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Solving Fuzzy Fractional IVPs of order 2β by Residual Power Series Algorithm

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***Abstract*—In this paper, an efficient numeric-analytic algorithm has been applied based on the residual power series approach to solve fuzzy fractional initial value problems of order 2β, $0 < \beta \leq 1$, under the strongly generalized differentiability. The present method relies basically upon the concept of the residual functions and generalized Taylor formula that constructs analytical and approximate solutions in the form of rapidly convergent series according to their parametric form. To validate the efficiency, reliability, and applicability of the proposed approach, the experimental data has been presented.**

Keywords—strongly generalized differentiability, Caputo fractional derivative, residual power series algorithm, fuzzy fractional initial value problems.

I. INTRODUCTION

Fuzzy fractional differential equation is a novel and important part of the fuzzy mathematical analysis. It has ample applications because of that many phenomena and problems in engineering disciplines, computer science, physics, artificial intelligence, thermal systems, and operations research can be modeled very accurately, and can be converted to uncertain process issues of fractional order [1-4]. In optimization and simulation analysis, ordinary differential equations (ODEs) play a vital role in various fields. But in several cases, information about these problems often intersects uncertainty. This uncertainty resulted from different reasons, such as measurement errors, missing data, cumulative errors, or even if the conditions of restrictions were specified. So, it is necessary to have some mathematical tools to understand this uncertainty. Consequently, the formation of a convenient and applicable algorithm is important to achieve a computational structure that will adequately process the fuzzy fractional initial value problems (FFIVPs) and resolve it appropriately [5-10].

The main purpose of this paper is to provide analytic-numeric solutions of (FFIVPs) of order 2β under the strongly generalized differentiability by utilizing a recent efficient algorithm, the residual power series (RPS) algorithm. The RPS method was proposed, at the first time, and successfully applied by Abu Arqub as a new optimization technique to determine the unknown coefficients of the power series of fuzzy IVPs [11-13]. The RPS is an effective and easy approach to construct fractional power series solution (FPS) for both linear and nonlinear problems without linearization, perturbation or discretization [14-24]. Unlike the classical power series method, the RPS technique neither requires comparing the corresponding coefficients nor is a recursion relation needed as well.

The remainder of this paper is organized as follows. In Section II, basic definitions and theories related to fuzzy analysis and fractional power series representations are given. In Section III, the RPS procedure for solving fuzzy fractional IVPs is formulated and constructed. In Section IV, construction of the FFIVPs solutions of order 2β is presented by using the RPS algorithm. In Section V, numerical application is performed to show capability, potentiality and simplicity of the proposed approach. Concluding remarks are shown in Section VI.

II. PRELIMINARIES

In the present section, we review some basic concepts, definitions and results related to fuzzy analysis and fractional power series representations. For more details, we refer to [25-33].

Definition 1. A fuzzy number ρ is a mapping such that $\rho: \mathbb{R} \to [0,1]$, satisfying the following conditions:

1. $\rho(\lambda s + (1-\lambda)t) \geq min\{\rho(s), \rho(t)\}$, for all $s, t \in \mathbb{R}$, and $0 < \lambda \leq 1$, then ρ is fuzzy convex.
2. If there is $s_* \in \mathbb{R}$, such that $\rho(s_*) = 1$, i.e. ρ is normal.
3. $\rho(s_*) \geq lim_{s \to s_*^+} \rho(s)$, for any $s_* \in \mathbb{R}$, i.e. ρ is upper-semi continuous.
4. $supp(\rho) = \{s \in \mathbb{R}: \rho(s) > 0\}$ is the support of ρ, and $\overline{\{s \in \mathbb{R}: \rho(s) > 0\}}$ is compact, where $\overline{\{*\}}$ denotes the closure of subset.

The parametric form or the r-cut representation of a fuzzy number $\rho \in \mathbb{R}_{\mathcal{F}}$, is defined as: $[\rho]^r = \{s \in \mathbb{R}: \rho(s) \geq r\}$, if $r \in (0,1]$, and $[\rho]^r = \overline{supp(\rho)}$, if $r = 0$. Clearly, the parametric form of ρ is closed and bounded interval $[\rho_{1r}, \rho_{2r}]$ in which ρ_{1r} is the lower r-cut representation of ρ, and ρ_{2r} is the upper r-cut representation of ρ.

The metric structure of $\mathbb{R}_{\mathcal{F}}$ is defined by the Hausdorff distance mapping $\mathcal{D}_H: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$ such that $\mathcal{D}_H(v, w) = sup_{0 \leq r \leq 1} max\{|\rho_{1r} - \omega_{1r}|, |\rho_{2r} - \omega_{2r}|\}$, for arbitrary fuzzy numbers ρ and ω. The metric $(\mathbb{R}_{\mathcal{F}}, \mathcal{D}_H)$ has been proved as complete metric space [21-23].

