,1]\} = [-\epsilon_0, 0]$, which implies that

$$\zeta(\nu) = \sup_{\pi\in K}(-\pi^T\nu) \geq 0.$$

If $\epsilon_0 < 0$, then

$$\{\alpha\pi_0^T\nu : \alpha \in [0,1]\} = [\epsilon_0, 0].$$

Therefore, $\{-\alpha\pi_0^T\nu : \alpha \in [0,1]\} = [0, \epsilon_0]$, which implies that

$$\zeta(\nu) = \sup_{\pi\in K}(-\pi^T\nu) \geq 0.$$

□

The following corollaries follow from the proof of Lemma 3.3.

Corollary 3.1.1. *If $K \cap \bar{B}(0,\delta) \neq \emptyset$ for any $\delta > 0$, then there exists $\pi \in K$ such that $\alpha\pi \in K$ for all $\alpha \in [0,1]$.*

Corollary 3.1.2. *If $K \cap \bar{B}(0,\delta) \neq \emptyset$ for any $\delta > 0$, then $\zeta(\kappa) \geq 0$ for all $\kappa \in \widetilde{K}$.*

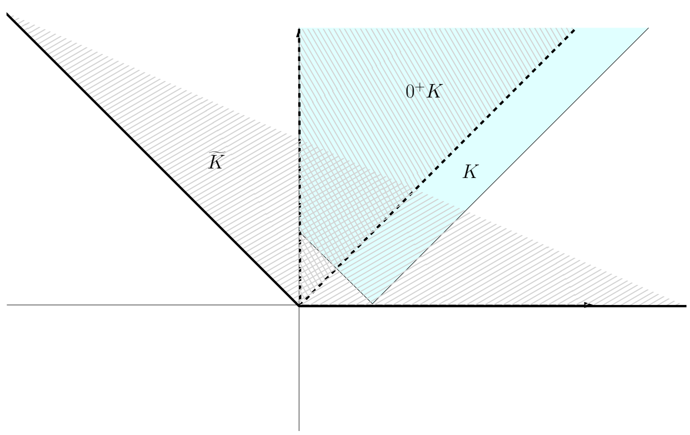
For example, if $K = [0,\infty)^n$, then $\widetilde{K} = [0,\infty)^n$ and $\zeta(\kappa) = 0$ for all $\kappa \in \widetilde{K}$. On the other hand, if $K = [c,\infty)^n$, for some $c > 0$, then $\widetilde{K} = [0,\infty)^n$ and $\zeta(\kappa) = -c(\sum_i \kappa_i)$, as shown in (5.79). Thus $\zeta(\cdot)$ is locally bounded in this case, but not bounded from below.

We will see afterwards, that having negative values of $\zeta(\cdot)$ will be useful in eliminating relative arbitrage opportunities. Thus, getting rid of the assumption of boundedness from below of $\zeta(\cdot)$ helps us significantly. Also, from Lemma 3.1.2 and (2.19) it is clear that if we assume $\zeta(\cdot)$ to be bounded from below, then we cannot study the case where investment strategies are constrained to be portfolios.

The set $\widetilde{K}$ is the dual cone to the recession cone 0^+K (see Section 8 of Rockafellar (1970) for a definition of recession cones) of K, i.e.

$$\widetilde{K} = \{\kappa \in \mathbb{R}^n \ : \ \forall\, w \in 0^+K, \kappa^T w \geq 0\}. \tag{3.4}$$

$0 \in \widetilde{K}$ and for $\kappa \neq 0$, $\kappa \in \widetilde{K}$ iff $-\kappa$ makes an angle of more than 90^o with any direction of recession of K. Figure 3.1 illustrates this in the case when K is a cone and thus itself is its own recession cone.



The following lemma follows immediately from (3.4).

Lemma 3.1.3. *Both* $\kappa, -\kappa \in \mathbb{R}^n$ *are in* $\widetilde{K}$ *if and only if* $\kappa \perp$ *vect*(0^+K).

3.2 Dual processes and upper hedging price

We introduce now the dual processes, mentioned before. They act as Lagrange multipliers penalising strategies which do not remain constrained in K at all times. Cvitanić and Karatzas (1993) constructed auxilliary markets based on these dual processes and showed that the upper hedging price of a contingent claim in the constrained market can be expressed as the supremum of the upper hedging prices of the contingent claim in the unconstrained auxiliary markets. We will follow their approach with suitable twists. We will restrict ourselves to Markovian markets and show that the dual processes can be restricted to be Markovian. This will come at a price though. We will have to restrict the contingent claims to satisfy certain conditions.

For $I \in \mathcal{I}$, let $\mathcal{H}_I$ denote the Hilbert space of $\{\mathcal{F}_t\}$-progressively measurable processes

$$\begin{gathered} \mathfrak{V} : I \times \Omega \to \mathbb{R}^n \text{ such that } \exists\, \phi^{\mathfrak{V}} : I \times \Omega \to \mathbb{R}^n \\ \text{which satisfies } \mathfrak{s}\phi^{\mathfrak{V}} = \mathfrak{V} \text{ and } \int_I \|\phi^{\mathfrak{V}}\|^2 \mathrm{d}t < \infty, \end{gathered} \tag{3.5}$$

with norm $[[\mathfrak{V}]]$ given by

$$[[\mathfrak{V}]]_I^2 \triangleq \mathbb{E}\left[\int_I \|\mathfrak{V}(t)\|^2 \mathrm{d}t\right] < \infty.$$

We define the inner product

$$\langle \mathfrak{V}_1, \mathfrak{V}_2 \rangle_I = \mathbb{E}\left[\int_I \mathfrak{V}_1'(t)\mathfrak{V}_2(t)\mathrm{d}t\right]$$

on this space, and define

$$\begin{aligned} \mathcal{D}_I(K) :=\ & \text{the subset of } \mathcal{H}_I \text{ consisting of processes } \mathfrak{V} : I \times \Omega \to \widetilde{K} \\ & \qquad\qquad \text{with } \mathbb{E}\int_I \zeta(\mathfrak{V}(t))\mathrm{d}t < \infty. \end{aligned} \tag{3.6}$$

$$\mathcal{D}_I^{(b)}(K) := \text{ the set of uniformly bounded processes in } \mathcal{D}_I(K). \tag{3.7}$$

$$\mathcal{D}_I^M(K) := \text{ the set of processes in } \mathcal{D}_I^{(b)}(K) \text{ which are of the form } \mathfrak{V}(t) \equiv \nu(t, \mathcal{X}(t)). \tag{3.8}$$

For each $\mathfrak{V} \in \mathcal{D}_I(K)$, consider an auxiliary interest rate process

$$\mathfrak{r}_{\mathfrak{V}}(t) \triangleq \mathfrak{r}(t) + \zeta(\mathfrak{V}(t)),\ t \in I, \tag{3.9}$$

as well as an auxiliary mean rate of return vector process

$$\mathfrak{b}_{\mathfrak{V}}(t) \triangleq \mathfrak{b}(t) + \mathfrak{V}(t) + \zeta(\mathfrak{V}(t))\mathbf{1},\ t \in I. \tag{3.10}$$

Having introduced these auxiliary processes, Cvitanić and Karatzas (1993) construct a new auxilliary market

$$M_{\mathfrak{V}} = (\mathfrak{r}_{\mathfrak{V}}(\cdot), \mathfrak{b}_{\mathfrak{V}}(\cdot), \mathfrak{s}(\cdot), S(0)).$$

Though we will make use of these auxiliary processes, we will stop short of constructing auxiliary markets because in a Markovian framework, this will introduce the unnecessary hassle of making assumptions that unique-in-distribution weak solutions exist for the corresponding stochastic differential equations for the stock prices in the auxiliary markets. The price we have to pay is not having a nice characterization. [1]

The assumption of the existence of a square-integrable market-price-of-risk implies the existence of a square-integrable process $\Theta_{\mathfrak{V}}$ for each $\mathfrak{V}$, such that

$$\mathfrak{s}\Theta_{\mathfrak{V}} = \mathfrak{b}_{\mathfrak{V}} - \mathfrak{r}_{\mathfrak{V}}\mathbf{1}.$$

In particular, we can take

$$\Theta_{\mathfrak{V}} = \Theta + \phi_{\mathfrak{V}},$$

[1] As pointed out by Ioannis Karatzas in a personal communication (2012), the assumption of uniqueness in distribution may not be needed for our work. In the case of multitude of solutions, we might be able to use the techniques for selecting Markov solutions as in Krylov (1973) and Chapter 12 of Stroock and Varadhan (1979). See also page 15 of Ruf (2011). This remains to be worked upon.

where $\phi_{\mathfrak{V}}$ is as defined in (3.5).

For each $\mathfrak{V} \in \mathcal{D}_I(K)$, consider now the process $S_0^{\mathfrak{V}}(\cdot), Z^{\Theta_{\mathfrak{V}}}(\cdot), H^{\Theta_{\mathfrak{V}}}(\cdot)$ defined as

$$\begin{aligned} S_0^{\mathfrak{V}}(t) &= S_0(t)\exp\left[\int_0^t \zeta(\mathfrak{V}(s))\mathrm{d}s\right], \\ Z^{\Theta_{\mathfrak{V}}}(t) &= \exp\left[-\int_0^t \Theta'_{\mathfrak{V}}(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t \|\Theta_{\mathfrak{V}}(s)\|^2\mathrm{d}s\right], \\ H^{\mathfrak{V},\Theta_{\mathfrak{V}}}(t) &= Z^{\Theta_{\mathfrak{V}}}(t)/S_0^{\mathfrak{V}}(t) \end{aligned} \tag{3.11}$$

By assumption (2.6.2), $\mathfrak{s}$ is invertible. Hence for each $\mathfrak{V} \in \mathcal{D}$, $\Theta_{\mathfrak{V}}$ is uniquely defined as

$$\Theta_{\mathfrak{V}}(t) = (\mathfrak{s}(t))^{-1}(\mathfrak{b}_{\mathfrak{V}}(t) - \mathfrak{r}_{\mathfrak{V}}(t)\mathbf{1}).$$

Hence, here we will drop $\Theta_{\mathfrak{V}}$ from the superscript and denote

$$H^{\mathfrak{V}}(t) := H^{\mathfrak{V},\Theta_{\mathfrak{V}}}(t); \quad \widetilde{H}^{\mathfrak{V}}(t,s) := H^{\mathfrak{V},\Theta_{\mathfrak{V}}}(s)/H^{\mathfrak{V},\Theta_{\mathfrak{V}}}(t),\ 0 \le t \le s \le T. \tag{3.12}$$

For an investment strategy $\mathfrak{p}$, the quantity $\widetilde{H}^{\mathfrak{V}}(t,T)V^{\mathfrak{p}}(t,T)$ will be of interest often. For it, we introduce the notation,

$$U^{\mathfrak{V},\mathfrak{p}}(t,s) := \widetilde{H}^{\mathfrak{V}}(t,s)V^{\mathfrak{p}}(t,s),\ 0 \le t \le s \le T. \tag{3.13}$$

3.3 Upper hedging price under constraints

Under assumptions (2.6.1) and (2.6.2), we know that for $K = \mathbb{R}^n$,

$$u^Y(t,T,\mathbb{R}^n) = \mathbb{E}[\widetilde{H}^{0,\Theta}(t,T)Y\big|\mathcal{F}_t].$$

For any closed convex set $K \subseteq \mathbb{R}^n$, Theorem 6.4 of Cvitanić and Karatzas (1993) shows that the upper hedging price $u^Y(t,T,K)$, defined in (2.23), satisfies

$$u^Y(t,T,K) = \operatorname{esssup}_{\mathfrak{V}\in\mathcal{D}_{(t,T]}(K)}\mathbb{E}\left[\widetilde{H}^{\mathfrak{V}}(t,T)Y\big|\mathcal{F}_t\right].$$

If this random variable is a.s. finite, then there exists a strategy $\mathfrak{p} \in \mathcal{H}((t,T],K)$ satisfying

$$u^Y(t,T,K)V^{\mathfrak{p}}(t,T) \geq Y; \tag{3.14}$$

we say then that $\mathfrak{p}(\cdot)$ is a "super-replicating strategy"; if equality holds in (3.14), we say that Y is "K-attainable", and that $\mathfrak{p}(\cdot)$ is a "replicating strategy".

Now assume that the market model is Markovian, i.e., it is of the form

$$\mathfrak{b}(t) = b(t, \mathcal{X}(t)), \;\; \mathfrak{s}(t) = \sigma(t, \mathcal{X}(t)), \;\; \mathfrak{r}(t) = r(t, \mathcal{X}(t)). \tag{3.15}$$

Assumption 3.3.1. *For a Markovian market, suppose that the eigenvalues of σ are locally bounded away from zero and infinity uniformly on $[t,T]$, and also the interest rate process r and the drift parameter b are locally bounded uniformly on $[t,T]$, i.e. for each compact set $C \subset \mathbb{R}^n_+$, there exists $k_c > 0$, such that*

$$\frac{1}{k_C} I \leq \sigma(s,x) \leq k_C I, \;\; r(s,x) + |b(s,x)| \leq k_C \;\; \forall \;\; s \in [t,T], x \in C. \tag{3.16}$$

Assumption 3.3.2. *The contingent claim Y is the terminal wealth generated by a Markovian strategy $\pi(\cdot,\cdot)$ which is locally bounded as a function of the stock price uniformly on $[0,T]$, i.e. for each compact set $C \subset \mathbb{R}^n_+$, there exists a constant $k_C > 0$, such that $|\pi(t,x)| \leq k_C$ if $x \in C$ and $t \in [0,T]$.*

We state now the main theorem of this chapter.

Theorem 3.3.1. *Suppose that the market model is Markovian (i.e, of the form* (3.15)*) and satisfies Assumptions 2.6.1, 2.6.2 and 3.3.1. Suppose that the stock price process takes values in $(0,\infty)^n$. Suppose also that the contingent claim Y satisfies Assumption 3.3.2, and the constraint set is such that Assumption 3.1.1 holds. Then,*

$$u^Y(t,T,K) = esssup_{\mathfrak{V} \in \mathcal{D}^{(b)}_{(t,T]}(K)} \mathbb{E}\left[\widetilde{H}^{\mathfrak{V}}(t,T) Y \middle| \mathcal{F}_t \right]$$

$$= esssup_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}\left[\widetilde{H}^\nu(t,T)Y \middle| \mathcal{F}_t\right] \triangleq \widehat{u}_M(t,T,K), \tag{3.17}$$

where $u^Y(t,T,K)$ is the upper hedging price defined in (2.23) *and the class of processes $\mathcal{D}^{(b)}_{(t,T]}(K)$ and $\mathcal{D}^{(M)}_{(t,T]}(K)$ are as defined in* (3.7) *and* (3.8)*, respectively.*
If $\widehat{u}_M(t,T,K)$ is a.s. finite, then there exists a Markovian strategy $\widehat{p}\ (\cdot,\cdot) \in \mathcal{H}((t,T],K)$ such that $\widehat{u}_M(t,T,K)V^{\widehat{p}}(t,T) \geq Y$ a.s.

Proof. See Section 3.4. □

Remark 3.3.1. In the framework of stochastic control theory, we can frame our problem as maximizing the reward $\mathbb{E}\left[\widetilde{H}^\nu(t,T)Y \middle| \mathcal{F}_t\right]$ over control process $\nu(\cdot,\cdot)$, which take values in an "action space" A. When $b(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$, $r(\cdot,\cdot)$, $\pi(\cdot,\cdot)$ are bounded and continuous, and the action space A is compact, it follows from El Karoui et al. (1987) that there exists an optimal Markovian "relaxed control", i.e. a control which takes values in the space of probability measures on the compact action space A. If we further assume that the drift and dispersion of $\widetilde{H}^\nu(t,T)Y$, under any values of the control process, take values in a fixed convex set, then it would follow from Haussmann (1986) that there exists an optimal Markovian control. In both these papers, the authors use the compactness of the action space to argue the compactness of the space of laws of the controlled processes $\widetilde{H}^\nu(t,T)Y$. Under assumptions which lead to upper-semicontinuity of the reward function, Haussmann (1986) shows the existence of optimal laws, and then uses Markovian selection techniques as in Krylov (1973) and Chapter 12 of Stroock and Varadhan (1979) to select an optimal strong Markov process, which is shown to correspond to a Markovian control. However, in our case, the action space $\widetilde{K}$ is not necessarily compact. Even then, if we could assume that $b(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$, $r(\cdot,\cdot)$, $\pi(\cdot,\cdot)$ are bounded and continuous, then we could have used these results to conclude that the optimum reward function is the supremum of the reward functions over Markovian controls. Soner and Touzi (2003) has used this line of argument for their problem. We do not make these assumptions, and use completely different argument and the special structure of our problem, to prove our claim.

Using Theorem 3.3.1, we can write,

$$\begin{aligned} u^Y(t,T,K) &= wV^\pi(0,t)\ \text{esssup}_{\nu\in\mathcal{D}^M}\mathbb{E}[\widetilde{H}^\nu(t,T)V^\pi(t,T)|\mathcal{F}_t] \\ &= wV^\pi(0,t)\ \text{esssup}_{\nu\in\mathcal{D}^M}\mathbb{E}[U^{\nu,\pi}(t,T)|\mathcal{F}_t], \end{aligned} \tag{3.18}$$

where $U^{\nu,\pi}(\cdot,\cdot)$ is as defined in (3.13). In the following, we will work with a fixed strategy π, and hence, unless there is scope for confusion, we will write

$$U^\nu(\cdot,\cdot) \equiv U^{\nu,\pi}(\cdot,\cdot). \tag{3.19}$$

We now define,

$$\mathfrak{z}_K(t,T) := \text{esssup}_{\nu\in\mathcal{D}^M(K)}\mathbb{E}[U^\nu(t,T)\big|\mathcal{X}(t)], \tag{3.20}$$

$$\mathfrak{z}_K^\nu(t,T) := \mathbb{E}[U^\nu(t,T)\big|\mathcal{X}(t)],\ \ \nu\in\mathcal{D}^M(K). \tag{3.21}$$

When the constraint set K is clear from the context, we will drop the subscript K and denote $\mathfrak{z}_K$ simply by $\mathfrak{z}$.

We note that

$$u^Y(t,T,K) = wV^\pi(0,t)\mathfrak{z}_K(t,T),\ \ \text{a.s.} \tag{3.22}$$

We can replicate the claim Y by starting with $wV^\pi(0,t)$ dollars at time t and following the investment strategy π on the time interval $[t,T]$. However, this need not be the minimum amount of wealth needed to super-replicate Y. If the super-replicating strategies are constrained to take values in K, then the minimum amount of wealth needed for super-replication at time t is $wV^\pi(0,t)\mathfrak{z}_K(t,T)$. The random variable $\mathfrak{z}_K(\cdot,T)$ takes values in $(0,\infty]$. It is clear that if $\pi(s,x)\in K,\ (s,x)\in[t,T]\times\mathbb{R}_+^n$, then $0<\mathfrak{z}_K(\cdot,T)\le 1$ on $[t,T]$. On the other hand, if K is such that Y cannot be super-replicated by any strategy constrained to take values in K starting with any finite amount of initial wealth, then $\mathfrak{z}_K = \infty$. This would happen, for example, if the stock market consists of two stocks with the price processes X_1, X_2 being independent geometric Brownian motions, the contingent claim is $Y = X_1(T)$ and $K = \{0\} \times \mathbb{R}$, i.e. only the second stock can be traded. On the other hand, $\mathfrak{z}_K(t,T) < 1$ would imply that if $\mathcal{X}(t) = x$, then there exists a strategy constrained to take values in K, which starts with less than $wV^\pi(0,t)$ dollars at time t and super-replicates Y, while the strategy π starts with $wV^\pi(0,t)$ dollars at time t and replicates Y, i.e. there exists a strategy constrained to take values in K which presents a relative arbitrage opportunity with respect to π over the time horizon $[t,T]$.

It is clear intuitively, as also from (3.20), that if $K_2 \subset K_1 \subset \mathbb{R}^n_+$ are two closed convex sets, then $\mathfrak{z}_{\mathbb{R}^n}(\cdot,T) \leq \mathfrak{z}_{K_1}(\cdot,T) \leq \mathfrak{z}_{K_2}(\cdot,T)$. Suppose now that $\mathfrak{z}_{\mathbb{R}^n}(t,T) < 1$, i.e. there exists relative arbitrage with respect to the strategy π. We would then be interested in knowing for what constraint sets K_1 do we still have $\mathfrak{z}_{K_1}(t,T) < 1$. And then for what kind of constraint sets K_2 do we have $\mathfrak{z}_{K_2}(t,T) \geq 1$.

Thus, given a closed convex set K, the random variable $\mathfrak{z}_K(\cdot,T)$ will be our litmus test for the existence of arbitrage opportunities relative to π among strategies constrained to take values in K. We call $\mathfrak{z}_K(\cdot,T)$ the *arbitrage coefficient* of π under K.

3.4 Proof of Theorem 3.3.1

We prove the theorem along the lines of the proof of Theorem 5.6.2 of Karatzas and Shreve (1998) but with suitable modifications. We have elaborated upon the parts which are different and have referred the reader to Karatzas and Shreve (1998) for the details of the parts which are similar.

To alleviate the notation we will write $\mathcal{D}^M$ for $\mathcal{D}^M_{(t,T]}(K)$ in the proof with the understanding that throughout the proof t, T and K remain fixed.

We call the non-negative process

$$\widehat{X}(s) = \text{esssup}_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}\left[\widetilde{H}^{\nu}(s,T)Y|\mathcal{F}_s\right] \tag{3.23}$$

the *upper hedging value process* for the contingent claim Y. Before starting the proof of Theorem 3.3.1 we will study a few properties of the hedging value process.
For $k \in \mathbb{N}, k \geq 1$, denote

$$T^{\mathcal{X}}_k := \inf\{t \leq s \leq T : \mathcal{X}_s \notin [\frac{1}{k}, k]^n\},$$
$$T^{\vartheta}_k := \inf\{t \leq s \leq T : \int_t^s \|\vartheta\|^2 > k\},$$
$$T_k := T^{\mathcal{X}}_k \wedge T^{\vartheta}_k.$$

The square-integrability of ϑ and the continuity of the stock price process ensures that $T_k \uparrow \infty$ a.s.
For $t \leq s \leq T$, denote

$$\widehat{X}^k(s) = \text{esssup}_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}\left[\widetilde{H}^{\nu}(s,T)Y\mathbf{1}_{T<T_k}|\mathcal{F}_s\right],$$
$$J_{\nu}(s) = \mathbb{E}[\widetilde{H}^{\nu}(s,T)Y|\mathcal{F}_s],$$
$$J^k_{\nu}(s) = \mathbb{E}[\widetilde{H}^{\nu}(s,T)Y\mathbf{1}_{T<T_k}|\mathcal{F}_s].$$

so that

$$\widehat{X}^k(s) = \text{esssup}_{\nu \in \mathcal{D}^M} J^k_{\nu}(s) \tag{3.24}$$

With $\nu \in \mathcal{D}^M$ fixed, we denote by $\mathcal{D}^M_{s,\nu}$ the set of all processes $\mu(\cdot) \in \mathcal{D}^M$ that agree with $\nu(\cdot)$ on $[t,s] \times \Omega$. Since $\widetilde{H}^{\mu}(s,T)$ depends only on the values of $\mu(v)$ of $\mu(\cdot)$ for $s \leq v \leq T$, we may rewrite (3.24) as

$$\widehat{X}^k(s) = \text{esssup}_{\mu \in \mathcal{D}^M_{s,\nu}} J^k_{\mu}(s).$$

We then have the following lemma.

Lemma 3.4.1. *For any $t \leq s \leq T$ and any integer $k \geq 1$, the collection $\{J^k_\mu(s)\}_{\mu \in \mathcal{D}^M_{s,\nu}}$ is such that for any $\mu_1, \mu_2 \in \mathcal{D}^M_{s,\nu}$, there exists $\mu^k \in \mathcal{D}^M_{s,\nu}$ such that $J^k_{\mu^k}(s) \geq J^k_{\mu_1}(s) \vee J^k_{\mu_2}(s)$.*

Proof. See Section 3.5. □

We also have the following technical result.

Proposition 3.4.1. *Under the assumption $\widehat{u}_M(t, T, K) < \infty$, the upper hedging value process $\widehat{X}(\cdot)$ of (3.23) is finite and satisfies the dynamic programming equation*

$$\widehat{X}(s) = esssup_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}\left[\widetilde{H}^\nu(s, s')\widehat{X}(s')|\mathcal{F}_s\right], \quad t \leq s \leq s' \leq T \tag{3.25}$$

Furthermore, $\widehat{X}(\cdot)$ has an RCLL modification; choosing this modification, we have that the process $\widetilde{H}^\nu(t, \cdot)\widehat{X}(\cdot)$ is an RCLL supermartingale for every $\nu(\cdot) \in \mathcal{D}^M_{(t,T]}(K)$.

Proof. For any arbitrary but fixed process $\nu \in \mathcal{D}^M$ we will first show that

$$\widehat{X}(s) \geq \mathbb{E}[\widetilde{H}^\nu(s, s')\widehat{X}(s')|\mathcal{F}_s] \text{ a.s.} \tag{3.26}$$

This is the supermartingale property for $H^\nu(s)\widehat{X}(s), t \leq s \leq T$.

With very slight change in the proof of Theorem A.3, Appendix A Karatzas and Shreve (1998) we can prove the slightly more general statement: Let $\mathcal{X}$ be a nonempty family of nonnegative random variables. Then $\mathcal{X}^* = \text{esssup}\mathcal{X}$ exists. Furthermore if $\mathcal{X}$ is such that $X, Y \in \mathcal{X}$ implies that there exists $Z \in \mathcal{X}$ with $Z \geq X \vee Y$ a.s., then there is a non-decreasing sequence $\{Z_n\}_{n=1}^\infty$ of random variables in $\mathcal{X}$ satisfying $\mathcal{X}^* = \lim_{n\to\infty} Z_n$ a.s.

It follows from the above statement and Lemma 3.4.1 that for each integer $k \geq 1$, there is a sequence $\{\mu^k_n(\cdot)\}_{n=1}^\infty$ in $\mathcal{D}^M_{s,\nu}$ such that $\{J^k_{\mu^k_n}(s')\}_{n=1}^\infty$ is nondecreasing and $\widehat{X}^k(s') = \lim_{n\to\infty} J^k_{\mu^k_n}(s')$. The monotone convergence theorem now implies

$$\mathbb{E}\left[\widetilde{H}^\nu(s, s')\widehat{X}(s')|\mathcal{F}_s\right] = \lim_{k\to\infty} \mathbb{E}\left[\widetilde{H}^\nu(s, s')\widehat{X}^k(s')|\mathcal{F}_s\right]$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \lim_{n\to\infty} \mathbb{E}\left[\widetilde{H}^{\nu}(s,s') J^k_{\mu^k_n}(s') | \mathcal{F}_s\right] \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \mathbb{E}\left[\widetilde{H}^{\mu^k_n}(s,T) Y \mathbf{1}_{T<T_k} | \mathcal{F}_s\right] \\
&\le \lim_{k\to\infty} \lim_{n\to\infty} \mathbb{E}\left[\widetilde{H}^{\mu^k_n}(s,T) Y | \mathcal{F}_s\right] \le \widehat{X}(s)
\end{aligned}
$$

and (3.26) is established.

The rest of the proof follows as in the proof of Proposition 5.6.5 in Karatzas and Shreve (1998). □

Remark 3.4.1. The supermartingale property for the nonnegative RCLL process $\widetilde{H}^{\nu}(t,\cdot)\widehat{X}(\cdot)$ implies that we have

$$\widehat{X}(s) = 0, \forall s \in [\widehat{\tau}, T]$$

almost surely on $\{\widehat{\tau} < T\}$, where

$$\widehat{\tau} \triangleq \inf\left\{s \in [t,T) : \widehat{X}(s) = 0\right\} \vee T \tag{3.27}$$

and $\widehat{X}(\cdot)$ is as defined in (3.23). Since we consider only strictly positive contingent claim Y, $\widehat{X}(\cdot)$ is strictly positive in $[t,T]$ and $\widehat{\tau} = T$ almost surely.

Lemma 3.4.2. *For any $\{\mathcal{F}_t\}$-stopping time τ taking values in $[t,T]$, the collection $\{J^k_{\mu}(\tau)\}_{\mu\in\mathcal{D}^M_{\tau,\nu}}$ is such that for any two given processes $\mu_1(\cdot)$ and $\mu_2(\cdot)$ in $\mathcal{D}^M_{\tau,\nu}$, there exists $\mu^k(\cdot) \in \mathcal{D}^M_{\tau,\nu}$ such that $J^k_{\mu^k}(\tau) \ge J^k_{\mu_1}(\tau) \vee J^k_{\mu_2}(\tau)$.*

Proof. See Section 3.6. □

Remark 3.4.2. For the contingent claim Y, and for each $\{\mathcal{F}_t\}$-stopping time τ taking values in $[t,T]$, let us define

$$\widetilde{X}(\tau) \triangleq \operatorname{esssup}_{\nu\in\mathcal{D}^M} \frac{\mathbb{E}\left[\widetilde{H}^{\nu}(t,T) Y | \mathcal{F}_{\tau}\right]}{\widetilde{H}^{\nu}(t,\tau)}.$$

It follows from the discussion in Remark 5.6.7 in Karatzas and Shreve (1998) and Lemma 3.4.2 that, when we take a right-continuous modification of $\widehat{X}(\cdot)$, we have $\widehat{X}(\tau) = \widetilde{X}(\tau)$, a.s.

