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ON THE ZEROS OF A CLASS OF BIVARIATE FIBONACCI POLYNOMIALS

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Abstract

In this article, we study the zeros of a class of bivariate Fibonacci polynomials and investigate their relationship with the eigenvalues of a certain tridiagonal matrix. Then, based on this study, we give a full description of the zeros of such polynomials.

1. Introduction

In [1] and [2], the author defined the bivariate Fibonacci polynomials and gave some properties of these polynomials. In [3], Catalani defined the generalized bivariate Fibonacci polynomial. Also, in [4], the authors have defined various types of bivariate Fibonacci polynomials. We would like to refer the interested reader to [5-7] and [8], where the authors have investigated some fundamental properties of bivariate Fibonacci polynomials.

In this paper, we consider the bivariate Fibonacci polynomials defined as

$$g_n(x, y) = xg_{n-1}(x, y) + yg_{n-2}(x, y), \quad g_0(x, y) = x, \quad g_1(x, y) = y. \quad (1)$$

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If $x = y = 1,$ then the resulting sequence is the Fibonacci numbers. Applying the recurrence relation (1) to obtain the exact form of $g_n(x, y)$ for $n = 2, 3$ and 4 as follows:

$$g_2(x, y) = 2yx,$$

$$g_3(x, y) = 2x^2y + y^2 = 2y\begin{vmatrix} x & -y \\ \frac{1}{2} & x \end{vmatrix},$$

$$g_4(x, y) = 2x^3y + 3xy^2 = 2y\begin{vmatrix} x & -y & 0 \\ 1 & x & -y \\ 0 & \frac{1}{2} & x \end{vmatrix},$$

where $|A|$ denotes the determinant of the matrix A.

For a complex number $y \neq 0$, let $T_m(y)$ be the $m \times m$ tridiagonal matrix

$$T_m(y) = \begin{bmatrix} 0 & y & 0 & \cdots & 0 \\ -1 & 0 & y & \ddots & \vdots \\ 0 & -1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & y \\ 0 & \cdots & 0 & -\frac{1}{2} & 0 \end{bmatrix},$$

then from the recurrence relation (1) and using induction, it can be easily shown that for $n \geq 2,$

$$g_n(x, y) = 2y\begin{vmatrix} x & -y & 0 & \cdots & 0 \\ 1 & x & -y & \ddots & \vdots \\ 0 & 1 & x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -y \\ 0 & \cdots & 0 & \frac{1}{2} & x \end{vmatrix} = 2y|\, xI_{n-1} - T_{n-1}(y)\,|,$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Since $|\, xI_{n-1} - T_{n-1}(y)\,|$ can be considered as the characteristic polynomial of $T_{n-1}(y),$ one can easily see that for $n \geq 2,$ the set of zeros of $g_n(x, y)$ is

$$\{(x, 0) | x \in C\} \cup \{(x, y) | y \neq 0 \text{ and } x \text{ is an eigenvalue of } T_{n-1}(y)\}, \quad (2)$$

where C denotes the set of all complex numbers.

In this paper, we study the eigenvalue problem for the matrix $T_m(y)$, where we investigate the relationship between the eigenvalues of $T_m(y)$ and the zeros of a certain monic polynomial which will be appearing in the next section and be denoted by $p(w)$. So, the main task in this work aims at locating the zeros of $p(w)$. Then, based on our study, we are able to give a complete description of the zeros of the Fibonacci polynomials $g_n(x, y)$.

2. Main Results

First, we need to consider the eigenvalue problem for the matrix $T_m(y)$.

