Fast Computation of Polynomial Data Points over Simplicial Face Values

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Abstract

Polynomial functions F of degree m have a form in the Bernstein basis defined over $l-$dimensional simplex W. The Bernstein coefficients exhibit a number of special properties. The function F can be optimized by the smallest and largest Bernstein coefficients (enclosure bound) over W. By a proper choice of barycentric subdivision steps of W, we prove the inclusion property of Bernstein enclosure bounds. To this end, we provide an algorithm that computes the Bernstein coefficients over subsimplices. These coefficients are collected in an $l-$dimensional array in the field of computer aided geometric design such a construct is typically classified as a patch. We show that the Bernstein coefficients of F over the faces of a simplex are coincide with the coefficients contained in the patch.

Keywords: Bernstein expansion; simplex; computing of range values; inclusion of bounds; face values.

1 Introduction

This paper considers the computation of bounding functions for multivariate polynomials over simplices. Bounding the range of functions is an important issue in many areas of applied mathematics and computation like global optimization, computer aided geometric design and robust control. The expansion of a (multivariate) polynomial function F is given into the so-called *simplicial Bernstein basis* over a simplex (triangles), [2, 3], [5], [15, 16]. The Bernstein expansion is used due to the tightness of the enclosure and its rate of convergence properties to the exact range [9]. The expansion was extended to rational polynomial functions in [6], [10] and [21]. The Bernstein basis is also used to certify whether a given function is positive over a simplex [15]. Our main goal with computing enclosure bounds for polynomials is to certify the positivity of polynomials over subsimplices. Certifying the positivity of functions can be improved by subdividing the given simplex at a specific point. This can be satisfied if the inclusion property of the enclosure bound holds. This property, which is important in optimization and interval computations [17, Section 1.4], states that the Bernstein enclosure bound over a subdomian is contained in the enclosure bound over the whole domain. Specifically, if the domain is shrunk or subdivided to a smaller subdomain, then the Bernstein bound shrinks too. The inclusion property is widely studied in the literature for related functions over some domains. The proof of this property for a univariate polynomial over one dimensional interval was given in [13]. In [7], a short proof for the tensorial case over a box is given. In [15], Leroy investigated a particular case of the property by reparametrization of simplices. The same problem was addressed over different domains by other authors, see [1] and [19]. On the other hand of applications, in [11, 12], the key to finding a Lyapunov function for a polynomial linear system is to find positivity certificates, where the inclusion isotonicity satisfies the local positivity certificates over subsimplices. However, computing Bernstein coefficients on the face values of a simplex is the fastest way for computing the minimum bound. The face values property for polynomials over boxes was addressed in [8]. In this paper, we extend this property to the simplicial case together with proving further important properties of the Bernstein basis over a simplex. Subdivision of a simplex is a widely applied scheme, wherein a starting domain is subdivided into subsimplices. It follows that the simplicial Bernstein form is inclusion isotone and the positivity certificates are local. First, we provide a representation for the Bernstein coefficients and the enclosure bound of polynomials over a simplex. Subsequently, we provide a fast algorithm for computing the Bernstein coefficients over subsimplices. With this method, we prove that the enclosure bound over subsimplices is optimized by the enclosure bound over the whole simplex. Finally, the simplicial face values property holds, and number of calculation steps needed for our algorithm depends on the degree of F and number of dimensions.

The organization of this paper is as follows. In the next section, we recall the main basics and background of Bernstein expansion. In Section 3, we start our contributions with providing a representation for the Bernstein coefficients and the inclusion isotonicity property over a simplex. In Section 4, we prove the simplicial face values property. Applications of our algorithm and face values property are given in Section 5. Section 6 comprises the future work. Last, conclusions of the main contributions are given in Section 7.

2 Background

In this section, we introduce some notation and necessary material about the simplicial Bernstein basis. Let $w_0, \dots, w_l$ be $l+1$ points of $\mathbb{R}^l$ $(l \geq 1)$, the ordered list $W = [w_0, \dots, w_l]$ is called simplex of vertices $w_0, \dots, w_l$. Throughout the paper, $W = [w_0, \dots, w_l]$ will be denoted as a non-degenerate simplex of $\mathbb{R}^l$; viz the points $w_0, \dots, w_l$ are affinely independent. Let $\lambda_0, \dots, \lambda_l$ be the associated barycentric coordinates of W, i.e., the linear polynomials of $\mathbb{R}[X] = \mathbb{R}[X_1, \dots, X_l]$ such that $\sum_{i=0}^{l} \lambda_i(x) = 1$, and $\forall x \in \mathbb{R}^l$, $x = \sum_{i=0}^{l} \lambda_i(x) w_i$. Define the convex hull $|W|$ of the points $w_0, \dots, w_l$. We also refer to the multi-index $\alpha = (\alpha_0, \dots, \alpha_l) \in \mathbb{N}^{l+1}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_l$.

