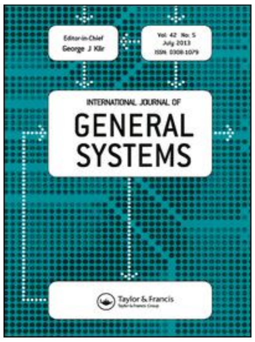

International Journal of General Systems

ISSN: 0308-1079 (Print) 1563-5104 (Online) Journal homepage: https://www.tandfonline.com/loi/ggen20

Stability of mixed type functional equation in normed spaces using fuzzy concept

Pasupathi Narasimman, Hemen Dutta & Iqbal H. Jebril

To cite this article: Pasupathi Narasimman, Hemen Dutta & Iqbal H. Jebril (2019): Stability of mixed type functional equation in normed spaces using fuzzy concept, International Journal of General Systems, DOI: 10.1080/03081079.2019.1586683

To link to this article: https://doi.org/10.1080/03081079.2019.1586683

Published online: 21 Mar 2019.

INTERNATIONAL JOURNAL OF GENERAL SYSTEMS
https://doi.org/10.1080/03081079.2019.1586683

Stability of mixed type functional equation in normed spaces using fuzzy concept

Pasupathi Narasimman[a], Hemen Dutta[b] and Iqbal H. Jebril[c,d]

[a]Department of Mathematics, Thiruvalluvar University College of Arts and Science, Tirupattur, India; [b]Department of Mathematics, Gauhati University, Guwahati, India; [c]Department of Mathematics, Taibah University, Almunawwarah, Saudi Arabia; [d]Department of Mathematics, Al-Zaytoonah University of Jordan, Amman, Jordan

ABSTRACT
S. M. Ulam once addressed the problem 'when is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?'. This problem was solved by D. H. Hyers in 1941 using the functional equation and thereafter numerous research papers and monographs have been published for various types of functional equations in different spaces. The solution proposed by D. H. Hyers (1941) later developed into the famous generalized Hyers-Ulam-Rassias stability of functional equations. In this paper, we intend to attain the general solution to a new mixed type functional equation and interrogate the generalized Hyers–Ulam–Rassias stability in fuzzy normed spaces. Also, we seek to provide its application for generating secret keys in client–server environment.

ARTICLE HISTORY
Received 26 September 2018
Accepted 6 February 2019

KEYWORDS
Additive and quadratic functional equations; generalized Hyers-Ulam-Rassias stability; fuzzy normed space

1. Introduction

The stability of following functional equations

$$h(t_1 + t_2) = h(t_1) + h(t_2),$$
$$f(mt_1 + nt_2) + f(mt_1 - nt_2) = 2m^2 f(t_1) + 2n^2 f(t_2),$$

for nonzero real numbers m,n with $m \neq \pm 1$ using fuzzy concepts has been investigated in Mirmostafaee and Moslehian (2008) and Lee et al. (2010), respectively. The following mixed type functional equations

$$g(2t_1 + t_2) + g(2t_1 - t_2) = g(t_1 + t_2) + g(t_1 - t_2) + 2g(2t_1) - 2g(t_1)$$

and

$$g(-t_1) + g(2t_1 - t_2) + g(2t_2) + g(t_1 + t_2) - g(-t_1 + t_2) \\ - g(t_1 - t_2) - g(-t_1 - t_2) = 3g(t_1) + 3g(t_2),$$

CONTACT Hemen Dutta hemen_dutta08@rediffmail.com

have been established in Gordji, Ghobadipour, and Rassias (2009) and Ravi, Rassias, and Narasimman (2010), respectively.

Recently, Ravi and Kodandan Ravi and Kodandan (2010) obtained the stability of the functional equation

$$g\left(\frac{t_1t_3}{t_2}+\frac{t_2t_4}{t_1}\right)+g\left(\frac{t_1t_3}{t_2}-\frac{t_2t_4}{t_1}\right)=g\left(\frac{t_2t_4}{t_1}\right)+2g\left(\frac{t_1t_3}{t_2}\right)+g\left(\frac{-t_2t_4}{t_1}\right),$$

where $t_1, t_2 \neq 0$ in Non-Archimedean spaces.

In 2016, Wongkum and Kumam (2016) proved the stability of sextic functional equations by using the fixed point technique in the framework of fuzzy modular spaces with the lower semi continuous (briefly, l.s.c.) and β-homogeneous conditions.

In 2018, Nazarianpoor, Rassias, and Sadeghi (2018) investigated the general solution and the generalized Hyers-Ulam stability of a new functional equation satisfied by $f(x) = x^{24}$, which is called the quattuorvigintic functional equation in intuitionistic fuzzy normed spaces by using the fixed point method.

In 2019, Kumar and Dutta (2019) investigated the solution and various stabilities of a rational functional equation involving two variables by means of fixed point tactic in the vicinity of Felbin's type fuzzy normed spaces with real-time examples.

For the generalized Hyers–Ulam–Rassias stability, one can refer to Gavruta (1994); Rassias (1982, 1978); Ulam (1960).

Definition 1.1: Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:

- $N(x, c) = 0$ for $c \leq 0$;
- $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
- $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
- $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
- $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
- For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement 'the norm of x is less than or equal to the real number t'.

The readers are expected to be familiar with examples of fuzzy norm (Bag and Samanta 2003) and fuzzy normed space.

In this work, we examine the general solution in Section 2, thrash out the generalized Hyers–Ulam–Rassias stability in Section 3 for a new mixed type functional equation

$$\begin{aligned}\mathrm{h}\left(-\frac{tv}{u}\right)+\mathrm{h}\left(\frac{2tv}{u}-\frac{uw}{t}\right)+\mathrm{h}\left(2\frac{uw}{t}\right)+\mathrm{h}\left(\frac{tv}{u}+\frac{uw}{t}\right)-\mathrm{h}\left(-\frac{tv}{u}+\frac{uw}{t}\right)\\ -\mathrm{h}\left(\frac{tv}{u}-\frac{uw}{t}\right)-\mathrm{h}\left(-\frac{tv}{u}-\frac{uw}{t}\right)=3\mathrm{h}\left(\frac{tv}{u}\right)+3\mathrm{h}\left(\frac{uw}{t}\right),\end{aligned} \tag{1}$$

for all $t \neq 0, u \neq 0, v, w \in R$ in normed spaces using the fuzzy concept. We provide the application of functional equation (1) in Section 4 and conclusion in Section 5.

2. The general solution of the functional equation (1)

In this section, we reveal the general solution of (1) and let T and U be linear spaces.

