RULED SURFACES OF FINITE II-TYPE

HASSAN AL-ZOUBI, AMER DABABNEH, AND MUTAZ AL-SABBAGH

ABSTRACT. In this paper, we consider surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ without parabolic points which are of finite II-type, that is, they are of finite type, in the sense of B.-Y. Chen, with respect to the second fundamental form. We study an important family of surfaces, namely, ruled surfaces in $\mathbb{E}^3$. We show that ruled surfaces are of infinite II-type.

1. Introduction

Euclidean immersions of finite type were introduced by B.-Y. Chen about four decades ago and it has been a topic of active research by many differential geometers since then. Many results on this subject have been collected in [6]. A submanifold M^m is said to be of finite type with respect to the first fundamental form I, if the position vector $\boldsymbol{x}$ of M^m can be written as a finite sum of nonconstant eigenvectors of the Laplacian Δ^I, that is,

$$\boldsymbol{x} = \boldsymbol{x}_0 + \sum_{i=1}^{k} \boldsymbol{x}_i, \tag{1.1}$$

where $\Delta^I \boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i, i = 1, ..., k$, $\boldsymbol{x}_0$ is a constant vector and $\lambda_1, \lambda_2, ..., \lambda_k$ are eigenvalues of Δ^I. Moreover, if there are exactly k nonconstant eigenvectors $\boldsymbol{x}_1, ..., \boldsymbol{x}_k$ appearing in (1.1) which all belong to different eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$, then M^m is said to be of I-type k. However, if $\lambda_i = 0$ for some $i = 1, ..., k$, then M^m is said to be of null I-type k , otherwise M^m is said to be of infinite type.

The class of finite type submanifolds in an arbitrary dimensional Euclidean space is very large, meanwhile very little is known about surfaces of finite type in the Euclidean 3-space with respect to the first fundamental form. Actually, so far, the only known surfaces of finite type in the Euclidean 3-space are the minimal surfaces, the circular cylinders and the spheres. So in [5] B.-Y. Chen mentions the following problem

Problem 1. *Determine all surfaces of finite Chen I-type in $\mathbb{E}^3$.*

With the aim of getting an answer to this problem, important families of surfaces were studied by different authors by proving that finite type ruled surfaces [8], finite type quadrics [9], finite type tubes [7], finite type cyclides of Dupin [10, 11], finite type cones [12], and finite type spiral surfaces [4] are the only known examples of surfaces in $\mathbb{E}^3$. However, for other classical families of surfaces, such as surfaces of revolution, translation surfaces as well as helicoidal surfaces, the classification of its finite type surfaces is not known yet.

2010 *Mathematics Subject Classification.* 53A05.
Key words and phrases. Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Ruled surface, Beltrami operator.

In this area, S. Stamatakis and H. Al-Zoubi restored attention to this theme by introducing the notion of surfaces of finite type with respect to the second or third fundamental forms (see [15]). As an extension of the above problem, we raise the following two questions which seem to be interesting:

Problem 2. *Classify all surfaces of finite II-type in $\mathbb{E}^3$.*

Problem 3. *Classify all surfaces of finite III-type in $\mathbb{E}^3$.*

Therefore, in order to give an answer to the second and third problem, it is worthwhile investigating the classification of surfaces in the Euclidean space $\mathbb{E}^3$ in terms of finite J-type, $(J = II, III)$ by studying the families of surfaces mentioned above.

According to problem (2), in [1] H. Al-Zoubi studied finite type tubes with respect to the second fundamental form and he proved that: All tubes in $\mathbb{E}^3$ are of infinite type. However, for all other classical families of surfaces, the classification of its finite type surfaces is not known yet.

Concerning problem (3), ruled surfaces and tubes are the only families studied according to its finite type classification. More specifically, in [3] authors have shown that all tubes in $\mathbb{E}^3$ are of infinite type, while in [2], H. Al-Zoubi and others proved that: Helicoids are the only ruled surfaces of finite III-type in the 3-dimensional Euclidean space.

In this paper we will pay attention to surfaces of finite II-type. First, we will establish a formula for $\Delta^{II}\boldsymbol{x}$ and $\Delta^{II}\boldsymbol{n}$ by using tensors calculations. Further, we continue our study by proving finite type surfaces for an important class of surfaces, namely, ruled surfaces in the Euclidean 3-space.