Henceforth, we shall always take the RCLL modification of $\widehat{X}(\cdot)$.

Proof of Theorem 3.3.1. We start with the proof of the inequality $u^Y(t,T,K) \geq \widehat{u}_M(t,T,K)$ which is obvious if $u^Y(t,T,K) = \infty$. Now assume $u^Y(t,T,K) < \infty$, and consider an arbitrary $\mathcal{F}_t$-measurable random variable X_t taking values in $[0,\infty)$ for which there exists a strategy $\pi \in \mathcal{H}((t,T],K)$ whose associated wealth process satisfies $X_t V^\pi(t,T) \geq Y$ a.s. Let $\nu \in \mathcal{D}^M_{(t,T]}(K)$ be given. For $t \leq s \leq T,\ x \in \mathbb{R}^n_+$, we denote

$$B_\nu(s,x) = -\pi'(s,x)\nu(s,x) - \zeta\left(\nu(s,x)\right), \tag{3.28}$$

$$S_\nu(s,x) = \sigma'(s,x)\pi(s,x) - \vartheta(s,x) - \sigma^{-1}(s,x)\nu(s,x). \tag{3.29}$$

Then,

$$\begin{aligned}\widetilde{H}^\nu(t,T)V^\pi(t,T) = 1 + &\int_t^T \widetilde{H}^\nu(t,s)V^\pi(t,s)S'_\nu(s,\mathcal{X}(s))\mathrm{d}W_s \\ &+ \int_t^T \widetilde{H}^\nu(t,s)V^\pi(t,s)B_\nu(s,\mathcal{X}(s))\mathrm{d}s.\end{aligned} \tag{3.30}$$

Denote, for $t \leq s \leq T$,

$$M(s) = \int_t^s \widetilde{H}^\nu(t,u)V^\pi(t,u)S'_\nu(u,\mathcal{X}(u))\mathrm{d}W_u, \tag{3.31}$$

$$A(s) = \int_t^s \widetilde{H}^\nu(t,u)V^\pi(t,u)B_\nu(u,\mathcal{X}(u))\mathrm{d}u. \tag{3.32}$$

The process $\{M(s)\}_{t\leq s\leq T}$ is a continuous local martingale. Then there exists a sequence of stopping times $T_n, n \in \mathbb{N}$, such that $T_n \uparrow \infty$ and M^{T_n} is a martingale for each n. By optional sampling theorem, we see that $\mathbb{E}\left(M(T_n \wedge T)\right) = M(t) = 0$. $\pi'\nu + \zeta(\nu) \geq 0$ since $\pi(s) \in K$. Hence $A(s), t \leq s \leq T$ is an increasing nonnegative process. Therefore,

$$\mathbb{E}\left[\widetilde{H}^\nu(t,T_n \wedge T)V^\pi(t,T_n \wedge T)\right] = 1 - \mathbb{E}\left[A(T_n \wedge T)\right] \leq 1. \tag{3.33}$$

By an application of Fatou's lemma, we get from (3.33), that

$$\mathbb{E}\left[\widetilde{H}^{\nu}(t,T)V^{\pi}(t,T)\right] \leq \liminf_{n\to\infty}\mathbb{E}\left[\widetilde{H}^{\nu}(t,T_n\wedge T)V^{\pi}(t,T_n\wedge T)\right] \leq 1. \tag{3.34}$$

and hence,

$$\mathbb{E}[\widetilde{H}^{\nu}(t,T)Y|\mathcal{F}_t] \leq X_t\mathbb{E}[\widetilde{H}^{\nu}(t,T)V^{\pi}(t,T)] \leq X_t.$$

Thus if X_t is such that there exists a strategy $\pi \in \mathcal{H}((t,T],K)$ with $X_tV^{\pi}(t,T) \geq Y$, then for any $\nu \in \mathcal{D}^M_{(t,T]}(K)$, we have

$$X_t \geq \mathbb{E}[\widetilde{H}^{\nu}(t,T)Y|\mathcal{F}_t]. \tag{3.35}$$

Taking the essential supremum over all $\nu \in \mathcal{D}^M_{(t,T]}(K)$ of the right hand side of (3.35), we get

$$u^Y(t,T,K) \geq \widehat{u}_M(t,T,K). \tag{3.36}$$

We turn to the reverse inequality $u^Y(t,T,K) \leq \widehat{u}_M(t,T,K)$ which is obvious if $\widehat{u}_M(t,T,K) = \infty$. So we will assume for the remainder that $\widehat{u}_M(t,T,K) < \infty$ and show that there exists a strategy $\widehat{\pi} \in \mathcal{H}([t,T],K)$ such that $\widehat{u}_M(t,T,K)V^{\widehat{\pi}}(t,T) \geq \widehat{X}(T)$. We will mention some of the important steps that prove our claim. The left-out details follow exactly as in the proof of Theorem 5.6.2 in Karatzas and Shreve (1998), even though we have restricted the class of dual processes to $\mathcal{D}^M$. All that is needed is the supermartingale property of $\widetilde{H}^{\nu}(t,\cdot)\widehat{X}(\cdot)$ for any $\nu \in \mathcal{D}^M$. We proceed as follows. Fix $\nu(\cdot) \in \mathcal{D}^M$. It follows that the non-negative supermartingale $\widetilde{H}^{\nu}(t,\cdot)\widehat{X}(\cdot)$ has a unique Doob-Meyer decomposition

$$\widetilde{H}^{\nu}(t,s)\widehat{X}(s) = \widehat{u}_M(t,T,K) + \int_t^s \psi'_{\nu,t}(u)\mathrm{d}W(u) - A_{\nu,t}(s),\ t \leq s \leq T \tag{3.37}$$

almost surely, where,

i. $A_{\nu,t}(\cdot)$ is an adapted, natural process with nondecreasing, right-continuous paths almost surely, $\mathbb{E}[A_{\nu,t}(T)] < \infty, A(0) = 0$;

ii. $\psi_{\nu,t}(\cdot)$ is a progressively measurable, $\mathbb{R}^n$-valued process satisfying the square-integrability condition $\int_t^T \|\psi_{\nu,t}(u)\|^2 \mathrm{d}u < \infty$ almost surely.

We denote

$$\varphi(t,s) \triangleq \frac{\psi_{\nu,t}(s)}{\widetilde{H}^{\nu}(t,s)} + \widehat{X}(s)\vartheta_{\nu}(s), \tag{3.38}$$

$$\widehat{C}(t,s) \triangleq \int_{(t,s]} \frac{\mathrm{d}A_{\nu,t}(s)}{\widetilde{H}^{\nu}(t,s)} - \int_t^s \left[\widehat{X}(u)\zeta(\nu(u)) + \varphi'(t,u)\sigma^{-1}(u)\nu(u)\right] \mathrm{d}u \tag{3.39}$$

It turns out that the processes $\varphi(t,\cdot)$ and $\widehat{C}(t,\cdot)$ defined in (3.38) and (3.39) do not depend on $\nu(\cdot) \in \mathcal{D}^M$. We also have $\int_t^T \|\varphi(t,s)\|^2 \mathrm{d}s < \infty$ almost surely.

The process $\widehat{C}(t,\cdot)$ of (3.39) is adapted, with RCLL paths. Writing (3.39) with $\nu(\cdot) \equiv 0$, we obtain

$$\widehat{C}(t,s) = \int_{(t,s]} \frac{\mathrm{d}A_{0,t}(s)}{\widetilde{H}^{0}(t,s)} \tag{3.40}$$

which shows that $\widehat{C}(\cdot)$ is nondecreasing. Consider the portfolio process

$$\widehat{\mathfrak{p}}(s) \triangleq \begin{cases} \frac{1}{\widehat{X}(s)} \left(\sigma'(s)\right)^{-1} \varphi(s), & \text{if } \widehat{X}(s) \neq 0, \\ \mathfrak{p}_*, & \text{if } \widehat{X}(s) = 0, \end{cases} \tag{3.41}$$

where $\mathfrak{p}_*$ is an arbitrary but fixed vector in K. From (3.37), (3.38), (3.41) and (3.40) with $\nu \equiv 0$, we have

$$\begin{aligned} \widetilde{H}^0(t,s)\widehat{X}(s) &= \widehat{u} + \int_t^s \psi'(s)\mathrm{d}W(s) - A(s) \\ &= \widehat{u} + \int_t^s \widetilde{H}^0(t,u)\widehat{X}(u)\left[\sigma'(u)\widehat{\mathfrak{p}}(u) - \vartheta(u)\right]' \mathrm{d}W(u) - \int_{(t,s]} \widetilde{H}^0(t,s)\mathrm{d}\widehat{C}(t,u) \end{aligned}$$

Comparing this with (2.25) we can see $\widehat{u}(t,T,K)V^{\widehat{\mathfrak{p}}}(t,T) \geq \widehat{X}(T)$.

In order to conclude the proof of Theorem 3.3.1 we need to show that

$$\widehat{\mathfrak{p}}(s) \in K \text{ for Lebesgue a.e. } s \in [t,T] \tag{3.42}$$

holds almost surely.

We denote

$$R(s) = \text{esssup}_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}\left[\widetilde{H}^\nu(s,T)V^\pi(s,T)|\mathcal{F}_s\right] = \frac{\widehat{X}(s)}{wV^\pi(0,s)}. \tag{3.43}$$

It follows from (3.37) and (**??**), that for any $\nu \in \mathcal{D}^M$ and $t \in [0,T]$,

$$\begin{aligned}\frac{\mathrm{d}R(s)}{R(s)} &= -\left[B_\nu(s,\mathcal{X}(s)) + \left(\frac{\psi'_{\nu,t}(s)}{\widetilde{H}^\nu(t,s)\widehat{X}(s)} - S'_\nu(s,\mathcal{X}(s))\right)S_\nu(s)\right]\mathrm{d}s \\ &+ \left(\frac{\psi'_{\nu,t}(s)}{\widetilde{H}^\nu(t,s)\widehat{X}(s)} - S'_\nu(s,\mathcal{X}(s))\right)\mathrm{d}W(s) - \frac{\mathrm{d}A_{\nu,t}(s)}{\widetilde{H}^\nu(t,s)\widehat{X}(s)}, \quad t \le s \le T.\end{aligned} \tag{3.44}$$

Since the dynamics of $R(\cdot)$ satisfies (3.44) for any $t \in [0,T]$, it is clear that

$$\frac{\psi'_{\nu,t}(s)}{\widetilde{H}^\nu(t,s)\widehat{X}(s)} - S'_\nu(s,\mathcal{X}(s)) = \frac{\phi'(t,s)}{\widehat{X}(s)} - \sigma'(s,\mathcal{X}(s))\pi(s,\mathcal{X}(s) \tag{3.45}$$

is independent of t and $\nu \in \mathcal{D}^M$. It is also clear from (3.43) that for any $s \in [0,T]$, $\omega \in \Omega$, $R(s,\omega)$ is a function of s and $\mathcal{X}(s,\omega)$; and from (3.44) we see that $\log R(\cdot)$ is an additive semimartingale. It now follows from Theorem 6.27 in Çinlar et al. (1980) and (3.45) that $\widehat{\mathfrak{p}}(\cdot)$ defined in (3.41) satisfies

$$\widehat{\mathfrak{p}}(s) = \widehat{p}(s,\mathcal{X}(s)), \quad s \in [0,T], \tag{3.46}$$

for some Borel measurable function $\widehat{p} : [0,T] \times \mathbb{R}^n_+ \to \mathbb{R}^n$. Following the proof of Lemma 5.4.2 in Karatzas and Shreve (1998), we can then find a Borel measurable function $\nu : [0,T] \times \mathbb{R}^n_+ \to \widetilde{K}$, such that $\nu \in \mathcal{D}^M$,

$$\left\|\nu(t,\mathcal{X}(t))\right\| \le 1, \quad \left|\zeta(\nu(t,\mathcal{X}(t)))\right| \le 1, \quad 0 \le t \le T,$$

almost surely, and for all $t \in [0,T]$ we have

$$\widehat{p}\,(t,\mathcal{X}(t)) \in K \Leftrightarrow \nu\,(t,\mathcal{X}(t)) = 0, \tag{3.47}$$

$$\widehat{p}\,(t,\mathcal{X}(t)) \notin K \Leftrightarrow \zeta\,(\nu(t,\mathcal{X}(t))) + \widehat{p}'\,(t,\mathcal{X}(t))\,\nu\,(t,\mathcal{X}(t)) < 0 \tag{3.48}$$

almost surely. The important point to note here is that the arguments and the construction presented in the proof of Lemma 5.4.2 in Karatzas and Shreve (1998) continue to hold even when $\zeta(\cdot)$ is only assumed to be locally bounded on $\widetilde{K}$ instead of being bounded from below as in Cvitanić and Karatzas (1993).

For any positive integer k, the process $k\nu(\cdot)$ must also be in $\mathcal{D}^M$, and (**??**) gives,

$$0 \leq \int_{(t,\widehat{\tau}]} \frac{\mathrm{d}A_{k\nu}(s)}{\widetilde{H}^{k\nu}(t,s)} = \widehat{C}(t,\widehat{\tau}) + k\int_t^{\widehat{\tau}} \left[\widehat{X}(u)\zeta(\nu(u)) + \widehat{\pi}'(u)\nu(u)\right] \mathrm{d}u$$

almost surely. Because $\nu(\cdot)$ satisfy (3.47) and (3.48), the integrand on the right-hand side of this inequality is nonpositive, and by choosing k sufficiently large the right-hand side can be made negative with positive probability, unless

$$\zeta\left(\nu(s)\right) + \pi'(s)\nu(s) = 0 \text{ for Lebesgue-a.e. } s \in [t,\widehat{\tau}] \tag{3.49}$$

holds almost surely. Thus (3.49) must hold, and with it, (2.18) must hold as well. This completes the proof of Theorem 3.3.1. □

3.5 Proof of Lemma 3.4.1

Proof of Lemma 3.4.1. For each integer $n \geq 0$, we define

$$t_i^n = s + i\frac{T-s}{2^n}, \quad i = 0, 1, 2, \ldots, 2^n, \tag{3.50}$$

thus dividing the time interval $[s,T]$ into 2^n intervals of equal length. Define the family of binary operators $\{\otimes_i^n\}_{n\in\mathbb{N}}$, $\otimes_i^n : \mathcal{D}_{s,\nu}^M \times \mathcal{D}_{s,\nu}^M \to \mathcal{D}_{s,\nu}^M$ as

$$\left(\mu_1 \otimes_i^n \mu_2\right)(u) = \mu_1(u)\mathbf{1}_{u<t_i^n} + \mu_2(u)\mathbf{1}_{u\geq t_i^n}, \quad s \leq u \leq T, \quad \mu_1, \mu_2 \in \mathcal{D}_{s,\nu}^M. \tag{3.51}$$

Now, fix any two processes $\mu_1(\cdot)$ and $\mu_2(\cdot)$ in $\mathcal{D}_{s,\nu}^M$. Define

$$A_i^{n,k} = \left\{J_{\mu_1}^k(t_i^n) > J_{\mu_2}^k(t_i^n)\right\}, \ i = 0, 1, 2, \ldots, 2^n. \tag{3.52}$$

Define the process ν_n^k on $[t,T]$ as

$$\nu_n^k(u) = \nu(u)\mathbf{1}_{t\le u<s} + \sum_{i=0}^{2^n-1}\left\{\mu_1(u)\mathbf{1}_{A_i^{n,k}} + \mu_2(u)\mathbf{1}_{\left(A_i^{n,k}\right)^c}\right\}\mathbf{1}_{t_i^n\le u<t_{i+1}^n}. \tag{3.53}$$

For any $n\in\mathbb{N}$, for $i=2^n-1$, we see that,

$$\begin{aligned} J^k_{\nu_n^k}(s) &= \mathbb{E}\left[\widetilde{H}^{\nu_n^k\otimes_i^n\mu_1}(s,T)Y\mathbf{1}_{A_i^{n,k}} + \widetilde{H}^{\nu_n^k\otimes_i^n\mu_2}Y\mathbf{1}_{\left(A_i^{n,k}\right)^c}\middle|\mathcal{F}_s\right] &(3.54)\\ &= \mathbb{E}\left[\widetilde{H}^{\nu_n^k}(s,t_i^n)\left(J^k_{\mu_1}(t_i^n)\mathbf{1}_{A_i^{n,k}} + J^k_{\mu_2}(t_i^n)\mathbf{1}_{\left(A_i^{n,k}\right)^c}\right)\middle|\mathcal{F}_s\right] &(3.55)\\ &\ge J^k_{\nu_n^k\otimes_i^n\mu_1}(s)\vee J^k_{\nu_n^k\otimes_i^n\mu_2}(s) &(3.56)\end{aligned}$$

Suppose that we have shown that for some $i\in\{1,2,2^n-1\}$,

$$J^k_{\nu_n^k}(s) \ge \mathbb{E}\left[\widetilde{H}^{\nu_n^k}(s,t_i^n)\left(J^k_{\mu_1}(t_i^n)\mathbf{1}_{A_i^{n,k}} + J^k_{\mu_2}(t_i^n)\mathbf{1}_{\left(A_i^{n,k}\right)^c}\right)\middle|\mathcal{F}_s\right]. \tag{3.57}$$

Then we can write,

$$\begin{aligned} J^k_{\nu_n^k}(s) &\ge \mathbb{E}\left[\widetilde{H}^{\nu_n^k}(s,t_i^n)\left(J^k_{\mu_1}(t_i^n)\mathbf{1}_{A_i^{n,k}} + J^k_{\mu_2}(t_i^n)\mathbf{1}_{\left(A_i^{n,k}\right)^c}\right)\middle|\mathcal{F}_s\right] &(3.58)\\ &= \mathbb{E}\Big[\widetilde{H}^{\nu_n^k}(s,t_{i-1}^n)\Big(J^k_{\mu_1}(t_{i-1}^n)\mathbf{1}_{A_{i-1}^{n,k}}\mathbf{1}_{A_i^{n,k}} + J^k_{\mu_2\otimes_i^n\mu_1}(t_{i-1}^n)\mathbf{1}_{\left(A_{i-1}^{n,k}\right)^c}\mathbf{1}_{A_i^{n,k}} \\ &\quad + J^k_{\mu_1\otimes_i^n\mu_2}(t_{i-1}^n)\mathbf{1}_{A_{i-1}^{n,k}}\mathbf{1}_{\left(A_i^{n,k}\right)^c} + J^k_{\mu_2}(t_{i-1}^n)\mathbf{1}_{\left(A_{i-1}^{n,k}\right)^c}\mathbf{1}_{\left(A_i^{n,k}\right)^c}\Big)\Big|\mathcal{F}_s\Big] &(3.59)\end{aligned}$$

But,

$$J^k_{\mu_2\otimes_i^n\mu_1}(t_{i-1}^n)\mathbf{1}_{\left(A_{i-1}^{n,k}\right)^c}\mathbf{1}_{A_i^{n,k}} \ge J^k_{\mu_2}(t_{i-1}^n)\mathbf{1}_{\left(A_{i-1}^{n,k}\right)^c}\mathbf{1}_{A_i^{n,k}},$$

and

$$J^k_{\mu_1\otimes_i^n\mu_2}(t_{i-1}^n)\mathbf{1}_{A_{i-1}^{n,k}}\mathbf{1}_{\left(A_i^{n,k}\right)^c} \ge J^k_{\mu_1}(t_{i-1}^n)\mathbf{1}_{A_{i-1}^{n,k}}\mathbf{1}_{\left(A_i^{n,k}\right)^c}.$$

Plugging them into (3.59) gives,

$$J^k_{\nu_n^k}(s) \ge \mathbb{E}\left[\widetilde{H}^{\nu_n^k}(s,t_{i-1}^n)\left(J^k_{\mu_1}(t_{i-1}^n)\mathbf{1}_{A_{i-1}^{n,k}} + J^k_{\mu_2}(t_{i-1}^n)\mathbf{1}_{\left(A_{i-1}^{n,k}\right)^c}\right)\middle|\mathcal{F}_s\right]. \tag{3.60}$$

By induction, this shows that,

$$J^k_{\nu_n^k}(s) \ge \left(J^k_{\mu_1}(s)\mathbf{1}_{A_0^{n,k}} + J^k_{\mu_2}(s)\mathbf{1}_{\left(A_0^{n,k}\right)^c}\right) = J^k_{\mu_1}(s)\vee J^k_{\mu_2}(s). \tag{3.61}$$

We define the function $p_n : [s,T] \to [s,T]$,

$$p_n(u) = \sum_{i=0}^{2^n-1} i \mathbf{1}_{t_i^n \le u < t_{i+1}^n}.$$

Then, from (3.53), we see that,

$$\nu_n^k(u) = \nu(u)\mathbf{1}_{u<s} + \left(\mu_1(u)\mathbf{1}_{A^{n,k}_{p_n(u)}} + \mu_2(u)\mathbf{1}_{\left(A^{n,k}_{p_n(u)}\right)^c}\right)\mathbf{1}_{u\ge s}.$$

On the other hand,

$$t^n_{p_n(u)} \uparrow u, \text{ as } n \uparrow \infty.$$

Therefore,

$$J^k_{\mu_1}\left(t^n_{p_n(u)}\right) = \frac{\mathbb{E}\left[\widetilde{H}^{\mu_1}(t,T)Y\mathbf{1}_{T<T_k}\middle|\mathcal{F}_{t^n_{p_n(u)}}\right]}{\widetilde{H}^{\mu_1}\left(t, t^n_{p_n(u)}\right)} \to \frac{\mathbb{E}\left[\widetilde{H}^{\mu_1}(t,T)Y\mathbf{1}_{T<T_k}\middle|\mathcal{F}_{u-}\right]}{\widetilde{H}^{\mu_1}\left(t,u-\right)} = J^k_{\mu_1}(u),$$

since $\mathbb{F}$, being the filtration generated by the stock price process $\mathcal{X}$ is left-continuous. Similarly,

$$J^k_{\mu_2}\left(t^n_{p_n(u)}\right) \to J^k_{\mu_2}(u).$$

Therefore,

$$\nu_n^k(u) \to \nu(u)\mathbf{1}_{u<s} + \left(\mu_1(u)\mathbf{1}_{J^k_{\mu_1}(u)>J^k_{\mu_2}(u)} + \mu_2(u)\mathbf{1}_{J^k_{\mu_1}(u)\le J^k_{\mu_2}(u)}\right)\mathbf{1}_{u\ge s} =: \mu^k(u). \tag{3.62}$$

With μ^k as defined in (3.62), we see that $\mu^k \in \mathcal{D}^M_{\nu,s}$. Also, since μ_1 and μ_2 are uniformly bounded on $[t,T]$, so are ν_n^k and μ^k.

For any $\mu \in \mathcal{D}^M$, we have

$$\mathrm{d}H^\mu = -H^\mu\left(r_\mu \mathrm{d}t + \vartheta'_\mu \mathrm{d}W\right),$$

where

$$r_\mu = r + \zeta(\mu), \;\; \vartheta_\mu = \vartheta + \sigma^{-1}\mu.$$

Therefore,

$$\mathrm{d}\log H^{\mu} = -\left(r_{\mu} - \frac{1}{2}\vartheta_{\mu}'\vartheta_{\mu}\right)\mathrm{d}t - \vartheta_{\mu}'\mathrm{d}W, \tag{3.63}$$

and hence

$$\log\frac{\widetilde{H}^{\nu_n^k}(s,T_k)}{\widetilde{H}^{\mu^k}(s,T_k)} = \int_s^{T_k}\left(\zeta(\mu^k) - \zeta(\nu_n^k) + \frac{1}{2}\left\|\vartheta_{\mu^k}\right\|^2 - \frac{1}{2}\left\|\vartheta_{\nu_n^k}\right\|^2\right)\mathrm{d}u + \int_s^{T_k}\left(\vartheta_{\mu^k} - \vartheta_{\nu_n^k}\right)'\mathrm{d}W_u$$

From assumption 3.3.1, assumption 3.1.1 and (3.62), it follows by an application of DCT, that

$$\log\widetilde{H}^{\nu_n^k}(s,T_k) - \log\widetilde{H}^{\mu^k}(s,T_k) \xrightarrow{P} 0 \tag{3.64}$$

(3.64) implies that there is a subsequence of $\{\nu_n^k\}_{n\in\mathbb{N}}$ again denoted by $\{\nu_n^k\}_{n\in\mathbb{N}}$, such that

$$\log\widetilde{H}^{\nu_n^k}(s,T_k) \xrightarrow{n\to\infty} \log\widetilde{H}^{\mu^k}(s,T_k), \text{ a.s.},$$

or, in other words,

$$\widetilde{H}^{\nu_n^k}(s,T_k)Y \xrightarrow{n\to\infty} \widetilde{H}^{\mu^k}(s,T_k)Y, \text{ a.s.} \tag{3.65}$$

By Assumption 3.3.2, the contingent claim Y is the wealth generated by a strategy π. It follows from (3.30) that

$$\widetilde{H}^{\nu_n^k}(s,T_k)V^{\pi}(s,T_k) = \exp\left\{\int_s^{T_k}\left(B_{\nu_n^k} - \frac{1}{2}S_{\nu_n^k}'S_{\nu_n^k}\right)\mathrm{d}t + \int_s^{T_k}S_{\nu_n^k}'\mathrm{d}W\right\} \tag{3.66}$$

Therefore,

$$\begin{aligned}
&\left(\widetilde{H}^{\nu_n^k}(s,T_k)V^{\pi}(s,T_k)\right)^2 \\
&= \exp\left\{\int_s^{T_k}\left(2B_{\nu_n^k} + S_{\nu_n^k}'S_{\nu_n^k}\right)\mathrm{d}t\right\}\exp\left\{\int_s^{T_k}2S_{\nu_n^k}'\mathrm{d}W - \int_s^{T_k}2S_{\nu_n^k}'S_{\nu_n^k}\mathrm{d}t\right\} \\
&\leq C\exp\left\{\int_s^{T_k}2S_{\nu_n^k}'\mathrm{d}W - \int_s^{T_k}2S_{\nu_n^k}'S_{\nu_n^k}\mathrm{d}t\right\}
\end{aligned} \tag{3.67}$$

and hence,

$$\mathbb{E}\left[\left(\widetilde{H}^{\nu_n^k}(s,T_k)V^{\pi}(s,T_k)\right)^2\right] \leq C, \tag{3.68}$$

which implies that the family $\left\{\widetilde{H}^{\nu_n^k}(s,T_k)V^\pi(s,T_k)\right\}_{n\in\mathbb{N}}$ is uniformly integrable. Along with (3.65), this implies that

$$\mathbb{E}^{s,x}\left[\widetilde{H}^{\nu_n^k}(s,T)V^\pi(0,T)\right] \to \mathbb{E}^{s,x}\left[\widetilde{H}^{\mu^k}(s,T)V^\pi(0,T)\right]. \tag{3.69}$$

(3.61) and (3.69) implies that

$$J^k_{\mu^k}(s) \geq J^k_{\mu_1}(s) \vee J^k_{\mu_2}(s).$$

□

3.6 Proof of Lemma 3.4.2

Proof of Lemma 3.4.2. Define

$$\mu^k(s) = \nu(s)1_{\{s<\tau\}} + \left[\mu_1(s)1_{\{J^k_{\mu_1}(s)\geq J^k_{\mu_2}(s)\}} + \mu_2(s)1_{\{J^k_{\mu_1}(s)<J^k_{\mu_2}(s)\}}\right]1_{\{s\geq\tau\}}, \tag{3.70}$$

It is easy to see that $\mu^k \in \mathcal{D}^M_{\tau,\nu}$.

For any deterministic time $s \in [t,T]$, it follows from Lemma 3.4.1 that $J^k_{\mu^k}(s) \geq J^k_{\mu_1}(s) \vee J^k_{\mu_2}(s)$ a.s. for each s. Since this holds for any deterministic time $s \in [t,T]$, hence it holds for any stopping time τ taking finitely many values in $[t,T]$.

Let τ be an arbitrary stopping time taking values in $[t,T]$. Define

$$\tau_n(\omega) = \begin{cases} \frac{j}{2^n}T & \text{if } \tau(\omega) \in \left(\frac{j-1}{2^n}T, \frac{j}{2^n}T\right] \\ 0 & \text{o.w.} \end{cases}$$

Then $\tau_n(\omega) \downarrow \tau(\omega)$ a.s., τ_n are stopping times for all n.