Without loss of generality, we will often consider the standard simplex $\Delta = [e_0, e_1, \dots, e_l]$, where $(e_1, \dots, e_l)$ denotes the canonical basis of $\mathbb{R}^l$, and $e_0 = (0, \dots, 0)$ the origin. This is not a restriction since any simplex W in $\mathbb{R}^l$ can be mapped linearly upon Δ. Hence, we assume that $W = \Delta$. It follows that if $x = (x_1, \dots, x_l) \in \Delta$, then $(\lambda_0, \dots, \lambda_l) = (1 - \sum_{i=1}^{l} x_i, x_1, \dots, x_l)$. For $\hat{\beta}, \hat{\alpha} \in \mathbb{N}^l$ with $\hat{\beta} \leq \hat{\alpha}$, we define

$$\binom{\hat{\alpha}}{\hat{\beta}} := \prod_{i=1}^{l} \binom{\alpha_i}{\beta_i}.$$

If $M \in \mathbb{N}$ so that $|\hat{\beta}| \leq M$, we use the notation

$$\binom{M}{\hat{\beta}} := \frac{M!}{\beta_1! \dots \beta_l! . (M - |\hat{\beta}|)!}.$$

The Bernstein basis of degree M over W is defined by $(S_{\alpha}^{(M)})_{|\alpha|=M}$, where

$$S_\alpha^{(M)}(\lambda) = \binom{M}{\alpha}\lambda^\alpha.$$

For $x \in \mathbb{R}^l$ its multi-powers are $x^{\hat{\beta}} := \prod_{i=1}^{l} x_i^{\beta_i}$. Let a (power form) polynomial function F of degree m,

$$F(x) = \sum_{|\hat{\beta}| \le m} a_{\hat{\beta}} x^{\hat{\beta}}, \tag{1}$$

F can be expanded in the simplicial Bernstein form of degree $m \le M$ as

$$F(x) = \sum_{|\alpha|=M} C_\alpha(F, M, \Delta) S_\alpha^{(M)},$$

where $C_\alpha(F, M, \Delta)$ are called the Bernstein coefficients of F of degree M over Δ.

The grid points of degree M associated to W are the points

$$w_\alpha(M, W) = \frac{\alpha_0 w_0 + \cdots + \alpha_l w_l}{M} \in \mathbb{R}^l \ \ (|\alpha| = M)$$

which leads us to the control points of F of degree M over W

$$(w_\alpha(M, W),\ b_\alpha(F, M, W)) \in \ \mathbb{R}^{l+1} \ \ (|\alpha| = M). \tag{2}$$

The control points of F form its control net of degree M.

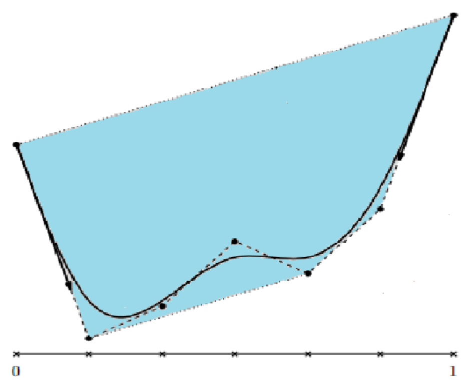

Figure 1. The curve of a univariate polynomial of degree 6, and the convex hull (shaded) of its control points (marked by points) optimizes the curve by the minimum and maximum control points.

