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ON FINDING THE EIGENVALUES OF CERTAIN TRIDIAGONAL MATRICES

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Abstract

In this article, we establish a relationship between finding the eigenvalues of a class of certain tridiagonal matrices and finding the zeros of certain twinned polynomials. Then, based on this study, we determine the exact number of real eigenvalues that such matrices can have and give conditions for the existence of such a number of real eigenvalues. Our findings were evidenced by solving numerically some examples using Matlab.

1. Introduction

Let T_m be the $m \times m$ tridiagonal matrix of the form

$$T_m = \begin{bmatrix} a & b & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix},$$

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where a and b are real numbers with $ab \neq 0$. Since T_m and its transpose have the same set of eigenvalues, according to the well-known cycle theorem due to Gerschgorin (see, e.g., [3], Theorem 9.1, p. 500), each eigenvalue λ of T_m must belong to one of the circles

$$|\lambda - b| \leq 1, \quad |\lambda| \leq |a| + 1, \quad |\lambda| \leq 2, \quad |\lambda| \leq 1.$$

In [5], the author has obtained the result

$$|\lambda| \leq \max\{2, |a| + |b|\}.$$

It has been shown in [8] that

$$|\lambda| \leq 1 + \max\{|a|, |b|\}.$$

The authors in [8] have dealt with such a tridiagonal matrix because its eigenvalues match with the zeros of a certain Fibonacci-like polynomial. We refer to [1] where the author has discussed upper bounds for the eigenvalues of T_m but for the more general case where a and b may be complex numbers.

In this paper, we study the eigenvalue problem for the matrix T_m where we investigate the relationship between the eigenvalues of T_m and the zeros of a certain monic polynomial which will be appearing in the next section and be denoted by $P(r)$. So, the main task in this work aims at locating the zeros of $P(r)$. Then, based on our study, we are able to specify the exact number of real eigenvalues that T_m can have and give conditions for the existence of such a number.

We would like to note that some ordinary algebra used will be omitted here for the sake of brevity. Finally, we solve numerically some examples and present some graphs using Matlab in order to illustrate our ideas.

2. Properties of the Eigenvalues of T_m

In this section, we establish a relation between the eigenvalues of T_m

and the zeros of a twinned polynomial so that we will be able to conclude some basic properties of the eigenvalues of T_m.

First, we need to consider the eigenvalue problem for the matrix T_m. Writing out the eigenvalue problem

$$T_m Z = \lambda Z,$$

where λ is an eigenvalue of T_m and $Z = [z_1, \ldots, z_m]^t \neq 0$ is a corresponding eigenvector, we obtain that λ and Z satisfy the following recurrence relation (1) with the artificial initial condition (2) and the two boundary conditions (3) and (4):

$$z_{j-1} - z_{j+1} = \lambda z_j, \quad j = 1, \ldots, m, \tag{1}$$

$$z_0 = 1, \tag{2}$$

$$az_1 + bz_2 = \lambda z_1, \tag{3}$$

$$z_{m-1} = \lambda z_m. \tag{4}$$

The general solution to the recurrence relation (1) is

$$z_j = \alpha r_1^j + \beta r_2^j, \tag{5}$$

where r_1 and r_2 are the roots of the characteristic equation

$$r^2 + \lambda r - 1 = 0, \tag{6}$$

which are

$$r_1 = \frac{1}{2}(-\lambda + \sqrt{\lambda^2 + 4}), \quad r_2 = \frac{1}{2}(-\lambda - \sqrt{\lambda^2 + 4}).$$

It can be easily shown that the following relations hold:

$$r_1 r_2 = -1 \text{ and } r_1 + r_2 = -\lambda. \tag{7}$$

Using the artificial initial condition (2) in the explicit formula (5), we obtain $\beta = 1 - \alpha,$ so, (5) can be rewritten as

$$z_j = \alpha r_1^j + (1 - \alpha) r_2^j. \tag{8}$$

Using the explicit formula (8) with $j = 1$ and the relation (6), we get

$$-r_1 z_1 = \alpha(-r_1^2) + (1 - \alpha). \tag{9}$$

Applying the recurrence relation (1) with $j = 1$ implies

$$1 - z_2 = \lambda z_1. \tag{10}$$

Multiplying (10) by b and adding the boundary condition (3) yields

$$z_1[\lambda(1 + b) - a] = b. \tag{11}$$

Now, multiplying (9) by $[\lambda(1 + b) - a]$ and (11) by $r_1,$ then adding imply