Lemma 2.1: *A mapping* $\mathrm{h}: T \to U$ *is additive if and only if* h *is odd and satisfies the functional equation with* $\mathrm{h}(0) = 0$

$$\mathrm{h}\left(\frac{2tv}{u} - \frac{uw}{t}\right) + 2\mathrm{h}\left(\frac{tv}{u} + \frac{uw}{t}\right) = 4\mathrm{h}\left(\frac{tv}{u}\right) + \mathrm{h}\left(\frac{uw}{t}\right), \tag{2}$$

for all $t, u, v, w \in T$.

Proof: Pretend that h is additive, then take up the classic additive functional equation

$$\mathrm{h}(t+u) = \mathrm{h}(t) + \mathrm{h}(u), \tag{3}$$

true for all $t, u \in T$. By insert $t = u = 0$ in (3), we look at $\mathrm{h}(0) = 0$, and replacing (t, u) by (t, t) in (3), we attain

$$\mathrm{h}(2t) = 2\mathrm{h}(t), \tag{4}$$

for all $t \in T$. Letting $(t, u) = (t, 2t)$ in (3) and using (4), we get

$$\mathrm{h}(3t) = 3\mathrm{h}(t), \tag{5}$$

for all $t \in T$. Assuming $(t, u) = (t, -t)$ in (3), we get

$$\mathrm{h}(-t) = -\mathrm{h}(t),$$

for all $t \in T$. whence, h is odd. By $(t, u) = (2(tv/u), -uw/t)$ in (3), we arrive

$$\mathrm{h}\left(\frac{2tv}{u} - \frac{uw}{t}\right) = 2\mathrm{h}\left(\frac{tv}{u}\right) - \mathrm{h}\left(\frac{uw}{t}\right), \tag{6}$$

for all $t, u, v, w \in T$. Putting $(t, u) = (tv/u, uw/t)$ in (3) and multiply by 2, we arrive

$$2\mathrm{h}\left(\frac{tv}{u} + \frac{uw}{t}\right) = 2\mathrm{h}\left(\frac{tv}{u}\right) + 2\mathrm{h}\left(\frac{uw}{t}\right), \tag{7}$$

for all $t, u, v, w \in T$. Adding (6) and (7), we turn up (2).

Conversely, suppose that h is odd and satisfying the functional equation (2) with $\mathrm{h}(0) = 0$.

Changing (t, u, v, w) into (t, t, v, v) and $(t, t, v, 2v)$ in (2), we get

$$\mathrm{h}(2t) = 2\mathrm{h}(t) \quad \textit{and} \quad \mathrm{h}(3t) = 3\mathrm{h}(t), \tag{8}$$

respectively, for all $t \in T$. Setting $(t, u, v, w) = (t, t, v, -2w)$ in (2) and employing (8), we obtain

$$2\mathrm{h}(v + w) + 2\mathrm{h}(v - 2w) = 4\mathrm{h}(v) - 2\mathrm{h}(w), \tag{9}$$

for all $v, w \in T$. Replacing v by w and w by v in (9), we obtain

$$2\mathrm{h}(v + w) - 2\mathrm{h}(2v - w) = 4\mathrm{h}(w) - 2\mathrm{h}(v), \tag{10}$$

for all $v, w \in T$. Letting $(t, u, v, w) = (t, t, v, w)$ in (2) and multiplying the resultant by 2, we get

$$2\mathrm{h}(2v - w) + 4\mathrm{h}(v + w) = 8\mathrm{h}(v) + 2\mathrm{h}(w), \tag{11}$$

for all $v, w \in T$. Adding (10) and (11), we arrive (3) and in consequence h is additive function. ■

Lemma 2.2: *A mapping* $\mathrm{h} : T \to U$ *is quadratic if and only if* h *is even and satisfies the functional equation with* $\mathrm{h}(0) = 0$

$$\mathrm{h}\left(\frac{2tv}{u} - \frac{uw}{t}\right) - 2\mathrm{h}\left(\frac{tv}{u} - \frac{uw}{t}\right) = 2\mathrm{h}\left(\frac{tv}{u}\right) - \mathrm{h}\left(\frac{uw}{t}\right), \tag{12}$$

for all $t, u, v, w \in T$.

Proof: Surmise that h is quadratic, then the popular quadratic functional equation

$$\mathrm{h}(t + u) + \mathrm{h}(t - u) = 2\mathrm{h}(t) + 2\mathrm{h}(u), \tag{13}$$

holds for all $t, u \in T$. By $t = u = 0$ in (13), we get $\mathrm{h}(0) = 0$, and setting $(t, u) = (0, t)$ in (13), we arrive $\mathrm{h}(-t) = \mathrm{h}(t)$ accordingly h is even. Setting $(t, u) = (t, t)$ and $(t, u) = (t, t - u)$ in (13), we arrive $\mathrm{h}(2t) = 4\mathrm{h}(t)$ and

$$\mathrm{h}(2t - u) - 2\mathrm{h}(t - u) = 2\mathrm{h}(t) - \mathrm{h}(u), \tag{14}$$

for all $t \in T$, respectively. Substituting $(t, u) = (tv/u, uw/t)$ in (14), we obtain (12).

Take h is even and satisfies the functional equation (12) with $\mathrm{h}(0) = 0$. Setting $(t, u, v, w) = (t, t, v, 2v)$ in (12), we obtain

$$\mathrm{h}(2t) = 4\mathrm{h}(t), \tag{15}$$

for all $t \in T$. Setting $(t, u, v, w) = (t, t, v, 3v)$ in (12) and applying (15), we arrive

$$\mathrm{h}(3t) = 9\mathrm{h}(t), \tag{16}$$

for all $t \in T$. By $(t, u, v, w) = (t, t, v, v - w)$ in (12), we arrive (13) in consequence h is a quadratic function.

With the use of Lemmas 2.1 and 2.2, we will forthwith prove our main results. ■

Theorem 2.3: *A mapping* $\mathrm{h} : T \to U$ *pacifies the functional equation* (1) *if and only if there exist additive mapping* $F : T \to U$ *with* $F(0) = 0$ *and quadratic mapping* $G : T \to U$ *with* $G(0) = 0$ *such that* $\mathrm{h}(t) = F(t) + G(t)$ *for all* $t \in T$.