Proposition 2.1. *[16, Proposition 2.7] For a polynomial $F \in \mathbb{R}_m[X]$ and $M \ge m$, the following properties hold.*

(i) Linear precision: If degree $F \le 1$, then

$$C_\alpha(F, M, W) = F(w_\alpha(M, W)),\ \forall |\alpha| = M;$$

(ii) Interpolation at the vertices: If $(e_0, \ldots, e_l)$ denotes the canonical basis of $\mathbb{R}^{l+1}$, then

$$C_{Me_i} = F(w_i),\ 0 \le i \le l; \tag{3}$$

(iii) Convex hull property: The graph of F over W is contained in the convex hull of its associated control points, see Figure 1;

(iv) Range enclosing property:

$$\min_{|\alpha|=M} C_\alpha(F, M, W) \le F(x) \le \max_{|\alpha|=M} C_\alpha(F, M, W),\ \forall x \in W. \tag{4}$$

Put the enclosure bound $G(F, M, W)$,

$$G(F, M, W) := [\min C_\alpha(F, M, W), \max C_\alpha(F, M, W)]. \tag{5}$$

It follows from (iv) in Proposition 2.1 that $G(F, M, W)$ optimizes F over W, Figure 1.

3 Bounding Values

Bernstein form refers to the expansion of a polynomial in Bernstein basis, a procedure which can be employed in computation and optimization of functions. In the following remark, we provide the simplicial Bernstein function of F on Δ.

Remark 3.1. *For $\hat{\alpha}, \hat{\beta} \in \mathbb{N}^l$, let F be a power form polynomial of degree m. The Bernstein polynomial form of F of degree $m \leq M$ over Δ is presented by*

$$F(x) = \sum_{|\hat{\alpha}|+\alpha_0=M} C_{(\hat{\alpha},\alpha_0)}(F, M, \Delta) S^{(M)}_{(\hat{\alpha},\alpha_0)}(x), \tag{6}$$

where

$$S^{(M)}_{(\hat{\alpha},\alpha_0)}(x) = \binom{M}{\hat{\alpha}, \alpha_0} x^{\hat{\alpha}} (1 - |x|)^{\alpha_0}, \; |\hat{\alpha}| + \alpha_0 = M$$

and

$$C_{(\hat{\alpha},\alpha_0)}(F, M, \Delta) = \sum_{\hat{\beta} \leq \hat{\alpha}} \frac{\binom{\hat{\alpha}}{\hat{\beta}}}{\binom{M}{\hat{\beta}}} a_{\hat{\beta}}. \tag{7}$$

The Bernstein coefficients of degree M $(m \leq M)$ can be given as linear combinations of Bernstein coefficients of degree m, see [15, Proposition 1.12].

Let $(\hat{e}_0, \ldots, \hat{e}_l)$ be points of $\mathbb{R}^{l+1}$, $\hat{e}_i = (\underbrace{0, \ldots, 0}_{i}, 1, \underbrace{0, \ldots, 0}_{l-i})$, $i = 0, \ldots, l$. Multiplication of (6) with $1 = (|x| + 1 - |x|)^{M+1}$ and rearranging the output, we get, see [4, Lemma 1.1],

$$F = \sum_{|\beta|=M+1} C_\beta(F, M+1, \Delta) S_\beta^{M+1},$$

where

$$C_\beta(F, M+1, \Delta) = \frac{1}{M+1} \sum_{i=0}^{l} \beta_i C_{\beta - \hat{e}_i}(F, M, \Delta).$$

Hence, the range of F of degree $M+1$ over Δ can be optimized by

$$G(F, M+1, \Delta) \subseteq G(F, M, \Delta). \tag{8}$$

Finally, $F(x)$ in (4) can be tightly optimized by $G(F, M+1, \Delta)$.

Remark 3.2 *The number of Bernstein coefficients of an $l-$variate polynomial of degree M is equal* $\binom{M+l}{M}$.

3.1 De Casteljau Algorithm

The simplicial Bernstein coefficients are found in all coordinate directions by application of De Casteljau algorithm. Assume that Δ has been subdivided at a point $\hat{w} \in \mathbb{R}^l$, i.e., $\Delta = W^{[1]} \cup \ldots \cup W^{[l]}$. Then, the following algorithm computes the coefficients over $W^{[i]}$, $i \in \{0, \ldots, l\}$.

Algorithm 3.1. (De Casteljau [14], [18])
Input: The standard simplex Δ, the Bernstein coefficients $C_\alpha(F, m, \Delta)$ of degree m, and $\widehat{w} \in \mathbb{R}^l$.
Output: The Bernstein coefficients $C_\alpha(F, m, W^{[i]})$, forevery $i \in \{0, \dots, l\}$.
Initialization: $\forall |\alpha| = m,\ C_\alpha^{(0)} := C_\alpha(F, m, \Delta)$.

for $j = 1, \dots, m$ ***do***

for $|\alpha| = m - j$ ***do***

$$C_\alpha^{(j)} = \sum_{i=0}^{l} \lambda_i(\widehat{w}) C_{\alpha+\hat{e}_i}^{(j-1)} \tag{9}$$

end for

end for

return $C_\alpha(F, m, W^{[i]}) = C_{\alpha^{[i]}}^{(\alpha_i)} \quad (|\alpha| = m,\ i = 0, \dots, l)$.