2. Preliminaries

In the three-dimensional Euclidean space $\mathbb{E}^3$ let S be a C^r-surface, $r \geq 3$, defined on a region U of $\mathbb{R}^2$, by an injective C^r-immersion $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$, whose Gaussian curvature K never vanishes. We denote by

$$I = g_{ij}\, du^i du^j, \quad II = b_{ij}\, du^i du^j, \quad III = e_{ij}\, du^i du^j, \quad i, j = 1, 2,$$

the first, second and third fundamental forms of S respectively. For two sufficiently differentiable functions $f(u^1, u^2)$ and $g(u^1, u^2)$ on S, the first differential parameter of Beltrami with respect to the fundamental form $J = I, II, III$ is defined by [13]

$$\nabla^J(f, g) := a^{ij} f_{/i} g_{/j} \tag{2.1}$$

where $f_{/i} := \frac{\partial f}{\partial u^i}$, and (a^{ij}) denotes the inverse tensor of $(g_{ij}), (b_{ij})$ and (e_{ij}) for $J = I, II$ and III respectively. The second differential parameter of Beltrami with respect to the fundamental form $J = I, II, III$ of S is defined by [13]

$$\triangle^J f := -a^{ij} \nabla^J_i f_{/j}, \tag{2.2}$$

where ∇^J_i is the covariant derivative in the u^i direction with respect to the fundamental form J and (a^{ij}) stands, as in definition (2.1), for the inverse tensor of $(g_{ij}), (b_{ij})$ and (e_{ij}) for $J = I, II$ and III respectively.

We first mention the following two relations for later use [15]:

$$\nabla^{II}(f, \boldsymbol{n}) + grad^I f = 0, \tag{2.3}$$

$$\nabla^{II}(f, \boldsymbol{x}) + grad^{III} f = 0, \tag{2.4}$$

where $grad^I f := \nabla^I(f, \boldsymbol{x})$, $grad^{III} f := \nabla^{III}(f, \boldsymbol{n})$ and $\boldsymbol{n}$ denotes the Gauss map of S.

Applying (2.2) for the position vector $\boldsymbol{x}$ of S we have

$$\triangle^{II}\boldsymbol{x} = -b^{ij}\nabla^{II}_j \boldsymbol{x}_{/i}. \tag{2.5}$$

Recalling the equations

$$\nabla^{II}_j \boldsymbol{x}_{/i} = -\frac{1}{2}b^{kr}(\nabla^I_k b_{ij})\boldsymbol{x}_{/r} + b_{ij}\boldsymbol{n},$$

(see [13], p.128) and inserting these into (2.5), one finds

$$\triangle^{II}\boldsymbol{x} = \frac{1}{2}b^{kr}b^{ij}(\nabla^I_k b_{ij})\boldsymbol{x}_{/r} - b^{ij}b_{ij}\boldsymbol{n}, \tag{2.6}$$

By using the Mainardi-Codazzi equations (see [13], p.128)

$$\nabla^I_k b_{ij} - \nabla^I_i b_{jk} = 0, \tag{2.7}$$

relation (2.6) becomes

$$\triangle^{II}\boldsymbol{x} = \frac{1}{2}b^{kr}b^{ij}\nabla^I_i b_{jk}\boldsymbol{x}_{/r} - 2\boldsymbol{n}. \tag{2.8}$$

We consider the Christoffel symbols of the second kind with respect to the first, second and third fundamental form, respectively

$$\Gamma^k_{ij} := \frac{1}{2}g^{kr}(-g_{ij/r} + g_{ir/j} + g_{jr/i}),$$

$$\Pi^k_{ij} := \frac{1}{2}b^{kr}(-b_{ij/r} + b_{ir/j} + b_{jr/i}),$$

$$\Lambda^k_{ij} := \frac{1}{2}e^{kr}(-e_{ij/r} + e_{ir/j} + e_{jr/i}),$$

and we put

$$T^k_{ij} := \Gamma^k_{ij} - \Pi^k_{ij}, \tag{2.9}$$

$$\widetilde{T}^k_{ij} := \Lambda^k_{ij} - \Pi^k_{ij}. \tag{2.10}$$

It is known that (see [13], p.22)

$$T^k_{ij} := -\frac{1}{2}b^{kr}\nabla^I_r b_{ij}, \tag{2.11}$$

$$\widetilde{T}^k_{ij} := -\frac{1}{2}b^{kr}\nabla^{III}_r b_{ij}, \tag{2.12}$$

and

$$\widetilde{T}^k_{ij} + T^k_{ij} = 0. \tag{2.13}$$

Using (2.9) and (2.11), relation (2.8) becomes

$$\triangle^{II}\boldsymbol{x} = -b^{kr}T^j_{kj}\boldsymbol{x}_{/r} - 2\boldsymbol{n} = -b^{kr}(\Gamma^j_{kj} - \Pi^j_{kj})\boldsymbol{x}_{/r} - 2\boldsymbol{n}. \tag{2.14}$$

