Numerical solutions of linear time-fractional Klein-Gordon equation by using power series approach

Mohammad Alaroud[1,*], Mohammed Al-Smadi[2], Omar Abu Arqub[3], Rokiah Rozita Ahmad[1], Ma'mon Abu Hammad[4], Shaher Momani[4]

[1]*School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor,Malaysia*
[2]*Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan*
[3]*Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan*
[4]*Department of Mathematics, Faculty of Science, The University of Jordan, Amman, 11942, Jordan*

Abstract. In this paper, we provide approximate solution to linear time-fractional Kline-Gordon equations (FKGEs) with initial conditions by using the residual power series (RPS) method. The proposed technique relies on generalized Taylor formula under Caputo sense aiming at extracting a supportive analytical solution in convergent series form. Graphical results show the geometric behaviors to the approximate solutions at different values of fraction order γ. The numerical analysis detects that the RPS technique is an efficient, simple and powerful tool to determine the solutions of the time-fractional KGE.

Keywords: Fractional partial differential equations; Klein–Gordon equations; Caputo fractional derivative.

1. Introduction

In the present study, we consider the following Klein–Gordon equation with time-fractional model

$$D^{\gamma}\rho(x,t)=\frac{\partial^2}{\partial x^2}\rho(x,t)+\lambda\rho(x,t)+\mu\rho^2(x,t)+\sigma\rho^3(x,t), 0<\gamma\leq 1, \quad (1)$$

subject to the initial condition

$$\rho(x,0)=\rho_0(x). \quad (2)$$

where λ,μ and σ are constants, γ is a parameter describe the order of the time-Caputo fractional derivative, $\rho(x,t)$ is an unknown analytical function to be determined, and $\rho_0(x)$ is known analytical function. In case $\gamma=1$, we get the classical Klein–Gordon equation.

The Klein–Gordon equation is considered one of the basic nonlinear equations that have been gained a much of attention where are used to describe relativistic electrons and the spineless pion as well as in discussing several phenomena appearing in classical quantum and relativistic mechanics, which plays an important role in many mathematical applications including optics and solid-state problems [1,2]. The Fractional Klein-Gordon equations (FKGEs) arise in various areas, although investigations on the numerical methods for these equations are scarce [3-6].

The basic target of this study is to present a novel treatment technique to solve FKGEs subject to fit initial condition based on fractional RPS method. This method is developed and applied successfully in [7]. The RPS is characterized as an applicable and easy technique to create power series solutions for strongly linear and nonlinear equations without being linearized, discretized or exposed to perturbation [8-15].

The present study is arranged as follow. Some main notations, definitions, and preliminary results are given in Section 2. Fractional RPS scheme for FKGE model is presented in Section 3. Illustrative example is given in Section 4. Finally, conclusions are drawn in Section 5.

2. Preliminaries

In this section, we recall some the main definitions and the basic results related to factional calculus and the fractional power series representations [16-21].

Definition 2.1 [17] For $\gamma>0$, and m is the smallest integer that exceeds γ, the fractional derivative

$$D^{\gamma}\rho(x,t)=\begin{cases}\frac{1}{\Gamma(m-\gamma)}\int_0^t\frac{\partial^m\rho(x,\varepsilon)}{\partial\varepsilon^m}(t-\varepsilon)^{m-\gamma-1}d\zeta, & m-1<\gamma<m\\ \frac{\partial^m\rho(x,\varepsilon)}{\partial\varepsilon^m}, & \gamma=m\in\mathbb{N}\end{cases}.$$

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is referred to as the Caputo time-fractional derivative operator of order γ.

Definition 2.2 [15] A fractional power series (FPS) representation at $t = t_0$ has the form

$$\sum_{m=0}^{\infty} \rho_m(x)(t - t_0)^{m\gamma} = \rho_0(x) + \rho_1(x)(t - t_0)^{\gamma} + \rho_2(x)(t - t_0)^{2\gamma} + \cdots,$$

where $0 \leq n - 1 < \gamma \leq n$ and $t \geq t_0$ is called multiple fractional power series (MFPS) about t_0.

