Minimization and Positivity of the Tensorial Rational Bernstein Form

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***Abstract*—Polynomials and rational functions of total degree l defined on n dimensional box have a representation in the Bernstein form. The range of these functions is bounded by the smallest and the largest Bernstein coefficients. In this paper, bounding properties of the range of monomials are extended to the multivariate rational Bernstein case. First, algebraic identities certifying the positivity of a given rational function over a box are addressed. Subsequently, we investigate certificates of positivity by minimization, and bounding functions which are independent of the given dimension.**

***Index Terms*—Bernstein polynomials, rational functions, range bounding, certificates of positivity, optimization.**

I. Introduction

The problem to decide whether a given rational function in $n-$variables is positive, in the sense that all its Bernstein coefficients are positive, goes back to [21], [22], [23]. The problem was investigated for polynomials in [1] and [19]. Certificates of positivity for polynomials in the Bernstein basis was addressed in [3], [14] and [19]. In this paper, we extend certificates of positivity to the tensorial multivariate rational Bernstein form. On the other hand, the problem of minimizing and approximating the minimum value of a polynomial over simplices was extensively studied, see [2], [5], [8], [10], [13], [16] and [18]. The expansion of a given (multivariate) polynomial p into Bernstein polynomials is used over a simplex, the so-called *simplicial Bernstein form*, see [2], [5], [6], [15] and [16]. The approach was extended to the rational case in [9], [10], [17] and [24]. Here, we extend the approximation of lower bounds to the rational Bernstein form over boxes. In [15], [16] and [24], the authors gave results on convergence properties and minimization under subdivision of the given domain, however, without studying certificates of positivity or minimizing the given lower bound. In this paper, we extend properties under the same approach to the tensorial rational case. We decide if a rational function is positive and give certificates of positivity in the rational Bernstein form by sharpness, degree elevation (global certificates), subdivision (local certificates) and minimization of a rational function. At the last, we provide a bound that does not depend on the number of variables of f.

The organization of this paper is as follows:In Section 2, we recall the most important background of the tensorial Bernstein expansion. In Section 3, we provide certificates of positivity for rational functions. Minimization and independent bounds of the rational case are given in Section 3. Finally, Section 4 comprises the conclusion.

A. Bernstein Properties

We recall some essential properties of the Bernstein expansion, which will be used for the tensorial case over a box. In many cases, it is desired to calculate the Bernstein expansion of a polynomial function p over a general n-dimensional box X in the set of real intervals $\mathbb{I}(\mathbb{R})^n$,

$$X = [\underline{x}_1, \overline{x}_1] \times .... \times [\underline{x}_n, \overline{x}_n]$$

with

$$\underline{x}_\mu \leq \overline{x}_\mu,\ \mu = 1, ..., n.$$

The *width* of $X_\mu = [\underline{x}_\mu, \overline{x}_\mu]$ is denoted by $w(X_\mu)$,

$$w(X_\mu) := \overline{x}_\mu - \underline{x}_\mu.$$

Let $||w(X)||_\infty := \max\{|X_1|, ..., |X_n|\}$, where $|X_\mu| = \max\{|\underline{x}_\mu|, |\overline{x}_\mu|,\ \mu \in \{1, ..., n\}\}$.

Comparisons and arithmetic operations on multiindices $i = (i_1, ..., i_n)^T$ are defined component-wise. For $x \in \mathbb{R}^n$ its monomials are $x^j := x_1^{j_1} ... x_n^{j_n}$. Using the compact notation, $k = (k_1, ..., k_n)$,

$$\sum_{j=0}^{k} := \sum_{j_1=0}^{k_1} \cdots \sum_{j_n=0}^{k_n},\quad \binom{k}{i} := \prod_{\mu=1}^{n} \binom{k_\mu}{i_\mu},$$

an n-variate polynomial function p, $p(x) = \sum_{j=0}^{l} a_j x^j$, $l = (l_1, ..., l_n)$, can be represented in the Bernstein form as

$$p(x) = \sum_{i=0}^{k} b_i^{(k)}(p)\ B_i^{(k)}(x),\ x \in X, \tag{1}$$

where, the ith Bernstein polynomial of degree $k \geq l$ is given by

$$B_i^{(k)}(x) = \binom{k}{i}(x-\underline{x})^i(\overline{x}-x)^{k-i}w(X)^{-k},\ \ 0 \leq i \leq k, \tag{2}$$

and the so-called *Bernstein coefficients* $b_i^{(k)}$ of p of degree k over X are given by

$$b_i^{(k)}(p) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{k}{j}}\ c_j,\ \ 0 \leq i \leq k, \tag{3}$$

$$c_j = w(X)^j \sum_{\tau=j}^{l} \binom{\tau}{j} a_\tau \underline{x}^{\tau-j},\ \ a_j = 0 \text{ for } l < j. \tag{4}$$

The vector 0 here denotes the multiindex with all components equal to 0, which should not cause ambiguity.