3.2 Inclusion Property

Here, we show the inclusion isotonicity property of the simplicial Bernstein form. We apply the *barycentric subdivision strategy*, which is a particular method of subdividing Δ at $\widehat{w} \in \mathbb{R}^l$ into subsimplices, e.g., Figure 2.

Let

$$\widehat{w} = \sum_{i=0}^{l} \lambda_i(\widehat{w}) e_i. \tag{10}$$

Specifically, we aim at computing the Bernstein coefficients over a subsimplex $U = [\widehat{w}_0, \dots, \widehat{w}_l]$, which is extracted from Δ by the barycentric subdivision strategy, as convex combinations of the Bernstein coefficients over Δ. In order to do so, we compute the Bernstein coefficients in a particular coordinate direction, r, since the De Casteljau algorithm computes the coefficients in all coordinate directions: Let $\widehat{w}_r,\ r \in \{0, \dots, l\}$, be a non edge point with respect to Δ. Then we set

$$U^{[\widehat{w}_r]} = [e_0, \dots, \widehat{w}_r, \dots, e_l], \tag{11}$$

$$(e_{l+1} := e_0,\ e_{-1} := e_l)$$

where

$$\widehat{w}_r = \lambda_0(\widehat{w}_r)e_0 + \dots + \lambda_i(\widehat{w}_r)e_i + \dots + \lambda_l(\widehat{w}_r)e_l. \tag{12}$$

Let $\widehat{w}_r$ be an edge point with respect to Δ. Then

$$\widehat{w}_r = \lambda_i(\widehat{w}_r)e_i + (1 - \lambda_i(\widehat{w}_r))e_{i+1}. \tag{13}$$

If we subdivide Δ at an edge point, $\widehat{w}_r$, then Δ will be subdivided into two subsimplices, $U^{[r]},\ U^{[r+1]}$, extracted from $\lambda_i(\widehat{w}_r), \lambda_{i+1}(\widehat{w}_r)$, and we call them the extracted subsimplices. Otherwise (non-edge point), Δ can be subdivided into $\leq l + 1$ (constructed) subsimplices, e.g., Figure 2.

It is sufficient to show that the inclusion isotonicity holds true if we compute the coefficients in r^{th} coordinate direction, $0 < \lambda_r(\widehat{w}_r) < 1$, with respect to the extracted simplex. In other words, by subdivision at $\widehat{w}_0$ we proceed into λ_0, at $\widehat{w}_1$ we proceed into λ_1, and so on (need not to be successively).
Let r be integer such that for some $r \in \{0, \dots, l\}$, $0 < \lambda_r(\widehat{w}_r) < 1$, and for all $i,\ i \neq r,\ 1 > \lambda_i(\widehat{w}_r) \geq 0$. The following algorithm computes the Bernstein coefficients in the r^{th} coordinate direction to extract a new subsimplex U, where the barycentric subdivision method is applied.

Algorithm 3.2. (Computing of Bernstein coefficients over a subsimplex)

Given: Simplices $U^{[\hat{w}_r]}$ is contained in Δ, and the Bernstein coefficients on Δ.
Wanted: The Bernstein coefficients on U as convex combinations of the ones on Δ.
Ensure: $1 \geq \lambda_i \geq 0,\ i = 0, \ldots, l$, with $1 > \lambda_{i_0} > 0$, for some $i_0 \in \{0, \ldots., l\}$.

Initialize: $\forall |\alpha| = m,\ C_\alpha^{(0)} := C_\alpha(F, m, \Delta)$.
Choose: $r \in \{0, \ldots, l\},\ r \neq i_0,\ 1 > \lambda_r(\hat{w}_r) > 0$.

