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A Certain Class of Deferred Weighted Statistical $\mathcal{B}$-Summability Involving (p,q)-Integers and Analogous Approximation Theorems

Amjed Zraiqat[a], S.K. Paikray[b], Hemen Dutta[c]

[a]*Department of Mathematics, Al-Zaytoonah University of Jordan, Amman 11733, Jordan*
[b]*Department of Mathematics, Veer Surendra Sai University of Technology, Burla 768018, Odisha, India*
[c]*Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India*

Abstract. The preliminary idea of statistical weighted $\mathcal{B}$-summability was introduced by Kadak et al. [27]. Subsequently, deferred weighted statistical $\mathcal{B}$-summability has recently been studied by Pradhan et al. [38]. In this paper, we study statistical versions of deferred weighted $\mathcal{B}$-summability as well as deferred weighted $\mathcal{B}$-convergence with respect to the difference sequence of order r (> 0) involving (p,q)-integers and accordingly established an inclusion between them. Moreover, based upon our proposed methods, we prove an approximation theorem (Korovkin-type) for functions of two variables defined on a Banach space $C_B(\mathcal{D})$ and demonstrated that, our theorem effectively improves and generalizes most (if not all) of the existing results depending on the choice of (p,q)-integers. Finally, with the help of the modulus of continuity we estimate the rate of convergence for our proposed methods. Also, an illustrative example is provided here by generalized (p,q)-analogue of Bernstein operators of two variables to demonstrate that our theorem is stronger than its traditional and statistical versions.

1. Introduction, Preliminaries and Motivation

Let ω be the set of all real valued sequences and call any subspace of ω the sequence space. Let (x_k) be a sequence with real or complex terms. Suppose ℓ_∞ is the class of all bounded linear sequence spaces and let c, c_0 be the respective classes for convergent and null sequences with real or complex terms. We have,

$$\|x\|_\infty = \sup_k |x_k| \quad (k \in \mathbb{N}),$$

and we recall here that under this norm, the above mentioned spaces are all Banach spaces.

The space of difference sequence was initially studied by Kızmaz [30] and then it was extended to the difference sequence of natural order r $(r \in \mathbb{N}_0 := \{0\} \cup \mathbb{N})$ by defining

$$\lambda(\Delta^r) = \{x = (x_k) : \ \Delta^r(x) \in \lambda, \ \ \lambda \in (\ell_\infty, c_0, c)\} ;$$

$$\Delta^0 x = (x_k); \Delta^r x = \left(\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}\right)$$

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Communicated by Ljubiša D.R. Kočinac
Email addresses: amjad@zuj.edu.jo (Amjed Zraiqat), skpaikray_math@vssut.ac.in (S.K. Paikray), hemen_dutta08@rediffmail.com (Hemen Dutta)

and

$$\Delta^r x_k = \sum_{i=1}^{r} (-1)^i \binom{r}{i} x_{k+i}$$

(see [20]). Also, these are all Banach spaces under the norm defined by

$$\|x\|_{\Delta^r} = \sum_{i=0}^{r} |x_i| + \sup_k |\Delta^m x_k|.$$

For more details, see the recent works [6, 9, 11, 12, 26].

The basic idea of statistical convergence was initially studied by Fast [21] and Steinhaus [44]. Statistical convergence being more general than usual convergence, it has so recently been an attractive research area of current researchers and scope of such theory has been studied in the different areas of (for instance) Number Theory, Fourier Analysis, Functional Analysis, Topology, and Approximation Theory. For the study in this direction, one may refer to the current works [4, 13, 16–19, 24, 25, 38–42].

Let $K \subseteq \mathbb{N}$ (the set of naturals) and let

$$K_n = \{k : k \leqq n \quad \text{and} \quad k \in K\}.$$

The natural (asymptotic) density of K is given by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

provided the limit exists.

Recall that, a sequence (x_n) is statistically convergent (or stat-convergent) to L, if for every $\epsilon > 0$

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}$$

has natural density zero (see [21, 44]). That is, for each $\epsilon > 0$,

$$d(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} x_n = L.$$

Consider the following example:

Example 1.1. Let $x = (x_n)$ be a sequence given by

$$x_n = \begin{cases} \frac{1}{2} & (n = m^2, \;\; m \in \mathbb{N}) \\ \frac{n}{n+1} & (\text{otherwise}). \end{cases}$$

Observe that the sequence (x_n) is statistically convergent to l but it is not convergent in the usual sense. Also, every convergent sequence is statistically convergent in the sense that, the subset to be discarded has natural density zero. Thus, statistical convergence is more general than usual convergence.

The basic concept of weighted statistical convergence was initially studied by Karakaya and Chishti [29]. Gradually, it was improved by Mursaleen et al. (see [36]) and accordingly some important approximation results were proved. For more results in this direction one may refer to the following works [13, 18, 40].

Suppose that (p_k) is a sequence of nonnegative numbers and

$$P_n = \sum_{k=0}^{n} p_k \quad (p_0 > 0; \;\; n \to \infty).$$

Then, upon setting

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k x_k \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

(x_n) is weighted statistically convergent (or $\text{stat}_{\bar{N}}$-convergent) to a number L, if for every $\epsilon > 0$

$$\{k : k \leqq P_n \quad \text{and} \quad p_k|x_k - L| \geqq \epsilon\}$$

has weighted density zero (see [36]). That is, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{P_n}|\{k : k \leqq P_n \quad \text{and} \quad p_k|x_k - L| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat}_{\bar{N}} \lim x_n = L.$$

Let X and Y be two sequence spaces and let $\mathcal{A} = (a_{n,k})$ be a regular summability matrix (with non-negative entries). If for each $x_k \in X$ and for all $(n \in \mathbb{N})$, the series

$$\mathcal{A}_n x = \sum_{k=1}^{\infty} a_{n,k} x_k$$

converges and the sequence $(\mathcal{A}_n x)$ belongs to Y, then the matrix $\mathcal{A}$ maps X into Y (denoted as (X, Y)).

Now, under the regularity condition (see Silverman-Toeplitz theorem [15]), $\mathcal{A}$ is known to be regular if

$$\lim_{n\to\infty} \mathcal{A}_n x = L \quad \text{whenever} \quad \lim_{k\to\infty} x_k = L.$$

In 1981, Freedman and Sember [22] introduced $\mathcal{A}$-statistical convergence by considering a regular summability matrix (with non-negative entries) $\mathcal{A} = (a_{n,k})$. We recall here that for any regular summability matrix (with non-negative entries) $\mathcal{A}$, a sequence (x_n) is $\mathcal{A}$-statistically convergent (or $\text{stat}_{\mathcal{A}}$-convergent) to a number L if for every $\epsilon > 0$

$$d_{\mathcal{A}}(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}.$$

That means, for every $\epsilon > 0$

$$\lim_{n\to\infty} \sum_{k:|x_k - L|\geqq\epsilon} a_{n,k} = 0.$$

Here, we write

$$\text{stat}_{\mathcal{A}} \lim x_n = L.$$

Subsequently, the idea of $\mathcal{A}$-statistical convergence was improved and enhanced to $\mathcal{B}$-statistical convergence by Kolk [31] with respect to $F_{\mathcal{B}}$-convergence or $\mathcal{B}$-summability (also, see [43]).