Consider any $\mu \in \mathcal{D}^M_{\tau,\nu}$. Since, $\widetilde{H}^\mu(t,s)J_\mu(s) = \mathbb{E}[\widetilde{H}^\mu(t,T)Y|\mathcal{F}_s]$, hence $\{\widetilde{H}^\mu(t,s)J_\mu(s)\}_{s\in[t,T]}$ is a martingale. Therefore, by optional sampling theorem $\widetilde{H}^\mu(t,\tau)J_\mu(\tau) = \mathbb{E}[\widetilde{H}^\mu(t,T)J_\mu(T)|\mathcal{F}_\tau] = \mathbb{E}[\widetilde{H}^\mu(t,T)Y|\mathcal{F}_\tau]$.

Therefore,

$$\begin{aligned}\widetilde{H}^{\mu}(t,\tau)J_{\mu}(\tau) &= \mathbb{E}[\widetilde{H}^{\mu}(t,T)Y|\mathcal{F}_{\tau}] = \lim_{n\to\infty}\mathbb{E}[\widetilde{H}^{\mu}(t,T)Y|\mathcal{F}_{\tau_n}] \\ &= \lim_{n\to\infty}\widetilde{H}^{\mu}(t,\tau_n)J_{\mu}(\tau_n) = \widetilde{H}^{\mu}(t,\tau)\lim_{n\to\infty}J_{\mu}(\tau_n)\end{aligned}$$

The second equality holds by Levy's convergence result for backward martingales. The last equality holds because $\widetilde{H}^{\widehat{\mu}}(t,\cdot)$ is continuous a.s.
Therefore, for any $\mu \in \mathcal{D}^{M}_{\tau,\nu}$, we have $J_{\mu}(\tau) = \lim_{n\to\infty} J_{\mu}(\tau_n)$. In particular,

$$\begin{aligned}J_{\mu^k}(\tau) = \lim_{n\to\infty} J_{\mu^k}(\tau_n) \geq \lim_{n\to\infty} J_{\mu_1}(\tau_n) \vee J_{\mu_2}(\tau_n) &\geq \Big(\lim_{n\to\infty} J_{\mu_1}(\tau_n)\Big) \vee \Big(\lim_{n\to\infty} J_{\mu_2}(\tau_n)\Big) \\ &= J_{\mu_1}(\tau) \vee J_{\mu_2}(\tau)\end{aligned}$$

Hence, $J_{\mu^k}(\tau) \geq J_{\mu_1}(\tau) \vee J_{\mu_2}(\tau)$ a.s. $\square$

Chapter 4

Upper Hedging Price under Constraints: Viscosity Solutions Characterization

We recall first, an observation made in Subsection 2.6.1. We saw that if the market is Markovian, the contingent claim Y is the terminal wealth generated by a Markovian strategy π and if $z_{\mathbb{R}^n}(\cdot,\cdot)$ (defined in (2.31)) is locally $C^{1,2}$, then $z_{\mathbb{R}^n}(\cdot,\cdot)$ satisfies the partial differential equation (2.32). The interested reader can see Janson and Tysk (2006) for a nice discussion on similar Feynman-Kac type theorems and their converse. This connection with partial differential equations is extremely useful as under suitable conditions one can use the standard theory of parabolic partial differential equations to come to interesting conclusions. For example, Proposition 2 and its corollary in Section 9 of Fernholz and Karatzas (2010), use the maximum principle for parabolic equations to conclude that under certain conditions on the volatility matrix, "short-term arbitrage" with respect to the market portfolio implies "long-term arbitrage" with respect to it. This motivates us to get a similar PDE characterization for $\widehat{u}_M(t,T,K)$ defined in

(3.17). However, it would be too much to expect $\widehat{u}_M(t, T, K)$ to be in $C^{1,2}$. We will therefore, take a much more general approach, viz. that of viscosity solutions, which was introduced in the early 1980s by Pierre-Louis Lions and Michael Crandall as a generalization of the classical concept of what is meant by a "solution" to a partial differential equation. The survey paper Crandall et al. (1992) is an excellent resource for the study of viscosity solutions.

Throughout this chapter, we will assume that the conditions in Theorem 3.3.1 hold.

In a Markovian market and for contingent claims which depend only on the final stock price, Soner and Touzi (2003) used the dynamic programming principle followed by the upper hedging price process to characterize it in terms of viscosity solution of certain variational inequalities. We will follow an approach similar to theirs, but for a class of contingent claims, which can be replicated by Markovian investment strategies. We will also work under more general market conditions. For example, unlike Soner and Touzi (2003), we do not assume $b(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ to be bounded and continuous. Nevertheless, we will use our results to study how relative arbitrage opportunities disappear as the constraints become stricter.

In Section 4.1 we will present a dynamic programming principle for the process $z_K(\cdot, \mathcal{X}(\cdot))$, defined in (4.3), and in Section 4.2 we will use this dynamic programming principle to characterize the function $z_K(\cdot, \cdot)$ in terms of viscosity solutions to some partial differential equations. Sections 4.3, 4.4 and 4.5 contain the technical proofs of the results presented in Section 4.2. Section 4.6 contains an auxiliary result about estimates of moments of the stock price process and the discounted wealth process. Section 4.2, and hence the related sections following it, stand upon the assumption that the function $z_K(\cdot, \cdot)$ is lower-semicontinuous. Section 4.7 presents sufficient conditions for lower-semicontinuity of $z_K(\cdot,\cdot)$.

4.1 Dynamic programming principle

We recall from Theorem 3.3.1 that for a contingent claim $Y = V^\pi(0,T)$, the upper hedging price $u^Y(t,T,K)$ (defined in (2.23)) can be expressed as

$$u^Y(t,T,K) = V^\pi(0,t)\text{esssup}_{\nu\in\mathcal{D}^M_{(t,T]}(K)}\mathbb{E}\left[U^\nu(t,T)\big|\mathcal{F}_t\right].$$

The following dynamic programming principle (which is the pivot on which this Chapter rests) follows from the proof of Theorem 3.3.1.

Theorem 4.1.1. *For $\{\mathcal{F}_t\}$-stopping times ρ and τ satisfying $t \le \rho \le \tau \le T$ a.s., and under the conditions of Theorem 3.3.1, the following dynamic programming principle holds.*

$$u^Y(\rho,T,K) = esssup_{\nu\in\mathcal{D}^M_{(t,T]}(K)}\mathbb{E}\left[\widetilde{H}^\nu(\rho,\tau)u^Y(\tau,T,K)\big|\mathcal{F}_\rho\right].$$

$$\mathfrak{z}_K(\rho) = esssup_{\nu\in\mathcal{D}^M_{(t,T]}(K)}\mathbb{E}\left[U^\nu(\rho,\tau)\mathfrak{z}_K(\tau)\big|\mathcal{F}_\rho\right].$$

We will now assume that

Assumption 4.1.1. *For every initial condition $x \in (0,\infty)^n$, there exists a unique-in-distribution weak solution to the SDE* (2.6).

We will denote by $(\mathcal{X}^{s,x}(u))_{u\in[s,\infty)}$ the solution to the stochatic differential equation

$$\begin{aligned} dX_i(t) &= X_i(t)\left(b_i(t,\mathcal{X}(t))dt + \sum_{\nu=1}^d \sigma_{i\nu}(t,\mathcal{X}(t))dW_\nu(t)\right), \quad t \ge s \\ X_i(s) &= x_i > 0,\ i = 1,2,\dots,n. \end{aligned} \tag{4.1}$$

$\widetilde{H}^{\nu,s,x}(\cdot,\cdot), V^{\pi,s,x}(\cdot,\cdot)$, $U^{\nu,s,x}(\cdot,\cdot)$ and $u^{Y,s,x}(\cdot,T,K)$ will denote the corresponding processes when the stock price process $\mathcal{X}(\cdot)$ satisfies (4.1).

We will, henceforth, restrict attention to the case where $(\Omega, \mathcal{F})$ is a separable standard Borel space; this will be the case if, for example, $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical probability space with $\Omega = C\left([0,\infty)^n\right)$ and $\mathcal{F}$ is the sigma algebra generated by all continuous functions on $[0,\infty)^n$. It then follows from Theorem V.8.1 in Parthasarathy (1967) that there exists a regular conditional probability measure $\mathbb{P}^{s,x}$, defined to be the conditional probability

$$\mathbb{P}^{s,x}(B) = \mathbb{P}\left[B \middle| \mathcal{X}(s) = x\right], B \in \mathcal{F}.$$

For $B \in \mathcal{B}(C\left([0,\infty)^n\right))$, we can now write

$$\mathbb{P}\left[\mathcal{X}(s+\cdot) \in B \middle| \mathcal{X}(s) = x\right] = \mathbb{P}^{s,x}\left[\mathcal{X}(s+\cdot) \in B\right] = \mathbb{P}\left[\mathcal{X}^{s,x}(s+\cdot) \in B\right].$$

Under Assumption 3.3.1, Theorem 5.4.20 in Karatzas and Shreve (1991) gives us the strong Markov property, that for any stopping time $\tau \geq s$ and $B \in \mathcal{B}(C\left([0,\infty)^n\right))$,

$$\mathbb{P}^{s,x}\left[\mathcal{X}(\tau+\cdot) \in B \middle| \mathcal{X}(\tau)\right] = \mathbb{P}^{\tau,\mathcal{X}(\tau)}\left[\mathcal{X}(\tau+\cdot) \in B\right].$$

We will denote by $\mathbb{E}^{s,x}$ the expectation with respect to the probability measure $\mathbb{P}^{s,x}$. We can now write,

$$\mathfrak{z}_K(t) = \operatorname{esssup}_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}\left[U^\nu(t,T) \middle| \mathcal{X}(t)\right] = \operatorname{esssup}_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}^{t,\mathcal{X}(t)}\left[U^\nu(t,T)\right]. \tag{4.2}$$

We now define,

$$z_K(t,x) := \sup_{\nu \in \mathcal{D}^M(K)} \mathbb{E}^{t,x}[U^\nu(t,T)] = \sup_{\nu \in \mathcal{D}^M(K)} \mathbb{E}[U^{\nu,t,x}(t,T)], \tag{4.3}$$

When the constraint set K is clear from the context, we will drop the subscript K and denote z_K simply by z.

For the rest of this chapter we will assume the following:

Assumption 4.1.2. *$z(\cdot,\cdot)$ is lower-semicontinuous on $[0,T) \times \mathbb{R}^n_+$.*

Sufficient conditions for the lower-semicontinuity of $z(\cdot,\cdot)$ will be presented in Section 4.7.

For any $s \in [t,T]$, the dynamic programming principle from Theorem 4.1.1 now allows us to write

$$z_K(t,x) = \sup_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}\left[U^\nu(t,s) z_K(s,\mathcal{X}_s) \middle| \mathcal{X}_t = x \right]. \tag{4.4}$$

4.2 Viscosity solution characterization

In this section we will use the dynamic programming principle of Theorem 4.1.1 to characterize the function $z_K : [0,T] \times \mathbb{R}^n_+ \to \mathbb{R}$ as a viscosity solution to certain variational inequalities. We will present the results here and immediately move on to its implications in the next section, leaving the technical proofs for later sections.

For the convenience of the reader, we start by presenting the definition of viscosity solutions. We denote by $\mathcal{S}(n)$, the set of all symmetric $n \times n$ matrices. Consider a function $G : [0,\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ which satisfies the monotonicity condition

$$G(t,x,r,q,p,X) \le G(t,x,s,q,p,Y) \text{ whenever } r \le s \text{ and } Y \le X; \tag{4.5}$$

where $(t,x,q,p) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $r,s \in \mathbb{R}$, $X,Y \in \mathcal{S}(n)$ and $\mathcal{S}(n)$ is equipped with its usual order.

Given an open set $V \subset [0,\infty) \times \mathbb{R}^n$, we say that an upper semicontinuous function $u : V \to \mathbb{R}$ is a *viscosity subsolution* of

$$G\left(t,x,u(t,x),\frac{\partial u}{\partial t}(t,x),\nabla_x u(t,x),D^2u(t,x)\right) = 0, \quad (t,x) \in V, \tag{4.6}$$

if, for any $(\bar{t},\bar{x}) \in V$ and any function $\phi \in C^{1,2}(V)$ satisfying

$$0 = (\phi - u)(\bar{t},\bar{x}) = \min_{(t,x)\in V} (\phi - u)(t,x),$$

we have

$$G\left(\bar{t},\bar{x},\phi(\bar{t},\bar{x}),\frac{\partial\phi}{\partial t}(\bar{t},\bar{x}),\nabla_x\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\le 0.$$

We say that a lower semicontinuous function $v : V \to \mathbb{R}$ is a *viscosity supersolution* of

$$G\left(t,x,v(t,x),\frac{\partial v}{\partial t}(t,x),\nabla_x v(t,x),D^2 v(t,x)\right)=0,\quad (t,x)\in V,$$

if, for any $(\bar{t},\bar{x})\in V$ and any function $\phi\in C^{1,2}(V)$ satisfying

$$0=(v-\phi)(\bar{t},\bar{x})=\min_{(t,x)\in V}(v-\phi)(t,x),$$

we have

$$G\left(\bar{t},\bar{x},\phi(\bar{t},\bar{x}),\frac{\partial\phi}{\partial t}(\bar{t},\bar{x}),\nabla_x\phi(\bar{t},\bar{x}),D^2\phi(\bar{t},\bar{x})\right)\ge 0.$$

We say that a continuous function $w : V \to \mathbb{R}$ is a viscosity solution to (4.6) if it is both a supersolution and a subsolution to (4.6).

We now give an equivalent definition of viscosity solutions which will be useful in some cases. Given $(q,p,X)\in\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}(n)$, we say that $(q,p,X)\in\mathcal{J}_V^{2,+}u(\bar{t},\bar{x})$ (the "superjet" of u at $(\bar{t},\bar{x})$) if

$$u(t,x)\le u(\bar{t},\bar{x})+q(t-\bar{t})+\langle p,x-\bar{x}\rangle+\frac{1}{2}\langle X(x-\bar{x}),x-\bar{x}\rangle+o\left(|t-\bar{t}|+|x-\bar{x}|^2\right),$$
$$\text{as } V\ni(t,x)\to(\bar{t},\bar{x}).$$

We say that $(q,p,X)\in\mathcal{J}_V^{2,-}v(\bar{t},\bar{x})$ (the "subjet" of v at $(\bar{t},\bar{x})$) if

$$v(t,x)\ge v(\bar{t},\bar{x})+q(t-\bar{t})+\langle p,x-\bar{x}\rangle+\frac{1}{2}\langle X(x-\bar{x}),x-\bar{x}\rangle+o\left(|t-\bar{t}|+|x-\bar{x}|^2\right),$$
$$\text{as } V\ni(t,x)\to(\bar{t},\bar{x}).$$

u is said to be a viscosity subsolution of (4.6) if

$$G(t,x,u(t,x),q,p,X)\le 0,\quad\text{for all } (t,x)\in V \text{ and } (q,p,X)\in\mathcal{J}_V^{2,+}u(t,x).$$

Similarly, v is said to be a viscosity supersolution of (4.6) if

$$G(t,x,v(t,x),q,p,X) \geq 0, \quad \text{for all } (t,x) \in V \text{ and } (q,p,X) \in \mathcal{J}_V^{2,-} v(t,x).$$

We introduce some more notations now. We denote

$$\mathcal{O}_K := \begin{cases} \text{the set of all points } (t,x) \in (0,T) \times \mathbb{R}_+^n, \text{ such that there exists} \\ \text{an open neighborhood } V_{(t,x)} \text{ of } (t,x) \text{ on which } z(\cdot,\cdot) \text{ is bounded.} \end{cases} \tag{4.7}$$

As usual, we will drop the subscript K, and use it only when there is scope for confusion.

Even though the set $\mathcal{O}_K$ has been introduced for technical reasons, it has some economic significance also. Suppose $(t,x) \notin \mathcal{O}_K$ for some $(t,x) \in (0,T) \times \mathbb{R}_+^n$. Suppose that the stock price process takes values in any arbitrary open neighborhood of x in any arbitrary open time interval containing t, with positive probability. If an investor plans to start investing in strategies constrained to take values in K, at some undetermined time close to t, then he should be aware that he might need any arbitrarily large multiple of the wealth generated by π up to that time, in order to be able to super-replicate the claim $Y = V^\pi(0,T)$ at time T.

On the closure $\overline{\mathcal{O}}$ of the set of (4.7), we define the upper-semicontinuous envelope z^* and lower-semicontinuous envelope z_* of z by

$$z^*(\bar{t},\bar{x}) = \limsup_{\mathcal{O} \ni (t',x') \to (\bar{t},\bar{x})} z(t',x'), \quad z_*(\bar{t},\bar{x}) = \liminf_{\mathcal{O} \ni (t',x') \to (\bar{t},\bar{x})} z(t',x'). \tag{4.8}$$

Note that by Assumption 4.1.2

$$z_*(\bar{t},\bar{x}) = z(\bar{t},\bar{x}), \ (\bar{t},\bar{x}) \in \mathcal{O}.$$

Let $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{M}^\nu$ be the operators defined by

$$\mathcal{L}w = \frac{\partial w}{\partial t} + \frac{1}{2} \sum_i x_i \frac{\partial w}{\partial x_i} \left(a_{i*}\pi + r\right) + \frac{1}{2} \sum_{i,j} x_i x_j a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j}, \tag{4.9}$$

$$\mathcal{M}w = \inf_{\nu \in \widetilde{K}} \left[w\left(\pi'\nu + \zeta(\nu)\right) + \sum \nu_i x_i \frac{\partial w}{\partial x_i} \right], \tag{4.10}$$

$$\mathcal{M}^{\nu} w = w\left(\pi'\nu + \zeta(\nu)\right) + \sum \nu_i x_i \frac{\partial w}{\partial x_i}, \tag{4.11}$$

for $w \in C^{1,2}\big((0,T) \times \mathbb{R}^n_+\big)$.

We now have the following theorem.

Theorem 4.2.1. *Under the conditions of Theorem 3.3.1 and Assumption 4.1.2, z is a viscosity supersolution to the equation*

$$-\mathcal{L}z(s,x) + \mathcal{M}z(s,x) = 0,\ (s,x) \in \mathcal{O}. \tag{4.12}$$

We will present the proof of Theorem 4.2.1 in Section 4.3.

Since $0 \in \widetilde{K}$, hence for any $w \in C^{1,2}$,

$$\mathcal{M}w(s,x) \leq 0,\quad s \in [t,T], x \in \mathbb{R}^n_+.$$

On the other hand, if there exists $\nu \in \widetilde{K}, x \in \mathbb{R}^n_+$ such that

$$w(s,x)\left(\pi'(s,x)\nu + \zeta(\nu)\right) + \sum \nu_i x_i \frac{\partial w}{\partial x_i}(s,x) < 0,$$

then from the linear homogeneity of ζ it follows that

$$\mathcal{M}w(s,x) = -\infty.$$

Thus, for any $w \in C^{1,2}$

$$\mathcal{M}w(s,x) = \begin{cases} -\infty, & \text{if } \pi'(s,x)\nu + \zeta(\nu) < 0 \text{ for some } \nu \in \widetilde{K}, \\ 0, & \text{o.w.} \end{cases} \tag{4.13}$$

Hence, for a fixed $(\bar{t},\bar{x}) \in \mathcal{O}$, if $\phi \in C^{1,2}$ is such that

$$0 = (z-\phi)\left(\bar{t},\bar{x}\right) = \min_{(s,x)\in\mathcal{O}} (z-\phi)\left(s,x\right),$$

then, by Theorem 4.2.1 and (4.13), we have

$$-\mathcal{L}\phi(\bar{t},\bar{x}) \geq 0,\quad \mathcal{M}\phi(\bar{t},\bar{x}) = 0. \tag{4.14}$$

Thus, we see that properties of viscosity supersolutions to the partial differential equation in (4.12) can be studied by studying the two separate partial differential equations,

$$-\mathcal{L}\phi(t,x) = 0, \quad (t,x) \in \mathcal{O}, \tag{4.15}$$

$$\mathcal{M}\phi(\bar{t},\bar{x}) = 0, \quad (t,x) \in \mathcal{O}. \tag{4.16}$$

The parabolic PDE (4.15) does not depend on the constraint set K, and comparison principles can be easily written down for its supersolutions. It can have multiple supersolutions. In fact, it has all constant functions as trivial solutions. The choice of the right supersolution is dictated by (4.16), the terminal conditions to be derived soon and by $z^*(\cdot,\cdot)$, since $z \le z^*$.

We now derive a subsolution property of z^*. Let $\mathcal{M}_1$ be the operator defined by

$$\mathcal{M}_1 w = \inf_{\nu \in \widetilde{K}_1} \left[w\left(\pi'\nu + \zeta(\nu)\right) + \frac{1}{2} \sum \nu_i x_i \frac{\partial w}{\partial x_i} \right], \; w \in C^{1,2}, \tag{4.17}$$

where,

$$\widetilde{K}_1 := \widetilde{K} \cap \left\{ |\nu| = 1 \text{ and } \zeta(\nu) + \zeta(-\nu) \neq 0 \right\}.$$

A known result of convex analysis is that

$$x \in \mathrm{ri}(K) \text{ if and only if } x \in K \text{ and } \inf_{y \in \widetilde{K}_1} \left(x'y + \zeta(y)\right) > 0. \tag{4.18}$$

Let $\mathcal{P}$ be the operator defined by

$$\mathcal{P}w(s,x) = \limsup_{\delta \downarrow 0} \sup_{\eta \in (\mathrm{vect}(K))^{\perp}} \left\{ \eta' w(t,y) : (t,y) \in (s-\delta, s+\delta) \times B(x,\delta) \right\}. \tag{4.19}$$

It is easy to see that

$$\mathcal{P}w(s,x) = 0 \iff w(t,y) \in \mathrm{vect}(K) \; \forall \; (t,y) \text{ sufficiently close to } (s,x).$$

Theorem 4.2.2. *Under the conditions of Theorem 4.2.1, z^* defined in* (4.8) *is a viscosity subsolution of the equation*

$$\min\left\{-\mathcal{L}z^*(s,x), \mathcal{M}_1 z^*(s,x)\right\} \mathbf{1}_{\mathcal{P}\pi(s,x)=0} + \mathcal{M}z^*(s,x) = 0,\ (s,x) \in \mathcal{O}. \tag{4.20}$$

If $\pi(s,x) = diag(x)\nabla\Pi(s,x)$, for some $\Pi : [0,T] \times \mathbb{R}^n_+ \to \mathbb{R}^n$ which is differentiable w.r.t the space variable, then z^* is a viscosity subsolution to the equation

$$\min\left\{-\mathcal{L}z^*(s,x), \mathcal{M}_1 z^*(s,x)\right\} + \mathcal{M}z^*(s,x) = 0,\ (s,x) \in \mathcal{O}. \tag{4.21}$$

We will present the proof of Theorem 4.2.2 in Section 4.4

Remark 4.2.1. Unless $(\mathrm{diag}(x))^{-1}\pi(t,x)$ is a conservative vector field, Theorem 4.2.2 gives us useful information about the value of $z(\cdot,\cdot)$ at (t,x) only if the investment strategy π takes values in K in a neighborhood of (t,x).

We denote

$$\overline{\mathcal{O}}_T := \left\{x \in \mathbb{R}^n_+ : (T,x) \in \overline{\mathcal{O}}\right\}. \tag{4.22}$$

Theorem 4.2.3. *Under the conditions of Theorem 4.2.1, $z_*(T,\cdot)$ is a viscosity supersolution in $\overline{\mathcal{O}}_T$ of*

$$z_*(T,x) - 1 + \mathcal{M}z_*(T,x) = 0,\ \ x \in \overline{\mathcal{O}}_T,$$
$$\textit{or},\ \min\left\{z_*(T,x) - 1, \mathcal{M}z_*(T,x)\right\} = 0,\ \ x \in \overline{\mathcal{O}}_T.$$

We denote

$$\widetilde{O} := \begin{cases} \text{the set of points } x \in \mathbb{R}^n_+ \text{ such that } z(\cdot,\cdot) \text{ is} \\ \text{bounded on } (T-\delta,T] \times B(x,\delta) \text{ for some } \delta > 0. \end{cases} \tag{4.23}$$

We also denote,

$$\widetilde{Q} := \begin{cases} \text{the set of points } y \in \mathbb{R}^n_+ \text{ for which there exists a } \delta > 0,\ \text{such that} \\ \sup_{[T-\delta,T]\times B(y,\delta)} \left(z(t,x) - \sup_{\nu\in\mathcal{D},\|\nu\|\le m} \mathbb{E}^{t,x}\left[U^\nu(t,T)\right]\right) \to 0,\ \text{as } m \uparrow \infty \end{cases} \tag{4.24}$$

Equivalently,

$$\widetilde{Q} := \begin{cases} \text{the set of points } y \in \mathbb{R}^n_+ \text{ for which there exists a } \delta > 0, \\ \text{such that given any } \epsilon > 0, \text{ we can find a } M \text{ such that} \\ z(t,x) - \sup_{\nu \in \mathcal{D}, \|\nu\| \le M} \mathbb{E}^{t,x}\left[U^\nu(t,T)\right] < \epsilon, \ \forall\ (t,x) \in [T-\delta, T] \times B(y,\delta). \end{cases} \tag{4.25}$$

Theorem 4.2.4. *Under the conditions of Theorem 4.2.1, $z^*(T,\cdot)$ is a viscosity subsolution of the equation*

$$\min\left\{z^*(T,x) - 1, \mathcal{M}_1 z^*(T,x)\right\} \mathbf{1}_{\mathcal{P}\pi(T,x)=0} + \mathcal{M} z^*(T,x) = 0, \ x \in \widetilde{Q} \cap \widetilde{O}. \tag{4.26}$$

Remark 4.2.2. Theorem 4.2.2 gives us useful information about the value of $z(T-,x)$ only if the investment strategy π takes values in K in a neighborhood of x and at times close to the terminal time T. However, $x \in \widetilde{O}$ is not an easily tractable condition. Thus, the usefulness of Theorem 4.2.2 is questionable.