Let the grid point $x_i^{(k)}$ of the μth component in X is given by

$$x_{i,\mu}^{(k_\mu)} = \underline{x}_\mu + \frac{i_\mu}{k_\mu}(\overline{x}_\mu - \underline{x}_\mu),\ \ \mu = 1, \ldots, n. \tag{5}$$

Since any compact non-empty box in $\mathbb{R}^n$ can be mapped thereupon by an affine transformation, without loss of generality, we may consider the unit box $I := [0,1]^n$. Hence, $p(x)$ can be expressed as (1) with

$$B_i^{(k)}(x) = \binom{k}{i} x^i(1-x)^{k-i},\ \ x \in I, \tag{6}$$

and

$$b_i^{(k)}(p) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{k}{j}} a_j,\ \ 0 \leq i \leq k. \tag{7}$$

In particular, we have the *endpoint interpolation property*

$$b_i^{(k)}(p) = p(\frac{\hat{i}}{k}), \quad \text{for all } \hat{i},\ \ 0 \leq \hat{i} \leq k, \tag{8a}$$

$$\text{with} \quad \hat{i}_\mu \in \{0, k_\mu\}. \tag{8b}$$

The *convex hull* is a generalization of the range enclosing property, which states that the graph of p over I is contained within the convex hull of the control points derived from the Bernstein coefficients, i.e.,

$$\left\{ \begin{pmatrix} x \\ p(x) \end{pmatrix} : x \in I \right\} \subseteq conv \left\{ \begin{pmatrix} \frac{i}{l} \\ b_i^{(l)} \end{pmatrix} : 0 \leq i \leq l \right\}, \tag{9}$$

where *conv* denotes the convex hull, see Figure 1. This implies the *interval enclosing property* [4]

The implied *enclosing property* is given as

$$\min_{0 \leq i \leq k} b_i^{(k)}(p) \leq p(x) \leq \max_{0 \leq i \leq k} b_i^{(k)}(p), \ \ \text{for all}\ \ x \in I. \tag{10}$$

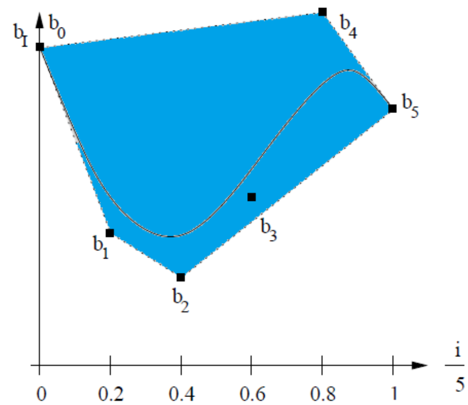

Figure 1. The curve of a polynomial of degree 5, the convex hull (colored blue) of its control points (marked by squares).

We define the total degree of a polynomial p as

$$K = \max\{k_\mu :\ \ \mu = 1, ..., n\}.$$

Denote the enclosure bound of a polynomial $B^{(k)}(p, X) := [\min_{0 \leq i \leq k} b_i^{(k)}(p), \max_{0 \leq i \leq k} b_i^{(k)}(p)]$, and the range $P(X) := [\min p(x), \max p(x)]$. Finally, recall the (Haussdorff) distance H between $B^{(k)}(p, X)$ and $P(X)$

$$H(P(X), B(p, \boldsymbol{X})) =$$

$$\max\{|\min_{0 \leq i \leq k} b_i^{(k)}(p) - \min p(x)|, |\max_{0 \leq i \leq k} b_i^{(k)}(p) - \max p(x)|\}. \tag{11}$$

B. Rational Bernstein Form

We may assume a rational function $f := p/q$, where both p and q have the same degree l since otherwise we can elevate the degree of the Bernstein expansion of either polynomial by component where necessary to ensure that their Bernstein coefficients are of the same order $l \leq k$. Since any box can be mapped upon the unit box by an affine transformation, we will extend results from polynomials to rational functions over the unit box I. Let the range of f over I is given by $F(I) := [\min f(x), \max f(x)] =: [\underline{f}, \overline{f}]$. The *tensorial rational Bernstein coefficients* of f of degree k are given by

$$b_i^{(k)}(f) = \frac{b_i^{(k)}(p)}{b_i^{(k)}(q)},\ \ 0 \leq i \leq k. \tag{12}$$

Throughout the paper, without loss of generality, we assume that $b_i^{(k)}(q) > 0$, $\forall 0 \leq i \leq k$.

