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Essential Electromagnetism: Solutions

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Solutions

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Preface

This book gives the solutions to the exercises at the end of each chapter of my book "Essential Electromagnetism" (also published by Ventus). I recommend that you attempt a particular exercise after reading the relevant chapter, and before looking at the solutions published here. Often there is more than one way to solve a problem, and obviously one should use any valid method that gets the result with the least effort. Usually this means looking for symmetry in the problem – for example from the information given can we say that from symmetry arguments the field we need to derive can only be pointing in a certain direction. If so, we only need to calculate the component of the field in that direction, or we may be able to use Gauss' law or Ampère's law to enable us to write down the result. In some of these exercise solutions the simplest route to the solution is deliberately not taken in order to illustrate other methods of solving a problem, but in these cases the simpler method is pointed out.

The solutions to the exercise problems for Each chapter of "Essential Electromagnetism" are presented here in the corresponding chapters of "Essential Electromagnetism - Solutions".

I hope you find these exercises useful. If you find typos or errors I would appreciate you letting me know. Suggestions for improvement are also welcome – please email them to me at protheroe.essentialphysics@gmail.com.

Raymond J. Protheroe, January 2013
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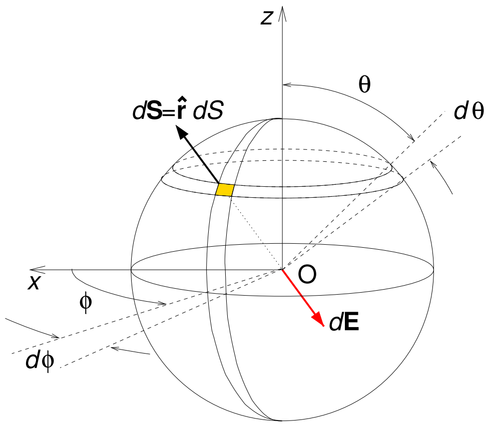
1 Electrostatics

1–1 The surface of a non-conducting sphere of radius a centred on the origin has surface charge density $\sigma(a,\theta,\phi) = \sigma_0 \cos\theta$ and is uniformly filled with charge of density ρ_0. Find the electric field at the origin.

Solution



At the centre of the sphere the electric field due to the volume charge will be zero because the contribution of a volume element located at $\mathbf{r}'$ will be exactly cancelled by that of an equivalent volume element at $-\mathbf{r}'$, so we only need to consider the surface charge.

$$\mathbf{E}(0,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(a,\theta,\phi)}{a^2}(-\widehat{\mathbf{r}})dS, \tag{1.1}$$

$$= \frac{1}{4\pi\varepsilon_0 a^2}\int_0^{2\pi}\int_0^{\pi} \sigma(a,\theta,\phi)(-\widehat{\mathbf{r}})[a^2\sin\theta d\theta d\phi], \tag{1.2}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta(\sigma_0\cos\theta)(-\widehat{\mathbf{r}}). \tag{1.3}$$

Because of the symmetry of the problem, the electric field at the centre can only be in the $\pm z$ direction, and so we only need to find the z-component

$$\mathbf{E}(0,\theta,\phi)\cdot\widehat{\mathbf{z}} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta(\sigma_0\cos\theta)(-\widehat{\mathbf{r}})\cdot\widehat{\mathbf{z}}, \tag{1.4}$$

$$= \frac{1}{4\pi\varepsilon_0}2\pi\int_{-1}^{1} d\cos\theta(\sigma_0\cos\theta)(-\cos\theta), \tag{1.5}$$

$$\therefore \quad \mathbf{E}(0,\theta,\phi)\cdot\widehat{\mathbf{z}} = -\frac{\sigma_0}{2\varepsilon_0}\int_{-1}^{1} d\cos\theta\cos^2\theta \;=\; -\frac{\sigma_0}{3\varepsilon_0}. \tag{1.6}$$

$$\therefore \quad \mathbf{E}(0,\theta,\phi) = -\frac{\sigma_0}{3\varepsilon_0}\,\widehat{\mathbf{z}}. \tag{1.7}$$

1–2 A spherically symmetric charge distribution has the following charge density profile

$$\rho(r,\theta,\phi) = \begin{cases} \rho_0 & (r < a) \\ \rho_0 (r/a)^{-\beta} & (r \geq a) \end{cases} \tag{1.8}$$

where β is a constant $(2 < \beta < 3)$. Find the electric field and electrostatic potential everywhere.

<u>Solution</u>

The charge density is spherically symmetric, with no dependence on θ or ϕ, so the electric field must be in the radial direction and depend only on r. This is the ideal case to exploit Gauss' law in integral form,

$$\oint \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int \rho d^3 r. \tag{1.9}$$

For $r < a$

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho_0, \qquad \therefore \quad \mathbf{E}(\mathbf{r}) = \frac{\rho_0 r}{3\varepsilon_0}\widehat{\mathbf{r}}. \tag{1.10}$$

For $r > a$

$$4\pi r^2 E_r \;=\; \frac{1}{\varepsilon_0}\frac{4}{3}\pi a^3\rho_0 \;+\; \frac{1}{\varepsilon_0}\int_a^r 4\pi (r')^2 \rho_0 a^\beta (r')^{-\beta} dr', \tag{1.11}$$

$$=\; \frac{1}{\varepsilon_0}\frac{4}{3}\pi a^3\rho_0 \;+\; \frac{4\pi\rho_0 a^\beta}{\varepsilon_0}\left[\frac{(r')^{3-\beta}}{3-\beta}\right]_a^r, \tag{1.12}$$

$$\therefore \quad 4\pi r^2 E_r \;=\; \frac{4\pi\rho_0 a^3}{\varepsilon_0}\left(\frac{1}{3} \;+\; \frac{(r/a)^{3-\beta}-1}{3-\beta}\right). \tag{1.13}$$

$$\therefore \quad E_r \;=\; \frac{\rho_0 a^3}{(3-\beta)\varepsilon_0 r^2}\left(\left(\frac{r}{a}\right)^{3-\beta} - \frac{\beta}{3}\right). \tag{1.14}$$

This electric field is due entirely to the charge distribution, and so must be conservative, and we would expect that $\boldsymbol{\nabla} \times \mathbf{E} = 0$ as $\mathbf{E}$ is directed radially outward and so has no circulation. It follows that:

$$\begin{aligned}
V(r \geq a) &= -\frac{\rho_0 a^3}{(3-\beta)\varepsilon_0} \int_\infty^r \left((r')^{1-\beta} a^{\beta-3} - \frac{\beta (r')^{-2}}{3} \right) dr', && (1.15)\\
&= -\frac{\rho_0 a^3}{(3-\beta)\varepsilon_0} \left[\frac{(r')^{2-\beta} a^{\beta-3}}{2-\beta} + \frac{\beta (r')^{-1}}{3} \right]_\infty^r, && (1.16)\\
&= \frac{\rho_0 a^3}{3(3-\beta)(\beta-2)\varepsilon_0} \left(3r^{2-\beta} a^{\beta-3} - \beta(\beta-2) r^{-1} \right). && (1.17)\\
V(r \leq a) &= V(a) - \frac{\rho_0}{3\varepsilon_0} \int_a^r r' \, dr', && (1.18)\\
&= \frac{(3+2\beta-\beta^2)\rho_0 a^2}{3(3-\beta)(\beta-2)\varepsilon_0} - \frac{\rho_0}{3\varepsilon_0} \left[\frac{(r')^2}{2} \right]_a^r, && (1.19)\\
&= \frac{(3+2\beta-\beta^2)\rho_0 a^2}{3(3-\beta)(\beta-2)\varepsilon_0} + \frac{\rho_0}{6\varepsilon_0} (a^2 - r^2). && (1.20)
\end{aligned}$$

1–3 The electric field is given by $\mathbf{E}(\mathbf{r}) = E_0 \cos(z/z_0) \exp(-r/r_0)\,\widehat{\mathbf{r}}$, where z_0 and r_0 are constants. Find the charge density.

Solution

In this problem the electric field is given in terms of z and r. We will need to write $\mathbf{E}$ in terms of either Cartesian or spherical coordinates, and then use Gauss' law in differential form. Choosing spherical coordinates because $\mathbf{E}$ is in the radial direction,

$$\rho(\mathbf{r}) = \varepsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E}, \tag{1.21}$$

$$= \varepsilon_0 E_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \cos\left(\frac{r\cos\theta}{z_0}\right)\exp\left(-\frac{r}{r_0}\right)\right], \tag{1.22}$$

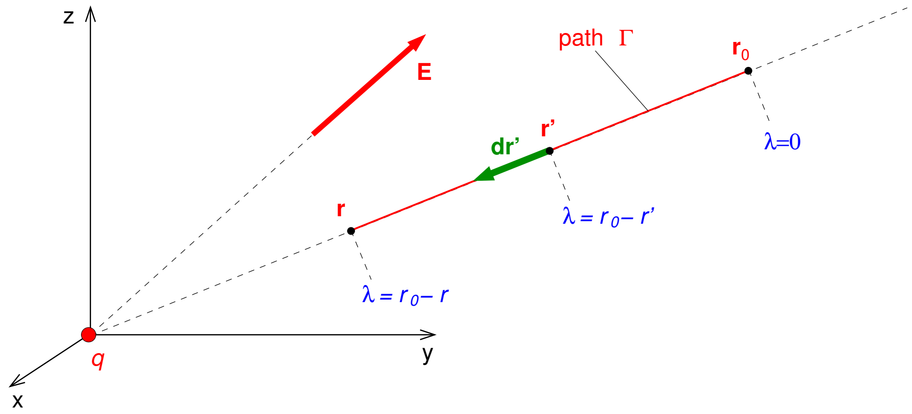
$$\begin{aligned} = \frac{\varepsilon_0 E_0}{r^2}\Bigg[&2r\cos\left(\frac{r\cos\theta}{z_0}\right)\exp\left(-\frac{r}{r_0}\right) - r^2\sin\left(\frac{r\cos\theta}{z_0}\right)\frac{\cos\theta}{z_0}\exp\left(-\frac{r}{r_0}\right) + \\ &r^2\cos\left(\frac{r\cos\theta}{z_0}\right)\exp\left(-\frac{r}{r_0}\right)\left(\frac{-1}{r_0}\right)\Bigg], \end{aligned} \tag{1.23}$$

$$= \frac{\varepsilon_0 E_0}{r}\cos\left(\frac{r\cos\theta}{z_0}\right)\exp\left(-\frac{r}{r_0}\right)\left[2 - \tan\left(\frac{r\cos\theta}{z_0}\right)\frac{r\cos\theta}{z_0} - \frac{r}{r_0}\right], \tag{1.24}$$

$$= \frac{\varepsilon_0 E_0}{r}\cos\left(\frac{z}{z_0}\right)\exp\left(-\frac{r}{r_0}\right)\left[2 - \tan\left(\frac{z}{z_0}\right)\frac{z}{z_0} - \frac{r}{r_0}\right]. \tag{1.25}$$

1–4 If we had a point charge q at the origin we might choose the reference point to be some point at an arbitrary distance r_0 (usually infinity) from the origin. Then if we wish to find $V(r,\theta,\phi)$ it would be convenient to have the reference point at $\mathbf{r}_0 = (r_0,\theta,\phi)$. Although obtaining the potential in this case is trivial, and one would usually just write it down, obtain the potential by carrying out explicitly the line integral for an appropriately parameterised curve.

Solution



We start by parameterising the path from $\mathbf{r}_0$ to $\mathbf{r}$:

$$\mathbf{r}'(\lambda) = (r_0 - \lambda)\,\widehat{\mathbf{r}}; \quad d\mathbf{r}' = -d\lambda\,\widehat{\mathbf{r}}; \quad (0 < \lambda < r_0 - r). \tag{1.26}$$

Then,

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}', \tag{1.27}$$

$$= -\int_{\mathbf{r}(\lambda=0)}^{\mathbf{r}(\lambda=r_0-r)} \mathbf{E}(\mathbf{r}'(\lambda)) \cdot d\mathbf{r}', \tag{1.28}$$

$$= -\int_{\mathbf{r}(\lambda=0)}^{\mathbf{r}(\lambda=r_0-r)} \frac{q}{4\pi\varepsilon_0(r_0-\lambda)^2}\widehat{\mathbf{r}} \cdot (-d\lambda\,\widehat{\mathbf{r}}), \tag{1.29}$$

$$= \int_0^{r_0-r} \frac{q}{4\pi\varepsilon_0(r_0-\lambda)^2} d\lambda, \tag{1.30}$$

$$= \left[\frac{q}{4\pi\varepsilon_0(r_0-\lambda)}\right]_0^{r_0-r}, \tag{1.31}$$

$$= \frac{q}{4\pi\varepsilon_0[r_0-(r_0-r)]} - \frac{q}{4\pi\varepsilon_0(r_0-0)}, \tag{1.32}$$

$$= \frac{q}{4\pi\varepsilon_0 r} - \frac{q}{4\pi\varepsilon_0 r_0}. \tag{1.33}$$

Hence, if we set $r_0 = \infty$ we get the usual potential for a point charge q at the origin

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}. \tag{1.34}$$

1–5 The electric field is given by $\mathbf{E}(\mathbf{r}) = E_0 \cos(z/z_0) \exp(-r/r_0)\,\widehat{\mathbf{r}}$, where z_0 and r_0 are constants. Check whether or not the electric field is conservative. If it is conservative find the potential, if it isn't suggest how it may be possible to find the electrostatic part of the electric field (if present) and the corresponding electrostatic potential $V(\mathbf{r})$.

Solution

First we need to test whether or not the field is purely electrostatic, i.e. whether or not it is conservative. If $\boldsymbol{\nabla} \times \mathbf{E} = 0$ then $\mathbf{E}$ is conservative. First write the field in spherical coordinates

$$\mathbf{E}(r, \theta, \phi) = E_0 \cos(r \cos\theta/z_0) \exp(-r/r_0)\,\widehat{\mathbf{r}} \tag{1.35}$$

and use

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{A} = {} & \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\widehat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r\,A_\phi)\right]\widehat{\boldsymbol{\theta}} \\ & + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\widehat{\boldsymbol{\phi}}.\end{aligned} \tag{1.36}$$

$$\therefore \quad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{r}\frac{\partial E_r}{\partial \theta}\,\widehat{\boldsymbol{\phi}}, \tag{1.37}$$

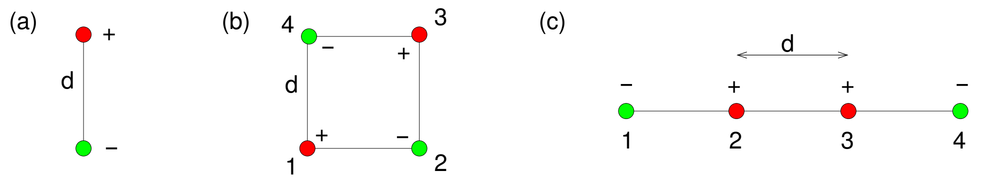
$$= -\frac{E_0}{r}\exp(-r/r_0)\frac{\partial}{\partial \theta}\,\cos(r\cos\theta/z_0)\,\widehat{\boldsymbol{\phi}}, \tag{1.38}$$

$$= -\frac{E_0}{r}\exp(-r/r_0)\,[-\sin(r\cos\theta/z_0)][-r\sin\theta/z_0]\,\widehat{\boldsymbol{\phi}}, \tag{1.39}$$

$$= -\frac{E_0}{z_0}\exp(-r/r_0)\,\sin(r\cos\theta/z_0)\sin\theta\,\widehat{\boldsymbol{\phi}}. \tag{1.40}$$

Since $\boldsymbol{\nabla} \times \mathbf{E} \neq 0$ the electric field is not purely electrostatic. However, from Exercise 1—3 we see that there is a non-zero charge density $\rho(r, \theta, \phi)$, and so there must be an electrostatic component of the electric field. This electrostatic field and potential could be computed from ρ using Coulomb's law.

1–6 How much work must be done to assemble: (a) a physical dipole made of charge $+q$ and charge $-q$ separated by distance d, (b) a physical quadrupole made up of four charges $+q$, $-q$, $+q$ and $-q$ on successive corners of a square of side d, and (c) a physical quadrupole made up of four charges $-q$, $+q$, $+q$ and $-q$ equally spaced apart by distance d on a straight line (see diagram below).



Solution

The work done to bring together a group of N charges is

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i V(\mathbf{r}_i). \tag{1.41}$$

(a)

$$W \;=\; \frac{1}{2}\left((-q)\frac{(+q)}{4\pi\varepsilon_0 d} \;+\; (+q)\frac{(-q)}{4\pi\varepsilon_0 d}\right) \;=\; -\frac{q^2}{4\pi\varepsilon_0 d}. \tag{1.42}$$

(b)

$$V_1 \;=\; V_3 \;=\; \frac{q}{4\pi\varepsilon_0}\left(\frac{-1}{d}+\frac{\sqrt{2}}{d}+\frac{-1}{d}\right) \;=\; \frac{q}{4\pi\varepsilon_0 d}\left(\sqrt{2}-2\right), \tag{1.43}$$

$$\therefore\; q_1V_1 \;=\; q_3V_3 \;=\; \frac{q^2}{4\pi\varepsilon_0 d}\left(\sqrt{2}-2\right). \tag{1.44}$$

$$V_2 \;=\; V_4 \;=\; \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{d}+\frac{-\sqrt{2}}{d}+\frac{1}{d}\right) \;=\; \frac{q}{4\pi\varepsilon_0 d}\left(2-\sqrt{2}\right), \tag{1.45}$$

$$\therefore\; q_2V_2 \;=\; q_4V_4 \;=\; \frac{q^2}{4\pi\varepsilon_0 d}\left(\sqrt{2}-2\right). \tag{1.46}$$

Hence,

$$W \;=\; \frac{1}{2}\times 4\times\frac{q^2}{4\pi\varepsilon_0 d}\left(\sqrt{2}-2\right) \;=\; -\frac{q^2}{2\pi\varepsilon_0 d}\left(2-\sqrt{2}\right). \tag{1.47}$$

(c)

$$V_1 \;=\; V_4 \;=\; \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{d}+\frac{1}{2d}+\frac{-1}{3d}\right) \;=\; \frac{7}{6}\,\frac{q}{4\pi\varepsilon_0 d}, \tag{1.48}$$

$$\therefore\; q_1V_1 \;=\; q_4V_4 \;=\; -\frac{7}{6}\,\frac{q^2}{4\pi\varepsilon_0 d}. \tag{1.49}$$

$$V_2 \;=\; V_3 \;=\; \frac{q}{4\pi\varepsilon_0}\left(\frac{-1}{d}+\frac{1}{d}+\frac{-1}{2d}\right) \;=\; -\frac{1}{2}\,\frac{q}{4\pi\varepsilon_0 d}, \tag{1.50}$$

$$\therefore\; q_2V_2 \;=\; q_3V_3 \;=\; -\frac{1}{2}\,\frac{q^2}{4\pi\varepsilon_0 d}. \tag{1.51}$$

Hence,

$$W \;=\; \frac{1}{2}\times\frac{q^2}{4\pi\varepsilon_0 d}\left(-\frac{7}{6}-\frac{1}{2}-\frac{1}{2}-\frac{7}{6}\right) \;=\; -\frac{10}{3}\,\frac{q^2}{4\pi\varepsilon_0 d}. \tag{1.52}$$

1–7 (a) Use Gauss' law in integral form to find the electric field due to charge density $\rho(\mathbf{r}) = \rho_0 \exp(-r/r_0)$, and (b) check that you obtain the original charge density by taking the divergence of the electric field you find.