Recall the following. Let $\mathcal{B} = (\mathcal{B}_i)$ be a sequence of matrices (infinite) with $\mathcal{B}_i = (b_{n,k}(i))$ and for $x_n \in \ell_\infty$, (x_n) is $\mathcal{B}$-summable to the value $\mathcal{B}\text{-}\lim_{n\to\infty}(x_n)$, if

$$\lim_{n\to\infty}(\mathcal{B}_i x)_n = \lim_{n\to\infty}\sum_{k=0}^{\infty} b_{n,k}(i)(x)_k = \mathcal{B}\text{-}\lim_{n\to\infty}(x_n) \quad \text{uniformly in } i \ \ (i = 0, 1, 2, ...).$$

The method $(\mathcal{B}_i)$ is regular (see [14, 43]) if and only if it satisfies the conditions:

(i) $\|\mathcal{B}\| = \sup_{n,i\to\infty} \sum_{k=0}^{\infty} |b_{n,k}(i)| < \infty$ (uniformly in i);

(ii) $\lim_{n\to\infty} b_{n,k}(i) = 0$ for each $k \in \mathbb{N}$ (uniformly in i);

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} b_{n,k}(i) = 1$ (uniformly in i).

Let $K = \{k_i\} \subset \mathbb{N}$ $(k_i < k_{i+1})$ for all i, then the $\mathcal{B}$-density of K is given by

$$d_{\mathcal{B}}(K) = \lim_{n\to\infty} \sum_{k=0}^{\infty} b_{n,k}(i) \text{ uniformly in } i.$$

Let $\mathcal{R}^+$ denotes the set of all regular methods $\mathcal{B}$ with $b_{n,k}(i) \geqq 0$ $(\forall\, n,k,i \in \mathbb{N})$ and suppose $\mathcal{B} \in \mathcal{R}^+$. Recall that (x_n) is $\mathcal{B}$-statistically convergent (or stat$_{\mathcal{B}}$-convergent) to a number L, if for each $\epsilon > 0$

$$d_{\mathcal{B}}(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}.$$

This means that for each $\epsilon > 0$, we have

$$\lim_{n\to\infty} \sum_{k:|x_k - L|\geqq\epsilon} b_{n,k}(i) = 0 \text{ uniformly in } i.$$

Here we write

$$\mathrm{stat}_{\mathcal{B}} \lim x_n = L.$$

Subsequently, with the development of q-calculus, various researchers worked on certain new generalizations of positive linear operators based on q-integers (see [3, 7, 8, 23, 33]). Recently, Mursaleen et al. [35] introduced the (p,q)-analogue of Bernstein operators in connection with (p,q)-integers and later on, some approximation results for Baskakov operators and Bernstein-Schurer operators were studied for (p,q)-integers by [1] and [37].

We now recall some definitions and basic notations on (p,q)-integers for our present study.

Let $\mathbb{N}$ be the set of naturals and for $n \in \mathbb{N}$, the (p,q)-integer $[n]_{p,q}$ is given by

$$[n]_{p,q} = \begin{cases} \frac{p^n - q^n}{p-q} & (n \geq 1) \\ 0 & (n = 0) \end{cases}$$

where $0 < p < q \leq 1$.

The (p,q)-factorial is given by

$$[n]!_{p,q} = \begin{cases} [1]_{p,q}[2]_{p,q}...[n]_{p,q} & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

The (p,q)-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]!_{p,q}}{[k]!_{p,q}\,[n-k]!_{p,q}} \qquad \text{for all } n,k \in \mathbb{N} \text{ and } n \geq k.$$

We also recall that, supposing that $0 < q < p \leq 1$ and r is a non-negative integer, the operator

$$\Delta^{[r]}_{p,q} : \omega \to \omega$$

is given by

$$\Delta^{[r]}_{p,q}(x_n) = \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i}$$

(see [10, 25, 28]). That is,

$$\Delta^{[r]}_{p,q}(x_n) = \begin{bmatrix} r \\ 0 \end{bmatrix}_{p,q} x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + \ldots + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r}$$

$$= x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + \ldots + (-1)^r x_{n-r}$$

$$= x_n - \left(\frac{p^r - q^r}{p-q} \right) x_{n-1} + \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})}{(p-q)^2(p+q)} \right) x_{n-2}$$

$$- \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})(p^{r-2} - q^{r-2})}{(p-q)^3(p^2+pq+q^2)(p+q)} \right) x_{n-3} + \ldots + (-1)^m x_{n-r}.$$

Now we present below the following example to see that a sequence is not convergent; however the associated difference sequence is convergent.

Example 1.2. Suppose $(x_n) = n + 1$ $(n \in \mathbb{N})$ is a sequence. It is clear that the sequence (x_n) is not convergent in the usual sense.

Also, we see that

$$\Delta^{[3]}(x_n) = x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} \quad (x_n = n+1)$$

converges to 0 $(n \to \infty)$.

For $r = 3$, we get

$$\begin{aligned} \Delta^{[3]}_{p,q}(x_n) &= x_n - [3]_{p,q} x_{n-1} + [3]_{p,q} x_{n-2} - x_{n-3} \quad (x_n = n+1) \\ &= x_n - (p_n^2 + p_n q_n + q_n^2) x_{n-1} + (p_n^2 + p_n q_n + q_n^2) x_{n-2} - x_{n-3} \\ &= n + 1 - (p_n^2 + p_n q_n + q_n^2) n + (p_n^2 + p_n q_n + q_n^2)(n-1) - (n-2) \quad (x_n = n+1) \\ &= 3 - (\beta^2 + \alpha\beta + \alpha^2). \end{aligned}$$

Clearly, based on the different choice of the values of p and q, the difference sequence $\Delta^{[3]}_{p,q}(x_n)$ of third order has different limits. This fact is due to the usual definition of (p,q)-integers. However, in order to obtain a criterion of convergence for all the values of p and q belonging to the operator $\Delta^{[r]}_{p,q}$, we must have to overcome this difficulty. This type of difficulties can be avoided in the following two ways. The first one is taking $p = q = 1$ and thus the operator reduces to the usual difference sequence. Next, the second way is to replace $p = p_n$ and $q = q_n$ under the limits, $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 \leq \alpha, \beta \leq 1)$ where $0 < q_n < p_n \leq 1$, for all $(n \in \mathbb{N})$. Afterwards, the difference sequence $\Delta^{[3]}_{p,q}(x_n)$ of order 3 converges to the value $3 - (\beta^2 + \alpha\beta + \alpha^2)$. Thus, if we take $q_n = \left(\frac{n+1}{n+1+s}\right) < \left(\frac{n+1}{n+1+t}\right) = p_n$ such that $0 < q_n < p_n \leq 1$ $(s > t > 0)$, then $\lim_n q_n = 1 = \lim_n p_n$. Hence, $\Delta^{[3]}_{p,q}(x_n) \to 0 \quad (n \to \infty)$.

Remark 1.3. If $r = 1$, $\lim_n q_n = 1$ and $\lim_n p_n = 1$, then the difference operator $\Delta^{[r]}_{p,q}$ reduces to the $\Delta^{[1]}$ (see [5]). Also, if $r = 0$, $\lim_n q_n = 1$ and $\lim_n p_n = 1$, then the difference operator $\Delta^{[r]}_{p,q}$ reduces to the general sequence (x_n).