The *range enclosing property* for the rational function is given from [17, Theorem 3.1] as

$$m^{(k)} := \min b_i^{(k)}(f) \leq f(x) \leq \max b_i^{(k)}(f) =: M^{(k)}. \tag{13}$$

Remark I.1. *The interval*

$$B^{(k)}(f, I) := [\min b_i^{(k)}(f), \max b_i^{(k)}(f)]$$

encloses the range of f over I.

By application of (12) to (13), the following theorem provides the sharpness property of f with respect to its enclosure bound.

Proposition I.1. *[9, Proposition 3] For $x \in \mathbb{R}^n$ it holds that $m^{(k)} = \underline{f}$ $(M^{(k)} = \overline{f})$ if and only if $m^{(k)}$ $(M^{(k)}) = b_{\hat{i}}^{(k)}(f)$ with $\hat{i}$ satisfying (8b).*

II. Certificates of Positivity

In this section, we study the positivity of the tensorial rational Bernstein form over I. Certifying the positivity of rational functions is widely applied in stability of dynamic systems, optimization and control theory, see [11], [12]. The enclosure property of f shows that if all Bernstein coefficients of f over I are positive, then the rational function f is positive over I. The converse need not to be true. There are rational functions which are positive over I and some Bernstein coefficients $b_i^{(k)}(f)$ are negative.

Example II.1. *Let*

$$f(x) = \frac{7x^2 - 5x + 1}{x^2 - 2x + 7},$$

which is positive over $[-1, 1]$, but the ordered list of Bernstein coefficients $b^{(2)}(f) = (1.3, -1, 0.5)$ has a negative value at $b_1(f) = -1$.

The (univariate) Bernstein polynomials of p of degree k on $[\underline{x}, \overline{x}]$

$$B_i^{(k)}(x) = \binom{k}{i} \frac{(\overline{x} - x)^{k-i}(x - \underline{x})^i}{w(X)^k}, \; i = 0, ..., k,$$

take positive values over $(\underline{x}, \overline{x})$. Note that $B_0^{(k)}(x)$ is positive at $\underline{x}$, and $B_k^{(k)}(x)$ is positive at $\overline{x}$. The Bernstein coefficient $b_0^{(k)}$ is the value of p at $\underline{x}$, and $b_k^{(k)}$ is the value at $\overline{x}$. Hence, if all Bernstein coefficients of $f = p/q$ are positive, the rational Bernstein form of f over a given domain provides certificates of positivity for f over the same domain. Without loss of generality, we assume that the (multivariate) rational case is studied on I. Denote by $b^{(k)}(f)$ the ordered list of Bernstein coefficients of a rational function f over I, we define the certificate of positivity $Cert(b^{(k)}(p))$ by:

$$Cert(b^{(k)}(p)) : \begin{cases} b_i^{(k)}(p) \geq 0 & \text{for all } 0 \leq i \leq k \\ b_{\hat{i}}^{(k)}(p) > 0 & \text{for } \hat{i}, \; \hat{i}_\mu \in \{0, k_\mu\}. \end{cases}$$

The rational Bernstein form of f of degree k is positive on I if $\min b_i^{(k)}(f) > 0$.

A. Certificates by Sharpness

The sharpness property in Proposition I.1 satisfies the certificate of positivity of a rational function over I. The equality holds in the left hand side of (13) if

$$\min b_i^{(k)}(f) = b_{\hat{i}}(f) \text{ for some } \hat{i}, \text{ with } \hat{i}_\mu \in \{0, k\}.$$

This implies the following proposition.

Proposition II.1. *Given f is positive on I. If $\min b_i^{(k)}(f) = b_{\hat{i}}(f)$ for some $\hat{i}$, with $\hat{i}_\mu \in \{0, k_\mu\}$. Then f satisfies the certificate of positivity.*

B. Global Certificates

If $k \geq l$ big enough, the minimum rational Bernstein coefficient of f converges linearly to the minimum range $\underline{f}$ over I. We show that the positive rational function has a global certificate of positivity in degree k over I. In the following theorem, the linear convergence of the range of a rational function to the enclosure bound under degree elevation is given.