Definition 2. Let $u: (a, b) \to \mathbb{R}_{\mathcal{F}}$ and for fixed $t_0 \in [a, b]$. One can say u is a strongly generalized differentiable at t_0, if there is an element $u'(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that either:

i. The H-differences $u(t_0 + \xi) \ominus u(t_0)$, $u(t_0) \ominus u(t_0 - \xi)$ exist, for each $\xi > 0$, sufficiently tends to 0, and $\lim_{\xi \to 0^+} \frac{u(t_0+\xi) \ominus u(t_0)}{\xi} = u'(t_0) = \lim_{\xi \to 0^+} \frac{u(t_0) \ominus u(t_0 - \xi)}{\xi}$, or

ii. The H-differences $u(t_0) \ominus u(t_0 + \xi), \varphi(t_0 - \xi) \ominus u(t_0)$ exist, for each $\xi > 0$, sufficiently tends to 0, and $\lim_{\xi \to 0^+} \frac{u(t_0) \ominus u(t_0 + \xi)}{-\xi} = u'(t_0) = \lim_{\xi \to 0^+} \frac{u(t_0 - \xi) \ominus u(t_0)}{-\xi}$.

Remark 1. If u is differentiable in terms of the first condition of Definition 2, then we say that u is (1)-differentiable on (a, b). Similarly, if u is differentiable in terms of second condition of Definition 2, then we say that u is (2)-differentiable on (a, b).

Definition 3. The Caputo fractional derivative of order $\beta > 0$ is defined by

$$D_{a^+}^{m\beta} f(t) = \frac{1}{\Gamma(m - \beta)} \int_a^t \frac{f^{(m)}(s)}{(t - s)^{\beta - m + 1}} ds,$$

where $m - 1 < \beta \leq m,\ m \in \mathbb{N}$.

Definition 4. Let $u: [a, b] \to \mathbb{R}_{\mathcal{F}}$ and $u \in C^{\mathcal{F}}[a, b] \cap L^{\mathcal{F}}[a, b]$. One can say u is Caputo fuzzy H-differentiable at t when $D_{a^+}^{\beta} u(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{u'(\tau)}{(t-\tau)^{\beta}} d\tau$ exists, where $0 < \beta \leq 1$. Then, we say that u is Caputo $[(1) - \beta]$differentiable when as u *is* (1)-differentiable and u is Caputo $[(2) - \beta]$ differentiable when as u is (2)-differentiable.

Definition 5. A fractional power series (FPS) representation at $t = a$ has the following form

$$\sum_{m=0}^{\infty} c_m (t - a)^{m\beta} = c_0 + c_1 (t - a)^{\beta} + c_2 (t - a)^{2\beta} + \cdots.$$

Theorem 2. Suppose that $f(t)$ has the following FPS representation at $t = a$

$$f(t) = \sum_{m=0}^{\infty} \frac{D_{a^+}^{m\beta} f(a)}{\Gamma(m\beta + 1)} (t - a)^{m\beta},$$

where $n - 1 < \beta \leq n, a < t < a + R,\ f(t) \in C[a, a + R)$ and $D_{a^+}^{m\beta} f(t) \in C(a, a + R)$ for $m = 0,1,2, \ldots$, and $D_{a^+}^{m\beta} = D_{a^+}^{\beta} \cdot D_{a^+}^{\beta} \cdot \ldots \cdot D_{a^+}^{\beta}$ (m-times).

III. FORMULATION OF THE MODEL

Let us consider the FFIVPs of the following form

$$D_{a^+}^{2\beta} u(t) = f(t) D_{a^+}^{\beta} u(t) + F(t, u(t)),\ 0 < \beta \leq 1, \quad (1)$$

with the fuzzy initial conditions

$$u(a) = \rho,\ D_{a^+}^{\beta} u(a) = \delta. \quad (2)$$

where $\rho, \delta \in \mathbb{R}_{\mathcal{F}}, F: [a, b] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a linear or nonlinear continuous fuzzy-valued function, $f(t)$ is a continuous real valued function with nonnegative values on $[a, b]$ and $u(t)$ is unknown fuzzy function to be determined.