$$[\lambda(1 + b) - a][-\alpha r_1^2 + (1 - \alpha)] + b r_1 = 0. \tag{12}$$

Using the two relations (6) and (7) implies

$$\lambda = \frac{1 - r_1^2}{r_1} = \frac{1 - r_2^2}{r_2}. \tag{13}$$

Substituting in (12) implies

$$\left[\left(\frac{1 - r_1^2}{r_1}\right)(b + 1) - a r_1\right][-\alpha r_1^2 + (1 - \alpha)] + b r_1 = 0$$

which implies

$$[(1 - r_1^2)(b + 1) - a r_1][-\alpha r_1^2 + (1 - \alpha)] + b r_1 = 0. \tag{14}$$

If we apply the explicit solution (8) in the boundary condition (4), then we obtain

$$\alpha r^{m-1} + (1 - \alpha)\left(-\frac{1}{r}\right)^{m-1} = \lambda\left[\alpha r^m + (1 - \alpha)\left(-\frac{1}{r}\right)^m\right], \tag{15}$$

where we let r to stand for r_1. With some ordinary algebra, (15) can be simplified into

$$(1-\alpha)+\alpha(-r^2)^{m+1}=0. \tag{16}$$

Remark 1. Note that we must have $(-r^2)^{m+1}\neq 1$ otherwise, $1=0$, so, $r\neq \pm i$ and therefore, according to relation (14), $\pm 2i$ are not eigenvalues of T_m.

So, (16) yields

$$\frac{\alpha-1}{\alpha}=(-r^2)^{m+1} \text{ and } \alpha=\frac{1}{1-(-r^2)^{m+1}}.$$

Using the above two relations in (14) implies

$$[(1-r^2)(b+1)-ar][1-(-r^2)^m]+b[(-r^2)^{m+1}-1]=0$$

which can be written as

$$-(-r^2)^{m+1}+ar(-r^2)^m-(b+1)[(-r^2)^m+(b+1)(-r^2)-ar+1]=0. \tag{17}$$

We can split (17) according to the parity of m as

$$P(r)=r^{2m+2}+ar^{2m+1}-(b+1)r^{2m}-(b+1)r^2-ar+1=0 \tag{18}$$

when m is even and

$$P(r)=r^{2m+2}+ar^{2m+1}-(b+1)r^{2m}+(b+1)r^2+ar-1=0 \tag{19}$$

when m is odd.

Remark 2. Note that the polynomial $P(r)$ in (18) is semi-palindromic and in (19) is semi-antipalindromic. (A polynomial $p(w)=\sum_{j=0}^{n}a_jw^j$ is called *palindromic* when $a_j=a_{n-j}$ and *antipalindromic* when $a_j=-a_{n-j}$).

Remark 3. Now, λ can be found by solving for r in (18) or (19), then using the relation (13).

In the following lemmas, we investigate some basic properties of the zeros of $P(r)$ when m is even. It can be easily verified that the same also applies when m is odd.

Lemma 1. *If r is a zero of $P(r)$, then $-\frac{1}{r}$ is also a zero of $P(r)$.*

Proof. It can be easily verified that

$$P(r) = 0 \Rightarrow P\left(-\frac{1}{r}\right) = 0. \qquad \square$$

Lemma 2. *All zeros of $P(r)$ are simple.*

Proof. Suppose that r is a multiple zero of $P(r)$. Then r satisfies

$$P(r) = r^{2m+2} + ar^{2m+1} - (b+1)r^{2m} - (b+1)r^2 - ar + 1 = 0 \tag{20}$$

and

$$\begin{aligned} P'(r) = {} & (2m+2)r^{2m+1} + (2m+1)ar^{2m} \\ & - 2m(b+1)r^{2m-1} - 2(b+1)r - a = 0. \end{aligned} \tag{21}$$