Remark 2.1 [15] Suppose that $\rho(x, t)$ has the MFPS representation at t_0 as follows

$$\rho(x, t) = \sum_{m=0}^{\infty} \rho_m(x)(t - t_0)^{m\gamma}.$$

Then, if $\rho(x, t) \in I \times C[t_0, t_0 + R)$, and $D^{m\gamma}\rho(x, t) \in I \times C(t_0, t_0 + R)$, for $m = 0,1,2, \ldots$, the coefficients $\rho_m(x)$ will be given as $\rho_m(x) = \frac{D^{m\gamma}\rho(x,t_0)}{\Gamma(m\gamma+1)}$ such that $D^{m\gamma} = D^{\gamma} \cdot D^{\gamma} \cdot \ldots \cdot D^{\gamma}$ (m-times).

3. Fractional RPS method of KGE model

The current section is devoted to show the methodology of fractional RPS technique for obtaining multiple FPS solution for time-fractional KGEs (1) and (2) through substituting the expansion of multiple FPS among the truncated residual functions. To perform that, let the solution for Eqs. (1) and (2) has a multiple FPS at $t_0 = 0$ as follows:

$$\rho(x, t) = \sum_{n=0}^{\infty} \rho_n(x)\, t^{n\gamma}, 0 < \gamma \leq 1, t \geq 0\,, x \in \mathbb{R}. \tag{3}$$

Since $\rho(x, 0) = \rho_0(x)$, the series solution can be written as:

$$\rho(x, t) = \rho_0(x) + \sum_{n=1}^{\infty} \rho_n(x)\, t^{n\gamma}. \tag{4}$$

Thereafter, we suppose that $\rho_j(x, t)$ indicate the j^{th}-approximate solution of $\rho(x, t)$ such that

$$\rho_j(x, t) = \rho_0(x) + \sum_{n=1}^{j} \rho_n(x)\, t^{n\gamma}. \tag{5}$$

According to applying the fractional RPS algorithm, the residual function is defined by

$$Res_\rho(x, t) = D^{\gamma}\rho(x, t) - \frac{\partial^2}{\partial x^2}\rho(x, t) - \lambda\rho(x, t) - \mu\rho^2(x, t) - \sigma\rho^3(x, t),$$

and the j^{th}-residual function Res_ρ^j for $j = 1,2,3, \ldots$, is defined as

$$Res_\rho^j(x, t) = D^{\gamma}\rho_j(x, t) - \frac{\partial^2}{\partial x^2}\rho_j(x, t) - \lambda\rho_j(x, t) - \mu\rho_j^2(x, t) - \sigma\rho_j^3(x, t). \tag{6}$$

As described in [7-13], $Res_\rho(x, t) = 0$ and $\lim_{j\to\infty} Res_\rho^j(x, t) = 0$ for each $x \in \mathbb{R}$ and $t > 0$. Thus, $D^{i\gamma}Res_\rho(x, t) = 0$. Also, $D^{(i-1)\gamma}Res_\rho(x, 0) = D^{(i-1)\gamma}Res_\rho^j(x, 0)$ for each $i = 1,2, \ldots, j$. This fact is considered the essential rule in the fractional RPS scheme.

In view of that, to determine the unknown coefficients $\rho_n(x)$ of Eq. (5) for $n = 1,2,3, \ldots, j$. Perform the next manner; write the j^{th}-approximate solution of $\rho(x, t)$ into $Res_\rho^j(x, t)$ in Eq. (6), then compute $D^{(j-1)\gamma}Res_\rho^j(x, t)$, thereafter solve the resulting fractional equation at $t = 0$, that is, $D^{(j-1)\gamma}Res_\rho^j(x, 0) = 0$. For other numerical methods, see [22-27].