In the year of 2016, Kadak [25] introduced weighted statistical convergence involving (p, q)-integers and proved some approximation theorems for functions of two variables. Subsequently, it was extended to the generalized difference sequences involving (p, q)-gamma function and accordingly associated approximation theorems were proved (see [24]). Mohiuddine [34] also introduced weighted $\mathcal{A}$-summability as well as weighted $\mathcal{A}$-statistical convergence and accordingly proved certain approximation theorems (Korovkin-type). Furthermore, Kadak et al. [27] presented the idea of statistical weighted $\mathcal{B}$-summability and established some approximation theorems on that basis. Very recently, Srivastava et al. [41] introduced the deferred weighted (Nörlund) summability of a sequence and accordingly proved Korvokin-type approximation theorems based on equi-statistical convergence.

Essentially motivated by the above cited works, here we introduce the (presumably new) notion of deferred weighted $\mathcal{B}$-statistical convergence and statistical deferred weighted $\mathcal{B}$-summability with respect to the generalized difference sequences of order r involving (p, q)-integers, and establish some new approximation results on that basis.

2. Definitions, Notations and Regular Methods

In this section, we introduce some definitions (presumably new) those are required for our proposed study. Also, we present here certain inclusion relations with regard to regular methods.

Let (a_n) and (b_n) be sequences of integers (non-negative) such that, the conditions of regularity for the deferred weighted mean (see Agnew [2]) can be viewed as:

(i) $a_n < b_n \quad (n \in \mathbb{N})$

and

(ii) $\lim\limits_{n\to\infty} b_n = \infty.$

Now, we suppose that (s_n) is the sequence of real numbers (non-negative) such that

$$S_n = \sum_{m=a_n+1}^{b_n} s_m.$$

To define the deferred weighted mean $D_a^b(\overline{N}, s)$ by the difference operator $(\Delta_{p,q}^r)$, we first set

$$\Phi_n^{p,q}(\Delta x) = \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} s_m(\Delta_{p,q}^{[r]} x_m) \qquad (0 < q < p \leq 1) \ \ (r \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}).$$

The given sequence (x_n) is deferred weighted summable (or $c^{D(\bar{N})}$-summable) to L under the mean of the difference operator $(\Delta_{p,q}^{[r]})$, if

$$\lim_{n\to\infty} \Phi_n^{p,q}(\Delta x) = L.$$

Here we write

$$c_\Delta^{D(\bar{N})} \lim x_n = L.$$

We denote the deferred weighted summable sequences under the difference operator $(\Delta_{p,q}^{[r]})$ by $c^{D(\bar{N})}$.

Definition 2.1. Let $\mathcal{B} \in \mathcal{R}^+$, $0 < q_n < p_n \leq 1$ be such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and suppose that r is a non-negative integer. Also let (a_n) and (b_n) be sequences of non-negative integers. A sequence (x_n) is deferred weighted $\mathcal{B}$-summable (or $[D(\bar{N})_A; s_n]$-summable) to a number L with respect to the difference operator $\Delta_{p,q}^{[r]}$, if the $\mathcal{B}$-transform of (x_n) is deferred weighted summable to L (the same number) under the difference operator $\Delta_{p,q}^{[r]}$, that is,

$$\lim_{n\to\infty} \Psi_n^{p,q}(\Delta x) = \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta_{p,q}^{[r]} x_k) = L.$$

Here, we write

$$[D(\bar{N})^{\Delta}_{A}; s_n] \lim_{n\to\infty} x_n = L.$$

We denote the set of all sequences which are deferred weighted summable by the difference operator $(\Delta^{[r]}_{p,q})$ by $[D(\bar{N})^{\Delta}_{A}; s_n]$.

Definition 2.3 below is a generalization of many known definitions as discussed in Remark 2.2 below.

Remark 2.2. If $a_n + 1 = a_n$, $\lim_n q_n = 1$, $\lim_n p_n = 1$ and $r = 0$, then $\Psi^{p,q}_n(\Delta x)$ mean reduces to $\mathcal{B}^{a,b}_n(x_n)$ mean (see [25]), and if $\mathcal{B} = I$, then $\Psi^{p,q}_n(\Delta x)$ mean reduces to $\Lambda^{n}_{p,q}(x_n)$-mean (see [23]). Finally, if $\lim_n q_n = 1$, $\lim_n p_n = 1$, $r = 0$, $a_n = 0$, $b_n = n$ and $\mathcal{B} = \mathcal{A}$, then $\Psi^{p,q}_n(\Delta x)$ mean reduces to $\mathcal{A}^{\bar{N}}_n$ mean (see [31]).

Definition 2.3. Let $\mathcal{B} = (b_{n,k}(i))$ be a matrix, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r be a non-negative integer. Suppose that (a_n) and (b_n) are the sequences of integers (non-negative). The matrix $\mathcal{B} = (b_{n,k}(i))$ is a regular deferred weighted matrix (or deferred weighted regular method), if

$$\mathcal{B}x \in c^{D(\bar{N})}_{\Delta} \quad (\forall\ x_n \in c)$$

with

$$c^{D(\bar{N})}_{\Delta} \lim \mathcal{B}x_n = \mathcal{B} \lim(x_n)$$

and we denote it by $\mathcal{B} \in \left(c : c^{D(\bar{N})}_{\Delta}\right)$.

This means that $\Psi^{p,q}_n(\Delta x)$ exists for each $n \in \mathbb{N}$, $x_n \in c$ and

$$\lim_{n\to\infty} \Psi^{p,q}_n(\Delta x) \to L \quad \text{whenever} \quad \lim_{n\to\infty} x_n \to L.$$

We denote the set of all deferred weighted regular matrices (methods) by $\mathcal{R}^{+}_{D(w)}$.

We now present the following theorem as a characterization of deferred weighted regular methods.

Theorem 2.4. *Let $\mathcal{B} = (b_{n,k}(i))$ be a sequence of infinite matrices, $0 < q_n < p_n \leq 1$ be such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a integer (non-negative). Let (a_n) and (b_n) be sequences of integers (non-negative). Then $\mathcal{B} \in \left(c : c^{D(\bar{N})}_{\Delta}\right)$ if and only if*

$$\sup_n \sum_{k=1}^{\infty} \frac{1}{S_n} \left| \sum_{m=a_n+1}^{b_n} s_m b_{m,k}(i) \right| < \infty \ \ \textit{uniformly in } i; \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} s_m b_{m,k}(i) = 0 \ \ \textit{uniformly in } i \ \ (\textit{for each } \ k \in \mathbb{N}) \tag{2.2}$$

and

$$\lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k=1}^{\infty} s_m b_{m,k}(i) = 1 \ \ \textit{uniformly in } i. \tag{2.3}$$