If we multiply (20) by $2m + 2$, multiply (21) by r and subtract, then we obtain

$$r^{2m}[ar - 2(b+1)] = 2m(b+1)r^2 + a(2m+1)r - (2m+2). \tag{22}$$

Solving for r^{2m} in (22), then substituting r^{2m} in (20) imply

$$cr^4 + er^3 + fr^2 + er + c = 0, \tag{23}$$

where

$$c = -2m(b+1),$$

$$e = a[(2m+1) + (b+1)(2m-1)],$$

$$f = a^2(2m+2) + (b+1)^2(2m-2) + (2m+2).$$

So, r is a zero of the polynomial in (23) which is a palindromic polynomial, so is $-\frac{1}{r}$, since it is a zero of $P(r)$ with same multiplicity as that of r but as it can be easily verified that $-\frac{1}{r}$ is not a zero of the palindromic polynomial in (23). Hence, the assertion holds. □

Lemma 3. *All eigenvalues of T_m are distinct.*

Proof. Other than the two roots $\pm i$ of $P(r)$, the complex root of $P(r)$ can be divided into quadriples $\left\{r, -\frac{1}{r}, \bar{r}, -\frac{1}{\bar{r}}\right\}$, where the pair $\left\{r, -\frac{1}{r}\right\}$ gives a complex eigenvalue λ through the relation (13) and the other pair $\left\{\bar{r}, -\frac{1}{\bar{r}}\right\}$ gives the conjugate eigenvalue $\bar{\lambda}$ of T_m. The real zeros of $P(r)$ can be divided into pairs $\left\{r, -\frac{1}{r}\right\}$ which leads to the same real eigenvalue λ of T_m. Since, by Lemma 2, all of these $2m$ zeros of $P(r)$ are simple, all the corresponding eigenvalues of T_m are distinct. □

3. Existence of Real Eigenvalues

In the following main result, we use Descartes' rule of signs (see, e.g., [2]) to determine the exact number of real zeros of the polynomial $P(r)$ and hence to determine the exact number of real eigenvalues of T_m. The rule tells us that the number of positive real zeroes in a polynomial $Q(x)$ is the same or less than by an even number as the number of changes in the sign of the coefficients. The number of negative real zeroes of $Q(x)$ is the same as the number of changes in sign of the coefficients of the terms of $Q(x)$ or less than this by an even number.

Theorem 1. *T_m has either exactly two or no real eigenvalues when m is even and has exactly one or three real eigenvalues when m is odd.*

Proof. When m is even, we have, as can be easily seen in (18), exactly two changes of signs. So, by Descartes' rule of sign, the number of positive zeros of $P(r)$ must be either 0, or 2 and the number of negative zeros must follow to be either 0 or 2 (in fact, the negative real zeros are the negatives of the reciprocals of the positive real zeros, see Lemma 1). Similarly, when m is odd, we have, as can be easily seen in (19), exactly three changes of signs, so, the number of positive zeros of $P(r)$ must be either 1 or 3 and the number of negative zeros of $P(r)$ must be correspondingly 1 or 3.

Hence, according to the correspondence between the zeros of $P(r)$ and the eigenvalues of T_m (see the proof of Lemma 3), the assertion in this theorem holds. □

Remark 4. In fact, when m is odd, the existence of a real zero of $P(r)$ can be established by noting that in (19), $P(0) = -1 < 0$ and $P(1) = 2a > 0$ when $a > 0$ or $P(-1) = -2a > 0$ when $a < 0$.

As we have mentioned in the introduction, one of the aims in this article is to investigate the conditions on a and b to have a certain number of real eigenvalues. Here, we investigate the case when m is even leaving the case when m is odd for the interested reader.

Remark 5. It can be easily seen in (18) and (19) that

$$P(a, b, r) = (-1)^m P(-a, b, -r).$$

So, it suffices to consider the case $a > 0$.

Lemma 4. *If m is even and $b > 0$, then T_m has exactly two real eigenvalues.*

Proof. Since in (18), $P(0) = 1 > 0$ and $P(1) = -2b < 0$, $P(r)$ has exactly two pairs $\left\{r, -\frac{1}{r}\right\}$ of real zeros and, correspondingly, T_m will have exactly two real eigenvalues. □

Now, we are left with the case m even and $b < 0$. Recall that in Remark 5, we have decided that it suffices to consider only the case $a > 0$.

In (18), $P(r)$ can be written as

$$P(r) = r^{2m} Q_1(r) + Q_2(r), \tag{24}$$

where

$$Q_1(r) = r^2 + ar - (b+1) \text{ and } Q_2(r) = -(b+1)r^2 - ar + 1. \tag{25}$$

We first consider the case $\underline{b < -1}$. Note that the discriminant for both $Q_1(r)$ and $Q_2(r)$ is $\Delta = a^2 + 4b + 4$. If $\Delta < 0$, then, since $Q_1(0) = -(b+1) > 0$ and $Q_2(0) = 1 > 0$, then $P(r) > 0, \forall r$ and therefore $P(r)$ has no real zero.