For the Christoffel symbols Γ^j_{kj} and Π^j_{kj} we have (see [13], p.125)

$$\Gamma^j_{ij} := \frac{g_{/i}}{2g}, \qquad \Pi^j_{ij} := \frac{b_{/i}}{2b}, \tag{2.15}$$

where $g := det(g_{ij})$ and $b := det(b_{ij})$. Thus, relation (2.14) becomes

$$\triangle^{II}\boldsymbol{x} = -\frac{1}{2}b^{kr}(\frac{g_{/k}}{g} - \frac{b_{/k}}{b})\boldsymbol{x}_{/r} - 2\boldsymbol{n}. \tag{2.16}$$

On the other hand, the Gauss curvature K of S is given by

$$K = \frac{b}{g}.$$

Once, we have

$$\frac{K_{/k}}{K} = \frac{b_{/k}}{b} - \frac{g_{/k}}{g}, \tag{2.17}$$

it follows that

$$\triangle^{II}\boldsymbol{x} = \frac{1}{2K} b^{kr} K_{/k}\boldsymbol{x}_{/r} - 2\boldsymbol{n} = \frac{1}{2K}\nabla^{II}(K, \boldsymbol{x}) - 2\boldsymbol{n}. \tag{2.18}$$

Hence, we obtain, in view of (2.4), the following relation

$$\triangle^{II}\boldsymbol{x} = -\frac{1}{2K} grad^{III}(K) - 2\boldsymbol{n}. \tag{2.19}$$

We focus now our interest to the computation of $\triangle^{II}\boldsymbol{n}$. Taking into consideration the equations ([13], p.128)

$$\nabla_i^{II}\boldsymbol{n}_{/j} = -\frac{1}{2} b^{kr}(\nabla_r^{III} b_{ij})\boldsymbol{n}_{/k} - e_{ij}\boldsymbol{n},$$

so that

$$\triangle^{II}\boldsymbol{n} = -b^{ij}(\nabla_i^{II})\boldsymbol{n}_{/j},$$

takes the form

$$\triangle^{II}\boldsymbol{n} = \frac{1}{2} b^{kr} b^{ij}(\nabla_r^{III} b_{ij})\boldsymbol{n}_{/k} + b^{ij} e_{ij}\boldsymbol{n}.$$

On account of

$$2H = b_{ij} g^{ij} = e_{ij} b^{ij},$$

and (2.12) we obtain

$$\triangle^{II}\boldsymbol{n} = -b^{ij}\widetilde{T}_{ij}^{k}\boldsymbol{n}_{/k} + 2H\boldsymbol{n}.$$

On use of (2.7), (2.11) and (2.13) we have

$$\triangle^{II}\boldsymbol{n} = b^{kr} T_{rj}^{j}\boldsymbol{n}_{/k} + 2H\boldsymbol{n}. \tag{2.20}$$

On the other hand using (2.9), (2.11), (2.15) and (2.17) we have

$$b^{kr} T_{rj}^{j}\boldsymbol{n}_{/k} = -\frac{1}{2K} b^{kr} K_{/r}\boldsymbol{n}_{/k} = -\frac{1}{2K}\nabla^{II}(K, \boldsymbol{n}).$$

Inserting this in (2.20) we get in view of (2.3)

$$\triangle^{II}\boldsymbol{n} = \frac{1}{2K} grad^{I}(K) + 2H\boldsymbol{n}. \tag{2.21}$$

We now prove the following two relations:

$$\triangle^{II}(f\boldsymbol{x}) = (\triangle^{II} f)\boldsymbol{x} + f\triangle^{II}\boldsymbol{x} + 2grad^{III} f. \tag{2.22}$$

$$\triangle^{II}(f\boldsymbol{n}) = (\triangle^{II} f)\boldsymbol{n} + f\triangle^{II}\boldsymbol{n} + 2grad^{I} f. \tag{2.23}$$