Writing out the eigenvalue problem

$$T_m(y)Z = xZ,$$

where x is an eigenvalue of $T_m(y)$ and $Z = [z_1, \ldots, z_m]^t \neq 0$ is a corresponding eigenvector, we obtain that x and Z satisfy the following recurrence relation (3) with boundary conditions (4) and (5):

$$-z_{j-1} + yz_{j+1} = xz_j, \quad j = 1, \ldots, m-1, \quad (3)$$

$$z_0 = 0, \quad (4)$$

$$-\frac{1}{2} z_{m-1} = xz_m. \quad (5)$$

The general solution to the recurrence relation (3) is

$$z_j = \alpha r_1^j + \beta r_2^j, \quad j = 1, \ldots, m,$$

where r_1 and r_2 are the roots of the characteristic equation

$$yr^2 - xr - 1 = 0,$$

which are

$$r_1 = \frac{1}{2y}\left(x + \sqrt{x^2 + 4y}\right), \quad r_2 = \frac{1}{2y}\left(x - \sqrt{x^2 + 4y}\right).$$

Using the boundary condition (4), we obtain $\alpha = -\beta,$ which gives

$$z_j = \alpha(r_1^j - r_2^j), \quad j = 1, \ldots, m.$$

It can be easily shown that the following relations hold:

$$r_1 r_2 = -\frac{1}{y} \tag{6}$$

and

$$r_1 + r_2 = \frac{x}{y}. \tag{7}$$

Using the boundary condition (5), we obtain

$$-\frac{1}{2}\alpha(r_1^{m-1} - r_2^{m-1}) = x\alpha(r_1^m - r_2^m). \tag{8}$$

Since $Z \neq 0$ we must have $\alpha \neq 0$. Eliminating α then using identities (6) and (7) to eliminate r_2 and x one can verify that equation (8) is equivalent to

$$(-r_1^2 y)^m = \frac{-r_1^2 y + 2}{-2r_1^2 y + 1}. \tag{9}$$

So, one can solve for r_1 in terms of y through the identity (9), then solve for x in terms of y through the identity (7). Let $w = -r_1^2 y.$ Then equation (9) can be written as

$$w^m = \frac{w + 2}{2w + 1}. \tag{10}$$

Thus solving (9) for r_1 in terms of y requires finding the zeros of the polynomial

$$p(w) = 2w^{m+1} + w^m - w - 2.$$

Note that $p(w)$ is anti-palindromic. (A polynomial $p(w) = \sum_{j=0}^{n} a_j w^j$ is called *palindromic* when $a_j = a_{n-j}$ and *anti-palindromic* when $a_j = -a_{n-j}$.)

In the following lemmas, we investigate some basic properties of the zeros of $p(w)$.

Lemma 1. (i) $w = 1$ *is a simple root of* $p(w)$.

(ii) *If m is odd, then* $w = -1$ *is a simple root of* $p(w)$.

(iii) *If w is a root, then* $\frac{1}{w}$ *is also a root*.

Proof. It can be easily verified that

(i) $p(1) = 0$ and $p'(1) \neq 0$.

(ii) If m is odd, then $p(-1) = 0$ and $p'(-1) \neq 0$.

(iii) $p(w) = 0 \Rightarrow p\left(\frac{1}{w}\right) = 0.$ □

Lemma 2. $p(w)$ *has all its roots on the unit circle*.

Proof. Let $w = a + ib$ be a root of $p(w)$ and let $c^2 = |2w + 1|^2$. Then from (10) and by direct calculation, we have

$$|w|^{2m} = \frac{|w|^2 + 4a + 4}{c^2} = \frac{c^2 - 3|w|^2 + 3}{c^2} = 1 - \frac{3(|w|^2 - 1)}{c^2}.$$

One can see that this equation is satisfied if and only if $|w| = 1$. □

Lemma 3. *All roots of* $p(w)$ *are simple*.

Proof. Suppose that $w = e^{i\theta}$ is a multiple root of $p(w)$, then w satisfies

$$p(w) = 2w^{m+1} + w^m - w - 2 = 0 \tag{11}$$

and

$$p'(w) = 2(m+1)w^m + mw^{m-1} - 1 = 0. \tag{12}$$

If we multiply (11) by $m+1$, multiply (12) by w and subtract, then we obtain

$$w^m = mw + 2(m+1).$$

Substitute in (11) to obtain

$$2mw^2 + (5m+3)w + 2m = 0.$$

The discriminant of this equation in the variable w is greater than zero, so the unimodular root is real. Therefore $w = \pm 1$ but from Lemma 1, we have ± 1 are simple roots. Hence the assertion holds. □

As we have mentioned earlier, we aim at locating the roots of the polynomial $p(w)$. In the next theorem and the following three remarks we investigate the location of the zeros of $p(w)$.