Solution

Gauss' law

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}Q_{\text{enc}}, \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{1.53}$$

(a)

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\int_0^r \rho_0 \exp(-r'/r_0) 4\pi (r')^2\, dr', \tag{1.54}$$

$$\therefore\ E_r = \frac{\rho_0}{\varepsilon_0 r^2}\left[r_0\left(-e^{-r'/r_0}\right)\left(2r_0^2 + 2r_0 r' + (r')^2\right)\right]_0^r, \tag{1.55}$$

$$= \frac{\rho_0}{\varepsilon_0 r^2}\left[2r_0^3 - r_0 e^{-r/r_0}\left(2r_0^2 + 2r_0 r + r^2\right)\right]. \tag{1.56}$$

(b)

$$\rho = \varepsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E} \tag{1.57}$$

$$= \varepsilon_0 \frac{1}{r^2}\frac{d}{dr} r^2 \left\{ \frac{\rho_0}{\varepsilon_0 r^2} \left[2r_0^3 - r_0 e^{-r/r_0} \left(2r_0^2 + 2r_0 r + r^2 \right) \right] \right\}, \tag{1.58}$$

$$= \frac{\rho_0}{r^2} \left[-r_0 \left(\frac{-1}{r_0} \right) \left(2r_0^2 + 2r_0 r + r^2 \right) - r_0 \left(2r_0 + 2r \right) \right] e^{-r/r_0}, \tag{1.59}$$

$$= \rho_0 \exp(-r/r_0). \tag{1.60}$$

1–8 An isolated conducting sphere of radius a has net charge Q. Find how much work was done to charge the sphere using two different methods: (a) from the charge on the sphere and its potential, (b) by finding the energy stored in the electric field.

Solution

From Gauss' law

$$E_r(r) = \begin{cases} 0 & (r < a) \\ Q/4\pi\varepsilon_0 r^2 & (r \geq a) \end{cases}, \qquad V(r) = \begin{cases} Q/4\pi\varepsilon_0 a & (r < a) \\ Q/4\pi\varepsilon_0 r & (r \geq a) \end{cases}. \tag{1.61}$$

(a) The work done to bring together a group of N charges, or a continuous charge distribution $\rho(\mathbf{r})$ is

$$W = \frac{1}{2} \int_{\text{all space}} \rho(\mathbf{r}) V(\mathbf{r}).\, d^3 r. \tag{1.62}$$

We can re-write this for a surface charge density

$$W = \frac{1}{2} \int_S \sigma(\mathbf{r}) V(\mathbf{r})\, dS, \tag{1.63}$$

$$= \frac{1}{2} \frac{Q}{4\pi a^2} \frac{Q}{4\pi\varepsilon_0 a} 4\pi a^2, \tag{1.64}$$

$$= \frac{Q^2}{8\pi\varepsilon_0 a}. \tag{1.65}$$

(b) The energy density stored in an electric field is

$$u_E(\mathbf{r}) = \frac{\varepsilon_0}{2} \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}). \tag{1.66}$$

Since the sphere is conducting $\mathbf{E} = 0$ for $r < a$, and so

$$W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 d^3 r, \tag{1.67}$$

$$= \frac{\varepsilon_0}{2} \int_a^\infty \left(\frac{Q}{4\pi\varepsilon_0 r^2} \right)^2 4\pi r^2 \, dr, \tag{1.68}$$

$$= \frac{\varepsilon_0}{2} \frac{Q^2}{(4\pi\varepsilon_0)^2} 4\pi \int_a^\infty r^{-2} \, dr, \tag{1.69}$$

$$= \frac{Q^2}{8\pi\varepsilon_0 a}. \tag{1.70}$$

2 Poisson's and Laplace's equations

2–1 Charge $+q$ is located on the z axis a distance $d/2$ from a grounded plane conductor in the x–y plane. Find how much work was done to bring the charge to its current location using two different approaches: (a) the work done against the electrostatic force if the image charge were real and there was no grounded conductor, (b) the work done against the electrostatic force due to the induced surface charge

$$\sigma(x, y, 0) = \frac{-q}{2\pi} \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \tag{2.1}$$

where z is the height of the charge above the plane.

Solution

(a) The force on charge $+q$ at height $+z$ due to image charge $-q$ at height $-z$ is

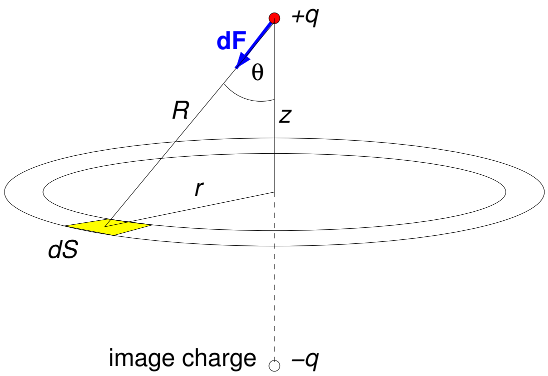
$$\mathbf{F}(z) = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2z)^2}\widehat{\mathbf{z}}. \tag{2.2}$$

$$W(z = d/2) = -\int_{(0,0,\infty)}^{(0,0,d/2)} \mathbf{F} \cdot d\mathbf{r}, \tag{2.3}$$

$$= -\frac{q^2}{16\pi\varepsilon_0}\int_{d/2}^{\infty} z^{-2} dz, \tag{2.4}$$

$$= -\frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 d}. \tag{2.5}$$

$$\tag{2.6}$$



(b) From symmetry arguments, the force on charge $+q$ at $(0, 0, z)$ due to the real surface charge density $\sigma(r)$ will be in the z direction,

$$dF_z(z) = dF(z)\cos\theta, \tag{2.7}$$

$$= \frac{(+q)[\sigma(r)dS]}{4\pi\varepsilon_0 R^2}\,\frac{z}{R}, \tag{2.8}$$

$$= \frac{(+q)}{4\pi\varepsilon_0 R^2}\,\frac{-q}{2\pi}\,\frac{z}{(R^2)^{3/2}}\,\frac{z}{R}\,2\pi r\,dr, \tag{2.9}$$

$$\therefore\quad dF_z(z) = \frac{-q^2 z^2}{4\pi\varepsilon_0 R^6} r\,dr. \tag{2.10}$$

$$F_z(z) = \frac{-q^2}{4\pi\varepsilon_0}\int_0^\infty \frac{z^2 r}{(r^2+z^2)^3}\,dr, \tag{2.11}$$

$$F_z(z) = \frac{-q^2}{4\pi\varepsilon_0}\left[-\frac{z^2}{4(r^2+z^2)^2}\right]_0^\infty, \tag{2.12}$$

$$\therefore\quad \mathbf{F}(z) = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2z)^2}\widehat{\mathbf{z}}. \tag{2.13}$$

This is identical to the force on charge $+q$ at height $+z$ due to image charge $-q$, and so the work done will be identical to that calculated in part (a).

2–2 Charge $+q$ is brought near to two orthogonal grounded conducting planes, one corresponding to the x–z plane and the other to the y–z plane. The charge is located at $(a, b, 0)$. Find the work done in bringing the charge from infinity to its current location (a) by using the method of images to find the potential at the location of the real charge, and (b) by considering the force on the charge as it is brought from infinity.

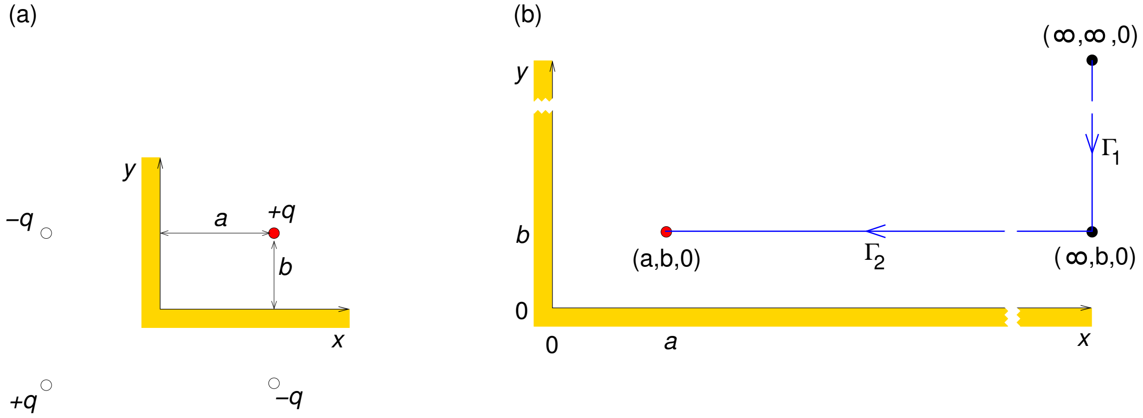
Solution

(a) At the location of charge $+q$ the potential can be calculated as if it were due to the three image charges as in part (a) of the diagram below,

$$V = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{2a} + \frac{1}{[(2a)^2+(2b)^2]^{1/2}} - \frac{1}{2b}\right], \tag{2.14}$$

$$\therefore\quad W = \frac{1}{2}\sum_{i=1}^{N} q_i V(\mathbf{r}_i) \;=\; \frac{q^2}{16\pi\varepsilon_0}\left[-\frac{1}{a} + \frac{1}{(a^2+b)^2]^{1/2}} - \frac{1}{b}\right]. \tag{2.15}$$

Note, it is only the real charge that enters into the sum above.



(b) We first calculate the force on charge $+q$ at its final position due to the induced surface charge on the conductor as if it were due instead to the image charges as in part (a) of diagram above,

$$\mathbf{F}(a,b,z) = \frac{(+q)}{4\pi\varepsilon_0}\left[\frac{-q\widehat{\mathbf{x}}}{(2a)^2} + \frac{+q}{[(2a)^2+(2b)^2]}\left(\frac{a\widehat{\mathbf{x}}}{(a^2+b^2)^{1/2}} + \frac{b\widehat{\mathbf{y}}}{(a^2+b^2)^{1/2}}\right) + \frac{-q\widehat{\mathbf{y}}}{(2b)^2}\right],$$

$$\therefore \quad \mathbf{F}(a,b,z) = \frac{q^2}{16\pi\varepsilon_0}\left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\widehat{\mathbf{x}} + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\widehat{\mathbf{y}}\right]. \qquad (2.16)$$

Similarly, for the charge at some arbitrary position (x, y, z) the force is

$$\mathbf{F}(x,y,z) = \frac{q^2}{16\pi\varepsilon_0}\left[\left(\frac{x}{(x^2+y^2)^{3/2}} - \frac{1}{x^2}\right)\widehat{\mathbf{x}} + \left(\frac{y}{(x^2+y^2)^{3/2}} - \frac{1}{y^2}\right)\widehat{\mathbf{y}}\right]. \tag{2.17}$$

The work done to move a charge from $\mathbf{r}_1 = (\infty, \infty, 0)$ to $\mathbf{r}_2 = (a, b, 0)$ is $W = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}$, and because the electrostatic field is conservative, this is independent of the path taken. For convenience we split the path into two parts as in part (b) of the diagram above. Then

$$W = -\int_{\Gamma_1} \mathbf{F}(\infty, y, 0)\cdot(-dy\,\widehat{\mathbf{y}}) - \int_{\Gamma_2} \mathbf{F}(x, b, 0)\cdot(-dx\,\widehat{\mathbf{x}}), \tag{2.18}$$

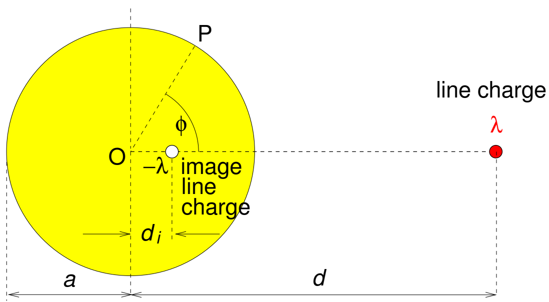
$$= \frac{q^2}{16\pi\varepsilon_0}\left[\int_b^\infty \left(-\frac{1}{y^2}\right) dy + \int_a^\infty \left(\frac{x}{(x^2+b^2)^{3/2}} - \frac{1}{x^2}\right) dx\right], \tag{2.19}$$

$$= \frac{q^2}{16\pi\varepsilon_0}\left\{\left[\frac{1}{y}\right]_b^\infty + \left[-\frac{1}{\sqrt{x^2+b^2}} + \frac{1}{x}\right]_a^\infty\right\}, \tag{2.20}$$

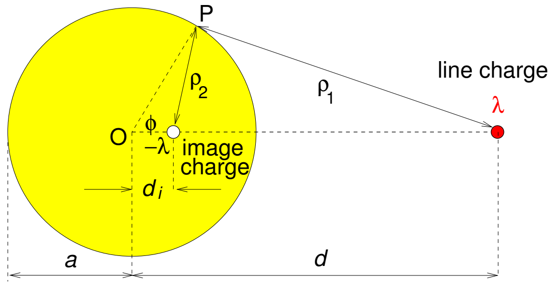
$$= \frac{q^2}{16\pi\varepsilon_0}\left[-\frac{1}{b} + \frac{1}{(a^2+b)^{1/2}} - \frac{1}{a}\right], \tag{2.21}$$

which is the same as found in part (a).

2–3 Show that the potential outside a long conducting cylinder of radius a in the presence of a long parallel line charge $+\lambda$ at distance d is identical to the potential of the line charge and a parallel image line charge $-\lambda$ at distance d_i from the cylinder's axis towards the real line charge (see diagram below). [Hint: draw lines to point P from the two line charges. Use the cosine rule of triangles to write the two distances in terms of a, d_i, d and ϕ and use the formula the for potential due to a line charge, and superposition, to write a formula for the potential at P. Finally require that V does not change if ϕ changes.]



Solution



Using the cosine law:

$$\rho_1^2 = a^2 + d^2 - 2ad\cos\phi, \quad \rho_2^2 = a^2 + d_i^2 - 2ad_i\cos\phi. \tag{2.22}$$

Adding the potentials at P of the real and image line charges,

$$V(a,\phi) = -\frac{1}{2\pi\varepsilon_0}\left[(+\lambda)\ln\rho_1 \,+\, (-\lambda)\ln\rho_2\right], \tag{2.23}$$

$$= -\frac{1}{2\pi\varepsilon_0}\ln\left(\frac{\rho_2}{\rho_1}\right), \tag{2.24}$$

$$= -\frac{1}{2\pi\varepsilon_0}\frac{1}{2}\ln\left(\frac{a^2+d_i^2-2ad_i\cos\phi}{a^2+d^2-2ad\cos\phi}\right). \tag{2.25}$$

For this to be constant on the cylinder's surface, $\partial V/\partial\phi = 0$, i.e.

$$\frac{\partial}{\partial\phi}\left(\frac{a^2+d_i^2-2ad_i\cos\phi}{a^2+d^2-2ad\cos\phi}\right) = 0, \tag{2.26}$$

$$\therefore \quad \frac{2ad_i\sin\phi}{(a^2+d^2-2ad\cos\phi)} - \frac{(a^2+d_i^2-2ad_i\cos\phi)2ad\sin\phi}{(a^2+d^2-2ad\cos\phi)^2} = 0. \tag{2.27}$$

$$\therefore \quad \frac{2a\sin\phi\left[d_i(a^2+d^2-2ad\cos\phi)-(a^2+d_i^2-2ad_i\cos\phi)d\right]}{(a^2+d^2-2ad\cos\phi)^2} = 0. \tag{2.28}$$

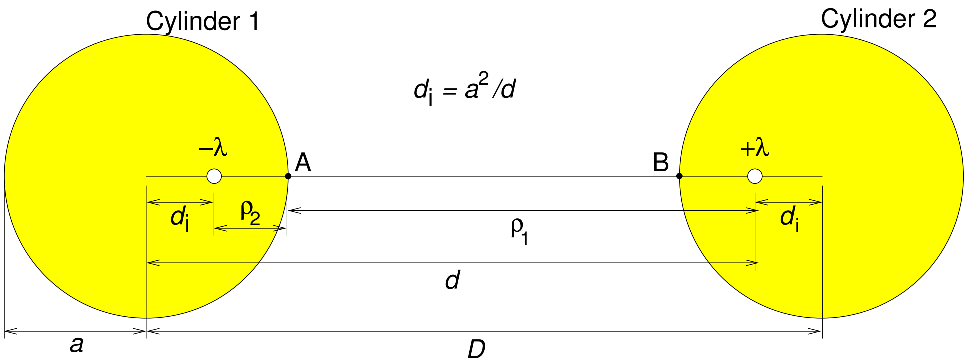
$$\therefore \quad (-d)d_i^2 + (d^2+a^2)d_i + (-a^2d) = 0. \tag{2.29}$$

The solution of this quadratic equation is

$$d_i = a^2/d \quad \text{or} \quad d_i = d. \tag{2.30}$$

The physical solution is $d_i = a^2/d$ as $d_i = d$ corresponds to the image line charge $-\lambda$ being co-located with the real line charge $+\lambda$ — it is nevertheless a solution as $V = 0$ on the cylinder's surface, as well as everywhere else!

2–4 Find the capacitance of a two-wire transmission line comprising two identical parallel cylindrical conductors of radius a whose axes are separated by distance D (see diagram below). You may use the result for the potential due to a line charge near a single cylindrical conductor to find the potential difference by replacing the cylinders by equal but opposite image line charges, $+\lambda$ and $-\lambda$ (C m^{-1}). The capacitance of two conductors with potential difference V and having charge $+q$ on one and $-q$ on the other is $C = q/V$



Solution

The potential at A (and conductor 1 surface) due to the image line charges is,

$$V_A = -\frac{1}{2\pi\varepsilon_0}\left[(+\lambda)\ln\rho_1 \;+\; (-\lambda)\ln\rho_2\right], \tag{2.31}$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\rho_1}{\rho_2}\right), \tag{2.32}$$

$$= \; -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{d-a}{a-d_i}\right) \;=\; -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{d-a}{a-a^2/d}\right) \;=\; -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{d}{a}\right). \tag{2.33}$$

Similarly, the potential at B (and conductor 2 surface) due to image line-charge $-\lambda$ is

$$V_B \;=\; +\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{d}{a}\right). \tag{2.34}$$

Hence the potential difference between the two conductors is

$$V_{BA} \;=\; (V_B - V_A) \;=\; \frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{d}{a}\right). \tag{2.35}$$

But $d = D - d_i$ and $d_i = a^2/d$, so

$$d^2 - D\,d + a^2 \;=\; 0 \quad \text{(quadratic equation)}, \quad \therefore\; d = \frac{1}{2}(D + \sqrt{D^2 - 4a^2}) \tag{2.36}$$

since the other solution, $d = \frac{1}{2}(D - \sqrt{D^2 - 4a^2})$ is discarded because usually $D \gg a$ and $d \gg a$.

Hence the potential difference is

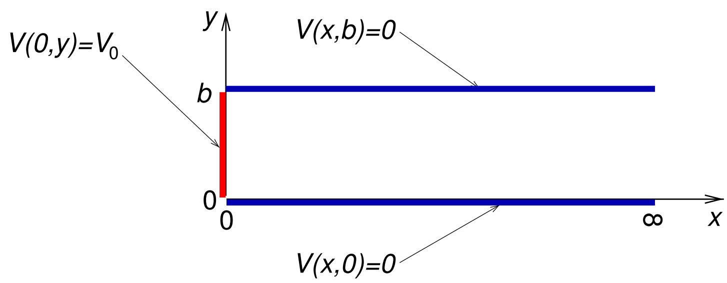
$$V_{BA} = \frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{d}{a}\right) \;=\; \frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{\frac{1}{2}(D + \sqrt{D^2 - 4a^2})}{a}\right). \tag{2.37}$$

By definition the charge per unit length is λ, and the capacitance per unit length (F m^{-1}) is the charge per unit length divided by the potential difference, so that

$$C = \frac{\lambda}{V_{BA}} \;=\; \frac{\pi\varepsilon_0}{\ln\left(\frac{1}{2a}(D + \sqrt{D^2 - 4a^2})\right)} \;\approx\; \frac{\pi\varepsilon_0}{\ln(D/a)}, \tag{2.38}$$

where the approximate result is valid for $D \gg a$.

2–5 A region of space is bounded by three plane conductors as illustrated. Find the potential everywhere between the conductors.