Let $[D_{a^+}^{2\beta} u_{1r}(t), D_{a^+}^{2\beta} u_{2r}(t)]$, $[u_{1r}(t), u_{2r}(t)]$, $[D_{a^+}^{\beta} u_{1r}(t), D_{a^+}^{\beta} u_{2r}(t)]$, and $[F_{1r}(\cdot), F_{2r}(\cdot)]$ be the parametric forms of $D_{a^+}^{2\beta} u(t)$, $u(t)$, $D_{a^+}^{\beta} u(t)$ and $F(\cdot)$, respectively. Let r-cut representation of F be $[F(t, u(t))]^r = [F_{1r}(t, [u(t)]^r), F_{2r}(t, [u(t)]^r)]$, so, the parametric form of FFIVPs (1) and (2) can be given by

$$[D_{a^+}^{2\beta} u(t)]^r = f(t) [D_{a^+}^{\beta} u(t)]^r + [F(t, u(t))]^r, \quad (3)$$

with the fuzzy initial conditions

$$[\varphi(a)]^r = [\rho_{1r}, \rho_{2r}],\ [D_{a^+}^{\beta} \varphi(a)]^r = [\delta_{1r}, \delta_{2r}]. \quad (4)$$

The (n)-solution of FFIVPs (1) and (2) is a function $u: [a, b] \to \mathbb{R}_{\mathcal{F}}$ that has Caputo $[(n, m) - \beta]$-derivative and satisfies the FFIVPs (1) and (2).

Algorithm 1. To obtain the (n, m)-solution of the FFIVPs (1) and (2), there are four cases:

Case I: If $u(t)$ is Caputo [(1,1)- β]-differentiable, we convert the FFIVPs (1) and (2) to the following ordinary fractional differential equations (OFDEs) system:

$$\begin{aligned} D_{a^+}^{2\beta} u_{1r}(t) &= f(t) D_{a^+}^{\beta} u_{1r}(t) + F_{1r}(t, u_{1r}(t), u_{2r}(t)), \\ D_{a^+}^{2\beta} u_{2r}(t) &= f(t) D_{a^+}^{\beta} u_{2r}(t) + F_{2r}(t, u_{1r}(t), u_{2r}(t)), \\ u_{1r}(a) &= \rho_{1r}, D_{a^+}^{\beta} u_{1r}(a) = \delta_{1r}, \\ u_{2r}(a) &= \rho_{2r}, D_{a^+}^{\beta} u_{2r}(a) = \delta_{2r}, \end{aligned} \quad (5)$$

Then do the following steps:

Step 1: Solve the system (5) for $u_{1r}(t)$ and $u_{2r}(t)$.

Step 2: Ensure that $[u_{1r}(t), u_{2r}(t)]$, $[D_{a^+}^{\beta} u_{1r}(t), D_{a^+}^{\beta} u_{2r}(t)]$ and $[D_{a^+}^{2\beta} u_{1r}(t), D_{a^+}^{2\beta} u_{2r}(t)]$ are valid level sets for each $r \in [0,1]$.

Step 3: Construct the (1,1)-solution $u(t)$ whose r-cut representation is $[u_{1r}(t), u_{2r}(t)]$.

Case II: If $u(t)$ is Caputo [(1,2)- β]-differentiable, we convert the FFIVPs (1) and (2) to the following (OFDEs):

$$\begin{aligned} D_{a^+}^{2\beta} u_{2r}(t) &= f(t) D_{a^+}^{\beta} u_{1r}(t) + F_{1r}(t, u_{1r}(t), u_{2r}(t)), \\ D_{a^+}^{2\beta} u_{1r}(t) &= f(t) D_{a^+}^{\beta} u_{2r}(t) + F_{2r}(t, u_{1r}(t), u_{2r}(t)), \\ u_{1r}(a) &= \rho_{1r}, D_{a^+}^{\beta} u_{1r}(a) = \delta_{1r}, \\ u_{2r}(a) &= \rho_{2r}, D_{a^+}^{\beta} u_{2r}(a) = \delta_{2r}, \end{aligned} \quad (6)$$

Then do the following steps:

Step 1: Solve the system (6) for $u_{1r}(t)$ and $u_{2r}(t)$.

Step 2: Ensure that $[u_{1r}(t), u_{2r}(t)]$, $\left[D_{a^+}^{\beta}u_{1r}(t), D_{a^+}^{\beta}u_{2r}(t)\right]$ and $\left[D_{a^+}^{2\beta}u_{2r}(t), D_{a^+}^{2\beta}u_{1r}(t)\right]$ are valid level sets $\forall r \in [0,1]$.

Step 3: Construct the (1,2)-solution $u(t)$ whose r-cut representation is $[u_{1r}(t), u_{2r}(t)]$.

Case III: If $u(t)$ is Caputo $[(2,1)\text{-}\beta]$-differentiable, we convert the FFIVPs (1) and (2) to the following (OFDEs):

$$\begin{aligned} D_{a^+}^{2\beta}u_{2r}(t) &= f(t)D_{a^+}^{\beta}u_{2r}(t) + F_{1r}\big(t, u_{1r}(t), u_{2r}(t)\big), \\ D_{a^+}^{2\beta}u_{1r}(t) &= f(t)D_{a^+}^{\beta}u_{1r}(t) + F_{2r}\big(t, u_{1r}(t), u_{2r}(t)\big), \\ u_{1r}(a) &= \rho_{1r}, D_{a^+}^{\beta}u_{1r}(a) = \delta_{1r}, \\ u_{2r}(a) &= \rho_{2r}, D_{a^+}^{\beta}u_{2r}(a) = \delta_{2r}, \end{aligned} \tag{7}$$

Then do the following steps:

Step 1: Solve the system (7) for $u_{1r}(t)$ and $u_{2r}(t)$.