Proof: Ascertain the mappings by $F, G : T \to U$

$$F(t) = \frac{1}{2}\left[\mathrm{h}(t) - \mathrm{h}(-t)\right] \tag{17}$$

and

$$G(t) = \frac{1}{2}\left[\mathrm{h}(t) + \mathrm{h}(-t)\right], \tag{18}$$

for all $t \in T$, respectively. Then, letting t by $-t$ in (17) and (18), we arrive

$$F(-t) = -F(t) \quad \textit{and} \quad G(-t) = G(t), \tag{19}$$

for all $t \in T$. Practicing the values of $F(t)$ and $G(t)$ and assigning Equation (1), we obtain the following equations:

$$\begin{aligned} &F\left(-\frac{tv}{u}\right) + F\left(\frac{2tv}{u} - \frac{uw}{t}\right) + F\left(2\frac{uw}{t}\right) + F\left(\frac{tv}{u} + \frac{uw}{t}\right) - F\left(-\frac{tv}{u} + \frac{uw}{t}\right) \\ &\quad - F\left(\frac{tv}{u} - \frac{uw}{t}\right) - F\left(-\frac{tv}{u} - \frac{uw}{t}\right) = 3F\left(\frac{tv}{u}\right) + 3F\left(\frac{uw}{t}\right), \end{aligned} \tag{20}$$

$$\begin{aligned} &G\left(-\frac{tv}{u}\right) + G\left(\frac{2tv}{u} - \frac{uw}{t}\right) + G\left(2\frac{uw}{t}\right) + G\left(\frac{tv}{u} + \frac{uw}{t}\right) - G\left(-\frac{tv}{u} + \frac{uw}{t}\right) \\ &\quad - G\left(\frac{tv}{u} - \frac{uw}{t}\right) - G\left(-\frac{tv}{u} - \frac{uw}{t}\right) = 3G\left(\frac{tv}{u}\right) + 3G\left(\frac{uw}{t}\right), \end{aligned} \tag{21}$$

for all $t, u \in T$. Setting $(t, u, v, w) = (t, t, v, 0)$ in (20), we obtain $F(2t) = 2F(t)$ for all $t \in T$. Hence, Equation (20) is of the form

$$F\left(\frac{2tv}{u} - \frac{uw}{t}\right) + 2F\left(\frac{tv}{u} + \frac{uw}{t}\right) = 4F\left(\frac{tv}{u}\right) + F\left(\frac{uw}{t}\right)$$

and by Lemma 2.1, F is additive. By setting $(t, u, v, w) = (t, t, v, 0)$ in (21), we obtain $G(2t) = 4G(t)$ for all $t \in T$. Hence, Equation (21) is of the form

$$G\left(\frac{2tv}{u} - \frac{uw}{t}\right) - 2G\left(\frac{tv}{u} - \frac{uw}{t}\right) = 2G\left(\frac{tv}{u}\right) - G\left(\frac{uw}{t}\right)$$

and by Lemma 2.2, G is quadratic. Thus, if $\mathrm{h} : T \to U$ satisfies Equation (1), then we have $\mathrm{h}(t) = F(t) + G(t)$ for all $t \in T$. Suppose that there exist a additive mapping $F : T \to U$ and a quadratic mapping $G : T \to U$ with $F(0) = 0$ and $G(0) = 0$ such that $\mathrm{h}(t) = F(t) + G(t)$ for all $t \in T$. Then, by Lemmas 2.1 , 2.2 and Equation (19), we arrive

$$\begin{aligned} &\mathrm{h}\left(-\frac{tv}{u}\right) + \mathrm{h}\left(\frac{2tv}{u} - \frac{uw}{t}\right) + \mathrm{h}\left(2\frac{uw}{t}\right) + \mathrm{h}\left(\frac{tv}{u} + \frac{uw}{t}\right) - \mathrm{h}\left(-\frac{tv}{u} + \frac{uw}{t}\right) \\ &\quad - \mathrm{h}\left(\frac{tv}{u} - \frac{uw}{t}\right) - \mathrm{h}\left(-\frac{tv}{u} - \frac{uw}{t}\right) - 3\mathrm{h}\left(\frac{tv}{u}\right) - 3\mathrm{h}\left(\frac{uw}{t}\right) = 0, \end{aligned}$$

for all $t, u, v, w \in T$. ■

3. Stability of the functional equation (1)

All through this section, consider T, (V, S') and (U, S) are linear space, fuzzy normed space and fuzzy Banach space, respectively. For appropriateness, we follow the abbreviation for a given mapping $\mathrm{h} : T \to U$:

$$E_{\mathrm{h}}(t,u,v,w) = \mathrm{h}\left(-\frac{tv}{u}\right) + \mathrm{h}\left(\frac{2tv}{u} - \frac{uw}{t}\right) + \mathrm{h}\left(2\frac{uw}{t}\right) + \mathrm{h}\left(\frac{tv}{u} + \frac{uw}{t}\right)$$
$$- \mathrm{h}\left(-\frac{tv}{u} + \frac{uw}{t}\right) - \mathrm{h}\left(\frac{tv}{u} - \frac{uw}{t}\right) - \mathrm{h}\left(-\frac{tv}{u} - \frac{uw}{t}\right) - 3\mathrm{h}\left(\frac{tv}{u}\right) - 3\mathrm{h}\left(\frac{uw}{t}\right),$$

for all $t, u, v, w \in T$. We now interrogate the generalized Hyers–Ulam–Rassias stability for (1).

Theorem 3.1: *Let $\eta \in \{1, -1\}$ be fixed and let $\phi_1 : T \times T \to V$ be a mapping such that for some $\kappa > 0$ with $(\kappa/4)^\eta < 1$*

$$S'(\phi_1(2^\eta t, 2^\eta t, 2^\eta t, 2^\eta t), a) \geq S'(\kappa^\eta \phi_1(t,t,t,t), a), \tag{22}$$

for all $t \in T$, $a > 0$ and

$$\lim_{s\to\infty} S'(\phi_1(2^{\eta s}t, 2^{\eta s}u, 2^{\eta s}v, 2^{\eta s}w), 4^{\eta s}a) = 1,$$

for all $t, u, v, w \in T$ and all $a > 0$. Pretend that an even mapping $\mathrm{h} : T \to U$ with $\mathrm{h}(0) = 0$ satisfies the inequality

$$S(E_{\mathrm{h}}(t,u,v,w), a) \geq S'(\phi_1(t,u,v,w), a), \tag{23}$$

for all $a > 0$ and all $t, u, v, w \in T$. Then the limit

$$G(t) = S - \lim_{s\to\infty} \frac{1}{4^{\eta s}} \mathrm{h}(2^{\eta s}t)$$

exists for all $t \in T$ and the mapping $G : T \to U$ is the unique quadratic mapping satisfying

$$S(\mathrm{h}(t) - G(t), a) \geq S'(\phi_1(t,t,t,t), a\,|4-\kappa|), \tag{24}$$

for all $t \in T$ and all $a > 0$.