Solution

The potential must be finite at $x = 0$ and drop to zero as $x \to \infty$, so we need the negative exponentials for the functions in x. At $x = 0$ the potential must be zero at $y = 0$ and so we need the sine functions for the functions in y. Furthermore $V(0, b) = 0$ requires $k = n\pi/b$ so that the solution is

$$V(x, y) = \sum_{n=0}^{\infty} A_n \exp\left(-\frac{n\pi}{b}x\right) \sin\left(\frac{n\pi}{b}y\right). \tag{2.39}$$

The boundary conditions at $x = 0$ determine the coefficients A_n in this Fourier sine series

$$\sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{b}y\right) = V_0. \tag{2.40}$$

$$\therefore \quad A_n = \frac{2}{b}\int_0^b V_0 \sin\left(\frac{n\pi}{b}y\right)\, dy, \tag{2.41}$$

$$= \frac{2}{b} V_0 \left[-\frac{b}{n\pi}\cos\left(\frac{n\pi}{b}y\right)\right]_0^b \tag{2.42}$$

$$= \begin{cases} 0 & n \text{ even}, \\ 4V_0/n\pi & n \text{ odd}. \end{cases} \tag{2.43}$$

2–6 Find the potential inside the rectangular region, $0 < x < a$, $0 < y < b$ and $0 < z < c$ with $V(x, y, c) = V_0(x, y)$, and V=0 on the other 5 sides, where

$$V_0(x, y) = V_1 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right). \tag{2.44}$$

Solution

The potential for this case is of the form

$$V\left(x, y, z\right) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_{kl} \sinh\left(\gamma_{kl}\, z\right)\, \sin\left(\alpha_k\, x\right)\, \sin\left(\beta_l\, y\right), \tag{2.45}$$

$$\alpha_k \equiv \frac{k\pi}{a}, \quad \beta_l \equiv \frac{l\pi}{b} \quad \text{and} \quad \gamma_{kl}^2 \equiv \alpha_k^2 + \beta_l^2. \tag{2.46}$$

The potential at $z = c$ may be written

$$V_0(x, y) = V_1 \sin\left(\alpha_1 x\right) \sin\left(\beta_3 y\right) \tag{2.47}$$

and so the coefficients in the series are

$$A_{kl} = \frac{4V_1}{a\, b\, \sinh\left(\gamma_{kl}\, c\right)} \int_0^a \int_0^b \sin\left(\alpha_1 x\right) \sin\left(\alpha_k\, x\right)\, \sin\left(\beta_3 y\right) \sin\left(\beta_l\, y\right)\, dx\, dy, \tag{2.48}$$

and in this case there is only one non-zero coefficient

$$A_{13} = \frac{4V_1}{a\,b\,\sinh(\gamma_{13}\,c)} \int_0^a \sin^2(\alpha_1 x)\;dx \int_0^b \sin^2(\beta_3 y)\;dy, \tag{2.49}$$

$$= \frac{4V_1}{a\,b\,\sinh(\gamma_{13}\,c)} \left[\frac{x}{2} - \frac{\sin(2\alpha_1 x)}{4\alpha_1}\right]_0^a \times \left[\frac{y}{2} - \frac{\sin(2\beta_3 y)}{4\beta_3}\right]_0^b \tag{2.50}$$

$$= \frac{4V_1}{a\,b\,\sinh(\gamma_{13}\,c)} \left[\frac{x}{2} - \frac{\sin(2\pi x/a)}{4\pi/a}\right]_0^a \times \left[\frac{y}{2} - \frac{\sin(6\pi y/b)}{12\pi/b}\right]_0^b \tag{2.51}$$

$$= \frac{4V_1}{a\,b\,\sinh(\gamma_{13}\,c)} \left[\frac{a}{2} - \frac{\sin(2\pi)}{4\pi/a}\right] \times \left[\frac{b}{2} - \frac{\sin(6\pi)}{12\pi/b}\right] \tag{2.52}$$

$$A_{13} = \frac{V_1}{\sinh(\gamma_{13}\,c)}. \tag{2.53}$$

$$\therefore\;\; V(x,y,z) = V_1 \frac{\sinh\left(\sqrt{(\pi/a)^2 + (3\pi/b)^2}\;z\right)}{\sinh\left(\sqrt{(\pi/a)^2 + (3\pi/b)^2}\;c\right)} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right). \tag{2.54}$$

Actually, we could have written down this answer straight away after recognising that $V_0(x,y)$ was the product of one of the allowed functions of x having $\alpha = \alpha_1$ with one of the allowed functions of y having $\beta = \beta_3$, from which we obtain immediately the solution in z with $\gamma = \gamma_{13}$.

2–7 The potential on a non-conducting sphere of radius a is given by

$$V = V_0(3\cos^2\theta + \cos\theta - 1). \tag{2.55}$$

(a) Find the potential and electric field inside the sphere.
(b) Find the potential and electric field outside the sphere.
(c) Find the surface charge density on the sphere as a function of θ.

Solution

(a) Clearly we have spherical symmetry and no dependence on azimuthal coordinate ϕ. The general solution of Laplace's equation with axial symmetry is

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \left(A_\ell\, r^\ell + B_\ell\, r^{-(\ell+1)}\right) P_\ell(\cos\theta). \tag{2.56}$$

Since the potential must be finite as $r \to 0$ we must have $B_\ell = 0$ for all ℓ. Before applying the boundary condition it will simplify our working if we write it in terms of Legendre polynomials

$$V(a,\theta,\phi) = V_0[P_1(\cos\theta) + 2P_2(\cos\theta)]. \tag{2.57}$$

Then, applying the boundary condition,

$$\sum_{\ell=0}^{\infty} A_\ell\, a^\ell P_\ell(\cos\theta) = V_0[P_1(\cos\theta) + 2P_2(\cos\theta)], \tag{2.58}$$

and by equating coefficients of $P_\ell(\cos\theta)$ we see that $A_1 = V_0/a$ and $A_2 = 2V_0/a^2$, giving

$$V^{\text{in}}(r,\theta,\phi) = \left[\frac{r}{a}P_1(\cos\theta) + 2\frac{r^2}{a^2}P_2(\cos\theta)\right]V_0, \tag{2.59}$$

$$\therefore\quad V^{\text{in}}(r,\theta,\phi) = \left[\frac{r}{a}\cos\theta + \frac{r^2}{a^2}(3\cos^2\theta - 1)\right]V_0. \tag{2.60}$$

The electric field is

$$\mathbf{E}^{\text{in}}(\mathbf{r}) = -\left(\frac{\partial V}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\widehat{\boldsymbol{\phi}}\right), \tag{2.61}$$

$$= -\left[\frac{1}{a}\cos\theta + \frac{2r}{a^2}(3\cos^2\theta - 1)\right]V_0\,\widehat{\mathbf{r}} + \left[\frac{1}{a}\sin\theta + \frac{6r}{a^2}\cos\theta\sin\theta\right]V_0\,\widehat{\boldsymbol{\theta}}. \tag{2.62}$$

(b) Since the potential must tend to zero as $r \to \infty$ we must have $A_\ell = 0$ for all ℓ. Again, we write the boundary condition in terms of Legendre polynomials

$$V(a,\theta,\phi) = V_0[P_1(\cos\theta) + 2P_2(\cos\theta)]. \tag{2.63}$$

Then, applying the boundary condition,

$$\sum_{\ell=0}^{\infty} B_\ell\, a^{-(\ell+1)} P_\ell(\cos\theta) = V_0[P_1(\cos\theta) + 2P_2(\cos\theta)], \tag{2.64}$$

we see that $B_1 = a^2 V_0$ and $B_2 = 2a^3 V_0$, giving

$$V^{\text{out}}(r,\theta,\phi) = \left[\frac{a^2}{r^2} P_1(\cos\theta) + 2\frac{a^3}{r^3} P_2(\cos\theta)\right] V_0, \tag{2.65}$$

$$\therefore \quad V^{\text{out}}(r,\theta,\phi) = \left[\frac{a^2}{r^2}\cos\theta + \frac{a^3}{r^3}(3\cos^2\theta - 1)\right] V_0. \tag{2.66}$$

The electric field is

$$\mathbf{E}^{\text{out}}(\mathbf{r}) = -\left(\frac{\partial V}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\widehat{\boldsymbol{\phi}}\right), \tag{2.67}$$

$$= \left[2\frac{a^2}{r^3}\cos\theta + 3\frac{a^3}{r^4}(3\cos^2\theta - 1)\right] V_0 \widehat{\mathbf{r}} + \left[\frac{a^2}{r^3}\sin\theta + 6\frac{a^3}{r^4}\cos\theta\sin\theta\right] V_0 \widehat{\boldsymbol{\theta}}. \tag{2.68}$$

(c) We use Gauss' law in integral form for a small section of the sphere of area δS located at (a,θ,ϕ) inside an infinitesimally thin gaussian pill box having an upper surface of area

δS just outside the sphere and a lower surface just inside the sphere. For the upper surface of the pill box the normal unit vector outwards from the pill box is $\widehat{\mathbf{n}} = \widehat{\mathbf{r}}$, whereas for the lower surface of the pill box the normal unit vector outwards from the pill box is $\widehat{\mathbf{n}} = -\widehat{\mathbf{r}}$. Applying Gauss law in integral form

$$\mathbf{E}^{\text{out}}(a,\theta,\phi)\cdot(\delta S\,\widehat{\mathbf{r}}) + \mathbf{E}^{\text{in}}(a,\theta,\phi)\cdot(-\delta S\,\widehat{\mathbf{r}}) \quad = \quad \sigma(\theta)\delta S/\varepsilon_0. \tag{2.69}$$

Hence,

$$\sigma(\theta) = \varepsilon_0[E_r^{\text{out}}(a,\theta,\phi) - E_r^{\text{in}}(a,\theta,\phi)], \tag{2.70}$$

$$= \varepsilon_0\frac{V_0}{a}\left(\left[2\cos\theta + 3(3\cos^2\theta - 1)\right] + \left[\cos\theta + 2(3\cos^2\theta - 1)\right]\right), \tag{2.71}$$

$$= \varepsilon_0\frac{V_0}{a}\left(3\cos\theta + 15\cos^2\theta - 5\right). \tag{2.72}$$

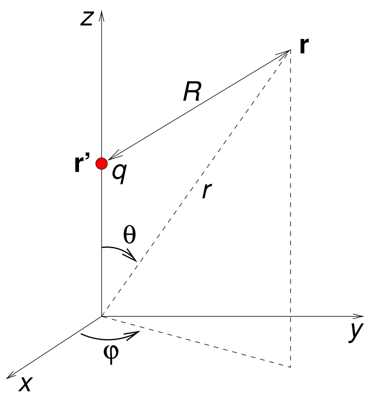
2–8 Consider a point charge on the z-axis at $z = r'$. Find $V(r,\theta,\phi)$ in terms of Legendre polynomials for $r > r'$.

Solution

There is no dependence of V on ϕ, so

$$V(r,\theta,\phi) = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-(n+1)})P_n(\cos\theta). \tag{2.73}$$

The diagram shows the geometry for this problem.



The boundary condition for this problem will be the potential along the z axis for $z > r'$, which we can obtain from Coulomb's law

$$V(r,0,\phi) = \frac{q}{4\pi\varepsilon_0}(r-r')^{-1}, \tag{2.74}$$

$$= \frac{q}{4\pi\varepsilon_0}\, r^{-1}\left[1-\left(\frac{r'}{r}\right)\right]^{-1}, \tag{2.75}$$

$$= \frac{q}{4\pi\varepsilon_0}\, r^{-1}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \quad \text{(binomial series)}. \tag{2.76}$$

On the z-axis $\theta = 0$ so $P_n(\cos\theta) = P_n(1) = 1$, and the general solution for the potential is

$$V(r,0,\phi) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right). \tag{2.77}$$

Hence,

$$\sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1}) = \frac{q}{4\pi\varepsilon_0} r^{-1}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \tag{2.78}$$

giving

$$A_n = 0, \qquad B_n = \frac{(r')^n q}{4\pi\varepsilon_0}, \tag{2.79}$$

$$V(r>r',\theta,\phi) = \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}(r')^n r^{-(n+1)} P_n(\cos\theta). \tag{2.80}$$

2–9 The potential on the surface of a sphere is

$$V(a,\theta,\phi) = V_1\sin\theta\sin\phi + V_2\sin\theta\cos\theta\sin\phi. \tag{2.81}$$

Find the potential inside the sphere.

Solution

The solution is of the form

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell,m}\, r^{\ell} + B_{\ell,m}\, r^{-(\ell+1)}\right) Y_{\ell,m}(\theta,\phi). \tag{2.82}$$

The requirement that $V(0,\theta,\phi)$ is finite gives $B_\ell = 0$ for all ℓ. The boundary condition at $r = a$ can be re-written in terms of spherical harmonics as follows

$$V(a,\theta,\phi) = V_1\sqrt{\frac{2\pi}{3}}\left[Y_{1,1}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right] + V_2\sqrt{\frac{2\pi}{15}}\left[Y_{2,1}(\theta,\phi) - Y_{2,-1}(\theta,\phi)\right]. \tag{2.83}$$

Hence, comparing coefficients we find

$$A_{1,1} = V_1\sqrt{\frac{2\pi}{3}}\,a^{-1}, \qquad A_{1,-1} = -V_1\sqrt{\frac{2\pi}{3}}\,a^{-1}, \tag{2.84}$$

$$A_{2,1} = V_2\sqrt{\frac{2\pi}{15}}\,a^{-2}, \qquad A_{2,-1} = -V_2\sqrt{\frac{2\pi}{15}}\,a^{-2}, \tag{2.85}$$

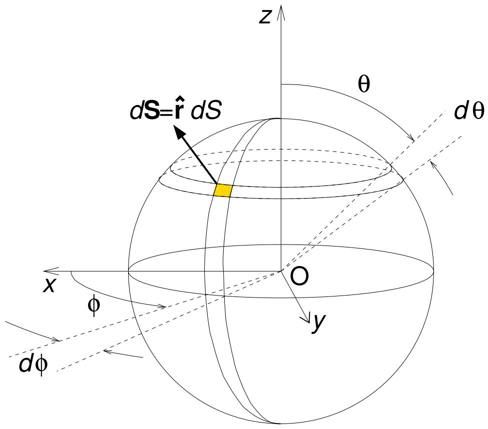
giving

$$\begin{aligned} V(r,\theta,\phi) &= V_1\frac{r}{a}\sqrt{\frac{2\pi}{3}}\left[Y_{1,1}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right] + V_2\frac{r^2}{a^2}\sqrt{\frac{2\pi}{15}}\left[Y_{2,1}(\theta,\phi) - Y_{2,-1}(\theta,\phi)\right], \\ &= V_1\frac{r}{a}\sin\theta\cos\phi + V_2\frac{r^2}{a^2}\sin\theta\cos\theta\cos\phi. \end{aligned} \tag{2.86}$$

3 Multipole expansion for localised charge distribution

3–1 On the surface of a non-conducting sphere of radius a is surface charge density $\sigma(a,\theta,\phi) = \sigma_0 \cos^3\theta$. Find the dipole moment of the sphere.

Solution



For a surface charge density $\sigma(\mathbf{r})$ the dipole moment is

$$\mathbf{p} = \int \sigma(\mathbf{r})\,\mathbf{r}\,dS. \tag{3.1}$$

In this example, the surface charge density depends only on θ and so $\mathbf{p} = p_z\widehat{\mathbf{z}}$ where

$$\begin{aligned}
p_z &= \int \sigma(\mathbf{r})\,z\,dS && (3.2)\\
&= \int_0^\pi \sigma(\theta)\,z\,2\pi a^2 \sin\theta\,d\theta, && (3.3)\\
&= \int_{-1}^1 (\sigma_0\cos^3\theta)\,(a\cos\theta)\,2\pi a^2\,d(\cos\theta), && (3.4)\\
&= 2\pi a^3\sigma_0\left[\frac{\cos^5\theta}{5}\right]_{-1}^1, && (3.5)\\
p_z &= \frac{4}{5}\pi a^3\sigma_0. && (3.6)\\
\therefore\quad \mathbf{p} &= \frac{4}{5}\pi a^3\sigma_0\widehat{\mathbf{z}}. && (3.7)
\end{aligned}$$

3–2 The quadrupole potential is

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int \rho(\mathbf{r}')(r')^2 \frac{1}{2}\left[3(\widehat{\mathbf{r}}\cdot\widehat{\mathbf{r}}')^2 - 1\right] d^3r'. \tag{3.8}$$

Show that it can be written as

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^5}\,\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij}\, r_i r_j \tag{3.9}$$

where the quadrupole moment tensor is

$$Q_{ij} = \int \rho(\mathbf{r}')\left[3r_i' r_j' - \delta_{ij}\,(r')^2\right] d^3r'. \tag{3.10}$$

[This exercise is easy using index notation.]

<u>Solution</u>
The quadrupole potential is

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int \rho(\mathbf{r}')(r')^2 \frac{1}{2}\left[3(\widehat{\mathbf{r}}\cdot\widehat{\mathbf{r}}')^2 - 1\right] d^3r', \tag{3.11}$$

$$= \frac{1}{4\pi\varepsilon_0 r^5}\int \rho(\mathbf{r}') r^2\,(r')^2 \frac{1}{2}\left[3(\widehat{\mathbf{r}}\cdot\widehat{\mathbf{r}}')^2 - 1\right] d^3r', \tag{3.12}$$

$$= \frac{1}{4\pi\varepsilon_0 r^5}\int \rho(\mathbf{r}') \frac{1}{2}\left[3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2\,(r')^2\right] d^3r', \tag{3.13}$$

$$= \frac{1}{4\pi\varepsilon_0 r^5}\int \rho(\mathbf{r}') \frac{1}{2}\left[3\, r_i r_i'\, r_j r_j' - r_i r_i\,(r')^2\right] d^3r', \tag{3.14}$$

$$= \frac{1}{4\pi\varepsilon_0 r^5}\int \rho(\mathbf{r}') r_i r_j \frac{1}{2}\left[3r_i' r_j' - \delta_{ij}\,(r')^2\right] d^3r', \tag{3.15}$$

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^5}\,\frac{1}{2}\, Q_{ij}\, r_i r_j. \tag{3.16}$$

Index notation and the Einstein summation convention has been used above, but writing the summation explicitly we have

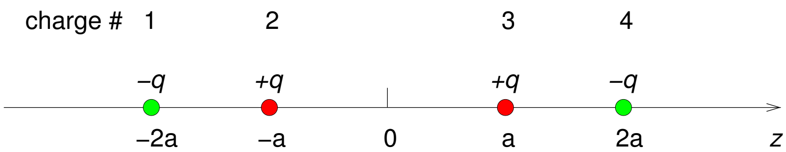
$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^5}\,\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij}\, r_i r_j \tag{3.17}$$

where

$$Q_{ij} = \int \rho(\mathbf{r}') \left[3r_i' r_j' - \delta_{ij} (r')^2\right] d^3r'. \tag{3.18}$$

3–3 A physical quadrupole is made up of four charges lined up along the z axis: -q_0 at (0,0,$-2a$), +q_0 at (0,0,$-a$), +q_0 at (0,0,a) and -q_0 at (0,0,$2a$). (a) Obtain the quadrupole moment. (b) Find the potential at $\mathbf{r} = (b, b, 0)$ for $b \gg a$.

