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Application of Power Series Method for Solving Obstacle Problem of Fractional Order

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***Abstract*—An effective numerical method depends on the fractional power series is applied to solve a class of boundary value problems associated with obstacle, unilateral, and contact problems of fractional order $2\alpha, 0 < \alpha \le 1$. The fractional derivative is considered in the Caputo sense. This method constructs a convergent sequence of approximate solutions for the obstacle problem. A numerical example is given to illustrate the higher accuracy of this technique**.

***Keywords*—fractional residual power series, boundary value problem, obstacle problem, Caputo derivative**.

I. INTRODUCTION

The theory of variational inequalities is a powerful tool in the study of obstacle and unilateral problems that arise in mathematical and engineering sciences. It is effective in studying fluid flow through porous media, elasticity, transportation, and economics equilibrium, see [1-3]. For example, in [2], Kikuchi and Oden have shown that the equilibrium problems for elastic objects touching a rigid base can be handled in the context of the theory of variational inequality problem (VIP). The obstacle model is essential in the development of the VIPs theory that arises in a variety of pure and differential applied sciences. Because of their importance, various numerical methods have been developed and applied to find approximate solutions of the second order obstacle problems. Some of these methods are, the finite difference method, spline method, and collocation method [4-8].

In the last few decades, fractional calculus attracted the attention of many researchers for its considerable importance in many applications in fluid dynamics, viscoelasticity, optical technology, entropy theory and engineering. Many mathematicians provide a brief history, theoretical developments, and applications of factional calculus, see [9-13]. Therefore, most of the initial and boundary value problems (BVPs) of integer order were generalized to fractional order and various methods were modified to solve them.

The basic motivation of this paper is to solve the following generalized obstacle system of fractional order α:

$$D_a^{2\alpha} u(x) = \begin{cases} f(x), & a \le x < c, \\ g(x)u(x) + f(x) + r, & c \le x < d, \\ f(x), & d \le x \le b, \end{cases} \quad (1)$$

subject to the boundary conditions

$$u(a) = \mu_1, \ \ u(b) = \mu_2, \quad (2)$$

where $0 < \alpha \le 1$, $D_a^{2\alpha}$ is the Caputo-fractional derivative, $\mu_1, \mu_2 \in \mathbb{R}$, the parameter r is real finite constant, $g(x)$ is an analytical continuous function on $[c, d]$, $f(x)$is a continuous on $[a, b]$, the function $u(x)$ is unknown smooth function to be determined such that $u^{(i)}(x), i = 0,1$, are continuous functions at the points c and d in $[a, b]$.

The BVP in (1) and (2) is the generalized fractional form of the second order obstacle problem which results if we put $\alpha = 1$. Many techniques were applied to solve (1) and (2) in the integer order case. Some of these techniques are; the collocation method [4], second and fourth order finite difference and spline methods [5], quadratic and cubic spline methods, parametric cubic spline method, using quadratic non-polynomial splines, and cubic non-polynomial splines [6-8].

In this paper, we present numerical solution for the fractional obstacle problem (1) and (2) via fractional residual power series method (RPSM). This solution is given in the form of rapid convergent series with easily computable components. The residual power series method was first proposed in 2013 by the Jordanian mathematician Omar Abu Arqub [14] to solve fuzzy differential equations. After that, it has been successfully applied to different types of problems. For instance, Lane-Emden equation, higher-order regular differential equations, nonlinear fractional KdV-Burgers equation, and nonlinear time-fractional dispersive PDEs [15-19]. This method ensures the convergence of the approximate series solution because it depends on minimizing residual errors. For more details, see [20-23].

This paper is organized in five sections including the introduction, which appear as follows. In Section II, some fundamental concepts of fractional calculus and the power series method are given. In Section III, a description the FRPSM is introduced by applying it to solve the fractional obstacle BVP in (1) and (2). The numerical example is presented in Section IV. This article ends in Section V with some conclusions.

II. Fundamental Concepts

In this section, main concepts, definitions and results about the fractional calculus and power series in Caputo sense are given briefly. For more details, we refer to [24-33].