Proof: Let $\eta = 1$. Letting $(t,u,v,w) = (t,t,v,v)$ in (23) and in the resultant replacing v by t, we get

$$S(\mathrm{h}(2t) - 4\mathrm{h}(t), a) \geq S'(\phi_1(t,t,t,t), a), \tag{25}$$

for all $t \in T$ and all $a > 0$. Replacing t by $2^s t$ in (25), we obtain

$$S\left(\frac{\mathrm{h}(2^{s+1}t)}{4} - \mathrm{h}(2^s t), \frac{a}{4}\right) \geq S'(\phi_1(2^s t, 2^s t, 2^s t, 2^s t), a), \tag{26}$$

for all $t \in T$ and all $a > 0$. Using (22), we get

$$S\left(\frac{\mathrm{h}(2^{s+1}t)}{4} - \mathrm{h}(2^s t), \frac{a}{4}\right) \geq S'\left(\phi_1(t,t,t,t), \frac{a}{\kappa^s}\right), \tag{27}$$

for all $t \in T$ and all $a > 0$. Replacing a by $a\kappa^s$ in (27), we get

$$S\left(\frac{\mathrm{h}(2^{s+1}t)}{4^{s+1}} - \frac{\mathrm{h}(2^s t)}{4^s}, \frac{a\kappa^s}{4(4^s)}\right) \geq S'(\phi_1(t,t,t,t),a), \tag{28}$$

for all $t \in T$ and all $a > 0$. It follows from

$$\frac{\mathrm{h}(2^s t)}{4^s} - \mathrm{h}(t) = \sum_{i=0}^{s-1} \frac{\mathrm{h}(2^{i+1}t)}{4^{i+1}} - \frac{\mathrm{h}(2^i t)}{4^i}$$

and (28) that

$$\begin{aligned} &S\left(\frac{f(2^s t)}{4^s} - \mathrm{h}(t), \sum_{i=0}^{s-1} \frac{a\kappa^i}{4(4^i)}\right) \\ &\geq \min\left\{S\left(\frac{\mathrm{h}(2^{i+1}t)}{4^{i+1}} - \frac{\mathrm{h}(2^i t)}{4^i}, \frac{a\kappa^i}{4(4^i)}\right) : i = 0,1,\ldots,s-1\right\} \\ &\geq S'(\phi_1(t,t,t,t),a), \end{aligned} \tag{29}$$

for all $t \in T$ and all $a > 0$. Replacing t by $2^c t$ in (29), we get

$$\begin{aligned} S\left(\frac{\mathrm{h}(2^{s+c}t)}{4^{s+c}} - \frac{\mathrm{h}(2^c t)}{4^c}, \sum_{i=0}^{s-1} \frac{a\kappa^i}{4(4^i)(4^c)}\right) &\geq S'(\phi_1(2^c t, 2^c t, 2^c t, 2^c t), a) \\ &\geq S'\left(\phi_1(t,t,t,t), \frac{a}{\kappa^c}\right) \end{aligned}$$

and so

$$S\left(\frac{\mathrm{h}(2^{s+c}t)}{4^{s+c}} - \frac{\mathrm{h}(2^c t)}{4^c}, \sum_{i=c}^{s+c-1} \frac{a\kappa^i}{4(4^i)}\right) \geq S'(\phi_1(t,t,t,t),a), \tag{30}$$

for all $t \in T$, $a > 0$ and all $c, s \geq 0$. Replacing a by $a/\sum_{i=c}^{s+c-1} \kappa^i/4(4^i)$, we obtain

$$S\left(\frac{\mathrm{h}(2^{s+c}t)}{4^{s+c}} - \frac{\mathrm{h}(2^c t)}{4^c}, a\right) \geq S'\left(\phi_1(t,t,t,t), \frac{a}{\sum_{i=c}^{s+c-1} \frac{\kappa^i}{4(4^i)}}\right), \tag{31}$$

for all $t \in T$, $a > 0$ and all $c, s \geq 0$. Since $0 < \kappa < 4$ and $\sum_{i=0}^{\infty} (\kappa/4)^i < \infty$, the Cauchy criterion for convergence and (N_5) imply that $\{\mathrm{h}(2^s t)/4^s\}$ is a Cauchy sequence in (U, S). Since (U, S) is a fuzzy Banach space, this sequence converges to some point $G(t) \in U$. Define the mapping $G : T \to U$ by

$$G(t) := S - \lim_{s\to\infty} \frac{\mathrm{h}(2^s t)}{4^s},$$

for all $t \in T$. Since h is even, G is even. Letting $c=0$ in (31), we get

$$S\left(\frac{\mathrm{h}(2^s t)}{4^s} - \mathrm{h}(t), a\right) \geq S'\left(\phi_1(t,t,t,t), \frac{a}{\sum_{i=0}^{s-1} \frac{\kappa^i}{4(4^i)}}\right), \tag{32}$$

for all $t \in T$ and all $a > 0$. Taking the limit as $s \to \infty$ and using (N_6), we get

$$S(\mathrm{h}(t) - G(t), a) \geq S'(\phi_1(t,t,t,t), a(4-\kappa)),$$

for all $t \in T$ and all $a > 0$. Now, we claim that G is quadratic. Replacing (t,u,v,w) by $(2^s t, 2^s u, 2^s v, 2^s w)$ in (23), we get

$$S\left(\frac{1}{4^s} E_{\mathrm{h}}(2^s t, 2^s u, 2^s v, 2^s w), a\right) \geq S'(\phi_1(2^s t, 2^s u, 2^s v, 2^s w), 4^s a), \tag{33}$$

for all $t, u, v, w \in T$ and all $a > 0$. Since

$$\lim_{s\to\infty} S'(\phi_1(2^s t, 2^s u, 2^s v, 2^s w), 4^s a) = 1,$$

G satisfies the functional equation (1). Hence, $G : T \to U$ is quadratic. To prove the uniqueness of G, Let $G' : T \to U$ be another quadratic mapping satisfying (24). Fix $t \in T$, clearly $G(2^s t) = 4^s G(t)$ and $G'(2^s t) = 4^s G'(t)$ for all $t \in T$and all $s \in S$. It follows from (24) that

$$\begin{aligned}
&S(G(t) - G'(t), a) \\
&\quad = S\left(\frac{G(2^s t)}{4^s} - \frac{G'(2^s t)}{4^s}, a\right) \\
&\quad \geq \min\left\{S\left(\frac{G(2^s t)}{4^s} - \frac{\mathrm{h}(2^s t)}{4^s}, \frac{a}{2}\right), S\left(\frac{\mathrm{h}(2^s t)}{4^s} - \frac{G'(2^s t)}{4^s}, \frac{a}{2}\right)\right\} \\
&\quad \geq S'\left(\phi_1(2^s t, 2^s t, 2^s t, 2^s t), \frac{4^s a(4-\kappa)}{2}\right) \\
&\quad \geq S'\left(\phi_1(t,t,t,t), \frac{4^s a(4-\kappa)}{2\kappa^s}\right),
\end{aligned}$$

for all $t \in T$ and all $a > 0$. Since $\lim_{s\to\infty} 4^s a(4-\kappa)/2\kappa^s = \infty$, we obtain

$$\lim_{s\to\infty} S'\left(\phi_1(t,t,t,t), \frac{4^s a(4-\kappa)}{2\kappa^s}\right) = 1.$$