Theorem II.2. *[9, Theorem 5] For $l \leq k$ it holds that*

$$H(F(\boldsymbol{I}), B^{(k)}(f, \boldsymbol{I})) \leq \frac{\delta}{K}, \tag{14}$$

where δ is a constant can be given explicitly not depending on the total degree K.

The Bernstein degree is estimated in the following proposition.

Proposition II.3. *Given f is a positive rational function of degree l over I. If*

$$K > \frac{\delta}{\underline{f}},$$

where δ is the constant in (14), then f satisfies the global certificate of positivity.

Proof. Let $k \geq l$ so that

$$\underline{f} - m^{(k)} \leq \underline{f}.$$

Then $b_i^{(k)}(f)$ are nonnegative. Theorem II.2 implies that

$$\underline{f} - m^{(k)} \leq \frac{\delta}{K},$$

and the interpolation property (8a), (8b), shows that $b_{\hat{i}}$, $0 \leq \hat{i} \leq \hat{k}$, are positive. $\square$

Observing the obtained global certificate of positivity, we give the following corollary.

Corollary 1. *If f is a rational function of degree l positive over I, then there exist $k \geq l$ such that the minimum rational Bernstein coefficient of f of degree k is positive.*

Example II.2. *Let a rational function*

$$f(x) = \frac{5x^2 - 3x + 1}{x^2 + 1} \tag{15}$$

of total degree $L = 2$, which is positive over $[0, 1]$. Note that $\min b_i^{(2)}(f) = -0.5$ is negative. Since $\min b_i^{(3)}(f) = 0$, $b_0^{(3)}(f) = 1$ and $b_3^{(3)}(f) = 1.5$, the rational Bernstein form of $f(x)$ (15) has a local certificate of positivity at $K = 3$.

C. Certificates by the Width of a Box

We aim at deciding if f is positive on any box X and obtaining the certificate of positivity.

Theorem II.4. *[9, Theorem 6] Let $A \in \mathbb{I}(\mathbb{R})^n$ be fixed. Then for all $X \in \mathbb{I}(\mathbb{R})^n$, $X \subseteq A$, and $l \leq k$,*

$$H(F(X), B^{(k)}(f, X)) \leq \ \delta' \ ||w(X)||_\infty^2, \qquad (16)$$

where δ' is a constant which can be given explicitly independent of X.

The interpolation property shows that $b_{\hat{i}}^{(k,X)}(f) > 0$ for all $\hat{i}$, $\hat{i}_\mu \in \{0, k_\mu\}$. The following corollary is an immediate consequence of Theorem II.4 with (11) and the interpolation property.

Corollary 2. *Let f be a rational function of degree l, positive on X. Let $\underline{f}$ be the minimum of the range of f on X. Assume that*

$$||w(X)||_\infty^2 < \frac{\underline{f}}{\delta'}.$$

Then f satisfies the certificate of positivity.

D. Local Certificates

We will not increase the degree in this section. We will choose $k = l$. This will lead to the quadratic convergence with respect to subdivision, and the certificates of positivity are local. We also consider the unit box I. Repeated bisection of $I^{(0,1)} := I$ in all n coordinate directions results at subdivision level $1 \leq d$ in subinterval $I^{(d,\nu)}$ of edge length 2^{-d}, $\nu = 1, \dots, 2^{nd}$, see [7], [8]. The Bernstein coefficients of f over $I^{(d,\nu)}$ are given by $b_i^{(d,\nu)}(f)$. An n-variate polynomial p, $p(x) = \sum_{j=0}^{l} c_j x^j$, can be represented as

$$p(x) = \sum_{i=0}^{l} b_i^{(d,\nu)}(p) \ B_i^{(l,I^{(d,\nu)})}(x), \ x \in I^{(d,\nu)}, \qquad (17)$$

where the *Bernstein coefficients* $b_i^{(d,\nu)}(p)$ of degree l over $I^{(d,\nu)}$ are given by

$$b_i^{(d,\nu)}(p) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} \ c_j^{(d,\nu)}, \ 0 \leq i \leq l, \qquad (18)$$

and

$$c_j^{(d,\nu)} = w(I^{(d,\nu)})^j \sum_{\tau=j}^{l} \binom{\tau}{j} a_\tau \underline{x}_{(d,\nu)}^{\tau-j}. \qquad (19)$$

Assume that

$$B^{(d)}(f) := [\min_{\substack{0 \leq i \leq l, \\ 1 \leq \nu \leq 2^{nd}}} b_i^{(d,\nu)}(f), \ \max_{\substack{0 \leq i \leq l, \\ 1 \leq \nu \leq 2^{nd}}} b_i^{(d,\nu)}(f)].$$

The following theorem provides the quadratic convergence with respect to subdivision.