Step 2: Ensure that $[u_{1r}(t), u_{2r}(t)]$, $\left[D_{a^+}^{\beta}u_{2r}(t), D_{a^+}^{\beta}u_{1r}(t)\right]$ and $\left[D_{a^+}^{2\beta}u_{2r}(t), D_{a^+}^{2\beta}u_{1r}(t)\right]$ are valid level sets $\forall r \in [0,1]$.

Step 3: Construct the (2,1)-solution $u(t)$ whose r-cut representation is $[u_{1r}(t), u_{2r}(t)]$.

Case IV: If $u(t)$ is Caputo $[(2,2)\text{-}\beta]$-differentiable, we convert the FFIVPs (1) and (2) to the following (OFDEs):

$$\begin{aligned} D_{a^+}^{2\beta}u_{1r}(t) &= f(t)D_{a^+}^{\beta}u_{2r}(t) + F_{1r}\big(t, u_{1r}(t), u_{2r}(t)\big), \\ D_{a^+}^{2\beta}u_{2r}(t) &= f(t)D_{a^+}^{\beta}u_{1r}(t) + F_{2r}\big(t, u_{1r}(t), u_{2r}(t)\big), \\ u_{1r}(a) &= \rho_{1r}, D_{a^+}^{\beta}u_{1r}(a) = \delta_{1r}, \\ u_{2r}(a) &= \rho_{2r}, D_{a^+}^{\beta}u_{2r}(a) = \delta_{2r}, \end{aligned} \tag{8}$$

Then do the following steps:

Step 1: Solve the system (8) for $u_{1r}(t)$ and $u_{2r}(t)$.

Step 2: Ensure that $[u_{1r}(t), u_{2r}(t)]$, $\left[D_{a^+}^{\beta}u_{2r}(t), D_{a^+}^{\beta}u_{1r}(t)\right]$ and $\left[D_{a^+}^{2\beta}u_{1r}(t), D_{a^+}^{2\beta}u_{2r}(t)\right]$ are valid level sets $\forall r \in [0,1]$.

Step 3: Construct the (2,2)-solution $u(t)$ whose r-cut representation is $[u_{1r}(t), u_{2r}(t)]$.

IV. Analysis of the RPS Algorithm

In this section, we apply the RPS method to construct FPS (1,1)-solution of (1) under Caputo $[(1,1)\text{-}\beta]$-differentiable.

Depending on the RPS algorithm, the j^{th}-truncated FPS solutions of (5) about $a = 0$ can be given by

$$\begin{aligned} u_{j,1r}(t) &= \textstyle\sum_{m=0}^{j} a_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \\ u_{j,2r}(t) &= \textstyle\sum_{m=0}^{j} b_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}. \end{aligned} \tag{9}$$

Utilizing the initial conditions of (5), then the 1st-FPS approximated solutions of $u_{1r}(t)$, and $u_{2r}(t)$ can be, respectively, given by $u_{1,1r}(t) = \rho_{1r} + \delta_{1r}\frac{t^{\beta}}{\Gamma(\beta+1)}$ and $u_{1,2r}(t) = \rho_{2r} + \delta_{2r}\frac{t^{\beta}}{\Gamma(\beta+1)}$. Thus, (9) can be written as

$$\begin{aligned} u_{j,1r}(t) &= \rho_{1r} + \delta_{1r}\frac{t^{\beta}}{\Gamma(\beta+1)} + \textstyle\sum_{m=2}^{j} a_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \\ u_{j,2r}(t) &= \rho_{2r} + \delta_{2r}\frac{t^{\beta}}{\Gamma(\beta+1)} + \textstyle\sum_{m=2}^{j} b_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}. \end{aligned} \tag{10}$$

Define the so-called j^{th}-residual functions, $Res_{j,1r}$ and $Res_{j,2r}$, $j = 2,3, \dots m$ as follow

$$\begin{aligned} Res_{j,1r}(t) = D_{0^+}^{2\beta}u_{j,1r}(t) - f(t)D_{0^+}^{\beta}u_{j,1r}(t) \\ - F_{1r}\left(t, u_{j,1r}(t), u_{j,2r}(t)\right) \end{aligned}$$

$$\begin{aligned} Res_{j,2r}(t) = D_{0^+}^{2\beta}u_{j,2r}(t) - f(t)D_{0^+}^{\beta}u_{j,2r}(t) \\ - F_{2r}\left(t, u_{j1r}(t), u_{j,2r}(t)\right) \end{aligned}$$

The values of a_m and b_m, $m = 2,3,4, \dots, j$, can be obtained through following fashion: write the m^{th}-truncated FPS solutions into m^{th}-residual functions, compute $D_{0^+}^{(j-2)\beta}Res_{j,1r}(t)$ and $D_{0^+}^{(j-2)\beta}Res_{j,2r}(t)$, then solve the resulting equations at $t = 0$.

