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Differential calculus on multiple products

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Abstract

This paper introduced the calculus of mappings on finite product of spaces, allowing different degrees of differentiability in the different factors. This enables us to prove an important feature in the infinite-dimensional Lie theory, the exponential law in generalized setting for locally convex spaces and for manifolds modelled on locally convex spaces.

Keywords: Differential calculus; Generalized differential calculus; Differentiation theory; Locally convex spaces; Infinite-dimensional manifolds; k-space; Metrizability; Smooth compact-open topology; Evaluation map; Exponential law; Function space

1. Introduction and statement of results

The object of this paper is to introduce and study the differential calculus of mappings on finite product of locally convex spaces (resp. manifolds modelled on locally convex spaces) called (C^α-maps), where calculus in each folds is based on differentiability in the sense of Michal and Bastiani (C^r-maps), also known as Keller's C^r_c-maps (see [3,9,16,17,19,20]; cf. [4]; see [14,22] for maps on suitable non-open domains).

For all $i \in \{1, \ldots, n\}$. Let E_i and F be locally convex spaces, U_i be an open subset of E_i and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ such that $\alpha := (\alpha_1, \ldots, \alpha_n)$. Suppose that $\breve{D}_{w_i}$ is the directional derivative in the ith component, we say that a map $f : U_1 \times \cdots \times U_n \to F$ is C^α if the directional derivative

$$(\breve{D}_{w_1} \cdots \breve{D}_{w_n} f)(x)$$

exists and is continuous function on $U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$ such that $\beta_i \in \mathbb{N}_0,\ \beta_i \leq \alpha_i$ (see Definition 3.1 for details). To establish context for compact sets and manifolds with

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boundary, we consider C^α-maps on non-open locally convex domains (see Definition 3.2). We topologize the spaces of C^α-mapping with the compact-open C^α-topology (see Definitions 3.18 and 4.2) that is analogous to the compact-open C^r-topology (as recalled in Definition 2.6).

In order to provide a scheme for handling a variety of problems in infinite dimensional analysis and geometry arising more from a sort of mapping on finite product of locally convex spaces or on finite product of mapping spaces than from product of two spaces, this paper generalizes the main result of [2] which introduced the differential calculus of mappings on products of two locally convex spaces. We shall be concerned with the Schwarz theorem, the chain rule, and also the key attribute of C^α-theory, the exponential law (Theorem 3.22) which is utilizable tool in infinite dimensional Lie theory. For instance, establishing regularity in Milnor's sense for some classes of Lie groups. The following are some sample of applications:

(a) C^α-Theory used in [15] to show that the group of all smooth diffeomorphisms of compact convex subset with non-empty interior of $\mathbb{R}^n$ is a C^0-regular infinite dimensional Lie group and also for the results concerning solutions to ordinary differential equations on compact convex sets.
(b) In [1], it used to construct Lie group structure on mapping spaces of the form $C^k(M, K)$, where M is a non-compact smooth manifold and K is a Lie group.

Recall that a Hausdorff topological space X is called a *k-space* if a subset of X is closed whenever its intersection with every compact subset of X is closed (see [18] and [21]). For example, locally compact spaces, topological manifolds, first-countable spaces and metrizable topological spaces are k-space. The main result of Section 3 (Theorems 3.20 and 3.22) is the following exponential law.

Theorem A. *For all $i \in \{1, \dots, n\}$, let E_i and F be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. For $j \in \{2, \dots, n\}$ define $U := U_1 \times \cdots \times U_{j-1}$, $V := U_j \times \cdots \times U_n$, $\gamma := (\alpha_1, \dots, \alpha_{j-1})$ and $\eta := (\alpha_j, \dots, \alpha_n)$. If $f\colon U \times V \to F$ is $C^{(\gamma,\eta)}$. Then $g^\vee\colon U \to C^\eta(V, F)$, $x \mapsto \gamma(x, \bullet)$ is C^γ for each $g \in C^{(\gamma,\eta)}(U \times V, F)$, and the map*

$$\Phi\colon C^{(\gamma,\eta)}(U \times V, F) \to C^\gamma(U, C^\eta(V, F)), \quad g \mapsto g^\vee \tag{1.1}$$

is a linear topological embedding. Let $X_i := \{0\}$ if $\alpha_i = 0$, otherwise $X_i := E_i$. If $U \times V \times X_1 \times X_2 \times \cdots \times X_n$ is a k-space or V is locally compact, then Φ is an isomorphism of topological vector spaces.