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ over the interval $[a,b]$ for a function g is defined by $(J_{a+}^{\alpha} g)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g(z)}{(x-z)^{1-\alpha}} dz$, $x > a$. For $\alpha = 0$, J_{a+}^{α} is the identity operator.

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ is defined by $D_a^{\alpha} g(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{g^{(n)}(z)}{(x-z)^{\alpha-n+1}} dz$, $x > a$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$.

Definition 3. A power series expansion of the form $\sum_{m=0}^{\infty} c_m (x-a)^{m\alpha}$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$, is called fractional power series about $x = a$.

Theorem 1. Suppose that f has a fractional power series representation at $x = a$ of the form

$u(x) = \sum_{m=0}^{\infty} c_m (x-a)^{m\alpha}$; $a \le x < a+R$,

and if $D_a^{m\alpha} u(x), m = 0,1,2,\ldots$ are continuous on $(a, a+R)$, then $c_m = \frac{D_a^{m\alpha} u(a)}{\Gamma(1+m\alpha)}$.

III. RPSM for Solving Fractional Obstacle System

The fractional RPS technique can be applied for the obstacle problem (1) and (2) to obtain the approximate solution $u_n(x)$ as follows: we consider three cases depending on the corresponding intervals. These cases are:

- Case I: The RPS solution, $u_1(x)$, on $[a,c]$ can be obtained using the following procedure:
  Let $D_a^{2\alpha} u_1(x) = f(x)$ on $[a,c]$ and let the solution $u_1(x)$ has the FPS expansion about the initial point a such as

  $$u_1(x) = \sum_{n=0}^{\infty} c_n (x-a)^{n\alpha}, \qquad (3)$$

  and the k-th truncated series

  $$u_{1,k}(x) = \sum_{n=0}^{k} c_n (x-a)^{n\alpha}. \qquad (4)$$

  Since $u_1(x)$ satisfy the initial condition $u_1(a) = \mu_1 = c_0$, then $u_{1,k}(x)$ can be rewritten as

  $$u_{1,k}(x) = \mu_1 + c_1 (x-a)^{\alpha} + \sum_{n=2}^{k} c_n (x-a)^{n\alpha}. \qquad (5)$$

  According the RPS method, the kth-residual error function, $Res_{u_1}^k(x)$, can be defined by

  $$Res_{u_1}^k(x) = D_a^{2\alpha} u_{1,k}(x) - f(x), \qquad (6)$$

where the residual error function, $Res_{u_1}(x)$, can be given as follows

$$Res_{u_1}(x) = \lim_{k\to\infty} Res_{u_1}^k(x).$$

Consequently, we need to minimize $Res_{u_1}^k(x)$ and utilize the relation $D_a^{(k-2)\alpha} Res_{u_1}^k(x)\big|_{x=a} = 0$, $k = 2,3,\ldots$, to determine the unknown coefficients $c_n, n = 2,3,\ldots,k$, of (5). In this point, the value of $c_1 = A$ will be determined later by using the continuity conditions of Eq. (1).

Now, to illustrate the main steps of the RPS algorithm in finding the unknown coefficients $c_n, n = 2,3,\ldots,k$, let $k = 2$ and substitute the approximation

$$u_{1,2}(x) = \mu_1 + A(x-a)^{\alpha} + c_2(x-a)^{2\alpha}$$

into the kth-residual error function, $Res_{u_1}^2(x)$, such that

$$\begin{aligned} Res_{u_1}^2(x) &= D_a^{2\alpha} u_{1,2}(x) - f(x) \\ &= D_a^{2\alpha}(\mu_1 + A(x-a)^{\alpha} \\ &\quad + c_2(x-a)^{2\alpha}) - f(x) \\ &= c_2 \Gamma(2\alpha+1) - f(x), \end{aligned}$$

and then by the fact $D_a^{(k-2)\alpha} Res_{u_1}^k(x)\big|_{x=a} = 0$, $k = 2$, we get $c_2 \Gamma(2\alpha+1) - f(a) = 0$, that is, $c_2 = \frac{f(a)}{\Gamma(2\alpha+1)}$.