V. Simulations and Numerical Results

The aim of this section is to demonstrate the superiority of the proposed algorithm in solving the following FFIVP:

$$\begin{aligned} D_{0^+}^{2\beta}u(t) + u(t) = [r, 2 - r], 0 < \beta \leq 1, t \geq 0, \\ u(0) = D_{0^+}^{\beta}u(0) = [r - 1, 1 - r]. \end{aligned} \tag{11}$$

Based upon differentiability types, then FFIVPs (11) can be converted into one of the following systems of OFDEs:

Case I: If $u(t)$ is Caputo $[(1,1)\text{-}\beta]$-differentiable, then the corresponding (1,1)-system will be in the form

$$\begin{aligned} D_{0^+}^{2\beta}u_{1r}(t) + u_{1r}(t) &= r, \\ D_{0^+}^{2\beta}u_{2r}(t) + u_{2r}(t) &= 2 - r, \\ u_{1r}(0) = D_{0^+}^{\beta}u_{1r}(0) &= r - 1, \\ u_{2r}(0) = D_{0^+}^{\beta}u_{2r}(0) &= 1 - r, \end{aligned} \tag{12}$$

To apply the RPS scheme, starting with the initial conditions of (12) and based upon (10), then the j^{th}-FPS approximated solutions for system (12) will be as

$$\begin{aligned} u_{j,1r}(t) &= (r - 1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \textstyle\sum_{m=2}^{j} a_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \\ u_{j,2r}(t) &= (1 - r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \textstyle\sum_{m=2}^{j} b_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \end{aligned} \tag{13}$$

where the values of a_m and b_m, $m = 2,3,4, \dots,$, can be obtained through construction $Res_{j,1r}$ and $Res_{j,2r}$.

$$\begin{aligned} Res_{j,1r}(t) &= D_{0^+}^{2\beta}u_{j,1r}(t) + u_{j,1r}(t) - r, \\ Res_{j,2r}(t) &= D_{0^+}^{2\beta}u_{j,2r}(t) + u_{j,2r}(t) - 2 + r. \end{aligned} \tag{14}$$

Now, in order to determine the 2nd-FPS approximate solutions, let $j = 2$, in (14) to get

$$Res_{2,1r}(t) = D_{0^+}^{2\beta}\left(r - 1 + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{a_2 t^{2\beta}}{\Gamma(2\beta+1)}\right) + \left(r - 1 + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{a_2 t^{2\beta}}{\Gamma(2\beta+1)}\right) - r = a_2 + \left(r - 1 + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{a_2 t^{2\beta}}{\Gamma(2\beta+1)}\right) - r$$ and

$Res_{2,2r}(t) = D_{0^+}^{2\beta}\left(1 - r + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{b_2 t^{2\beta}}{\Gamma(2\beta+1)}\right) + \left(1 - r + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{b_2 t^{2\beta}}{\Gamma(2\beta+1)}\right) - 2 + r = b_2 + \left(1 - r + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{b_2 t^{2\beta}}{\Gamma(2\beta+1)}\right) - 2 + r.$

Depending on $Res_{2,1r}(0) = Res_{2,2r}(0) = 0$, it yields that $a_2 = 1$, $b_2 = 1$. So, the 2nd-FPS approximated solutions of FFIVPs (12) are given by $u_{2,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)}$, and $u_{2,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)}$.

For obtaining the 3rd-FPS approximate solutions, consider $j = 3$, through (14), and apply the operator $D_{0^+}^{\beta}$ of $Res_{3,1r}(t)$, and $Res_{3,2r}(t)$, such that

$$D_{0^+}^{\beta} Res_{3,1r}(t) = D_{0^+}^{\beta}\left(D_{0^+}^{2\beta}\left(r - 1 + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{a_3 t^{3\beta}}{\Gamma(3\beta+1)}\right) + \left(r - 1 + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{a_3 t^{3\beta}}{\Gamma(3\beta+1)}\right) - r\right),$$

$$D_{0^+}^{\beta} Res_{3,2r}(t) = D_{0^+}^{\beta}\left(D_{0^+}^{2\beta}\left(1 - r + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{b_3 t^{3\beta}}{\Gamma(3\beta+1)}\right) + \left(1 - r + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{b_3 t^{3\beta}}{\Gamma(3\beta+1)}\right) - 2 + r\right).$$

Then, by solving the above resultant fractional equations, we get $a_3 = 1 - r$, and $b_3 = r - 1$. So, the 3rd-FPS approximated solutions can be written as $u_{3,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(1-r)t^{3\beta}}{\Gamma(3\beta+1)}$ and $u_{3,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(r-1)t^{3\beta}}{\Gamma(3\beta+1)}$.