Solution



(a) The quadrupole moment tensor is

$$Q_{ij} = \int \rho(\mathbf{r}') \left[3r_i' r_j' - \delta_{ij} (r')^2\right] d^3r'. \tag{3.19}$$

For N point charges this becomes

$$Q_{ij} = \sum_{k=1}^{N} q_k \left[3 r_i^{[k]} r_j^{[k]} - \delta_{ij}\,(r^{[k]})^2\right] \tag{3.20}$$

where $\mathbf{r}_k = (r_1^{[k]}, r_2^{[k]}, r_3^{[k]})$ is postion of charge q_k.
Since $r_1^{[k]} = r_2^{[k]} = 0$ for all four charges as they are on the z axis, the quadrupole moment tensor is diagonal with only Q_{11}, Q_{22} and Q_{33} being non-zero,

$$Q_{11} = \sum_{k=1}^{4} q_k \left[3 \times 0 \times 0 - \delta_{11}\,(r^{[k]})^2\right], \tag{3.21}$$

$$= (-q)\left[-(-2a)^2\right] \;+\; (+q)\left[-(-a)^2\right] \;+\; (+q)\left[-(a)^2\right] \;+\; (-q)[-(2a)^2], \tag{3.22}$$

$$\therefore \quad Q_{11} = 6qa^2. \tag{3.23}$$

Similarly,

$$Q_{22} = 6qa^2. \tag{3.24}$$

$$Q_{33} = \sum_{k=1}^{4} q_k \left[3 \times r_3^{[k]} \times r_3^{[k]} - \delta_{33}\,(r^{[k]})^2\right], \tag{3.25}$$

$$= (-q)\left[3(-2a)^2 - (2a)^2\right] \;+\; (+q)\left[3(-a)^2 - (a)^2\right] \;+\; (+q)\left[3(a)^2 - (a)^2\right] \;+\; (-q)\left[3(2a)^2 - (2a)^2\right], \tag{3.26}$$

$$\therefore \quad Q_{33} = -12qa^2. \tag{3.27}$$

$$\therefore \quad Q_{ij} \;=\; \begin{bmatrix} 6qa^2 & 0 & 0 \\ 0 & 6qa^2 & 0 \\ 0 & 0 & -12qa^2 \end{bmatrix}. \tag{3.28}$$

(b) The potential for $r \gg a$ can be approximated by the quadrupole potential

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r^5}\,\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} Q_{ij}\, r_i r_j. \tag{3.29}$$

Only Q_{11}, Q_{22} and Q_{33} are non-zero, and for $\mathbf{r} = (b, b, 0)$ the distance from the origin is

$r = \sqrt{2}b$, so that

$$V_{\rm quad}(b,b,0) = \frac{1}{4\pi\varepsilon_0(\sqrt{2}b)^5}\,\frac{1}{2}\,\left[6qa^2 \times b \times b \;+\; 6qa^2 \times b \times b \;-\; 12qa^2 \times 0 \times 0\right], \tag{3.30}$$

$$= \frac{3qa^2}{8\pi\sqrt{2}\varepsilon_0 b^3}. \tag{3.31}$$

3–4 Charge $-q$ is located at the origin and charge $+q$ is located at $(x,y,z) = (a\sin\theta_0\cos\phi_0, a\sin\theta_0\sin\phi_0, a\cos\theta_0)$.

(a) Find the the non-zero moments of the multipole expansion of the potential in Cartesian coordinates, i.e. q, $\mathbf{p}$, Q_{ij} (if non-zero), and use these moments in the multipole expansion in Cartesian coordinates to find the potential at $(x,y,z) = (r\sin\theta\,\cos\phi, r\sin\theta\,\sin\phi, r\cos\theta)$ where $r \gg a$.

(b) Find the non-zero moments of the multipole expansion of the potential in spherical coordinates, i.e.

$$q_{\ell m} = \int Y^*_{\ell m}(\theta',\phi')r'^{\ell}\rho(\mathbf{r}')d^3r', \tag{3.32}$$

and use these moments in the multipole expansion in spherical coordinates to find the potential at (r,θ,ϕ) where $r \gg a$. Compare the result with that from part (a).

Solution

The net charge (monopole moment) is zero. There are two equal but opposite charges and so we have an electric dipole moment, and no higher moments.

(a) In Cartesian coordinates

$$\mathbf{p} = (-q)(0,0,0) + (+q)(a\sin\theta_0\cos\phi_0, a\sin\theta_0\sin\phi_0, a\cos\theta_0), \tag{3.33}$$

$$= p_0 \times (\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0), \tag{3.34}$$

where $p_0 = qa$. The potential at $(x,y,z) = (r\sin\theta\,\cos\phi, r\sin\theta\,\sin\phi, r\cos\theta)$ where $r \gg a$

is

$$
\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0 r^2}\mathbf{p}\cdot\widehat{\mathbf{r}}, && (3.35)\\
&= \frac{p_0}{4\pi\varepsilon_0 r^2}(\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0)\cdot(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), && (3.36)\\
&= \frac{p_0}{4\pi\varepsilon_0 r^2}\left(\sin\theta_0\cos\phi_0\sin\theta\cos\phi + \sin\theta_0\sin\phi_0\sin\theta\ \sin\phi + \cos\theta_0\cos\theta\right), && (3.37)\\
&= \frac{p_0}{4\pi\varepsilon_0 r^2}\left[\sin\theta_0\sin\theta\left(\cos\phi_0\cos\phi + \sin\phi_0\sin\phi\right) + \cos\theta_0\cos\theta\right], && (3.38)\\
&= \frac{p_0}{4\pi\varepsilon_0 r^2}\left[\sin\theta_0\sin\theta\cos(\phi_0 - \phi) + \cos\theta_0\cos\theta\right]. && (3.39)
\end{aligned}
$$

(a) In spherical coordinates the multipole moments are given by

$$
q_{\ell,m} = \int Y^*_{\ell,m}(\theta',\phi')r'^{\ell}\rho(\mathbf{r}')d^3r'. \qquad (3.40)
$$

Writing the charge density using Dirac delta functions in spherical coordinates we have

$$\rho(\mathbf{r}) = -q\delta(\mathbf{r}) + q\frac{\delta(r-a)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta}. \tag{3.41}$$

There will only be dipole ($\ell = 1$) multipole moments

$$q_{1-1} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\, r\, \frac{q\delta(r-a)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta} r^2\sin\theta dr\, d\theta\, d\phi,$$

$$\therefore \quad q_{1-1} = aq\sqrt{\frac{3}{8\pi}}\sin\theta_0 e^{i\phi_0}. \tag{3.42}$$

$$q_{10} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sqrt{\frac{3}{4\pi}}\cos\theta\, r\, \frac{q\delta(r-a)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta} r^2\sin\theta dr\, d\theta\, d\phi,$$

$$\therefore \quad q_{10} = aq\sqrt{\frac{3}{4\pi}}\cos\theta_0. \tag{3.43}$$

$$q_{11} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} -\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\, r\, \frac{q\delta(r-a)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta} r^2\sin\theta dr\, d\theta\, d\phi,$$

$$\therefore \quad q_{11} = -aq\sqrt{\frac{3}{8\pi}}\sin\theta_0 e^{-i\phi_0}. \tag{3.44}$$

The potential for $r \gg a$ will then be the same as in part (a),

$$V(r,\theta,\phi) = \frac{1}{\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{q_{\ell,m}}{2\ell+1}r^{-(\ell+1)}Y_{\ell,m}(\theta,\phi), \tag{3.45}$$

$$= \frac{aq}{3\varepsilon_0 r^2}\left[\sqrt{\frac{3}{8\pi}}\sin\theta_0 e^{i\phi_0}\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} + \sqrt{\frac{3}{4\pi}}\cos\theta_0\sqrt{\frac{3}{4\pi}}\cos\theta \right.$$
$$\left. + \sqrt{\frac{3}{8\pi}}\sin\theta_0 e^{-i\phi_0}\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right], \tag{3.46}$$

$$= \frac{aq}{3\varepsilon_0 r^2}\left[\frac{3}{8\pi}\sin\theta_0\sin\theta e^{i(\phi_0-\phi)} + \frac{3}{4\pi}\cos\theta_0\cos\theta \right.$$
$$\left. + \frac{3}{8\pi}\sin\theta_0\sin\theta e^{-i(\phi_0-\phi)}\right], \tag{3.47}$$

$$= \frac{aq}{4\pi\varepsilon_0 r^2}\left[\sin\theta_0\sin\theta\cos(\phi_0-\phi) + \cos\theta_0\cos\theta\right]. \tag{3.48}$$

4 Macroscopic and microscopic dielectric theory

4–1 A dielectric sphere (dielectric constant K) of radius a is placed in an initially uniform electric field $\mathbf{E}_0$. (a) What are the boundary conditions on V, $\mathbf{E}$ and $\mathbf{D}$ for this problem. (b) Find the potential everywhere. (c) Find $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{P}$ everywhere. (d) Find the dipole moment of the sphere and the surface polarisation charge density.

Solution

(a) The boundary conditions at the surface of the sphere are that $E_{\|}$, $D_{\perp}$ and V are continuous across the boundary. In addition the electric field very far from the sphere must equal the initial field. Defining this to be in the z-direction,

$$\mathbf{E}(r \gg a, \theta, \phi) = E_0\widehat{\mathbf{z}}, \tag{4.1}$$

$$\therefore \quad V(r \gg a, \theta, \phi) = -E_0 z \;=\; -E_0 r\cos\theta \;=\; -E_0 r P_1(\cos\theta). \tag{4.2}$$

Since the potential has not been specified anywhere, we are free for convenience to set $V(0,\theta,\phi) = 0$.

(b) This is a problem with spherical symmetry but with no dependence on ϕ. Hence, we can write down the form of the potential

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\left(A_\ell\, r^\ell + B_\ell\, r^{-(\ell+1)}\right) P_\ell(\cos\theta). \tag{4.3}$$

So that V_{in} is finite inside the sphere (containing $r = 0$), we must have all $B_\ell^{\text{in}} = 0$. Similarly, so that V_{out} is finite outside the sphere we must have all $A_\ell^{\text{out}} = 0$, except as needed to give $V(r \gg a, \theta, \phi) = -E_0 r P_1(\cos\theta)$, i.e. $A_1^{\text{out}} = -E_0$. Hence

$$V_{\text{in}}(r,\theta,\phi) = \sum_{\ell=0}^{\infty} A_\ell^{\text{in}}\, r^\ell P_\ell(\cos\theta), \tag{4.4}$$

$$V_{\text{out}}(r,\theta,\phi) = -E_0 r^1 P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell^{\text{out}}\, r^{-(\ell+1)} P_\ell(\cos\theta). \tag{4.5}$$

Applying the boundary condition on V at $r = a$, and remembering that we set $V(0,\theta,\phi) =$

0,

$$\sum_{\ell=0}^{\infty} A_\ell^{\text{in}}\, a^\ell P_\ell(\cos\theta) = -E_0 a^1 P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell^{\text{out}}\, a^{-(\ell+1)} P_\ell(\cos\theta), \tag{4.6}$$

$$\therefore \quad A_1^{\text{in}} a = -E_0 a + B_1^{\text{out}} a^{-2}. \tag{4.7}$$

with all other coefficients zero. Hence, we can now write the form of the solution as

$$V_{\text{in}}(r,\theta,\phi) = A_1^{\text{in}} r\cos\theta, \tag{4.8}$$

$$V_{\text{out}}(r,\theta,\phi) = -E_0 r\cos\theta + B_1^{\text{out}} r^{-2}\cos\theta. \tag{4.9}$$

Next we apply the boundary conditions on $\mathbf{E}$ and $\mathbf{D}$ at $r = a$,

$$\mathbf{E}_\parallel = -\frac{1}{r}\frac{\partial V}{\partial\theta}\bigg|_{r=a} \widehat{\boldsymbol{\theta}}, \tag{4.10}$$

$$\therefore \quad A_1^{\text{in}} \sin\theta = -E_0 \sin\theta + B_1^{\text{out}} a^{-3} \sin\theta. \tag{4.11}$$

$$\mathbf{D}_{\perp} = -\varepsilon \left.\frac{\partial V}{\partial r}\right|_{r=a} \widehat{\mathbf{r}}, \tag{4.12}$$

$$\therefore \quad -\varepsilon A_1^{\mathrm{in}} \cos\theta = \varepsilon_0 E_0 \cos\theta + 2\varepsilon_0 B_1^{\mathrm{out}} a^{-3} \cos\theta. \tag{4.13}$$

Equations 4.7, 4.11 and 4.13 are three equations in two unknowns, but we only need two equations. Solving Eqs. 4.11 and 4.13 gives

$$B_1^{\mathrm{out}} = E_0 a^3 \frac{(\varepsilon - \varepsilon_0)}{(\varepsilon + 2\varepsilon_0)}, \quad A_1^{\mathrm{in}} = -E_0 \frac{3\varepsilon_0}{(\varepsilon + 2\varepsilon_0)}. \tag{4.14}$$

Hence,

$$V_{\mathrm{in}}(r,\theta,\phi) = -E_0 \frac{3\varepsilon_0}{(\varepsilon + 2\varepsilon_0)} r\cos\theta = -E_0 \frac{3\varepsilon_0}{(\varepsilon + 2\varepsilon_0)} z, \tag{4.15}$$

$$V_{\mathrm{out}}(r,\theta,\phi) = -E_0 r\cos\theta + E_0 a^3 \frac{(\varepsilon - \varepsilon_0)}{(\varepsilon + 2\varepsilon_0)} r^{-2} \cos\theta. \tag{4.16}$$

(c) The electric field is $\mathbf{E} = -\boldsymbol{\nabla} V$ and the displacement field is $\mathbf{D} = \varepsilon\mathbf{E}$

$$\mathbf{E}(r,\theta,\phi) = -\frac{\partial V}{\partial r}\widehat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial\theta}\widehat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\widehat{\boldsymbol{\phi}}. \tag{4.17}$$

$$\therefore \quad \mathbf{E}_{\mathrm{in}} = \frac{3\varepsilon_0}{(\varepsilon + 2\varepsilon_0)}\mathbf{E}_0, \qquad \mathbf{D}_{\mathrm{in}} = \varepsilon\mathbf{E}_{\mathrm{in}}. \tag{4.18}$$

$$\mathbf{E}_{\mathrm{out}} = \mathbf{E}_0 + \frac{(\varepsilon - \varepsilon_0)}{(\varepsilon + 2\varepsilon_0)} a^3 E_0 \left[2\cos\theta\,\widehat{\mathbf{r}} + \sin\theta\,\widehat{\boldsymbol{\theta}}\right] r^{-3}, \quad \mathbf{D}_{\mathrm{out}} = \varepsilon_0\mathbf{E}_{\mathrm{out}}. \tag{4.19}$$

The polarisation field inside the dielectric is obtained from $\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E}$ giving

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E} = \frac{3(\varepsilon - \varepsilon_0)\varepsilon_0}{(\varepsilon + 2\varepsilon_0)}\mathbf{E}_0. \tag{4.20}$$

(d) Since the polarisation field is uniform the dipole moment is

$$\mathbf{p} = \frac{4}{3}\pi a^3 \mathbf{P} = \frac{4}{3}\pi a^3 \frac{3(\varepsilon - \varepsilon_0)\varepsilon_0}{(\varepsilon + 2\varepsilon_0)}\mathbf{E}_0. \tag{4.21}$$

The surface polarisation charge density on the sphere is

$$\sigma_p(a,\theta,\phi) = \mathbf{P}\cdot\widehat{\mathbf{n}} = \frac{3(\varepsilon-\varepsilon_0)\varepsilon_0}{(\varepsilon+2\varepsilon_0)}E_0\cos\theta. \tag{4.22}$$

4–2 An electret, i.e. a piece of material with a permanent electric polarisation, is in the shape of a sphere of radius a and has $\mathbf{P}(\mathbf{r}) = \mathbf{P}_0$. (a) Find the surface polarisation charge density and the dipole moment of the sphere, (b) find V, $\mathbf{E}$, and $\mathbf{D}$ everywhere, and (c) sketch the field lines of $\mathbf{E}$ and $\mathbf{D}$.

Solution

(a) We are free to choose the sphere to be polarised in the $\widehat{\mathbf{z}}$ direction. Then

$$\sigma_{\text{pol}}(a,\theta,\phi) = \mathbf{P}\cdot\widehat{\mathbf{n}} = (P_0\,\widehat{\mathbf{z}})\cdot\widehat{\mathbf{r}} = P_0\cos\theta = P_0\,P_1(\cos\theta). \tag{4.23}$$

As the polarisation is uniform, we can obtain the dipole moment directly from $\mathbf{P}$ and the sphere's volume

$$\mathbf{p} = \frac{4}{3}\pi a^3 P_0\,\widehat{\mathbf{z}}. \tag{4.24}$$

(b) We first need to write down the form of the solution for the potential (inside and outside the sphere), and then apply the boundary conditions to fix the coefficients in the series for V.

The form of the solution for the potential is

$$V(r,\theta,\phi) = \sum_{\ell=0}^{\infty}(A_\ell\, r^\ell + B_\ell\, r^{-(\ell+1)})P_\ell(\cos\theta). \tag{4.25}$$

Examining the angular dependence of σ_{pol} we realise that the solution will only involve terms with $\ell \leq 1$, then

$$V^{\text{in}}(r,\theta,\phi) = A_0 + A_1 r\cos\theta, \tag{4.26}$$

$$V^{\text{out}}(r,\theta,\phi) = B_0 r^{-1} + B_1 r^{-2}\cos\theta. \tag{4.27}$$

The potential must be continuous ar $r = a$ so that

$$A_0 = B_0 a^{-1}, \qquad A_1 = B_1 a^{-3}. \tag{4.28}$$

Gauss' law can be used to provide a boundary condition on $\mathbf{E}$, and for this we will need the normal (in this case radial) components of the electric field, $E_r = -\partial V / \partial r$,

$$E_r^{\text{in}}(r,\theta,\phi) = -A_1 \cos\theta, \tag{4.29}$$

$$E_r^{\text{out}}(r,\theta,\phi) = B_0 r^{-2} + 2B_1 r^{-3} \cos\theta. \tag{4.30}$$

Gauss' law applied to the a pillbox spanning $r = a$ at (a,θ,ϕ) is then,

$$E_r^{\text{out}}(a,\theta,\phi) - E_r^{\text{in}}(a,\theta,\phi) = \sigma_{\text{pol}}(a,\theta,\phi)/\varepsilon_o, \tag{4.31}$$

$$B_0 a^{-2} + 2B_1 a^{-3} \cos\theta + A_1 \cos\theta = P_0 \cos\theta/\varepsilon_0. \tag{4.32}$$

From this we see that $B_0 = 0$, and then from Eq. 4.28 that $A_0 = 0$, and that

$$A_1 = P_0/3\varepsilon_0, \qquad B_1 = a^3 P_0/3\varepsilon_0. \tag{4.33}$$

Hence, the potential is

$$V^{\text{in}}(r,\theta,\phi) = \frac{P_0}{3\varepsilon_0} r\cos\theta = \frac{P_0}{3\varepsilon_0} z, \tag{4.34}$$

$$V^{\text{out}}(r,\theta,\phi) = \frac{a^3 P_0}{3\varepsilon_0} r^{-2}\cos\theta. \tag{4.35}$$

The electric field is

$$\mathbf{E}(r,\theta,\phi) = -\frac{\partial V}{\partial r}\,\widehat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\,\widehat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\widehat{\boldsymbol{\phi}}. \tag{4.36}$$

$$\therefore\ \mathbf{E}^{\text{in}}(r,\theta,\phi) = -\frac{P_0}{3\varepsilon_0}\,\widehat{\mathbf{z}} = \frac{P_0}{3\varepsilon_0}\left(-\cos\theta\,\widehat{\mathbf{r}} + \sin\theta\,\widehat{\boldsymbol{\theta}}\right), \tag{4.37}$$

$$\mathbf{E}^{\text{out}}(r,\theta,\phi) = \frac{a^3 P_0}{3\varepsilon_0} r^{-3}\left(2\cos\theta\,\widehat{\mathbf{r}} + \sin\theta\,\widehat{\boldsymbol{\theta}}\right). \tag{4.38}$$

The electric potential and field outside the sphere is identical to that of a dipole

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\widehat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}, \tag{4.39}$$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\widehat{\mathbf{r}} + \sin\theta\,\widehat{\boldsymbol{\theta}}\right). \tag{4.40}$$

and is consistent with $\mathbf{p}$ calculated earlier from $\mathbf{P}_0$ and the volume.