Thus, $S(G(t) - G'(t), a) = 1$ for all $t \in T$ and all $a > 0$, and so $G(t) = G'(t)$. For $\eta = -1$, we can prove the result by a similar method. ■

Generalized Hyers–Ulam, Ulam–Gavruta–Rassias and J. M.Rassias stabilities of functional equation (1) are obtained in the following Corollaries 3.2, 3.5 and 3.7 from Theorems 3.1, 3.4 and 3.6, respectively.

Corollary 3.2: *Consider an even mapping* $\mathrm{h}: T \to U$ *with* $\mathrm{h}(0) = 0$ *satisfying all the conditions in Theorem* 3.1. *Then, there exists quadratic mapping* $G: T \to U$ *satisfying*

$$S(\mathrm{h}(t) - G(t), a) = \begin{cases} S'\left(\dfrac{4\gamma}{|4-2^{\alpha}|}\|t\|^{\alpha}, a\right), \\ \quad 0 \le \alpha < 2 \text{ or } \alpha > 2; \\ S'\left(\dfrac{\gamma}{|4-2^{4\alpha}|}\|t\|^{4\alpha}, a\right), \\ \quad 0 \le \alpha < \dfrac{1}{2} \text{ or } \alpha > \dfrac{1}{2}; \\ S'\left(\dfrac{5\gamma}{|4-2^{4\alpha}|}\|t\|^{4\alpha}, a\right), \\ \quad 0 \le \alpha < \dfrac{1}{2} \text{ or } \alpha > \dfrac{1}{2}, \end{cases} \tag{34}$$

for all $t \in T$ *and* $a, \gamma > 0$.

Proof: We prove this corollary, using Theorem 3.1 by replacing $\phi_1(t, u, v, w) = \gamma(\|t\,|^{\alpha} + \|u\|^{\alpha} + \|v\|^{\alpha} + \|w\|^{\alpha}), \gamma(\|t\|^{\alpha}\|u\|^{\alpha}\|v\|^{\alpha}\|w\|^{\alpha})$ and $\gamma(\|t\|^{4\alpha} + \|u\|^{4\alpha} + \|v\|^{4\alpha} + \|w\|^{4\alpha} + \|t\|^{\alpha}\|u\|^{\alpha}\|v\|^{\alpha}\|w\|^{\alpha})$, respectively. ■

Example 3.3: Let T be a Banach space and $\mathrm{h}: T \to U$ be an even mapping with $\mathrm{h}(0) = 0$ satisfies the inequality (23) and the mapping $G: T \to U$ is the unique quadratic mapping. In Theorem 3.1, put

$$(\mathrm{h}(t), \phi_1(t, u, v, w)) = \begin{cases} \mu(t + t^2) + \nu\|t\|^{\alpha} t_0, \\ \quad \gamma(\|t\|^{\alpha} + \|u\|^{\alpha} + \|v\|^{\alpha} + \|w\|^{\alpha}); \\ \mu(t + t^2) + \nu\|t\|^{4\alpha} t_0, \\ \quad \gamma(\|t\|^{\alpha}\|u\|^{\alpha}\|v\|^{\alpha}\|w\|^{\alpha}); \\ \mu(t + t^2) + \nu\|t\|^{4\alpha} t_0, \\ \quad \gamma(\|t\|^{4\alpha} + \|u\|^{4\alpha} + \|v\|^{4\alpha} + \|w\|^{4\alpha} \\ \quad + \|t\|^{\alpha}\|u\|^{\alpha}\|v\|^{\alpha}\|w\|^{\alpha}), \end{cases}$$

for all $t, t_0 \in T$ and $\mu, \nu \in R$ and in the resultant replacing (t, u, v, w) by (t, t, t, t), then the equation (34) of corollary 3.2 is satisfied.

Theorem 3.4: *Let* $\eta \in \{1, -1\}$ *be fixed and let* $\phi_2 : T \times T \to V$ *be a mapping such that for some* $\kappa > 0$ *with* $(\frac{\kappa}{2})^{\eta} < 1$

$$S'(\phi_2(2^{\eta}t, 2^{\eta}t, 2^{\eta}t, 2^{\eta}t), a) \ge S'(\kappa^{\eta}\phi_2(t, t, t, t), a), \tag{35}$$

for all $t \in T$ *and all* $a > 0$, *and*

$$\lim_{s\to\infty} S'\left(\phi_2(2^{\eta s}t, 2^{\eta s}u, 2^{\eta s}v, 2^{\eta s}w), 2^{\eta s}a\right) = 1,$$

for all $t, u, v, w \in T$ and all $a > 0$. Suppose that an odd mapping $\mathrm{h} : T \to U$ *with* $\mathrm{h}(0) = 0$ *satisfies the inequality*

$$S(E_{\mathrm{h}}(t, u, v, w), a) \geq S'(\phi_2(t, u, v, w), a), \tag{36}$$

for all $t, u, v, w \in T$ and all $a > 0$. Then the limit

$$F(t) = S - \lim_{s \to \infty} \frac{1}{2^{\eta s}} \mathrm{h}(2^{\eta s} t)$$

exists for all $t \in T$ and the mapping $F : T \to U$ is the unique additive mapping satisfying

$$S(\mathrm{h}(t) - F(t), a) \geq S'(\phi_2(t, t, t, t), 3a\,|2 - \kappa|), \tag{37}$$

for all $t \in T$ and all $a > 0$.

