A New Impulsive Sequential Multi-Orders Fractional Differential Equation Involving Multipoint Fractional Integral Boundary Conditions

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Abstract

A new impulsive sequential multi-orders fractional differential equation is studied. The existence and uniqueness results are obtained for a nonlinear problem with fractional integral boundary conditions applying standard fixed point theorems. An example for the illustration of the main result is given.

1 Introduction

Nowadays, fractional differential equations have attracted a lot of attention due to its wide range of applications in many practical problems such as in physics, engineering, economics, and so on; see [1–5]. Impulsive sequential differential equations have extensively been studied in the past two decades. Indeed impulsive differential equations are used to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in harvesting, earthquakes, diseases, and so forth. Recently, fractional impulsive differential equations have attracted the attention of many researchers. For the general theory and applications of such equations we refer the interested reader to see [6] and [7–18] and the references therein. Based on previous studies, in this topic we concentrate on the existence results of solutions of the following problem:

$$
\begin{gathered}
({}^cD^{\beta_k}_{t_k^+} + \lambda {}^cD^{\beta_k - 1}_{t_k^+})x(t) = f\left(t, x(t)\right), 1 < \beta_k \leq 2, k = 0, 1, ...q; t \in J', \\
\Delta x(t_k) = \psi_k(x(t_k)), \Delta x'(t_k) = \psi^*_k(x(t_k)), k = 1, ..., q, \\
x(0) = \sum_{k=0}^{q} \lambda_k \mathbf{I}^{\alpha_k}_{t_k^+} x\left(\eta_k\right), x'(0) = 0.
\end{gathered} \tag{1}
$$

where ${}^c\mathrm{D}^{\beta_k}_{t_k^+}$ is the Caputo fractional derivative of order $\beta_k \in (1, 2]$ and $\mathbf{I}^{\alpha_k}_{t_k^+}$is fractional Riemann-Liouville integral of order $\alpha_k > 0$, $f \in (J \times \mathbb{R}, \mathbb{R})$, $\psi_k, \psi^*_k \in C\left(\mathbb{R} \times \mathbb{R}\right)$, $\lambda_k \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, $J = [0, T]$, $J' = J \backslash \{t_1, ..., t_q\}$, $0 = t_0 < t_1 < \cdots < t_k ... < t_q < t_{q+1} = T$, $\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right)$ and $\Delta x'(t_k) = x^{'}\left(t_k^+\right) - x'\left(t_k^-\right)$. Here, respectively, the right and the left limits of $x(t)$ at $t = t_k^+$ are represented by $x(t_k^+)$ and $x(t_k^-)$.

2 Basic materials

We introduce throughout this section preliminary facts that will be used in this paper. We fix $J_0 = [0, t_1]$, $J_{k-1} = (t_{k-1}, t_k]$, and $k = 1, 2, ..., q + 1$ with $t_{q+1} = T$ and define the Banach space

$$PC(J) = \left\{x : J \to \mathbb{R} \mid x \in C(J'), \text{and } x(t_k^+), x(t_k^-) \text{ exist, and } x(t_k^+) = x(t_k), 1 \leq k \leq q\right\},$$

with the norm $\|x\|_{PC} = \sup \{|x(t)| : t \in J\}$.

Definition 1 *The fractional integral of order β with the lower limit zero for a function $f : [x, \infty) \to \mathbb{R}$ is defined as*

$$\mathbf{I}^{\beta}_{x^+} f(t) = \frac{1}{\Gamma(\beta)} \int_x^t (t - s)^{\beta - 1} f(s) ds, \ t > 0 \ , \beta > 0,$$

provided the right side is point-wise defined on $[x, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2 *The Caputo fractional derivative of order $\beta > 0$, of a function $f : [x, \infty) \to \mathbb{R}$ can be written as*

$$\mathbf{D}^{\beta}_{x^+} f(t) = \frac{1}{\Gamma(n - \beta)} \left(\frac{d}{dt}\right)^n \int_x^t (t - s)^{n - \beta - 1} f\left(s\right) ds, t > 0, n - 1 < \beta < n.$$

where $n = [\beta] + 1$(the notation $[\beta]$ stands for the largest integer not greater than β).

Lemma 3 *For a given $h \in PC[0,T]$ a function $x \in PC(J,\mathbb{R})$ is a solution of the impulsive sequential fractional differential equation*

$$({}^{c}D_{t_k^+}^{\beta_k}+\lambda{}^{c}D_{t_k^+}^{\beta_k-1})u(t)=h(t),1<\beta_k\le 2,k=0,1,2,...,q,t\in J', \tag{2}$$

$$\Delta x(t_k)=\psi_k(x(t_k)),\Delta x'(t_k)=\psi_k^*(x(t_k)),k=1,...,q, \tag{3}$$

with boundary condition

$$x(0)=\sum_{k=0}^{q}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}x(\eta_k),x'(0)=0, \tag{4}$$

if and only if

$$x(t)=\begin{cases}\int_0^t e^{-\lambda(t-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds+\wp,t\in J_0;\\ \int_{t_k}^t e^{-\lambda(t-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\\ +\sum_{j=1}^k e^{-\lambda(t-t_j)}\left[\int_{t_{j-1}}^{t_j}e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(s)ds\right.\\ \left.-\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)-\frac{1}{\lambda}\psi_j^*(x(t_j))\right]\\ +\sum_{j=1}^k\left[\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)+\psi_j(x(t_j))+\frac{1}{\lambda}\psi_j^*(x(t_j))\right]\\ +\wp,t\in J_k,k=1,...,q,\end{cases} \tag{5}$$

where

$$\begin{aligned}\wp=&\left(1-\sum_{k=0}^{q}\frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}\right)^{-1}\\ &\times\left\{\sum_{k=0}^{q}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}\left(\int_{t_k}^{\eta_k}e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right)(\eta_k)\right.\\ &+\sum_{k=0}^{q}\sum_{j=1}^{k}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}e^{-\lambda(\eta_k-t_j)}\\ &\left[\int_{t_{j-1}}^{t_j}e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)-\frac{1}{\lambda}\psi_j^*(x(t_j))\right.\\ &\left.\left.+\sum_{k=0}^{q}\sum_{j=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}\left[\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)+\psi_j(x(t_j))+\frac{1}{\lambda}\psi_j^*(x(t_j))\right]\right\}.\end{aligned}$$