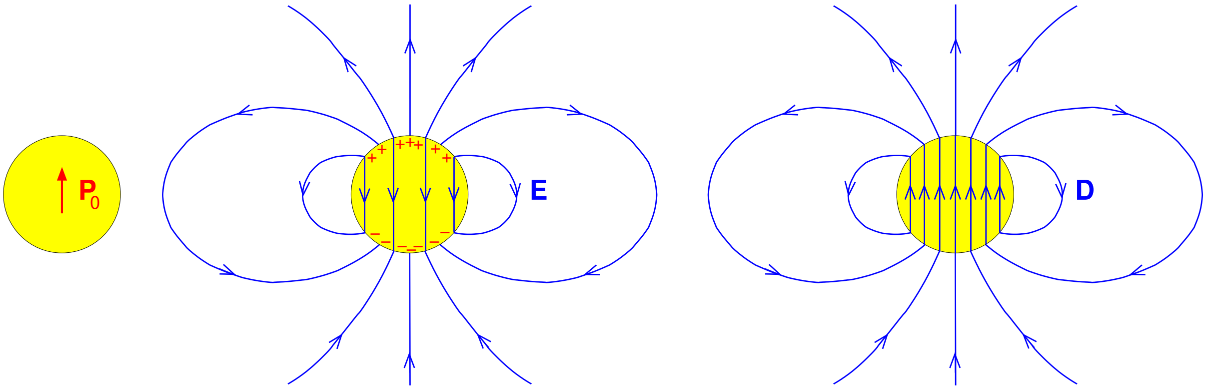
Finally, the displacement field is given by $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$,

$$\mathbf{D}^{\text{in}}(r,\theta,\phi) = \frac{2}{3}\mathbf{P}_0, \tag{4.41}$$

$$\mathbf{D}^{\text{out}}(r,\theta,\phi) = \frac{a^3 P_0}{3} r^{-3}\left(2\cos\theta\,\widehat{\mathbf{r}} + \sin\theta\,\widehat{\boldsymbol{\theta}}\right). \tag{4.42}$$

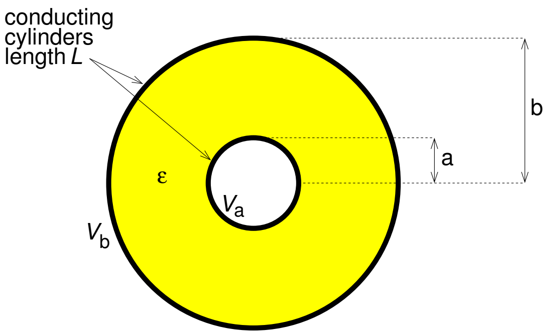
(c) To sketch the electric field lines we notice that inside the sphere $\mathbf{E}$ is constant and in the $-\widehat{\mathbf{z}}$ direction, and that at the poles $|E^{\text{in}}| = |E^{\text{out}}|/2$, and that outside the sphere it has a dipole field. Also, electric field lines start on positive charge (either free or polarisation charge) and end on negative charge (free or polarisation charge).

To sketch the displacement field we notice that inside the sphere $\mathbf{D}$ is constant and in the $+\widehat{\mathbf{z}}$ direction, at the poles that $|D^{\text{in}}| = |D^{\text{out}}|$, and that outside the sphere it has a dipole field. Also, since there is no free charge present the field lines of $\mathbf{D}$ must form closed loops.



4–3 The space between two concentric conducting cylinders of radius a and $b > a$ and length $L \gg b$ is filled with a dielectric with permittivity ε. The inner and outer conductors are held at potentials V_a and V_b, respectively. Find: (a) $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{P}$ everywhere; (b) the polarisation surface and volume charge density everywhere, and the net polarisation charge; (c) the free charge on the inner and outer conductors, and the capacitance.

<u>Solution</u>



(a) We start by solving Laplace's equation in cylindrical coordinates with no dependence on ϕ and z

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0. \tag{4.43}$$

Integrating, we get

$$\rho\frac{dV}{d\rho} = A, \qquad \int dV = A\int\frac{d\rho}{\rho}, \qquad \therefore\ \ V = A\ln\rho + B. \tag{4.44}$$

A and B are integration constants to be determined from the boundary conditions at $\rho = a$ and $\rho = b$,

$$V_a = A \ln a + B, \quad V_b = A \ln b + B. \tag{4.45}$$

Solving for A and B,

$$A = \frac{(V_b - V_a)}{\ln(b/a)}, \quad B = V_a + \frac{(V_a - V_b)}{\ln(b/a)} \ln a. \tag{4.46}$$

$$\therefore \quad V(\rho) = V_a + (V_b - V_a)\frac{\ln(\rho/a)}{\ln(b/a)}. \tag{4.47}$$

The electric field will be present only between the inner and outer conductors

$$\mathbf{E}(\rho) \;=\; -\frac{dV}{d\rho}\widehat{\boldsymbol{\rho}} \;=\; -\frac{(V_b - V_a)}{\ln(b/a)}\frac{1}{\rho}\widehat{\boldsymbol{\rho}}. \tag{4.48}$$

Since the dielectric is linear the displacement and polarisation fields, again only between the inner and outer conductors, are

$$\mathbf{D} = \varepsilon\mathbf{E} \;=\; \varepsilon\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{\rho}\widehat{\boldsymbol{\rho}}, \tag{4.49}$$

$$\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E} \;=\; (\varepsilon - \varepsilon_0)\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{\rho}\widehat{\boldsymbol{\rho}}. \tag{4.50}$$

(b) The volume polarisation charge density is

$$\rho_{\rm pol} = -\boldsymbol{\nabla}\cdot\mathbf{P} \;=\; -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\, P_\rho) \;=\; 0. \tag{4.51}$$

The surface polarisation charge density is $\sigma_{\rm pol} = \mathbf{P}\cdot\widehat{\mathbf{n}}$,

$$\sigma_{\rm pol}(a) \;=\; (\varepsilon - \varepsilon_0)\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{a}\widehat{\boldsymbol{\rho}}\cdot\widehat{\boldsymbol{\rho}} \;=\; (\varepsilon - \varepsilon_0)\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{a}, \tag{4.52}$$

$$\sigma_{\rm pol}(b) = (\varepsilon - \varepsilon_0)\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{b}\widehat{\boldsymbol{\rho}}\cdot(-\widehat{\boldsymbol{\rho}}) \;=\; -(\varepsilon - \varepsilon_0)\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{b}. \tag{4.53}$$

The net polarisation charge is

$$q_{\rm pol} \;=\; L\times\left[2\pi a\times\sigma_{\rm pol}(a) + 2\pi b\times\sigma_{\rm pol}(b)\right], \tag{4.54}$$

$$\;=\; L\,(\varepsilon - \varepsilon_0)\frac{(V_a - V_b)}{\ln(b/a)}\left(\frac{2\pi a}{a} - \frac{2\pi b}{b}\right) \;=\; 0. \tag{4.55}$$

(c) We obtain the free charge present on the conductors using Gauss' law from which $\sigma_f = \mathbf{D}\cdot\widehat{\mathbf{n}}$,

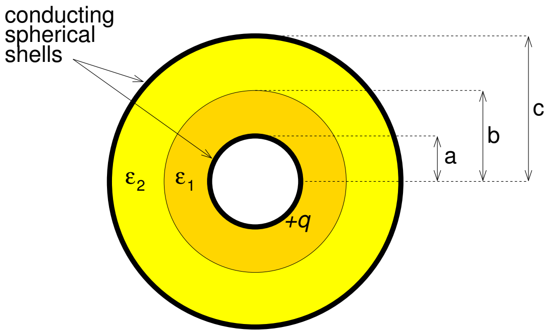
$$\sigma_f(a) \;=\; \varepsilon\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{a}\widehat{\boldsymbol{\rho}}\cdot\widehat{\boldsymbol{\rho}} \;=\; \varepsilon\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{a}, \quad q_f(a) \;=\; 2\pi L\varepsilon\frac{(V_a - V_b)}{\ln(b/a)}, \tag{4.56}$$

$$\sigma_f(b) \;=\; \varepsilon\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{b}\widehat{\boldsymbol{\rho}}\cdot(-\widehat{\boldsymbol{\rho}}) \;=\; -\,\varepsilon\frac{(V_a - V_b)}{\ln(b/a)}\,\frac{1}{b}, \quad q_f(b) \;=\; -2\pi L\varepsilon\frac{(V_a - V_b)}{\ln(b/a)}. \tag{4.57}$$

Hence, the capacitance is $C = q_f/(V_a - V_b)$,

$$C = \frac{2\pi L\varepsilon}{\ln(b/a)}. \tag{4.58}$$

4–4 A spherical capacitor is filled with two different dielectrics with permittivities ε_1 and ε_2 as shown in the diagram. The capacitor is charged such that charge $+q$ is on the inner conductor. Find: (a) $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{P}$ everywhere; (b) the polarisation surface and volume charge density everywhere; (c) the net polarisation charge; (d) the potential difference between the inner and outer conductor, and the capacitance of the capacitor.



Solution

Because of the spherical symmetry we can use Gauss' law in integral form to find the displacement field between the conductors

$$4\pi r^2 D_r = +q, \qquad \mathbf{D}(a<r<c,\theta,\phi) = \frac{q}{4\pi r^2}\widehat{\mathbf{r}}. \tag{4.59}$$

The electric field is $\mathbf{E} = \mathbf{D}/\varepsilon$,

$$\mathbf{E}(r,\theta,\phi) = \begin{cases} \frac{q}{4\pi\varepsilon_1 r^2}\widehat{\mathbf{r}} & (a<r<b) \\ \frac{q}{4\pi\varepsilon_2 r^2}\widehat{\mathbf{r}} & (b<r<c) \end{cases}. \tag{4.60}$$

The polarisation field is $\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E}$,

$$\mathbf{P}(r,\theta,\phi) = \begin{cases} \frac{q}{4\pi r^2}\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right)\widehat{\mathbf{r}} & (a<r<b) \\ \frac{q}{4\pi r^2}\left(1-\frac{\varepsilon_0}{\varepsilon_2}\right)\widehat{\mathbf{r}} & (b<r<c) \end{cases}. \tag{4.61}$$

(b) The volume polarisation charge density is

$$\rho_{\text{pol}} \;=\; -\boldsymbol{\nabla}\cdot\mathbf{P} \;=\; -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) \;=\; 0. \tag{4.62}$$

The surface polarisation charge density is $\sigma_{\text{pol}} = \mathbf{P}\cdot\widehat{\mathbf{n}}$, and each of the two dielectrics will have surface polarisation charge at its inner and outer radii. Hence,

$$\sigma_{\text{pol}}^{(1)}(a) = \frac{q}{4\pi a^2}\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right)\widehat{\mathbf{r}}\cdot(-\widehat{\mathbf{r}}) = -\frac{q}{4\pi a^2}\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right), \tag{4.63}$$

$$\sigma_{\text{pol}}^{(1)}(b) = \frac{q}{4\pi b^2}\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right)\widehat{\mathbf{r}}\cdot(+\widehat{\mathbf{r}}) = \frac{q}{4\pi b^2}\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right), \tag{4.64}$$

$$\sigma_{\text{pol}}^{(2)}(b) = \frac{q}{4\pi b^2}\left(1-\frac{\varepsilon_0}{\varepsilon_2}\right)\widehat{\mathbf{r}}\cdot(-\widehat{\mathbf{r}}) = -\frac{q}{4\pi b^2}\left(1-\frac{\varepsilon_0}{\varepsilon_2}\right), \tag{4.65}$$

$$\sigma_{\text{pol}}^{(2)}(c) = \frac{q}{4\pi c^2}\left(1-\frac{\varepsilon_0}{\varepsilon_2}\right)\widehat{\mathbf{r}}\cdot(+\widehat{\mathbf{r}}) = \frac{q}{4\pi c^2}\left(1-\frac{\varepsilon_0}{\varepsilon_2}\right). \tag{4.66}$$

(c) The net polarisation charge is

$$q_{\text{pol}} = 4\pi a^2\sigma_{\text{pol}}^{(1)}(a) + 4\pi b^2\left[\sigma_{\text{pol}}^{(1)}(b) + \sigma_{\text{pol}}^{(2)}(b)\right] + 4\pi c^2\sigma_{\text{pol}}^{(2)}(c), \tag{4.67}$$

$$= q\left[-\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right)+\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right)-\left(1-\frac{\varepsilon_0}{\varepsilon_2}\right)+\left(1-\frac{\varepsilon_0}{\varepsilon_2}\right)\right] = 0. \tag{4.68}$$

(d) The potential difference is $V_a - V_c = -\int_c^a \mathbf{E}\cdot d\mathbf{r}$,

$$V_b - V_c = -\int_c^b \frac{q}{4\pi\varepsilon_2 r^2}dr = \int_c^b \frac{q}{4\pi\varepsilon_2 r^2}dr = \frac{q}{4\pi\varepsilon_2}\left(\frac{1}{b}-\frac{1}{c}\right), \tag{4.69}$$

$$V_a - V_b = -\int_b^a \frac{q}{4\pi\varepsilon_1 r^2}dr = \int_b^a \frac{q}{4\pi\varepsilon_1 r^2}dr = \frac{q}{4\pi\varepsilon_1}\left(\frac{1}{a}-\frac{1}{b}\right). \tag{4.70}$$

$$\therefore\ V_a - V_c = \frac{q}{4\pi}\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{\varepsilon_2}\left(\frac{1}{b}-\frac{1}{c}\right)\right]. \tag{4.71}$$

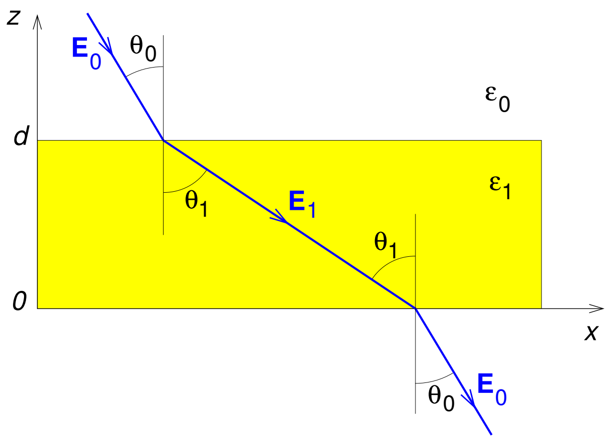
Hence, the capacitance is

$$C = 4\pi\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{\varepsilon_2}\left(\frac{1}{b}-\frac{1}{c}\right)\right]^{-1}. \tag{4.72}$$

4–5 A uniform slab of material with permittivity ε_1 is suspended parallel to the xy-plane, and has its lower surface at $z = 0$ and its upper surface at $z = d$. Outside the slab there is a uniform electric field $\mathbf{E}_0 = E_0(\sin\theta_0\,\widehat{\mathbf{x}} - \cos\theta_0\,\widehat{\mathbf{z}})$. (a) Find formulae for $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{P}$ in the dielectric, the angle between $\mathbf{E}$ in the dielectric and the normal to the surface, and the surface polarisation charge density at $z = 0$ and $z = d$. (b) Find numerical values for the case of $E_0 = 1000$ V m^{-1}, $\theta_0 = 45°$ and $\varepsilon_r = 2$, and include a sketch showing field directions.

Solution

A field line will bend as in the diagram.



(a) The component of $\mathbf{E}$ parallel to the boundary is unchanged, and since there is no free charge the component of $\mathbf{D}$ normal to the boundary is unchanged

$$E_0 \sin\theta_0 = E_1 \sin\theta_1, \qquad \varepsilon_0 E_0 \cos\theta_0 = \varepsilon_1 E_1 \cos\theta_1. \tag{4.73}$$

The electric field is

$$\mathbf{E}_1 = E_1 \sin\theta_1 \widehat{\mathbf{x}} - E_1 \cos\theta_1 \widehat{\mathbf{z}}. \tag{4.74}$$

$$\therefore \quad \mathbf{E}_1 = E_0 \sin\theta_0 \widehat{\mathbf{x}} - \frac{\varepsilon_0}{\varepsilon_1} E_0 \cos\theta_0 \widehat{\mathbf{z}}. \tag{4.75}$$

Hence, from Eqs. 4.75 the magnitude and direction

$$E_1 = \left[\sin^2\theta_0 + \left(\frac{\varepsilon_0}{\varepsilon_1}\right)^2 \cos^2\theta_0\right]^{1/2} E_0, \qquad \theta_1 = \arctan\left(\frac{\varepsilon_1}{\varepsilon_0}\tan\theta_0\right). \tag{4.76}$$

The displacement and polarisation fileds are

$$\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = \varepsilon_1 E_0 \sin\theta_0\, \widehat{\mathbf{x}} - \varepsilon_0 E_0 \cos\theta_0\, \widehat{\mathbf{z}}, \tag{4.77}$$

$$\mathbf{P}_1 = (\mathbf{D}_1 - \varepsilon_0 \mathbf{E}_1) = (\varepsilon_1 - \varepsilon_0) E_0 \sin\theta_0\, \widehat{\mathbf{x}} - \frac{(\varepsilon_1 - \varepsilon_0)\varepsilon_0}{\varepsilon_1} E_0 \cos\theta_0\, \widehat{\mathbf{z}}. \tag{4.78}$$

The surface polarisation charge is

$$\sigma_{\text{pol}}(x, y, 0) = +P_1 \cos\theta_1, \qquad \sigma_{\text{pol}}(x, y, d) = -P_1 \cos\theta_1. \tag{4.79}$$

(b) Substituting for the case of $E_0 = 1000$ V m^{-1}, $\theta_0 = 45°$ and $\varepsilon_r = 2$, i.e. $\varepsilon_1 = 2\varepsilon_0$ we find

$$\theta_1 = 63.4°, \quad E_1 = 791 \text{ V m}^{-1}, \quad D_1 = 1.40 \times 10^{-8} \text{ C m}^{-2}, \tag{4.80}$$

$$P_1 = 7.00 \times 10^{-9} \text{ C m}^{-2}, \tag{4.81}$$

$$\sigma_{\text{pol}}(x, y, z{=}0) = +3.13 \times 10^{-9} \text{ C m}^{-2}, \tag{4.82}$$

$$\sigma_{\text{pol}}(x, y, z{=}d) = -3.13 \times 10^{-9} \text{ C m}^{-2}. \tag{4.83}$$

4–6 Derive the force on an electric dipole in a non-uniform electric field.

Solution

Consider a physical dipole consisting of charge $+q$ located at $\mathbf{r}_{\rm pos}$ and charge $-q$ located at $\mathbf{r}_{\rm neg}$. Its dipole moment is $\mathbf{p} = q\mathbf{d}$ where $\mathbf{d} = (\mathbf{r}_{\rm pos} - \mathbf{r}_{\rm neg})$. It follows that the force is

$$\mathbf{F} = (+q)\mathbf{E}(\mathbf{r}_{\rm pos}) + (-q)\mathbf{E}(\mathbf{r}_{\rm neg}), \tag{4.84}$$

$$= q\Delta\mathbf{E} \tag{4.85}$$

$$= q(\widehat{\mathbf{x}}\,\Delta E_x + \widehat{\mathbf{y}}\,\Delta E_y + \widehat{\mathbf{z}}\,\Delta E_z), \tag{4.86}$$

$$= q\left[\widehat{\mathbf{x}}\,(\mathbf{d}\cdot\boldsymbol{\nabla}E_x) + \widehat{\mathbf{y}}\,(\mathbf{d}\cdot\boldsymbol{\nabla}E_y) + \widehat{\mathbf{z}}\,(\mathbf{d}\cdot\boldsymbol{\nabla}E_z)\right], \tag{4.87}$$

$$= q\left[(\mathbf{d}\cdot\boldsymbol{\nabla}E_x\,\widehat{\mathbf{x}}) + (\mathbf{d}\cdot\boldsymbol{\nabla}E_y\,\widehat{\mathbf{y}}) + (\mathbf{d}\cdot\boldsymbol{\nabla}E_z\,\widehat{\mathbf{z}})\right], \tag{4.88}$$

$$= q(\mathbf{d}\cdot\boldsymbol{\nabla}\mathbf{E}), \tag{4.89}$$

$$= (q\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}, \tag{4.90}$$

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{4.91}$$

4–7 The relative permittivities of Nitrogen, Argon and Hydrogen in gas (at 20°C) and liquid phases are given below.