Proof. Assuming (2.1)-(2.3) are true and suppose that $(\Delta^{[r]}_{p,q}x_k) \to L \quad (n \to \infty)$, then for every $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $|(\Delta^{[r]}_{p,q}x_k) - L| \leqq \epsilon \ (m > m_0)$. Thus, we have

$$\begin{aligned}
\left|\mathcal{B}_n^{(a_n,b_n)}(\Delta^{[r]}_{p,q}x_k) - L\right| &= \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k) - L\right| \\
&= \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k - L) + L\left(\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i) - 1\right)\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k - L)\right| + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i) - 1\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{b_{n-2}} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k - L)\right| \\
&\quad + \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=b_{n-1}}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k - 1)\right| + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i) - 1\right| \\
&\leqq \sup_k |\Delta^{[r]}_{p,q}x_k - L| \sum_{k=1}^{b_{n-2}} \frac{1}{S_n}\sum_{m=a_n+1}^{b_n} s_m a_{m,k} + \epsilon \frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i) \\
&\quad + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i) - 1\right|.
\end{aligned}$$

For $n \to \infty$ and using (2.2) and (2.3), we obtain

$$\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k) - L\right| \leqq \epsilon.$$

Since $\epsilon > 0$ is arbitrary small, so it clearly implies that

$$\lim_{n\to\infty} \frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k) = L = \lim(x_n).$$

Conversely, suppose that $\mathcal{B} \in \left(c : c_\Delta^{D(\bar{N})}\right)$ and $x_n \in c$. As $\mathcal{B}x$ exists, so we fairly have the inclusion

$$\left(c : c_\Delta^{D(\bar{N})}\right) \subset (c : L_\infty).$$

Thus, there exists a constant M satisfying

$$\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)\right| \leqq M \quad (\forall\ m, n)$$

and the associated series,

$$\left|\frac{1}{P_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)\right|$$

converges for each n (uniformly in i). Therefore, (2.1) is valid.

Next, consider $x^{(n)} = (x_k^{(n)}) \in c_0$ given by

$$x_k^{(n)} = \begin{cases} 1 & (n = k) \\ 0 & (n \neq k); \end{cases}$$

for every $n \in \mathbb{N}$ and $y = (y_n) = (1, 1, 1, ...) \in c$. Moreover, as $\mathcal{B}x^{(n)}$ and $\mathcal{B}y$ are in $c_{\Delta}^{D(\bar{N})}$, so (2.2) and (2.3) are trivially true. □

Furthermore, we consider the following definitions for our study.

Definition 2.5. Let $\mathcal{B} \in \mathcal{R}^+_{D(w)}$, $0 < q_n < p_n \leq 1$ be such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and suppose that r is a integer. Let (a_n) and (b_n) be sequences of integers (non-negative) and also let $K = (k_i) \subset \mathbb{N}$ $(k_i \leq k_{i+1})$ for each i, then the deferred weighted $\mathcal{B}$-density of K is given by

$$d^{\mathcal{B}}_{D(\bar{N})}(K) = \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in K} s_m b_{m,k}(i) \quad \text{(uniformly in } i\text{)},$$

subject to the existence of limit. A sequence (x_n) is said to be deferred weighted $\mathcal{B}$-statistical convergent to a number L with respect to the difference operator $\Delta^{[r]}_{p,q}$, if for each $\epsilon > 0$

$$d^{\mathcal{B}}_{D(\bar{N})}(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \text{ and } |\Delta^{[r]}_{p,q}(x_k) - L| \geqq \epsilon\}.$$

Here, we write

$$\text{stat}^{p,q}_{\Psi_\Delta} \lim_{n\to\infty}(x_n) = L.$$

Definition 2.6. Let $\mathcal{B} \in \mathcal{R}^+_{D(w)}$, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r be a integer (non-negative). Let (a_n) and (b_n) be sequences of integers (non-negative). Then, the sequence is statistical deferred weighted $\mathcal{B}$-summable to a number L under the operator $\Delta^{[r]}_{p,q}$, if for every $\epsilon > 0$

$$d(E_\epsilon) = 0,$$

where

$$E_\epsilon = \{k : k \in \mathbb{N} \text{ and } |\Psi^{p,q}_n(\Delta x) - L| \geqq \epsilon\}.$$

Here, we write

$$\text{stat}D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_{n\to\infty}(x_n) = L.$$

The following theorem provides a relation between statistical deferred weighted $\mathcal{B}$-summability and deferred weighted $\mathcal{B}$ statistical convergence.

Theorem 2.7. *Suppose that*

$$s_n b_{n,k}(i)\left|\Delta^{[r]}_{p,q}x_n - L\right| \leqq M \quad (n \in \mathbb{N})$$

and $0 < q_n < p_n \leq 1$ $(\forall n \in \mathbb{N})$ *such that* $\lim_n q_n = \alpha$ *and* $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$*. If a sequence* (x_n) *is deferred weighted* $\mathcal{B}$*-statistical convergent to a number L, then it is statistical deferred weighted* $\mathcal{B}$*-summable to L (the same number), but the converse is not necessarily true.*

Proof. Let

$$s_n b_{n,k}(i)\left|\Delta^{[r]}_{p,q}x_n - L\right| \leqq M \quad (n \in \mathbb{N}) \text{ and } \lim_n q_n = \alpha, \quad \lim_n p_n = \beta \;\; (0 < \alpha, \beta \leq 1).$$

Also let (x_n) be deferred weighted $\mathcal{B}$-statistical convergent to L with respect to the operator $\Delta^{[r]}_{p,q}$. We have,

$$d^{\mathcal{B}}_{D(\bar{N})}(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \text{ and } |\Delta^{[r]}_{p,q}(x_k) - L| \geqq \epsilon\}.$$

Thus, we have

$$\begin{aligned}
\left|\Psi^{p,q}_n(\Delta x) - L\right| &= \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q}x_k - L)\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i)\left(\Delta^{[r]}_{p,q}x_k - L\right)\right| + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i) - 1\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k\in K_\epsilon} s_m b_{m,k}(i)\left(\Delta^{[r]}_{p,q}x_k - L\right)\right| \\
&\quad + \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k\notin K_\epsilon}^{\infty} s_m b_{m,k}(i)\left(\Delta^{[r]}_{p,q}x_k - L\right)\right| + \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m b_{m,k}(i) - 1\right| \\
&\leqq \sup_{k\to\infty}\left|\Delta^{[r]}_{p,q}x_k - L\right|\frac{1}{S_n}\sum_{k\in K_\epsilon}\sum_{m=a_n+1}^{b_n} s_m b_{m,k}(i) + \epsilon\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k\notin K_\epsilon} s_m b_{m,k}(i) \\
&\quad + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k\in K_\epsilon} s_m b_{m,k}(i) - 1\right| \to \epsilon \quad (n \to \infty).
\end{aligned}$$

which implies that $\Psi^{p,q}_n(\Delta x) \to L$. That is, the sequence (x_n) is deferred weighted $\mathcal{B}$-summable to L under the difference operator $\Delta^{[r]}_{p,q}$. Hence, the sequence (x_n) is statistical deferred weighted $\mathcal{B}$-summable to L (the same number) with respect to the same difference operator $\Delta^{[r]}_{p,q}$. □

To prove falsity of converse part, we are presenting the following example.