See [12] for finite-dimensional vector spaces over a complete ultrametric field. The preceding exponential law (1.1) could also be extended to a finite product of locally convex manifolds (possibly with boundary, corners or rough boundary). To make this more explicit we remind the reader (as in [14] and [19]) that an ordinary manifold (without boundary) modelled on a locally convex space E is a Hausdorff topological space M with an atlas of smoothly compatible homeomorphisms $\phi\colon U_\phi \to V_\phi$ from open subsets U_ϕ of M onto open subsets $V_\phi \subseteq E$. If each V_ϕ is locally convex subsets with dense interior, M is a *manifold with rough boundary*. If each V_ϕ is a relatively open subset of $\lambda_1^{-1}([0,\infty[) \cap \cdots \cap \lambda_n^{-1}([0,\infty[)$, for suitable $n \in \mathbb{N}$ and linearly independent $\lambda_1, \dots, \lambda_n \in E'$ (the space of continuous linear functional on E), then M is a *manifold with corners*. In the case of a *manifold with smooth boundary*, each V_ϕ is relatively open in a closed hyperplane $\lambda^{-1}([0,\infty[)$, where $\lambda \in E'$. The main results of Section 4 is the following exponential law (Theorem 4.4).

Theorem B. *For all $i \in \{1, \dots, n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on locally convex space E_i. Let F be a locally convex space and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. For $j \in \{2, \dots, n\}$ define $M := M_1 \times \cdots \times M_{j-1}$, $N := M_j \times \cdots \times M_n$, $\gamma := (\alpha_1, \dots, \alpha_{j-1})$ and $\eta := (\alpha_j, \dots, \alpha_n)$. Then $g^\vee \in C^\gamma(M, C^\eta(N, F))$ for all $g \in C^{(\gamma,\eta)}(M \times N, F)$, and the map*

$$\Phi \colon C^{(\gamma,\eta)}(M \times N, F) \to C^\gamma(M, C^\eta(N, F)), \quad g \mapsto g^\vee \tag{1.2}$$

is a linear topological embedding. If E_i is metrizable for all $i \in \{1, \dots, n\}$, then Φ is an isomorphism of topological vector spaces.

Along the same lines one can also show that Theorem B holds in the following situations:

1. $M_j, \dots, M_n$ are finite-dimensional manifolds with smooth boundary, with corners or without boundary (then N is a locally compact).
2. $M_1, \dots, M_n$ are manifolds with smooth boundary, with corners or without boundary and $E_1 \times \cdots \times E_n \times X_1 \times \cdots \times X_n$ is a k-space.
3. $M_1, \dots, M_n$ are manifolds with smooth boundary, with corners or without boundary and E_i and X_i are hemicompact k-spaces (recall that a space X is called hemicompact or k_ω-space if there exists a sequence of compact sets $\{A_n : n \in \mathbb{N}\}$ in X such that for any compact subset A of X, $A \subseteq A_n$ holds for some n (see [8], cf. [13] for locally k_ω-spaces)).

Note that C^α-maps $U_1 \times \cdots \times U_n \to F$ can be defined just as well if U_i is a Hausdorff topological space, for all $i \in \{1, \dots, n\}$ with $\alpha_i = 0$. In such case, all results hold with obvious modification. Also, we obtain analogous results if F is a complex locally convex space and E_i is a locally convex space over $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}\}$, and all directional derivatives in the ith variable are considered as derivatives over the ground field $\mathbb{K}_i$. The corresponding maps could be called $C^\alpha_{\mathbb{K}_1,\dots,\mathbb{K}_n}$-maps.

The proofs of Theorems A, B and 3.22 (and Proposition 2.4) are modelled after the arguments of chapters (3 and 4) of [1], published as part of [2], with non-trivial modifications.