Proof: Let $\eta = 1$. Letting $(t, u, v, w) = (t, t, v, v)$ in (36), using oddness of h and in the resultant replacing v by t, we get

$$S\left(\mathrm{h}(2t) - 2\mathrm{h}(t), \frac{a}{3}\right) \geq S'(\phi_2(t, t, t, t), a), \tag{38}$$

for all $t \in T$ and all $a > 0$. Replacing t by $2^s t$ in (38), we obtain

$$S\left(\frac{\mathrm{h}(2^{s+1} t)}{2} - \mathrm{h}(2^s t), \frac{a}{6}\right) \geq S'(\phi_2(2^s t, 2^s t, 2^s t, 2^s t), a), \tag{39}$$

for all $t \in T$ and all $a > 0$. Using (35), we get

$$S\left(\frac{\mathrm{h}(2^{s+1} t)}{2} - \mathrm{h}(2^s t), \frac{a}{6}\right) \geq S'\left(\phi_2(t, t, t, t), \frac{a}{\kappa^s}\right), \tag{40}$$

for all $t \in T$ and all $a > 0$. Rest of the proof is similar to that of Theorem 3.1. ■

Corollary 3.5: *Consider an odd mapping* $\mathrm{h} : T \to U$ *with* $\mathrm{h}(0) = 0$ *satisfies all the conditions in Theorem* 3.4. *Then there exists additive mapping $F : T \to U$ satisfying*

$$S(\mathrm{h}(t) - F(t), a) = \begin{cases} S'\left(\dfrac{4\gamma}{|3(2 - 2^{\alpha})|} \|t\|^{\alpha}, a\right), \\ \quad 0 \leq \alpha < 1 \text{ or } \alpha > 1; \\ S'\left(\dfrac{\gamma}{|3(2 - 2^{4\alpha})|} \|t\|^{4\alpha}, a\right), \\ \quad 0 \leq \alpha < \dfrac{1}{4} \text{ or } \alpha > \dfrac{1}{4}; \\ S'\left(\dfrac{5\gamma}{|3(2 - 2^{4\alpha})|} \|t\|^{4\alpha}, a\right), \\ \quad 0 \leq \alpha < \dfrac{1}{4} \text{ or } \alpha > \dfrac{1}{4}, \end{cases}$$

for all $t \in T$ and $a, \gamma > 0$.

Proof: We prove this corollary, using Theorem 3.4 by replacing $\phi_2(t,u,v,w) = \gamma(\|t\|^\alpha + \|u\|^\alpha + \|v\|^\alpha + \|w\|^\alpha), \gamma(\|t\|^\alpha \|u\|^\alpha \|v\|^\alpha \|w\|^\alpha)$ and $\gamma(\|t\|^{4\alpha} + \|u\|^{4\alpha} + \|v\|^{4\alpha} + \|w\|^{4\alpha} + \|t\|^\alpha \|u\|^\alpha \|v\|^\alpha \|w\|^\alpha)$, respectively. ■

Theorem 3.6: *Let $\eta \in \{1,-1\}$ be fixed and let $\phi : T \times T \to V$ be a mapping such that for some $\kappa > 0$ with $\kappa^\eta < (-\eta+3)^\eta$*

$$S'(\phi(2^\eta t, 2^\eta t, 2^\eta t, 2^\eta t), a) \geq S'(\kappa^\eta \phi(t,t,t,t), a), \tag{41}$$

for all $t \in T$ and all $a > 0$, and

$$\lim_{s\to\infty} S'\left(\phi(2^{\eta s}t, 2^{\eta s}u, 2^{\eta s}v, 2^{\eta s}w), \left[(|\eta|+\eta)\, 2^{2\eta s-1} + (|\eta|-\eta)\, 2^{\eta s}\right] a\right) = 1,$$

for all $a > 0$ and all $t, u \in T$. Suppose that a mapping $\mathrm{h} : T \to U$ with $\mathrm{h}(0) = 0$ satisfies the inequality

$$S(E_{\mathrm{h}}(t,u,v,w), a) \geq S'(\phi(t,u,v,w), a), \tag{42}$$

for all $t, u \in T$ and all $a > 0$. Then there exists a unique quadratic mapping $G : T \to U$ and a unique additive mapping $F : T \to U$ such that

$$S(\mathrm{h}(t) - G(t) - F(t), a) \geq S''(t,a), \tag{43}$$

for all $t \in T$ and all $a > 0$. Where

$$S''(t,a) := \min\left\{S'\left(\phi(t,t,t,t), \frac{a(4-\kappa)}{2}\right), S'\left(\phi(t,t,t,t), \frac{3a(2-\kappa)}{2}\right)\right\}.$$

Proof: Assume $\eta = 1$. Then we have $\kappa < 2$. Let

$$\mathrm{h}_e(t) = \frac{\mathrm{h}(t) + \mathrm{h}(-t)}{2},$$

for all $t \in T$. Then $\mathrm{h}_e(0) = 0, \mathrm{h}_e(-t) = \mathrm{h}_e(t)$ and

$$\begin{aligned} S(E_{\mathrm{h}_e}(t,u,v,w), a) &= S\left(\frac{1}{2}\left[E_{\mathrm{h}}(t,u,v,w) + E_{\mathrm{h}}(-t,-u,-v,-w)\right], a\right) \\ &\geq \min\left\{S(E_{\mathrm{h}}(t,u,v,w), a), S(E_{\mathrm{h}}(-t,-u,-v,-w), a)\right\}, \end{aligned}$$

for all $t, u, v, w \in T$ and all $a > 0$. Hence by Theorem 3.1, there exist a unique quadratic mapping $G : T \to U$ satisfying

$$S(\mathrm{h}_e(t) - G(t), a) \geq S'(\phi(t,t,t,t), a(4-\kappa)), \tag{44}$$

for all $t \in T$ and all $a > 0$. Let

$$\mathrm{h}_o(t) = \frac{\mathrm{h}(t) - \mathrm{h}(-t)}{2},$$

for all $t \in T$. Then $\mathrm{h}_o(0) = 0, \mathrm{h}_o(-t) = -\mathrm{h}_o(t)$ and

$$\begin{aligned} S(E_{\mathrm{h}_o}(t,u,v,w), a) &= S\left(\frac{1}{2}\left[E_{\mathrm{h}}(t,u,v,w) - E_{\mathrm{h}}(-t,-u,-v,-w)\right], a\right) \\ &\geq \min\left\{S(E_{\mathrm{h}}(t,u,v,w), a), S(E_{\mathrm{h}}(-t,-u,-v,-w), a)\right\}, \end{aligned}$$

for all $t, u \in T$ and all $a > 0$. Hence by Theorem 3.4, there exist a unique additive mapping $F : T \to U$ satisfying