Example 2.8. For $\lim_n q_n = 1$, $\lim_n p_n = 1$, $s_n = 1$, $a_n = 0$ and $b_n = n$ $(\forall n \in \mathbb{N})$, consider the sequence $x = (x_n)$ given by

$$x_n = \begin{cases} \frac{1}{m^2} & (n = m^2 - m, m^2 - m + 1, ..., m^2 - 1) \\ -\frac{1}{m^3} & (n = m^2, \ \ m > 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Consider infinite matrices $\mathcal{B} = (\mathcal{B}_i)$ with $B_i = (b_{n,k}(i))$ given by (see [25])

$$x_n = \begin{cases} \frac{1}{n+1} & (i \leqq k \leqq i + n) \\ 0 & (\text{otherwise}). \end{cases}$$

Since, we have

$$\begin{aligned}
\sum_{k=i}^{i+n} \Delta^{[r]}_{p,q} x_k &= \sum_{k=i}^{i+n} \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i} \\
&= \sum_{k=i}^{i+n} \left\{ x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + ... + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r} \right\} \\
&= \sum_{k=i}^{i+n} \left\{ x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + ... + (-1)^r x_{n-r} \right\} \\
&= \{(i+n)x_{i+n} + (1 - [r]_{p,q})(i+n-1)x_{i+n-1} + \left(1 - [r]_{p,q} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!}(i+n-2)x_{i+n-2}\right) \\
&\quad +... + \left(1 - [r]_{p,q} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} - \frac{[r]_{p,q}[r-1]_{p,q}[m-2]_{p,q}}{[3]_{p,q}!} + ...\right)(i+n-k)x_{i+n-k}\} \qquad (2.4)
\end{aligned}$$

so,

$$\Psi^{p,q}_n(\Delta x) = \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k=1}^{\infty} s_m b_{m,k}(i)(\Delta^{[r]}_{p,q} x_k) = \frac{1}{n} \sum_{m=1}^{n} \frac{1}{m+1} \sum_{k=i}^{i+n} \Delta^{[r]}_{p,q} x_k.$$

Morwover, each part on the right hand side of (2.4) being convergent to zero ($n \to \infty$), thus we obtain

$$\Psi^{p,q}_n(\Delta x) \to 0.$$

It implies that

$$\text{stat}\Psi^{p,q}_n(\Delta x) \to 0.$$

Hence, (x_n) is not deferred weighted $\mathcal{B}$-statistical convergenct, even if it is statistical deferred weighted $\mathcal{B}$-summable.

3. A Korovkin-Type Theorem via Statistical Deferred Weighted $\mathcal{B}$-Summability

In this section, by using the idea of deferred weighted statistical $\mathcal{B}$-summability with respect to the difference sequence of order r based on (p,q)-integers, we prove a Korovkin type approximation theorem (see for details [32]) for a function of two variables. Furthermore, we use (p,q)-analogue of Bernstein operators for two variables and show that our proposed method is stronger than that of traditional and statistical versions of Korovkin-type theorems.

Let $C_B(\mathcal{D})$ be the space of all real valued functions (continuous) on $\mathcal{D}$ equipped with the norm

$$\|f\|_{C_B(\mathcal{D})} = \sup\{|f(x,y)| : (x,y) \in \mathcal{D}\}, \quad f \in C_B(\mathcal{D}),$$

where $\mathcal{D}$ is any compact subset.

Let $T : C_B(\mathcal{D}) \to C_B(\mathcal{D})$ be a linear operator and let

$$f \geqq 0 \ \text{ implies } \ T(f) \geqq 0,$$

that is, T is a positive linear operator. Also, we use the notation $T(f; x, y)$ for the values of $T(f)$ at the a point $(x, y) \in \mathcal{D}$.

Theorem 3.1. *Let $\mathcal{B} \in \mathcal{R}^+$, (a_n) and (b_n) be the sequences of non-negative integers, r be a non-negative integer, and $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$. Let $T_n : C_B(\mathcal{D}) \to C_B(\mathcal{D})$ be a sequence of linear operators (positive) and let $f \in C_B(\mathcal{D})$. Then*

$$stat\ D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(f(s,t); x, y) - f(x,y)\|_{C_B(\mathcal{D})} = 0, \quad f \in C_B(\mathcal{D}) \tag{3.1}$$

if and only if

$$stat\ D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(1; x, y) - 1\|_{C_B(\mathcal{D})} = 0; \tag{3.2}$$

$$stat\ D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(s; x, y) - x\|_{C_B(\mathcal{D})} = 0; \tag{3.3}$$

$$stat\ D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(t; x, y) - y\|_{C_B(\mathcal{D})} = 0; \tag{3.4}$$

$$stat\ D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(s^2 + t^2; x, y) - (s^2 + t^2)\|_{C_B(\mathcal{D})} = 0. \tag{3.5}$$

Proof. Since each of the functions given by

$$f_0(s,t) = 1, \qquad f_1(s,t) = s, \qquad f_2(s,t) = t \quad \text{and} \quad f_3(s,t) = s^2 + t^2$$

is in $C_B(\mathcal{D})$, the following implication:

$$(3.1) \Longrightarrow (3.2) - (3.5)$$

is trivial. To complete the proof, we have to first assume that (3.2)-(3.5) hold true. Let $f \in C_B(\mathcal{D})$, for all $(x, y) \in \mathcal{D}$. Since $f(x, y)$ is bounded on $\mathcal{D}$, then there exists a constant $M > 0$ such that $|f(x, y)| \leqq \mathcal{K}$ for all $x, y \in \mathcal{D}$, which implies that

$$|f(s,t) - f(x,y)| \leqq 2\mathcal{K}. \tag{3.6}$$

Clearly, for a given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(s,t) - f(x,y)| < \epsilon \quad \text{whenever} \quad |s - x| < \delta \quad \text{and} \quad |t - y| < \delta, \tag{3.7}$$

for all $s, t, x, y \in \mathcal{D}$.

From equation (3.6) and (3.7), we get

$$|f(s,t) - f(x,y)| < \epsilon + \frac{2\mathcal{K}}{\delta^2}\left([\varphi(s,x)]^2 + [\varphi(t,y)]^2\right) \tag{3.8}$$

where

$$\varphi(s,x) = s - x \ \text{ and } \ \varphi(t,y) = t - y.$$

Further, as $f \in C_B(\mathcal{D})$, the inequality (3.8) holds for $s, t, x, y \in \mathcal{D}$.