2. Preliminaries

The letter $\mathbb{K}$ always stands for $\mathbb{R}$ or $\mathbb{C}$. All vector spaces will be $\mathbb{K}$-vector spaces and all linear maps will be $\mathbb{K}$-linear, unless the contrary is stated. We write $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

Definition 2.1. Let E_1, E_2 and F be locally convex spaces, U and V open subsets of E_1 and E_2 respectively, $r, s \in \mathbb{N}_0 \cup \{\infty\}$ and $i, j \in \mathbb{N}_0$ such that $i \le r, j \le s$. For all $x \in U, y \in V, w_1, \dots, w_i \in E_1, v_1, \dots, v_j \in E_2$, we say that

1. A mapping $f \colon U \to F$ is called a C^r,[1] if the iterated directional derivatives
$$d^{(i)} f(x, w_1, \dots, w_i) := (D_{w_i} D_{w_{i-1}} \cdots D_{w_1} f)(x)$$
exist and define continuous maps $d^{(i)} f \colon U \times E_1^i \to F$. If f is C^∞ it is also called smooth. We abbreviate $df := d^{(1)} f$.
2. A mapping $f \colon U \times V \to F$ is a $C^{(r,s)}$-map, if the iterated directional derivative
$$\begin{aligned} &d^{(i,j)} f(x, y, w_1, \dots, w_i, v_1, \dots, v_j) \\ &:= (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y) \end{aligned}$$
exists and $d^{(i,j)} f : U \times V \times E_1^i \times E_2^j \to F$, is a continuous map.

[1] For the theory of C^r-maps, the reader is referred to [9,14,16,19,20] (cf. also [4])

Definition 2.2 (*Differentials on Non-open Domains*).

1. The set $U \subseteq E_1$ is called *locally convex* if every $x \in U$ has a convex neighbourhood W in U.
2. Let $U \subseteq E_1$ be a locally convex subset with dense interior. A mapping $f: U \to F$ is called C^r if $f|_{U^\circ}: U^\circ \to F$ is C^r and each of the maps $d^{(i)}(f|_{U^\circ}): U^\circ \times E_1^i \to F$ admits a (unique) continuous extension $d^{(i)}f: U \times E_1^i \to F$. If $U \subseteq \mathbb{R}$ and f is C^1, we obtain a continuous map $f': U \to E_1$, $f'(x) := df(x)(1)$.
   In particular if f is of class C^r, we define recursively
   $$f^{(i)}(x) = (f^{(i-1)})'(x)$$
   for $i \in \mathbb{N}_0$, such that $i \le r$ where $f^{(0)} := f$.
3. Let $U \subseteq E_1$, $V \subseteq E_2$ be locally convex subsets with dense interior. A mapping $f: U \times V \to F$ is a $C^{(r,s)}$-map, if $f|_{U^0 \times V^0}: U^0 \times V^0 \to F$ is $C^{(r,s)}$-map and for all $i, j \in \mathbb{N}_0$ such that $i \le r$, $j \le s$, the map $d^{(i,j)}(f|_{U^0 \times V^0}): U^0 \times V^0 \times E_1^i \times E_2^j \to F$ extends to a continuous map $d^{(i,j)}f: U \times V \times E_1^i \times E_2^j \to F$.

Remark 2.3 (*[9]*). If E_1, E_2 and F are locally convex topological spaces, $U \subseteq E_1$ and $V \subseteq E_2$ open subsets and the map $f: U \times V \to F$ is continuous, then f is C^1 if and only if the directional derivatives $d^{(1,0)}f(x, y; h_1) := D_{(h_1,0)}f(x, y)$ and $d^{(0,1)}f(x, y; h_2) := D_{(0,h_2)}f(x, y)$ exist for all $x \in U$, $y \in V$, $h_1 \in E_1$ and $h_2 \in E_2$, and define continuous functions $d^{(1,0)}f(x, y; h_1) : U \times V \times E_1 \to F$ and $d^{(0,1)}f(x, y; h_2) : U \times V \times E_2 \to F$. In this case,

$$df((x, y), (h_1, h_2)) = d^{(1,0)}f(x, y, h_1) + d^{(0,1)}f(x, y, h_2) \tag{2.0.1}$$

for all $(x, y) \in U \times V$ and $(h_1, h_2) \in E_1 \times E_2$.