4.3 Viscosity supersolution property

Proof of Theorem 4.2.1. For a fixed $(\bar{t},\bar{x}) \in \mathcal{O}$, let $\phi \in C^{1,2}$ be such that

$$0 = (z-\phi)(\bar{t},\bar{x}) = \min_{(s,x)\in\mathcal{O}} (z-\phi)(s,x). \tag{4.27}$$

Fix any $\nu \in \mathcal{D}^M_{[\bar{t},T]}$. It follows from (4.4) and (4.27) that, for any $\{\mathcal{F}_t\}$-stopping time θ, taking values in $[\bar{t},T]$,

$$\begin{aligned} z(\bar{t},\bar{x}) &\ge \mathbb{E}\left[U^\nu(t,\theta) z(\theta,\mathcal{X}_\theta) | \mathcal{X}_{\bar{t}} = \bar{x}\right], \\ \implies \phi(\bar{t},\bar{x}) &\ge \mathbb{E}\left[U^\nu(t,\theta)\phi(\theta,\mathcal{X}_\theta) | \mathcal{X}_{\bar{t}} = \bar{x}\right] \end{aligned} \tag{4.28}$$

From (3.30) and by an application of Itô's lemma, we get

$$\begin{aligned}
\mathrm{d}U^\nu\phi &= U^\nu\left[\left(\frac{\partial\phi}{\partial t}+\sum_{i=1}^n\frac{\partial\phi}{\partial x_i}X_ib_i+\frac{1}{2}\sum_{i,j}\frac{\partial^2\phi}{\partial x_i\partial x_j}X_iX_ja_{ij}\right)\mathrm{d}t+\sum_{i=1}^n\frac{\partial\phi}{\partial x_i}X_i\sigma_{i*}\mathrm{d}W\right]\\
&\quad+\phi U^\nu\left(B_\nu\mathrm{d}t+S_\nu'\mathrm{d}W\right)+\sum_{i=1}^n\frac{\partial\phi}{\partial x_i}U^\nu X_i\sigma_{i*}S_\nu\mathrm{d}t\\
&=U^\nu\left[\frac{\partial\phi}{\partial t}+\sum_{i=1}^n\frac{\partial\phi}{\partial x_i}X_i\left(b_i+\sigma_{i*}S_\nu\right)+\frac{1}{2}\sum_{i,j}\frac{\partial^2\phi}{\partial x_i\partial x_j}X_iX_ja_{ij}+\phi B_\nu\right]\mathrm{d}t\\
&\quad+U^\nu\left[\sum_{i=1}^n\frac{\partial\phi}{\partial x_i}X_i\sigma_{i*}+\phi S_\nu'\right]\mathrm{d}W
\end{aligned} \tag{4.29}$$

Then (4.29) can be written as

$$\mathrm{d}U^\nu\phi=U^\nu\left[\mathcal{L}\phi-\mathcal{M}^\nu\phi\right]\mathrm{d}t+U^\nu\left[\sum_{i=1}^n\frac{\partial\phi}{\partial x_i}X_i\sigma_{i*}+\phi\left(\sigma'\pi-\vartheta-\sigma^{-1}\nu\right)'\right]\mathrm{d}W \tag{4.30}$$

For some constants $\eta>0, Q>0$, let

$$\tau^\nu:=\inf\left\{s\geq t:|\mathcal{X}_s-x|>\eta \text{ or } U^\nu(t,s)>Q \text{ or } U^\nu(t,s)<1/Q\right\}.$$

Let $\{h_m\}_{m\in\mathbb{N}}$ be a sequence of positive numbers such that $h_m\to 0$. Consider the stopping times

$$\theta_m^\nu:=\tau^\nu\wedge(t+h_m).$$

From (4.28), it follows that

$$\begin{aligned}
&0\geq\mathbb{E}\left[U^\nu(t,\theta_m^\nu)\phi(\theta_m^\nu,\mathcal{X}_{\theta_m^\nu})-\phi(t,x)\;\middle|\;\mathcal{X}_t=x\right]\\
\implies&0\geq\mathbb{E}\left[\int_t^{\theta_m^\nu}U^\nu(t,s)\left(\mathcal{L}\phi-\mathcal{M}^\nu\phi\right)(s,\mathcal{X}_s)\mathrm{d}s\;\middle|\;\mathcal{X}_t=x\right]
\end{aligned}\tag{4.31}$$

The continuity of the paths of $\mathcal{X}$ and U^ν imply that for a.e. $\omega\in\Omega$, $\theta_m^\nu=t+h_m$ for large enough m. If we take ν to be a constant process, then for a.e. ω, we have

$$U^\nu(t,s)\left(\mathcal{L}\phi-\mathcal{M}^\nu\phi\right)(s,\mathcal{X}_s)\mathbf{1}_{t\leq s\leq\theta_m^\nu\leq t+h_m}-\left(\mathcal{L}\phi-\mathcal{M}^\nu\phi\right)(\bar{t},\bar{x})\mathbf{1}_{t\leq s\leq t+h_m}\to 0.$$

Applying the DCT twice, we get

$$\mathbb{E}\left[\frac{1}{h_m}\int_t^{\theta_m^\nu} U^\nu(t,s)\left(\mathcal{L}\phi - \mathcal{M}^\nu\phi\right)(s,\mathcal{X}_s)\mathrm{d}s \;\middle|\; \mathcal{X}_{\bar{t}} = \bar{x}\right] \to \left(\mathcal{L}\phi - \mathcal{M}^\nu\phi\right)(\bar{t},\bar{x}).$$

(4.31) gives us

$$\left(\mathcal{L}\phi - \mathcal{M}^\nu\phi\right)(\bar{t},\bar{x}) \leq 0.$$

Since the above holds for every $\nu \in \widetilde{K}$, hence

$$\sup_{\nu\in\widetilde{K}}\left(\mathcal{L}\phi - \mathcal{M}^\nu\phi\right)(\bar{t},\bar{x}) \leq 0,$$

$$\text{i.e. } \mathcal{L}\phi(\bar{t},\bar{x}) - \mathcal{M}\phi(\bar{t},\bar{x}) \leq 0$$

Thus, z is seen to be a viscosity supersolution to the equation

$$-\mathcal{L}z(s,x) + \mathcal{M}z(s,x) = 0,\ (s,x) \in \mathcal{O}. \tag{4.32}$$

□

4.4 Viscosity subsolution property

We denote

$$\mathcal{O}_{sub} := \left\{(s,x) \in \mathcal{O}\middle| z^*(\cdot,\cdot) \text{ is a subsolution to } -\mathcal{L}z^*(s,x) = 0\right\}. \tag{4.33}$$

Remark 4.4.1. Note that if $(t,x) \in \mathcal{O}$, is such that $\mathcal{P}\pi(s,y) = 0,\ \mathcal{M}z^*(s,y) = 0,\ \mathcal{M}_1 z^*(s,y) > 0$, for all (s,y) in a neighborhood of (t,x), then from the proof of Theorem 5.2.1 it follows that strict inequality holds in (5.16) with $z(\cdot,\cdot)$ replaced by $z^*(\cdot,\cdot)$. Similarly strict inequality holds in (5.32) with $z(\cdot,\cdot)$ replaced by $z^*(\cdot,\cdot)$.

Remark 4.4.2. Unless $(\mathrm{diag}(x))^{-1}\pi(t,x)$ is a conservative vector field, Theorem 4.2.2 gives us useful information about the value of $z(\cdot,\cdot)$ at (t,x) only if the investment strategy π takes values in K in a neighborhood of (t,x).

Proof of Theorem 4.2.2. When $w > 0$, we can rewrite (4.17) as

$$\mathcal{M}_1 w = \inf_{\nu \in \widetilde{K}_1} w \left[(\pi + \operatorname{diag}(x) \nabla \log w)' \nu + \zeta(\nu) \right], \ w \in C^{1,2},$$

Therefore, if $w > 0$, then

$$\mathcal{M}(w) = 0, \mathcal{M}_1(w) > 0 \iff \pi + \operatorname{diag}(x) \nabla \log w \in \operatorname{ri}(K). \tag{4.34}$$

For a fixed $(\bar{t}, \bar{x}) \in \mathcal{O}$, let $\phi \in C^{1,2}$ be such that

$$0 = (z^* - \phi)(\bar{t}, \bar{x}) = \max_{(s,x) \in \mathcal{O}} (z^* - \phi)(s, x). \tag{4.35}$$

As we have seen earlier, $\mathcal{M}\phi(\bar{t}, \bar{x})$ can be either 0 or $-\infty$. If it takes the value $-\infty$, then it trivially follows that

$$\min \{ -\mathcal{L}\phi(\bar{t}, \bar{x}), \mathcal{M}_1 \phi(\bar{t}, \bar{x}) \} \mathbf{1}_{\mathcal{P}\pi(s,x)=0} + \mathcal{M}\phi(\bar{t}, \bar{x}) \leq 0.$$

So let us suppose that

$$\mathcal{M}\phi(\bar{t}, \bar{x}) = 0. \tag{4.36}$$

We will prove our claim by contradiction. Suppose that

$$\min \{ -\mathcal{L}\phi(\bar{t}, \bar{x}), \mathcal{M}_1 \phi(\bar{t}, \bar{x}) \} \mathbf{1}_{\mathcal{P}\pi(\bar{t},\bar{x})=0} > 0. \tag{4.37}$$

(4.36) and (4.37) together imply that

$$\Big(\pi + \operatorname{diag}(x) \nabla \log \phi \Big)(\bar{t}, \bar{x}) \in \operatorname{ri}(K).$$

$\mathbf{1}_{\mathcal{P}\pi(\bar{t},\bar{x})=0}$ implies that for (s, x) in a neighbourhood of $(\bar{t}, \bar{x})$, $\pi(s, x) \in \operatorname{vect}(K)$. Theorem 5.2.1 then implies that for any $\nu \in (\operatorname{vect}(K))^{\perp}$, $z(\bar{t}, \bar{x}e^{r\nu})$ is constant for r in a neighbourhood of 0. Hence, we can also take $\phi(\bar{t}, \bar{x}e^{r\nu})$ to be constant for r in a neighbourhood of 0 and $\nu \in (\operatorname{vect}(K))^{\perp}$. It follows that $\operatorname{diag}(x)\nabla \log \phi$ and hence $(\pi + \operatorname{diag}(x) \nabla \log \phi)(s, x)$ lies in $\operatorname{vect}(K)$ for (s, x) in a neighbourhood of $(\bar{t}, \bar{x})$. Therefore, there exists $\delta > 0$ such that

$$(\pi + \operatorname{diag}(x) \nabla \log \phi)(s, x) \in \operatorname{ri}(K), \ (s, x) \in \mathrm{B}((\bar{t}, \bar{x}), \delta) \subset [t, T] \times \mathbb{R}^n_+$$

and hence that there exist $\delta > 0, \epsilon > 0$, such that

$$\mathcal{M}\phi(s,y) \geq 0,\ -\mathcal{L}\phi(s,y) + \mathcal{M}_1\phi(s,y) > \epsilon, \forall\ (s,y) \in \mathrm{B}\left((\bar{t},\bar{x}),\delta\right) \subset [t,T] \times \mathbb{R}^n_+. \tag{4.38}$$

By definition of z^*, there exists a sequence $\{(t_m,x_m)\}_{m\in\mathbb{N}}$ in $B\left((\bar{t},\bar{x}),\delta\right)$ such that

$$(t_m,x_m) \to (\bar{t},\bar{x}),\ z(t_m,x_m) < \infty,\ z(t_m,x_m) \to z^*(\bar{t},\bar{x}).$$

If we define

$$\gamma_m := z(t_m,x_m) - \phi(t_m,x_m),$$

then by the continuity of ϕ and (4.35), it follows that

$$\gamma_m \to 0,\ \text{as } m \to \infty.$$

Let $\{h_m\}_{m\in\mathbb{N}}$ be a sequence of strictly positive numbers such that

$$h_m \to 0,\ \frac{\gamma_m}{h_m} \to 0.$$

Let

$$\rho_m := \inf\left\{T \geq s \geq t_m : \mathcal{X}_s \notin \mathrm{B}\left((\bar{t},\bar{x}),\delta\right)\right\} \wedge \left(t_m + h_m\right).$$

From (4.4), we know that there exists $\nu_m \in \mathcal{D}^M_{[t,T]}$ such that

$$z(t_m,x_m) \leq \mathbb{E}\left[U^{\nu_m}(t_m,\rho_m) z(\rho_m,\mathcal{X}_{\rho_m}) | \mathcal{X}_{t_m} = x_m\right] + \frac{\epsilon h_m}{4}.$$

Let

$$\rho_{m,\nu_m} := \inf\left\{T \geq s \geq t_m : U^{\nu_m}(t_m,s) \notin \left(\frac{1}{2},2\right)\right\} \wedge \rho_m.$$

It is easy to see that, then,

$$z(t_m,x_m) \leq \mathbb{E}\left[U^{\nu_m}(t_m,\rho_{m,\nu_m}) z(\rho_{m,\nu_m},\mathcal{X}_{\rho_{m,\nu_m}}) | \mathcal{X}_{t_m} = x_m\right] + \frac{\epsilon h_m}{4}.$$

Hence,

$$\phi(t_m,x_m) + \gamma_m - \frac{\epsilon h_m}{4} \leq \mathbb{E}\left[U^{\nu_m}(t_m,\rho_{m,\nu_m}) \phi(\rho_{m,\nu_m},\mathcal{X}_{\rho_{m,\nu_m}}) | \mathcal{X}_{t_m} = x_m\right].$$

An application of Ito's lemma gives us,

$$\begin{aligned}\gamma_m - \frac{\epsilon h_m}{4} &\leq \mathbb{E}\left[\int_{t_m}^{\rho_{m,\nu_m}} U^{\nu_m}(t_m,s)\left(\mathcal{L}\phi - \mathcal{M}^{\nu_m}\phi\right)(s,\mathcal{X}_s)\mathrm{d}s \,\middle|\, \mathcal{X}_{t_m} = x_m\right] \\ &\leq -\epsilon\, \mathbb{E}\left[\int_{t_m}^{\rho_{m,\nu_m}} U^{\nu_m}(t_m,s)\mathrm{d}s \,\middle|\, \mathcal{X}_{t_m} = x_m\right] \\ &\leq -\frac{\epsilon}{2}\, \mathbb{E}\left[\rho_{m,\nu_m} - t_m \,\middle|\, \mathcal{X}_{t_m} = x_m\right]. \end{aligned} \tag{4.39}$$

Now,

$$\begin{aligned}\mathbb{P}\left[\rho_{m,\nu_m} - t_m \leq h_m\right] &\leq \mathbb{P}\left[\sup_{s\in[t_m,\rho_{m,\nu_m}]} |\mathcal{X}_s - x_m| \vee |U^{\nu_m}(t_m,s) - 1| \geq \delta \wedge \frac{1}{2}\right] \\ &\leq \left(\delta \wedge \frac{1}{2}\right)^{-2} \mathbb{E}\left[\sup_{s\in[t_m,\rho_{m,\nu_m}]} |\mathcal{X}_s - x_m|^2 + |U^{\nu_m}(t_m,s) - 1|^2\right] \end{aligned} \tag{4.40}$$

$$\longrightarrow 0, \text{ as } m \to \infty. \tag{4.41}$$

The second inequality follows from Tchebyshev's inequality. The convergence to zero of the expectation in (4.40) follows from Lemma 4.6.1.

Moreover, since

$$\mathbb{P}\left[\rho_{m,\nu_m} - t_m > h_m\right] \leq \frac{1}{h_m}\mathbb{E}\left[\rho_{m,\nu_m} - t_m\right] \leq 1,$$

(4.41) implies that

$$\frac{1}{h_m}\mathbb{E}\left[\rho_{m,\nu_m} - t_m \,\middle|\, \mathcal{X}_{t_m} = x_m\right] \to 1, \text{ as } h_m \to 0.$$

Dividing both sides of (4.39) by h_m, we get

$$\frac{\gamma_m}{h_m} - \frac{\epsilon}{4} \leq -\frac{\epsilon}{2h_m}\mathbb{E}\left[\rho_{m,\nu_m} - t_m \,\middle|\, \mathcal{X}_{t_m} = x_m\right].$$

By letting $m \uparrow \infty$, we get,

$$-\frac{\epsilon}{4} \leq -\frac{\epsilon}{2} \qquad \rightarrow\leftarrow$$

This proves our first claim.

In the special case when $\pi = \text{diag}(x)\nabla\Pi$, instead of (4.37), we suppose that

$$\min\left\{-\mathcal{L}\phi(\bar{t},\bar{x}), \mathcal{M}_1\phi(\bar{t},\bar{x})\right\} > 0. \tag{4.37'}$$

and will show that such a supposition leads to contradiction. Corollary 5.2.4 tells us that for any $\nu \in (\text{vect}(K))^{\perp}$,

$$\Pi\left(\bar{t}, \bar{x}e^{r\nu}\right) + \ln z\left(\bar{t}, \bar{x}e^{r\nu}\right) = \Pi\left(\bar{t}, \bar{x}\right) + \ln z\left(\bar{t}, \bar{x}\right), \ \forall r \in \mathbb{R}.$$

Therefore, without loss of generality, we can take ϕ defined in (4.35) to satisfy the condition

$$\Pi\left(\bar{t}, \bar{x}e^{r\nu}\right) + \ln\phi\left(\bar{t}, \bar{x}e^{r\nu}\right) = \Pi\left(\bar{t}, \bar{x}\right) + \ln\phi\left(\bar{t}, \bar{x}\right), \ \forall r \in \mathbb{R}.$$

This implies that

$$\Big(\pi + \text{diag}(x)\nabla\log\phi\Big)(s,x) \in \text{vect}(K), \ \ \forall (s,x) \in (0,T) \times \mathbb{R}^n_+,$$

and hence there exists $\delta > 0$ such that

$$\Big(\pi + \text{diag}(x)\nabla\log\phi\Big)(s,x) \in K, \ \ \forall (s,x) \in \text{B}\left((\bar{t},\bar{x}),\delta\right) \cap [t,T] \times \mathbb{R}^n_+$$

and hence that there exist $\delta > 0, \epsilon > 0$, such that (4.38) holds. The same arguments that we made after (4.38), proves our second claim. □

4.5 Terminal condition

Proof of Theorem 4.2.3. Suppose $(T,\bar{x}) \in \overline{\mathcal{O}}_T$. Then by definition of z_*, there exists a sequence $\{(t_m,x_m)\}_{m\in\mathbb{N}}$ in $\mathcal{O}$ such that

$$(t_m,x_m) \to (T,\bar{x}), \ t_m < T, \ z(t_m,x_m) < \infty, \ z(t_m,x_m) \to z_*(T,\bar{x}).$$

For any $\nu \in \mathcal{D}^{(b)}_{(t,T]}(K)$, by Lemma 4.7.1,

$$U^{\nu}(t_m, x_m) \to 1, \text{ as } m \to \infty.$$

Hence, by Fatou's lemma,

$$\begin{aligned} z_*(T, \bar{x}) = \liminf_{m\to\infty} z(t_m, x_m) &\geq \liminf_{m\to\infty} \sup_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \mathbb{E}^{t_m, x_m}\left[U^{\nu}(t_m, T)\right] \\ &\geq \sup_{\nu \in \mathcal{D}^M_{(t,T]}(K)} \liminf_{m\to\infty} \mathbb{E}^{t_m, x_m}\left[U^{\nu}(t_m, T)\right] \geq 1. \end{aligned}$$

It follows exactly as in the proof of Proposition 4.3 in Soner and Touzi (2003), that $z_*(T, \cdot)$ is a viscosity supersolution of $\mathcal{M}z_*(T, \cdot) = 0$. □

Proof of Theorem 4.2.4. Let $\bar{x} \in \widetilde{O}$. By definition of $\widetilde{O}$, there exists $\delta_1 > 0$, such that $z(\cdot, \cdot)$ is bounded on $(T - \delta_1, T] \times B(\bar{x}, \delta_1)$.
Let ψ be a smooth function on $\mathbb{R}^n$ such that

$$0 = \left(z^*(T, \cdot) - \psi\right)(\bar{x}) = \max_{\mathbb{R}^n_+} \left(z^*(T, \cdot) - \psi\right)(x). \tag{4.42}$$

As we have seen in (4.13), $\mathcal{M}\psi(\bar{x}) \in \{0, -\infty\}$.
If $\mathcal{M}\psi(\bar{x}) = -\infty$, or if $\mathcal{P}\pi(T, \bar{x}) \neq 0$, it follows trivially that

$$\min\left\{\psi(\bar{x}) - 1, \mathcal{M}_1\psi(\bar{x})\right\} \mathbf{1}_{\mathcal{P}\pi(T,\bar{x})=0} + \mathcal{M}\psi(\bar{x}) \leq 0.$$

So let us suppose that $\mathcal{M}\psi(\bar{x}) = 0$ and $\mathcal{P}\pi(T, \bar{x}) = 0$. The latter implies that for some $\delta_1 > \delta > 0$,

$$\pi(s, y) \in \text{vect}(K), \text{ for all } (s, y) \in (T - \delta, T] \times B(\bar{x}, \delta). \tag{4.43}$$

Therefore, it follows from Theorem 5.2.1, that for any $\nu \in (\text{vect}(K))^{\perp}$,

$$z\left(s, xe^{r\nu}\right) = z(s, x), \text{ for } s > T - \delta, \{x, xe^{r\nu}\} \subset B(\bar{x}, \delta).$$

Without loss of generality, we can assume that for any $\nu \in (\text{vect}(K))^{\perp}$,

$$\psi\left(xe^{r\nu}\right) = \psi(x), \text{ for } \{x, xe^{r\nu}\} \subset B(\bar{x}, \delta). \tag{4.44}$$

(4.44) implies that

$$\text{diag}(x)\nabla \log \psi(x) \in \text{vect}(K), \; x \in B(\bar{x}, \delta). \tag{4.45}$$

To prove the theorem, we need to show that if $\psi(\bar{x}) > 1$ then $\mathcal{M}_1\psi(\bar{x}) \le 0$. We will prove this by contradiction.

We assume that

$$\psi(\bar{x}) > 1, \; \mathcal{M}_1\psi(\bar{x}) > 0. \tag{4.46}$$

$\mathcal{M}\psi(\bar{x}) = 0, \mathcal{M}_1\psi(\bar{x}) > 0$ together imply that

$$\left(\pi(T, \cdot) + \text{diag}(x)\nabla \log \psi\right)(\bar{x}) \in \text{ri}(K). \tag{4.47}$$

Let P_K denote the projection operator such that for any $x \in \mathbb{R}^n$, $P_K(x)$ is the orthogonal projection of x onto $\text{vect}(K)$.

For $m \in \mathbb{N}$, define

$$\phi_m(t, x) = \psi(x) + \|P_K(\log x - \log \bar{x})\|^4 + m(T - t).$$

It is easy to see that

$$\text{diag}(x)\nabla \log \phi_m(t, x) = \text{diag}(x)\nabla \log \psi(x) + 4\|P_K(\log x - \log \bar{x})\|^2 P_K(\log x - \log \bar{x}). \tag{4.48}$$

(4.43), (4.45), (4.47) and (4.48) imply that

$$\pi(t, x) + \text{diag}(x)\nabla \log \phi_m(t, x) \in \text{ri}(K), \; (t, x) \in (T - \delta, T] \times B(\bar{x}, \delta).$$

and hence

$$\mathcal{M}\phi_m(t, x) = 0, \mathcal{M}_1\phi_m(t, x) > 0, \quad (t, x) \in (T - \delta, T] \times B(\bar{x}, \delta) \tag{4.49}$$

Since by definition, $z^*(T,\cdot) \geq z_*(T,\cdot)$, we have

$$z^*(T,\cdot) \geq 1.$$

Hence, for all $x \in \mathbb{R}^n_+$,

$$\begin{aligned}(z-\phi_m)(T,x) &= 1 - \psi(x) - \|P_K(\log x - \log \bar{x})\|^4 \\ &\leq \left(z^*(T,\cdot) - \psi\right)(x) - \|P_K(\log x - \log \bar{x})\|^4 \\ &\leq -\|P_K(\log x - \log \bar{x})\|^4 \leq 0 \end{aligned} \tag{4.50}$$

Now, we take

$$0 < \delta_0 < \delta/2.$$

We define

$$C(\bar{x},\delta_0) := \left\{\bar{x}e^{\nu_K + \nu_{K^\perp}} : \|\nu_K\| \vee \|\nu_{K^\perp}\| \leq \delta_0, \nu_K \in \text{vect}(K), \nu_{K^\perp} \in (\text{vect}(K))^\perp\right\}$$
$$E(\bar{x},\delta_0) := \left\{\bar{x}e^{\nu_K + \nu_{K^\perp}} : \|\nu_K\| = \|\nu_{K^\perp}\| = \delta_0, \nu_K \in \text{vect}(K), \nu_{K^\perp} \in (\text{vect}(K))^\perp\right\}$$
$$D(\bar{x}) := \left\{\bar{x}e^{\nu_{K^\perp}} : \nu_{K^\perp} \in (\text{vect}(K))^\perp\right\}$$

(4.50) implies that

$$\sup_{C(\bar{x},\delta_0)} (z-\phi_m)(T,\cdot) \leq 0.$$

We claim that

$$\limsup_{m\to\infty} \sup_{C(\bar{x},\delta_0)} (z-\phi_m)(T,\cdot) < 0. \tag{4.51}$$

On the contrary, there exists a subsequence of (ϕ_m) still denoted (ϕ_m), such that

$$\lim_{m\to\infty} \sup_{C(\bar{x},\delta_0)} (z-\phi_m)(T,\cdot) = 0.$$

For each m, let $\left(x_m^k\right)_k$ be a maximizing sequence of $(z-\phi_m)(T,\cdot)$ on $C(\bar{x},\delta_0)$, i.e.,

$$\lim_{m\to\infty} \lim_{k\to\infty} (z-\phi_m)(T,x_m^k) = 0,$$

which combined with (4.50) implies

$$\lim_{m\to\infty}\lim_{k\to\infty} x_m^k \in D(\bar{x}).$$

Hence,

$$0 = \lim_{m\to\infty}\lim_{k\to\infty}(z-\phi_m)(T,x_m^k) = 1-\psi(\bar{x}) < (z^*(T,\cdot)-\psi)(\bar{x}). \tag{4.52}$$

(4.52) contradicts (4.42), and thus verifies (4.51). Suppose that

$$\limsup_{m\to\infty}\sup_{C(\bar{x},\delta_0)}(z-\phi_m)(T,\cdot) \leq -\kappa(\delta_0) < 0. \tag{4.53}$$

Obviously, $\kappa(\delta_0)$ is a decreasing function of δ_0.

Next, we take a sequence (s_m) converging to T with $s_0 \leq s_m < T$. Let us consider a maximizing sequence (t_m, x_m) of $z^* - \phi_m$ on $[s_m, T] \times E(\bar{x}, \delta_0)$. Then,

$$\limsup_{m\to\infty}\sup_{[s_m,T]\times E(\bar{x},\delta_0)}(z^*-\phi_m) \leq \limsup_{m\to\infty}(z^*(t_m,x_m)-\psi(x_m)) - \delta_0^4.$$

Since, t_m converges to T, and x_m upto a subsequence converges to some $x_0 \in E(\bar{x},\delta_0)$, we have by definition of z^*,

$$\limsup_{m\to\infty}\sup_{[s_m,T]\times E(\bar{x},\delta_0)}(z^*-\phi_m) \leq (z^*(T,\cdot)-\psi)(x_0) - \delta_0^4.$$

Also,

$$(z^*-\phi_m)(T,\bar{x}) = (z^*(T,\cdot)-\psi)(\bar{x}) = 0.$$

Therefore, for m large enough,

$$\sup_{[s_m,T]\times E(\bar{x},\delta_0)}(z-\phi_m) \leq -\delta_0^4 < 0 = (z^*-\phi_m)(T,\bar{x}) \leq \max_{[s_m,T]\times C(\bar{x},\delta_0)}(z^*-\phi_m). \tag{4.54}$$

(4.53) and (4.54) together imply that for m large enough,

$$\sup_{\partial_p([s_m,T]\times C(\bar{x},\delta_0))}(z-\phi_m) \leq -4\beta(\delta_0) < 0 \leq \max_{[s_m,T]\times C(\bar{x},\delta_0)}(z^*-\phi_m). \tag{4.55}$$

where

$$4\beta(\delta_0) = \kappa(\delta_0) \wedge \delta_0^4 > 0.$$

Since, $\kappa(\delta_0)$ is a decreasing function of δ_0, hence for small enough δ_0, we will have

$$4\beta(\delta_0) = \delta_0^4.$$

We can choose $(u_m, v_m) \in (s_m, T) \times C(\bar{x}, \delta_0)$, such that ,

$$(z - \phi_m)(u_m, v_m) \geq -2\beta(\delta_0) + \max_{[s_m,T]\times C(\bar{x},\delta_0)} (z^* - \phi_m). \tag{4.56}$$

Hence, by (4.55),

$$(z - \phi_m)(u_m, v_m) \geq 2\beta(\delta_0) + \sup_{\partial_p([s_m,T]\times C(\bar{x},\delta_0))} (z - \phi_m).$$

We define the stopping time

$$\theta_m := \inf \left\{ s \geq u_m : (s, \mathcal{X}(s)) \in \partial_p \left([u_m, T] \times C(\bar{x}, \delta_0) \right) \right\}.$$

We can find $\nu_m(\delta_0) \in \mathcal{D}^{\mathcal{M}}$, such that

$$z(u_m, v_m) - \beta(\delta_0) \leq \mathbb{E}^{u_m,v_m} \left[U^{\nu_m(\delta_0)} \left(u_m, \theta_m \right) z \left(\theta_m, \mathcal{X}_{\theta_m}^{(u_m,v_m)} \right) \right] \tag{4.57}$$

By continuity of $\mathcal{X}^{u_m,v_m}$, we have $\left(\theta_m, \mathcal{X}_{\theta_m}^{u_m,v_m}\right) \in \partial_p \left([u_m, T] \times C(\bar{x}, \delta_0) \right)$. Therefore, from (**??**), it follows that

$$z(\theta_m, \mathcal{X}_{\theta_m}^{u_m,v_m}) \leq \phi_m(\theta_m, \mathcal{X}_{\theta_m}^{u_m,v_m}) + (z - \phi_m) \left(u_m, v_m \right) - 2\beta(\delta_0).$$

$$\begin{aligned} \mathcal{L}\phi_m &= \frac{\partial \phi_m}{\partial t} + \frac{1}{2} \sum_i x_i \frac{\partial \phi_m}{\partial x_i} \left(a_{i*}\pi + r \right) + \frac{1}{2} \sum_{i,j} x_i x_j a_{ij} \frac{\partial^2 \phi_m}{\partial x_i \partial x_j} \\ &= -m + \ \left(\text{a locally bounded function of time and space, independent of } m \right) \end{aligned}$$

Thus, for any compact set $A \subset [0, T] \times \mathbb{R}_+^n$,

$$\mathcal{L}\phi_m(t, x) < 0, \ \forall \ (t, x) \in A, m \text{ large enough}. \tag{4.58}$$