Proof. Assume that x is a solution of (1). For any $t\in J_0$, we have

$$x(t)=\int_0^t e^{-\lambda(t-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds+e^{-\lambda t}a_1+a_2,t\in J_0, \tag{6}$$

where a_1 and $a_2\in\mathbb{R}$. Differentiating the obtained linear equation (6) on J_0, leads to

$$x'(t)=-\lambda\int_0^t e^{-\lambda(t-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds+\mathbf{I}_{0^+}^{\beta_0-1}h(t)-\lambda e^{-\lambda t}a_1. \tag{7}$$

If $t\in J_1$, then

$$\begin{aligned}x(t)&=\int_{t_1}^t e^{-\lambda(t-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds+e^{-\lambda(t-t_1)}b_1+b_2,\\ x'(t)&=-\lambda\int_{t_1}^t e^{-\lambda(t-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds+\mathbf{I}_{t_1^+}^{\beta_1-1}h(t)-\lambda e^{-\lambda(t-t_1)}b_1,\end{aligned} \tag{8}$$

for some $b_1,b_2\in\mathbb{R}$. Thus,

$$\begin{aligned}x(t_1^-)&=\int_0^{t_1}e^{-\lambda(t_1-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds+e^{-\lambda t_1}a_1+a_2,\\ x'(t_1^-)&=-\lambda\int_0^{t_1}e^{-\lambda(t_1-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds+\mathbf{I}_{0^+}^{\beta_0-1}h(t_1)-\lambda e^{-\lambda t_1}a_1,\\ x(t_1^+)&=b_1+b_2,\\ x'(t_1^+)&=-\lambda b_1.\end{aligned} \tag{9}$$

Now, by the following impulsive conditions

$$\Delta x(t_1) = x(t_1^+) - x(t_1^-) = \psi_1(x(t_1)),\\ \Delta x'(t_1) = x'(t_1^+) - x'(t_1^-) = \psi_1^*(x(t_1)), \tag{10}$$

we can get that

$$b_1 = \int_0^{t_1} e^{-\lambda(t_1-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds - \frac{1}{\lambda}\mathbf{I}_{0^+}^{\beta_0-1}h(t_1) + e^{-\lambda t_1}a_1 - \frac{1}{\lambda}\psi_1^*(x(t_1)),\\ b_2 = \frac{1}{\lambda}\mathbf{I}_{0^+}^{\beta_0-1}h(t_1) + \psi_1(x(t_1)) + \frac{1}{\lambda}\psi_1^*(x(t_1)) + a_2. \tag{11}$$

Consequently,

$$\begin{aligned} x(t) = &\int_{t_1}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds + e^{-\lambda(t-t_1)}\left[\int_0^{t_1} e^{-\lambda(t_1-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds\right.\\ &\left.-\frac{1}{\lambda}\mathbf{I}_{0^+}^{\beta_0-1}h(t_1) - \frac{1}{\lambda}\psi_1^*(x(t_1))\right] + \left[\frac{1}{\lambda}\mathbf{I}_{0^+}^{\beta_0-1}h(t_1)\right.\\ &\left.+\psi_1(x(t_1)) + \frac{1}{\lambda}\psi_1^*(x(t_1))\right] + e^{-\lambda t}a_1 + a_2, t \in J_1. \end{aligned}$$

If $t \in J_2$, then

$$x(t) = \int_{t_2}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_2^+}^{\beta_2-1}h(s)ds + e^{-\lambda(t-t_2)}c_1 + c_2,\\ x'(t) = -\lambda\int_{t_2}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_2^+}^{\beta_2-1}h(s)ds + \mathbf{I}_{t_2^+}^{\beta_1-1}h(t) - \lambda e^{-\lambda(t-t_2)}c_1, \tag{12}$$

for some $c_1, c_2 \in \mathbb{R}$. Thus,

$$\begin{aligned} x(t_2^-) &= \int_{t_1}^{t_2} e^{-\lambda(t_2-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds + e^{-\lambda(t_2-t_1)}b_1 + b_2,\\ x'(t_2^-) &= -\lambda\int_{t_1}^{t_2} e^{-\lambda(t_2-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds + \mathbf{I}_{t_1^+}^{\beta_1-1}h(t_2) - \lambda e^{-\lambda(t_2-t_1)}b_1,\\ x(t_2^+) &= c_1 + c_2,\\ x'(t_2^+) &= -\lambda c_1. \end{aligned} \tag{13}$$

Now, by the following impulsive conditions

$$\Delta x(t_2) = x(t_2^+) - x(t_2^-) = \psi_2(x(t_2)),\\ \Delta x'(t_2) = x'(t_2^+) - x'(t_2^-) = \psi_2^*(x(t_2)), \tag{14}$$

we can get that

$$\begin{aligned} c_1 = &\int_{t_1}^{t_2} e^{-\lambda(t_2-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds - \frac{1}{\lambda}\mathbf{I}_{t_1^+}^{\beta_1-1}h(t_2) + e^{-\lambda(t_2-t_1)}b_1\\ &- \frac{1}{\lambda}\psi_2^*(x(t_2)),\\ c_2 = &\frac{1}{\lambda}\mathbf{I}_{t_1^+}^{\beta_1-1}h(t_2) + \psi_2(x(t_2)) + \frac{1}{\lambda}\psi_2^*(x(t_2)) + b_2. \end{aligned} \tag{15}$$

Consequently,

$$x(t)=\int_{t_2}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_2^+}^{\beta_2-1}h(s)ds+e^{-\lambda(t-t_2)} \tag{16}$$
$$\times\left[\int_{t_1}^{t_2} e^{-\lambda(t_2-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{t_1^+}^{\beta_1-1}h(t_2)\right.$$
$$\left.-\frac{1}{\lambda}\psi_2^*(x(t_2))\right]+\frac{1}{\lambda}\mathbf{I}_{t_1^+}^{\beta_1-1}h(t_2)+\psi_2(x(t_2))$$
$$+\frac{1}{\lambda}\psi_2^*(x(t_2))+e^{-\lambda(t-t_1)}b_1+b_2$$
$$=\int_{t_2}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_2^+}^{\beta_2-1}h(s)ds+e^{-\lambda(t-t_2)}$$
$$\left[\int_{t_1}^{t_2} e^{-\lambda(t_2-s)}\mathbf{I}_{t_1^+}^{\beta_1-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{t_1^+}^{\beta_1-1}h(t_2)\right.$$
$$\left.-\frac{1}{\lambda}\psi_2^*(x(t_2))\right]+\frac{1}{\lambda}\mathbf{I}_{t_1^+}^{\beta_1-1}h(t_2)+\psi_2(x(t_2))+\frac{1}{\lambda}\psi_2^*(x(t_2))$$
$$+e^{-\lambda(t-t_1)}\left[\int_0^{t_1} e^{-\lambda(t_1-s)}\mathbf{I}_{0^+}^{\beta_0-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{0^+}^{\beta_0-1}h(t_1)-\frac{1}{\lambda}\psi_1^*(x(t_1))\right]$$
$$+\frac{1}{\lambda}\mathbf{I}_{0^+}^{\beta_0-1}h(t_1)+\psi_1(x(t_1))+\frac{1}{\lambda}\psi_1^*(x(t_1))+e^{-\lambda t}a_1+a_2, t\in J_2.$$