Theorem 1. *A complex number* $w = e^{i\theta}$ *is a zero of* $p(w)$ *if and only if* θ *is a solution to the equation*

$$f(\theta) = \sin(m-1)\frac{\theta}{2} + 2\sin(m+1)\frac{\theta}{2} = 0. \tag{13}$$

Proof. We shall group the complex roots of $p(w)$ according to the parity of m. In case $m = 2k$, then, by Lemma 1, $p(w)$ has 1 as one of its roots. Let

$$q(w) = \frac{p(w)}{w-1} = 2w^m + 3(w + \cdots + w^{m-1}) + 2$$

and let

$$u_j = w^j + \frac{1}{w^j}.$$

Then $u_j = 2\cos j\theta$ and

$$w^{-k}q(w) = 2u_k + 3 + 3\sum_{j=1}^{k-1} u_j.$$

Thus, $w = e^{i\theta}$ is a complex root of $p(w)$ if and only if

$$4\cos k\theta + 3 + 3\sum_{j=1}^{k-1} 2\cos j\theta = 0. \tag{14}$$

Using the identity

$$\frac{1}{2} + \sum_{j=1}^{n} \cos jt = \frac{\sin(2n+1)t/2}{2\sin t/2}, \tag{15}$$

we can rewrite (14) as

$$2\cos k\theta + 3\left(\frac{\sin(2k-1)\theta/2}{2\sin\theta/2}\right) = 0,$$

which can be easily shown to be equivalent to

$$\cos k\theta \sin\frac{\theta}{2} + 3\sin k\theta \cos\frac{\theta}{2} = 0. \tag{16}$$

Using the identity $\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$, one can verify that (16) is equivalent to (13).

In case $m = 2k+1$ is odd, then, by Lemma 1, $p(w)$ has ± 1 as two of its roots.

Let

$$q(w) = \frac{p(w)}{w^2 - 1} = 2w^{m-1} + w^{m-2} + 2w^{m-3} + \cdots + w + 2.$$

Let $u_j = w^j + \frac{1}{w^j}$. Then $u_j = 2\cos j\theta$ and

$$w^{-k}q(w) = \begin{cases} 2 + 2\displaystyle\sum_{\substack{j=2 \\ j \text{ even}}}^{k} u_j + \sum_{\substack{j=1 \\ j \text{ odd}}}^{k-1} u_j & \text{if } k \text{ is even,} \\ 1 + \displaystyle\sum_{\substack{j=2 \\ j \text{ even}}}^{k-1} v_j + 2\sum_{\substack{j=1 \\ j \text{ odd}}}^{k} v_j & \text{if } k \text{ is odd} \end{cases} \tag{17}$$

$$= \begin{cases} 2 + 4\displaystyle\sum_{j=1}^{k} \cos j\theta - 2\sum_{j=1}^{\frac{k}{2}} \cos(2j-1)\theta & \text{if } k \text{ is even,} \\ 1 + 2\displaystyle\sum_{j=1}^{k} \cos j\theta + 2\sum_{j=1}^{\frac{k+1}{2}} \cos(2j-1)\theta & \text{if } k \text{ is odd.} \end{cases} \tag{18}$$

Using identity (15) and the identity $\sum_{j=1}^{n} \cos(2j-1)t = \frac{\sin 2nt}{2\sin t}$ in (17) and (18), one can verify that $w^{-k}q(w) = 0$ is equivalent to (13). Hence, the statement of the theorem holds. □

Remark 1. Observe that in (13) when $m = 2k$ is even, $f(\theta) > 0$ whenever $\sin(m+1)\frac{\theta}{2} = 1$ and $f(\theta) < 0$ whenever $\sin(m+1)\frac{\theta}{2} = -1$. Therefore, equation (13) has a root between every consecutive pairs of values of $\theta \in (0, \pi]$ with $\sin(2k+1)\frac{\theta}{2} = \pm 1$. There are precisely $k+1$ values of θ for which $\sin(2k+1)\frac{\theta}{2} = \pm 1$, namely $\frac{2j-1}{2k+1}\pi$, $j = 1, \ldots, k+1$. This renders the k roots of equation (13) and hence the k zeros of $p(w)$ on the upper half of the unit circle. Taking complex conjugates yields another k zeros of $p(w)$ and with the root 1 always present, we have all of the $m+1 = 2k+1$ roots of $p(w)$.