For the proof of (2.22) we use (2.2) to obtain

$$\begin{aligned}
\triangle^{II}(f\boldsymbol{x}) &= -b^{ik}\nabla_k^{II}(f\boldsymbol{x})_{/i} = -b^{ik}\nabla_k^{II}(f_{/i}\boldsymbol{x} + f\boldsymbol{x}_{/i}) \\
&= -(b^{ik}\nabla_k^{II} f_{/i})\boldsymbol{x} - b^{ik} f_{/i}\nabla_k^{II}\boldsymbol{x} - b^{ik}(\nabla_k^{II} f)\boldsymbol{x}_{/i} - b^{ik} f(\nabla_k^{II}\boldsymbol{x}_{/i}) \\
&= (\triangle^{II} f)\boldsymbol{x} + f\triangle^{II}\boldsymbol{x} - 2b^{ik} f_{/i}\boldsymbol{x}_{/k} \\
&= (\triangle^{II} f)\boldsymbol{x} + f\triangle^{II}\boldsymbol{x} - 2\nabla^{II}(f, \boldsymbol{x}).
\end{aligned}$$

On account of (2.4) we arrive to (2.22).

We have similarly

$$\begin{aligned}
\triangle^{II}(f\boldsymbol{n}) &= -b^{ik}\nabla^{II}_k(f\boldsymbol{n})_{/i} = -b^{ik}\nabla^{II}_k(f_{/i}\boldsymbol{n} + f\boldsymbol{n}_{/i}) \\
&= -(b^{ik}\nabla^{II}_k f_{/i})\boldsymbol{n} - b^{ik}f_{/i}\nabla^{II}_k\boldsymbol{n} - b^{ik}(\nabla^{II}_k f)\boldsymbol{n}_{/i} - b^{ik}f(\nabla^{II}_k\boldsymbol{n}_{/i}) \\
&= (\triangle^{II}f)\boldsymbol{n} + f\triangle^{II}\boldsymbol{n} - 2b^{ik}f_{/i}\boldsymbol{n}_{/k} \\
&= (\triangle^{II}f)\boldsymbol{n} + f\triangle^{II}\boldsymbol{n} - 2\nabla^{II}(f,\boldsymbol{n}),
\end{aligned}$$

which is (2.23) in view of (2.3).

From (2.19) and (2.21) we obtain the following two results which were proved in [15]:

Theorem 1. *A surface S in $\mathbb{E}^3$ is of finite II-type 1 if and only if S is part of a sphere.*

Theorem 2. *The Gauss map of a surface S in $\mathbb{E}^3$ is of finite II-type 1 if and only if S is part of a sphere.*

Up to now, the only known surfaces of finite II-type in $\mathbb{E}^3$ are parts of spheres. In the next section we focus our attention on the class of ruled surfaces. Our main result is the following

Theorem 3. *All ruled surfaces in the three-dimensional Euclidean space are of infinite II-type.*

3. Proof of Theorem 3

In the three-dimensional Euclidean space $\mathbb{E}^3$ let S be a ruled C^r-surface, $r \geq 3$, of nonvanishing Gaussian curvature defined by an injective C^r-immersion $\boldsymbol{x} = \boldsymbol{x}(s,t)$ on a region $U := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) of $\mathbb{R}^2$.[1] The surface S can be expressed in terms of a directrix curve $\Gamma\colon \boldsymbol{\sigma} = \boldsymbol{\sigma}(s)$ and a unit vector field $\boldsymbol{\rho}(s)$ pointing along the rulings as follows

$$S\colon \boldsymbol{x}(s,t) = \boldsymbol{\sigma}(s) + t\,\boldsymbol{\rho}(s), \quad s \in I, t \in \mathbb{R}. \tag{3.1}$$

Moreover, we can take the parameter s to be the arc length along the spherical curve $\boldsymbol{\rho}(s)$. Then we have

$$\langle \boldsymbol{\sigma}', \boldsymbol{\rho} \rangle = 0, \quad \langle \boldsymbol{\rho}, \boldsymbol{\rho} \rangle = 1, \quad \langle \boldsymbol{\rho}', \boldsymbol{\rho}' \rangle = 1,$$

where the differentiation with respect to s is denoted by a prime and $\langle\, ,\rangle$ denotes the standard scalar product in $\mathbb{E}^3$. It is easily verified that the first and the second fundamental forms of S are given by

$$I = n\,ds^2 + dt^2,$$
$$II = \frac{m}{\sqrt{n}}\,ds^2 + \frac{2A}{\sqrt{n}}\,ds\,dt,$$