Repeating the process in this way, we get

$$x(t)=\int_{t_k}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds \tag{17}$$
$$+\sum_{j=1}^{k} e^{-\lambda(t-t_j)}$$
$$\times\left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)-\frac{1}{\lambda}\psi_j^*(x(t_j))\right]$$
$$+\sum_{j=1}^{k}\left[\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)+\psi_j(x(t_j))+\frac{1}{\lambda}\psi_j^*(x(t_j))\right]+e^{-\lambda t}a_1+a_2,$$
$$t\in J_k, k=1,2,...,q.$$

Taking (6), (7) and (17) to the boundary conditions,

$$x(0)=\sum_{k=0}^{q}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}x\left(\eta_k\right), x'(0)=0,$$

implies $a_1=0$. For $t\in J_k$, we have

$$\mathbf{I}_{t_k^+}^{\alpha_k}x(t)=\mathbf{I}_{t_k^+}^{\alpha_k}\left(\int_{t_k}^{r} e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right)(t)$$
$$+\sum_{j=1}^{k}\mathbf{I}_{t_k^+}^{\alpha_k}e^{-\lambda(t-t_j)}$$
$$\times\left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)-\frac{1}{\lambda}\psi_j^*(x(t_j))\right]$$
$$+\sum_{j=1}^{k}\frac{\left(t-t_k\right)^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\left[\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)+\psi_j(x(t_j))+\frac{1}{\lambda}\psi_j^*(x(t_j))\right]$$
$$+\frac{\left(t-t_k\right)^{\alpha_k}a_2}{\Gamma\left(\alpha_k+1\right)},$$

$$\sum_{k=0}^{q}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}x(\eta_k)=\sum_{k=0}^{q}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}\left(\int_{t_k}^{r}e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right)(\eta_k) \tag{18}$$
$$+\sum_{k=0}^{q}\sum_{j=1}^{k}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}e^{-\lambda(\eta_k-t_j)}$$
$$\times\left[\int_{t_{j-1}}^{t_j}e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)-\frac{1}{\lambda}\psi_j^*(x(t_j))\right]$$
$$+\sum_{k=0}^{q}\sum_{j=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}$$
$$\times\left[\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)+\psi_j(x(t_j))+\frac{1}{\lambda}\psi_j^*(x(t_j))\right]$$
$$+\sum_{k=0}^{q}.\frac{\lambda_k(\eta_k-t_k)^{\alpha_k}a_2}{\Gamma(\alpha_k+1)}.$$

and

$$a_2=\left(1-\sum_{k=0}^{q}\frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}\right)^{-1} \tag{19}$$
$$\times\left\{\sum_{k=0}^{q}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}\left(\int_{t_k}^{r}e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right)(\eta_k)\right.$$
$$+\sum_{k=0}^{q}\sum_{j=1}^{k}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}e^{-\lambda(\eta_k-t_j)}$$
$$\times\left[\int_{t_{j-1}}^{t_j}e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(s)ds-\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)-\frac{1}{\lambda}\psi_j^*(x(t_j))\right]$$
$$\left.+\sum_{k=0}^{q}\sum_{j=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}\left[\frac{1}{\lambda}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)+\psi_j(x(t_j))+\frac{1}{\lambda}\psi_j^*(x(t_j))\right]\right\}.$$

Substituting the value of a_j $(j=1,2)$ in (6) and (17), we obtain (5) . Conversely, assume that x is a solution of the impulsive sequential fractional integral equation (5) ; then by a direct computation, it follows that the solution given by (5) satisfies (4). This completes the proof. ■

We presented some estimations that used in the forthcoming theorems.

Theorem 4 *For any $h\in PC([0,T],\mathbb{R})$ with $\|h\|=\sup_{t\in[0,T]}|h(t)|$, we have*

(i)

$$\left|\mathbf{I}_{t_k^+}^{\alpha_k}\left(\int_{t_k}^{r}e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right)(t)\right|\leq\mathbf{I}_{t_k^+}^{\alpha_k}\left|\int_{t_k}^{r}e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right|(t)$$
$$\leq\frac{(\eta_k-t_k)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k)\Gamma(\beta_k)}\|h\|_{PC}.$$

(ii)

$$\left|\int_{t_k}^{t}e^{-\lambda(t-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)\,ds\right|\leq\frac{(t-t_k)^{\beta_k-1}}{\lambda\Gamma(\beta_k)}\left(1-e^{-\lambda(t-t_k)}\right)\|h\|_{PC},$$

(iii)

$$\left|\int_{t_{j-1}}^{t_j}e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(s)\,ds\right|\leq\frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})}\left(1-e^{-\lambda(t_j-t_{j-1})}\right)\|h\|_{PC},$$

(iv)

$$\left|\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\beta_{j-1}-2}}{\Gamma(\beta_{j-1}-1)}h(s)\,ds\right|\leq\frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\Gamma(\beta_{j-1})}\|h\|_{PC},$$

(v)

$$\left|\int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)}\mathbf{I}_{t_k^+}^{\beta_k+\alpha_k-1}h\left(s\right)ds\right| \leq \frac{\left(\eta_k-t_k\right)^{\beta_k+\alpha_k-1}}{\lambda\Gamma\left(\beta_k+\alpha_k\right)}\left(1-e^{-\lambda\left(\eta_k-t_k^+\right)}\right)\|h\|_{PC}$$

(vi)

$$\left|\int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)}\frac{\left(\eta_k-s\right)^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)}h\left(s\right)ds\right| \leq \frac{\left(\eta_k-t_k\right)^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\|h\|_{PC}.$$