If $\Delta = 0$, then Q_1 and Q_2 have the two repeated real zeros $-\frac{a}{2}$ and $\frac{a}{-2(b+1)}$, respectively, and since the two differ in sign, $P(r)$ has no real zero in this case too. If $\Delta > 0$, then Q_1 and Q_2 have the two pairs of real zeros

$$\sigma_1 = \frac{-a+\Delta}{2}, \quad \sigma_2 = \frac{-a-\Delta}{2},$$

$$\eta_1 = \frac{a+\Delta}{-2(b+1)}, \quad \eta_2 = \frac{a-\Delta}{-2(b+1)},$$

respectively.

Note that, since $|a| > \Delta$, we must have both of σ_1 and σ_2 negative and both of η_1 and η_2 positive. Now, if $\sigma_1 < -1$, then $\exists x < -1 \ni Q_1(x) < 0$, so the negativity of $x^{2m}Q_1(x)$ will exceed the positivity of $Q_2(x)$ for sufficiently large m and then $P(r) < 0$, so, $P(r)$ will have a real zero. Or if $\eta_2 \le 1$, then $\exists x \ni 0 < x < 1$ and $Q_2(x) < 0$, so the negativity of $Q_2(x)$

will exceed the positivity of $x^{2m}Q_1(x)$ for sufficiently large m and then $P(r) < 0,$ so, $P(r)$ will have a real zero. With some ordinary algebra, which has been omitted here for the sake of brevity, we arrive at the condition that $a \geq 2$ or $a > -b$.

In case $\underline{b = -1}$, then Q_1 will have the two zeros, 0 and $-a$, and Q_2 will be linear and has the single zero $\frac{1}{a}$. This leads to the condition $a > -b$ for $P(r)$ to have a real zero for a sufficiently large m.

We remain with the case $\underline{-1 < b < 0}$. Here, $\Delta > 0$ and $|a| < \Delta$, so Q_1 will have the two zeros $\sigma_1 > 0$ and $\sigma_2 < 0$. Similarly, Q_2 will have the two zeros $\eta_1 < 0$ and $\eta_2 > 0$. With the same kind of analysis done above, we arrive at the condition $a > -b$ for $P(r)$ to have a real zero for a sufficiently large m.

Now, we are in a position to state our second main result.

Theorem 2. *For all sufficiently large even* m, T_m *has exactly two real eigenvalues if and only if one of the following conditions holds*:

(i) $b > 0,$

(ii) $-1 \leq b < 0$ *and* $|a| > |b|,$

(iii) $b < -1$ *and* $\Delta > 0$ *and* $(a \geq 2$ *or* $|a| > |b|).$

Otherwise, T_m *will have no real eigenvalues*.

4. Numerical Evidence

As mentioned in the introduction, numerical experiments have been carried out using Matlab. First, we have computed the eigenvalues of T_m directly when $m = 6,$ $a = 3$ and $b = -8,$ then we obtained the eigenvalues of T_m as follows:

$$1.285663172214663 - 2.490713065658173i,$$
$$1.285663172214663 + 2.490713065658173i,$$
$$0.145225125471461 - 0.607364515842368i,$$
$$0.145225125471461 + 0.607364515842368i,$$
$$0.069111702313877 - 1.614389459303184i,$$
$$0.069111702313877 + 1.614389459303184i.$$

Afterwards, we have computed the eigenvalues of T_m indirectly by first finding the zeros of $P(r)$, then using the relation (13) to find the eigenvalues of T_m to be as follows:

$$1.285663172214663 + 2.490713065658173i,$$
$$1.285663172214663 - 2.490713065658173i,$$
$$0.069111702313876 + 1.614389459303185i,$$
$$0.069111702313876 - 1.614389459303185i,$$
$$0.145225125471460 + 0.607364515842368i,$$
$$0.145225125471460 - 0.607364515842368i.$$

Note that the two sets of eigenvalues obtained are the same. Also, note that we have no real eigenvalues in this case, since the discriminant $\Delta = a^2 + 4b + 4 < 0$.