Therefore,

$$u_{1,2}(x) = \mu_1 + A(x-a)^{\alpha} + \frac{f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha}.$$

Likewise, to find the unknown coefficient c_3, substitute the third truncatedseries

$$u_{1,3}(x) = \mu_1 + A(x-a)^{\alpha} + \frac{f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha} + c_3(x-a)^{3\alpha}$$

into $Res_{u_1}^3(x)$ such that

$$Res_{u_1}^3(x) = D_a^{2\alpha} u_{1,3}(x) - f(x) = D_a^{2\alpha}\left(\mu_1 + A(x-a)^{\alpha} + \frac{f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha} + c_3(x-a)^{3\alpha}\right) - f(x) = f(a) + c_3 \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}(x-a)^{\alpha} - f(x),$$

and then by using $D_a^{\alpha} Res_{u_1}^3(x)\big|_{x=a} = 0$, we obtain $c_3 \Gamma(3\alpha+1) - D_a^{\alpha} f(a) = 0$, that is, $c_3 = \frac{D_a^{\alpha} f(a)}{\Gamma(3\alpha+1)}$.

Therefore,

$$u_{1,3}(x) = \mu_1 + A(x-a)^{\alpha} + \frac{f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha} + \frac{D_a^{\alpha} f(a)}{\Gamma(3\alpha+1)}(x-a)^{3\alpha}.$$

Now, to find the unknown coefficient c_4, substitute the fourth truncated series $u_{1,4}(x) = \mu_1 + A(x-a)^{\alpha} + \frac{f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha} + \frac{D_a^{\alpha} f(a)}{\Gamma(3\alpha+1)}(x-a)^{3\alpha} + c_4(x-a)^{4\alpha}$ into $Res_{u_1}^4(x)$ such that

$$Res_{u_1}^4(x) = D_a^{2\alpha} u_{1,4}(x) - f(x) = D_a^{2\alpha}\left(\mu_1 + A(x-a)^{\alpha} + \frac{f(a)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha} + \frac{D_a^{\alpha} f(a)}{\Gamma(3\alpha+1)}(x-a)^{3\alpha} + c_4(x-a)^{4\alpha}\right) - f(x)$$

$$= f(a) + \frac{D_a^{\alpha} f(a)}{\Gamma(\alpha+1)}(x-a)^{\alpha} + c_4 \frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)}(x-a)^{2\alpha} - f(x),$$

and then by using $D_a^{2\alpha} Res_{u_1}^4(x)\big|_{x=a} = 0$, we obtain $c_4\Gamma(4\alpha+1) - D_a^{2\alpha} f(a) = 0$, that is, $c_4 = \frac{D_a^{2\alpha} f(a)}{\Gamma(4\alpha+1)}$. Therefore, the fourth RPS-approximation is given by

$$u_{1,4}(x) = \mu_1 + A(x-a)^{\alpha} + \sum_{n=2}^{4} \frac{D_a^{(n-2)\alpha} f(a)}{\Gamma(n\alpha+1)}(x-a)^{n\alpha} \quad (7)$$

- Case II: The RPS solution, $u_2(x)$, on $[c,d]$ can be presented as follows:

  Let $D_a^{2\alpha} u_2(x) = g(x)u_2(x) + f(x) + r$ on $[c,d]$ and let the solution $u_2(x)$ has the k-th truncated series expansion about the initial point c such that

$$u_{2,k}(x) = \sum_{n=0}^{k} c_n (x-c)^{n\alpha}. \quad (8)$$

Since there is no condition at the initial point c, $u_{2,k}(x)$ can be written as

$$u_{2,k}(x) = c_0 + c_1(x-c)^{\alpha} + \sum_{n=2}^{k} c_n (x-c)^{n\alpha}. \quad (9)$$

According the RPS method, the kth-residual error function, $Res_{u_2}^k(x)$, can be defined by

$$Res_{u_2}^k(x) = D_a^{2\alpha} u_{2,k}(x) - g(x)u_{2,k}(x) - f(x) - r. \quad (10)$$

Consequently, to obtain the unknown coefficients $c_n, n = 2,3,\ldots,k$, of Eq. (9), we need to minimize $Res_{u_2}^k(x)$ and utilize the relation $D_a^{(k-2)\alpha} Res_{u_2}^k(x)\big|_{x=c} = 0, k = 2,3,\ldots$. In this point, the values of $c_0 = B$ and $c_1 = C$ will be determined later by using the continuity conditions of Eq. (1).