$$S(\mathrm{h}_o(t) - F(t), a) \geq S'(\phi(t,t,t,t), 3a(2-\kappa)), \tag{45}$$

for all $t \in T$ and all $a > 0$. Using Equations (44) and (45), we obtain

$$\begin{aligned}
&S(\mathrm{h}(t) - G(t) - F(t), a)\\
&\quad \geq S(\mathrm{h}_e(t) + \mathrm{h}_o(t) - G(t) - F(t), a)\\
&\quad \geq \min\left\{S\left(\mathrm{h}_e(t) - G(t), \frac{a}{2}\right), S\left(\mathrm{h}_o(t) - F(t), \frac{a}{2}\right)\right\}\\
&\quad \geq \min\left\{S'\left(\phi(t,t,t,t), \frac{a(4-\kappa)}{2}\right), S'\left(\phi(t,t,t,t), \frac{3a(2-\kappa)}{2}\right)\right\}\\
&\quad \geq S''(t,a),
\end{aligned}$$

which follows Equation (43). If $\eta = -1$, we can prove the result by a similar method. ■

Corollary 3.7: *Consider mapping* $\mathrm{h} : T \to U$ *with* $\mathrm{h}(0) = 0$ *satisfying all the conditions in Theorem* 3.6. *Then, there exists quadratic mapping* $G : T \to U$ *and additive mapping* $F : T \to U$ *satisfying*

$$S(\mathrm{h}(t) - G(t) - F(t), a) = \begin{cases}
\min\left\{S'\left(\dfrac{8\gamma}{|4-2^{\alpha}|}\|t\|^{\alpha}, a\right),\right.\\
\quad \left. S'\left(\dfrac{8\gamma}{|3(2-2^{\alpha})|}\|t\|^{\alpha}, a\right)\right\}, 0 \leq \alpha < 1\\
\quad or\ \alpha > 1;\\
\min\left\{S'\left(\dfrac{2\gamma}{|4-2^{4\alpha}|}\|t\|^{4\alpha}, a\right),\right.\\
\quad \left. S'\left(\dfrac{2\gamma}{|3(2-2^{4\alpha})|}\|t\|^{4\alpha}, a\right)\right\}, 0 \leq \alpha < \dfrac{1}{4}\\
\quad or\ \alpha > \dfrac{1}{4};\\
\min\left\{S'\left(\dfrac{10\gamma}{|4-2^{4\alpha}|}\|t\|^{4\alpha}, a\right),\right.\\
\quad \left. S'\left(\dfrac{10\gamma}{|3(2-2^{4\alpha})|}\|t\|^{4\alpha}, a\right)\right\}, 0 \leq \alpha < \dfrac{1}{4}\\
\quad or\ \alpha > \dfrac{1}{4},
\end{cases}$$

for all $t \in T$ *and* $a, \gamma > 0$.

Proof: We prove this corollary, using Theorem 3.6 by replacing $\phi(t,u,v,w) = \gamma(\|t\|^{\alpha} + \|u\|^{\alpha} + \|v\|^{\alpha} + \|w\|^{\alpha}), \gamma(\|t\|^{\alpha}\|u\|^{\alpha}\|v\|^{\alpha}\|w\|^{\alpha})$ and $\gamma(\|t\|^{4\alpha} + \|u\|^{4\alpha} + \|v\|^{4\alpha} + \|w\|^{4\alpha} + \|t\|^{\alpha}\|u\|^{\alpha}\|v\|^{\alpha}\|w\|^{\alpha})$, respectively. ■

4. Application: secret key generation using functional equations

In order to access any client server authentication system, a client must enter an user ID and password, which is the first influence of authentication and then enter the PIN or OTP (secret key), which is provided by the corresponding server as an authentication factor, see Chou, Draper, and Sayeed (2012).

We introduce a new two-factor authenticated secret key system using functional equations, where values of variables in functional equations become a secret key.

The secret key is verified and granted to the client using functional equation (1) as shown in Figure 1.

At the client side, the 4 digit values which are received from the server is applied in the left-hand side of the functional equation (1) and the solution $f(x) = x + x^2$ of functional equation (1) is sent to the server for verification.

At the server side, the same 4 digit values applied in the right-hand side of functional equation (1), then the result is compared with the client side result. If both the results are equal, the server granted the secret key to the client. Otherwise, the secret key is denied.

The characteristics of Figure1 are implemented in JAVA program and shows the outputs with the client and server IP addresses in Figures 2–4.

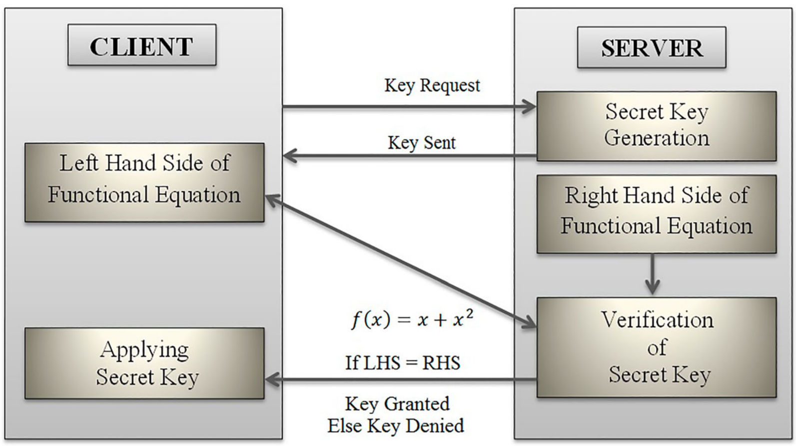

Figure 1. Secret key generation using functional equation (1).

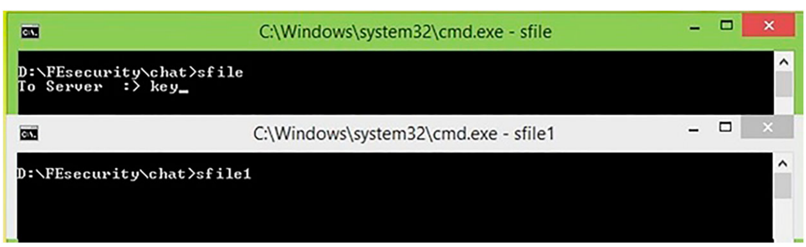

Figure 2. Key request from client.