Moreover, by using the fact $D_{0^+}^{(j-2)\beta} Res_{j,1r}(0) = D_{0^+}^{(j-2)\beta} Res_{j,2r}(0) = 0$, $j = 4,5,\ldots,8$, the 8th-FPS approximated solutions for system (12) are given by

$$u_{8,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(1-r)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(r-1)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(1-r)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)},$$

$$u_{8,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(r-1)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(1-r)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(r-1)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)}.$$

Then, the approximation of (12) can be expressed as

$$u_{1r}(t) = \lim_{j\to\infty} u_{j,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(1-r)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(r-1)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(1-r)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)} + \cdots,$$

$$u_{2r}(t) = \lim_{j\to\infty} u_{j,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(r-1)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(1-r)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(r-1)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)} + \cdots.$$

Particularly, for $\beta = 1$, the approximate solutions corresponding to (1,1)-system are in good agreement with Taylor series expansion of the exact solutions $u_{1r}(t) = r - \cos t + (r-1)\sin t$, and $u_{2r}(t) = 2 - r - \cos t + (1-r)\sin t$.

Case II: If $u(t)$ is Caputo $[(1,2)\text{-}\,\beta]$-differentiable, then the corresponding (1,2)-system will be

$$\begin{aligned} &D_{0^+}^{2\beta} u_{1r}(t) + u_{2r}(t) = 2 - r, \\ &D_{0^+}^{2\beta} u_{2r}(t) + u_{1r}(t) = r, \\ &u_{1r}(0) = D_{0^+}^{\beta} u_{1r}(0) = r - 1, \\ &u_{2r}(0) = D_{0^+}^{\beta} u_{2r}(0) = 1 - r. \end{aligned} \tag{15}$$

If we choose the initial guesses approximations as $u_{1,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)}$ and $u_{1,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)}$, then the jth FPS solutions of (15) are given by

$$\begin{aligned} &u_{j,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \sum_{m=2}^{j} a_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \\ &u_{j,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \sum_{m=2}^{j} b_m \frac{t^{m\beta}}{\Gamma(m\beta+1)}. \end{aligned} \tag{16}$$

Apparently, the jth-residual functions of (15) will be

$$\begin{aligned} &Res_{j,1r}(t) = D_{0^+}^{2\beta} u_{j,1r}(t) + u_{j,2r}(t) - 2 + r, \\ &Res_{j,2r}(t) = D_{0^+}^{2\beta} u_{j,2r}(t) + u_{j,1r}(t) - r. \end{aligned} \tag{17}$$

Following the argument of the method and based upon $D_{0^+}^{(j-2)\beta} Res_{j,1r}(0) = D_{0^+}^{(j-2)\beta} Res_{j,2r}(0) = 0$, $j = 2,3,\ldots,8$, the 8th-FPS solutions for system (15) can be written as

$$u_{8,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(r-1)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(r-1)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(r-1)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)},$$

$$u_{8,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(1-r)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(1-r)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(1-r)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)}.$$

Thus, the approximation of (15) can be expressed as

$$u_{1r}(t) = \lim_{j\to\infty} u_{j,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(r-1)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(r-1)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(r-1)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)} + \cdots,$$

$$u_{2r}(t) = \lim_{j\to\infty} u_{j,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(1-r)t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(1-r)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(1-r)t^{7\beta}}{\Gamma(7\beta+1)} - \frac{t^{8\beta}}{\Gamma(8\beta+1)} + \cdots.$$

which are in good agreement with Taylor series expansion of the exact solutions $u_{1r}(t) = r - \cos t + (r-1)\sinh t$, and $u_{2r}(t) = 2 - r - \cos t + (1-r)\sinh t$, in case $\beta = 1$.

Case III: If $u(t)$ is Caputo $[(2,1)\text{-}\,\beta]$-differentiable, then the corresponding (2,1)-system will be

$$\begin{aligned} &D_{0^+}^{2\beta} u_{1r}(t) + u_{1r}(t) = 2 - r, \\ &D_{0^+}^{2\beta} u_{2r}(t) + u_{2r}(t) = r, \\ &u_{1r}(0) = D_{0^+}^{\beta} u_{1r}(0) = r - 1, \\ &u_{2r}(0) = D_{0^+}^{\beta} u_{2r}(0) = 1 - r, \end{aligned} \tag{18}$$

If $\beta = 1$, then the exact solutions of (18) are $u_{1r}(t) = 2 - r + (2r-3)\cos t + (r-1)\sin t$, and $u_{2r}(t) = r + (1-2r)\cos t + (1-r)\sin t$.