Element	N_2 gas	Ar gas	H_2 gas	N_2 liquid	Ar liquid	H_2 liquid
ε_r	1.000546	1.000517	1.000272	1.45	1.53	1.22

http://www.kayelaby.npl.co.uk/general_physics/2_2/2_2_1.html

Use the Clausius-Mossotti formula to find the electronic polarisability, and compare the results for the same elements in the liquid and gas phases. [You will need to look up any constants and the atomic weights and densities required.]

Solution

We need to use the Clausius-Mossotti formula

$$\alpha_{\text{pol}} = \frac{3\varepsilon_0}{N}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) \tag{4.92}$$

where $N = \rho/(\bar{A}\,u)$, ρ is the density, $\bar{A}$ is the mean atomic mass, $u = 1.66 \times 10^{-27}$ kg and $\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}.

Values of density in the liquid and gas phases, and the mean molecular weight have been looked up in tables of physical/chemical constants and have been added to the table, and the polarisability calculated using Eq. 4.92.

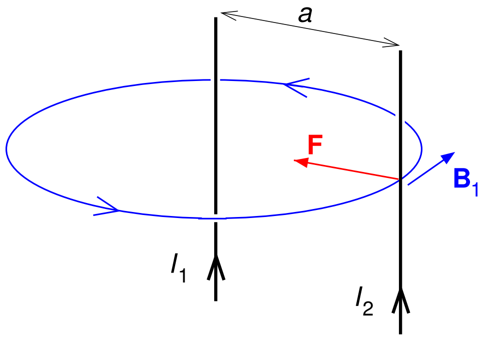
Element	N_2 gas	Ar gas	H_2 gas	N_2 liquid	Ar liquid	H_2 liquid
ε_r	1.000546	1.000517	1.000272	1.45	1.53	1.22
ρ (kg m^{-3})	1.25	1.78	0.089	800.	1393.	67.8
$\bar{A}$	14.	39.95	1.	14.	39.95	1.
α_{pol} (C m^{2}V^{-1})	8.98×10^{-41}	1.70×10^{-40}	4.49×10^{-41}	1.01×10^{-40}	1.89×10^{-40}	4.44×10^{-41}

The agreement in α_{pol} for the same element between liquid and gas phases is quite good, the difference being at most 10%.

5 Magnetic field and vector potential

5–1 Two parallel wires are separated by distance a and carry currents I_1 and I_2 in the same direction. Find the force per unit length of wire. Include a diagram showing the direction of the force. If $I_1 = I_2 = 1$ A and $a = 1$ m, what is the magnitude of the force per unit length?

Solution



The force on a circuit in a magnetic field is given by Ampère's force law

$$\mathbf{F}_{\text{mag}} = \oint (I\, d\mathbf{r} \times \mathbf{B}). \tag{5.1}$$

For parallel currents as in the diagram, the force on length L of wire 2 is

$$F = I_2 L B_1 = I_2 L \frac{\mu_0 I_1}{2\pi a} \tag{5.2}$$

and is directed towards wire 1 as shown. The force per unit length is

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi a}. \tag{5.3}$$

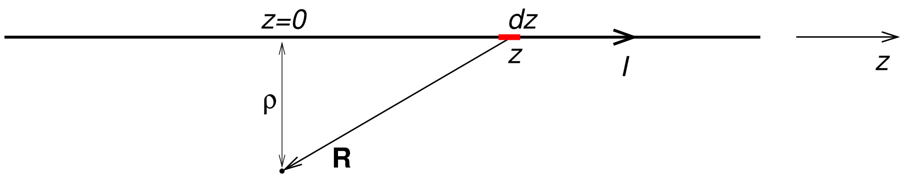
If the wires are 1 m apart and each carry 1 A,

$$\frac{F}{L} = \frac{4\pi \times 10^{-7}}{2\pi} = 2 \times 10^{-7} \ \ \text{N m}^{-1}. \tag{5.4}$$

The amp is defined as the current flowing in two parallel wires 1 m apart such that the force beween them per unit length is 2×10^{-7} N m^{-1}.

5–2 Using the equation for the vector potential in terms of the current density find the vector potential of an infinite straight wire along the z axis carrying current I.

Solution



We shall consider a long straight current along the z axis $(-L < z < L)$ and obtain the vector potential for cylindrical coordinate radii $\rho \ll L$, which is a good approximation for the case $L \to \infty$. Then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint \frac{I d\mathbf{r}'}{R} \tag{5.5}$$

$$= \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz\, \widehat{\mathbf{z}}}{\sqrt{z^2 + \rho^2}} \tag{5.6}$$

$$= \widehat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \left[\ln(\sqrt{z^2 + \rho^2} + z) \right]_{-L}^{L} \tag{5.7}$$

$$= \widehat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi}\ln\left(\frac{\sqrt{L^2+\rho^2}+L}{\sqrt{L^2+\rho^2}-L}\right) \tag{5.8}$$

$$= \widehat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi}\ln\left(\frac{\sqrt{1+\rho^2/L^2}+1}{\sqrt{1+\rho^2/L^2}-1}\right) \tag{5.9}$$

$$\approx \widehat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi}\ln\left(\frac{1+\rho^2/2L^2+1}{1+\rho^2/2L^2-1}\right) \quad \text{(1st 2 terms of series expansion)}$$

$$= \widehat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi}\ln\left(\frac{2}{\rho^2/2L^2}\right) \quad (\text{provided } \rho \ll L) \tag{5.10}$$

$$= \widehat{\mathbf{z}}\,\frac{\mu_0 I}{4\pi}\left[\ln(4L^2) - 2\ln(\rho)\right] \tag{5.11}$$

$$\therefore \quad \mathbf{A}(\mathbf{r}) = -\,\widehat{\mathbf{z}}\,\frac{\mu_0 I}{2\pi}\ln(\rho) \;+\; \widehat{\mathbf{z}}C. \quad (C \text{ is a constant.}). \tag{5.12}$$

Note that the value of C has no effect on the magnetic field, and so it is convenient to write $\mathbf{A}(\mathbf{r}) = -\,\widehat{\mathbf{z}}\,(\mu_0 I/2\pi)\ln(\rho)$.

5–3 We can add the gradient of a scalar field $U(\mathbf{r})$ to $\mathbf{A}(\mathbf{r})$ without changing $\mathbf{B}(\mathbf{r})$. This is called a gauge transformation. The “gauge” of the vector potential is determined by the value of $\boldsymbol{\nabla}\cdot\mathbf{A}$. Show that in magnetostatics $\boldsymbol{\nabla}\cdot\mathbf{A} = \nabla^2 U(\mathbf{r})$.

Solution

$$\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{r}) = \boldsymbol{\nabla}\cdot\left[\left(\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{R}\,d^3r'\right) + \boldsymbol{\nabla}U(\mathbf{r})\right], \tag{5.13}$$

$$= \frac{\mu_0}{4\pi}\int\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}(\mathbf{r}')}{R}\right)\,d^3r' \;+\; \nabla^2 U(\mathbf{r}). \tag{5.14}$$

We shall show that the integral is zero for a finite current distribution, i.e. for $\mathbf{J}(\mathbf{r}\to\infty) = \mathbf{0}$. We can use the product rule $\boldsymbol{\nabla}\cdot(a\mathbf{V}) = \boldsymbol{\nabla}a\cdot\mathbf{V} + a\boldsymbol{\nabla}\cdot\mathbf{V}$ to expand the integrand

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}(\mathbf{r}')}{R}\right) = \boldsymbol{\nabla}\left(\frac{1}{R}\right)\cdot\mathbf{J}(\mathbf{r}') \;+\; \frac{1}{R}\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r}'), \tag{5.15}$$

$$= \boldsymbol{\nabla}\left(\frac{1}{R}\right)\cdot\mathbf{J}(\mathbf{r}') \;+\; 0 \quad \text{(diff. is w.r.t. unprimed coords.)}, \tag{5.16}$$

$$= -\boldsymbol{\nabla}'\left(\frac{1}{R}\right)\cdot\mathbf{J}(\mathbf{r}') \quad (\boldsymbol{\nabla}R^n = -\boldsymbol{\nabla}'R^n = nR^{n-1}\widehat{\mathbf{R}}). \tag{5.17}$$

Next use the same product rule but for primed coordinates $\boldsymbol{\nabla}' \cdot (a\mathbf{V}) = \boldsymbol{\nabla}' a \cdot \mathbf{V} + a\boldsymbol{\nabla}' \cdot \mathbf{V}$ to get

$$\boldsymbol{\nabla}' \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{R} \right) = \boldsymbol{\nabla}' \left(\frac{1}{R} \right) \cdot \mathbf{J}(\mathbf{r}') \; + \; \frac{1}{R} \boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{r}'), \tag{5.18}$$

$$= \boldsymbol{\nabla}' \left(\frac{1}{R} \right) \cdot \mathbf{J}(\mathbf{r}') \; + \; 0, \tag{5.19}$$

since charge conservation in magnetostatics requires $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$.

Then

$$\int_{\text{all space}} \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{R} \right) \, d^3r' = - \int_{\text{all space}} \boldsymbol{\nabla}' \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{R} \right) d^3r', \tag{5.20}$$

$$= - \oint_{S \text{ at } \infty} \left(\frac{\mathbf{J}(\mathbf{r}')}{R} \right) \cdot d\mathbf{S}', \tag{5.21}$$

$$= 0, \tag{5.22}$$

since $\mathbf{J}(\mathbf{r}' \to \infty) = 0$ for a localised charge distribution. Hence,

$$\boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{r}) = \nabla^2 U(\mathbf{r}). \tag{5.23}$$

In magnetostatics, it is convenient to choose $U(\mathbf{r})$ such that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ (Coulomb gauge).

5–4 (a) Find the vector potential of the constant magnetic field $\mathbf{B}_0 = (B_x^0, B_y^0, B_z^0)$, (b) check that the vector potential you find does give the desired magnetic field, (c) find $\boldsymbol{\nabla} \cdot \mathbf{A}$ and check it is what is expected in magnetostatics. [Hint: first find the vector potential $\mathbf{A}^{(z)}(\mathbf{r})$ of the simpler constant field $B_z^0 \, \widehat{\mathbf{z}}$ by writing down the components of $\boldsymbol{\nabla} \times \mathbf{A}^{(z)}$ in Cartesian coordinates before appealing to the symmetry of the problem, and then integrating.]

Solution

(a) We first attempt to find $\mathbf{A}^{(z)}(\mathbf{r})$ which must satisfy

$$\left(\frac{\partial A_y^{(z)}}{\partial x} - \frac{\partial A_x^{(z)}}{\partial y} \right) = B_z^0. \tag{5.24}$$

$\mathbf{A}^{(z)}(\mathbf{r})$ must be symmetrical about the z axis which suggests that

$$\frac{\partial A_y^{(z)}}{\partial x} = \frac{1}{2} B_z^0, \qquad -\frac{\partial A_x^{(z)}}{\partial y} = \frac{1}{2} B_z^0. \tag{5.25}$$

Integrating we get

$$A_y^{(z)} = \frac{1}{2} B_z^0 \, x, \quad A_x^{(z)} = -\frac{1}{2} B_z^0 \, y. \tag{5.26}$$

Hence,

$$\mathbf{A}^{(z)} = \frac{1}{2} \left(B_z^0 x \, \widehat{\mathbf{y}} - B_z^0 y \, \widehat{\mathbf{x}} \right). \tag{5.27}$$

Similarly, or by cyclic permutation of x, y and z above,

$$\mathbf{A}^{(x)} = \frac{1}{2} \left(B_x^0 y \, \widehat{\mathbf{z}} - B_x^0 z \, \widehat{\mathbf{y}} \right), \tag{5.28}$$

$$\mathbf{A}^{(y)} = \frac{1}{2} \left(B_y^0 z \, \widehat{\mathbf{x}} - B_y^0 x \, \widehat{\mathbf{z}} \right), \tag{5.29}$$

so that the vector potential of $\mathbf{B}_0 = (B_x^0, B_y^0, B_z^0)$ is

$$\mathbf{A} = (\mathbf{A}^{(x)} + \mathbf{A}^{(y)} + \mathbf{A}^{(z)}), \tag{5.30}$$
$$= \frac{1}{2}\left[\widehat{\mathbf{x}}\left(B_y^0 z - B_z^0 y\right) + \widehat{\mathbf{y}}\left(B_z^0 x - B_x^0 z\right) + \widehat{\mathbf{z}}\left(B_x^0 y - B_y^0 x\right)\right], \tag{5.31}$$
$$= \frac{1}{2}\mathbf{B}_0 \times \mathbf{r} \quad = \quad -\frac{1}{2}\mathbf{r} \times \mathbf{B}_0. \tag{5.32}$$

This $\mathbf{A}(\mathbf{r})$ is not unique – we could add the gradient of any scalar field to $\mathbf{A}(\mathbf{r})$ without changing $\mathbf{B}(\mathbf{r})$.

(b) Taking curl of the vector potential found above, this time using index notation,

$$\boldsymbol{\nabla} \times \mathbf{A} = -\frac{1}{2}\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{B}^0), \tag{5.33}$$
$$[\boldsymbol{\nabla} \times \mathbf{A}]_i = -\frac{1}{2}\varepsilon_{ijk}\nabla_j \varepsilon_{klm} r_l B_m^0, \tag{5.34}$$
$$= -\frac{1}{2}\varepsilon_{kij}\varepsilon_{klm}\nabla_j r_l B_m^0, \tag{5.35}$$
$$= -\frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\nabla_j r_l B_m^0, \tag{5.36}$$
$$= -\frac{1}{2}(\delta_{il}\delta_{jm}\nabla_j r_l B_m^0 - \delta_{im}\delta_{jl}\nabla_j r_l B_m^0), \tag{5.37}$$
$$= -\frac{1}{2}(\nabla_m r_i B_m^0 - \nabla_l r_l B_i^0), \tag{5.38}$$
$$= -\frac{1}{2}(r_i \nabla_m B_m^0 + B_m^0 \nabla_m r_i - r_l \nabla_l B_i^0 - B_i^0 \nabla_l r_l), \tag{5.39}$$
$$= -\frac{1}{2}(0 + B_i^0 - 0 - 3B_i^0), \tag{5.40}$$
$$= -\frac{1}{2}(-2B_i^0), \tag{5.41}$$
$$[\boldsymbol{\nabla} \times \mathbf{A}]_i = B_i^0, \tag{5.42}$$
$$\therefore \quad \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}^0 \quad \text{(as required)}. \tag{5.43}$$

We have used $\boldsymbol{\nabla} \cdot \mathbf{B}^0 = 0$, $\nabla_m r_i = \delta_{mi}$, $\mathbf{B}^0$ constant, and $\boldsymbol{\nabla} \cdot \mathbf{r} = 3$ used in Eq. 5.39 above.

(c) the divergence is

$$\nabla \cdot \mathbf{A} = -\frac{1}{2}\nabla \cdot (\mathbf{r} \times \mathbf{B}^0), \tag{5.44}$$

$$= -\frac{1}{2}\nabla_i \, \varepsilon_{ijk} r_j B^0_k, \tag{5.45}$$

$$= -\frac{1}{2}\varepsilon_{ijk}(B^0_k \nabla_i r_j + r_j \nabla_i B^0_k), \tag{5.46}$$

$$= -\frac{1}{2}\varepsilon_{kij} B^0_k \nabla_i r_j + 0, \tag{5.47}$$

$$= -\frac{1}{2}B^0_k \varepsilon_{kij} \nabla_i r_j, \tag{5.48}$$

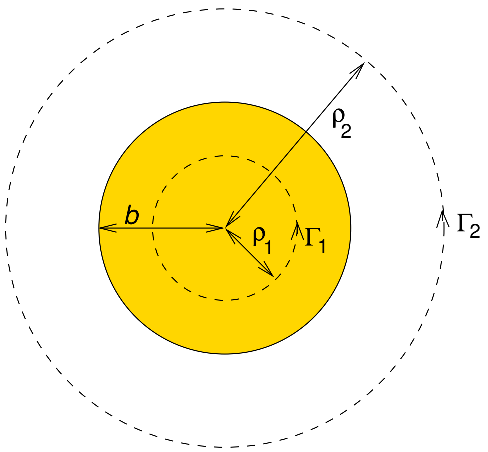
$$= -\frac{1}{2}B^0_k [\nabla \times \mathbf{r}]_k, \tag{5.49}$$

$$\therefore \quad \nabla \cdot \mathbf{A} = 0 \quad (\text{because } \nabla \times \mathbf{r} = 0), \tag{5.50}$$

and the vector potential is seen to satisfy Coulomb gauge.

5–5 A steady current I flows down a long cylindrical wire of radius b. Find the magnetic field both inside and outside the wire.

Solution



We define the z axis to correspond the axis of the wire, and point out of the screen/page. Then assuming the current density is constant inside the wire

$$\mathbf{J}(\mathbf{r}) = \begin{cases} I/(\pi b^2)\,\hat{\mathbf{z}} & (0 < \rho < b) \\ 0 & (\rho > b) \end{cases} \tag{5.51}$$

From symmetry arguments $\mathbf{B}$ must be azimuthal, i.e. $\mathbf{B}(\mathbf{r}) = B(\rho)\widehat{\boldsymbol{\phi}}$. We apply Ampère's law to loops Γ_1 and Γ_2 in the diagram,

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{\text{encl}}. \tag{5.52}$$

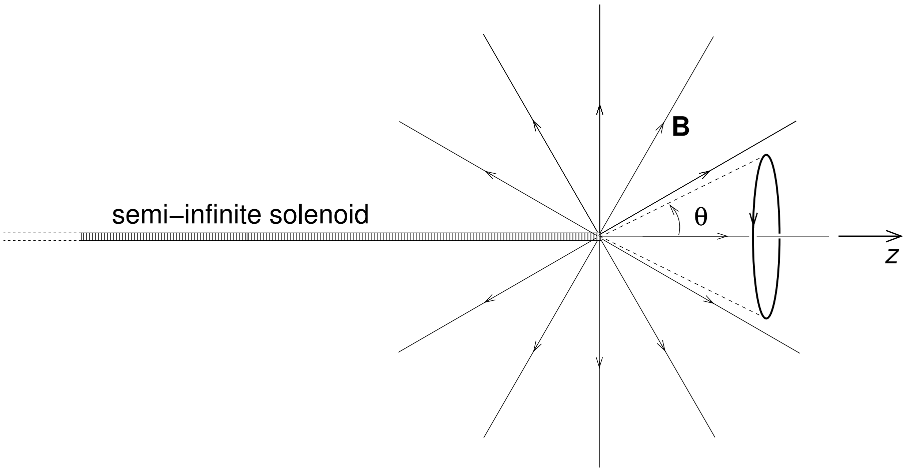
$$\therefore \quad 2\pi\rho_1 B(\rho_1) = \mu_0 \pi \rho_1^2 \frac{I}{\pi b^2} \qquad (\text{loop } \Gamma_1), \tag{5.53}$$

$$\therefore \quad 2\pi\rho_2 B(\rho_2) = \mu_0 I \qquad (\text{loop } \Gamma_2). \tag{5.54}$$

$$\therefore \quad \mathbf{B}(\mathbf{r}) = \begin{cases} (\mu_0 \rho I / 2\pi b^2)\,\widehat{\boldsymbol{\phi}} & (0 < \rho < b), \\ (\mu_0 I / 2\pi\rho)\,\widehat{\boldsymbol{\phi}} & (\rho > b). \end{cases} \tag{5.55}$$

5–6 A semi-infinite solenoid of radius a, has n turns per unit length, extends from $z = -\infty$ to $z = 0$ along the z axis and carries current I in the $+\widehat{\boldsymbol{\phi}}$ direction. Magnetic flux is confined to the solenoid, but emerges isotropically from its end at the origin as shown below.