for $d = 1, \ldots, m$ ***do***
for $|\alpha| = m - d$ ***do***
if $\lambda_r + \lambda_{i_0} = 1$ ***then***

$$C_\alpha^{(m)} = \lambda_r C_{\alpha+\hat{e}_r}^{(m-1)} + (1 - \lambda_r) C_{\alpha+\hat{e}_{r+1}}^{(m-1)}$$

else

$$C_\alpha^{(m)} = \lambda_0 C_{\alpha+\hat{e}_0}^{(m-1)} + \ldots + \lambda_r C_{\alpha+\hat{e}_r}^{(m-1)} + \ldots + \lambda_l C_{\alpha+\hat{e}_l}^{(m-1)}$$

end if
end for
end for
return

$$C_\alpha(F, m, U^{[\hat{w}_r]}) = C_{\alpha^{[r]}}^{(\alpha_r)}\ \ (|\alpha| = m).$$

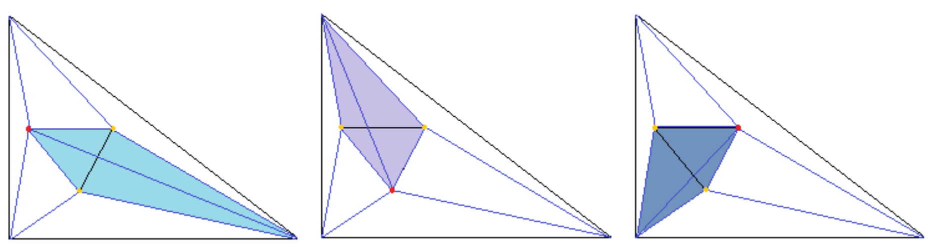

Figure 2: Subsimplices are extracted by barycentric subdivision steps at edge and inner points.

Theorem 3.1. *Let* U *be a subsimplex, which is extracted by the barycentric subdivision strategy from* Δ*. Then*

$$G(F, m, U) \subseteq G(F, m, \Delta).$$

Proof. Let $U^{[\hat{w}_r]}$ be a subsimplex extracted at $\lambda_r(\hat{w}_r)$ by subdividing at $\hat{w}_r,\ r \in \{0, \ldots, l\}$. We proceed into r^{th} coordinate direction and return to Algorithm 3.2 for all r (full algorithm). Let (for simplicity) $r = 0$, then we have

$$U^{[\hat{w}_0]} = [\hat{w}_0, e_1, \ldots, e_l]$$

be a subsimplex of Δ. By Algorithm 3.2, the Bernstein coefficients on $U^{[\hat{w}_0]}$ are convex combinations of the coefficients on Δ. Repeatedly splitting at the remaining $\hat{w}_r,\ r = 1, \ldots, l$, with respect to the extracted simplices, then we will have finally at $\hat{w}_l$ the Bernstein coefficients over

$$U = [\hat{w}_0, \ldots, \hat{w}_l]$$

are given as convex combinations of the coefficients over $U^{[\hat{w}_{l-1}]}$, which completes the proof.

Corollary 1. *Denote the union of enclosure bounds over* $W^{[i]}$, $i = 0, \ldots, l$, *by* $G(F, M, W^{[\Delta]})$. *For all* $x \in W^{[\Delta]}$ *and* $W^{[0]} \cup \ldots \cup W^{[l]} \subseteq \Delta$, *it follows from Theorem 3.1 that*

$$[minF(x), maxF(x)] \subseteq G(F, M, W^{[\Delta]}) \subseteq G(F, M, \Delta).$$

Example 3.1. *Given the polynomial* $F = 5x^2 - 2x + 1$ *is over* $W = [-1,1]$, *where the ordered list of Bernstein coefficients* $C(F, 2, [-1,1]) = (8, -4, 4)$. *The enclosure bound is* $G(F, M, W) = [-4, 8]$. *However, by the first binary splitting of* Δ, *the lists of Bernstein coefficients over each subsimplex are* $C(F, 2, [-1,0]) = (8,2,1)$ *and* $C(F, 2, [0,1]) = (1,0,4)$. *Hence the union of enclosure bounds* $G(F, M, W^{[\Delta]}) = [0,8]$ *is contained in* $G(F, M, W)$.

Remark 3.3. *From the edge splitting of a simplex, we have just two barycentric coordinates of* $\widehat{w}$ *with respect to* Δ, $h = 2$ *say. We have at the non edge splitting step* $h \leq l + 1$.

The Complexity (number of Bernstein coefficients and the computation steps) needed to perform one call to Algorithm 3.2 is given in the following lemma.