More generally, using the method of the proof of Rule on Partial Differentials as in [14], one obtains the following proposition.

Proposition 2.4 (*Rule on Partial Differentials*). *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$, $U := U_1 \times \cdots \times U_n$ and $f : U_1 \times \cdots \times U_n \to F$ be a continuous map. Assume that there exist continuous functions $d_i f : U_1 \times \cdots \times U_n \times E_i \to F$ such that $D_{(w_i)^*}f(x_1, \ldots, x_n)$ exists and coincides with $d_i f(x_1, \ldots, x_n, w_i)$ for all $i \in \{1, \ldots, n\}$ and for all $(x_1, \ldots, x_n) \in U^0$, $w_i \in E_i$ and the corresponding element $(w_i)^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$. Then f is C^1 and*

$$df((x_1, \ldots, x_n), (w_1, \ldots, w_n)) = \sum_{i=1}^{n} d_i f(x_1, \ldots, x_n, w_i). \tag{2.0.2}$$

Proof. It is obvious that $d_i f$ exists for all $i \in \{1, \ldots, n\}$ if f is C^1. Conversely, assume that $d_i f$ exists for all $i \in \{1, \ldots, n\}$. If we can show that $f|_{U^0}$ is C^1 and (2.0.2) holds for $f|_{U^0}$, then the right hand side of (2.0.2) provides a continuous extension of $d(f|_{U^0})$ to $U_1 \times \cdots \times U_n \times (E_1 \times \cdots \times E_n)$, whence f is C^1 and (2.0.2) holds. We may therefore assume that $U_1 \times \cdots \times U_n$ is open in $E_1 \times \cdots \times E_n$. Given $(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$ and $w_i \in E_i$ for all $i \in \{1, \ldots, n\}$, there exists $\epsilon > 0$ such that $(x_1, \ldots, x_n) + \mathbb{D}_\epsilon w_1 \times \cdots \times \mathbb{D}_\epsilon w_n \subseteq U_1 \times \cdots \times U_n$, where $\mathbb{D}_\epsilon := \{z \in \mathbb{K} : |z| \le \epsilon\}$. Then $(x_1, \ldots, x_n) + [0, 1]tw_1 \times \cdots \times [0, 1]tw_n \subseteq U_1 \times \cdots \times U_n$ for each $0 \ne t \in \mathbb{D}_\epsilon$. By the Mean Value Theorem (see [14]), we obtain

$$\frac{1}{t}(f((x_1,\ldots,x_n)+t(w_1,\ldots,w_n))-f(x_1,\ldots,x_n))$$
$$=\sum_{j=1}^{n}\frac{1}{t}f(x_1+tw_1,\ldots,x_j+tw_j,x_{j+1},\ldots,x_n)$$
$$-\sum_{j=2}^{n}\frac{1}{t}f(x_1+tw_1,\ldots,x_{j-1}+tw_{j-1},x_j,\ldots,x_n)-\frac{1}{t}f(x_1\ldots,x_n)$$
$$=\sum_{j=1}^{n}\int_0^1 d_jf(x_1+tw_1,\ldots,x_{j-1}+tw_{j-1},x_j+\sigma tw_j,x_{j+1},\ldots,x_n,w_j)\,d\sigma. \tag{2.0.3}$$

Note that the integrals in (2.0.3) make sense also for $t=0$ (the integrands are then constants), and hence define mappings $I_1,\ldots,I_n:\mathbb{D}_\epsilon\to F$. The map $\mathbb{D}_\epsilon\times[0,1]\to F,\ (t,\sigma)\mapsto d_if(x_1+tw_1,\ldots,x_i+\sigma tw_i,x_{i+1},\ldots,x_n,w_i)$ being continuous for all $i\in\{1,\ldots,n\}$, the parameter-dependent integral I_i is continuous (see [14]). Hence the right hand side of (2.0.3) converges as $t\to 0$, with limit $I_1(0)+\cdots+I_2(0)=d_1f(x_1,\ldots,x_n,w_1)+\cdots+d_nf(x_1,\ldots,x_n,w_n)$. Hence df exists and is given by the right-hand side of (2.0.2) and hence continuous, whence f is C^1. $\square$

Remark 2.5. We shall use the following fundamental facts of C^r-maps.