4. Illustrative Example

The target of the present section is to show the efficiency and applicability of the fractional RPS technique by applying it to the following fractional linear Klein–Gordon equation [28]

$$D^{\gamma}\rho(x, t) = \frac{\partial^2}{\partial x^2}\rho(x, t) + \rho(x, t), 0 < \gamma \leq 1, t \geq 0, x \in \mathbb{R}, \tag{7}$$

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with the initial condition

$$\rho(x,0) = 1 + \sin(x). \tag{8}$$

which has exact solution $\rho(x,t) = \sin x + e^{t}$ at $\gamma = 1$.

Based on the last description of fractional RPS scheme, starting with initial condition $\rho_0(x) = \rho(x,0) = 1 + \sin(x)$ to get the j^{th}-approximate solution $\rho_j(x,t)$ for Eq. (7) as follows

$$\rho_j(x,t) = 1 + \sin(x) + \sum_{n=1}^{j} \rho_n(x)\, t^{n\gamma}. \tag{9}$$

According to Eq. (6), the j^{th}-residual function is given by

$$Res_\rho^j(x,t) = D^\gamma \rho_j(x,t) - \frac{\partial^2}{\partial x^2}\rho_j(x,t) + \rho_j(x,t). \tag{10}$$

Now, to obtain the first unknown coefficient, $\rho_1(x)$, we substitute $\rho_1(x,t) = 1 + \sin(x) + \rho_1(x)t^\gamma$ into $Res_\rho^1(x,t)$ of Eq. (10) such that

$$\begin{aligned} Res_\rho^1(x,t) &= D^\gamma \rho_1(x,t) - \frac{\partial^2}{\partial x^2}\rho_1(x,t) + \rho_1(x,t) \\ &= \Gamma(\gamma+1)\rho_1(x) - (-\sin(x) + \rho_1''(x)t^\gamma) + (1 + \sin(x) + \rho_1(x)t^\gamma). \end{aligned} \tag{11}$$

By using the fact $D^{(j-1)\gamma} Res_\rho^j(x,0) = 0$, for $j = 1$, the first coefficient is $\rho_1(x) = \frac{1}{\Gamma(\gamma+1)}$.

Next, to determine the second unknown coefficient, $\rho_2(x)$, use the 2nd residual function

$$\begin{aligned} Res_\rho^2(x,t) &= D^\gamma \rho_2(x,t) - \frac{\partial^2}{\partial x^2}\rho_2(x,t) + \rho_2(x,t) \\ &= \Gamma(\gamma+1)\rho_1(x) + \frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}\rho_2(x)t^\gamma - (-\sin(x) + \rho_1''(x)t^\gamma + \rho_2''(x)t^{2\gamma}) + (1 + \sin(x) + \\ &\quad \rho_1(x)t^\gamma + \rho_2(x)t^{2\gamma}), \end{aligned} \tag{12}$$

and consider the fact $D^\gamma Res_\rho^2(x,0) = 0$ to get that $\rho_2(x) = \frac{1}{\Gamma(2\gamma+1)}$.

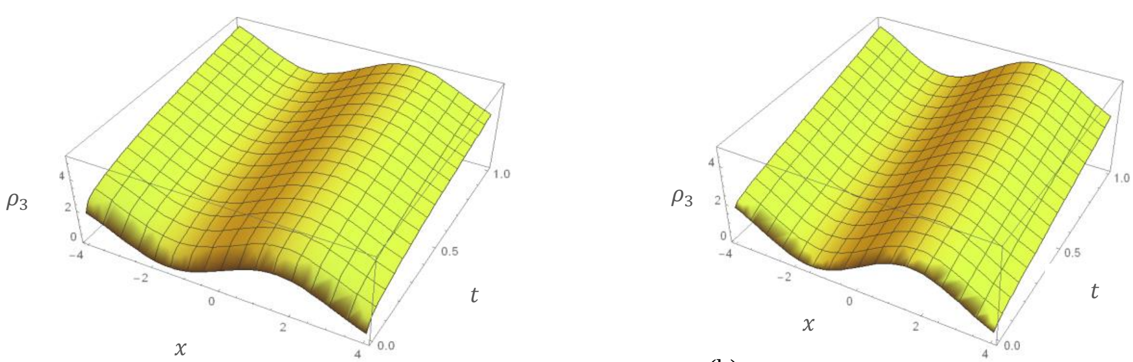
For $j = 3$, Using the same manner to obtain the 3rd unknown coefficient, $\rho_3(x)$, in multiple FPS such as $\rho_3(x) = \frac{1}{\Gamma(3\gamma+1)}$.