where

$$\begin{aligned}
n &= \langle \boldsymbol{\sigma}', \boldsymbol{\sigma}' \rangle + 2\langle \boldsymbol{\sigma}', \boldsymbol{\rho}' \rangle t + t^2, \\
m &= (\boldsymbol{\sigma}', \boldsymbol{\rho}, \boldsymbol{\sigma}'') + [(\boldsymbol{\sigma}', \boldsymbol{\rho}, \boldsymbol{\rho}'') + (\boldsymbol{\rho}', \boldsymbol{\rho}, \boldsymbol{\sigma}'')]\,t + (\boldsymbol{\rho}', \boldsymbol{\rho}, \boldsymbol{\rho}'')\,t^2, \\
A &= (\boldsymbol{\sigma}', \boldsymbol{\rho}, \boldsymbol{\rho}')\,.
\end{aligned}$$

[1] The reader is referred to [14] for definitions and formulae on ruled surfaces.

If, for simplicity, we put

$$\zeta := \langle \boldsymbol{\sigma}', \boldsymbol{\sigma}' \rangle, \qquad \eta := \langle \boldsymbol{\sigma}', \boldsymbol{\rho}' \rangle,$$
$$\mu := (\boldsymbol{\rho}', \boldsymbol{\rho}, \boldsymbol{\rho}''), \quad \nu := (\boldsymbol{\sigma}', \boldsymbol{\rho}, \boldsymbol{\rho}'') + (\boldsymbol{\rho}', \boldsymbol{\rho}, \boldsymbol{\sigma}''), \quad \xi := (\boldsymbol{\sigma}', \boldsymbol{\rho}, \boldsymbol{\sigma}''),$$

we have

$$n = t^2 + 2\eta\, t + \zeta, \quad m = \mu\, t^2 + \nu\, t + \xi.$$

For the Gauss curvature K of S we find

$$K = -\frac{A^2}{n^2}.$$

The second Beltrami differential operator with respect to the second fundamental form after a long computation is given by [16]

$$\triangle^{II} = -\frac{\sqrt{n}}{A}\Big(-2\frac{\partial^2}{\partial s \partial t} + \frac{m}{A}\frac{\partial^2}{\partial t^2} + \frac{m_t}{A}\frac{\partial}{\partial t}\Big), \tag{3.2}$$

where $m_t := \frac{\partial m}{\partial t}$.

Applying (3.2) for the position vector $\boldsymbol{x}$, it follows:

$$\triangle^{II}\boldsymbol{x} \quad = \quad -\frac{1}{\sqrt{n}}\Big(-\frac{2n}{A}\boldsymbol{\rho}' + \frac{n m_t}{A^2}\boldsymbol{\rho}\Big) = \frac{1}{\sqrt{n}}\boldsymbol{P_1(t)} \tag{3.3}$$

where $\boldsymbol{P_1(t)}$ is a vector whose components are polynomials in t of degree less than or equal 3 with functions in s as coefficients. More precisely, we have

$$\boldsymbol{P_1(t)} = \frac{1}{A^2}\big[2\mu\boldsymbol{\rho}t^3 + \big((4\mu\eta+\nu)\boldsymbol{\rho} + 2A\boldsymbol{\rho}'\big)t^2 + \big((2\zeta\mu+2\eta\nu)\boldsymbol{\rho} + 4\eta A\boldsymbol{\rho}'\big) + (\zeta\eta\boldsymbol{\rho} + 2\zeta A\boldsymbol{\rho}')\big].$$

Before we start the proof of our theorem we give the following Lemma which can be proved by a straightforward computation.

Lemma 1. *Let g be a polynomial in t with functions in s as coefficients and $\deg(g) = d$. Then $\triangle^{II}\big(\frac{g}{n^r}\big) = \frac{\widehat{g}}{n^{r+\frac{3}{2}}}$, where $\widehat{g}$ is a polynomial in t with functions in s as coefficients and $\deg(\widehat{g}) \le d + 4$.*