From (4.57), we see that

$$-\beta(\delta_0) \leq \mathbb{E}^{u_m,v_m} \left[U^{\nu_m(\delta_0)} \left(u_m, \theta_m \right) z \left(\theta_m, \mathcal{X}_{\theta_m}^{u_m,v_m} \right) - z(u_m, v_m) \right]$$

$$
\begin{aligned}
&\le \mathbb{E}^{u_m,v_m}\left[U^{\nu_m(\delta_0)}(u_m,\theta_m)\left(\phi_m(\theta_m,\mathcal{X}^{u_m,v_m}_{\theta_m})+\right.\right.\\
&\qquad\left.\left.+\ (z-\phi_m)(u_m,v_m)-2\beta(\delta_0)\right)-z(u_m,v_m)\right]\\
&=\mathbb{E}^{u_m,v_m}\left[\left(U^{\nu_m(\delta_0)}(u_m,\theta_m)-1\right)\left(z(u_m,v_m)-\phi_m(u_m,v_m)-2\beta(\delta_0)\right)\right]\\
&\qquad+\mathbb{E}^{u_m,v_m}\left[U^{\nu_m(\delta_0)}(u_m,\theta_m)\phi_m(\theta_m,\mathcal{X}^{u_m,v_m}_{\theta_m})-\phi_m(u_m,v_m)\right]-2\beta(\delta_0)\\
&=\left|z(u_m,v_m)-\phi_m(u_m,v_m)-2\beta(\delta_0)\right|\mathbb{E}^{u_m,v_m}\left[\left|U^{\nu_m(\delta_0)}(u_m,\theta_m)-1\right|\right]\\
&\qquad+\mathbb{E}^{u_m,v_m}\left[\int_{u_m}^{\theta_m}\left(\mathcal{L}\phi_m-\mathcal{M}^{\nu_m(\delta_0)}\phi_m\right)\mathrm{d}t\right]-2\beta(\delta_0).
\end{aligned}\tag{4.59}
$$

By (4.49) and (4.58), it follows that

$$
\left(\mathcal{L}\phi_m-\mathcal{M}^{\nu_m(\delta_0)}\phi_m\right)(t,x)<0,\quad (t,x)\in[u_m,T]\times C(\bar{x},\delta_0).
$$

Hence, from (4.59), it follows that,

$$
-\beta(\delta_0)\le\left|z(u_m,v_m)-\phi_m(u_m,v_m)-2\beta(\delta_0)\right|\mathbb{E}^{u_m,v_m}\left[\left|U^{\nu_m(\delta_0)}(u_m,\theta_m)-1\right|\right]-2\beta(\delta_0).
$$

By Lemma 4.6.1, for some function $N:\mathbb{R}_+\to\mathbb{R}_+$, we have

$$
\begin{aligned}
&\mathbb{E}^{u_m,v_m}\left[\left|U^{\nu_m(\delta_0)}(u_m,\theta_m)-1\right|\right]\\
&\le N\left(\|\nu_m(\delta_0)\|_\infty\right)(T-u_m)^{1/2}e^{(T-u_m)N(\|\nu_m(\delta_0)\|_\infty)}\left(2+\|\bar{x}\|+\delta\right)\\
&=:\epsilon\left(\|\nu_m(\delta_0)\|_\infty,u_m\right)
\end{aligned}
$$

where p_K is as defined in (4.64). From (4.56) we have that

$$
\begin{aligned}
\max_{[s_m,T]\times C(\bar{x},\delta_0)}(z^*-\phi_m)&\ge z(u_m,v_m)-\phi_m(u_m,v_m)-2\beta(\delta_0)\\
&\ge -4\beta(\delta_0)+\max_{[s_m,T]\times C(\bar{x},\delta_0)}(z^*-\phi_m)
\end{aligned}
$$

Since $\max_{[s_m,T]\times C(\bar{x},\delta_0)}(z^*-\phi_m)\ge 0$ by (4.55), hence,

$$
\left|z(u_m,v_m)-\phi_m(u_m,v_m)-2\beta(\delta_0)\right|\le\max\left\{4\beta(\delta_0),\max_{[s_m,T]\times C(\bar{x},\delta_0)}(z^*-\phi_m)\right\}\tag{4.60}
$$

(4.59) and (4.60) imply that there will be contradiction if

$$
\epsilon\left(\|\nu_m(\delta_0)\|_\infty,u_m\right)\times\max\left\{4\beta(\delta_0),\max_{[s_m,T]\times C(\bar{x},\delta_0)}(z^*-\phi_m)\right\}<\frac{1}{2}\beta(\delta_0)\tag{4.61}
$$

Let us now assume that $\bar{x} \in \widetilde{Q}$. Then, there exists a finite $M > 0$, such that for any m, we can find $\nu_m(\delta_0)$ satisfying (4.57) and $\|\nu_m(\delta_0)\| < M$. Thus, from (4.59), we see that $\epsilon\left(\|\nu_m(\delta_0)\|_\infty, u_m\right) \to 0$, as $m \to \infty$, and hence for large enough m, (4.61) holds, giving us,

$$-\beta(\delta_0) < -\frac{3}{2}\beta(\delta_0). \quad \to\leftarrow$$

This proves that the supposition made in (4.46) is wrong, and this proves the theorem. $\square$

4.6 Estimates of moments

We will derive the moment estimates of $\mathcal{X}(\cdot)$ and $U^{\nu,\pi}(0,\cdot)$ at one go, viz. we will denote $Y(\cdot) = (\mathcal{X}(\cdot), U^{\nu,\pi}(0,\cdot))$, and derive estimates for the moments of Y. To that end, we will resort to Section 2.5 in Krylov (2009) and in particular its Corollary 2.5.12. The following lemma follows from there. Its proof is simple and only involves a straightforward casting of our problem in the proper framework and checking the conditions of the cited corollary.

Lemma 4.6.1. *Suppose Assumptions 3.1.1 and 3.3.1 hold. Suppose also that $\pi(\cdot,\cdot)$ is a Markovian strategy which is locally bounded in the space variable as in Assumption 3.3.2. Then for any compact set $A \subset \mathbb{R}^n_+$, any $q \geq 0$ and $\nu \in \mathcal{D}^{\mathcal{M}}$, there exists a constant $N(q, A, \|\nu\|_\infty)$, such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq\theta} \|(\mathcal{X}(s), U^{\nu,\pi}(0,s)) - (\mathcal{X}(0), 1)\|^q\right] \leq N t^{q/2} e^{Nt} \left(2 + \|\mathcal{X}(0)\|\right)^q, \quad \mathcal{X}(0) \in A$$

where

$$\theta := \inf\{0 \leq s \leq t : \mathcal{X}(\theta) \notin A\} \wedge t.$$

Proof. Y satisfies the system of SDEs

$$\mathrm{d}Y_i(t) = B_i(Y(t))\,\mathrm{d}t + S_{i*}(Y(t))\,\mathrm{d}W(t), \ i = 1, \ldots, n+1$$

where, B is a $n+1$ dimensional vector and S is an $(n+1)\times n$ dimensional matrix, with

$$B_i\left(Y\left(t\right)\right)=X_i\left(t\right)b_i\left(t,\mathcal{X}\left(t\right)\right),\ i=1,2,\ldots,n,$$
$$B_{n+1}\left(Y\left(t\right)\right)=-U^{\nu,\pi}\left(0,t\right)\left[\pi'\nu+\zeta\left(\nu\right)\right]\left(t,\mathcal{X}\left(t\right)\right),$$
$$S_{i*}\left(Y\left(t\right)\right)=X_i(t)\sigma_{i*}\left(t,\mathcal{X}\left(t\right)\right),\ i=1,\ldots,n,$$
$$S_{n+1,*}\left(Y\left(t\right)\right)=U^{\nu,\pi}\left(0,t\right)\left(\sigma'\pi-\vartheta-\sigma^{-1}\nu\right)'\left(t,\mathcal{X}\left(t\right)\right),$$

It follows that

$$\|B(y)\|\leq\|y\|\left(\|b(t,x)\|+\|\nu\|\left(\|\pi\|+\left|\zeta\left(\frac{\nu}{\|\nu\|}\right)\right|\right)\right),\tag{4.62}$$
$$\|S(y)\|\leq\|y\|\left(\|\sigma(t,x)\|\left(1+\|\pi\|\right)+\|\vartheta\|+\lambda_{(1)}^{-1}\|\nu\|\right),\tag{4.63}$$

where $\lambda_{(1)}$ denotes the smallest eigenvalue of σ.

We denote

$$p_K:=\sup_{\nu\in\widetilde{K}}\left|\zeta\left(\frac{\nu}{\|\nu\|}\right)\right|=\sup_{\nu\in\widetilde{K},\|\nu\|=1}\left|\zeta\left(\nu\right)\right|.\tag{4.64}$$

The equality in (4.64) holds because ζ is linearly homogeneous.

Then (4.62) and (4.63) imply

$$\|B(y)\|+\|S(y)\|\leq f(t,x)\|y\|,$$

where

$$f(t,x):=\left[\|b\|+\|\sigma\|\left(1+\|\pi\|\right)+\|\vartheta\|+\|\nu\|\left(\|\pi\|+\lambda_{(1)}^{-1}+p_K\right)\right](t,x).$$

Now, for some set $D\subset\mathbb{R}^n_+$, if $\mathcal{X}(s)\in D$ for all $0\leq s\leq t$ and if

$$f(s,x)\leq K_1,\ x\in D,0\leq s\leq t,\tag{4.65}$$

then by Corollary 2.5.12 of Krylov (2009) it follows that for each $q\geq 0$, there exists a constant $N(q,K_1)$, such that

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\|\mathcal{Y}(s)-\mathcal{Y}(0)\|^q\right]\leq Nt^{q/2}e^{Nt}\left(2+\|\mathcal{X}(0)\|\right)^q,$$

where $\mathcal{Y}(0) = (\mathcal{X}(0), 1)$. Since $\widetilde{K}$ is closed, $\left\{\nu \in \widetilde{K}, \|\nu\| = 1\right\}$ is compact, and hence by Assumption 3.1.1,

$$p_K < \infty.$$

Further, if Assumption 3.3.1 holds, and if $\pi(\cdot,\cdot)$ is locally bounded, then (4.65) is satisfied. □

4.7 Sufficient condition for lower-semicontinuity of $z(\cdot,\cdot)$

Assumption 4.7.1. *The strategy $\pi(\cdot,\cdot)$, volatility $\sigma(\cdot,\cdot)$, and the market price of risk $\vartheta(\cdot,\cdot)$ are locally Lipschitz continuous in the space variable.*

The theory of stochastic flows now leads us to Lemma 4.7.1 which essentially concludes that if two stock price processes start from close enough values at close enough times, then they will stay close to each other at all time points upto the finite terminal time T. Similar observation holds also for the discounted wealth process $U^\nu(\cdot)$, for any $\nu \in \mathcal{D}^{(b)}_{[0,T]}(K)$. We refer the reader to Kunita (1984) and Chapter V of Protter (2004) for an introduction to and further references on the theory of stochastic flows.

Lemma 4.7.1. *Fix a point $(t,x) \in [0,T] \times \mathbb{R}^n_+$. Then under Assumption 4.7.1, and for $\nu \in \mathcal{D}^{(b)}_{[0,T]}(K)$, we have for all sequences $(t_m, x_m)_{m\in\mathbb{N}} \subset [0,T] \times \mathbb{R}^n_+$ with $\lim_{m\to\infty}(t_m, x_m) = (t,x)$ that*

$$\lim_{m\to\infty} \sup_{u\in[t,T]} \left(\left\|\mathcal{X}^{t_m,x_m}(u) - \mathcal{X}^{t,x}(u)\right\| + \left|U^{\nu,t_m,x_m}(t_m,u) - U^{\nu,t,x}(t,u)\right| \right) = 0,$$

almost surely, where we set

$$\mathcal{X}^{t_m,x_m}(u) := x_m, \quad U^{\nu,t_m,x_m}(t_m,u) := 1, \quad \text{for } u \le t_m.$$

Lemma 4.7.1 follows immediately from Theorem V.37 of Protter (2004) and Lemma 1 and Lemma 2 of Ruf (2011), once the dynamics of the stock price process $\mathcal{X}(\cdot)$ and that of the discounted wealth process U^ν have been recast slightly.

The lower-semicontinuity of $z(\cdot,\cdot)$, defined in (4.3), now follows from Lemma 4.7.1.

Corollary 4.7.1. *Under Assumption 4.7.1, $z(\cdot,\cdot)$ is lower-semicontinuous on $[0,T]\times\mathbb{R}^n_+$.*

Proof. For any $\nu\in\mathcal{D}^{(b)}_{[0,T]}(K)$, Lemma 4.7.1 and Fatou's lemma implies

$$\mathbb{E}^{t,x}\left[U^\nu(t,T)\right]\le\liminf_{m\to\infty}\mathbb{E}^{t_m,x_m}\left[U^\nu(t_m,T)\right].$$

Hence,

$$\begin{aligned}z(t,x)&\le\sup_{\nu\in\mathcal{D}^M_{(t,T]}(K)}\liminf_{m\to\infty}\mathbb{E}^{t_m,x_m}\left[U^\nu(t_m,T)\right]\\&\le\liminf_{m\to\infty}\sup_{\nu\in\mathcal{D}^M_{(t,T]}(K)}\mathbb{E}^{t_m,x_m}\left[U^\nu(t_m,T)\right]=\liminf_{m\to\infty}z(t_m,x_m).\end{aligned}$$

□

Note that for any $s\in[t,T]$ and $x,y\in\mathbb{R}^n_+$,

$$\begin{aligned}\|\sigma^{-1}(s,x)-\sigma^{-1}(s,y)\|&\le\|\sigma^{-1}(s,x)\|\|I-\sigma(s,x)\sigma^{-1}(s,y)\|\\&\le\|\sigma^{-1}(s,x)\|\|\sigma^{-1}(s,y)\|\|\sigma(s,y)-\sigma(s,x)\|.\end{aligned}\tag{4.66}$$

On the other hand,

$$\|\sigma^{-1}(s,x)\|=\operatorname{tr}\left(\alpha^{-1}(s,x)\right)=\sum_{i=1}^n\frac{1}{\lambda^2_{(i)}}\le\frac{n}{\lambda^2_{(1)}},\tag{4.67}$$

where $\lambda_{(1)}\le\lambda_{(2)}\le\ldots\le\lambda_{(n)}$ are the eigenvalues of $\sigma(s,x)$. Under Assumptions 4.7.1 and 3.3.1, it follows from (4.66) and (4.67) that σ^{-1} is locally Lipschitz continuous in the space variable.

Now, for any $\nu \in \mathcal{D}^{(b)}_{(t,T]}(K)$, if we define the semimartingales

$$C_i(\cdot) = \int_t^{\cdot} \nu_i(s)\mathrm{d}s, \quad C_0(\cdot) = \int_t^{\cdot} \zeta\left(\nu(s)\right)\mathrm{d}s, \quad Z_{ij}(\cdot) = \int_t^{\cdot} \nu_j(s)\mathrm{d}W_i(s),$$

then we can write the dynamics of the stock price process $\mathcal{X}(\cdot)$ and that of the discounted wealth process U^{ν} as

$$\begin{aligned}
\mathrm{d}X_i(s) &= X_i(s)\left[b_i\left(s,\mathcal{X}(s)\right)\mathrm{d}s + \sigma_{i*}\left(s,\mathcal{X}(s)\right)\mathrm{d}W(s)\right], i = 1,2,\ldots,n, \\
\mathrm{d}U^{\nu}(s) &= U^{\nu}(s)\left[-\sum_{i=1}^{n}\pi_i\left(s,\mathcal{X}(s)\right)\mathrm{d}C_i(s) - \mathrm{d}C_0(s)\right] \\
&\quad + U^{\nu}(s)\left[\left(\pi'\sigma - \vartheta'\right)\left(s,\mathcal{X}(s)\right)\mathrm{d}W(s) - \sum_{i,j}\sigma^{ji}\left(s,\mathcal{X}(s)\right)\mathrm{d}Z_{ij}(s)\right],
\end{aligned}$$

where $\sigma^{ij} := \left(\sigma^{-1}\right)_{ij}$. Having cast the dynamics of $\mathcal{X}(\cdot)$ and U^{ν} in this form, the following lemma follows immediately from Theorem V.37 of Protter (2004) and Lemma 1 and Lemma 2 of Ruf (2011).

Chapter 5

Relative Arbitrage under Constraints

In Section 5.1 we will present some comparison results for these viscosity solutions and discuss their relevance in the discussion of relative arbitrage. Section 5.2 contains a very important part of the thesis, where we use the viscosity solution characterization to study how the constraints affect the value of $z(\cdot,\cdot)$ and thus govern the existence or absence of relative arbitrage. Convex polyhedral constraint sets are particularly amenable to our analysis and most of our examples will involve such constraint sets. In Section 5.3, we will discuss polyhedral convex sets. Section 5.4 contains some examples.

5.1 Comparison principles and relative arbitrage

In this section we will discuss comparison principles for viscosity super(sub)solutions to (4.15) and their implications about the existence of relative arbitrage opportunities. Note, that since the PDE (4.15) does not depend on the constraint set, hence, in this section, we will be looking only at how the market model dictates existence or absence of arbitrage relative to π, the strategy under consideration. Throughout the section, we will assume that the conditions of Theorem 4.2.1 hold.

In order to facilitate the subsequent discussion, we will first introduce some notations. Recall the operator $\mathcal{L}$ defined in (4.9). We rewrite it here as

$$\mathcal{L}w(t,x) = \frac{\partial w}{\partial t}(t,x) + \frac{1}{2}\left(a(t,x)\pi(t,x) + r(t,x)\mathbf{1}\right)' \operatorname{diag}(x)\nabla w(t,x) + \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(x)a(t,x)\operatorname{diag}(x)D^2 w(t,x)\right), \quad w \in C^{1,2}\left([0,T]\times\mathbb{R}^n_+\right).$$

For a fixed $\gamma \in \mathbb{R}$, we define the function $F_\gamma : [0,T]\times\mathbb{R}^n_+\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}(n)\to\mathbb{R}$ as

$$F_\gamma(t,x,z,p,X) = \gamma z - \frac{1}{2}\left(a(t,x)\pi(t,x) + r(t,x)\mathbf{1}\right)' \operatorname{diag}(x)p - \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(x)a(t,x)\operatorname{diag}(x)X\right). \tag{5.1}$$

We also define the function $F^{\mathcal{M}} : [0,T]\times\mathbb{R}^n_+\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ as

$$F^{\mathcal{M}}(t,x,z,p) = \inf_{\nu\in\widetilde{K}}\left[z\left(\pi'(t,x)\nu + \zeta(\nu)\right) + \nu'\operatorname{diag}(x)p\right] \tag{5.2}$$

In terms of the newly defined functions F_γ and $F^{\mathcal{M}}$, Theorem 4.2.1 tells us that z is a viscosity supersolution of

$$-\frac{\partial z}{\partial t}(t,x) + F_0(t,x,z,\nabla_x z, D^2 z) + F^{\mathcal{M}}(t,x,z,\nabla_x z) = 0, \quad (t,x)\in\mathcal{O}. \tag{5.3}$$

With F_γ as defined in (5.1), it is easy to see that for any $\gamma\in\mathbb{R}$ and $(t,x,z,p,X)\in[0,T]\times\mathbb{R}^n_+\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}(n)$,

$$F_\gamma \in C\left([0,T]\times\mathbb{R}^n_+\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}(n)\right),$$
$$\gamma(z_1 - z_2) \le F_\gamma(t,x,z_1,p,X) - F_\gamma(t,x,z_2,p,X), \quad z_1, z_2\in\mathbb{R},$$
$$F_\gamma(t,x,z,p,X) \le F_\gamma(t,x,z,p,Y), \text{ whenever } X \ge Y,\ X,Y\in\mathcal{S}(n).$$

Thus, for any $\gamma \ge 0$, F_γ is *proper* in the sense of Crandall et al. (1992), that is

$$F_\gamma(t,x,z_2,p,X) \le F_\gamma(t,x,z_1,p,Y), \text{ if } z_1 \ge z_2,\ X \ge Y. \tag{5.4}$$

In addition, we will assume one or more of the following conditions.

Assumption 5.1.1. *$\sigma(\cdot,\cdot)$, $\pi(\cdot,\cdot)$ and $r : [0,T] \times \mathbb{R}^n_+ \to \mathbb{R}$ are locally Lipschitz continuous in the space variable uniformly in time, i.e. given any compact set $K \subset \mathbb{R}^n_+$, there exists a constant C_K such that*

$$|r(t,x)-r(t,y)|+\|\sigma(t,x)-\sigma(t,y)\|+\|\pi(t,x)-\pi(t,y)\| \leq C_K\|x-y\|, \ \ t \in [0,T], \ x,y \in K.$$

Assumption 5.1.2. *For any fixed $x \in \mathbb{R}^n_+$, $\sigma(\cdot,x)$, $\pi(\cdot,x)$ and $r(\cdot,x)$ are bounded functions on $[0,T]$, i.e. there exists $C(x)$ such that*

$$\|\sigma(t,x)\| + \|\pi(t,x)\| + |r(t,x)| \leq C(x), \ \ \forall\, t \in [0,T].$$

Assumption 5.1.3. *$\sigma(\cdot,\cdot)$, $\pi(\cdot,\cdot)$ and $r(\cdot,\cdot)$ are locally Lipschitz continuous in both the time and space variable.*

Assumption 5.1.4. *$diag(x)\,(a(t,x)\pi(t,x) + r(t,x)\mathbf{1})$ and $diag(x)\sigma(t,x)$ are bounded functions on $[0,T] \times \mathbb{R}^n_+$.*

Suppose Γ is an open subset of $\mathbb{R}^n_+$, not necessarily bounded. For any $\gamma \in \mathbb{R}$, and $0 \leq s_1 < s_2 \leq T$, consider the partial differential equation,

$$-\frac{\partial w}{\partial t}(t,x) + F_\gamma(t,x,w(t,x),\nabla_x w(t,x),D^2_x w(t,x)) = 0, \ (t,x) \in (s_1,s_2)\times\Gamma, \quad (5.5)$$

and in particular,

$$-\frac{\partial w}{\partial t}(t,x) + F_0(t,x,w(t,x),\nabla_x w(t,x),D^2_x w(t,x)) = 0, \ (t,x) \in (s_1,s_2)\times\Gamma. \quad (5.6)$$

We then have the following comparison results.

Proposition 5.1.1. *Suppose that the conditions of Theorem 4.2.1 hold. Suppose u and v are viscosity subsolution and supersolution respectively, of* (5.5)*.*

i. If $\gamma > 0$, Γ is bounded and Assumption 5.1.1 and Assumption 5.1.2 hold, then

$$u(t,x) - v(t,x) \leq \sup_{\partial_p((s_1,s_2)\times\Gamma)} (u-v)^+, \ \ \forall\,(t,x) \in (s_1,s_2)\times\Gamma. \quad (5.7)$$

ii. *Suppose $\gamma \geq 0$ and Assumption 5.1.3 holds. Suppose u is of at most polynomial growth, and v is bounded from below. If $u(s_2, x) \leq v(s_2, x),\ x \in \Gamma$, then $u \leq v$ on $(s_1, s_2] \times \Gamma$.*

iii. *Suppose $\gamma > 0$ and Assumption 5.1.4 holds. Suppose also that $\lim_{|x|\to\infty} u < \infty$, $\lim_{|x|\to\infty} v > -\infty$ and $u(s_2, \cdot)$ or $v(s_2, \cdot)$ is uniformly continuous with modulus of continuity $m(\cdot)$. Then*

$$\sup_{(s_1,s_2)\times\Gamma} (u - v)(t, x) \leq e^{(\gamma+1)} \sup_{x\in\Gamma} (u(s_2, x) - v(s_2, x))^+ .$$

iv. *Suppose $\gamma \geq 0$ and Assumption 5.1.4 holds. Suppose also that u and v satisfy the following conditions:*

a. *$u(t, x) \leq K(|x| + 1),\ v(t, x) \geq -K(|x| + 1)$ for some $K > 0$ independent of $(t, x) \in (s_1, s_2) \times \Gamma$;*

b. *there is a modulus m_T such that*

$$u(t, x) - v(t, y) \leq m_T (|x - y|) \quad \text{for all } (t, x, y) \in \partial_p U,$$

where $U = (s_1, s_2) \times \Gamma \times \Gamma$;

c. *$u(t, x) - v(t, y) \leq K(|x - y| + 1)$ on $\partial_p U$ for some $K > 0$ independent of $(t, x, y) \in \partial_p U$.*

Then there is a modulus m such that

$$u(t, x) - v(t, y) \leq m (|x - y|) \ \text{ on } U.$$

Proof. (i) Since the class of locally Lipschitz continuous functions is preserved under addition and multiplication, it follows from Assumption 5.1.1, Assumption 5.1.2 and Example 3.6 in Crandall et al. (1992) that F_γ also satisfies condition (3.14) in

Crandall et al. (1992), i.e. there exists a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0+) = 0$ such that for any fixed $t \in [0,T]$,

$$\begin{aligned}
&F_\gamma(t,y,z,\alpha(x-y),Y) - F_\gamma(t,x,z,\alpha(x-y),X) \le \omega\left(\alpha|x-y|^2 + |x-y|\right) \\
&\qquad\qquad\text{whenever } x,y \in \mathbb{R}^n_+,\ X,Y \in \mathcal{S}(n),\ z \in \mathbb{R},\ \text{and} \\
&\qquad -3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\end{aligned} \tag{5.8}$$

We have already seen that F_γ satisfies (5.4) for any $\gamma > 0$. 5.7 now follows from Theorem 8.2 in Crandall et al. (1992).

(ii) This follows from Theorem 4.4.5 in Pham (2009).

(iii) This follows from Theorem 1.2.1 in Zhan (1999).

(iv) This follows from Theorem 4.2 and the discussion following it in Giga et al. (1991). □

Corollary 5.1.1. *Suppose the conditions of Theorem 4.2.1 and Assumption 5.1.3 hold. Suppose also that the function $z(\cdot,\cdot)$ is locally bounded on $(0,T] \times \mathbb{R}^n$. Then $z \ge 1$ on $(0,T) \times \mathbb{R}^n$ i.e. arbitrage relative to π is not possible on the time horizon $[t,T]$, for any value of the stock price $\mathcal{X}(t)$ at time $t > 0$.*

Proof. Under the conditions of Theorem 4.2.1, $z(\cdot,\cdot)$ is a supersolution and $u \equiv 1$ is a subsolution of (5.6) for $s_1 = 0, s_2 = T, \Gamma = \mathbb{R}^n_+$. The corollary now follows from Proposition 5.1.1 since $z(T,\cdot) = u(T,\cdot) \equiv 1$ on $\mathbb{R}^n_+$. □

Consider any $Y \in \mathcal{S}(n)$, $Y \ge 0$. Suppose $Y = Q\Lambda Q'$, is an eigendecomposition of Y, with $\Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ being a diagonal matrix with the eigenvalues of Y as its diagonal elements and Q is an orthonormal matrix. Then, for any $(t,x,z,p,X) \in [0,T] \times \mathbb{R}^n_+ \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$ and $\gamma \in \mathbb{R}$,

$$\begin{aligned}
&2\left(F_\gamma(t,x,z,p,X) - F_\gamma(t,x,z,p,X+Y)\right) \\
&= \operatorname{tr}\Big(\operatorname{diag}(x)a(t,x)\operatorname{diag}(x)Y\Big) = \operatorname{tr}\Big(\sigma'(t,x)\operatorname{diag}(x)Q\Lambda Q'\operatorname{diag}(x)\sigma\Big)
\end{aligned}$$

$$= \sum_i \lambda_i \sigma'(t,x)\text{diag}(x)\text{diag}(x)\sigma(t,x) = \sum_i \lambda_i x_i^2 a_{ii}(t,x)$$
$$\geq \left(\min_i x_i^2 a_{ii}(t,x)\right) \text{tr}(Y).$$

From Assumption 3.3.1 it follows that for any compact set $K \subset \mathbb{R}^n_+$, there exists $\mu(K) > 0$, such that

$$\left(\min_i x_i^2 a_{ii}(t,x)\right) \;\geq\; \mu(K), \quad (t,x) \in [0,T] \times K,$$

thus showing that F_γ considered as a function from $[0,T] \times K \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$ is "uniformly parabolic" in the sense of Da Lio (2004), i.e. for any $(t,x,z,p,X) \in [0,T] \times K \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$,

$$F_\gamma(t,x,z,p,X) - F_\gamma(t,x,z,p,X+Y) \;\geq\; \mu(K)\,\text{tr}(Y), \text{ if } Y \geq 0.$$

We need some more notations. For any set $\mathcal{U} \subset [0,T] \times \mathbb{R}^n_+$ and any point $P_0 = (t_0, x_0) \in \mathcal{U}$, we denote by $S(P_0, \mathcal{U})$ the set of all points $P \in \mathcal{U}$ which can be joined to P_0 by a simple continuous curve lying in $\mathcal{U}$ along which the time coordinate is non-increasing from P to P_0.

We have the following theorem.