Consequently, the solution $\rho(x,t)$ is given as

$$\begin{aligned} \rho(x,t) &= 1 + \sin(x) + \frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots + \frac{t^{j\gamma}}{\Gamma(j\gamma+1)} + \cdots \\ &= \sin(x) + \sum_{j=0}^{\infty} \frac{t^{j\gamma}}{\Gamma(j\gamma+1)} = \sin(x) + E_\gamma(t^\gamma)\,. \end{aligned} \tag{4.7}$$

where $E_\alpha(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m+1)}$ is the Mittag-Leffler Function.

Next, the FPS approximation solution, $\rho_3(x,t)$, has been studied in $3D$-space at the different values of γ for each $x \in [-4,4]$ and $t \in [0,1]$. Fig.1 shows schema of $\rho_3(x,t)$ when $\gamma = \{0.25, 0.5, 1\}$, and exact solution $\rho(x,t)$ for each $x \in [-4,4]$ and $t \in [0,1]$.



(a) (b)

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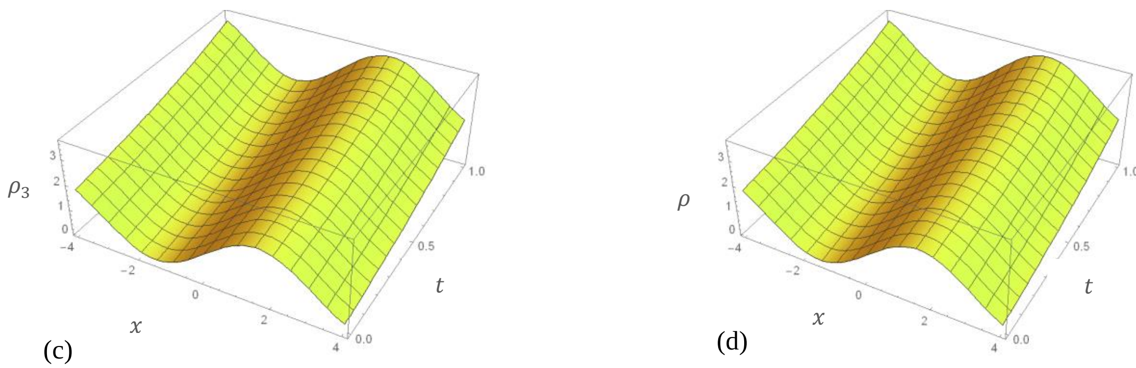

Fig.1: The surface plot of the 3rd FPS approximation and exact solution (a) $\gamma = 0.25$, (b) $\gamma = 0.5$, (c) $\gamma = 1$ (d) Exact solution.

5. Conclusions

In the present study, we have extended the application of the RPS algorithm to obtain the approximate solutions of time-fractional KGEs with the initial condition. The proposed method is utilized directly without being linearized, discretized or exposed to perturbation. Illustrative example is presented to show the effectiveness and ability of the proposed approach. Graphical results have revealed the validity and reliability of this technique with a great potential in scientific applications. The RPS method is considered a valuable tool, effective and straightforward to predict and construct numeric-analytic solutions of many problems related to fractional partial differential equations.

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