Lemma 3.2. *Let* F *of degree* m *be given over* Δ *and* $2 \leq h \leq l + 1$. *The number of computation steps needed for one call to Algorithm 3.2 at* $\widehat{w}$ *in* Δ *is*

$$\frac{h(m+l)!}{(m-1)!(l+1)!}.$$

Proof. The number of Bernstein coefficients of degree m over l-dimensional simplex is

$$R(m, l) := \binom{m+l}{l} = \frac{(m+l)!}{m!.l!}.$$

Note that in one call to Algorithm 3.2 over l-simplex, we need to compute the Bernstein coefficients of l variables of degrees 0 to $m - 1$. Therefore, the number of calculation steps for Bernstein coefficients is

$$H(j, l) := \sum_{j=0}^{m-1} R(j, l) = \binom{m+l}{m-1} = \frac{(m+l)!}{(m-1)!(l+1)!}.$$

Thus, the needed computation steps are

$$\frac{h(m+l)!}{(m-1)!.(l+1)!}.$$

4 Simplex Face Values

In this section, we mathematically show that the Bernstein coefficients of a polynomial F over the faces of a simplex Δ are the same as the coefficients located at the corresponding faces of the (patch) array of Bernstein coefficients over Δ. For $\hat{\alpha} \in \mathbb{N}^l$, let $\hat{\alpha}^{[i]} = (\alpha_1, .., \alpha_{i-1}, \alpha_{i+1}, .. \alpha_l)$ be a multi-index of $\mathbb{N}^{l-1}$. If $\boldsymbol{x}' = (x_1, \ldots, x_i, \ldots, x_l)$ is lying on the boundary of Δ then it is contained in a subsimplex of dimension $l - 1$, Δ' say.

Lemma 4.1. *Let* $\boldsymbol{x}' = (x_1, \ldots, x_i, \ldots, x_l)$ *be lying on the boundary of* Δ, $x_i = 1 - \sum_{j=1, j\neq i}^{l} x_j$. *Let* $0 < x_j < 1$, *for all* $j = 1, \ldots, l$. *Then the Bernstein coefficients of* F *over the* $(l-1)$*-dimensional simplex* Δ' *are given by*

$$C_{(\hat{\alpha}^{[i]}, m-|\hat{\alpha}^{[i]}|)}(F, m, \Delta') = \sum_{\hat{\beta}^{[i]} \leq \hat{\alpha}^{[i]}, \beta_i \leq m-|\hat{\alpha}^{[i]}|} \frac{\binom{\hat{\alpha}^{[i]}}{\hat{\beta}^{[i]}}\binom{m-|\hat{\alpha}^{[i]}|}{\beta_i}}{\binom{m}{\hat{\beta}}} a_{\hat{\beta}}. \tag{14}$$

Proof. Put $(\boldsymbol{x}^{[i]}) := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l)$ and $\hat{\beta}^{[i]} = (\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_l)$. Since $\boldsymbol{x}'$ is lying on the boundary of Δ and $0 < x_j < 1$, for all $j = 1, \ldots, l$, then $\lambda_0(\boldsymbol{x}') = 0$. The Bernstein expansion of F with respect to Δ' is given as follows:

$$p(x_1, \ldots, x_i, \ldots, x_l) = \sum_{|\hat{\beta}| \leq m} a_{\hat{\beta}} (\boldsymbol{x}^{[i]})^{\hat{\beta}^{[i]}} (1 - |(\boldsymbol{x}^{[i]})|)^{\beta_i} (|\boldsymbol{x}'| + 1 - |\boldsymbol{x}'|)^{m-|\hat{\beta}|}$$

$$= \sum_{|\hat{\beta}| \leq m} \sum_{|\hat{\gamma}^{[i]}| \leq m-|\hat{\beta}|} a_{\hat{\beta}} \binom{m-|\hat{\beta}|}{\gamma^{[i]}} (\boldsymbol{x}^{[i]})^{\hat{\beta}^{[i]}+\hat{\gamma}^{[i]}} (1 - |(\boldsymbol{x}^{[i]})|)^{m-|\hat{\beta}|-|\hat{\gamma}^{[i]}|+\beta_i}$$

$$= \sum_{|\hat{\beta}| \leq m} \sum_{|\hat{\gamma}^{[i]}| \leq m-|\hat{\beta}|} a_{\hat{\beta}} \binom{m-|\hat{\beta}|}{\hat{\gamma}^{[i]}} (\boldsymbol{x}^{[i]})^{\hat{\beta}^{[i]}+\hat{\gamma}^{[i]}} (1 - |(\boldsymbol{x}^{[i]})|)^{m-|\hat{\beta}^{[i]}|-|\hat{\gamma}^{[i]}|}$$