1. $d^{(i)}f(x,\bullet)\colon E^i\to F$ is symmetric i-linear, for each i as in Definition 2.2.
2. $f\colon E\supseteq U\to F$ is C^{r+1} if and only if f is C^1 and $df\colon U\times E\to F$ is C^r.
3. The compositions of C^r-maps are C^r-maps.
4. The parameter-dependent integrals Theorem for continuous and differentiable maps (as recorded in [5, Prop. 3.5]).

Definition 2.6. Let E,F be locally convex topological vector spaces and $U\subseteq E$ a locally convex subset, $r\in\mathbb{N}_0\cup\{\infty\}$. We endow the space of continuous maps from U to F denoted by $C(U,F)$ with the compact open topology. Furthermore we topologize the space of C^r-maps from U to F denoted by $C^r(U,F)$ with the *compact open C^r-topology*, that is the unique topology turning

$$(d^{(i)}(\bullet))_{\mathbb{N}_0\ni i\leq r}\colon C^r(U,F)\to\prod_{0\leq i\leq r}C(U\times E^i,F),\ f\mapsto(d^{(i)}f)$$

into a topological embedding (the initial topology with respect to the family of mappings $(d^{(i)}(\bullet))_{\mathbb{N}\ni i\leq r}$).

3. C^α-mappings

Definition 3.1. Let $E_1,\ldots,E_n$ and F be locally convex spaces, U_i be an open subset of E_i for all $i\in\{1,\ldots,n\}$ and $\alpha:=(\alpha_1,\ldots,\alpha_n)$ such that $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$. A mapping $f:U_1\times\cdots\times U_n\to F$ is called a C^α-map, if for all $\beta_i\in\mathbb{N}_0$ such that $\beta_i\leq\alpha_i$ and $\beta:=(\beta_1,\ldots,\beta_n)$ the iterated directional derivative

$$d^\beta f(x,w_1,\ldots,w_n):=(\breve{D}_{w_1}\cdots\breve{D}_{w_n}f)(x)$$

where $(\breve{D}_{w_i}f)(x):=(D_{(w_i)^*_{\beta_i}}\cdots D_{(w_i)^*_1}f)(x)$, exists for all $x:=(x_1,\ldots,x_n)$ where $x_i\in U_i,\ w_i:=((w_i)_1,\ldots,(w_i)_{\beta_i})$ such that $(w_i)_1,\ldots,(w_i)_{\beta_i}\in E_i$,

$(w_i)^*_1, \ldots, (w_i)^*_{\beta_i} \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$ and

$$d^\beta f : U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \longrightarrow F,$$
$$(x, w_1, \ldots, w_n) \longmapsto (\breve{D}_{w_1} \cdots \breve{D}_{w_n} f)(x)$$

is continuous.

More generally, the following definition allows us to speak about C^α-maps on compact intervals.

Definition 3.2. Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, then we say that $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map, if $f|_{U_1^0 \times \cdots \times U_n^0} : U_1^0 \times \cdots \times U_n^0 \to F$ is a C^α-map and for all $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \ldots, \beta_n)$, the map

$$d^\beta (f|_{U_1^0 \times \cdots \times U_n^0}) : U_1^0 \times \cdots \times U_n^0 \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \longrightarrow F$$

admits a continuous extension

$$d^\beta f : U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \longrightarrow F.$$

The following lemma provides an alternative formulation of Definitions 3.1 and 3.2.

Lemma 3.3. *For all $i \in \{1, \ldots, n\}$. Let E_i and F be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. For $j \in \mathbb{N}$, $2 \leq j \leq n$, let $x := (x_1, \ldots, x_{j-1}) \in U := U_1 \times \cdots \times U_{j-1}$, $y := (x_j, \ldots, x_n) \in V := U_j \times \cdots \times U_n$, $\gamma := (\alpha_1, \ldots, \alpha_{j-1})$ and $\eta := (\alpha_j, \ldots, \alpha_n)$. Fix $i \in \{1, \ldots, n\}$, then $f : U_1 \times \cdots \times U_n \to F$ is a $C^{(\gamma,\eta)}$-map if and only if f has the following Properties:*