Similarly, when $m = 2k+1$ is odd, we observe that equation (13) has a root between every consecutive pairs of values of $\theta \in (0, \pi)$ with $\sin(m+1)\frac{\theta}{2} = \sin(k+1)\theta = \pm 1$ which are precisely $k+1$ values, namely $\frac{2j-1}{2(k+1)}\pi,\ j = 1, ..., k+1.$ This renders the k zeros of $p(w)$ on the upper half of the unit circle. Taking complex conjugates yields another k zeros of $p(w)$ and with the two roots $\pm 1,$ we have all of the $m+1 = 2k+2$ roots of $p(w).$

Remark 2. Note the fact that the roots of $p(w)$ are interlaced between angles which are equally spaced, not only confirms that all the roots of $p(w)$ are simple but also gives some restriction on the distribution of the roots of $p(w).$ In particular, one can say that they are uniformly distributed on the unit circle.

Remark 3. One can see that equation (13) can be expressed as

$$U_{m-2}(t) + 2U_m(t) = 0,$$

where $t = \cos(\theta/2)$ and U_m is the Chebyshev polynomial of the second kind.

The interlacing of the roots of the polynomials $\{U_m\}$ can then be employed to conclude that the roots of $p(w)$ indeed are simple and are uniformly distributed on the unit circle. We leave the details for the interested reader.

We conclude the paper with a full description of the zeros of $g_n(x, y)$ as given in the following theorem.

Theorem 2. *The set of all zeros of* $g_n(x, y)$ *is*

$$\{(x, 0) | x \in C\} \cup \{(x, y) | y \neq 0 \text{ and } x = \pm 2i\sqrt{y}\cos t\}, \tag{19}$$

where t takes the values of all the solutions of the equation

$$\sin(n-2)t + 2\sin nt = 0. \tag{20}$$

Proof. For each zero $w = e^{i\theta}$ of $p(w)$, one can solve for x via the relation $w = -r_1^2 y$ and the two relations (6) and (7) to obtain the eigenvalues of $T_{n-1}(y)$ as $x = \pm 2i\sqrt{y}\cos\frac{\theta}{2}$. Setting $t = \frac{\theta}{2}$, one can see that the form (19) is equivalent to the form (2) of the set of all solutions of $g_n(x, y)$. Finally, the condition (20) on t is obtained by substituting $m = n - 1$ in (13). □

References

[1] M. Catalani, Some formulae for bivariate Fibonacci and Lucas polynomials, arXiv:math/0406323v1 [math.CO] 16 Jun 2004.

[2] M. Catalani, Identities for Fibonacci and Lucas polynomials derived from a book of Gould, arXiv:math/0407105v1 [math.CO] 7 Jul 2004.

[3] M. Catalani, Generalized bivariate Fibonacci polynomials, arXiv:math/0211366v2 [math.CO] 4 Jun 2004.

[4] F. R. V. Alves and P. M. M. C. Catarino, The bivariate (complex) Fibonacci and Lucas polynomials: an historical investigation with the Maple's help, Acta Didactica Napocensia 9(4) (2016), 71-95.

[5] M. Swammy, On a class of generalized polynomials, Fibonacci Quart. 35 (1997), 329-334.

[6] A. Horadam and E. Horadam, Roots of recurrence generated polynomials, Fibonacci Quart. 20(3) (1982), 219-226.

[7] Yu Hongquan and Liang Chuanguang, Identities involving partial derivatives of bivariate Fibonacci and Lucas polynomials, Fibonacci Quart. 35(1) (1997), 19-23.

[8] E. Gokcen Kocer, Bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials, IOSR J. Math. 12(4) (2016), 44-50.

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