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REDUCTION OF ORDER OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the solution of the second order fractional differential equation of the form $F(x,y,y^{(\alpha)},y^{(2\alpha)})=0$, in case either x is missing or in case y is missing.

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1. Introduction

There are many definitions available in the literature for fractional derivatives. The main ones are the Riemann Liouville definition and the Caputo definition, see [7] .

(i) Riemann - Liouville Definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

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(ii) Caputo Definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Such definitions have many setbacks such as

(i) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ ($D_a^{\alpha}(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^{\alpha}(fg) = fD_a^{\alpha}(g) + gD_a^{\alpha}(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^{\alpha}(f/g) = \frac{gD_a^{\alpha}(f) - fD_a^{\alpha}(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^{\alpha}(f \circ g)(t) = f^{(\alpha)}(g(t))\, g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^{\alpha}D^{\beta} f = D^{\alpha+\beta} f$, in general.

(vi) All fractional derivatives, specially Caputo definition, assumes that the function f is differentiable.

We refer the reader to [7] for more results on Caputo and Riemann - Liouville Definitions.

Recently, the authors in [5], gave a new definition of fractional derivative which is a natural extension to the usual first derivative. So many papers since then were written, and many equations were solved using such definition. We refer to [1-6] and references there in for recent results on conformable fractional derivative. The definition goes as follows:

Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. Then for all $t > 0$, $\alpha \in (0,1)$, let

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

T_{α} is called the **conformable fractional derivative of f of order α**.

Let $f^{(\alpha)}(t)$ stands for $T_{\alpha}(f)(t)$.

If f **is** $\alpha-$**differentiable in some** $(0,b),\ b>0$**, and** $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, **then define**

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t).$$

According to this definition, we have the following properties, see [5],

1. $T_\alpha(1)=0$,

2. $T_\alpha(t^p)=pt^{p-\alpha}$ for all $p\in\mathbb{R}$,

3. $T_\alpha(\sin at)=at^{1-\alpha}\cos at, \quad a\in\mathbb{R}$,

4. $T_\alpha(\cos at)=-at^{1-\alpha}\sin at, \quad a\in\mathbb{R}$

5. $T_\alpha(e^{at})=at^{1-\alpha}e^{at}, \quad a\in\mathbb{R}$.

Further, many functions behave as in the usual derivative. Here are some formulas:

$T_\alpha(\frac{1}{\alpha}t^\alpha)=1$

$T_\alpha(e^{\frac{1}{\alpha}t^\alpha})=e^{\frac{1}{\alpha}t^\alpha}$,

$T_\alpha(\sin\frac{1}{\alpha}t^\alpha)=\cos(\frac{1}{\alpha}t^\alpha)$,

$T_\alpha(\cos\frac{1}{\alpha}t^\alpha)=-\sin(\frac{1}{\alpha}t^\alpha)$.

2. Main Result

Consider a second order fractional differential equation of the form:

$$F(x,y,y^{(\alpha)},y^{(2\alpha)})=0\,, \tag{1}$$

where $y^{(\alpha)}$ is the $\alpha-$ conformable derivative of y with respect to x and $\alpha\in(0,1]$, and $y^{(2\alpha)}=D^\alpha D^\alpha y.$ Often, equation is not a standard equation in the sense it is not of any type that we can handle.

The object of this paper is to try to solve equation (1) in case either x is missing or y is missing using what we will call fractional reduction of order.

There are two cases to be considered:

(i) $F(x,y^{(\alpha)},y^{(2\alpha)})=0$, $\qquad y$ is missing

$(ii)\ F(y, y^{(\alpha)}, y^{(2\alpha)}) = 0$, x is missing

Case(i): y is missing

In this case put $y^{(\alpha)} = u$. Consequently, we get $y^{(2\alpha)} = u^{(\alpha)}$. This reduces the order of the equation from 2α to order α, which is much easier to handle.

Examples:

1- $y^{(2\alpha)} - (y^{(\alpha)})^2 = 1$.