Proof. *Obviously*

$$\begin{aligned}\left|\mathbf{I}_{t_k^+}^{\alpha_k}\left(\int_{t_k}^{r} e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right)(\eta_k)\right| &\leq \mathbf{I}_{t_k^+}^{\alpha_k}\left|\int_{t_k}^{r} e^{-\lambda(r-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h(s)ds\right|(\eta_k) \\ &\leq \int_{t_k}^{\eta_k}\frac{\left(\eta_k-r\right)^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)} \\ &\times\left(\left|\int_{t_k}^{r} e^{-\lambda(r-s)}\left(\int_{t_k}^{s}\frac{\left(s-u\right)^{\beta_k-2}}{\Gamma\left(\beta_k-1\right)}h\left(u\right)du\right)ds\right|\right)dr \\ &\leq \frac{\left(\eta_k-t_k\right)^{\beta_k-1}}{\Gamma\left(\beta_k\right)}\left(\int_{t_k}^{\eta_k}\frac{\left(\eta_k-r\right)^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)}e^{-\lambda(\eta_k-r)}dr\right)\|h\|_{PC} \\ &\leq \frac{\left(\eta_k-t_k\right)^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)}\frac{\left(\eta_k-t_k\right)^{\beta_k-1}}{\Gamma\left(\beta_k\right)}\left(\int_{t_k}^{\eta_k}e^{-\lambda(\eta_k-r)}dr\right)\|h\|_{PC} \\ &\leq \frac{\left(\eta_k-t_k\right)^{\alpha_k+\beta_k-1}}{\Gamma\left(\alpha_k\right)\Gamma\left(\beta_k\right)}\|h\|_{PC}.\end{aligned} \tag{i}$$

$$\begin{aligned}\left|\int_{t_k}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}h\left(s\right)ds\right| &= \left|\int_{t_k}^{t} e^{-\lambda(t-s)}\left(\int_{t_k}^{s}\frac{\left(s-u\right)^{\beta_k-2}}{\Gamma\left(\beta_k-1\right)}h\left(u\right)du\right)ds\right| \\ &\leq \frac{\left(t-t_k\right)^{\beta_k-1}}{\Gamma\left(\beta_k\right)}\left(\int_{t_k}^{t}e^{-\lambda(t-s)}ds\right)\|h\|_{PC} \\ &\leq \frac{\left(t-t_k\right)^{\beta_k-1}}{\lambda\Gamma\left(\beta_k\right)}\left(1-e^{-\lambda(t-t_k)}\right)\|h\|_{PC}, h\in PC\left(J,\mathbb{R}\right).\end{aligned} \tag{ii}$$

$$\begin{aligned}\left|\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h\left(s\right)ds\right| &= \left|\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)}\left(\int_{t_{j-1}}^{s}\frac{\left(s-u\right)^{\beta_{j-1}-2}}{\Gamma\left(\beta_{j-1}-1\right)}h\left(u\right)du\right)ds\right| \\ &\leq \frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\Gamma\left(\beta_k\right)}\left(\int_{t_{j-1}}^{t_j}e^{-\lambda(t_j-s)}ds\right)\|h\|_{PC} \\ &\leq \frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}\left(1-e^{-\lambda(t_j-t_{j-1})}\right)\|h\|_{PC}, h\in PC\left(J,\mathbb{R}\right).\end{aligned} \tag{iii}$$

$$\left|\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}h(t_j)\right| = \left|\int_{t_{j-1}}^{t_j}\frac{\left(t_j-s\right)^{\beta_{j-1}-2}}{\Gamma\left(\beta_{j-1}-1\right)}h\left(s\right)ds\right| \leq \frac{\left(t_j -_{j-1}\right)^{\beta_{j-1}-1}}{\Gamma\left(\beta_{j-1}\right)}\|h\|_{PC}, h\in PC\left(J,\mathbb{R}\right). \tag{iv}$$

$$\begin{aligned}\left|\int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)}\mathbf{I}_{t_k^+}^{\beta_k+\alpha_k-1}h\left(s\right)ds\right| &= \left|\int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)}\left(\int_{t_k}^{s}\frac{\left(s-u\right)^{\beta_k+\alpha_k-2}}{\Gamma\left(\beta_k+\alpha_k-1\right)}h\left(u\right)du\right)ds\right| \\ &\leq \frac{\left(\eta_k-t_k\right)^{\beta_k+\alpha_k-1}}{\Gamma\left(\beta_k+\alpha_k\right)}\left(\int_{t_k}^{\eta_k}e^{-\lambda(\eta_k-s)}ds\right)\|h\|_{PC} \\ &\leq \frac{\left(\eta_k-t_k\right)^{\beta_k+\alpha_k-1}}{\lambda\Gamma\left(\beta_k+\alpha_k\right)}\left(1-e^{-\lambda(\eta_k-t_k)}\right)\|x\|_{PC}, h\in PC\left(J,\mathbb{R}\right).\end{aligned} \tag{v}$$

$$\begin{aligned}\left|\mathbf{I}_{t_k^+}^{\alpha_k}e^{-\lambda(\eta_k-t_j)}\right| &= \left|\int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)}\frac{\left(\eta_k-s\right)^{\alpha_k-1}}{\Gamma\left(\alpha_k\right)}h\left(s\right)ds\right| \\ &\leq \frac{\left(\eta_k-t_k\right)^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\|h\|_{PC}, h\in PC\left(J,\mathbb{R}\right).\end{aligned} \tag{vi}$$

The proofs is completed. ■

3 Main results

This section deals with the existence and uniqueness of solutions for the problem (1). Before stating and proving the main results, we introduce the following hypotheses.

(H$_1$) there exist a nonnegative function $a(t) \in L(0,T)$ such that

$$|f(t,x)| \le a(t) + \xi |x|^{\sigma}, \sigma > 0,$$

where $\zeta$are nonnegative constant.

(H$_2$) there exists a constants L_ψ and L_{ψ^*} such that

$$|\psi_k(x)| \le L_\psi, |\psi_k^*(x)| \le L_{\psi^*}, t \in J, \quad x \in \mathbb{R}, k = 1,2,..,q.$$

(H$_3$) the function $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous

(H$_4$) there exists a constant $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \le L_f |x-y|, \quad t \in J, \quad x,y \in \mathbb{R}.$$

(H$_5$) there exist a positive constants L_ψ, L_{ψ^*} such that $|\psi_k(x) - \psi_k(y)| \le L_\psi |x-y|$, $|\psi_k^*(x) - \psi_k^*(y)| \le L_{\psi^*} |x-y|$.