The following four figures show graphs of the zeros of $P(r)$ and the corresponding eigenvalues of T_m. Note that the complex zeros ($\pm i$ excluded) of $P(r)$, plotted by $\times$, indeed form quadriples $\left\{r, -\frac{1}{r}, \bar{r}, -\frac{1}{\bar{r}}\right\}$, each corresponding to a pair of conjugate eigenvalues of T_m and the real zero of $P(r)$, in case they exist, plotted by $\circ$, indeed form pairs $\left\{r, -\frac{1}{r}\right\}$ with each pair corresponding to a real eigenvalue of T_m.

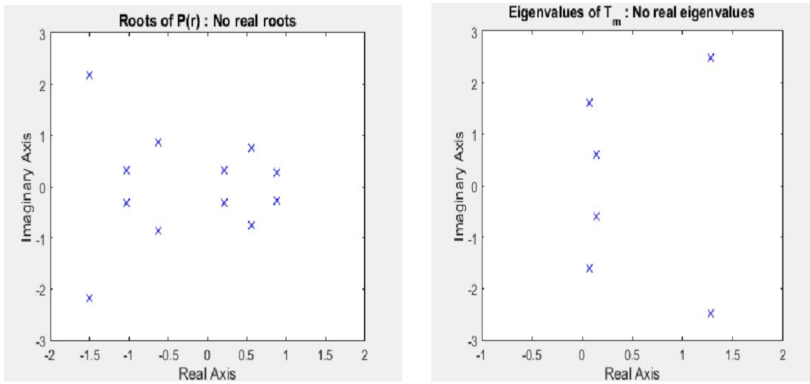

Figure 1. Roots of $P(r)$ and corresponding eigenvalues of T_m, $m = 6$, $a = 3$, $b = -8$.

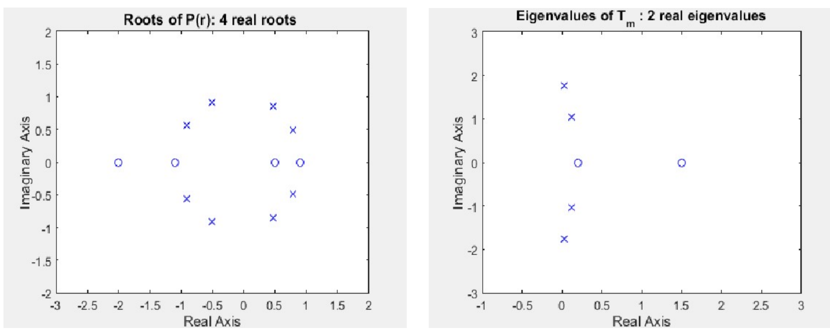

Figure 2. Roots of $P(r)$ and corresponding eigenvalues of T_m, $m = 6$, $a = 2$, $b = -1$.

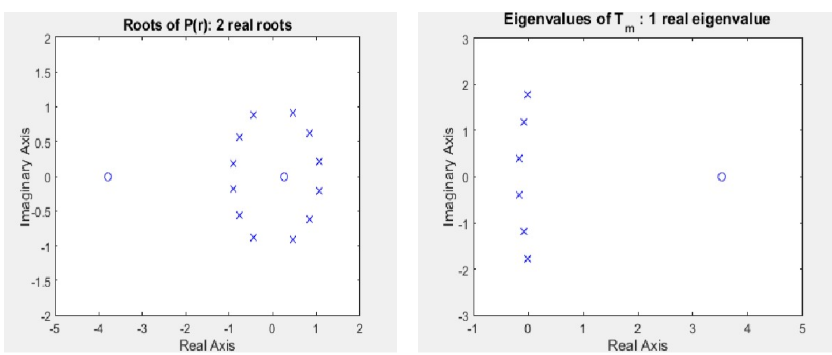

Figure 3. Roots of $P(r)$ and corresponding eigenvalues of T_m, $m = 7$, $a = 3$, $b = -2$.