Theorem II.5. *[9] For each $1 \leq d$, we have*

$$H(F(I), B^{(d)}(f)) \leq \delta''(2^{-d})^2, \qquad (20)$$

where δ'' is a constant which can be given explicitly independent of d.

Definition II.1. *If f satisfies the certificates of positivity $Cert(b^{(d,\nu)}(f))$ for all $\nu = 1, ..., 2^{nd}$, we say that f satisfies the local certificate of positivity associated to the subdivision level of $I^{(d,\nu)}$.*

From (11) and Theorem II.5, we directly get

$$\min_{\substack{0 \leq i \leq l, \\ 1 \leq \nu \leq 2^{nd}}} b_i^{(d,\nu)}(f) - \underline{f} \leq \delta''(2^{-d})^2$$

The following proposition can be similarly proven as Proposition II.3.

Proposition II.6. *Let a polynomial p of total degree l be positive on I. Assume that*

$$2^d > \frac{\sqrt{\delta''}}{\sqrt{\underline{f}}},$$

where $\underline{f}$ is the minimum of the range of f over I. Then f satisfies the local certificate of positivity associated to the subdivision level of $I^{(d,\nu)}$.

Example II.3. *We consider the rational function f (15) over $I = [-1, 1]$. The coefficients of f of degree 2 over subintervals $I_1 = [-1, -1/2]$, $I_2 = [-1/2, 0]$, $I_3 = [0, 1/2]$, $I_4 = [1/2, 1]$ of width $1/2$ are given as follows:*
$b_i^{(I_1)}(f) = (1.3, 0.91, 0.63)$, $b_i^{(I_2)}(f) = (0.63, 0.3, 0.14)$, $b_i^{(I_3)}(f) = (0.14, -0.03, 0.04)$, and $b_i^{(I_4)}(f) = (0.04, 0.12, 0.5)$. The rational Bernstein function still has a negative value over I_3. Therefore, halving the interval I_3 and finding the coefficients over the new subintervals of I_3

$$b_i^{([0,1/4])}(f) = (0.14, 0.05, 0.02) \text{ and } b_i^{([1/4,1/2])}(f) = (0.02, 0, 0.4),$$

satisfy the local certificate of positivity at the second subdivision step.

III. Minimization of Rational Functions

The enclosure bound lead us to a lower bound of f on I. We continue with choosing $k = l$. By repeatedly subdividing I, the minimum of f over I can then be approximated within any desired accuracy. In this section, we will bound the number of subdivision steps of a box. Again, we consider the unit box $I^{(0,1)}$ with subdivision level $1 \leq d$ in subboxes $I^{(d,\nu)}$ of edge length 2^{-d}, $\nu = 1, \dots, 2^{nd}$.

Remark III.1. *(cut-off-test) Let I' be a subbox of I, and f^* an upper bound on the minimum of the range of f over I. If $\min b_i^{(l,I')}(f) > f^*$, then the minimum of f can not occur in I'. Hence, I' can be deleted from the list of subboxes to be subdivided.*

Remark III.2. *Let the minimum rational Bernstein coefficient $\underline{m}^{(l)}$ of f of degree l on $I^{(d,\nu)}$, $\nu = 1, ..., 2^{nd}$, be attained at i_0 and ν_0, $1 \leq \nu_0 \leq 2^{nd}$, with the corresponding grid point $x_{i_0}^{(l,I^{(d,\nu_0)})}$ in $I^{(d,\nu_0)}$. Define the value m^* for $0 \leq i \leq l$ by*

$$m^* = \min\{f(x_{i_0}^{(l,I^{(d,\nu_0)})}), \ b_{\hat{i}}^{(I^{(d,\nu_0)})}(f), \ \hat{i}_\mu \in \{0, l_\mu\}, \ \forall \mu = 1, ..., n\}.$$