Define an operator $\mathcal{F} : PC(J) \to PC(J)$ by

$$\begin{aligned}
\mathcal{F}x(t) &= \int_{t_k}^{t} e^{-\lambda(t-s)} \mathbf{I}_{t_k^+}^{\beta_k - 1} f(s, x(s)) ds \\
&+ \sum_{j=1}^{k} e^{-\lambda(t-t_j)} \times \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_i - s)} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} f(s, x(s)) ds \right. \\
&\left. - \frac{1}{\lambda} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} f(t_j, x(t_j)) - \frac{1}{\lambda} \psi_j^*(x(t_j)) \right] \\
&+ \sum_{j=1}^{k} \left[\frac{1}{\lambda} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} f(t_j, x(t_j)) + \psi_j(x(t_j)) + \frac{1}{\lambda} \psi_j^*(x(t_j)) \right] \\
&+ \left(1 - \sum_{k=0}^{q} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k}}{\Gamma(\alpha_k + 1)} \right)^{-1} \\
&\times \left\{ \sum_{k=0}^{q} \lambda_k \mathbf{I}_{t_k^+}^{\alpha_k} \int_{t_k}^{\eta_k} e^{-\lambda(\eta_k - s)} \mathbf{I}_{t_k^+}^{\beta_k - 1} f(s, x(s)) ds \right. \\
&+ \sum_{k=0}^{q} \sum_{j=1}^{k} \lambda_k \mathbf{I}_{t_k^+}^{\alpha_k} e^{-\lambda(\eta_k - t_j)} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j - s)} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} f(s, x(s)) ds \right. \\
&\left. - \frac{1}{\lambda} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} f(t_j, x(t_j)) - \frac{1}{\lambda} \psi_j^*(x(t_j)) \right] \\
&\left. + \sum_{k=0}^{q} \sum_{j=1}^{k} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k}}{\Gamma(\alpha_k + 1)} \left[\frac{1}{\lambda} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} f(t_j, x(t_j)) + \psi_j(x(t_j)) + \frac{1}{\lambda} \psi_j^*(x(t_j)) \right] \right\},
\end{aligned} \tag{20}$$

For convenience, we will give some notations:

$$\begin{aligned}
\Delta &= \left| 1 - \sum_{k=0}^{q} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k}}{\Gamma(\alpha_k + 1)} \right|^{-1}, \\
\Lambda_1 &= \sum_{j=1}^{q+1} \frac{(t_j - t_{j-1})^{\beta_{j-1}-1}}{\lambda \Gamma(\beta_{j-1})}, \Lambda_2 = \sum_{j=1}^{q} \frac{(t_j - t_{j-1})^{\beta_{j-1}-1}}{\lambda \Gamma(\beta_{j-1})} \Lambda_3 = \sum_{k=0}^{q} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k + \beta_k - 1}}{\Gamma(\alpha_k) \Gamma(\beta_k)}, \\
\Lambda_4 &= \sum_{k=0}^{q} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k} (t_j - t_{j-1})^{\beta_{j-1}-1}}{\lambda \Gamma(\alpha_k + 1) \Gamma(\beta_{j-1})}, \Lambda_5 = \sum_{k=0}^{q} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k}}{\Gamma(\alpha_k + 1)},
\end{aligned}$$

$$\mu\left(u\right)=\Psi\left\|u\right\|+\left(1+\Delta\Lambda_{5}\right)qL_{\psi}+\left(\frac{T}{\lambda}+\frac{1}{\lambda}+\left(\frac{1}{\lambda}+\frac{1}{\lambda}\right)\Delta\Lambda_{5}\right)qL_{\psi^{*}}.$$

Theorem 5 *Suppose that (H_3), (H_4) and (H_5) hold. If*

$$L_{\mathcal{F}}<1 \tag{21}$$

then the equation (1) has a unique solution on J.

Proof. Show that $\mathcal{F}:PC\left(J\right)\rightarrow PC\left(J\right)$ is a completely continuous operator

$$\begin{aligned}
\left|\mathcal{F}x\left(t\right)\right| &\leq \int_{t_k}^{t} e^{-\lambda(t-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}\left|f(s,x\left(s\right))\right|ds \\
&+\sum_{j=1}^{k} e^{-\lambda(t-t_j)}\left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_i-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}\left|f(s,x\left(s\right))\right|ds\right. \\
&\left.+\left|\frac{1}{\lambda}\right|\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}\left|f(t_j,x\left(t_j\right))\right|+\left|\frac{1}{\lambda}\right|\left|\psi_j^*(x(t_j))\right|\right] \\
&+\sum_{j=1}^{k}\left[\left|\frac{1}{\lambda}\right|\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}\left|f(t_j,x\left(t_j\right))\right|+\left|\psi_j(x(t_j))\right|+\left|\frac{1}{\lambda}\right|\left|\psi_j^*(x(t_j))\right|\right] \\
&+\left|1-\sum_{k=0}^{q}\frac{\lambda_k\left(\eta_k-t_k\right)^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\right|^{-1} \\
&\times\left\{\sum_{k=0}^{q}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}\int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)}\mathbf{I}_{t_k^+}^{\beta_k-1}\left|f(s,x\left(s\right))\right|ds+\sum_{k=0}^{q}\sum_{j=1}^{k}\lambda_k\mathbf{I}_{t_k^+}^{\alpha_k}e^{-\lambda(\eta_k-t_j)}\right. \\
&\times\left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)}\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}\left|f(s,x\left(s\right))\right|ds+\left|\frac{1}{\lambda}\right|\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}\left|f(t_j,x\left(t_j\right))\right|+\left|\frac{1}{\lambda}\right|\left|\psi_j^*(x(t_j))\right|\right] \\
&\left.+\sum_{k=0}^{q}\sum_{j=1}^{k}\frac{\lambda_k\left(\eta_k-t_k\right)^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\left[\left|\frac{1}{\lambda}\right|\mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1}\left|f(t_j,x\left(t_j\right))\right|+\left|\psi_j(x(t_j))\right|+\left|\frac{1}{\lambda}\right|\left|\psi_j^*(x(t_j))\right|\right]\right\},
\end{aligned}$$