Now, since the operator $T_n(f;x,y)$ is monotone and linear, so under this operator the inequality in (3.8) follows:

$$\begin{aligned}
|T_n(f(s,t);x,y) - f(x,y)| &= |T_n(f(s,t) - f(x,y);x,y) + f(x,y)[T_k(f_0;x,y) - f_0]| \\
&\leqq |T_n(f(s,t) - f(x,y);x,y) + \mathcal{K}[T_k(1;x,y) - 1]| \\
&\leqq \left| T_n\left(\epsilon + \frac{2\mathcal{K}}{\delta^2}\left[\varphi(s,x)^2 + \varphi(t,y)^2\right];x,y\right)\right| + \mathcal{K}|T_n(1;x,y) - 1| \\
&\leqq \epsilon + (\epsilon + \mathcal{K})|T_n(f_0;x,y) - f_0(x,y)| + \frac{2\mathcal{K}}{\delta^2}|T_n(f_3;x,y) - f_3(x,y)| \\
&\quad - \frac{4\mathcal{K}}{\delta^2}x|T_n(f_1;x,y) - f_1(x,y)| - \frac{4\mathcal{K}}{\delta^2}y|T_n(f_2;x,y) - f_2(x,y)| \\
&\quad + \frac{2\mathcal{K}}{\delta^2}(x^2 + y^2)|T_n(f_0;x,y) - f_0(x,y)| \\
&\leqq \epsilon + \left(\epsilon + \mathcal{K} + \frac{2\mathcal{K}}{\delta^2}(|x|^2 + |y|^2)\right)|T_n(1;x,y) - 1| \\
&\quad + \frac{4\mathcal{K}}{\delta^2}|T_n(f_1;x,y) - f_1(x,y)| + \frac{4\mathcal{K}}{\delta^2}|T_n(f_3;x,y) - f_2(x,y)| \\
&\quad + \frac{2\mathcal{K}}{\delta^2}|T_n(f_3;x,y) - f_3(x,y)|.
\end{aligned} \tag{3.9}$$

Next, taking $\sup_{x,y\in\mathcal{D}}$ in both side of (3.9), we obtain

$$\|T_n(f(s,t);x,y) - f(x,y)\|_{C_B(\mathcal{D})} \leqq \epsilon + \mathcal{N}\sum_{j=0}^{3}\|T_n(f_j(s,t);x,y) - f_j(x,y)\|_{C_B(\mathcal{D})}, \tag{3.10}$$

where

$$\mathcal{N} = \left\{\epsilon + \mathcal{K} + \frac{2\mathcal{K}}{\delta^2}\right\}.$$

We now replace $T_n(f(s,t);x,y)$ by

$$\mathfrak{L}_n(f(s,t);x,y) = \frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=0}^{\infty} s_m b_{m,k}(i)\Delta_{p,q}^{[r]}(T_k(f;x,y)) \ \ (\forall\ i,m \in \mathbb{N})$$

in the equation (3.10).

Now, for a given $r > 0$, we choose $\epsilon' > 0$, such that $0 < \epsilon' < r$. Then, upon setting

$$A_n = |\{n : n \leqq \mathbb{N} \quad \text{and} \quad |\mathfrak{L}_n(f(s,t);x,y) - f(x,y)| \geqq r\}|$$

and

$$A_{j,n} = \left|\left\{n : n \leqq \mathbb{N} \quad \text{and} \quad |\mathfrak{L}_n(f_j(s,t);x,y) - f_j(x,y)| \geqq \frac{r - \epsilon'}{4N}\right\}\right|,$$

equation (3.10) implies

$$A_n \leqq \sum_{j=0}^{3} A_{j,n}.$$

Thus we have

$$\frac{\|A_n\|_{C_B(\mathcal{D})}}{n} \leqq \sum_{j=0}^{3} \frac{\|A_{j,n}\|_{C_B(\mathcal{D})}}{n}. \tag{3.11}$$

Finally, under the above assumption for the implication in (3.2)-(3.5) and Definition 2.6, the right-hand side of (3.11) seems to tend to zero ($n \to \infty$). Thus, we get

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_{n\to\infty} \|T_n(f_j(s,t);x,y) - f_j(x,y)\|_{C_B(\mathcal{D})} = 0.$$

Hence, the implication (3.1) holds true. The proof of Theorem 3.1 is thus completed. □

Remark 3.2. If we consider $\mathcal{B} = I$ (identity matrix), $s_n = 1$, $\lim_n q_n = 1$, $\lim_n p_n = 1$, $r = 0$, $a_n = 0$ and $b_n = n$ ($\forall\ n$) in our Theorem 3.1, then we obtain classical version of Korovkin type approximation theorem [32]. Also, if we put $\mathcal{B} = (C,1)$ (Cesàro matrix), $s_n = 1$, $\lim_n q_n = 1$, $\lim_n p_n = 1$, $r = 0$, $a_n = 0$ and $b_n = n$ ($\forall\ n$) in our Theorem 3.1, then we obtain statistical version of Korovkin-type approximation theorem [17]. Moreover, if we put $\mathcal{B} = (A)$, $s_n = 1$, $\lim_n q_n = 1$, $\lim_n p_n = 1$, $r = 0$, $a_n = 0$ and $b_n = n$ ($\forall\ n$) in our Theorem 3.1, then we obtain statistical weighted $\mathcal{A}$-summability version of approximation theorem (Korovkin-type) [34]. Finally, if we put $a_n + 1 = a_n$, $\lim_n q_n = 1$, $\lim_n p_n = 1$, $r = 0$ ($\forall\ n$) in our Theorem 3.1, then we obtain statistical weighted $\mathcal{B}$-summability version of Korovkin type approximation theorem (see [27]).

We now present below an illustrative example for Theorem 3.1 by using (p,q)-analogue of Bernstein operators of two variables (see [35]).

Example 3.3. Let $I = [0,1]$ and for a function $f \in C_B(\mathcal{D})$ on $\mathcal{D} = I \times I$, we have the operators

$$\mathfrak{B}_{n,p,q}(f;x,y) = \sum_{u=0}^{n}\sum_{v=0}^{m} f\left(\frac{[u]_{p,q}}{p_{u-n}[n]_{p,q}}, \frac{[v]_{p,q}}{p_{v-m}[m]_{p,q}}\right)\mathfrak{B}_{u,n}(x)\mathfrak{B}_{v,m}(y) \tag{3.12}$$

where

$$\mathfrak{B}_{u,n}(x) = \frac{1}{p^{\frac{n(n-1)}{2}}}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q} p^{\frac{u(u-1)}{2}} x^u \prod_{s=0}^{n-u-1}(p^s - q^s x)$$

and

$$\mathfrak{B}_{v,m}(y) = \frac{1}{p^{\frac{m(m-1)}{2}}}\begin{bmatrix} m \\ v \end{bmatrix}_{p,q} p^{\frac{v(v-1)}{2}} y^v \prod_{s=0}^{m-v-1}(p^s - q^s y).$$

Also, observe that

$$\mathfrak{B}_{n,p,q}(1;x,y) = 1, \quad \mathfrak{B}_{n,p,q}(s;x,y) = x, \quad \mathfrak{B}_{n,p,q}(t;x,y) = y \quad \text{and}$$

$$\mathfrak{B}_{n,p,q}(s^2+t^2;x,y) = \frac{p^{n-1}}{[n]_{p,q}}x + \frac{p^{m-1}}{[m]_{p,q}}y + \frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2 + \frac{q[m-1]_{p,q}}{[m]_{p,q}}y^2.$$

Let us consider a positive linear operator T_n as

$$T_n : C_B(\mathcal{D}) \to C_B(\mathcal{D})$$

such that

$$T_n(f;x,y) = (1+x_n)\mathfrak{B}_{n,p_n,q_n}(f;x,y) \quad (0 < q_n < p_n \leq 1,\ \forall\ n \in \mathbb{N}), \tag{3.13}$$

where (x_n) is a sequence as considered in Example 2.8. Clearly, (T_n) satisfies the conditions (3.2)-(3.5) of our Theorem 3.1, thus we fairly get:

$$\begin{aligned}
&\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} = 0;\\
&\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(s;x,y) - x\|_{C_B(\mathcal{D})} = 0;\\
&\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(t;x,y) - y\|_{C_B(\mathcal{D})} = 0;\\
&\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(s^2+t^2;x,y) - (s^2+t^2)\|_{C_B(\mathcal{D})} = 0.
\end{aligned}$$

Hence, from Theorem 3.1, we obtain

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_n \|T_n(f(s,t);x,y) - f(x,y)\|_{C_B(\mathcal{D})} = 0, \quad f \in C_B(\mathcal{D}).$$

Moreover, since (x_n) is not statistical weighted $\mathcal{B}$-summable, so the outcomes of Pradhan et al. [38], does not hold true for our operators defined by (3.13). Moreover, since (x_n) is statistical deferred weighted $\mathcal{B}$-summable with respect to the difference operator of order r based on (p,q)-integers, thus we conclude that our Theorem 3.1 fairly works for the same operators.