1. *For all $x \in U$, the map $f_x := f(x, \bullet) : V \to F$, $y \mapsto f_x(y) := f(x_1, \ldots, x_n)$ is C^η.*
2. *For all $y \in V$ and $w_i := ((w_i)_1, \ldots, (w_i)_{\beta_i}) \in E_i^{\beta_i}$, the map $U \to F$, $x \mapsto d^{(\beta_j, \ldots, \beta_n)} f_x(y, w_j, \ldots, w_n)$ is C^γ, where $\beta_i \in \mathbb{N}_0$, $\beta_i \leq \alpha_i$.*
3. *For $\beta := (\beta_1, \ldots, \beta_n)$, the map $d^\beta f : U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \to F$, $(x, y, w_1, \ldots, w_n) \mapsto$*
$$d^{(\beta_1, \ldots, \beta_{j-1})} (d^{(\beta_j, \ldots, \beta_n)} f_\bullet(y, w_j, \ldots, w_n))(x, w_1, \ldots, w_{j-1}),$$
is continuous.

Proof. *Step 1.* If $U_i \subseteq E_i$ is an open subset for all $i \in \{1, \ldots, n\}$ then the equivalence follows by the definition of the $C^{(\gamma,\eta)}$-map.

The general case. Assume that f is a $C^{(\gamma,\eta)}$-map.

Step 2. For all $k \in \{j, \ldots, n\}$, $v_k := ((v_k)_1, \ldots, (v_k)_{\beta_k}) \in E_k^{\beta_k}$ with corresponding elements $(v_k)^*_1, \ldots, (v_k)^*_{\beta_k} \in (\{0\})^{k-j} \times E_k \times (\{0\})^{n-k} \subseteq E_{j+1} \times \cdots \times E_n$ and for all $x \in U^0 := U_1^0 \times \cdots \times U_{j-1}^0$, then

$$(\breve{D}_{w_n} \cdots \breve{D}_{w_j}) f(x, y) = D_{(v_n)^*_{\beta_n}} \cdots D_{(v_j)^*_1} f_x(y)$$

exists for all $y \in V^0 := U_j^0 \times \cdots \times U_n^0$, with continuous extension

$$(y, v_j, v_{j+1}, \ldots, v_n) \mapsto d^{(0,\ldots,0,\beta_j,\ldots,\beta_n)} f(x, y, v_j, v_{j+1}, \ldots, v_n)$$

to $V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n} \to F$. Hence $f_x : V \to F$ is C^η. If $x \in U$ is arbitrary, $y \in V^0$, we show that $D_{(v_j)^*_1} f_x(y)$ exists and equals $d^{(0,\ldots,0,1,0\ldots,0)} f(x, y, (v_j)_1)$ with (j)-th entry 1. There exists $R > 0$ such that $y + t(v_j)_1^* \in V$ for all $t \in \mathbb{R}$, $|t| \leq R$ and there exists a relatively open convex neighbourhood $W \subseteq U$ of x in U. Because U^0 is dense, there exists $z \in U^0 \cap W$. Since W is convex, $x + \tau(z - x) \in W$ for all $\tau \in [0, 1]$. Moreover, since $z \in W^0$ we have for all $\tau \in (0, 1] \subseteq U^0$, $x + \tau(z - x) \in W^0$. Hence, for $\tau \in (0, 1]$, $f(x + \tau(z - x), y)$ is C^η in y, and thus by the Mean Value Theorem for $t \neq 0$

$$\frac{1}{t}(f(x + \tau(z - x), y + t(v_j)_1^*) - f(x + \tau(z - x), y))$$
$$= \int_0^1 d^{(0,\dots,0,1,0\dots,0)} f(x + \tau(z - x), y + \sigma t(v_j)_1^*, (v_j)_1^*)\, d\sigma.$$