$$(\hat{\beta}^{[i]} + \hat{\gamma}^{[i]} =: \hat{\alpha}^{[i]})$$

$$= \sum_{|\hat{\beta}| \leq m} \sum_{|\hat{\alpha}^{[i]}|+\alpha_i = m} a_{\hat{\beta}} \binom{m-|\hat{\beta}|}{(\hat{\alpha}^{[i]} - \beta^{[i]}), (\alpha_i - \beta_i)} (\boldsymbol{x}^{[i]})^{\hat{\alpha}^{[i]}} (1 - |(\boldsymbol{x}^{[i]})|)^{m-|\hat{\alpha}^{[i]}|}$$

$$= \sum_{|\hat{\beta}| \leq m} \sum_{|\hat{\alpha}^{[i]}|+\alpha_i = m} a_{\hat{\beta}} \frac{\binom{\hat{\alpha}^{[i]}}{\hat{\beta}^{[i]}}\binom{\alpha_i}{\beta_i}}{\binom{m}{\hat{\beta}}} \binom{m}{\hat{\alpha}^{[i]}, \alpha_i} (\boldsymbol{x}^{[i]})^{\hat{\alpha}^{[i]}} (1 - |(\boldsymbol{x}^{[i]})|)^{\alpha_i}$$

$$= \sum_{|\hat{\alpha}^{[i]}|+\alpha_i = m} \sum_{\hat{\beta} \leq \hat{\alpha}} a_{\hat{\beta}} \frac{\binom{\hat{\alpha}^{[i]}}{\hat{\beta}^{[i]}}\binom{m-|\hat{\alpha}^{[i]}|}{\beta_i}}{\binom{m}{\hat{\beta}}} \binom{m}{\hat{\alpha}^{[i]}, \alpha_i} (\boldsymbol{x}^{[i]})^{\hat{\alpha}^{[i]}} (1 - |(\boldsymbol{x}^{[i]})|)^{\alpha_i}.$$

We define the patch $H(\Delta) := (C_\alpha(F, m, \Delta))_{|\alpha|=m}$ and investigate the face values of the Bernstein form over simplices.

Proposition 4.2. *Let F be an l-variate polynomial and let $H(\Delta)$ be the patch of its Bernstein coefficients on Δ. Then Bernstein coefficients of F on m-dimensional faces of Δ are just the coefficients on the respective m-dimensional faces of $H(\Delta)$.*

Proof. Let $\boldsymbol{x}' \in \mathbb{R}^l$ be lying on a face of Δ. Then it is lying on a subsimplex of an $(l-1)$-dimensional simplex Δ'. Let $(\boldsymbol{x}^{[i]}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_l) \in \mathbb{R}^{l-1}$. If $\boldsymbol{x}' = (x_1, \ldots, x_i, \ldots, x_l)$, $x_i = 0$, then the polynomial

$$F(x_1, \ldots, 0, \ldots, x_l) = \sum_{|\hat{\alpha}| \leq m, \alpha_i = 0} a_{\hat{\alpha}} (\boldsymbol{x}^{[i]})^{\alpha^{[i]}}$$

has Bernstein coefficients

$$C_{(\hat{\alpha}, \alpha_0)}(F, m, \Delta') = \sum_{\hat{\beta} \leq \hat{\alpha}, \beta_i = 0} \frac{\binom{\hat{\alpha}^{[i]}}{\hat{\beta}^{[i]}}}{\binom{m}{\hat{\beta}}} a_{\hat{\beta}}$$

coincide with the respective coefficients contained in the part of array $C_{(\hat{\alpha},\alpha_0)}(F,m,\Delta) = \sum_{\hat{\beta}\leq\hat{\alpha},\alpha_i=0} \frac{\binom{\hat{\alpha}}{\hat{\beta}}}{\binom{m}{\hat{\beta}}} a_{\hat{\beta}}$ of the polynomial F.