$$\begin{aligned}
\left|\mathcal{F}x\left(t\right)\right| &\leq L_f\frac{\left(t-t_k\right)^{\beta_k-1}}{\lambda\Gamma\left(\beta_k\right)} \\
&+\sum_{j=1}^{k} e^{-\lambda(t-t_j)}\left[L_f\frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_f\frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_{\psi^*}\frac{1}{\lambda}\right] \\
&+\sum_{j=1}^{k}\left[L_f\frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_\psi+L_{\psi^*}\frac{1}{\lambda}\right]+\Delta\left\{L_f\sum_{k=0}^{q}\frac{\lambda_k\left(\eta_k-t_k\right)^{\alpha_k+\beta_k-1}}{\Gamma\left(\alpha_k\right)\Gamma\left(\beta_k\right)}.\right. \\
&+\sum_{k=0}^{q}\sum_{j=1}^{k}\frac{\lambda_k\left(\eta_k-t_k\right)^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\left[L_f\frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_f\frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_{\psi^*}\frac{1}{\lambda}\right] \\
&\left.+\sum_{k=0}^{q}\sum_{j=1}^{k}\frac{\lambda_k\left(\eta_k-t_k\right)^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}\left[L_f\frac{\left(t_j-t_{j-1}\right)^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_\psi+L_{\psi^*}\frac{1}{\lambda}\right]\right\},
\end{aligned}$$

$$
\begin{aligned}
|\mathcal{F}x(t)| \leq{} & L_f \frac{(t-t_k)^{\beta_k-1}}{\lambda\Gamma(\beta_k)} + L_f \sum_{j=1}^{k} e^{-\lambda(t-t_j)} \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} \\
& + L_f \sum_{j=1}^{k} e^{-\lambda(t-t_j)} \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} + \frac{1}{\lambda}\sum_{j=1}^{k} e^{-\lambda(t-t_j)} L_{\psi^*} \\
& + L_f \sum_{j=1}^{k} \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} + \sum_{j=1}^{k} L_\psi + \frac{1}{\lambda}\sum_{j=1}^{k} L_{\psi^*} \\
& + \Delta\left\{ L_f \sum_{k=0}^{q} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k)\Gamma(\beta_k)}. \right. \\
& + \sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} \left[\frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} L_f + L_f \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} + \frac{1}{\lambda}\frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} L_{\psi^*} \right] \\
& \left. + \sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} \left[\frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} L_f + L_\psi + \frac{1}{\lambda} L_{\psi^*} \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{F}x(t)| \leq{} & L_f \sum_{j=1}^{q+1} \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} + TL_f \sum_{j=1}^{q} \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} + TL_f \sum_{j=1}^{q} \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\beta_{j-1})} \\
& + \frac{qT}{\lambda} L_{\psi^*} + qL_\psi + \frac{q}{\lambda} L_{\psi^*} + \Delta\left\{ L_f \sum_{k=0}^{q} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k)\Gamma(\beta_k)}. \right. \\
& + L_f \sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\alpha_k+1)\Gamma(\beta_{j-1})} + L_f \sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\alpha_k+1)\Gamma(\beta_{j-1})} \\
& + \frac{1}{\lambda}\sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} L_{\psi^*} + L_f \sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma(\alpha_k+1)\Gamma(\beta_{j-1})} \\
& \left. + \sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} L_\psi + \frac{1}{\lambda}\sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} L_{\psi^*} \right\},
\end{aligned}
$$

$$
\begin{aligned}
& \leq L_f\{(1+q)\Lambda_1 + qT\Lambda_2 + qT\Lambda_2\} + \frac{qT}{\lambda} L_{\psi^*} + qL_\psi + \frac{q}{\lambda} L_{\psi^*} \\
& + \Delta\left\{ L_f\Lambda_3 + qL_f\Lambda_4 + qL_f\Lambda_4 + \frac{q}{\lambda}\Lambda_5 L_{\psi^*} + qL_f\Lambda_4 + q\Lambda_5 L_\psi + \frac{q}{\lambda}\Lambda_5 L_{\psi^*} \right\},
\end{aligned}
$$

$$
\begin{aligned}
& \leq L_f\{((1+q)\Lambda_1 + 2qT\Lambda_2) + \Delta(\Lambda_3 + 3q\Lambda_4)\} \\
& + (1+\Delta\Lambda_5)\, qL_\psi + \left(\frac{T}{\lambda} + \frac{1}{\lambda} + \left(\frac{1}{\lambda} + \frac{1}{\lambda}\right)\Delta\Lambda_5\right) qL_{\psi^*}
\end{aligned}
$$

which implies

$$
|\mathcal{F}x(t)| \leq \Psi L_f + (1+\Delta\Lambda_5)\, qL_\psi + \left(\frac{T}{\lambda} + \frac{1}{\lambda} + \left(\frac{1}{\lambda} + \frac{1}{\lambda}\right)\Delta\Lambda_5\right) qL_{\psi^*}.
$$

which implies that $\mathcal{F}x \in B$. Thus $\mathcal{F}B \subset B$. On the other hand, for any $t \in J_k, 0 \leq k \leq q$, we have

$$
\begin{aligned}
\left|(\mathcal{F}x)'(t)\right| \leq{} & \lambda \int_{t_k}^{t} e^{-\lambda(t-s)} \mathbf{I}_{t_k^+}^{\beta_k-1} |f(s,x(s))|\, ds + \int_{t_k}^{t} \frac{(t-s)^{\beta_k-2}}{\Gamma(\beta_k-1)} |f(s,x(s))|\, ds \\
& + \sum_{j=1}^{k} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} |f(s,x(s))|\, ds \right. \\
& \left. + \frac{1}{\lambda} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\beta_{j-1}-2}}{\Gamma(\beta_{j-1}-1)} |f(s,x(s))|\, ds + \left| \frac{1}{\lambda} \psi_j^*(x(t_j)) \right| \right],
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lambda L_f \int_{t_k}^{t} e^{-\lambda(t-s)} \mathbf{I}_{t_k^+}^{\beta_k-1} ds + L_f \int_{t_k}^{t} \frac{(t-s)^{\beta_k-2}}{\Gamma(\beta_k-1)} ds \\
&+ \sum_{j=1}^{q} \left[L_f \int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} ds + L_f \frac{1}{\lambda} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\beta_{j-1}-2}}{\Gamma(\beta_{j-1}-1)} ds + \frac{1}{\lambda} L_{\psi^*} \right],
\end{aligned}
$$

$$
\begin{aligned}
&\leq L_f \left[\frac{(t-t_k)^{\beta_k-1}}{\Gamma(\beta_k)} \left(\left(1 - e^{-\lambda(t-t_k)} \right) + 1 \right) \right] \\
&+ \frac{q}{\lambda} \left[L_f \left[\frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\Gamma(\beta_k)} \left(\left(1 - e^{-\lambda(t_j-t_{j-1})} \right) + 1 \right) \right] + L_{\psi^*} \right] \\
&= \mathcal{L}.
\end{aligned}
$$