Now let $\tilde{F}$ be a completion of F. By continuity of

$$h\colon\ [0, 1] \times [-R, R] \times [0, 1] \to \tilde{F},$$
$$(\tau, t, \sigma) \mapsto d^{(0,\dots,0,1,0\dots,0)} f(x + \tau(z - x), y + \sigma t(v_j)_1^*, (v_j)_1^*)$$

and continuity of the parameter-dependent integral

$$g\colon\ [0, 1] \times [-R, R] \to \tilde{F},\ \ g(\tau, t) := \int_0^1 h(\tau, t, \sigma)\, d\sigma.$$

Fix $t \neq 0$ in $[-R, R]$, then for all $\tau \in (0, 1]$

$$g(\tau, t) = \frac{1}{t}(f(x + \tau(z - x), y + t(v_j)_1^*) - f(x + \tau(z - x), y)). \tag{3.0.1}$$

Both sides are continuous in τ, (3.0.1) also holds for $\tau = 0$. Hence

$$g(0, t) = \frac{1}{t}(f(x, y + t(v_j)_1^*) - f(x, y)) \to g(0, 0)$$

as $t \to 0$. Thus $D_{(v_j)_1^*} f_x(y)$ exists and is given by

$$g(0, 0) = \int_0^1 d^{(0,\dots,0,1,0\dots,0)} f(x, y, (v_j)_1)\, d\sigma = d^{(0,\dots,0,1,0\dots,0)} f(x, y, (v_j)_1).$$

Fix $(v_j)_1$ and repeating the argument above, then for all $y \in V^0$ the directional derivative $D_{(v_j)_{\beta_j}^*} \cdots D_{(v_j)_1^*} f_x(y)$ exists and is given by

$$D_{(v_j)_{\beta_j}^*} \cdots D_{(v_j)_1^*} f_x(y) = d^{(0,\dots,0,\beta_j,0\dots,0)} f(x, y, v_j).$$

By repeating the argument again, then for all $v := (v_j, v_{j+1}, \dots, v_n) \in E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$ and $y \in V^0$ the directional derivative $D_{(v_n)_{\beta_n}^*} \cdots D_{(v_j)_1^*} f_x(y)$ exists and is given by

$$D_{(v_n)_{\beta_n}^*} \cdots D_{(v_j)_1^*} f_x(y) = d^{(0,\dots,0,\beta_j,\beta_{j+1},\dots,\beta_n)} f(x, y, v).$$

For $(y, v) \in V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$ the right hand side makes sense and is continuous, then f_x is C^η.

Step 3 Fixing $v \in E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$, the function

$$(x, y) \mapsto d^{(0,\dots,0,\beta_j,\beta_{j+1},\dots,\beta_n)} f(x, y, v)$$

is $C^{(\gamma,0)}$.

Applying Step 2 to the $C^{(0,\gamma)}$ function $(y, x) \mapsto d^{(0,\dots,0,\beta_j,\beta_{j+1},\dots,\beta_n)} f(x, y, v)$, then

$$U \to F,\ \ x \mapsto d^{(0,\dots,0,\beta_j,\beta j+1,\dots,\beta_n)} f(x, y, v)$$

is C^{γ} for each $y \in V$ and for $w \in E_1^{\beta_1} \times \cdots \times E_{j-1}^{\beta_{j-1}}$, we get,

$$d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}(d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} f_{\bullet}(y,v))(x,w) = d^{\beta} f(x,y,w,v),$$

which is continuous in $(x,y,w,v) \in U \times V \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$. Hence if f is $C^{(\gamma,\eta)}$, then (1), (2) and (3) hold.

Conversely, assume that (1), (2) and (3) hold. By step 1, $f|_{U^0 \times V^0}$ is $C^{(\gamma,\eta)}$ and for $(x,y) \in U^0 \times V^0$

$$d^{\beta} f|_{U^0 \times V^0}(x,y,w,v) = d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})}(d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} f_{\bullet}(y,v))(x,w). \tag{3.0.2}$$

By (3), the right hand side of (3.0.2) admits continuous extension to $d^{\beta} f : U \times V \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \to F$. Hence the map f is a $C^{(\gamma,\eta)}$. $\square$

The following lemma is a tool to prove the Schwarz theorem for C^{α}-maps.