Similarly, if

$$\boldsymbol{x}' = (x_1,\dots,x_i,\dots,x_l),\ x_i = 1 - \sum_{j=1,j\neq i}^{l} x_j, \tag{15}$$

where $\boldsymbol{x}' = \sum_{j=1}^{l} \lambda_j(\boldsymbol{x}')e_j$ and $\lambda_j \neq 0,\ \forall j = 1,\dots,l$. Then by Lemma 4.1 the polynomial

$$F(x_1,\dots,x_i,\dots,x_l) = \sum_{|\hat{\alpha}^{[i]}|\leq m,\alpha_i\leq m-|\hat{\alpha}^{[i]}|} a_{\hat{\alpha}} x_i^{\alpha_i}(\boldsymbol{x}^{[i]})^{\hat{\alpha}^{[i]}}$$

has Bernstein coefficients

$$C_{(\hat{\alpha}^{[i]},m-|\hat{\alpha}^{[i]}|)}(F,m,\Delta') = \sum_{\hat{\beta}^{[i]}\leq\hat{\alpha}^{[i]},\beta_i\leq m-|\hat{\alpha}^{[i]}|} \frac{\binom{\alpha^{[i]}}{\hat{\beta}^{[i]}}\binom{m-|\hat{\alpha}^{[i]}|}{\beta_i}}{\binom{m}{\hat{\beta}}} a_{\hat{\beta}} \tag{16}$$

coincide with the respective coefficients contained in the part of array

$$C_{(\hat{\alpha},0)}(F,m,\Delta) = \sum_{\hat{\beta}\leq\hat{\alpha},|\hat{\alpha}|=m} \frac{\binom{\hat{\alpha}}{\hat{\beta}}}{\binom{m}{\hat{\beta}}} a_{\hat{\beta}},\ \text{with}\ a_{\hat{\beta}} = 0\ \text{if}\ |\hat{\alpha}| < |\hat{\beta}|,$$

of the polynomial F. As before, start decreasing the dimension of simplices by applying the same arguments above to Δ' in order to investigate all the possible cases and arrive finally to $\boldsymbol{x}' = e_{i_0},\ i_0 \in \{0,\dots,l\}$, which by the interpolation property completes the proof.

5 Applications

An application of the face values of a box for bounding functions over a union of edges was given in [22]. In [8], the approach was used for solving systems of polynomial functions over boxes. An application of Proposition 4.2 for the control to facet problem for arbitrary polynomial vector fields on a simplex was addressed in [20]. Algorithm 3.2 for computing control (data) points can be used for checking positivity and stability of polynomial systems with coefficients depending on polynomial parameters. Specifically, our results lean upon positive certificates for polynomials over subsimplices, and stability of complex systems derived from control design, dynamics and machine learning. Using the face values property, we are then able to compute control points for polynomial systems over a simplex within a finite number of parameters. In the case of high polynomial parameters, the new algorithm based on the inclusion isotonicity of Bernstein bounds over a simplex is advantageous because of the requested coefficients are optimized and the face values property holds.

6 Future Work

Computation of bounding functions for positivity of a polynomial function is important in analysis and computer design. A simple method for this purpose can be considered by using the lower Bernstein bounding function of a polynomial function over a given domain. We aspire in our future work to find the set of all negative points of F over W. We will apply taking away from the domain all sub-domains generated for which a polynomial being negative. Apply for that the computed Bernstein coefficients over a simplex and a new subdivision strategy. It may be advantageous to sweep in a particular coordinate direction r to increase the possibility for finding non positive Bernstein coefficients. Using linear bounding functions decreases the domain and calculation steps we need to find non-positive coefficients. Moreover, Matlab toolbox for the developed algorithm should be designed. The new method will be implemented in such a way that data uncertainties can be taken into account and the enclosure of bounding functions can be guaranteed also in the presence of bounding errors appearing in the computations. This should be accomplished by the use of interval arithmetic and interval programming languages.

7 Conclusions

In this paper, the expansion of polynomials into Bernstein basis was applied. We provided an algorithm that computes Bernstein coefficients over substracted subsimplices from Δ in r^{th} direction. This algorithm can be used for fast computation of polynomial coefficients over any simplex. We proved the inclusion isotonicity of the Bernstein enclosure bounds. A property which is of fundamental important in optimization and computation of bounds for functions. To this end, subdividing the given domain was applied in r^{th} direction. Subsequently, we are able to certify the positivity of polynomials over a specific simplex. On the other hand, we have shown that the Bernstein coefficients on the face values of a simplex are coincide with the batch of Bernstein coefficients. With this property, computing the Bernstein coefficients is reduced to a finite number of face values, and the algorithm (De Casteljau) for computing of optimization bounds is developed. Finally, the Bernstein enclosure bound of F over a subsimplex U is optimized by the union of Bernstein enclosure bounds over the whole domain W.

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