Hence, for $\tau_1, \tau_2 \in J_k$ with $\tau_1 \leq \tau_2$ and $0 \leq k \leq q$, we have

$$
|(\mathcal{F}x)(\tau_2) - (\mathcal{F}x)(\tau_1)| \leq \int_{\tau_1}^{\tau_2} \left| (\mathcal{F}x)'(s) \right| ds \leq \mathcal{L}(\tau_2 - \tau_1).
$$

This implies that $\mathcal{F}x$ is equicontinuous on all $J_k, k = 0, 1, ..., q$. Consequently, Arzela-Ascoli theorem ensures the operator $\mathcal{F} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is a completely continuous operator. Next show that the operator $\mathcal{F}$ maps B into B. For that, let us choose $R \geq \max \left\{ 2\mu, (2L_\sigma)^{\frac{1}{1-\sigma}} \right\}$ and define a ball $B = \{ x \in PC(J, \mathbb{R}) : \|x\| \leq R \}$. For any $x \in B$, by the conditions (H_1) and (H_2), we have

$$
\begin{aligned}
|\mathcal{F}x(t)| \leq & \int_{t_k}^{t} e^{-\lambda(t-s)} \mathbf{I}_{t_k}^{\beta_k-1} \left[a(s) + \xi |x(s)|^\sigma \right] ds \\
&+ \sum_{j=1}^{k} e^{-\lambda(t-t_j)} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)} \mathbf{I}_{t_{j-1}}^{\beta_{j-1}-1} \left[a(s) + \xi |x(s)|^\sigma \right] ds \right. \\
&+ \left. \left| \frac{1}{\lambda} \right| \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} \left[a(t_j) + \xi |x(t_j)|^\sigma \right] + \left| \frac{1}{\lambda} \right| \left| \psi_j^*(x(t_j)) \right| \right] \\
&+ \sum_{j=1}^{k} \left[\left| \frac{1}{\lambda} \right| \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} \left[a(t_j) + \xi |x(t_j)|^\sigma \right] + \left| \psi_j(x(t_j)) \right| + \left| \frac{1}{\lambda} \right| \left| \psi_j^*(x(t_j)) \right| \right] \\
&+ \left| 1 - \sum_{k=0}^{q} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} \right|^{-1} \times \left\{ \sum_{k=0}^{q} \lambda_k \mathbf{I}_{t_k}^{\alpha_k} \int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)} \mathbf{I}_{t_k}^{\beta_k-1} \left[a(s) + \xi |x(s)|^\sigma \right] ds \right. \\
&+ \sum_{k=0}^{q} \sum_{j=1}^{k} \lambda_k \mathbf{I}_{t_k^+}^{\alpha_k} e^{-\lambda(\eta_k-t_j)} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)} \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} \left[a(s) + \xi |x(s)|^\sigma \right] ds \right. \\
&+ \left. \left| \frac{1}{\lambda} \right| \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} \left[a(t_j) + \xi |x(t_j)|^\sigma \right] + \left| \frac{1}{\lambda} \right| \left| \psi_j^*(x(t_j)) \right| \right] \\
&+ \left. \sum_{k=0}^{q} \sum_{j=1}^{k} \frac{\lambda_k (\eta_k - t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} \left[\left| \frac{1}{\lambda} \right| \mathbf{I}_{t_{j-1}^+}^{\beta_{j-1}-1} \left[a(t_j) + \xi |x(t_j)|^\sigma \right] + \left| \psi_j(x(t_j)) \right| + \left| \frac{1}{\lambda} \right| \left| \psi_j^*(x(t_j)) \right| \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{F}x(t)| \leq & \left[\|a\|+\xi\|x\|^{\sigma}\right] \frac{(t-t_k)^{\beta_k-1}}{\lambda\Gamma(\beta_k)}+\sum_{j=1}^{k} e^{-\lambda(t-t_j)} \\
& \times\left[\left[\|a\|+\xi\|x\|^{\sigma}\right] \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+\left[\|a\|+\xi\|x\|^{\sigma}\right] \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+\frac{1}{\lambda} L_{\psi^*}\right] \\
& +\sum_{j=1}^{k}\left[\left[\|a\|+\xi\|x\|^{\sigma}\right] \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_{\psi}+\frac{1}{\lambda} L_{\psi^*}\right] \\
& +\Delta\left\{\left[\|a\|+\xi\|x\|^{\sigma}\right] \sum_{k=0}^{q} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k)\Gamma(\beta_k)}+\sum_{k=0}^{q} \sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}\right. \\
& \times\left[\left[\|a\|+\xi\|x\|^{\sigma}\right] \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+\left[\|a\|+\xi\|x\|^{\sigma}\right] \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+\frac{1}{\lambda} L_{\psi^*}\right] \\
& \left.+\sum_{k=0}^{q} \sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}\left[\left[\|a\|+\xi\|x\|^{\sigma}\right] \frac{(t_j-t_{j-1})^{\beta_{j-1}-1}}{\lambda\Gamma\left(\beta_{j-1}\right)}+L_{\psi}+\frac{1}{\lambda} L_{\psi^*}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
& \leq \Psi\|a\|+(1+\Delta\Lambda_5) q L_{\psi}+\left(\frac{T}{\lambda}+\frac{1}{\lambda}+\left(\frac{1}{\lambda}+\frac{1}{\lambda}\right) \Delta\Lambda_5\right) q L_{\psi^*} \\
& +.\Psi\xi\|x\|^{\sigma}. \\
& \qquad \leq \mu(a)+\Psi\xi\|x\|^{\sigma}.
\end{aligned}
$$

Thus,

$$
|\mathcal{F}x(t)| \leq \mu(a)+\Psi\xi\|x\|^{\sigma} \leq \frac{R}{2}+\frac{R}{2}=R.
$$

This implies $\mathcal{F}: B \rightarrow B$. Hence, we conclude that $\mathcal{F}: B \rightarrow B$ is completely continuous. it is follows from the Schauder fixed point theorem that the operator $\mathcal{F}$ has at least one fixed point. That is problem (1) has at least one solution in B.

Theorem 6 *Assume that there exist a nonnegative function $W \in C(J, \mathbb{R}^{+})$)and nonnegative constants M, Z such that*

$$
\begin{gathered}
|f(t, x)-f(t, y)| \leq W(t)|x-y|, \quad t \in J, \quad x, y \in \mathbb{R}, \\
|\psi_k(x)-\psi_k(y)| \leq M|x-y|, \quad |\psi_k^*(x)-\psi_k^*(y)| \leq Z|x-y|,
\end{gathered}
$$

for $t \in J, x, y \in \mathbb{R}$ and $k=1,2,..,q$. furthermore, the assumption $\mu(W)<1$ holds. then the equation (1) has a unique solution on J.