(a) On the cone of half-angle θ with apex at the origin there is a circular loop (as shown) with all points on the loop being at distance $r \gg a$ from the origin. Find the magnetic flux passing through this loop.



(b) Find the magnetic vector potential at the point (r, θ, ϕ), and take it's curl to find the magnetic field. [The expression for magnetic flux through a loop in terms of the vector potential may be useful here.]

Solution

a) The magnetic flux emerging from the end of the solenoid is the flux inside the solenoid, i.e.

$$\Phi_0 = (\mu_0 n I)(\pi a^2). \tag{5.56}$$

The flux emerges isotropically from the end of the solenoid at $z = 0$, so we shall need the solid angle subtended at the origin by the circular loop, which is

$$\Omega \;=\; 2\pi \int_0^{\theta} \sin\theta' d\theta' \;=\; 2\pi[-\cos\theta']_0^{\theta} \;=\; 2\pi[(-\cos\theta) - (-1)] \;=\; 2\pi(1-\cos\theta). \tag{5.57}$$

Then the magnetic flux through the circular loop is

$$\Phi_B \;=\; \frac{\Omega}{4\pi}\Phi_0 \;=\; \frac{1}{2}(1-\cos\theta)\Phi_0 \;=\; \frac{1}{2}(1-\cos\theta)(\mu_0 n I)(\pi a^2). \tag{5.58}$$

(b) The magnetic flux through the loop is equal to the line-integral of the vector potential around the loop

$$\oint \mathbf{A} \cdot d\ell = \Phi_B. \tag{5.59}$$

From symmetry arguments, since the current around the solenoid is only in the $\widehat{\boldsymbol{\phi}}$ direction, the vector potential must also be in the $\widehat{\boldsymbol{\phi}}$ direction, and since the loop has radius $r\sin\theta$

$$A_\phi(r,\theta,\phi)(2\pi r\sin\theta) = \frac{1}{2}(1-\cos\theta)\Phi_0, \tag{5.60}$$

$$A_\phi(r,\theta,\phi) = \frac{1}{2}\frac{(1-\cos\theta)}{(2\pi r\sin\theta)}\Phi_0. \tag{5.61}$$

$$\therefore\quad \mathbf{A}(r,\theta,\phi) = \frac{(1-\cos\theta)}{(4\pi r\sin\theta)}\Phi_0\,\widehat{\boldsymbol{\phi}}. \tag{5.62}$$

Taking the curl,

$$\mathbf{B} = \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\widehat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r\,A_\phi)\right]\widehat{\boldsymbol{\theta}}$$
$$+\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\widehat{\boldsymbol{\phi}}, \tag{5.63}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,A_\phi)\,\widehat{\mathbf{r}}\ -\ \frac{1}{r}\frac{\partial}{\partial r}(r\,A_\phi)\,\widehat{\boldsymbol{\theta}}, \tag{5.64}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{(1-\cos\theta)}{(4\pi r)}\Phi_0\,\widehat{\mathbf{r}}\ +\ \mathbf{0}, \tag{5.65}$$

$$= \frac{1}{r\sin\theta}\frac{\sin\theta}{(4\pi r)}\Phi_0\,\widehat{\mathbf{r}}, \tag{5.66}$$

$$= \frac{\Phi_0}{4\pi r^2}\,\widehat{\mathbf{r}}, \tag{5.67}$$

$$= \frac{\mu_0(nI\pi a^2)}{4\pi r^2}\,\widehat{\mathbf{r}}. \tag{5.68}$$

This field has similar form to the electric field of a point electric charge. Hence, the end of a semi-infinite solenoid appears as if it were a magnetic monopole of "magnetic charge" $nI\pi a^2$.

5–7 By taking the curl of the vector potential for a magnetic dipole with moment $\mathbf{m}$ located at the origin, find it's magnetic field using index notation.

Solution

Using index notation,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\boldsymbol{\nabla}\times\left(\mathbf{m}\times\frac{\mathbf{r}}{r^3}\right). \tag{5.69}$$

$$B_i = \frac{\mu_0}{4\pi}\,\varepsilon_{ijk}\nabla_j\left(\frac{\varepsilon_{kln}m_l r_n}{r^3}\right), \tag{5.70}$$

$$= \frac{\mu_0}{4\pi}\,\varepsilon_{kji}\varepsilon_{kln}\left(\frac{\nabla_j m_l r_n}{r^3} + m_l r_n \nabla_j \frac{1}{r^3}\right), \tag{5.71}$$

$$= \frac{\mu_0}{4\pi}\,\varepsilon_{kji}\varepsilon_{kln}\left[\frac{m_l \nabla_j r_n}{r^3} + m_l r_n\left(-3\frac{\widehat{\mathbf{r}}}{r^4}\right)\cdot\widehat{\mathbf{e}}_j\right], \tag{5.72}$$

$$= \frac{\mu_0}{4\pi}\left(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl}\right)\left(\frac{m_l\delta_{jn}}{r^3} + m_l r_n(-3)r_j\frac{1}{r^5}\right), \tag{5.73}$$

$$= \frac{\mu_0}{4\pi}\left(\frac{m_i\delta_{nn}-m_j\delta_{ji}}{r^3} - \frac{3(m_i r_n r_n - m_j r_i r_j)}{r^5}\right), \tag{5.74}$$

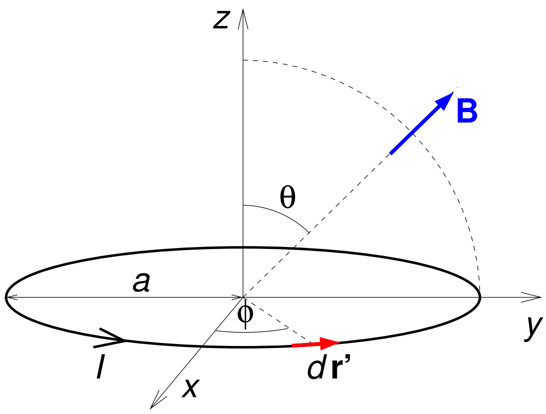
$$= \frac{\mu_0}{4\pi}\left(\frac{3m_i - m_i}{r^3} - \frac{3[m_i r^2 - (\mathbf{m}\cdot\mathbf{r})\,r_i]}{r^5}\right), \tag{5.75}$$

$$B_i = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\widehat{\mathbf{r}})(\widehat{\mathbf{r}}\cdot\widehat{\mathbf{e}}_i) - m_i}{r^3}\right). \tag{5.76}$$

$$\therefore\quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\widehat{\mathbf{r}})\widehat{\mathbf{r}} - \mathbf{m}}{r^3}\right). \tag{5.77}$$

5–8 A circular current loop in the xy plane has radius a and is centred on the origin. It carries current I in the ϕ-direction. There is a uniform magnetic field $\mathbf{B}(\mathbf{r}) = B_0(\cos\theta\widehat{\mathbf{z}} + \sin\theta\widehat{\mathbf{y}})$ present. By integrating the torque $d\mathbf{N} = \mathbf{r}' \times d\mathbf{F}$ on line element $d\mathbf{r}'$ of the current loop at $\mathbf{r}'$, find the torque on the entire current loop. Compare your result with the result you would get by first finding the current loop's dipole moment, and then applying the formula for the torque on a magnetic dipole in a magnetic field.

Solution



A point on the loop and the corresponding line element on the loop are

$$\mathbf{r}' = a(\cos\phi\,\widehat{\mathbf{x}} + \sin\phi\,\widehat{\mathbf{y}}), \tag{5.78}$$

$$d\mathbf{r}' = a\,d\phi(-\sin\phi\,\widehat{\mathbf{x}} + \cos\phi\,\widehat{\mathbf{y}}). \tag{5.79}$$

The force on that line element is

$$d\mathbf{F} = I d\mathbf{r}' \times \mathbf{B}, \tag{5.80}$$

$$= I \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ -a\,d\phi\,\sin\phi & a\,d\phi\,\cos\phi & 0 \\ 0 & B_0\sin\theta & B_0\cos\theta \end{vmatrix} \tag{5.81}$$

$$= IaB_0\,d\phi(\cos\phi\cos\theta\,\widehat{\mathbf{x}} + \sin\phi\cos\theta\,\widehat{\mathbf{y}} - \sin\phi\sin\theta\,\widehat{\mathbf{z}}). \tag{5.82}$$

The torque on the line element is

$$
\begin{aligned}
d\mathbf{N} &= \mathbf{r}' \times (d\mathbf{F}), && (5.83)\\
&= \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ a\cos\phi & a\sin\phi & 0 \\ IaB_0\, d\phi\, \cos\phi\cos\theta & IaB_0\, d\phi\, \sin\phi\cos\theta & -IaB_0\, d\phi\, \sin\phi\sin\theta \end{vmatrix} && (5.84)\\
&= Ia^2B_0\, d\phi\big[-\sin^2\phi\sin\theta\,\widehat{\mathbf{x}} + \cos\phi\sin\phi\sin\theta\,\widehat{\mathbf{y}} \\
&\qquad + (\cos\phi\sin\phi\cos\theta - \sin\phi\cos\phi\cos\theta)\,\widehat{\mathbf{z}}\big], && (5.85)\\
&= Ia^2B_0\, d\phi\,(-\sin\theta\,\sin^2\phi\,\widehat{\mathbf{x}} + \cos\phi\sin\phi\sin\theta\,\widehat{\mathbf{y}}). && (5.86)
\end{aligned}
$$

When integrating over ϕ we note that $\cos\phi\sin\phi$ is an odd function and its integral from 0 to 2π will be zero, and so

$$
\begin{aligned}
\mathbf{N} &= -Ia^2B_0\,\sin\theta\,\widehat{\mathbf{x}}\int_0^{2\pi}\sin^2\phi\,d\phi && (5.87)\\
&= -Ia^2B_0\,\sin\theta\,\widehat{\mathbf{x}}\left[\frac{1}{2}\phi - \frac{1}{2}\sin\phi\cos\phi\right]_0^{2\pi} && (5.88)\\
&= -I\pi a^2B_0\,\sin\theta\,\widehat{\mathbf{x}}. && (5.89)
\end{aligned}
$$

Given that the dipole moment is $\mathbf{m} = I\pi a^2\,\widehat{\mathbf{z}}$, we expect a torque

$$
\begin{aligned}
\mathbf{N} &= \mathbf{m}\times\mathbf{B}, && (5.90)\\
&= -I\pi a^2B_0\,\sin\theta\,\widehat{\mathbf{x}} && (5.91)
\end{aligned}
$$

as just found using the force law.

6 Magnetism of materials

6–1 Derive the following which are needed to obtain the magnetisation currents of a magnetised object: (a) Identity for $\boldsymbol{\nabla}' R^{-1}$,

$$\boldsymbol{\nabla}' R^{-1} = +R^{-2}\widehat{\mathbf{R}}. \tag{6.1}$$

(b) Product rule for $\boldsymbol{\nabla} \times (a\mathbf{F})$,

$$\boldsymbol{\nabla} \times (a\mathbf{F}) = (\boldsymbol{\nabla} a) \times \mathbf{F} + a\boldsymbol{\nabla} \times \mathbf{F}. \tag{6.2}$$

(c) Corollary to Gauss' Theorem,

$$\int_V \boldsymbol{\nabla} \times \mathbf{F}\, d^3r = -\oint_S \mathbf{F} \times d\mathbf{S}. \tag{6.3}$$

Solution

(a) First, for the identity for $\boldsymbol{\nabla}'R^{-1}$ we shall start by deriving the more general case,

$$\begin{aligned}
\boldsymbol{\nabla}'R^n &= \widehat{\mathbf{x}}\frac{\partial}{\partial x'}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2} \\
&\quad + \widehat{\mathbf{y}}\frac{\partial}{\partial y'}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2} \\
&\quad + \widehat{\mathbf{z}}\frac{\partial}{\partial z'}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{n/2}, \qquad (6.4)\\
&= \widehat{\mathbf{x}}\,\frac{n}{2}\,\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{(n/2-1)}\,2\,(x-x')(-1) \\
&\quad + \widehat{\mathbf{y}}\,\frac{n}{2}\,\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{(n/2-1)}\,2\,(y-y')(-1) \\
&\quad + \widehat{\mathbf{z}}\,\frac{n}{2}\,\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{(n/2-1)}\,2\,(z-z')(-1), \qquad (6.5)\\
&= -n\,\left[\widehat{\mathbf{x}}(x-x')+\widehat{\mathbf{y}}(y-y')+\widehat{\mathbf{z}}(z-z')\right]\,R^{n-2}, \qquad (6.6)\\
&= -n\,\mathbf{R}\,R^{n-2}, \qquad (6.7)\\
\therefore\quad \boldsymbol{\nabla}'R^n &= -n\,R^{n-1}\,\widehat{\mathbf{R}}. \qquad (6.8)
\end{aligned}$$

Thus, for this exercise,

$$\boldsymbol{\nabla}'R^{-1} = +R^{-2}\widehat{\mathbf{R}}. \qquad (6.9)$$

(b) The product rule for $\boldsymbol{\nabla}\times(a\mathbf{F})$ is derived as follows using index notation,

$$\begin{aligned}
[\boldsymbol{\nabla}\times(a\mathbf{F})]_i &= \varepsilon_{ijk}\nabla_j(aF_k), \qquad (6.10)\\
&= \varepsilon_{ijk}(F_k\nabla_j a \;+\; a\nabla_j F_k), \qquad (6.11)\\
&= \varepsilon_{ijk}(\boldsymbol{\nabla}a)_j F_k \;+\; a\varepsilon_{ijk}\nabla_j F_k, \qquad (6.12)\\
\therefore\quad [\boldsymbol{\nabla}\times(a\mathbf{F})]_i &= [(\boldsymbol{\nabla}a)\times\mathbf{F}]_i \;+\; [a\boldsymbol{\nabla}\times\mathbf{F}]_i. \qquad (6.13)\\
\therefore\quad \boldsymbol{\nabla}\times(a\mathbf{F}) &= (\boldsymbol{\nabla}a)\times\mathbf{F} \;+\; a\boldsymbol{\nabla}\times\mathbf{F}. \qquad (6.14)
\end{aligned}$$

Finally, re-arrange the product rule:

$$-(\boldsymbol{\nabla} a) \times \mathbf{F} \;=\; a\boldsymbol{\nabla} \times \mathbf{F} \;-\; \boldsymbol{\nabla} \times (a\mathbf{F}), \tag{6.15}$$

$$\therefore \quad \mathbf{F} \times (\boldsymbol{\nabla} a) \;=\; a\boldsymbol{\nabla} \times \mathbf{F} \;-\; \boldsymbol{\nabla} \times (a\mathbf{F}). \tag{6.16}$$

(c) We shall prove the corollary to Gauss' Theorem by applying Gauss' theorem for the vector field $\mathbf{c} \times \mathbf{F}(\mathbf{r})$ where $\mathbf{c}$ is a constant vector is

$$\int_V \boldsymbol{\nabla} \cdot (\mathbf{c} \times \mathbf{F})\, d^3r \;=\; \oint_S (\mathbf{c} \times \mathbf{F}) \cdot d\mathbf{S}. \tag{6.17}$$

Then we rearrange the right-hand side using the scalar triple product rule $(\mathbf{c} \times \mathbf{F}) \cdot d\mathbf{S} = (\mathbf{F} \times d\mathbf{S}) \cdot \mathbf{c}$, and on the left-hand side we can integrate by parts using the product rule $\boldsymbol{\nabla} \cdot (\mathbf{c} \times \mathbf{F}) = (\boldsymbol{\nabla} \times \mathbf{c}) \cdot \mathbf{F} - (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{c}$, giving

$$\int_V [(\boldsymbol{\nabla} \times \mathbf{c}) \cdot \mathbf{F} - (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{c}]\; d^3r \;=\; \oint_S (\mathbf{F} \times d\mathbf{S}) \cdot \mathbf{c}, \tag{6.18}$$

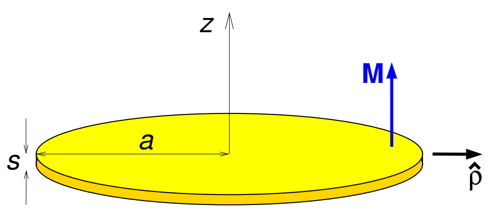
$$-\left[\int_V \boldsymbol{\nabla} \times \mathbf{F}\, d^3r\right] \cdot \mathbf{c} \;=\; \left[\oint_S \mathbf{F} \times d\mathbf{S}\right] \cdot \mathbf{c}, \tag{6.19}$$

$$\therefore \quad \int_V \boldsymbol{\nabla} \times \mathbf{F}\, d^3r = -\oint_S \mathbf{F} \times d\mathbf{S}, \tag{6.20}$$

as $\boldsymbol{\nabla} \times \mathbf{c} = \mathbf{0}$ for $\mathbf{c}$ being a constant vector.

6–2 A thin disc of magnetised material is coincident with the xy plane. It is of thickness s and radius a and has magnetisation $\mathbf{M} = M_0\hat{\mathbf{z}}$. Find the magnetisation current, and from this find the magnetic dipole moment of the disc. Compare this with what you would get by multiplying the disc's volume by $\mathbf{M}$.

Solution



The surface magnetisation current is

$$\mathbf{K}_{\text{mag}} \;=\; \mathbf{M}\times\widehat{\mathbf{n}} \;=\; M_0\,\widehat{\mathbf{z}}\times\widehat{\boldsymbol{\rho}} \;=\; M_0\,\widehat{\boldsymbol{\phi}}. \tag{6.21}$$

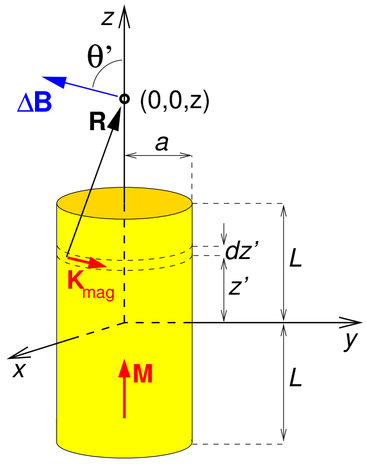
The net magnetisation current around the disc's circumference is $I_{\text{mag}} = sK_{\text{mag}} = sM_0$, and so the dipole moment is

$$\mathbf{m} \;=\; (\pi a^2) I_{\text{mag}}\,\widehat{\mathbf{z}} \;=\; \pi a^2 s\mathbf{M}. \tag{6.22}$$

This is just the volume multiplied by the magnetisation field.

6–3 Consider a permanent magnet in the form of a short cylinder of radius a extending along the z axis from $z = -L$ to $z = +L$ and having uniform magnetisation $\mathbf{M} = M_0\widehat{\mathbf{z}}$. (a) Find $\mathbf{B}$ and $\mathbf{H}$ at all points $(0, 0, z)$ on the cylinder's axis, and plot $\mathbf{B}(0, 0, z)$ and $\mathbf{H}(0, 0, z)$ vs. z. (b) Discuss whether the result obtained in part (a) obeys Ampère's law for $\mathbf{H}$ in integral form.

Solution



(a) Because the magnetisation field is uniform, there is no magnetisation volume current. The magnetisation surface current $\mathbf{K}_{\text{mag}}(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \widehat{\mathbf{n}}$ is zero on the two ends, and $\mathbf{K}_{\text{mag}}(\mathbf{r}) = M_0\widehat{\boldsymbol{\phi}}$ on the cylindrical surface.