Theorem 5.1.1. *Suppose that the conditions of Theorem 4.2.1 hold. Let Γ be an open bounded set in $\mathbb{R}^n_+$.*

i. If $(s_1,s_2) \times \overline{\Gamma} \subset \mathcal{O}$ and if $z(\cdot,\cdot)$ attains a minimum at a point $P_0 = (t_0,x_0) \in [s_1,s_2) \times \Gamma$, then $z(\cdot,\cdot)$ is constant in $S(P_0,[s_1,s_2)\times\Gamma)$. If $(s_1,s_2) \times \overline{\Gamma} \subset \mathcal{O}_{sub}$ and if $z^*(\cdot,\cdot)$ attains a maximum at a point $P_0 = (t_0,x_0) \in [s_1,s_2) \times \Gamma$, then $z^*(\cdot,\cdot)$ is constant in $S(P_0,[s_1,s_2)\times\Gamma)$.

ii. Suppose that Assumption 5.1.1 and Assumption 5.1.2 hold. If $(s_1,s_2) \times \Gamma \subset \mathcal{O}$, then

$$\inf_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z(t,x) \leq \inf_{(t,x)\in(s_1,s_2)\times\Gamma} z(t,x). \tag{5.9}$$

If $(s_1, s_2) \times \Gamma \subset \mathcal{O}_{sub}$, *then*

$$\sup_{(t,x)\in(s_1,s_2)\times\Gamma} z^*(t,x) \leq \sup_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z^*(t,x). \tag{5.10}$$

Proof. (i) Suppose that $(s_1, s_2)\times\overline{\Gamma} \subset \mathcal{O}_{sub}$. We recall from Theorem 4.2.2 that $z^*(\cdot,\cdot)$ is a viscosity subsolution to (5.6). It then follows from Corollary 2.4 in Da Lio (2004) that if $z^*(\cdot,\cdot)$ attains a maximum at a point $P_0 = (t_0, x_0) \in [s_1, s_2) \times \Gamma$, then $z^*(\cdot,\cdot)$ is constant in $S(P_0, [s_1, s_2) \times \Gamma)$.
Suppose that $(s_1, s_2)\times\overline{\Gamma} \subset \mathcal{O}$.We recall from Theorem 4.2.1 that $z(\cdot,\cdot)$ is a viscosity supersolution to (5.6). For any constant K, $z(\cdot,\cdot) - K$ is also a supersolution of (5.5) when $\gamma = 0$. Hence, it follows from Corollary 2.4 in Da Lio (2004) that if $z(\cdot,\cdot)$ attains a minimum at a point $P_0 = (t_0, x_0) \in [s_1, s_2) \times \Gamma$, then $z(\cdot,\cdot)$ is constant in $S(P_0, [s_1, s_2) \times \Gamma)$. This proves (i).

(ii) Suppose that $\widetilde{u} \geq 0$ and $\widetilde{v} \geq 0$ are viscosity subsolution and supersolution respectively, of (5.6). It is easy to see that for any $\gamma \leq \lambda,\ \gamma, \lambda \in \mathbb{R}$, $e^{\lambda t}\widetilde{u}$ is a viscosity subsolution to (5.5). Similarly, for any $\gamma \geq \lambda,\ \gamma, \lambda \in \mathbb{R}$, $e^{\lambda t}\widetilde{v}$ is a viscosity supersolution to (5.5). Therefore, for any $\lambda_1 \geq \lambda_2 > 0$, $e^{\lambda_1 t}\widetilde{u}$ and $e^{\lambda_2 t}\widetilde{v}$ are viscosity subsolution and supersolution respectively of (5.5) for $\gamma = \lambda_2$, and it follows from (5.7) that

$$e^{\lambda_1 t}\widetilde{u}(t,x) - e^{\lambda_2 t}\widetilde{v}(t,x) \leq \sup_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} \left(e^{\lambda_1 t}\widetilde{u}(t,x) - e^{\lambda_2 t}\widetilde{v}(t,x)\right)^+, \quad (t,x) \in (s_1,s_2)\times\Gamma. \tag{5.11}$$

If $(s_1, s_2) \times \Gamma \subset \mathcal{O}$, then $z(\cdot,\cdot)$ is a supersolution to (5.6) and hence $-z(\cdot,\cdot)$ is a subsolution to (5.6). $-\inf_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z(t,x)$ is a supersolution to (5.6). Hence, for any $\lambda_1 = \lambda_2 > 0$, (5.11) gives us

$$\inf_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z(t,x) \leq \inf_{(t,x)\in(s_1,s_2)\times\Gamma} z(t,x).$$

If $(s_1, s_2) \times \Gamma \subset \mathcal{O}_{sub}$, then $z^*(\cdot,\cdot)$ is a subsolution to (5.6). $\widetilde{v} \equiv 0$ is a supersolution to (5.6). Hence, for any $\lambda_1 \geq \lambda_2 > 0$, and $(t,x) \in (s_1,s_2) \times \Gamma$, (5.11) gives us

$$e^{\lambda_1 t} z^*(t,x) \leq \sup_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} e^{\lambda_1 t} z^*(t,x) \leq e^{\lambda_1 s_2} \sup_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z^*(t,x),$$

which implies

$$z^*(t,x) \leq e^{\lambda_1(s_2-t)} \sup_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z^*(t,x) \leq e^{\lambda_1 s_2} \max_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z^*(t,x). \quad (5.12)$$

Since this holds for each $\lambda_1 > 0$ we have

$$\sup_{(t,x)\in(s_1,s_2)\times\Gamma} z^*(t,x) \leq \sup_{(t,x)\in\partial_p((s_1,s_2)\times\Gamma)} z^*(t,x).$$

□

Corollary 5.1.2. *Suppose the conditions of Theorem 4.2.1 hold and for some* $C > 0$, $z(\cdot,\cdot) \leq C$ *on* $[0,T] \times \mathbb{R}^n_+$. *If* $z(t_0,x_0) = C$ *for some* $(t_0,x_0) \in \mathcal{O}^o_{sub}$, *then* $z(t,x) = C$ *for all* $(t,x) \in \mathcal{S}((t_0,x_0),\mathcal{O}^o_{sub})$.

Proof. Since $z(\cdot,\cdot) \leq C$ on $[0,T] \times \mathbb{R}^n_+$, hence $z^*(t_0,x_0) = C$ and (t_0,x_0) is a point of maximum for z^*. The corollary is now immediate from Theorem 5.1.1 (i). □

Suppose now that the market is time homogeneous, i.e. the functions b, σ and r in (2.6) are independent of time. Then, the function $z(\cdot,\cdot)$ will depend on t only through the time to maturity $T-t$ of the claim, and for any $0 \leq s_1 \leq s_2 \leq T$,

$$\begin{aligned}\sup_{\nu\in\mathcal{D}^M(K)} \mathbb{E}\left[U^\nu(s_1,s_2) \;\middle|\; \mathcal{X}(s_1) = x\right] &= \sup_{\nu\in\mathcal{D}^M(K)} \mathbb{E}\left[U^\nu(T-s_2+s_1,T) \;\middle|\; \mathcal{X}(T-s_2+s_1) = x\right] \\ &= z\left(T-s_2+s_1,x\right).\end{aligned}$$

Theorem 5.1.1, Corollary 5.1.2 and the same argument as in Proposition 2 in Fernholz and Karatzas (2010) now give us the following theorem.

Theorem 5.1.2. *Suppose that the market is time homogeneous. Suppose the conditions of Theorem 4.2.1 hold and for some* $C > 0$, $z(\cdot,\cdot) \leq C$ *on* $[0,T] \times \mathbb{R}^n_+$. *Suppose also that* $\mathcal{O}_{sub} = (0,T) \times \mathbb{R}^n_+$. *If* $z(t_0,x_0) < C$ *for some* $(t_0,x_0) \in (0,T) \times \mathbb{R}^n_+$, *then* $z(t,x) < C$ *for all* $(t,x) \in (0,T) \times \mathbb{R}^n_+$.

5.2 Constraint set and relative arbitrage

With the viscosity solution characterization of the function $z(\cdot,\cdot)$ and the comparison principles from the previous sections, we are now ready to investigate the presence of arbitrage opportunities relative to a given strategy π among strategies constrained to take values in given closed convex sets. Unlike in the previous section, we will now turn our attention to the PDE (4.16) and see how the constraint set comes into play.

Proposition 5.2.1. *Under the conditions of Theorem 4.2.1, if* $\phi \in C^{1,2}([0,T]\times\mathbb{R}^n_+)$ *satisfies*

$$-\mathcal{L}\phi(t,x)+\mathcal{M}\phi(t,x)\geq 0,\ (t,x)\in(0,T)\times\mathbb{R}^n+, \tag{5.13}$$

$$\phi(T,x)=1,\ x\in\mathbb{R}^n_+. \tag{5.14}$$

Then $\phi \geq z$ *in* $[0,T]\times\mathbb{R}^n_+$.

Proof. Consider any $\nu \in \mathcal{D}^M$. By an application of Itô's Theorem, we can see that the dynamics of $U^\nu(0,t)\phi(t,\mathcal{X}_t)$ is given by (4.30). (5.13) now inmplies that the process $\{U^\nu(0,t)\phi(t,\mathcal{X}_t)\}_{0\leq t\leq T}$ is a supermartingale. Hence, for any $(t,x)\in[0,T]\times\mathbb{R}^n_+$,

$$\phi(t,x)\geq \mathbb{E}^{t,x}\left[U^\nu(t,T)\phi(T)\right]=\mathbb{E}^{t,x}\left[U^\nu(t,T)\right].$$

Taking supremum over all $\nu\in\mathcal{D}^M$, we get $\phi(t,x)\geq z(t,x)$. □

Corollary 5.2.1. *Under the conditions of Theorem 4.2.1, if* $\pi(t,x)\in K$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^n_+$, *then* $z(t,x)\leq 1$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^n_+$.

Proof. If $\pi(t,x)\in K$ for all $(t,x)\in[0,T]\times\mathbb{R}^n_+$, then $\phi\equiv 1$ satisfies (5.13) and (5.14). □

Corollary 5.2.2. *Under the conditions of Theorem 4.2.1, if z takes the constant value $0 < C < \infty$ in a neighborhood of $(t, x) \in (0, T) \times \mathbb{R}^n_+$, then $\pi(t, x) \in K$.*

Proof. Since z is constant in a neighborhood of (t, x), hence $(0, 0, 0) \in \mathcal{J}^{2,-}_{\mathcal{O}} z(t, x)$. Since z is a supersolution to (5.3), we get

$$\inf_{\nu \in \widetilde{K}} \left[C \left(\pi'(t, x)\nu + \zeta(\nu) \right) \right] \geq 0,$$

which implies $\pi(t, x) \in K$. □

We recall from (4.14) that for a fixed $(\bar{t}, \bar{x}) \in \mathcal{O}$, if $\phi \in C^{1,2}$ is such that

$$0 = (z - \phi)(\bar{t}, \bar{x}) = \min_{(s,x) \in \mathcal{O}} (z - \phi)(s, x).$$

then,

$$\mathcal{M}\phi(\bar{t}, \bar{x}) = 0,$$

or in other words,

$$\left(\phi \left(\pi'\nu + \zeta(\nu) \right) + \sum \nu_i x_i \frac{\partial \phi}{\partial x_i} \right) (\bar{t}, \bar{x}) \geq 0 \text{ for all } \nu \in \widetilde{K}. \tag{5.15}$$

In the following, we will see that (5.15) gives us relationships in the form of inequalities between $z(\bar{t}, \bar{x})$ and the value of $z(\bar{t}, \cdot)$ at any other point, which can be joined to $(\bar{t}, \bar{x})$ by a smooth path which lies entirely in $\mathcal{O}$, has fixed time coordinate $\bar{t}$ and whose directions in space in the logarithmic scale always lie in $\widetilde{K}$. In other words, if we consider smooth paths which emanate from $(\bar{t}, \bar{x})$, always lie in $\mathcal{O}$ and along which the logarithm of the stock prices change in a direction belonging to $\widetilde{K}$, then the value of $z(\cdot, \cdot)$ at all points on this path will be shown to be related to $z(\bar{t}, \bar{x})$ through an inequality. We make the idea precise in the following theorem. We will denote by $\mathrm{vect}(K)$ the vector space generated by K.

Theorem 5.2.1. *Let $(\bar{t}, \bar{x}) \in \mathcal{O}$. Suppose that the conditions of Theorem 4.2.1 hold. Let $v : [0, 1] \to \mathbb{R}^n$ be a smooth function with*

$$v(0) = 0, \; v(1) = \nu_0,$$

such that

$$(\bar{t}, \bar{x}e^{v(u)}) \in \mathcal{O}, \ \ \frac{\partial}{\partial u} v(u) =: \nu(u) \in \widetilde{K}, \ \ 0 \le u \le 1.$$

Then,

$$\ln z\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \ln z\left(\bar{t}, \bar{x}\right) \ge -\int_0^1 \left(\pi'\left(\bar{t}, \bar{x}e^{v(u)}\right)\nu(u) + \zeta(\nu(u))\right) \mathrm{d}u, \tag{5.16}$$

If we further suppose that $\zeta(\nu(u)) + \zeta(-\nu(u)) = 0, \ 0 \le u \le 1$, *then*

$$\ln z\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \ln z\left(\bar{t}, \bar{x}\right) = -\int_0^1 \left(\pi'\left(\bar{t}, \bar{x}e^{v(u)}\right)\nu(u) + \zeta(\nu(u))\right) \mathrm{d}u. \tag{5.17}$$

If we suppose that $\nu(u) \in (\mathit{vect}(K))^{\perp}$ *and* $\pi\left(\bar{t}, \bar{x}e^{v(u)}\right) \in \mathit{vect}(K)$, $0 \le u \le 1$, *then*

$$z(\bar{t}, \bar{x}) = z\left(\bar{t}, \bar{x}e^{\nu_0}\right). \tag{5.18}$$

Note that, if $\nu \in (\mathrm{vect}(K))^{\perp}$, then both ν and $-\nu$ are in $\widetilde{K}$ and $\zeta(\nu) + \zeta(-\nu) = 0$. Also recall from Lemma 3.1.3 that both $\nu, -\nu \in \widetilde{K}$ if and only if $\nu \perp \mathrm{vect}(0^+K)$.

Proof. Fix any $\bar{u} \in (0,1)$. Let $\phi \in C^{1,2}$ be such that

$$0 = (z - \phi)\left(\bar{t}, \bar{x}e^{v(\bar{u})}\right) = \min_{(s,x)\in\mathcal{O}} (z - \phi)(s, x).$$

Consider the function

$$\psi_v(u) = \ln \phi\left(\bar{t}, \bar{x}e^{v(u)}\right), \ 0 \le u \le 1.$$

Then,

$$\frac{\partial}{\partial u}\psi_v(u) = (\nu(u))'\left(\mathrm{diag}(x)\nabla \ln \phi\right)\left(\bar{t}, \bar{x}e^{v(u)}\right).$$

(5.15) now implies that

$$\left.\frac{\partial}{\partial u}\psi_v(u)\right|_{u=\bar{u}} = (\nu(\bar{u}))'\left(\mathrm{diag}(x)\nabla \ln \phi\right)\left(\bar{t}, \bar{x}e^{v(\bar{u})}\right) \ge -\left(\pi'\left(\bar{t}, \bar{x}e^{v(\bar{u})}\right)\nu(\bar{u}) + \zeta(\nu(\bar{u})\right).$$

Thus, we can see that the function $w(u) := \ln z\left(\bar{t}, \bar{x}e^{v(u)}\right), \ 0 \le u \le 1$, is a viscosity supersolution to the equation

$$\frac{\partial}{\partial u}w(u) = -\left(\pi'\left(\bar{t}, \bar{x}e^{v(u)}\right)\nu(u) + \zeta(\nu(u))\right), 0 < u < 1.$$

This implies that,

$$\ln z\left(\bar{t},\bar{x}e^{\nu_0}\right)-\ln z\left(\bar{t},\bar{x}\right)\geq-\int_0^1\left(\pi'\left(\bar{t},\bar{x}e^{v(u)}\right)\nu(u)+\zeta(\nu(u))\right)\mathrm{d}u, \tag{5.19}$$

thus proving (5.16).

Now suppose further that both $\nu(u)$ and $-\nu(u)$ are in $\widetilde{K}$ for all $0\leq u\leq 1$, and

$$\zeta(\nu)=-\zeta(-\nu). \tag{5.20}$$

Consider the smooth function $\widetilde{v}:[0,1]\rightarrow\mathbb{R}^n$,

$$\widetilde{v}(u):=-\nu_0+v(1-u),\ \ \widetilde{\nu}(u):=\frac{\partial}{\partial u}\widetilde{v}(u)=-\nu(1-u),\ \ 0\leq u\leq 1.$$

Then, from (5.16),

$$\begin{aligned}\ln z\left(\bar{t},\bar{x}\right)-\ln z\left(\bar{t},\bar{x}e^{\nu_0}\right)&\geq-\int_0^1\pi'\left(\bar{t},\bar{x}e^{\nu_0+\widetilde{v}(u)}\right)\widetilde{\nu}(u)\mathrm{d}u-\int_0^1\zeta\left(\widetilde{\nu}(u)\right)\mathrm{d}u\\&=\int_0^1\pi'\left(\bar{t},\bar{x}e^{v(1-u)}\right)\nu(1-u)\mathrm{d}u-\int_0^1\zeta\left(-\nu(1-u)\right)\mathrm{d}u\\&=\int_0^1\pi'\left(\bar{t},\bar{x}e^{v(u)}\right)\nu(u)\mathrm{d}u-\int_0^1\zeta\left(-\nu(u)\right)\mathrm{d}u.\end{aligned} \tag{5.21}$$

(5.19), (5.21) and (5.20) together prove (5.17).

(5.18) is obvious from (5.17). □

Remark 5.2.1. It is not difficult to see that, for any function $v(\cdot)$ satisfying the conditions of Theorem 5.2.1,

$$v(u)\in\widetilde{K},\ u\in[0,1].$$

Remark 5.2.2. By definition of $\mathcal{O}$, there exists an open ball $B(\bar{x})\subset\mathbb{R}^n_+$ such that $\bar{t}\times B(\bar{x})\subset\mathcal{O}$. If $\bar{x}e^{\nu_0}\in B(\bar{x})$, then there exists at least one function v, which satisfies the conditions of Theorem 5.2.1, viz.

$$v(u)=u\nu_0,\ \ 0\leq u\leq 1.$$

We will present examples in a more complete way in Section 5.4. But, to keep the reader interested, let us present a small example of an application of Theorem 5.2.1.

Example 5.2.1. Consider the market portfolio given by $\pi_i(x) = x_i / \left(\sum x_j\right),\ i = 1, 2, \ldots, n$. Let the constraint set be $K = [0, \infty)^n$, i.e. only long-only strategies are admissible. It follows immediately from Corollary 3.1.2, that $\zeta(\nu) = 0$ for all $\nu \in \widetilde{K} = [0, \infty)^n$. Since $\pi \in K$ on $[0, T] \times \mathbb{R}_+^n$, hence by Corollary 5.2.1 we have that $z \leq 1$. Therefore $\mathcal{O} = (0, T) \times \mathbb{R}_+^n$. Now, for any $\nu \in \widetilde{K}$, Theorem 5.2.1 gives us that

$$\ln z\left(t, xe^{\nu}\right) - \ln z\left(t, x\right) \geq \log\left(\frac{\sum x_i}{\sum x_i e^{\nu_i}}\right),\ (t, x) \in (0, T) \times \mathbb{R}_+^n.$$

Therefore, if $z(t, x) < 1$, for some $(t, x) \in (0, T) \times \mathbb{R}_+^n$, then $z(t, xe^{-\nu}) < 1$ for all $\nu \in \widetilde{K}$ such that $(\sum x_i)/(\sum x_i e^{-\nu_i}) < 1/z(t, x)$. In other words, if $z(t, x) < 1$, then $z(t, y) < 1$ for all

$$y \in \mathbb{R}_+^n \text{ such that } 0 < y \leq x \text{ and } \sum_{i=1}^n y_i > z(t, x) \sum_{i=1}^n x_i. \tag{5.22}$$

This means that if there exists a long-only strategy which presents an arbitrage opportunity relative to the market portfolio on the time horizon $[t, T]$ given that the stock price $\mathcal{X}(t) = x$, then there exists similar relative arbitrage opportunity if the stock price $\mathcal{X}(t)$ takes any other value y satisfying (5.22). □

We denote

$$\mathcal{O}_K^{\pi} := \left\{(t, x) \in (0, T) \times \mathbb{R}_+^n \middle| \pi(t, x) \in K\right\}. \tag{5.23}$$

Proposition 5.2.2. *Suppose the conditions of Theorem 4.2.1 hold and the constraint set K is such that $K = vect(K)$. Then $z^*(\cdot, \cdot)$ is a subsolution to*

$$-\mathcal{L}z^*(t, x) = 0, \quad (t, x) \in \mathcal{O}_K^{\pi} \cap \mathcal{O}.$$

Proof. Consider any $(t,x) \in \mathcal{O}_K^{\pi}$. Let $\phi \in C^{1,2}$ be such that

$$0 = (\phi - z^*)(t,x) = \min_{(s,y)\in[0,T]\times\mathbb{R}_+^n} (\phi - z)(s,y).$$

Since $z > 0$, hence $\phi > 0$ on $[0,T] \times \mathbb{R}_+^n$. From Theorem 4.2.1, we see that if $\nu \in (\text{vect}(K))^{\perp}$, then $z(t,x) = z(t,xe^{\nu})$. Therefore, without loss of generality we can assume that $\phi(t,x) = \phi(t,xe^{\nu})$ for all $\nu \in (\text{vect}(K))^{\perp}$. This implies that $\text{diag}(x)\nabla\log\phi(t,x) \in \text{vect}(K)$. Hence, if $\pi(t,x) \in K$, then

$$(\pi + \text{diag}(x)\nabla\log\phi)(t,x) \in \text{vect}(K) = \text{ri}(K),$$

which implies that

$$M(\phi)(t,x) = 0, \mathcal{M}_1(\phi)(t,x) > 0.$$

The proposition then follows from Theorem 4.2.2. □

Corollary 5.2.3. *Suppose the conditions of Theorem 4.2.1 hold and the constraint set K is such that $K = vect(K)$. Suppose also that $\pi(t,x) \in K$ for all $(t,x) \in [0,T] \times \mathbb{R}_+^n$. If $z(t,x) < 1$ for some $(t,x) \in (0,T) \times \mathbb{R}_+^n$, then $z(t,x) < 1$ for all $(t,x) \in (t,T) \times \mathbb{R}_+^n$. If the market is time homogeneous, then $z(t,x) < 1$ for some $(t,x) \in (0,T) \times \mathbb{R}_+^n$ implies that $z(t,x) < 1$ for all $(t,x) \in (0,T) \times \mathbb{R}_+^n$.*

Proof. This follows from Corollary 5.1.2, Corollary 5.2.1, Corollary 5.2.2 and Theorem 5.1.2. □

The integral in (5.16) might depend on $v(u)$ for all values of u between 0 and 1. This motivates investigating when the integral depends on $v(\cdot)$ only through the terminal value $v(1)$. We have the following lemma.

Lemma 5.2.1. *Let $\mathfrak{S}$ be a vector subspace of $\mathbb{R}^n$ and $\pi(t_0,\cdot) : \mathbb{R}_+^n \to \mathbb{R}_+^n$ a continuous function. Let $v : [0,1] \to \mathbb{R}^n$ be a smooth function with*

$$v(0) = 0,\ v(1) = \nu_0 \in \mathfrak{S},$$

such that

$$\frac{\partial}{\partial u} v(u) =: \nu(u) \in \mathfrak{S}, \quad 0 \leq u \leq 1.$$

The path integral $\int_0^1 \pi' \left(t_0, x_0 e^{v(u)}\right) \nu(u) \mathrm{d}u$ *depends on* $v(\cdot)$ *only through the terminal point* $v(1)$*, if and only if there exists a scalar potential* $\Pi(t_0, x_0, \cdot) : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$D_x \nabla_x \Pi(t_0, x_0, x) = Proj_{\mathfrak{S}} \pi(t_0, x), \ \forall \ x \in \mathbb{R}^n_+, \tag{5.24}$$

where $Proj_{\mathfrak{S}}$ *denotes the projection operator into the vector space* $\mathfrak{S}$*.*

Proof. Let $\{s_1, s_2, \ldots, s_k\}$ be an orthonormal basis for $\mathfrak{S}$ and let S be a $n \times k$ matrix with $s_i, i = 1, 2, \ldots, k$ as its columns. Then for any vector $m \in \mathbb{R}^n$, we have

$$\mathrm{Proj}_{\mathfrak{S}} m = SS'm.$$

We denote

$$c(u) := S'v(u); \quad \widetilde{\pi}(t_0, c) = S'\pi\left(t_0, x_0 e^{Sc}\right).$$

Then, we can write the path integral

$$\begin{aligned}
\int_0^1 \pi' \left(t_0, x_0 e^{v(u)}\right) \nu(u) \mathrm{d}u &= \int_0^1 \pi' \left(t_0, x_0 e^{SS'v(u)}\right) SS'\nu(u) \mathrm{d}u \\
&= \int_0^1 \left(S'\pi \left(t_0, x_0 e^{SS'v(u)}\right)\right)' S'\nu(u) \mathrm{d}u \\
&= \int_0^1 \left(S'\pi \left(t_0, x_0 e^{Sc(u)}\right)\right)' \frac{\partial}{\partial u} c(u) \mathrm{d}u \\
&= \int_0^1 \widetilde{\pi}' \left(t_0, c(u)\right) \frac{\partial}{\partial u} c(u) \mathrm{d}u
\end{aligned} \tag{5.25}$$

The path integral in (5.25) depends on the path only through the terminal point $c(1)$ if and only if there exists a scalar potential $\widetilde{\Pi}(t_0, x_0, \cdot) : \mathbb{R}^k \to \mathbb{R}$, such that

$$\widetilde{\pi}\left(t_0, c\right) = \nabla_c \widetilde{\Pi}(t_0, x_0, c), \ \forall \ c \in \mathbb{R}^k. \tag{5.26}$$

We now need to show that such a $\widetilde{\Pi}(t_0, x_0, \cdot)$ exists if and only if there exists a $\Pi(t_0, x_0, \cdot)$ satisfying (5.24).

To prove the necessary condition, let us assume that there exists $\widetilde{\Pi}(t_0, x_0, \cdot)$ satisfying (5.26). We now define,

$$\Pi(t_0, x_0, x) := \widetilde{\Pi}\left(t_0, x_0, S'\left(\log x - \log x_0\right)\right). \tag{5.27}$$

It is now easy to see that

$$\begin{aligned}\nabla_x \Pi(t_0, x_0, x) &= D_x^{-1} S \nabla_c \widetilde{\Pi}(t_0, x_0, c) \quad \text{where } c = S'\left(\log x - \log x_0\right) \\ &= D_x^{-1} S \widetilde{\pi}(t_0, x_0, c) = D_x^{-1} S S' \pi(t_0, x),\end{aligned}$$

thus verifying (5.24).

To prove the sufficient condition, assume that there exists $\Pi(t_0, x_0, \cdot)$ satisfying (5.24). We then define,

$$\widetilde{\Pi}(t_0, x_0, c) := \Pi\left(t_0, x_0, x_0 e^{Sc}\right).$$

It is again easy to see that

$$\begin{aligned}\nabla_c \widetilde{\Pi}(t_0, x_0, c) &= S'\left(D_x \nabla_x \Pi\right)\left(t_0, x_0, x_0 e^{Sc}\right) \\ &= S' S S' \pi\left(t_0, x_0 e^{Sc}\right) = S' \pi\left(t_0, x_0 e^{Sc}\right),\end{aligned}$$

thus proving (5.26). This completes the proof of Lemma 5.2.1.

□

Remark 5.2.3. If $\dim(\mathfrak{S}) = 1$, then for any $\pi(t_0, \cdot)$, there exists $\Pi(t_0, x_0, \cdot)$ satisfying (5.24). Indeed, if s is a unit vector in $\mathfrak{S}$,

$$\Pi(t_0, x_0, x) := \int_0^{s'(\log x - \log x_0)} s' \pi\left(t_0, x_0 e^{us}\right) \mathrm{d}u,$$

satisfies (5.24). This is also clear intuitively, because if $\dim(\mathfrak{S}) = 1$, then there exists only one possible path $v(\cdot)$.

A special case which will be of interest to us is when

$$\pi(t, x) = \operatorname{diag}(x) \nabla_x \Pi(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}_+^n, \tag{5.28}$$

for some scalar potential $\Pi(t,\cdot) : \mathbb{R}^n_+ \to \mathbb{R}$, or in other words $\text{diag}(x)^{-1}\pi(t,\cdot)$ is a conservative vector filed. This is the situation presented in Lemma 5.2.1 with $\mathfrak{S} = \mathbb{R}^n$. The market portfolio, as defined in (2.20) and of significant importance to us, satisfies (5.28). Indeed, if we denote the market portfolio by π^M, then we can see that

$$\pi^M(t,x) = \frac{1}{\sum_i x_i} x = \left(\text{diag}(x)\nabla\Pi\right)(t,x),$$

where

$$\Pi(t,x) = \ln\left(\sum x_i\right), \quad (t,x) \in [0,T] \times \mathbb{R}^n_+.$$

Lemma 5.2.1 and Theorem 5.2.1 gives us the following corollary.