■

Proof. For $x, y \in B$ and for each $t \in J$, we have

$$\begin{aligned}
\left|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\right| \leq & \int_{t_k}^{t} e^{-\lambda(t-s)} \mathbf{I}_{t_k}^{\beta_k-1} \left|f(s,x(s))-f(s,y(s))\right| ds \\
& + \sum_{j=1}^{k} e^{-\lambda(t-t_j)} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)} \mathbf{I}_{t_{j-1}}^{\beta_{j-1}-1} \left|f(s,x(s))-f(s,y(s))\right| ds \right. \\
& \left. + \left|\frac{1}{\lambda}\right| \mathbf{I}_{t_{j-1}^{+}}^{\beta_{j-1}-1} \left|f(t_j,x(t_j))-f(t_j,y(t_j))\right| + \left|\frac{1}{\lambda}\right| \left|\psi_j^*(x(t_j))-\psi_j^*(y(t_j))\right| \right] \\
& + \sum_{j=1}^{k} \left[\left|\frac{1}{\lambda}\right| \mathbf{I}_{t_{j-1}^{+}}^{\beta_{j-1}-1} \left|f(t_j,x(t_j))-f(t_j,y(t_j))\right| + \left|\psi_j(x(t_j))-\psi_j(y(t_j))\right| \right. \\
& \left. + \left|\frac{1}{\lambda}\right| \left|\psi_j^*(x(t_j))-\psi_j^*(y(t_j))\right| \right] \\
& + \left|1-\sum_{k=0}^{q} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}\right|^{-1} \left\{ \sum_{k=0}^{q} \lambda_k \mathbf{I}_{t_k}^{\alpha_k} \int_{t_k}^{\eta_k} e^{-\lambda(\eta_k-s)} \mathbf{I}_{t_k}^{\beta_k-1} \left|f(s,x(s))-f(s,y(s))\right| ds \right. \\
& + \sum_{k=0}^{q}\sum_{j=1}^{k} \lambda_k \mathbf{I}_{t_k^{+}}^{\alpha_k} e^{-\lambda(\eta_k-t_j)} \left[\int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)} \mathbf{I}_{t_{j-1}}^{\beta_{j-1}-1} \left|f(s,x(s))-f(s,y(s))\right| ds \right. \\
& \left. + \left|\frac{1}{\lambda}\right| \mathbf{I}_{t_{j-1}^{+}}^{\beta_{j-1}-1} \left|f(t_j,x(t_j))-f(t_j,y(t_j))\right| + \left|\frac{1}{\lambda}\right| \left|\psi_j^*(x(t_j))-\psi_j^*(y(t_j))\right| \right] \\
& + \sum_{k=0}^{q}\sum_{j=1}^{k} \frac{\lambda_k(\eta_k-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)} \left[\left|\frac{1}{\lambda}\right| \mathbf{I}_{t_{j-1}^{+}}^{\beta_{j-1}-1} \left|f(t_j,x(t_j))-f(t_j,y(t_j))\right| \right. \\
& \left.\left. + \left|\psi_j(x(t_j))-\psi_j(y(t_j))\right| + \left|\frac{1}{\lambda}\right| \left|\psi_j^*(x(t_j))-\psi_j^*(y(t_j))\right| \right] \right\}, \\
\leq & \left\{ \Psi \|W\| + (1+\Delta\Lambda_5) q L_{\psi} + \left(\frac{T}{\lambda} + \frac{1}{\lambda} + \left(\frac{1}{\lambda} + \frac{1}{\lambda} \right) \Delta\Lambda_5 \right) q L_{\psi^*} \right\} \|x-y\|. \\
= & \mu(W) \|x-y\|.
\end{aligned}$$

As $\mu(W) \leq 1$, we have $\left|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\right| < \|x-y\|$. Therefore, $\mathcal{F}$ is a contraction mapping on $PC(J,\mathbb{R})$ due to condition (21). By applying the well-known Banach's contraction mapping we see that the operator $\mathcal{F}$ has a unique fixed point on $PC(J,\mathbb{R})$. Therefore, the problem (1) has a unique solution. This completes the proof. ■

Example 7 *consider the impulsive sequential fractional deferential equation*

$$\begin{gathered}
({}^{c}D_{t_k^{+}}^{\beta_k} + \lambda {}^{c}D_{t_k^{+}}^{\beta_k-1})x(t) = \frac{e^{t}\sin\left[3x(t)+e^{\left(\frac{1}{2}\right)x(t)}\right]}{2+x^4(t)} + \frac{\cos(2t+5)}{\sqrt{3+x^2(t)}}|x(t)|^{\sigma}, 0<t\leq 1, t\neq \frac{3}{4}, k=0,1,...q; \qquad (22) \\
\Delta x(\frac{3}{4}) = 11\sin^2 x\left(\frac{1}{4}\right), \Delta x'(\frac{3}{4}) = \frac{\left|x\left(\frac{3}{4}\right)\right|}{2\left(1+\left|x\left(\frac{3}{4}\right)\right|\right)}, \\
x(0) = \sum_{k=0}^{1} \lambda_k \mathbf{I}_{t_k^{+}}^{\alpha_k} x(\eta_k) + \frac{1}{2}, x'(0) = 0.
\end{gathered}$$

$t \in [0,1]$, let $\beta_0 = \frac{5}{4}, \beta_1 = 1$, $\beta = \frac{8}{5}, \alpha_0 = \frac{1}{2}, \alpha_1 = \frac{5}{3}, , \lambda_0 = \frac{2}{5}, \lambda_1 = \frac{3}{7}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{4}{5}$. Observe that

$$\begin{aligned}
|(t,x,y)| &= \left| \frac{e^{t}\sin\left[3x(t)+e^{\left(\frac{1}{2}\right)x(t)}\right]}{2+x^4(t)} + \frac{\cos(2t+5)}{\sqrt{3+x^2(t)}}|x(t)|^{\sigma} \right| \\
&\leq \frac{e^{t}}{2} + \frac{1}{\sqrt{3}}|x|\sigma
\end{aligned}$$

.

Clearly, $a(t) = \frac{e^t}{2}, \xi = \frac{1}{\sqrt{3}}, L_\psi = 11, L_{\psi^*} = \frac{1}{2}$, and the conditions of Theorem 4 hold. Thus, by Theorem, the impulsive sequential multi-orders fractional boundary value problem (22) has at least one solution.

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