From now on we suppose that S is of finite $II-type$ k. Hence there exist real numbers $c_1, c_2, \cdots, c_k$ such that

$$\left(\triangle^{II}\right)^{k+1}\boldsymbol{x} + c_1\left(\triangle^{II}\right)^{k}\boldsymbol{x} + \cdots + c_k\,\triangle^{II}\boldsymbol{x} = \boldsymbol{0}, \tag{3.4}$$

see [3]. By applying Lemma 1, we conclude that there is an $\mathbb{E}^3$-vector-valued function $\boldsymbol{P}_k$ in the variable t with some functions in s as coefficients, such that

$$\left(\triangle^{II}\right)^{k}\boldsymbol{x} = \boldsymbol{P}_k(t),$$

where $\deg(\boldsymbol{P}_k) \le 4k-1$ and $r = \frac{3}{2}k - 1$. Now, if k goes up by one, the degree of each component of $\boldsymbol{p}_k$ goes up at most by 4 while the degree of the denominator goes up by $\frac{3}{2}k - 1$. Therefore, the sum (3.4) can never be zero, unless of course

$$\triangle^{II}\boldsymbol{x} = \boldsymbol{P}_1 = \boldsymbol{0}. \tag{3.5}$$

But then

$$-2\boldsymbol{\rho}' + \frac{m_t}{A}\boldsymbol{\rho} = \boldsymbol{0}. \tag{3.6}$$

By taking the derivative of $\langle \boldsymbol{\rho}, \boldsymbol{\rho} \rangle = 1$, we observe that the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are linearly independent. Thus (3.6) cannot be achieved unless $\boldsymbol{\rho}$ is constant, which implies that $K \equiv 0$. This is clearly impossible for the surfaces under consideration. The proof of the theorem is completed.

References

[1] H. Al-Zoubi, Tubes of finite II-type in the Euclidean 3-space, Wseas Transaction on Math. **17** (2018), 1-5.

[2] H. Al-Zoubi, S. Al-Zubi, S. Stamatakis and H. Almimi, Ruled surfaces of finite Chen-type. J. Geom. and Graphics, accepted.

[3] H. Al-Zoubi, K. M. Jaber, S. Stamatakis, Tubes of finite Chen-type, Commun. Korean Math. Soc., **33** (2018), 581-590.

[4] Ch. Baikoussis, L. Verstraelen, The Chen-type of the spiral surfaces, Results. Math. **28** (1995), 214-223.

[5] B.-Y. Chen, Some open problems and conjectures on submanifolds of finite type, Soochow J. Math. **17** (1991), 169-188.

[6] B.-Y. Chen, Total mean curvature and submanifolds of finite type. Second edition, World Scientific Publisher, (2014).

[7] B.-Y. Chen, Surfaces of finite type in Euclidean 3-space, Bull.Soc. Math. Belg. **39** (1987), 243-254.

[8] B.-Y. Chen, F. Dillen, L. Verstraelen, L. Vrancken, Ruled surfaces of finite type, Bull. Austral. Math. Soc. **42** (1990), 447-553.

[9] B.-Y. Chen, F. Dillen, Quadrics of finite type, J. of Geom. **38** (1990), 16-22.

[10] F. Denever, R. Deszcz, L. Verstraelen, The compact cyclides of Dupin and a conjecture by B.-Y Chen, J. of Geom. **46** (1993), 33-38.

[11] F. Denever, R. Deszcz, L. Verstraelen, The Chen type of the noncompact cyclides of Dupin, Glasg. Math. J. **36** (1994), 71-75.

[12] O. Garay, Finite type cones shaped on spherical submanifolds, Proc. Amer. Math. Soc. **104** (1988), 868-870.

[13] H. Huck, U. Simon, R. Roitzsch, W. Vortisch, R. Walden, B. Wegner, and W. Wendl, Beweismethoden der Differentialgeometrie im Grossen, Lecture Notes in Mathematics. Vol. **335** (1973).

[14] H. Pottmann, J. Wallner, Computational Line Geometry. Springer-Verlag (2001).

[15] S. Stamatakis, H. Al-Zoubi, On surfaces of finite Chen-type, Results. Math. **43** (2003), 181-190.

[16] D. W. Yoon, Ruled surfaces whose mean curvature vector is an eigenvector of the Laplacian of the second fundamental form, International Mathematical Forum **1** (2006), 1783-1788.

Department of Mathematics, Al-Zaytoonah University of Jordan, P.O. Box 130, Amman, Jordan 11733

E-mail address: dr.hassanz@zuj.edu.jo

Department of Mathematics, Al-Zaytoonah University of Jordan

E-mail address: dababneh.amer@zuj.edu.jo

Department of Basic Sciences and Humanities, Imam Abdulrahman bin Faisal University

E-mail address: malsbbagh@iau.edu.sa