Corollary 5.2.4. *Let $(\bar{t},\bar{x}) \in \mathcal{O}$. Suppose that (5.28) holds. Let $v : [0,1] \to \mathbb{R}^n$ be a smooth function with*

$$v(0) = 0, \ v(1) = \nu_0,$$

such that

$$(\bar{t}, \bar{x}e^{v(u)}) \in \mathcal{O}, \quad \frac{\partial}{\partial u} v(u) =: \nu(u) \in \widetilde{K}, \quad 0 \le u \le 1.$$

Then, under the conditions of Theorem 4.2.1,

$$\ln z\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \ln z\left(\bar{t}, \bar{x}\right) \ge -\left(\Pi\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \Pi\left(\bar{t}, \bar{x}\right)\right) - \int_0^1 \zeta\left(\nu(u)\right)\mathrm{d}u. \tag{5.29}$$

If we further suppose that $\nu(u) \in (vect(K))^{\perp}$, $0 \le u \le 1$, then

$$\ln z\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \ln z\left(\bar{t}, \bar{x}\right) = -\left(\Pi\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \Pi\left(\bar{t}, \bar{x}\right)\right). \tag{5.30}$$

Furthermore, if we also suppose that $\pi\left(\bar{t}, \bar{x}e^{v(u)}\right) \in vect(K)$, $0 \le u \le 1$, then

$$z(\bar{t}, \bar{x}) = z\left(\bar{t}, \bar{x}e^{\nu_0}\right).$$

Since Theorem 5.2.1 holds for all possible smooth paths $v(\cdot)$ satisfying the specified conditions, we also have the following characterization. We denote

$$\mathcal{V}(\nu_0) := \begin{cases} \text{the collection of all smooth functions } v : [0,1] \to \mathbb{R}^n, \text{ such that} \\ v(0) = 0, \ v(1) = \nu_0; \ \left(\bar{t}, \bar{x}e^{v(u)}\right) \in \mathcal{O}, \ \frac{\partial}{\partial u} v(u) \in \widetilde{K}, \quad 0 \le u \le 1. \end{cases} \tag{5.31}$$

From Theorem 5.2.1 and Corollary 5.2.4, it follows that

Corollary 5.2.5. *Suppose the conditions of Theorem 4.2.1 hold. Let* $(\bar{t}, \bar{x}) \in \mathcal{O}$. *Then*

$$\ln z\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \ln z\left(\bar{t}, \bar{x}\right) \geq - \inf_{v \in \mathcal{V}(\nu_0)} \left\{ \int_0^1 \left(\pi'\left(\bar{t}, \bar{x}e^{v(u)}\right)\nu(u) + \zeta(\nu(u))\right) \mathrm{d}u \right\}. \quad (5.32)$$

In the special case that (5.28) *holds, we have*

$$\ln z\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \ln z\left(\bar{t}, \bar{x}\right) \geq -\left(\Pi\left(\bar{t}, \bar{x}e^{\nu_0}\right) - \Pi\left(\bar{t}, \bar{x}\right)\right) - \inf_{v \in \mathcal{V}(\nu_0)} \left\{ \int_0^1 \zeta\left(\nu(u)\right)\mathrm{d}u \right\}. \quad (5.33)$$

Here, we have followed the convention that the infimum of the empty set is ∞. For convenience of notation, we will denote

$$\mathfrak{I}^{\pi}(\nu_0, K, \bar{t}, \bar{x}) = \inf_{v \in \mathcal{V}(\nu_0)} \left\{ \int_0^1 \left(\pi'\left(\bar{t}, \bar{x}e^{v(u)}\right)\nu(u) + \zeta(\nu(u))\right) \mathrm{d}u \right\}, \quad (5.34)$$

$$\mathfrak{I}(\nu_0, K) = \inf_{v \in \mathcal{V}(\nu_0)} \left\{ \int_0^1 \zeta\left(\nu(u)\right)\mathrm{d}u \right\}. \quad (5.35)$$

All the results presented above hold for $(\bar{t}, \bar{x}) \in \mathcal{O}$. In the following we will give a necessary condition for $(\bar{t}, \bar{x})$ to belong to $\mathcal{O}$. For this we look at (5.17). Since the left hand side of (5.17) does not depend on $v(\cdot)$, the integral on the right hand side of (5.17) should also be independent of the path $v(\cdot) \in \mathcal{V}(\nu_0)$. Then this can be posed as a necessary condition for $(\bar{t}, \bar{x})$ to belong to $\mathcal{O}$. Thus we can say that under the conditions of Theorem 4.2.1, a necessary condition for $(\bar{t}, \bar{x}) \in \mathcal{O}$ is the existence of an open ball $B(\bar{x}) \subset \mathbb{R}^n$ containing $\bar{x}$, such that for any $\nu_0 \in (\mathrm{vect}(K))^{\perp}$ with $\bar{x}e^{\nu_0} \in B(\bar{x})$, the integral $\int_0^1 \pi'\left(\bar{t}, \bar{x}e^{v(u)}\right)\nu(u)\mathrm{d}u$ does not depend on the choice of $v(\cdot) \in \mathcal{V}(\nu_0)$. By Lemma 5.2.1, a necessary and sufficient condition for the independence of the path integral is the existence of a scalar potential $\Pi(\bar{t}, \bar{x}, \cdot) : \mathbb{R}^n_+ \to \mathbb{R}$ such that

$$Proj_{(\mathrm{vect}(K))^{\perp}} \pi(\bar{t}, x) = \mathrm{diag}(x) \nabla \Pi(\bar{t}, \bar{x}, x), \quad (5.36)$$

We can summarize this as a corollary.

Corollary 5.2.6. *Suppose that the conditions of Theorem 4.2.1 hold. A necessary condition for* $(\bar{t}, \bar{x}) \in \mathcal{O}$ *is that* (5.36) *holds in a neighborhood of* $(\bar{t}, \bar{x})$*.*

(5.36) will be satisfied vacuously if $(\text{vect}(K))^{\perp} = \emptyset$. By Remark 5.2.3, it will also be satisfied if $\dim((\text{vect}(K))^{\perp}) = 1$. For $\dim((\text{vect}(K))^{\perp}) \geq 2$, the condition will be satisfied if any of the following conditions are satisfied:

i. $\pi(\bar{t}, \bar{x}e^{\nu}) \in \text{vect}(K)$ whenever $\nu \in (\text{vect}(K))^{\perp}$ and $\bar{x}e^{\nu} \in B(\bar{x})$.

ii. $\text{diag}(x)^{-1}\pi(t,x)$ is a conservative vector field. See (5.28).

Suppose now that $\mathcal{O} = (0,T) \times \mathbb{R}_+^n$, i.e. $z(\cdot,\cdot)$ is locally bounded on $(0,T) \times \mathbb{R}_+^n$. Consider any $(t,x) \in (0,T) \times \mathbb{R}_+^n, \nu \in \widetilde{K}$. From (5.32), we see that for any $c > 0$,

$$\ln z\left(t, xe^{c\nu}\right) \geq \ln z(t,x) - \mathfrak{I}^{\pi}(c\nu, K, t, x). \tag{5.37}$$

Therefore, if $\mathfrak{I}^{\pi}(c\nu, K, t, x)$ is unbounded from below as a function of c on $(0,\infty)$, then we can choose $c_0 > 0$ such that $z\left(t, xe^{c_0\nu}\right) > 1$, thus proving the existence of a $(t,y) \in (0,T) \times \mathbb{R}_+^n$ such that arbitrage opportunities do not exist relative to π over the time horizon $[0,T]$, if the stock price $\mathcal{X}(t)$ at time t is y.
On the other hand, if $z(t,x) > 1$, then (5.37) yields that $z\left(t, xe^{c\nu}\right) \geq 1$ for all $\nu \in \widetilde{K}$ for which $\mathfrak{I}^{\pi}(c\nu, K, t, x) \leq \ln z(t,x)$.
Suppose now that $\widetilde{K} = \mathbb{R}^n$. Consider $0 \leq s_1 < s_2 \leq T$. For a fixed $x \in \mathbb{R}_+^n, \epsilon > 0$, denote

$$\delta_x := \sup_{s_1 < t < s_2} \{\ln z(t,x)\} + \epsilon.$$

If there exist $C > 0$, such that

$$\mathfrak{I}^{\pi}(c\nu, K, t, x) < \ln z(t,x) - \delta_x, \text{ for all } c \geq C, \nu \in \mathbb{R}^n, \text{ such that } \|\nu\| = 1, \tag{5.38}$$

then we will have

$$z(t, xe^{C\nu}) > e^{\epsilon} \sup_{s_1<t<s_2} z(t,x) \geq \inf_{[s_1,s_2)\times\Gamma} z(s,y),\ \nu \in \mathbb{R}^n,\ \|\nu\| = 1,\ t \in (s_1, s_2), \tag{5.39}$$

where

$$\Gamma := \left\{ xe^{c\nu} \mid \nu \in \mathbb{R}^n,\ \|\nu\| = 1,\ 0 \leq c < C \right\}.$$

(5.39) contradicts the comparison principle in Theorem 5.1.1 (i) and hence proves that, if there exists $C > 0$ satisfying (5.38), then $(s_1, s_2) \times \mathbb{R}^n_+ \not\subseteq \mathcal{O}$, i.e.

$$\text{there exists } (t, y) \in (s_1, s_2) \times \mathbb{R}^n_+ \text{ such that } z(t, y) = \infty, \tag{5.40}$$

i.e. if the stock price $\mathcal{X}(t)$ at time t is y, then the terminal wealth generated by π cannot be superreplicated by starting with any finite amount of wealth at time t and following a strategy which always takes values in K. In Example 5.4.3, we consider the portfolio defined as $\pi(t,x) = -2\ln x,\ (t,x) \in [0,T] \times \mathbb{R}^n_+$, and the constraint set $K = \{p : \sum p_i = 1, p \geq 0\}$ corresponding to the restriction that investments can be made only in long-only portfolios. We will show there that

$$\mathfrak{I}^{\pi}(c\nu, K, t, x) = -2c\nu' \ln x - c^2\|\nu\|^2 - c\nu_{(1)},$$

which can be made to take arbitrarily large negative values by increasing c and hence there exists a $C > 0$ satisfying (5.38). Therefore, (5.40) also holds in this case.

Similarly, if $\nu \in \widetilde{K}$, then writing (5.37) with x replaced by $xe^{-c\nu}$, we get

$$\ln z\,(t,x) + \mathfrak{I}^{\pi}(c\nu, K, t, xe^{-c\nu}) \geq \ln z(t, xe^{-c\nu}). \tag{5.41}$$

Therefore, if $\mathfrak{I}^{\pi}(c\nu, K, t, xe^{-c\nu})$ is unbounded from below as a function of c on $(0,\infty)$, then we can choose $c_0 > 0$ such that $z\,(t, xe^{-c_0\nu}) < 1$, thus proving the existence of a $(t,y) \in (0,T) \times \mathbb{R}^n_+$ such that arbitrage opportunities relative to π exist over the time horizon $[0,T]$, if the stock price $\mathcal{X}(t)$ at time t is y.

On the other hand, if $z(t,x) < 1$, then (5.37) yields that $z\left(t, xe^{-c\nu}\right) < 1$ for all $\nu \in \widetilde{K}$ for which $\mathfrak{I}^{\pi}(c\nu, K, t, xe^{-c\nu}) < -\ln z(t,x)$.

Suppose now that $\widetilde{K} = \mathbb{R}^n$. Consider $0 \le s_1 < s_2 \le T$. For a fixed $x \in \mathbb{R}^n_+, \epsilon > 0$, denote

$$\delta_x := \inf_{s_1 < t < s_2} \{\ln z(t,x)\} - \epsilon.$$

If there exist $C > 0$, such that

$$\mathfrak{I}^{\pi}(c\nu, K, t, x) < -\ln z(t,x) + \delta_x, \text{ for all } c \ge C, \nu \in \mathbb{R}^n, \text{ such that } \|\nu\| = 1, \quad (5.42)$$

then we will have

$$z^*(t, xe^{-(C+1)\nu}) < e^{-\epsilon} \inf_{s_1 < t < s_2} z(t,x) \le \sup_{[s_1,s_2)\times\Gamma} z^*(s,y),\ \nu \in \mathbb{R}^n,\ \|\nu\| = 1,\ t \in (s_1, s_2), \quad (5.43)$$

where

$$\Gamma := \left\{ xe^{-c\nu} \;\middle|\; \nu \in \mathbb{R}^n,\ \|\nu\| = 1,\ 0 \le c < C+1 \right\}.$$

(5.43) contradicts the comparison principle in Theorem 5.1.1 (i) and hence proves that, if there exists $C > 0$ satisfying (5.42), then $(s_1, s_2) \times \mathbb{R}^n_+ \not\subseteq \mathcal{O}$, i.e.

$$\text{there exists } (t,y) \in (s_1, s_2) \times \mathbb{R}^n_+ \text{ such that } z(t,y) = \infty, \quad (5.44)$$

i.e. if the stock price $\mathcal{X}(t)$ at time t is y, then the terminal wealth generated by π cannot be superreplicated by starting with any finite amount of wealth at time t and following a strategy which always takes values in K.

Consider two constraint sets $K_2 \subseteq K_1 \subseteq \mathbb{R}^n$. Then,

$$z_{K_2}(t,x) \ge z_{K_1}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n_+. \quad (5.45)$$

Suppose $\nu \in \mathbb{R}^n$ is such that

$$\nu \in \widetilde{K}_1 \subseteq \widetilde{K}_2,\ -\nu \in \widetilde{K}_2,\ \zeta_{K_2}(\nu) + \zeta_{K_2}(-\nu) = 0.$$

By Theorem 5.2.1 and (5.45), it follows that for any $c > 0$ and $(t, x) \in (0, T) \times \mathbb{R}^n_+$,

$$\begin{aligned} \ln z_{K_2}(t,x) - \mathfrak{I}^\pi(c\nu, K_2, t, x) &= \ln z_{K_2}\left(t, xe^{c\nu}\right) \\ &\geq \ln z_{K_1}\left(t, xe^{c\nu}\right) \geq \ln z_{K_1}(t,x) - \mathfrak{I}^\pi(c\nu, K_1, t, x). \end{aligned}$$

Therefore,

$$\ln z_{K_2}(t,x) - \ln z_{K_1}(t,x) \geq \sup_{c>0} \left[\mathfrak{I}^\pi(c\nu, K_2, t, x) - \mathfrak{I}^\pi(c\nu, K_1, t, x)\right]. \tag{5.46}$$

Now if $\sup_{c>0} [\mathfrak{I}^\pi(c\nu, K_2, t, x) - \mathfrak{I}^\pi(c\nu, K_1, t, x)] = \infty$ for some $(t, x) \in (0, T) \times \mathbb{R}^n_+$, then (5.46) would imply that $z_{K_2}(t, x) = \infty$.

When the constraint set K is a polyhedral convex set, $\zeta_K(\cdot)$ takes a simple form. Further, if $\pi(\cdot, \cdot)$ satisfies (5.28), then the function $\mathfrak{I}^\pi(\cdot, K, \cdot, \cdot)$ becomes much simpler and amenable to simple analyses yielding interesting results, which we will present after we discuss polyhedral convex constraint sets in the next section.

5.3 Convex polyhedral constraint set

As discussed in the end of Section 5.2, convex polyhedral constraint sets will be of special interest to us. We refer the reader to Section 19 in Rockafellar (1970) for a discussion on polyhedral convex sets. By Theorem 20.5 in Rockafellar (1970), every polyhedral convex set is locally simplical; hence satisfies Assumption 3.1.1 and thus fits into our framework. On the other hand, under fairly general conditions, convex sets can be approximated by polyhedral convex sets. We refer the reader to Theorem 2.1 in Ney and Robinson (1995) for polyhedral approximation of closed convex sets whose recession cone is polyhedral, and to Section 4 of Bronstein (2008) for approximation of compact convex sets with nonempty interior by polytopes (bounded polyhedral convex sets).

Suppose now that the constraint set K is a polyhedral convex set. By Theorem 19.1 in Rockafellar (1970), it is finitely generated, i.e. there exist vectors $a_1, a_2, \ldots, a_m$ such that, for a fixed integer k, $0 \leq k \leq m$, K is of the form

$$K = \left\{ \sum_{i=1}^{m} \beta_i a_i : \sum_{i=k+1}^{m} \beta_i = 1,\ \beta_i \geq 0,\ i = 1, 2, \ldots, m \right\}. \tag{5.47}$$

We will denote

$$S_0(K) := \{a_{k+1}, \ldots, a_m\}, \quad S_1(K) := \{a_1, \ldots, a_k\}.$$

Note that if K is a polyhedral cone, then $S_0(K) = \emptyset$, and the vectors in $S_1(K)$ are called the edges of the cone. By Theorem 19.5 in Rockafellar (1970), it follows that the recession cone 0^+K of K is also polyhedral, and in fact it can be written as

$$\left\{ \sum_{i=1}^{k} \beta_i a_i :\ \beta_i \geq 0,\ i = 1, \ldots, k \right\}.$$

Therefore, the set of edges of the recession cone 0^+K is $S_1(0^+K) = S_1(K)$.

The polyhedral convex cone 0^+K can also be seen as the intersection of closed half-spaces $H_1, H_2, \ldots, H_\ell$. Each half-space H_i has a corresponding inward pointing normal vector n_i, the face normal of the cone 0^+K, such that

$$H_i = \left\{ v \in \mathbb{R}^n \;\middle|\; v' n_i \geq 0 \right\}.$$

$\widetilde{K}$, the dual cone to 0^+K, will also be a polyhedral cone, and each of the face normals of 0^+K will be an edge of $\widetilde{K}$ and vice versa. Thus, $S_1(\widetilde{K}) = \{n_1, n_2, \ldots, n_\ell\}$, i.e.

$$\widetilde{K} := \left\{ \sum_{i=1}^{\ell} \alpha_i n_i :\ n_i \in S_1(\widetilde{K}),\ \alpha_i \geq 0,\ i = 1, 2, \ldots, \ell \right\} \tag{5.48}$$

With K as in (5.47), and since K is convex, it follows that

$$n_i' a_j \geq 0, \quad n_i \in S_1(\widetilde{K}),\ a_j \in S_1(K). \tag{5.49}$$

The edges of $\widetilde{K}$ can also be determined directly from the edges of 0^+K. It is particularly easy when $n = 2$ and $S_1(0^+K) = \{a_1, a_2\}$ for some $a_1, a_2 \in \mathbb{R}^2$. If $\widetilde{n}_1, \widetilde{n}_2 \in \mathbb{R}^2$ are perpendicular to a_1 and a_2 respectively, then $S_1(\widetilde{K}) = \{n_1, n_2\}$ where,

$$n_1 = \operatorname{sgn}(\widetilde{n}_1' a_2)\ \widetilde{n}_1, \quad n_2 = \operatorname{sgn}(\widetilde{n}_2' a_1)\ \widetilde{n}_2,$$

and $\operatorname{sgn}(x)$ is as defined in (1.1).

For any $\widetilde{K} \ni \nu = \sum_{i=1}^{\ell} \alpha_i n_i,\ \alpha \geq 0$, it follows that,

$$\begin{aligned}
\zeta(\nu) = -\inf_{\pi \in K} \pi'\nu &= -\inf\left\{ \sum_{j=1}^{m} \beta_j a_j' \left(\sum_{i=1}^{\ell} \alpha_i n_i \right) : \sum_{j=k+1}^{m} \beta_j = 1, \beta \geq 0 \right\} \\
&= -\inf\left\{ \sum_{j=k+1}^{m} \beta_j a_j' \left(\sum_{i=1}^{\ell} \alpha_i n_i \right) : \sum_{j=k+1}^{m} \beta_j = 1, \beta \geq 0 \right\} \\
&= -\min\left\{ a_j'\nu \ : j = k+1, \ldots, m \right\}
\end{aligned} \tag{5.50}$$

The third equality in the above, is a consequence of (5.49).
If $S_0(K) = \emptyset$, then from (5.50) we get

$$\zeta(\nu) = 0 \text{ for all } \nu \in \widetilde{K}, \tag{5.51}$$

as expected from Corollary 3.1.2.
If $k + 1 = m$, i.e. $S_0(K) = \{a_{k+1}\}$, then from (5.50) we get

$$\zeta(\nu) = -a_{k+1}'\nu \text{ for all } \nu \in \widetilde{K}. \tag{5.52}$$

From Lemma 3.1.3, we see that

$$\nu \in \widetilde{K}, -\nu \in \widetilde{K} \iff \nu \perp a_i,\ i = 1, 2, \ldots, k.$$

From (5.50), it follows that if $\nu, -\nu \in \widetilde{K}$, then

$$\zeta(\nu) + \zeta(-\nu) = 0 \iff \min_{k+1 \leq j \leq m} a_j'\nu = \max_{k+1 \leq j \leq m} a_j'\nu. \tag{5.53}$$

Therefore, a necessary and sufficient condition for $\zeta(\nu) + \zeta(-\nu) = 0$ is that ν is orthogonal to each vector in $S_1(K)$ and either of the following is satisfied:

1. $S_0(K) = \emptyset$.

2. $S_0(K)$ is a singleton set.

3. ν is orthogonal to each vector in $S_0(K)$, i.e. $\zeta(\nu) = 0$.

Suppose now that $S_1(\widetilde{K}) = \{n_1, \ldots, n_\ell\}$ are the distinct edges of $\widetilde{K}$. Consider any

$$\widetilde{K} \ni \nu_0 = \sum_{i=1}^{\ell} \alpha_i n_i.$$

Let $\mathfrak{L}(\alpha)$ denote the class of functions $\Lambda : [0,1] \to \mathbb{R}^n_+$ such that each component $\Lambda_i(\cdot), i = 1, \ldots, n$ is a smooth increasing function, with $\Lambda(0) = 0$ and $\Lambda(1) = \alpha$. It is easy to see that for any $\Lambda \in \mathfrak{L}(\alpha)$, the path $v(\cdot)$ defined by

$$v(u) = \sum_{i=1}^{\ell} \Lambda_i(u) n_i, \quad u \in [0,1], \tag{5.54}$$

belongs to $\mathcal{V}(\nu_0)$ defined as in (5.31). And for any path $v(\cdot) \in \mathcal{V}(\nu_0)$, there exists such a function $\Lambda(\cdot) \in \mathfrak{L}(\alpha)$, thus establishing a one-to-one correspondence between $\mathcal{V}(\nu_0)$ and $\mathfrak{L}(\alpha)$. If we denote $\lambda_i(u) = \partial/\partial u(\Lambda(u))$, then

$$\nu(u) = \frac{\partial}{\partial u} v(u) = \sum_{i=1}^{\ell} \lambda_i(u) n_i, \; u \in (0,1). \tag{5.55}$$

It follows from (5.50) that

$$\begin{aligned} \mathfrak{I}(\nu_0, K) &= \inf_{v \in \mathcal{V}(\nu_0)} \left\{ \int_0^1 \zeta\left(\nu(u)\right) \mathrm{d}u \right\} \\ &= - \sup_{\Lambda(\cdot) \in \mathcal{L}(\alpha)} \left\{ \int_0^1 \min_{k+1 \le j \le m} \left\{ \sum_{i=1}^{\ell} \lambda_i(u) \left(a_j' n_i\right) \right\} \mathrm{d}u \right\} \end{aligned} \tag{5.56}$$

Now, for any $\Lambda(\cdot) \in \mathcal{L}(\alpha)$,

$$\int_0^1 \min_{k+1\le j\le m} \left\{ \sum_{i=1}^{\ell} \lambda_i(u) \left(a_j' n_i\right) \right\} \mathrm{d}u \le \min_{k+1\le j\le m} \int_0^1 \left\{ \sum_{i=1}^{\ell} \lambda_i(u) \left(a_j' n_i\right) \right\} \mathrm{d}u = \min_{k+1\le j\le m} a_j'\nu_0.$$

Therefore,

$$\mathfrak{I}(\nu_0, K) \ge -\zeta(\nu_0). \tag{5.57}$$

On the other hand, taking $\Lambda(u) = u\alpha,\ 0 \le u \le 1$, we get

$$\int_0^1 \min_{k+1\le j\le m} \left\{ \sum_{i=1}^{\ell} \lambda_i(u) \left(a_j' n_i\right) \right\} \mathrm{d}u = \zeta(\nu_0) \int_0^1 u \,\mathrm{d}u = \zeta(\nu_0). \tag{5.58}$$

(5.57) and (5.58) together imply that

$$\mathfrak{I}(\nu_0, K) = \zeta(\nu). \tag{5.59}$$

Example 5.3.1. Suppose $K = \{p \in \mathbb{R}^n : \sum_i p_i \ge 1\}$. This would correspond to the constraint, that money can be borrowed from the bank, but the entire wealth has to be invested in stocks. Then,

$$S_0(K) = \left\{ \frac{1}{n}\mathbf{1} \right\};\ S_1(K) = S_1(0^+K) = \{\mathbf{1}, e_i - e_j\ i \ne j,\ i,j = 1,2,\ldots,n\},\ S_1(\widetilde{K}) = \{\mathbf{1}\}.$$

Any $\nu \in \widetilde{K}$ is of the form $\lambda\mathbf{1},\ \lambda \ge 0$. Therefore, for any $\widetilde{K} \ni \nu = \lambda\mathbf{1}$, by (5.52) and (5.59)

$$\mathfrak{I}(\nu, K) = \zeta(\nu) = -\lambda.$$

Example 5.3.2. If $K = \{p \in \mathbb{R}^n : \sum_i p_i \ge 1,\ p \ge 0\}$, then the constraint would be that money can be borrowed from the bank, but the entire wealth has to be used to take long positions in stocks. Then,

$$S_0(K) = S_1(K) = S_1(0^+K) = S_1(\widetilde{K}) = \{e_i\ i = 1,2,\ldots,n\}.$$

Therefore, for any $\nu^0 \in \widetilde{K}$, it follows from (5.50) and (5.59) that

$$\mathfrak{I}(\nu^0, K) = \zeta(\nu^0) = -\nu^0_{(1)}.$$

Example 5.3.3. If $K = \{p \in \mathbb{R}^n : \sum_i p_i \leq 1,\ p \geq 0\}$, then the constraint would be that money cannot be borrowed from the bank and investments can be made only in long-only strategies. Then,

$$S_0(K) = \{\mathbf{0}, e_i\ i = 1, 2, \ldots, n\}\,;\ S_1(K) = S_1(0^+K) = \emptyset;$$
$$S_1(\widetilde{K}) = \{e_i\ i = 1, 2, \ldots, n\}\,;\ \widetilde{K} = \mathbb{R}^n.$$

Therefore, for any $\nu^0 \in \widetilde{K}$, it follows from (5.50) and (5.59) that

$$\mathfrak{I}(\nu^0, K) = \zeta(\nu^0) = -\left(\nu^0_{(1)} \wedge 0\right).$$

Example 5.3.4. If $K = \{p \in \mathbb{R}^n : \sum_i p_i = 1,\ p \geq 0\}$, then the constraint would be that investments can be made only in long-only portfolios. Then,

$$S_0(K) = \{e_i\ i = 1, 2, \ldots, n\}\,;\ S_1(K) = S_1(0^+K) = \emptyset;$$
$$S_1(\widetilde{K}) = \{e_i\ i = 1, 2, \ldots, n\}\,;\ \widetilde{K} = \mathbb{R}^n.$$

Therefore, for any $\nu^0 \in \widetilde{K}$, it follows from (5.50) and (5.59) that

$$\mathfrak{I}(\nu^0, K) = \zeta(\nu^0) = -\nu^0_{(1)}.$$

Example 5.3.5. If $K = \{p \in \mathbb{R}^n : \sum_i p_i = 1\}$, then the constraint would be that investments can be made only in portfolios. Then,

$$S_0(K) = \left\{\frac{1}{n}\mathbf{1}\right\};\ S_1(K) = S_1(0^+K) = \{e_i - e_j\ i \neq j,\ i, j = 1, 2, \ldots, n\}\,,$$
$$S_1(\widetilde{K}) = \{\mathbf{1}, -\mathbf{1}\}\,;\ \widetilde{K} = \{\alpha\mathbf{1}\ :\ \alpha \in \mathbb{R}\}\,.$$

Therefore, for any $\widetilde{K} \ni \nu^0 = \alpha\mathbf{1}$, by (5.52) and (5.59)

$$\mathfrak{I}(\nu^0, K) = \zeta(\nu^0) = -\alpha.$$

Also, note that for any $\nu \in \widetilde{K}$, we will have

$$\zeta(\nu) + \zeta(-\nu) = 0.$$

Example 5.3.6. Let $n = 2$ and for $c \in \mathbb{R}^2$ and distinct points $a_1, a_2 \in \mathbb{R}^2$, suppose $S_0(K) = \{c\}$, $S_1(K) = \{a_1, a_2\}$, i.e.

$$K = \{C + \alpha_1 a_1 + \alpha_2 a_2,\ \alpha_1, \alpha_2 \geq 0\}.$$

Then, $S_1(0^+K) = \{a_1, a_2\}$. For any $\nu \in \widetilde{K}$,

$$\mathfrak{I}(\nu, K) = \zeta(\nu) = -\nu' c.$$

5.4 Examples

In this section, we will present several examples illustrating the use of the results developed so far. We start with the special case of convex polyhedral constraint sets and strategies which satisfy (5.28).