4. Rate of the Deferred Weighted $\mathcal{B}$-Statistical Convergence

In this section, we investigate the rate of the deferred weighted $\mathcal{B}$-statistical convergence of a sequence of linear operators (positive) of function of two variables defined on $C_B(\mathcal{D})$ into itself with the help of the modulus of continuity.

Definition 4.1. Let $\mathcal{B} \in \mathcal{R}^+_{D(w)}$, r be a non-negative integer, (a_n) and (b_n) be sequences of non-negative integers. Suppose that $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha < \beta \leq 1)$. Also, let (u_n) be a positive non-decreasing sequence. A sequence (x_n) is deferred weighted $\mathcal{B}$-statistical convergent to a number L with rate $o(u_n)$, if for each $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{u_n S_n} \sum_{m=a_n+1}^{b_n} \sum_{k \in K_\epsilon} s_m b_{m,k}(i) = 0 \ \text{(uniformly in } i\text{)},$$

where

$$K_\epsilon = \{k : k \leqq \mathbb{N} \quad \text{and} \quad |(\Delta^{[r]}_{p,q} x)_k - L| \geqq \epsilon\}.$$

Here, we may write

$$x_n - L = \text{stat}^{p,q}_{\Psi_\Delta} - o(u_n).$$

Let us now consider the following lemma:

Lemma 4.2. *Let (u_n) and (v_n) be two positive non-decreasing sequences. Assume that $\mathcal{B} \in \mathcal{R}^+_{D(w)}$, (a_n) and (b_n) are sequences of integers (non-negative), and let $x = (x_n)$ and $y = (y_n)$ be two sequences such that*

$$x_n - L_1 = stat^{p,q}_{\Psi_\Delta} - o(u_n)$$

and

$$y_n - L_2 = stat^{p,q}_{\Psi_\Delta} = o(v_n).$$

Then, each of the following assertions hold true:

(i) $(x_n - L_1) \pm (y_n - L_2) = stat^{p,q}_{\Psi_\Delta} - o(w_n)$;

(ii) $(x_n - L_1)(y_n - L_2) = stat^{p,q}_{\Psi_\Delta} - o(u_n v_n)$;
(iii) $\gamma(x_n - L_1) = stat^{p,q}_{\Psi_\Delta} - o(u_n)$ *(for any scalar* γ*)*;
(iv) $\sqrt{|x_n - L_1|} = stat^{p,q}_{\Psi_\Delta} - o(u_n)$,
where $w_n = \max\{u_n, v_n\}$.

Proof. To prove the assertion (i) of Lemma 4.2, we define the following sets for $\epsilon > 0$ and $x \in \mathcal{D}$:

$$\mathcal{N}_n = \left|\left\{k : k \leqq S_n \text{ and } \left|\left(\Delta^{[r]}_{p,q} x_k + \Delta^{[r]}_{p,q} y_k\right) - (L_1 + L_2)\right| \geqq \epsilon\right\}\right|,$$

$$\mathcal{N}_{0;n} = \left|\left\{k : k \leqq S_n \text{ and } |\Delta^{[r]}_{p,q} x_k - L_1| \geqq \frac{\epsilon}{2}\right\}\right|$$

and

$$\mathcal{N}_{1,n} = \left|\left\{k : k \leqq S_n \text{ and } |\Delta^{[r]}_{p,q} y_k - L_2| \geqq \frac{\epsilon}{2}\right\}\right|.$$

Clearly, we have

$$\mathcal{N}_n \subseteq \mathcal{N}_{0,n} \cup \mathcal{N}_{1,n}$$

and this implies that, for $n \in \mathbb{N}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in\mathcal{N}_n} s_m b_{m,k}(i) &\leqq \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in\mathcal{N}_{0,n}} s_m b_{m,k}(i) \\ &\leqq \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in\infty,\mathcal{N}_n} s_m b_{m,k}(i).\end{aligned} \tag{4.1}$$

Moreover, since

$$w_n = \max\{u_n, v_n\}, \tag{4.2}$$

so by (4.1), we get

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{w_n S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in\mathcal{N}_n} s_m b_{m,k}(i) &\leqq \lim_{n\to\infty} \frac{1}{u_n S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in\mathcal{N}_{0,n}} s_m b_{m,k}(i) \\ &\leqq \lim_{n\to\infty} \frac{1}{v_n S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in\mathcal{N}_{1,n}} s_m b_{m,k}(i).\end{aligned} \tag{4.3}$$

Further, by applying Theorem 3.1, we get

$$\lim_{n\to\infty} \frac{1}{w_n S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in\mathcal{N}_n} s_m b_{m,k}(i) = 0. \tag{4.4}$$

This proves the assertion (i) of Lemma 4.2.

Next, assertions (ii) to (iv) being similar to (i), so in the similar lines it can be proved. This completes the proof of Lemma 4.2. □

We now recall the modulus of continuity, $f(x, y) \in C_B(\mathcal{D})$ given by

$$\omega(f;\delta) = \sup_{(s,t),(x,y)\in\mathcal{D}} \left\{|f(s,t) - f(x,y)| : \sqrt{(s-x)^2 + (t-y)^2} \leqq \delta\right\} \qquad (\delta > 0), \tag{4.5}$$

and this implies that

$$|f(s,t) - f(x,y)| \leqq \omega\left[f;\ \sqrt{(s-x)^2 + (t-y)^2}\right]. \tag{4.6}$$

Now we prove the following theorem.

Theorem 4.3. *Let $\mathcal{B} \in \mathcal{R}^+_{D(W)}$ and (a_n) and (b_n) be sequences of integers (non-negative). Let $T_n : C_B(\mathcal{D}) \to C_B(\mathcal{D})$ be sequences of linear operators (positive). Also let (u_n) and (v_n) be the positive non-decreasing sequences. Suppose that the following conditions are satisfied:*
(i) $\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} = stat^{p,q}_{\Psi_\Delta} - o(u_n)$;
(ii) $\omega(f,\lambda_n) = stat^{p,q}_{\Psi_\Delta} - o(v_n)$ on $\mathcal{D}$,
where

$$\lambda_n = \sqrt{\|T_n(\psi^2(s,t), x, y)\|_{C_B(\mathcal{D})}} \quad \text{and} \quad \psi(s,t) = (s-x)^2 + (t-y)^2.$$

Then, for all $f \in C_B(\mathcal{D})$, the assertion as below holds true:

$$\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} = stat^{p,q}_{\Psi_\Delta} - o(w_n), \tag{4.7}$$

where (w_n) is given by (4.2).