Then by (7a), (7b) and (10), one can deduce that

$$\min_{\substack{0\le i\le l,\\ 1\le \nu\le 2^{nd}}} b_i^{(d,\nu)}(f) \le \min_{x\in I} f(x) \le m^*. \tag{21}$$

Theorem III.1. *Let $\epsilon > 0$ be a real number satisfying*

$$\frac{1}{(2^{-d})^2} > \frac{\delta''}{\epsilon}.$$

Then

$$m^* - \min_{\substack{0\le i\le l,\\ 1\le \nu\le 2^{nd}}} b_i^{(d,\nu)}(f) < \epsilon. \tag{22}$$

Proof. Let $x_{i_0}^{(l,I^{(d,\nu_0)})}$ be a grid point in $I^{(d,\nu_0)}$. Then

$$\begin{aligned} m^* - \min_{\substack{0\le i\le l,\\ 1\le \nu\le 2^{nd}}} b_i^{(d,\nu)}(f) &\le |f(x_{i_0}^{(l,I^{(d,\nu_0)})}) - \min_{\substack{0\le i\le l,\\ 1\le \nu\le 2^{nd}}} b_i^{(d,\nu)}(f)| \\ &= |f(x_{i_0}^{(l,I^{(d,\nu_0)})}) - b_{i_0}^{(d,\nu_0)}(f)| \\ &\le (2^{-d})^2 \delta'', \end{aligned}$$

where the last inequality follows by the proof of Theorem II.5. □

Corollary 3. *Under the assumptions of Theorem III.1, a given rational function f on I satisfies the local certificate of positivity if $\underline{f} \ge \epsilon$.*

A. Independent Bound

In this section, we provide a bound does not depend on the number of variables of $f = p/q$. Such this bound is very tight in high dimensions as explained in [10], [14] and [20].

The following theorem extends the bound found in [20] to the tensorial polynomial form.

Theorem III.2. *[19, Theorem 3] Let p be a polynomial of degree l, positive on I. Let $\underline{p}$ be the minimum of p on I. Then for*

$$k > \frac{l(l-1)}{2} \frac{\max |b_i^{(l)}(p)|}{\underline{p}}, \tag{23}$$

the Bernstein form of p of degree k has positive coefficients.

In the following corollary, we hold Theorem III.2, [20, Proposition 4] and results from [19] to the tensorial rational case.

Corollary 4. *Let $f = p/q$ be a positive rational function over I. If*

$$k > \frac{l(l-1)}{2} \frac{\max |b_i^{(l)}(p)|}{\underline{p}},$$

then f satisfies the global certificate of positivity.

Proof. Let k be large enough and $\frac{p(x)}{q(\boldsymbol{x})} > 0 =: a$. Hence, $h(\boldsymbol{x}) := p(\boldsymbol{x}) - aq(\boldsymbol{x}) > 0$ allows from Theorem III.2 the (global) certificate of positivity. Hence, If

$$k > \frac{l(l-1)}{2} \frac{\max |b_i^{(l)}(h)|}{\underline{h}},$$

then $b_i^{(l)}(p) - a.b_i^{(l)}(q) > 0$. It follows that $b_i^{(l)}(p)/b_i^{(l)}(q)$ are positive, $\forall\ 0 \le i \le l$. □

Corollary 5. *Let $D_1 = \frac{\delta}{\min f(x)}$ and $D_2 = \frac{l(l-1)}{2} \frac{\max |b_i^{(l)}(p)|}{\min f(x)}$, where δ is the explicit constant (14). Then the positive rational function $f = p/q$ satisfies the (global) certificate of positivity over I if $k > \max\{D_1, D_2\}$.*

Remark III.3. *Given f is a rational function of degree l, negative on I. Then f satisfies certificates of negativity by applying the same above arguments to the upper bounds.*

IV. Conclusions

In this paper, we considered the multivariate rational function $f = p/q$ in the tensorial Bernstein form. The expansion of the numerator and denominator polynomials into Bernstein polynomials was applied. The linear and quadratic convergence of the enclosure bound to the range of a rational function f improved the bounds of f over boxes. By repeatedly subdividing the box, the minimum of f over a box was approximated within a desired accuracy. We addressed a minimization of a (multivariate) rational function and bounded the number of subdivision steps. Subsequently, we extended certificates of positivity to the tensorial rational Bernstein form by sharpness, degree elevation and subdivision. Finally, we estimated the degree of the Bernstein expansion by a bound which is not depending on the given dimension.

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