Now, to apply the RPS algorithm in finding the coefficient c_2, substitute $u_{2,2}(x) = B + C(x-c)^{\alpha} + c_2(x-c)^{2\alpha}$ into $Res_{u_2}^2(x)$ such that

$$\begin{aligned} Res_{u_2}^2(x) &= D_a^{2\alpha} u_{2,2}(x) - g(x)u_{2,2}(x) - f(x) - r \\ &= D_a^{2\alpha}(B + C(x-c)^{\alpha} + c_2(x-c)^{2\alpha}) \\ &\quad - g(x)(B + C(x-c)^{\alpha} + c_2(x-c)^{2\alpha}) - f(x) - r \\ &= b_2\Gamma(2\alpha+1) - g(x)(B + C(x-c)^{\alpha} + c_2(x-c)^{2\alpha}) - f(x) - r, \end{aligned}$$

and then by using $Res_{u_2}^2(x)\big|_{x=c} = 0$, we obtain $c_2\Gamma(2\alpha+1) - Bg(c) - f(c) - r = 0$, that is, $c_2 = \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)}$. Therefore, the second approximation is

$$u_{2,2}(x) = B + C(x-c)^{\alpha} + \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)}(x-c)^{2\alpha}.$$

Again, the third approximation has the form

$$u_{2,3}(x) = B + C(x-c)^{\alpha} + \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + c_3(x-c)^{3\alpha}.$$

Thus, to obtain the value of c_3, substitute $u_{2,3}(x)$ into $Res_{u_2}^3(x)$ such that

$$\begin{aligned} Res_{u_2}^3(x) &= D_a^{2\alpha} u_{2,3}(x) - g(x)u_{2,3}(x) - f(x) - r \\ &= D_a^{2\alpha}\left(B + C(x-c)^{\alpha} + \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + c_3(x-c)^{3\alpha}\right) \\ &\quad - g(x)\left(B + C(x-c)^{\alpha} + \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + c_3(x-c)^{3\alpha}\right) - f(x) - r \\ &= (Bg(c) + f(c) + r) + c_3\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}(x-c)^{\alpha} \\ &\quad - g(x)\left(B + C(x-c)^{\alpha} + \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + c_3(x-c)^{3\alpha}\right) - f(x) - r, \end{aligned}$$

and then by using $D_a^{\alpha} Res_{u_2}^3(x)\big|_{x=c} = 0$, we obtain $c_3\Gamma(3\alpha+1) - BD_a^{\alpha} g(c) - CD_a^{\alpha}(g(x)(x-c)^{\alpha})|_{x=c} - \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)} D_a^{\alpha}(g(x)(x-c)^{2\alpha})|_{x=c} - c_3 D_a^{\alpha}(g(x)(x-c)^{3\alpha})|_{x=c} - D_a^{\alpha} f(c) = 0$, that is, $c_3 = \frac{\psi(c)}{\Gamma(3\alpha+1)}$, $\psi(c) = BD_a^{\alpha} g(c) + CD_a^{\alpha}(g(x)(x-c)^{\alpha})|_{x=c} + D_a^{\alpha} f(c)$. Therefore, $u_{2,3}(x) = B + C(x-c)^{\alpha} + \frac{Bg(c)+f(c)+r}{\Gamma(2\alpha+1)}(x-c)^{2\alpha} + \frac{\psi(c)}{\Gamma(3\alpha+1)}(x-c)^{3\alpha}$. Similarly, the fourth approximation $u_{2,4}(x)$ can be obtained.