Proof. Let $f \in C_B(\mathcal{D})$ and $(x,y) \in \mathcal{D}$. Using (4.6), we get

$$\begin{aligned}|T_n(f;x,y) - f(x,y)| &\leqq T_n(|f(s,t) - f(x,y)|;x,y) + |f(x,y)||T_n(1;x,y) - 1| \\ &\leqq T_n\left(\frac{\sqrt{(s-x)^2 + (t-y)^2}}{\delta} + 1; x, y\right)\omega(f,\delta) + M|T_n(1;x,y) - 1| \\ &\leqq \left(T_n(1;x,y) + \frac{1}{\delta^2}T_n(\psi(s,t);x,y)\right)\omega(f,\delta) + M|T_n(1;x,y) - 1|,\end{aligned}$$

where

$$M = \|f\|_{C_B(\mathcal{D})}.$$

Now, by taking supremum over $(x,y) \in \mathcal{D}$ on both sides, we get

$$\begin{aligned}\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} &\leqq \omega(f,\delta)\left\{\frac{1}{\delta^2}\|T_n(\psi(s,t);x,y)\|_{C_B(\mathcal{D})} + \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} + 1\right\} \\ &\quad + M\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})}.\end{aligned}$$

Now, substituting $\delta = \lambda_n = \sqrt{T_n(\psi^2;x,y)}$, we obtain

$$\begin{aligned}\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} &\leqq \omega(f,\lambda_n)\left\{\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} + 2\right\} + M\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} \\ &\leqq \omega(f,\lambda_n)\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} + 2\omega(f,\lambda_n) + M\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})}.\end{aligned}$$

So, we fairly get

$$\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \leqq \vartheta\left\{\omega(f,\lambda_n)\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} + \omega(f,\lambda_n) + \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})}\right\}.$$

(4.8)

Next, for a given $\epsilon > 0$, we consider the sets as follows:

$$\mathcal{H}_n = \left\{n : n \leqq S_n \text{ and } \|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \geqq \epsilon\right\}; \tag{4.9}$$

$$\mathcal{H}_{0,n} = \left\{n : n \leqq S_n \text{ and } \omega(f,\lambda_n)\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \geqq \frac{\epsilon}{3\mu}\right\}; \tag{4.10}$$

$$\mathcal{H}_{1,n} = \left\{n : n \leqq S_n \text{ and } \omega(f,\lambda_n) \geqq \frac{\epsilon}{3\mu}\right\} \tag{4.11}$$

and

$$\mathcal{H}_{2,n} = \left\{n : n \leqq S_n \text{ and } \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} \geqq \frac{\epsilon}{3\mu}\right\}. \tag{4.12}$$

Finally, by conditions (i) and (ii) of Theorem 4.3, together with Lemma 4.2, the last inequalities (4.9)-(4.12) lead us to the assertion (4.7) of Theorem 4.3 and it completes the proof of Theorem 4.3. □

5. Concluding Remarks and Observations

In this concluding section of our study, we present some further observations and remarks in relevance to different results that we have proved here.

Remark 5.1. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence given in Example 2.8. Then, since

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_{n\to\infty} x_n \to 0 \text{ on } \quad C_{\mathcal{B}}(\mathcal{D}),$$

we have

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_{n\to\infty} \|T_n(f_j;x,y) - f_j(x,y)\|_{C_{\mathcal{B}}(\mathcal{D})} = 0 \qquad (j = 0,1,2,3). \tag{5.1}$$

Hence, by applying Theorem 3.1, we have

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_{n\to\infty} \|T_n(f;x,y) - f(x,y)\|_{C_{\mathcal{B}}(\mathcal{D})} = 0, \qquad f \in C_{\mathcal{B}}(\mathcal{D}), \tag{5.2}$$

where

$$f_0(s,t) = 1, \;\; f_1(s,t) = s \;\; f_2(s,t) = t \text{ and } f_3(s,t) = s^2 + t^2.$$

However, since (x_n) is not ordinary convergent and so also it does not converge uniformly in the usual sense. Thus, the usual Korovkin-type theorem does not work here for the operators defined by (3.13). Hence, this application clearly indicates that our Theorem 3.1 is a non-trivial generalization of the traditional Korovkin-type theorem (see [32]).

Remark 5.2. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence as given in Example 2.8. Then, since

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_{n\to\infty} x_n \to 0 \text{ on } C_{\mathcal{B}}(\mathcal{D}),$$

so (5.1) holds. Now by applying (5.1) and our Theorem 3.1, condition (5.2) holds. However, since (x_n) does not deferred weighted $\mathcal{B}$-statistically converge, we can say that the result of Pradhan et al. ([38], p. 11, Theorem 3) does not hold true for our operator defined in (3.13). Thus, our Theorem 3.1 is also a non-trivial extension of Pradhan et al. ([38], p. 11, Theorem 3) and [36]. Based upon the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (3.13) and therefore, it is stronger than the classical and statistical versions of the Korovkin type approximation theorems (see [38] and [32]) established earlier.

Remark 5.3. If we replace the conditions (i) and (ii) in our Theorem 4.3 by the following condition:

$$|T_n(f_j;x,y) - f_j(x,y)|_{C_{\mathcal{B}}(\mathcal{D})} = \mathrm{stat}^{p,q}_{\Psi_\Delta} - o(u_{n_j}) \qquad (j = 0,1,2,3), \tag{5.3}$$

then we can write

$$T_n(\varphi^2;x,y) = N\sum_{j=0}^{3} \|T_n(f_j(s,t);x,y) - f_j(x,y)\|_{C_B(\mathcal{D})}, \tag{5.4}$$

where

$$N = \left\{\epsilon + M + \frac{2M}{\delta^2}\right\}, \quad (j = 0,1,2,3).$$

It thus follows from (5.3), (5.4) and Lemma 4.2 that

$$\lambda_n = \sqrt{T_n(\varphi^2)} = \mathrm{stat}^{p,q}_{\Psi_\Delta} - o(d_n) \;\text{ on }\; C_{\mathcal{B}}(\mathcal{D}), \tag{5.5}$$

where

$$o(d_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}, u_{n_3}\}.$$

Hence, we fairly get

$$\omega(f,\delta) = \mathrm{stat}^{p,q}_{\Psi_\Delta} - o(d_n) \;\text{ on }\; C_{\mathcal{B}}(\mathcal{D}).$$

By using (5.5) in Theorem 4.3, we immediately see for all $f \in C_{\mathcal{B}}(\mathcal{D})$ that

$$T_n(f;x,y) - f(x,y) = \mathrm{stat}^{p,q}_{\Psi_\Delta} - o(d_n) \;\text{ on }\; C_{\mathcal{B}}(\mathcal{D}). \tag{5.6}$$

Hence, if we use condition (5.3) in Theorem 4.3 in place of conditions (i) and (ii), then certainly we get the rates of the deferred weighted $\mathcal{B}$-statistical convergence of the sequence (T_n) of linear operators (positive) in Theorem 3.1.

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