This equation is not a linear equation. However, here y is missing. So put $y^{(\alpha)} = u$ and consequently, $y^{(2\alpha)} = u^{(\alpha)}$. The equation becomes

$$u^{(\alpha)} = u^2 + 1$$

which is a separable differential equation that can be solved as follows:

Since $u^{(\alpha)} = x^{1-\alpha}\dfrac{du}{dx}$, [3], the equation $u^{(\alpha)} = u^2 + 1$ can be written as:

$$x^{1-\alpha}\frac{du}{dx} = 1 + u^2$$

Thus $\tan^{-1}(u) = \frac{1}{\alpha}x^{\alpha} + a$. Consequently, $u = \tan\left(\frac{1}{\alpha}x^{\alpha} + a\right)$. Replacing u by $y^{(\alpha)}$ and then substituting $y^{(\alpha)}$ by $x^{(1-\alpha)}\frac{dy}{dx}$ and integrating we get:

$$y = -\ln\left|\cos(\frac{1}{\alpha}x^{\alpha} + a)\right| + b, \quad a,\ b \text{ are constants}.$$

2- $4x^{\alpha-1}(\cos x)y^{(2\alpha)} - (\sin x)(y^{(\alpha)})^2 = 4\sin x$

Here y is missing. Hence put $y^{(\alpha)} = u$. Then $y^{(2\alpha)} = u^{(\alpha)}$. The equation becomes

$$4x^{\alpha-1}(\cos x)\,u^{(\alpha)} - (\sin x)u^2 - 4\sin x = 0$$

which is a separable differential equation:

$$\frac{1}{u^2+4}\,du = \frac{\sin x}{\cos x}\,dx,$$

which can be solved to get

$$u = 2\tan 2\big(c_1 - \ln|\cos x|\big).$$

Replacing u by $y^{(\alpha)}$ and then substituting $y^{(\alpha)}$ by $x^{(1-\alpha)}\frac{dy}{dx}$, we get:

$$y = \int 2x^{\alpha-1}\tan 2\big(c_1 - \ln|\cos x|\big)\,dx$$

Case (ii): x is missing ,

In this case we put $y^{(\alpha)} = u$. Then $y^{(2\alpha)} = D_x^{\alpha} u = x^{1-\alpha}\, D_y u\, D_x y = y^{(\alpha)}\, D_y u = u D_y u,$ where $D_x^{\alpha} u$ represents the fractional derivative of u with respect to x. Similarly for y.

This reduces the equation to a new equation of lower order in u and y.

Example: Consider $y\, y^{(2\alpha)} + (\, y^{\alpha})^2 = 0$.

Put $y^{(\alpha)} = u$. So $y^{(2\alpha)} = y^{(\alpha)}\, \frac{du}{dy}$

Hence

$$yy^{(\alpha)}\frac{du}{dy} + u^2 = yu\frac{du}{dy} + u^2 = 0.$$

Solving this equation to get:

$yu = b$. Thus $yy^{(\alpha)} = b$,

This is again a separable equation in the form $y\,dy = bx^{\alpha-1}\,dx$. Solving that equation to get:

$$y^2 = \frac{2b}{\alpha}x^{\alpha} + c\,, \quad b\,,\, c \text{ are constants}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math. 279 (2015), 57–66.

[2] M. Abu Hammad and R. Khalil, Abel's formula and Wronskian for conformable fractional differential equations, Int. J. Differ. Equ. Appl. 13(3) (2014), 177–183.

[3] M. Abu Hammad and R. Khalil, Conformable fractional heat differential equation, Int. J. Pure. Appl. Math. 94(2)(2014), 215–221.

[4] M. Abu Hammad and R. Khalil, Fractional fourier series with applications, Amer. J. Comput. Appl. Math. 4(6)

[5] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., 264 (2014), 65-70.

[6] R. Khalil, M. Al Horani, and D. Anderson, Undetermined coefficients for local fractional differential equations, J. Math. Comput. Sci. 16 (2016), 140–146.

[7] A. Kilbas, H. Srivastava, J. Trujillo, Theory and applications of fractional differential equations, Elsevier Science B.V., Amsterdam, (2006).