By using the same manner as the previous cases, then the 8th-FPS approximated solutions of (18) are given by

$$u_{8,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{(3-2r)t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(1-r)t^{3\beta}}{\Gamma(3\beta+1)} + \frac{(2r-3)t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(r-1)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{(3-2r)t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(1-r)t^{7\beta}}{\Gamma(7\beta+1)} + \frac{(2r-3)t^{8\beta}}{\Gamma(8\beta+1)},$$

$$u_{8,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{(2r-1)t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(r-1)t^{3\beta}}{\Gamma(3\beta+1)} + \frac{(1-2r)t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(1-r)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{(2r-1)t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(r-1)t^{7\beta}}{\Gamma(7\beta+1)} + \frac{(1-2r)t^{8\beta}}{\Gamma(8\beta+1)}.$$

For $\beta = 1$, the general forms of the FPS approximated solutions of (18) can be written as

$$u_{1r}(t) = (r-1) + (r-1)t + \frac{(3-2r)t^2}{2} + \frac{(1-r)t^3}{3!} + \frac{(2r-3)t^4}{4!} + \frac{(r-1)t^5}{5!} + \frac{(3-2r)t^6}{6!} + \frac{(1-r)t^7}{7!} + \frac{(2r-3)t^8}{8!} + \cdots = 2 - r + (2r-3)\cos t + (r-1)\sin t,$$

$$u_{2r}(t) = (1-r) + (1-r)t + \frac{(2r-1)t^2}{2} + \frac{(r-1)t^3}{3!} + \frac{(1-2r)t^4}{4!} + \frac{(1-r)t^5}{5!} + \frac{(2r-1)t^6}{6!} + \frac{(r-1)t^7}{7!} + \frac{(1-2r)t^8}{8!} + \cdots = r + (1-2r)\cos t + (1-r)\sin t.$$

which are in good agreement with Taylor series expansion of the exact solutions.

Case IV: If $u(t)$ is Caputo [(2,2)- β]-differentiable, then the corresponding (2,2)-system is

$$\begin{aligned} &D_{0^+}^{2\beta} u_{1r}(t) + u_{2r}(t) = r, \\ &D_{0^+}^{2\beta} u_{2r}(t) + u_{1r}(t) = 2 - r, \\ &u_{1r}(0) = D_{0^+}^{\beta} u_{1r}(0) = r - 1, \\ &u_{2r}(0) = D_{0^+}^{\beta} u_{2r}(0) = 1 - r. \end{aligned} \quad (19)$$

If $\beta = 1$, then the exact solutions of (19) are $u_{1r}(t) = 2 - r - \cos t + 2(r-1)\cosh t + (r-1)\sinh t$, and $u_{2r}(t) = r - \cos t - 2(r-1)\cosh t + (1-r)\sinh t$.

Similarly, the 8th-FPS solutions of (19) will be as

$$u_{8,1r}(t) = (r-1) + \frac{(r-1)t^{\beta}}{\Gamma(\beta+1)} + \frac{(2r-1)t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(r-1)t^{3\beta}}{\Gamma(3\beta+1)} + \frac{(2r-3)t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(r-1)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{(2r-1)t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(r-1)t^{7\beta}}{\Gamma(7\beta+1)} + \frac{(2r-3)t^{8\beta}}{\Gamma(8\beta+1)},$$

$$u_{8,2r}(t) = (1-r) + \frac{(1-r)t^{\beta}}{\Gamma(\beta+1)} + \frac{(3-2r)t^{2\beta}}{\Gamma(2\beta+1)} + \frac{(1-r)t^{3\beta}}{\Gamma(3\beta+1)} + \frac{(1-2r)t^{4\beta}}{\Gamma(4\beta+1)} + \frac{(1-r)t^{5\beta}}{\Gamma(5\beta+1)} + \frac{(3-2r)t^{6\beta}}{\Gamma(6\beta+1)} + \frac{(1-r)t^{7\beta}}{\Gamma(7\beta+1)} + \frac{(1-2r)t^{8\beta}}{\Gamma(8\beta+1)}.$$

Hence, the approximated solutions of (19) have general form coinciding well with exact solutions at $\beta = 1$ such that

$$u_{1r}(t) = (r-1) + (r-1)t + \frac{(2r-1)t^2}{2} + \frac{(r-1)t^3}{3!} + \frac{(2r-3)t^4}{4!} + \frac{(r-1)t^5}{5!} + \frac{(2r-1)t^6}{6!} + \frac{(r-1)t^7}{7!} + \frac{(2r-3)t^8}{8!} + \cdots = 2 - r - \cos t + 2(r-1)\cosh t + (r-1)\sinh t,$$

$$u_{2r}(t) = (1-r) + (1-r)t + \frac{(3-2r)t^2}{2} + \frac{(1-r)t^3}{3!} + \frac{(1-2r)t^4}{4!} + \frac{(1-r)t^5}{5!} + \frac{(3-2r)t^6}{6!} + \frac{(1-r)t^7}{7!} + \frac{(1-2r)t^8}{8!} + \cdots = r - \cos t - 2(r-1)\cosh t + (1-r)\sinh t.$$