The Biot-Savart law gives the magnetic field due to an arbitrary surface current distribution:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{K}_{\text{mag}}(\mathbf{r}')\,dS'] \times \widehat{\mathbf{R}}}{R^2} \tag{6.23}$$

The diagram shows the contribution $\Delta\mathbf{B}$ to the magnetic field at $(0,0,z)$ due to the surface magnetisation current in a small patch of the surface making up part of the strip of thickness dz' at z'. The components of $\Delta\mathbf{B}$ due to the surface currents in different patches around the strip which are not in the z direction will cancel each other out. This leaves only a z-component for the contribution $d\mathbf{B}$, due to the entire strip, so that

$$d\mathbf{B}(0,0,z) = \frac{\mu_0}{4\pi}\frac{(K_{\text{mag}}dz')(2\pi a)}{R^2}\cos\theta'\,\widehat{\mathbf{z}}, \tag{6.24}$$

$$= \frac{\mu_0}{4\pi}\frac{(Mdz')(2\pi a)}{R^2}\frac{a}{R}\widehat{\mathbf{z}}, \tag{6.25}$$

$$\therefore\ d\mathbf{B}(0,0,z) = \frac{\mu_0 M a^2 dz'}{2[(z-z')^2+a^2]^{3/2}}\widehat{\mathbf{z}}. \tag{6.26}$$

Hence, integrating over the entire cylindrical surface,

$$\mathbf{B}(0,0,z) = \int_{-L}^{L} \frac{\mu_0 M a^2 dz'}{2[(z-z')^2+a^2]^{3/2}}\widehat{\mathbf{z}}, \tag{6.27}$$

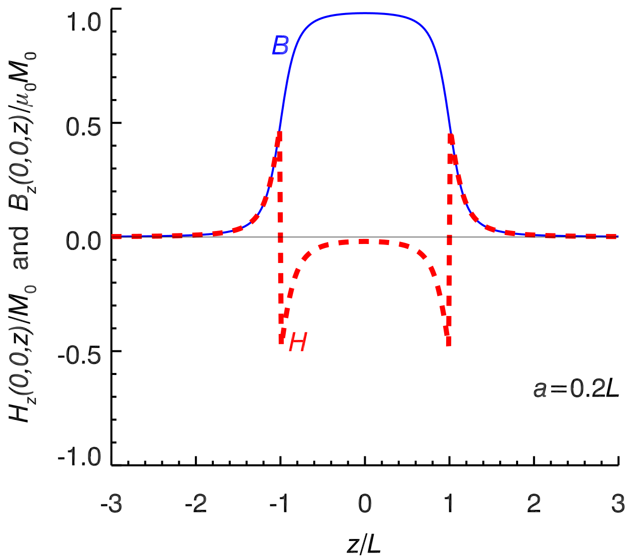
$$= \frac{\mu_0 M}{2}\left[\frac{z+z'}{\sqrt{(z+z')^2+a^2}}\right]_{-L}^{L}\widehat{\mathbf{z}}, \tag{6.28}$$

$$\therefore\ \mathbf{B}(0,0,z) = \frac{\mu_0 M}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z-L}{\sqrt{(z-L)^2+a^2}}\right]\widehat{\mathbf{z}}. \tag{6.29}$$

Now, $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, and so

$$\mathbf{H}(0,0,z) = \begin{cases} \frac{M_0}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z-L}{\sqrt{(z-L)^2+a^2}}\right]\widehat{\mathbf{z}} & (|z|>L) \\ \frac{M_0}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z-L}{\sqrt{(z-L)^2+a^2}}\right]\widehat{\mathbf{z}} - M_0\widehat{\mathbf{z}} & (|z|<L) \end{cases} \tag{6.30}$$

B and **H** are plotted below.



(b) Ampère's law in integral form is

$$\oint_{\Gamma} \mathbf{H}\cdot d\mathbf{r} = I_{f,\,\mathrm{encl.}} \tag{6.31}$$

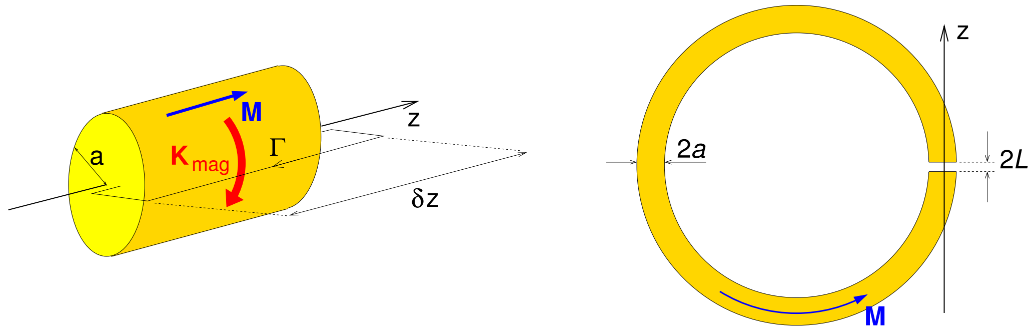
where $I_{f,\,\mathrm{encl.}}$ is the net free current through loop Γ. Since there are no free currents the integral must be zero for any closed path. We only know $\mathbf{H}$ on the z axis, and at $|\mathbf{r}| = \infty$ where $\mathbf{H}$ must be zero because the source of magnetic field, in this case the magnet, is localised near the origin. But we can construct a closed loop which has as part of it the entire z-axis as follows: $(0, 0, -\infty)$ to $(0, 0, +\infty)$ to $(0, +\infty, +\infty)$ to $(0, +\infty, -\infty)$ and back to $(0, 0, -\infty)$. The integrand is zero except along the z-axis, so in this case Ampère's law in integral form is satisfied provided

$$\int_{-\infty}^{\infty} H_z(0, 0, z)dz \;=\; 0. \tag{6.32}$$

Examining the plot of H_z vs. z above, it appears that the integral is indeed zero, as could easily be checked by numerical integration.

6–4 A cylindrical rod of radius a and length $h \gg a$ is permanently magnetised along its length which coincides with the z direction, i.e. $\mathbf{M} = M_0\widehat{\mathbf{z}}$. (a) Find the surface magnetisation current $\mathbf{K}_{\mathrm{mag}}(\mathbf{r})$, and use it together with Ampere's law to find $\mathbf{B}$ and $\mathbf{H}$ inside and outside the rod (assume $h \to \infty$). (b) The rod is now bent into a circle of circumference $(h + 2L)$ such that there is an air gap of width $2L < a$. Plot $\mathbf{B}$ and $\mathbf{H}$ along the axis of the magnet in the vicinity of the air gap for the case of $a = 0.5$ and $L = 0.2$.

Solution



(a) The surface magnetisation current density is $\mathbf{K}_{\mathrm{mag}}(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \widehat{\mathbf{n}}$, hence

$$\mathbf{K}_{\mathrm{mag}}(a, \phi, z) \;=\; M_0\,\widehat{\mathbf{z}} \times \widehat{\boldsymbol{\rho}} \;=\; M_0\,\widehat{\boldsymbol{\phi}}. \tag{6.33}$$

This surface magnetisation current is similar to the current in a tightly-wound solenoid, and the magnetic field inside the rod can be calculated in the same way using Ampère's law $\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{\mathrm{encl}}$. From the symmetry of the problem $\mathbf{B}$ inside the rod can only be in the $\widehat{\mathbf{z}}$ direction.

For Amperian rectangular loop Γ (see diagram) with one side of length δz inside the rod at cylindrical radius ρ_1 and one outside at cylindrical radius ρ_2

$$[B_z(\rho_1, \phi, z) - B_z(\rho_2, \phi, z)]\;\delta z \;=\; \mu_0 M_0 \delta z. \tag{6.34}$$

That this is independent of ρ_2 and applies equally to $\rho_2 \to \infty$ (where $\mathbf{B} = 0$) tells us that $\mathbf{B} = 0$ outside the rod. Again, since the integral is independent of ρ_1 the magnetic field inside the rod is constant,

$$\mathbf{B}(\rho < a, \phi, z) \;=\; \mu_0 M_0\,\widehat{\mathbf{z}} \;=\; \mu_0 \mathbf{M}. \tag{6.35}$$

Finally,

$$\mathbf{H} \;=\; \left(\frac{\mathbf{B}}{\mu_0}\right) - \mathbf{M} \;=\; 0 \tag{6.36}$$

everywhere. Note that this result is for an (unrealistic) infinite magnetised rod, and that near the two ends of the rod **B** would be different, and **H** would be non-zero.

(b) To find **B** and **H** on the axis of the magnet near the air gap we can use the information that $h \gg a$ and assume the magnet in this region is approximately straight, with its axis being along the z-axis, and with the air gap extending from $z = -L$ to $z = +L$. In that case we can imagine that a short magnet of length $2L$ has been removed from an infinite magnetised rod.

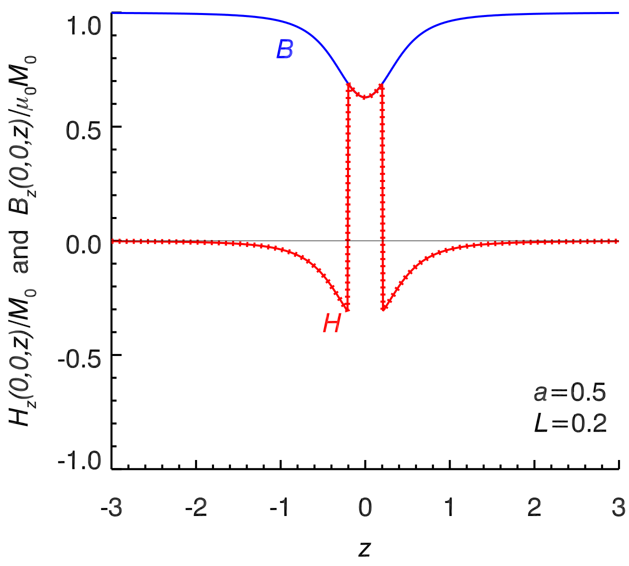
Using the principle of superposition, we can get **B** in the vicinity of the air gap by subtracting the field of the short magnet of length $2L$ from the field of an infinite straight magnetised rod with no air gap. For the short magnet we can use the results from Exercise 6–3 for the magnetic field of the short magnet. Thus,

$$\mathbf{B}(0,0,z) \;=\; \mu_0 M_0 \widehat{\mathbf{z}} \;-\; \frac{\mu_0 M_0}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z-L}{\sqrt{(z-L)^2+a^2}}\right]\widehat{\mathbf{z}}. \tag{6.37}$$

Now, $\mathbf{H} \;=\; \mathbf{B}/\mu_0 - \mathbf{M}$, and so

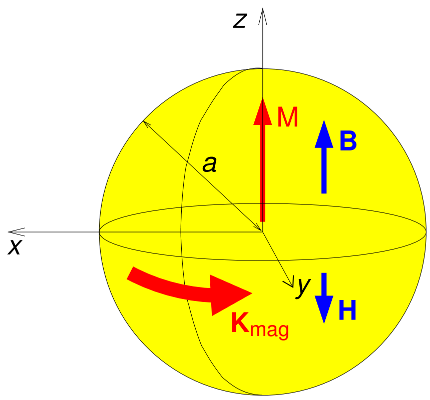
$$\mathbf{H}(0,0,z) \;=\; \begin{cases} -\,\frac{M_0}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z-L}{\sqrt{(z-L)^2+a^2}}\right]\widehat{\mathbf{z}} & (|z| > L) \\ M_0\widehat{\mathbf{z}} \;-\; \frac{M_0}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z-L}{\sqrt{(z-L)^2+a^2}}\right]\widehat{\mathbf{z}} & (|z| < L) \end{cases} \tag{6.38}$$

B and **H** are plotted below. Note that the area under the plot of $H_z(0,0,z)$ vs. z appears to be zero in agreement with Ampère's law in integral form for the case of no free currents.


1.0
0.5
0.0
-0.5
-1.0
-3
-2
-1
0
1
2
3
z
$H_z(0,0,z)/M_0$ and $B_z(0,0,z)/\mu_0 M_0$
B
H
a=0.5
L=0.2

6–5 Consider a permanently magnetised sphere of radius a with uniform magnetisation $\mathbf{M}(\mathbf{r}) = M_0\widehat{\mathbf{z}}$. (a) Find the surface magnetisation current density, and use this to find the magnetic dipole moment of the sphere. Compare this with what you expect given the volume of the sphere and the magnetisation field. (b) Find $\mathbf{B}$ and $\mathbf{H}$ at the centre of the sphere.

Solution



(a) The surface magnetisation current density is

$$\mathbf{K}_{\rm mag}(a,\phi,z) \;=\; \mathbf{M}\times\widehat{\mathbf{n}} \;=\; M_0\,\widehat{\mathbf{z}}\times\widehat{\mathbf{r}} \;=\; M_0\sin\theta\,\widehat{\boldsymbol{\phi}}. \tag{6.39}$$

The magnetic dipole moment of the surface magnetisation current distribution is

$$\mathbf{m} \;=\; \frac{1}{2}\oint \mathbf{r}\times\mathbf{K}_{\rm mag}(\mathbf{r})dS. \tag{6.40}$$

From symmetry arguments, the dipole moment must be $\mathbf{m} = m_z\,\widehat{\mathbf{z}}$ where

$$\begin{aligned}
m_z &= \widehat{\mathbf{z}}\cdot\frac{1}{2}\int_{-1}^{1} a\,\widehat{\mathbf{r}}\times(M_0\sin\theta\,\widehat{\boldsymbol{\phi}})2\pi a^2 d(\cos\theta) && (6.41)\\
&= \frac{2\pi a^3 M_0}{2}\int_{-1}^{1}\sin\theta\,\widehat{\mathbf{z}}\cdot(-\widehat{\boldsymbol{\theta}})\,d(\cos\theta) && (6.42)\\
&= \pi a^3 M_0\int_{-1}^{1}\sin^2\theta\,d(\cos\theta) && (6.43)\\
&= \pi a^3 M_0\int_{-1}^{1}(1-\cos^2\theta)\,d(\cos\theta) && (6.44)\\
&= \frac{4}{3}\pi a^3 M_0 && (6.45)
\end{aligned}$$

as expected.

(b) We can use the Biot-Savart law to obtain the magnetic field at the centre of the sphere

$$\mathbf{B}(0,0,0) = \frac{\mu_0}{4\pi}\oint \frac{\mathbf{K}_{\mathrm{mag}}(\mathbf{r}') \times \widehat{\mathbf{R}}}{R^2} dS'. \tag{6.46}$$

From symmetry, this must be $\mathbf{B}(0,0,0) = B(0,0,0)\,\widehat{\mathbf{z}}$ where

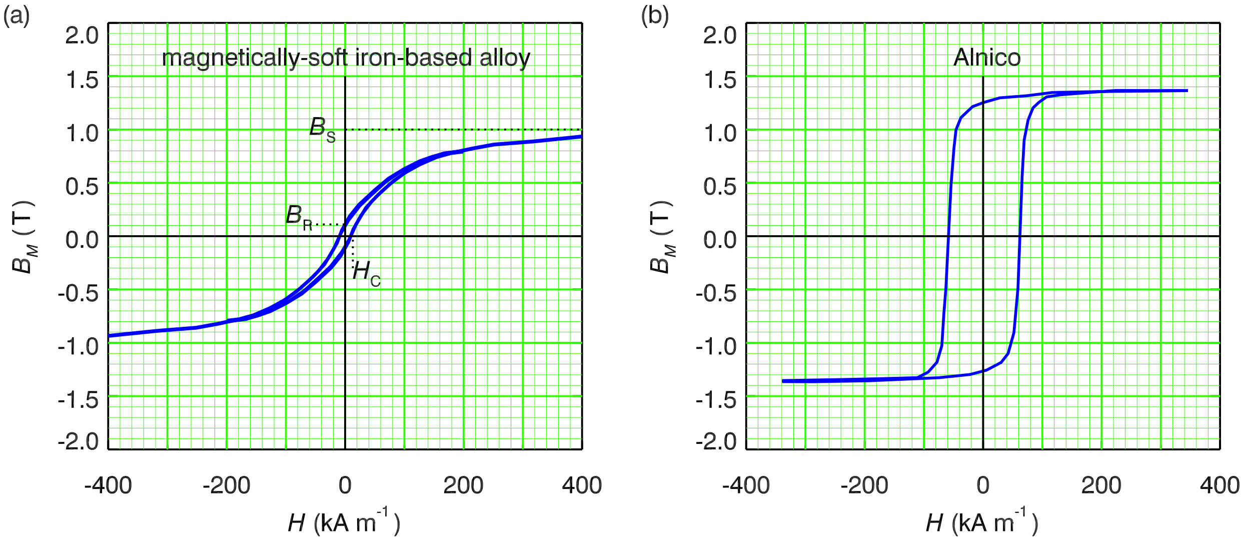
$$\begin{aligned}
B(0,0,0) &= \widehat{\mathbf{z}} \cdot \frac{\mu_0}{4\pi}\int_{-1}^{1} \frac{(M_0 \sin\theta\,\widehat{\boldsymbol{\phi}}) \times (-\widehat{\mathbf{r}})}{a^2} 2\pi a^2 d(\cos\theta), &(6.47)\\
&= \frac{\mu_0 M_0}{2}\int_{-1}^{1} \sin\theta\,\widehat{\mathbf{z}} \cdot (-\widehat{\boldsymbol{\theta}})\, d(\cos\theta), &(6.48)\\
&= \frac{\mu_0 M_0}{2}\int_{-1}^{1} \sin^2\theta\, d(\cos\theta), &(6.49)\\
&= \frac{\mu_0 M_0}{2}\int_{-1}^{1} (1-\cos^2\theta)\, d(\cos\theta), &(6.50)\\
\therefore\ B(0,0,0) &= \frac{2}{3}\mu_0 M_0. &(6.51)\\
\therefore\ \mathbf{B}(0,0,0) &= \frac{2}{3}\mu_0 \mathbf{M} &(6.52)
\end{aligned}$$

and is in the same direction as $\mathbf{M}$.

Now $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, so that $\mathbf{H}(0,0,0) = -\frac{1}{3}\mathbf{M}$ and it is in the opposite direction to $\mathbf{M}$.

6–6 Consider the hysteresis loops of the magnetically-soft iron-based amorphous alloy and the magnetically-hard alloy of iron, aluminium, nickel and cobalt shown in Chapter 6.
(a) Estimate the work done to bring 1 cm^3 of each material through one cycle of the hysteresis loop. (b) Two transformers operating at 50 Hz have magnetic cores of volume 100 cm^3 (one of each type of material) and are (unwisely) operated at a current at which saturation occurs. How much power is lost as heat in each case? [You could print the hysteresis plots and estimate the area by drawing over it a grid and measuring by hand sufficient points on the graph to get within say 20% accuracy for the area.]

Solution



Grid lines have been drawn over the hysteresis plots above. The area of the upper half of the hysteresis loop (for positive B_M) is identical to that of the lower half. Read off the (horizontal) "widths" in H at "heights" $B_M = 0, 0.1, 0.2, \ldots$ (T). The approximate values are tabulated below.

B_M (T)	iron-based alloy ΔH (kA m^{-1})	Alnico ΔH (kA m^{-1})
0.0	30	120
0.1	25	120
0.2	25	120
0.3	20	120
0.4	20	120
0.5	15	120
0.6	15	120
0.7	15	120
0.8	10	120
0.9	10	120
1.0	0	120
1.1		115
1.2		100
1.3		90
1.4		0
sum:	185	1625

The areas of the two loops are approximately:

$$\text{iron-based alloy:} \quad \oint B_M dH = 2 \times (185 \text{ kA m}^{-1}) \times (0.1 \text{ T}) \ = \ 3.7 \times 10^4 \text{ J}, \tag{6.53}$$

$$\text{Alnico:} \quad \oint B_M dH = 2 \times (1625 \text{ kA m}^{-1}) \times (0.1 \text{ T}) \ = \ 3.2 \times 10^5 \text{ J}. \tag{6.54}$$

Now, this is for a sample volume of 1 m^3. For a magnetic core of volume 100 cm^3=10^{-4} m^3, the energy to take the sample around one cycle is then 3.7 J (iron-based alloy) or 32 J (Alnico).

If the sample is used in a transformer operating at 50 Hz with the magnetic field saturating, the energy lost to heat in one second is 50 times the energy for one cycle, i.e. 185 W (iron-based alloy) or 1.6 kW (Alnico). If the core were made of Alnico, the rate of heating would be similar to that of a domestic electric room heater!