- Case III: The RPS solution, $u_3(x)$, on $[d,b]$ can be presented as follows:

  Let $D_a^{2\alpha} u_3(x) = f(x)$ on $[c,d]$ and let the solution, $u_3(x)$, has the k-th truncated series expansion at b in the form

$$u_{3,k}(x) = \sum_{n=0}^{k} c_n (x-b)^{n\alpha}. \quad (11)$$

Since $u_3(x)$ satisfy the condition $u_3(b) = \mu_2 = a_0$. Thus, $u_{3,k}(x)$ can be written as

$$u_{3,k}(x) = \mu_2 + c_1(x-b)^{\alpha} + \sum_{n=2}^{k} c_n (x-b)^{n\alpha}. \quad (12)$$

According the RPS method, the kth-residual error function, $Res_{u_3}^k(x)$, can be defined by

$$Res_{u_3}^{k}(x) = D_a^{2\alpha}u_{3,k}(x) - f(x). \quad (13)$$

However, to obtain the unknown coefficients $c_n, n = 2,3,\dots,k$, of Eq. (12), we need to minimize $Res_{u_3}^{k}(x)$ and utilize the relation $D_a^{(k-2)\alpha}Res_{u_3}^{k}(x)\big|_{x=b} = 0\,, k = 2,3,\dots$. In this point, the value of $c_1 = D$ will be determined later by using the continuity conditions of Eq. (1). Thus, to apply the FRPS algorithm in finding the coefficients a_2, substitute $u_{3,2}(x) = \mu_2 + D(x-b)^{\alpha} + c_2(x-b)^{2\alpha}$ into $Res_{u_3}^{2}(x)$ such that

$$\begin{aligned} Res_{u_3}^{2}(x) &= D_a^{2\alpha}u_{3,2}(x) - f(x) \\ &= D_a^{2\alpha}(\mu_2 + D(x-b)^{\alpha} + c_2(x-b)^{2\alpha}) \\ &- f(x) = c_2\Gamma(2\alpha+1) - f(x), \end{aligned}$$

and then by using $Res_{u_3}^{2}(x)\big|_{x=b} = 0$,we obtain $c_2\Gamma(2\alpha+1) - f(b) = 0$, that is, $c_2 = \frac{f(b)}{\Gamma(2\alpha+1)}$. Therefore, the second approximation is

$$u_{3,2}(x) = \mu_2 + D(x-b)^{\alpha} + \frac{f(b)}{\Gamma(2\alpha+1)}(x-b)^{2\alpha}.$$

In the same style, substitute the third truncated series $u_{3,3}(x) = \mu_2 + D(x-b)^{\alpha} + \frac{f(b)}{\Gamma(2\alpha+1)}(x-b)^{2\alpha} + c_3(x-b)^{3\alpha}$ into $Res_{u_3}^{3}(x)$ such that

$$\begin{aligned} Res_{u_3}^{3}(x) &= D_a^{2\alpha}u_{1,3}(x) - f(x) \\ &= D_a^{2\alpha}\Bigg(\mu_2 + D(x-b)^{\alpha} \\ &+ \frac{f(b)}{\Gamma(2\alpha+1)}(x-b)^{2\alpha} + c_3(x-b)^{3\alpha}\Bigg) \\ &- f(x) \\ &= f(b) + c_3\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}(x-b)^{\alpha} - f(x), \end{aligned}$$

and then by using $D_a^{\alpha}Res_{u_3}^{3}(x)\big|_{x=b} = 0$, we obtain $c_3\Gamma(3\alpha+1) - D_a^{\alpha}f(b) = 0$, that is, $c_3 = \frac{D_a^{\alpha}f(b)}{\Gamma(3\alpha+1)}$. Therefore, $u_{3,3}(x) = \mu_2 + D(x-b)^{\alpha} + \frac{f(b)}{\Gamma(2\alpha+1)}(x-b)^{2\alpha} + \frac{D_a^{\alpha}f(b)}{\Gamma(3\alpha+1)}(x-b)^{3\alpha}$. Hence, the fourth RPS-approximation on $[d,b]$ is given by

$$\begin{aligned} u_{3,4}(x) &= \mu_2 + D(x-b)^{\alpha} \\ &+ \sum_{n=2}^{4}\frac{D_a^{(n-2)\alpha}f(b)}{\Gamma(n\alpha+1)}(x-b)^{n\alpha}. \end{aligned} \quad (14)$$