Suppose that $\mathcal{O} = (0, T) \times \mathbb{R}^n_+$. We recall from (5.37) and (5.41) that for $\nu \in \widetilde{K}$ and $(t, x) \in (0, T) \times \mathbb{R}^n_+$,

$$\ln z\,(t, xe^{c\nu}) \geq \ln z(t, x) - \mathfrak{I}^\pi(c\nu, K, t, x), \quad c \geq 0, \tag{5.60}$$

$$\ln z\left(t, xe^{-c\nu}\right) \leq \ln z\,(t, x) + \mathfrak{I}^\pi(c\nu, K, t, xe^{-c\nu}), \quad c \geq 0,. \tag{5.61}$$

If $\nu \in \widetilde{K}$ is such that $-\nu \in \widetilde{K}$ and $\zeta(\nu) + \zeta(-\nu) = 0$, then

$$\ln z\,(t, xe^{c\nu}) = \ln z(t, x) - \mathfrak{I}^\pi(c\nu, K, t, x), \quad c \in \mathbb{R},$$

Suppose now that the strategy π is of the form

$$\pi(t, x) = \text{diag}(x)\nabla_x \Pi(t, x),\ (t, x) \in [0, T] \times \mathbb{R}^n_+,$$

and the constraint set K is a polyhedral convex set. Then, $\mathfrak{I}^\pi(c\nu, K, t, x)$ takes the form

$$\mathfrak{I}^\pi(c\nu, K, t, x) = \Pi\,(t, xe^{c\nu}) - \Pi\,(t, x) - c \min_{a \in S_0(K)} \{a'\nu\} \tag{5.62}$$

Given a $\nu \in \mathbb{R}^n$, suppose that we want a constraint set K so that arbitrage opportunities relative to π do not exist over the time horizon $[t,T]$, if the stock price $\mathcal{X}(t)$ at time t is $xe^{c\nu}$ for large enough c. From (5.60), we see that a sufficient condition for this would be

$$\inf_{c>0} \mathfrak{I}^{\pi}(c\nu, K, t, x) = -\infty. \tag{5.63}$$

From (5.62), we see that if

$$\Pi\left(t, xe^{c\nu}\right) - \Pi\left(t, x\right) \leq Ac, \text{ for some } A \in \mathbb{R}, \tag{5.64}$$

then (5.63) will be satisfied if $S_1(\widetilde{K}) = \{\nu\}$ and $S_0(K) = \{a_0\}$ for some $a_0 \in \mathbb{R}^n$ such that $a_0'\nu > A$. From (5.48), we see that if $S_1(\widetilde{K}) = \{\nu\}$, then $0^+K = \{p \in \mathbb{R}^n : p'\nu \geq 0\}$. Therefore, $K = \{a_0 + p : p \in \mathbb{R}^n, p'\nu \geq 0\}$. In particular, we can take $a_0 = \frac{1}{\|\nu\|^2}(A+1)\nu$.

Example 5.4.1. If π denotes the market portfolio, as defined in (2.20), then $\pi(t,x) = \left(\operatorname{diag}(x)\nabla\Pi\right)(t,x)$, where

$$\Pi(t,x) = \ln\left(\sum x_i\right), \quad (t,x) \in [0,T] \times \mathbb{R}^n_+.$$

Therefore,

$$\Pi\left(t, xe^{c\nu}\right) - \Pi\left(t, x\right) = \ln\left(\frac{\sum x_i e^{c\nu_i}}{\sum x_i}\right) \leq \ln\left(\frac{n x_{(n)} e^{c\nu_{(n)}}}{n x_{(1)}}\right) = \ln\frac{x_{(n)}}{x_{(1)}} + c\nu_{(n)}, \tag{5.65}$$

which satisfies (5.64). (5.63) will be satisfied if we take

$$a_0 = \gamma 1_{\nu_{(n)}>0}\left(1_{\nu_1=\nu_{(n)}}, \ldots, 1_{\nu_n=\nu_{(n)}}\right), \quad K = \{a_0 + p : p \in \mathbb{R}^n, p'\nu \geq 0\},$$

for any $\gamma > 1$. For any $x \in \mathbb{R}^n_+$, it follows from (5.60) that $z(t, xe^c\nu) \geq 1$ if

$$c \geq -\left(1_{\nu_{(n)}<0} + \frac{1}{\gamma-1}1_{\nu_{(n)}>0}\right)\frac{1}{\nu_{(n)}}\ln\left(\frac{x_{(1)}}{x_{(n)}}z(t,x)\right).$$

Thus, for example, if $\nu = e_1$, and if we take the constraint set to be $K = [\gamma, \infty) \times \mathbb{R}^{n-1}$, then for any $x \in \mathbb{R}^n_+$, we will have $z(t, xe^{ce_1}) > 1$ if $c \geq -\frac{1}{\gamma-1}\ln\left(\frac{x_{(1)}}{x_{(n)}}z(t,x)\right)$

Suppose now, that we want a constraint set K, such that for a fixed $(t,x) \in (0,T) \times \mathbb{R}^n_+$ and for large enough $c > 0$,

$$z_K\left(t, xe^{c\nu^i}\right) \geq 1, \;\; \nu^i \in \widetilde{K}, \; i = 1, 2, \ldots, \ell.$$

An easy solution would be to find K_i such that

$$z_{K_i}\left(t, xe^{c\nu^i}\right) \geq 1, \;\; \nu^i \in \widetilde{K}_i, \; i = 1, 2, \ldots, \ell,$$

for large enough $c > 0$, and then $K := \bigcap_i K_i$ will be our desired constraint set provided it is non-empty. We could also proceed as follows. Since $\nu_i \in \widetilde{K}, \; i = 1, 2, \ldots, \ell$, we should have

$$0^+K \subseteq \{p \in \mathbb{R}^n : p'\nu_i \geq 0, \; i = 1, 2, \ldots, \ell\}.$$

We would then need to determine $S_0(K)$ so that

$$\inf_{c>0}\left[\Pi\left(t, xe^{c\nu^i}\right) - \Pi\left(t, x\right) - c \min_{a \in S_0(K)}\left\{a'\nu^i\right\}\right] = -\infty, \;\; i = 1, 2, \ldots, \ell,$$

if it exists. We illustrate this in the following two examples.

Example 5.4.2. Let π denote the market portfolio. Given $(t,x) \in (0,T) \times \mathbb{R}^n_+$, suppose we want the constraint K to be such that for each $\nu \in \mathbb{R}^n$ with $\|\nu\| = 1$, $z(t, xe^{c\nu}) > 1$ for large enough $c > 0$. In order to be able to use the inequality (5.60), for any $\nu \in \mathbb{R}^n$, we need $\widetilde{K} = \mathbb{R}^n$, and hence $0^+K = \emptyset$. Therefore, K has to be a bounded set. From (5.65), we see that for any $\nu \in \mathbb{R}^n$,

$$\mathfrak{I}^\pi(c\nu, K, t, x) \leq \ln\frac{x_{(n)}}{x_{(1)}} + c\nu_{(n)} - c\min_{a \in S_0(K)}\{a'\nu\}. \tag{5.66}$$

In order to have the right hand side in (5.66) to go to $-\infty$ as $c \to \infty$ for any $\nu \in \mathbb{R}^n$, we would need to have

$$\nu_{(n)} < \min_{a \in S_0(K)}\{a'\nu\}, \tag{5.67}$$

for any $\nu \in \mathbb{R}^n$. For (5.67) to hold for $\nu = e_i,\ i = 1, 2, \ldots, n$, we require

$$\min_{a \in S_0(K)} a_i > 1, \quad i = 1, 2, \ldots, n. \tag{5.68}$$

On the contrary, for (5.67) to hold for $\nu = -e_i$ $i = 1, 2, \ldots, n$, we require

$$\max_{a \in S_0(K)} a_i < 1, \quad i = 1, 2, \ldots, n. \tag{5.69}$$

Therefore, we cannot find $S_0(K)$ as desired.

Example 5.4.3. Consider the portfolio π defined by $\pi(t, x) = -2 \ln x,\ (t, x) \in [0, T] \times \mathbb{R}^n_+$. Then $\pi(t, x) = (\text{diag}(x)\nabla\Pi)(t, x)$, where

$$\Pi(t, x) = -\sum (\ln x_i)^2\,,\ (t, x) \in [0, T] \times \mathbb{R}^n_+.$$

For a fixed $(t, x) \in (0, T) \times \mathbb{R}^n_+$, we want the constraint set K to be such that

$$\begin{aligned} &\text{for each } \nu \in \mathbb{R}^n \text{ with } \|\nu\| = 1,\ \exists\, c \geq 0, \text{ such} \\ &\text{that either } (t, xe^{c\nu}) \notin \mathcal{O} \text{ or, } z_K\left(t, xe^{c\nu}\right) > 1. \end{aligned} \tag{5.70}$$

It would be sufficient if we can find a constraint set K, such that for each $\nu \in \mathbb{R}^n$,

$$\text{if } (t, xe^{c\nu}) \in \mathcal{O} \text{ for all } c \geq 0, \text{ then } z\left(t, xe^{c\nu}\right) \geq 1 \text{ for some } c \geq 0. \tag{5.71}$$

In order to be able to use the inequality (5.60) for any $\nu \in \mathbb{R}^n$, we need $\widetilde{K} = \mathbb{R}^n$, and hence $0^+K = \emptyset$. Therefore, K has to be a bounded set.

Now, for any $\nu \in \mathbb{R}^n$, $\mathfrak{I}^\pi(c\nu, K, t, x)$ takes the form

$$\mathfrak{I}^\pi(c\nu, K, t, x) = -2c\nu' \ln x - c^2\|\nu\|^2 - c \min_{a \in S_0(K)} a'\nu. \tag{5.72}$$

For any finite set $S_0(K) \subset \mathbb{R}^n$, $\min_{a \in S_0(K)} a'\nu < \infty$ and hence it follows from (5.72) that

$$\mathfrak{I}^\pi(c\nu, K, t, x) \to -\infty \ \text{ as } c \to \infty \text{ for any } \nu \in \mathbb{R}^n.$$

Hence, we conclude that (5.70) will be satisfied if the constraint set K is any polytope (bounded polyhedral convex set) in $\mathbb{R}^n$.

In the next example, we will consider the constraint that investments can be made only in long-only portfolios and show that there exist time points $s \in (0,T)$ arbitrarily close to t such that for some $\nu \in \mathbb{R}^n$, $\|\nu\| = 1$ and $c \geq 0$, $(s, xe^{c\nu}) \notin \mathcal{O}$. However, the same argument would work if we constrain the investment strategies to take value in any other polytope in $\mathbb{R}^n$.

Example 5.4.4. (Continuation of Example 5.4.3)

Suppose $K = \{p : \sum p_i = 1, p \geq 0\}$. Then $\widetilde{K} = \mathbb{R}^n$. We will show that for any strip $(s_1, s_2) \times \mathbb{R}^n_+$, $0 < s_1 < s_2 \leq T$, there exists $(t, y) \in (s_1, s_2) \times \mathbb{R}^n_+$ such that $(t, y) \notin \mathcal{O}$.

On the contrary, suppose that there exist (s_1, s_2), $0 < s_1 < s_2 \leq T$, such that $(s_1, s_2) \times \mathbb{R}^n_+ \subset \mathcal{O}$. By Example 5.3.4, for any $\nu \in \mathbb{R}^n$, $c \in \mathbb{R}$, $(t, y) \in (0, T) \in \mathbb{R}^n_+$,

$$\mathfrak{I}(\nu, K) = \zeta(\nu) = -\nu_{(1)},$$
$$\mathfrak{I}^{\pi}(c\nu, K, t, y) = -2c\nu' \ln y - c^2\|\nu\|^2 - c\nu_{(1)}.$$

Fix $x \in \mathbb{R}^n_+$ and $\theta > \sup\left\{\ln z(t, x) \;\middle|\; t \in (s_1, s_2)\right\}$. By (??), we see that for any $\nu \in \mathbb{R}^n$ such that $\|\nu\| = 1$, we will have $z(t, xe^{c\nu}) > \theta$ if

$$-2c\nu' \ln x - c^2\|\nu\|^2 - c\nu_{(1)} < A$$
$$\text{i.e. } c^2 + c\left(\nu_{(1)} + 2\nu' \ln x\right) + A > 0$$

where $A = -\ln\theta + \ln z(t, x) < 0$. Denote

$$c(\nu) := \frac{1}{2}\left[-\left(\nu_{(1)} + 2\nu' \ln x\right) + \sqrt{\left(\nu_{(1)} + 2\nu' \ln x\right)^2 - 4A}\right].$$

By an application of Cauchy-Schwartz inequality it follows that,

$$0 < c(\nu) \leq \left|\nu_{(1)} + 2\nu' \ln x\right| + \sqrt{-A}$$

$$\leq |\nu_{(1)}| + 2\|\ln x\| + \sqrt{-A} \leq 1 + 2\|\ln x\| + \sqrt{-A}.$$

Therefore, if we fix $C > 1 + 2\|\ln x\| + \sqrt{-A}$, then for all $\nu \in \mathbb{R}^n$ with $\|\nu\| = 1$, we will have

$$z(t, xe^{C\nu}) > \theta > z(t,x) \geq \inf_{[s_1,s_2)\times\Gamma} z(s,y),\ \nu \in \mathbb{R}^n,\ \|\nu\| = 1,\ t \in (s_1, s_2),$$

where

$$\Gamma := \left\{ xe^{c\nu} \ \middle|\ \nu \in \mathbb{R}^n,\ \|\nu\| = 1,\ 0 \leq c < C \right\},$$

thus contradicting the comparison principle in Theorem 5.1.1 (i) and hence proving our claim.

We now turn our attention to (5.61). We see that if $(t,x) \in \mathcal{O}$, and if $\nu \in \mathbb{R}^n$ is such that

$$\inf_{c>0} \mathfrak{I}^\pi(c\nu, K, t, xe^{-c\nu}) = -\infty,$$

then for some $c > 0$ we will have $z(t, xe^{-c\nu}) < 1$. If $z(t,x) \leq 1$, and if $\mathfrak{I}^\pi(c\nu, K, t, xe^{-c\nu}) < 0$ for some $\nu \in \widetilde{K}$, $c > 0$, then $z(t, xe^{-c\nu}) < 1$. $z(t,x) \leq 1$, $(t,x) \in (0,T)\mathbb{R}^n_+$ if $\pi(t,x) \in K$, $(t,x) \in (0,T)\mathbb{R}^n_+$, a necessary condition for which is that

$$\pi'(t,x)\nu \geq \min_{a \in S_0(K)} a'\nu, \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^n_+. \tag{5.73}$$

On the other hand, for any $(t,x) \in (0,T) \times \mathbb{R}^n_+$, $\mathfrak{I}^\pi(c\nu, K, t, xe^{-c\nu}) < 0$ if and only if

$$c \min_{a \in S_0(K)} a'\nu > \Pi(t,x) - \Pi\left(t, xe^{-c\nu}\right) = c\nu'\pi\left(t, xe^{-b\nu}\right),\ b \in [0,c]. \tag{5.74}$$

The second equality follows by an application of the mean value theorem. (5.73) and (5.74) cannot hold simultaneously. Therefore, it is not possible that that there exist a constraint set K and a strategy π which is of the form (5.28), such that $\pi(\cdot,\cdot) \in K$ on $[0,T] \times \mathbb{R}^n_+$, and $\mathfrak{I}^\pi(c\nu, K, t, xe^{-c\nu}) < 0$ for some $(t,x) \in (0,T) \times \mathbb{R}^n_+$.

We need to look at strategies which are not of the form (5.28).

We will now demonstrate how introduction of suitable constraints K can make $\{t\} \times \mathbb{R}_+^n \nsubseteq \mathcal{O}_K$ for any $t \in (0,T)$. We refer the reader to the discussion after the definition of $\mathcal{O}_K$ in (4.7) for the economic significance of this. Suppose K_1 is a closed convex constraint set such that

$$S_0(K_1) = \emptyset,\ \widetilde{K}_1 \neq \{\mathbf{0}\} \text{ and } (0,T) \times \mathbb{R}_+^n \subseteq \mathcal{O}_{K_1}. \tag{5.75}$$

We want to find a constraint set $K_2 \subseteq K_1$ such that $\mathcal{O}_{K_2} \subsetneq (0,T) \times \mathbb{R}_+^n$. We choose any $\nu \in \widetilde{K}_1, \nu \neq 0$ and $a \in \mathbb{R}^n$ such that $a'\nu < 0$. We define

$$K_2 := a + 0^+ K_1 \cap \{p \in \mathbb{R}^n : p'\nu \leq 0\ \}. \tag{5.76}$$

It follows that $a \in S_0(K_2)$ and $\zeta_{K_2}(\nu) = -\zeta_{K_2}(-\nu) = -a'\nu > 0$. We will show that there does not exist any $t \in (0,T)$ such that $t \times \mathbb{R}_+^n \subseteq \mathcal{O}_{K_2}$. Suppose on the contrary that there exists a $t \in (0,T)$ such that $t \times \mathbb{R}_+^n \subseteq \mathcal{O}_{K_2}$. For any $x \in \mathbb{R}_+^n$ and $c > 0$,

$$\mathfrak{I}(c\nu, K_2, t, x) - \mathfrak{I}(c\nu, K_1, t, x) = \zeta_{K_2}(\nu) = -c\,(a'\nu) \to \infty,\ \text{as } c \to \infty.$$

It now follows from (5.46) that $z_{K_2}(t,x) = \infty$ contradicting our supposition. This proves our claim.

Example 5.4.5. Consider again the market portfolio π. Suppose $K_1 = [0,\infty)^n$. Since $\pi \in K_1$, hence $z_{K_1} \leq 1$ on $[0,T] \times \mathbb{R}_+^n$. Also, it is easily seen that the conditions in (5.75) hold with $\widetilde{K}_1 = [0,\infty)^n$. Suppose we choose $\nu = e_1$ and $a = -e_1$. Defining K_2 as in (5.76) gives us $K_2 = \{-1\} \times [0,\infty)^{n-1}$. It follows from (5.46), as discussed above, that there does not exist any $t \in (0,T)$ such that $t \times \mathbb{R}_+^n \subseteq \mathcal{O}_{K_2}$.

Example 5.4.6. If π denotes the market portfolio, as defined in (2.20), then we have seen that $\pi(t,x) = (\operatorname{diag}(x)\nabla\Pi)\,(t,x)$, where

$$\Pi(t,x) = \ln\left(\sum x_i\right),\quad (t,x) \in [0,T] \times \mathbb{R}_+^n.$$

Therefore,

$$\mathfrak{I}^{\pi}(\ln y - \ln x, K, t, x) = \ln \frac{\sum y_i}{\sum x_i} + \mathfrak{I}(\ln y - \ln x, K). \tag{5.77}$$

(i) If $K = [0,\infty)^n$, then $\widetilde{K} = [0,\infty)^n$. Corollary 5.2.1 implies that $z \leq 1$ on $[0,T] \times \mathbb{R}^n_+$, and hence $\mathcal{O} = (0,T) \times \mathbb{R}^n_+$. Lemma 3.1.1 and Corollary 3.1.2 imply that $\zeta(\nu) = 0$ and hence $\mathfrak{I}(\nu, K) = 0$ for all $\nu \in [0,\infty)^n$. By (5.77), for any $y \geq x \in \mathbb{R}^n_+$,

$$\mathfrak{I}^{\pi}(\ln y - \ln x, K, t, x) = \ln \frac{\sum y_i}{\sum x_i},$$

and hence by (5.60),

$$z\,(t,y) \geq \left(\frac{\sum x_i}{\sum y_i} \right) z\,(t,x)\,, \quad \text{if } y \geq x. \tag{5.78}$$

Therefore, if $x \in \mathbb{R}^n_+$ is such that $z(t,x) < 1$, then $z(t,y) < 1$ for all $\mathbb{R}^n_+ \ni y \leq x$ such that $\sum y_i > z(t,x)\,(\sum x_i)$. In particular, if $z(t,x) < 1$, then $z(t, e^c x) < 1$ for $0 \geq c > \ln z(t,x)$.

(ii) For some $c > 0$, suppose $K = [c,\infty)^n$. Then $\widetilde{K} = [0,\infty)^n$ and $S_0(K) = \{c\mathbf{1}\}$. By (5.52) and (5.59),

$$\mathfrak{I}(\nu, K) = \zeta(\nu) = -c\nu'\mathbf{1} < 0, \text{ for all } \nu \in \widetilde{K}. \tag{5.79}$$

Therefore, from (5.77), for any $y \geq x \in \mathbb{R}^n_+$,

$$\mathfrak{I}^{\pi}(\ln y - \ln x, K, t, x) = \ln \left(\frac{\sum y_i}{\sum x_i} \right) - c \sum_i \left(\ln y_i - \ln x_i \right), \tag{5.80}$$

and hence,

$$z\,(t,y) \geq \left(\frac{\Pi y_i}{\Pi x_i} \right)^c \left(\frac{\sum x_i}{\sum y_i} \right) z(t,x), \quad y \geq x \in \mathbb{R}^n_+. \tag{5.81}$$

Thus, for a fixed $x \in \mathbb{R}^n_+$, we see that

$$z(t,y) > 1, \text{ if } y > x \text{ and } \frac{(\Pi y_i)^c}{\sum y_i} > \frac{(\Pi x_i)^c}{\sum x_i} \frac{1}{z(t,x)},$$

$$\text{and, } z(t,y) < 1, \text{ if } y < x \text{ and } \frac{(\Pi y_i)^c}{\sum y_i} < \frac{(\Pi x_i)^c}{\sum x_i}\frac{1}{z(t,x)}.$$

Now, with $\nu = \ln y - \ln x$, we see from (5.80), (5.60) and (5.61),

$$\mathfrak{I}^\pi(\nu, K, t, x) \geq -c\sum \nu_i + \nu_{(n)},$$
$$z(t, xe^\nu) > 1, \text{ if } \nu > 0 \text{ and } c\sum \nu_i - \nu_{(n)} > -\ln z(t,x),$$
$$\text{and, } z(t, xe^\nu) < 1, \text{ if } \nu < 0 \text{ and } c\sum \nu_i - \nu_{(n)} < -\ln z(t,x).$$

(iii) Suppose $K = [c, 2c]^n$ for some $c > 0$. Then

$$\widetilde{K} = \mathbb{R}^n;\ S_0(K) = \{c\mathbf{1} + cv : v \in \{0,1\}^n\}.$$

By (5.50) and (5.59),

$$\mathfrak{I}(\nu, K) = \zeta(\nu) = -c\sum \nu_i - c\sum (\nu_i \wedge 0),\ \nu \in \mathbb{R}^n, \tag{5.82}$$
$$\mathfrak{I}^\pi(\ln y - \ln x, K, t, x) = \ln\left(\frac{\sum y_i}{\sum x_i}\right) - c\sum \ln\frac{y_i}{x_i} - c\sum \ln\left(\frac{y_i}{x_i} \wedge 1\right) \tag{5.83}$$

By (5.60),

$$z(t,y) \geq \left(\frac{\sum x_i}{\sum y_i}\right)\prod_{i=1}^n \left(\frac{y_i}{x_i}\right)^c \left(\frac{y_i}{x_i} \wedge 1\right)^c z(t,x).$$

(iv) If $K = \{p \in \mathbb{R}^n : p \geq 0, \sum_i p_i = 1\}$, then $\widetilde{K} = \mathbb{R}^n$. Since $\pi \in K$, hence Corollary 5.2.1 implies that $z \leq 1$. From Example 5.3.4, we see that,

$$\mathfrak{I}(\nu, K) = \zeta(\nu) = -\nu_{(1)}, \quad \text{for any } \nu \in \widetilde{K},$$
$$\mathfrak{I}^\pi(\ln y - \ln x, K, t, x) = \ln\left(\frac{\sum y_i}{\sum x_i}\right) - \min_i \ln\left(\frac{y_i}{x_i}\right), \quad x, y \in \mathbb{R}^n_+.$$

Therefore, by (5.60),

$$z(t,y) \geq \left(\min_i \frac{y_i}{x_i}\right)\left(\frac{\sum x_i}{\sum y_i}\right) z(t,x) \quad x, y \in \mathbb{R}^n_+. \tag{5.84}$$

If $z(t,x) < 1,\ (t,x) \in (0,T) \times \mathbb{R}_+^n$, then we will have $z(t,y) < 1$ if

$$\mathfrak{I}^\pi(\ln x - \ln y, K, t, y) < -\ln z(t,x).$$

(v) Suppose that the market is diverse, i.e. there exits a $1 > \delta' > 0$ such that $\pi_{(n)} \leq 1 - \delta'$. For some $0 < \delta < \delta'$, let the constraint set $K = [0, 1-\delta]^n$. Consider a strategy π which satisfies the conditions required for Theorem 4.2.1 to hold, $\pi \in K$ and is equal to the market portfolio on $[0, 1-\delta']^n$. By Corollary 5.2.1, $z \leq 1$ and hence $\mathcal{O} = (0,T) \times \mathbb{R}_+^n$. Also, by definition of δ', z is equal to that corresponding to the market portfolio in this diverse market.

Now,

$$\widetilde{K} = \mathbb{R}^n; \ \ S_0(K) = \{\mathbf{0}, (1-\delta)e_i, i = 1, 2, \ldots, n\}.$$

By (5.50) and (5.59), we get

$$\mathfrak{I}(\nu^0, K) = \zeta(\nu^0) = -(1-\delta)\left(\nu^0_{(1)} \wedge 0\right), \ \ \nu \in \mathbb{R}^n, \tag{5.85}$$

$$\mathfrak{I}^\pi(\ln y - \ln x, K, t, x) = \ln \frac{\sum y_i}{\sum x_i} - \ln \left[\left(\min_i \frac{y_i}{x_i}\right)^{1-\delta} \wedge 1\right], \ \ x, y \in \mathbb{R}_+^n. \tag{5.86}$$

Suppose now that $\mathcal{O}_{sub} = (0,T) \times \mathbb{R}_+^n$, i.e. z^* is a subsolution of $-\mathcal{L}z = 0$ on $(0,T) \times \mathbb{R}_+^n$. By Corollary 5.1.2, it then follows that if $z(t,x) = 1$ for some $x \in \mathbb{R}_+^n$, then $z(t,y) = 1$ for all $y \in \mathbb{R}_+^n$. Therefore, if we can show the existence of $x, y \in \mathbb{R}_+^n$ such that $z(t,y) < z(t,x)$, then that would prove that $z < 1$ on $(0,T) \times \mathbb{R}_+^n$.
We consider any fixed $x \in \mathbb{R}_+^n$ and $\nu = \mathbf{1} \in \mathbb{R}^n$, $y = xe^\nu = ex$. By (5.86), $\mathfrak{I}^\pi(\ln x - \ln y, K, t, y) = -\delta < 0$. From (5.61), it follows now that $z(t,y) < 1$, thus proving that $z < 1$ on $(0,T) \times \mathbb{R}_+^n$.
We have thus proved that in a diverse market satisfying the conditions of Theorem 4.2.1 and conditions which guarantee that $\mathcal{O}_{sub} = (0,T) \times \mathbb{R}_+^n$, there exists relative arbitrage opportunity with respect to the market portfolio over any time horizon $[t,T],\ 0 < t < T$, for $\mathcal{X}(t)$ belonging to an almost sure subset of $\mathbb{R}_+^n$.

Example 5.4.7. If $K = \{p \in \mathbb{R}^n : p_n = 0\}$, which corresponds to prohibition of trading in the n-th stock, then $\widetilde{K} = \{p \in \mathbb{R}^n : p_i = 0, i \neq n\}$, and $\zeta \equiv 0$ on $\widetilde{K}$. Taking $\nu' = (0, \ldots, 0, 1)$, we get that

$$z_*(t, x) = \frac{\sum_{i=1}^{n-1} x_i + 1}{\sum_{i=1}^{n} x_i} z_* \left(t, (x_1, \ldots, x_{n-1}, 1)\right).$$

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