To demonstrate the agreement between the exact and approximated solutions, the absolute error at some selected grid points with step size 0.16 of the 8th-approximated solutions for FFIVPs (11), case 1 and case 2, have been summarized respectively in Table I and Table II, for $\beta = 1$ and different values of r-levels.

TABLE I: ABSOLUTE ERROR FOR CASE 1.

	t_i	$r = 0$	$r = 0.5$	$r = 0.75$
$u_{1r}(t)$	0.16	1.86295×10^{-13}	9.15934×10^{-14}	4.42979×10^{-14}
	0.32	9.37681×10^{-11}	4.53340×10^{-11}	2.11168×10^{-11}
	0.48	3.54102×10^{-9}	1.68121×10^{-9}	7.51301×10^{-10}
	0.64	4.62912×10^{-8}	2.15619×10^{-8}	9.19732×10^{-9}
$u_{2r}(t)$	0.16	1.92513×10^{-13}	9.76996×10^{-14}	5.04041×10^{-14}
	0.32	9.99687×10^{-11}	5.15346×10^{-11}	2.73174×10^{-11}
	0.48	3.89823×10^{-9}	2.03842×10^{-9}	1.10851×10^{-9}
	0.64	5.26258×10^{-8}	2.78965×10^{-8}	1.55319×10^{-8}

TABLE II: ABSOLUTE ERROR FOR CASE 2.

	t_i	$r = 0$	$r = 0.5$	$r = 0.75$
$u_{1r}(t)$	0.16	1.86295×10^{-13}	9.15934×10^{-14}	4.42979×10^{-14}
	0.32	9.39486×10^{-11}	4.54242×10^{-11}	2.11620×10^{-11}
	0.48	3.55663×10^{-9}	1.68901×10^{-9}	7.55205×10^{-10}
	0.64	4.66609×10^{-8}	2.17468×10^{-8}	9.28974×10^{-9}
$u_{2r}(t)$	0.16	1.92513×10^{-13}	9.76996×10^{-14}	5.04041×10^{-14}
	0.32	1.00149×10^{-10}	5.16248×10^{-11}	2.73625×10^{-11}
	0.48	3.91384×10^{-9}	2.04622×10^{-9}	1.11241×10^{-9}
	0.64	5.29955×10^{-8}	2.80814×10^{-8}	1.56244×10^{-8}

The numerical results of the approximated solutions of FFIVPs (11), case 3 and case 4, for some selected grid points with step size 0.2 on interval [0,1], and fixed value of $r = 0.5$ at different valued of β's, where $\beta \in \{0.8, 0.9, 1\}$, are given, respectively, in Table III and Table IV.

TABLE III: NUMERICAL RESULTS FOR CASE 3.

	t_i	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$
$u_{1r}(t)$	0.2	−0.559468	−0.555199	−0.539605
	0.4	−0.536831	−0.494156	−0.430375
	0.6	−0.432992	−0.346611	−0.240665
	0.8	−0.252091	−0.129213	−0.001356
$u_{2r}(t)$	0.2	0.599335	0.620584	0.644647
	0.4	0.694709	0.717959	0.739847
	0.6	0.782321	0.799034	0.810146
	0.8	0.858678	0.863059	0.859013

TABLE IV: NUMERICAL RESULTS FOR CASE 4.

	t_i	$\beta = 1$	$\beta = 0.9$	$\beta = 0.8$
$u_{1r}(t)$	0.2	−0.600801	−0.624148	−0.653190
	0.4	−0.707510	−0.743682	−0.790807
	0.6	−0.829128	−0.883201	−0.959086
	0.8	−0.978194	−1.06150	−1.182760
$u_{2r}(t)$	0.2	0.640668	0.689533	0.758232
	0.4	0.865388	0.967485	1.100280
	0.6	1.178460	1.335620	1.528570
	0.8	1.584780	1.795350	2.043130

VI. CONCLUSIONS

This paper presents analytic-numeric solutions of the fuzzy fractional IVPs of order 2β, $0 < \beta \leq 1$, based on the RPS algorithm. The analytical solutions by proposed method coincide well with exact solutions. The RPS algorithm does not require linearization, limitation on the problem's nature, sort of classification or perturbation. The results refer that the RPS technique is an accurate, simple and powerful tool for solving of such problems. All computations are performed by using Mathematica 10 software package.

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