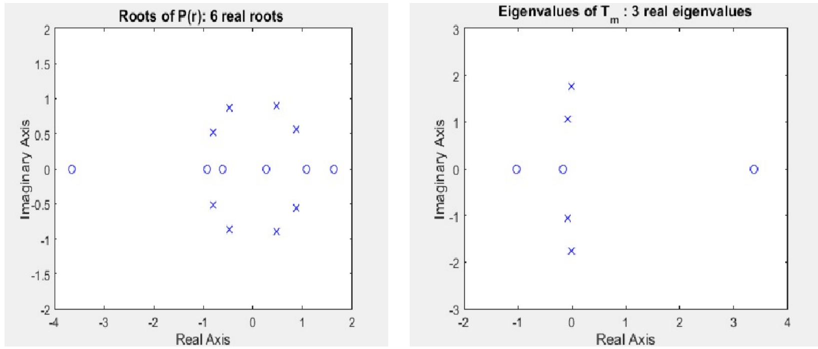

Figure 4. Roots of $P(r)$ and corresponding eigenvalues of T_m, $m = 7$, $a = 2$, $b = 5$.

Finally, we considered seven examples that cover seven possible cases for the values of a and b where we consider the case m even. We computed the zeros of $P(r)$ and then the corresponding eigenvalues of T_m via relation (14). We present our results in Table 1. For the sake of consistency, in all the examples, we have taken the size of m to be 20. In the first column of the table, we give the case, in the second column, we give the values of a and b used that agree with the case, and in the third column, we state the result whether there exist exactly two real eigenvalues or there are no real eigenvalues. We give only the two real eigenvalues in case of their existence, since it is not convenient to list all of the complex zeros.

Table 1

Case	Values of a and b	Existence of real eigenvalues
$b > 0$	$a = 0.8$, $b = 1.5$	$\lambda_1 = 1.53857$, $\lambda_2 = 0.41904$
$-1 \leq b < 0$ and $\lvert a \rvert > \lvert b \rvert$	$a = 0.8$, $b = -0.5$	$\lambda_1 = 0.38701$, $\lambda_2 = 0.08257$
$-1 \leq b < 0$ and $\lvert a \rvert \leq \lvert b \rvert$	$a = 0.48$, $b = -0.5$	No real eigenvalues
$b < -1$ and $\Delta > 0$ and $\lvert a \rvert \geq 2$ and $\lvert a \rvert \leq \lvert b \rvert$	$a = 3.48$, $b = -3.5$	$\lambda_1 = 2.06091$, $\lambda_2 = 0.20637$
$b < -1$ and $\Delta > 0$ and $\lvert a \rvert < 2$ and $\lvert a \rvert > \lvert b \rvert$	$a = 1.8$, $b = -1.5$	$\lambda_1 = 0.77033$, $\lambda_2 = 0.12563$
$b < -1$ and $\Delta > 0$ and $\lvert a \rvert < 2$ and $\lvert a \rvert \leq \lvert b \rvert$	$a = 1.48$, $b = -1.5$	No real eigenvalues
$b < -1$ and $\Delta \leq 0$	$a = 2.8$, $b = -3.5$	No real eigenvalues

As can be seen in the above figures and table, the computational results agree with our theoretical findings.

5. Conclusion

In this article, we have established a relation between the problem of finding all eigenvalues of a certain tridiagonal real matrix, T_m, and the problem of finding all zeros of a twined polynomial, $P(r)$. Also, we have investigated the existence of real eigenvalues of T_m, where we have proved the existence of exactly two real eigenvalues or no real eigenvalue when m is even and exactly one or three real eigenvalues when m is odd. Also, we have investigated the conditions on a and b for T_m to have either exactly two real eigenvalues or no real eigenvalue in case m is even. Our findings have been evidenced by solving some examples using Matlab. We leave it for the interested reader to look for the conditions on a and b for T_m to have either exactly one real eigenvalue or three real eigenvalues for the case m is odd. It is worth mentioning that in [8] the authors have investigated the conditions on a and b for T_m to have real eigenvalues but when a and b are integers by using a different approach and without deciding the exact number of real eigenvalues that one can have. We think that $P(r)$ being of even grade and a twined polynomial deserves further investigation to build a more efficient numerical algorithm where a symmetric division process may be employed to compute all zeros of $P(r)$ and hence to find all eigenvalues of T_m (see, e.g., [4] and [6]). The sparsity of $P(r)$ as well should be taken into account. In [7], the author suggests an algorithm to deal with sparse polynomials. We also think that this idea of establishing a relation between the problem of finding all eigenvalues of a tridiagonal matrix and finding all zeros of a twined polynomial can be used to find all eigenvalues of other types of tridiagonal matrices.

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