Moreover, the same routine can be repeated until an arbitrary order k, so the unknown coefficients $c_n, n = 4,5,6,\dots,k$, can be obtained. Furthermore, the values of the parameters A, B, C, and D can be found by utilizing the continuity conditions of Eq. (1) as well as solving the obtained system of algebraic equations,

$$\begin{aligned} u_{1,k}(c) = u_{2,k}(c), u_{2,k}(d) = u_{3,k}(d), \\ D_a^{\alpha}u_{1,k}(c) = D_a^{\alpha}u_{2,k}(c), D_a^{\alpha}u_{2,k}(d) = D_a^{\alpha}u_{3,k}(d), \end{aligned} \quad (15)$$

Therefore, the kth approximate solution on $[a,b]$ can be finally given by

$$u_k(x) = \begin{cases} u_{1,k}(x), a \le x \le c, \\ u_{2,k}(x), c \le x \le d, \\ u_{3,k}(x), d \le x \le b. \end{cases} \quad (16)$$

Hence, the kth RPS-approximate solution is completely constructed for the BVPs (1) and (2).

VI. NUMERICAL RESULTS

Consider the fractional obstacle system of differential equation (1) when $f(x) = 0$. So, the obstacle problem can be written by

$$(D_0^{2\alpha}u)(x) = \begin{cases} 0\,, & x \in \left[0,\frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4},\pi\right] \\ u(x) - 1\,, & x \in \left[\frac{\pi}{4},\frac{3\pi}{4}\right] \end{cases}, \alpha \in (0,1],$$

$$u(0) = u(\pi) = 0.$$

For $\alpha = 1$, the exact solution is

$$u(x) = \begin{cases} \frac{4x}{\pi + 4\coth\left(\frac{\pi}{4}\right)}, & x \in \left[0,\frac{\pi}{4}\right] \\ 1 - \frac{4\cosh\left(\frac{\pi}{2} - x\right)}{\pi\sinh\left(\frac{\pi}{4}\right) + 4\cosh\left(\frac{\pi}{4}\right)}, & x \in \left[\frac{\pi}{4},\frac{3\pi}{4}\right]. \\ \frac{4(\pi - x)}{\pi + 4\coth\left(\frac{\pi}{4}\right)}, & x \in \left[\frac{3\pi}{4},\pi\right] \end{cases}$$

To achieve our goal, divide the interval $[0,\pi]$ into n equal subintervals utilizing the standard grid points $x_i = ih, i = 0,,\dots,5, x_0 = 0, x_n = \pi$, and the step size $h = \pi/5$. Using the RPS algorithm, a numerical comparison of the obtained results with the exact solution at some selected grid points and the 8th-RPS solution of fractional-order $\alpha = 1$ are shown in Table I. While Figure 1 allocates of 2D plots associated with the 8th-RPS solution for different values of α with step size $h = 0.01$, and $\alpha \in [0.85,1]$.

TABLE I: NUMERICAL RESULTS AND ABSOLUTE ERROR

x_i	**8th-RPS solution for $\alpha = 1$**		
	Exact	***Approximation***	***Absolute Error***
$\frac{\pi}{5}$	0.271967952	0.2719679543	1.49519×10^{-12}
$\frac{2\pi}{5}$	0.476916995	0.4769169953	2.59801×10^{-12}
$\frac{3\pi}{5}$	0.476916995	0.4769169686	2.67096×10^{-8}
$\frac{4\pi}{5}$	0.271967954	0.2719679543	5.04818×10^{-13}

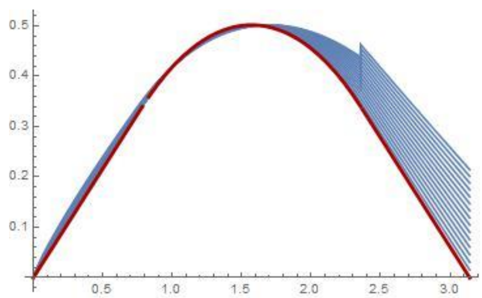

Fig. 1. *2D plots of RPS-solutions for different values of α.*

V. CONCLUSIONS

In this paper, we apply the fractional RPS method to solve a system of fractional order BVPs associated with obstacle, unilateral, and contact problems, and the approximate solution is obtained with a high degree of accuracy. Our method depends on minimizing the residual error, so we can ensure the convergence of the approximate solution series to the exact solution. The numerical results show that the present method is an accurate and reliable analytical technique for such systems.

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