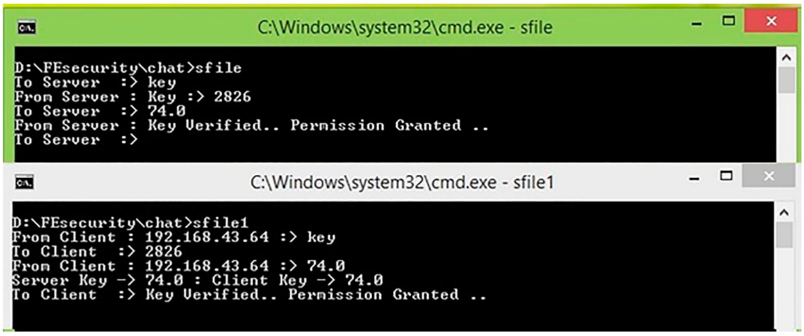

Figure 3. Key generation and verification.

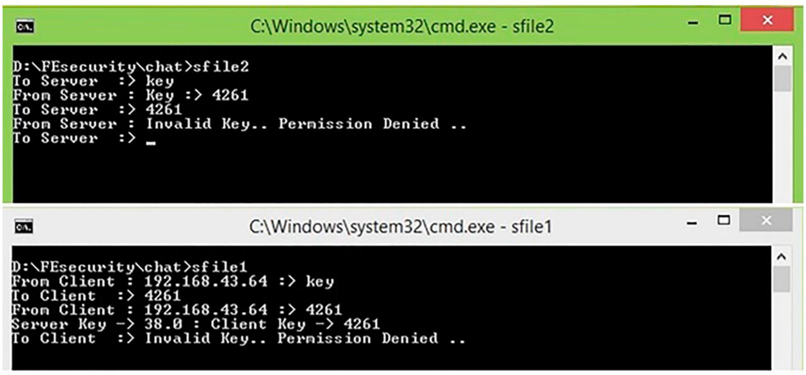

Figure 4. Key verified and denied.

The secrete key generated from functional equation (1) is strongly secured because we applied two factor for verifying keys, whereas traditional methods used single factor authentication for generating secret keys.

The secret key provided by the server is highly protected depending on how it is programmed. It makes hacker extremely difficult to use that secrete key to gain cruel access.

5. Conclusion

In this paper, general solution and generalized Hyers–Ulam–Rassias stability of newly introduced mixed type functional equation in fuzzy normed space have been obtained with suitable example. The results obtained are better than the results in other ordinary spaces and also with significant upper-bound. Importantly, we provided applications using functional equation (1) for generating secret keys in client–server environment. The findings should inspire to develop more mixed type functional equations and investigate their Hyers–Ulam–Rassias stability in various spaces. The functional equations can also be used in high security systems employed in Military and Banking sectors. Using mixed type functional equations, the security of these systems may be more effective than the systems with ordinary functional equations.

Acknowledgements

The authors are grateful to the anonymous referees for carefully reading the manuscript and suggestions made to improve the presentation of the paper.

Disclosure statement

The authors declare that there is no conflict of interest.

Notes on contributors

Dr. Pasupathi Narasimman is Assistant professor at Department of Mathematics, Thiruvalluvar University College of Arts and Science, Gajalnaickanpatti, Tirupattur-635901, Tamilnadu, India. His primary research area is functional equation. He has to his credit several research papers and a few book chapters.

Dr. Hemen Dutta is a faculty member of Mathematics at Gauhati University, Guwahati-781014, India. His primary research interests are in areas of Mathematical Analysis & Applications. He has to his credit several research papers, book chapters and a few books.

Dr. Iqbal H. Jebril is Professor at the Department of Mathematics, Taibah University, Almadinah, Almunawwarah, Kingdom of Saudi Arabia and Department of Mathematics, Al-Zaytoonah University of Jordan, Amman, Jordan. His main teaching and research interests include Functional Analysis, Operator Theory and Fuzzy Logic. He has published several research articles in international journals of mathematics.

References

Bag, T., and S. K. Samanta. 2003. "Finite Dimensional Fuzzy Normed Linear Spaces." *The Journal of Fuzzy Mathematics* 11: 687–705.

Chou, T. H., S. C. Draper, and A. M. Sayeed. 2012. "Key Generation using External Source Excitation: Capacity, Reliability and Secrecy Exponent." *IEEE Transactions on Information Theory* 58: 2455–2474.

Gavruta, P. 1994. "A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings." *Journal of Mathematical Analysis and Applications* 184: 431–436.

Gordji, M. E., N. Ghobadipour, and J. M. Rassias. 2009. *Fuzzy Stability of Additive-Quadratic Functional Equations*, arXiv:0903.0842v1 [math.FA] 4 Mar 2009.

Hyers, D. H. 1941. "On the Stability of the Linear Functional Equation." *Proceedings of the National Academy of Sciences USA* 27: 222–224.

Kumar, B. V. S., and H. Dutta. 2019. "Fuzzy Stability of a Rational Functional Equation and its Relevance to System Design." *International Journal of General Systems* 48 (2): 157–169.

Lee, J. R., S. Y. Jang, C. Park, and D. Y. Shin. 2010. "Fuzzy Stability of Quadratic Functional Equations." *Advances in Difference Equations*, Article ID 412160, 16 pages, doi:10.1155/2010/412160.

Mirmostafaee, A. K., and M. S. Moslehian. 2008. "Fuzzy Versions of Hyers-Ulam-Rassias Theorem." *Fuzzy Sets and Systems* 159: 720–729.

Nazarianpoor, M., J. M. Rassias, and G. H. Sadeghi. 2018. "Solution and Stability of Quattuorvigintic Functional Equation in Intutionistic Fuzzy Normed Spaces." *Iranian Journal of Fuzzy Systems* 15 (4): 13–30.

Rassias, Th. M. 1978. "On the Stability of the Linear Mapping in Banach Spaces." *Proceedings of the American Mathematical Society* 72: 297–300.

Rassias, J. M. 1982. "On Approximation of Approximately Linear Mappings by Linear Mapping." *Journal of Functional Analysis* 46: 126–130.

Ravi, K., and R. Kodandan. 2010. "Stability of Additive and Quadratic Functional Equation in Non-Archimedean Spaces." *International Journal of Pure and Applied Mathematics* 6: 149–160.

Ravi, K., J. M. Rassias, and P. Narasimman. 2010. "Fuzzy Stability and Solution of a Mixed Type Additive and Quadratic Functional Equation." *Oriental Journal of Mathematics* 4: 23–41.

Ulam, S. M. 1960. *A Collection of the Mathematical Problems*. New York: Interscience Publishers.

Wongkum, K., and P. Kumam. 2016. "The Stability of Sextic Functional Equation in Fuzzy Modular Spaces." *Journal of Nonlinear Sciences and Applications* 9 (6): 3555–3569.