Lemma 3.4. *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be an open subset of E_i, $x_i \in U_i$ for all $i \in \{1, \ldots, n\}$, $x := (x_1, \ldots, x_n)$ and $\alpha := (\alpha_1, \ldots, \alpha_{n-1}, 1)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. If $f : U_1 \times \cdots \times U_n \to F$ is a C^{α}-map, then*

$$D_{(w_n)_1^*} D_{(w_1)_{\beta_1}^*} \cdots D_{(w_{n-1})_1^*} f(x) \tag{3.0.3}$$

exists for all $\beta_i \in \mathbb{N}_0$, $\beta_i \le \alpha_i$ and for all $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$ and corresponding elements $(w_i)_1^*, \ldots, (w_i)_{\beta_i}^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$, and it coincides with

$$d^{(\beta_1,\ldots,\beta_{n-1},1)} f(x, (w_1)_1, \ldots, (w_{n-1})_{\beta_{n-1}}, (w_n)_1). \tag{3.0.4}$$

Proof. The proof is by induction on n. If $n = 1$, there is nothing to show. Let $n \ge 2$. Now the proof is by induction on β_1. If $\beta_1 = 0$, holding the first variable fixed, we see that (3.0.3) exists and coincides with (3.0.4), by the case $n-1$. Now assume that $\beta_1 \ge 1$. If $\beta_i = 0$ for $i = 2, \ldots, n-1$, the assertion follows from [2, Lemma 3.5]. Assume that at least one of the $\beta_i \ge 1$ for $i = 2, \ldots, n-1$. By induction, we know that

$$D_{(w_n)_1^*} D_{(w_1)_{\beta_1-1}^*} D_{(w_1)_{\beta_1-2}^*} \cdots D_{(w_{n-1})_1^*} f(x)$$

exists and coincides with

$$d^{(\beta_1-1,\beta_2,\ldots,\beta_{n-1},1)} f(x, (w_1)_1, \ldots, (w_1)_{\beta_1-1}, (w_2)_1, \ldots, (w_n)_1). \tag{3.0.5}$$

Define $g : U_1 \times \cdots \times U_n \to F$ via

$$\begin{aligned} g(x) &:= D_{(w_1)_{\beta_1-1}^*} D_{(w_1)_{\beta_1-2}^*} \cdots D_{(w_{n-1})_1^*} f(x) \\ &= d^{(\beta_1-1,\beta_2,\ldots,\beta_{n-1},0)} f(x, (w_1)_1, \ldots, (w_1)_{\beta_1-1}, (w_2)_1, (w_2)_2, \ldots, (w_{n-1})_{\beta_{n-1}}). \end{aligned}$$

By the preceding, g is differentiable in the nth variable and continuous in $((w_n)_1, x)$ with

$$D_{(w_n)_1^*} g(x) \tag{3.0.6}$$

$$= d^{(\beta_1-1,\beta_2,\ldots,\beta_{n-1},1)} f(x, (w_1)_1, \ldots, (w_1)_{\beta_1-1}, \ldots, (w_{n-1})_{\beta_{n-1}}, (w_n)_1) \tag{3.0.7}$$

$$= D_{(w_1)_{\beta_1-1}^*} D_{(w_1)_{\beta_1-2}^*} \cdots D_{(w_{n-1})_1^*} D_{(w_n)_1^*} f(x). \tag{3.0.8}$$

Hence g is $C^{(0,\ldots,0,1)}$ and $d^{(0,\ldots,0,1)} g(x, (w_n)_1)$ is given by (3.0.5). Because f is C^{α} and $\alpha_1 \ge \beta_1$, (3.0.7) can be differentiated once more in the first variable, hence also $D_{(w_n)_1^*} g(x)$, with

$$\begin{aligned}&d^{(1,0,0,\dots,0,1)}g(x,(w_n)_1,(w_1)_{\beta_1})\\&=D_{(w_1)^*_{\beta_1}}D_{(w_n)^*_1}g(x)\\&=D_{(w_1)^*_{\beta_1}}\cdots D_{(w_{n-1})^*_{\beta_{n-1}}}D_{(w_n)^*_1}f(x)\\&=d^{(\beta_1,\dots,\beta_{n-1},1)}f(x,(w_1)_1,\dots,(w_{n-1})_{\beta_{n-1}},(w_n)_1).\end{aligned}$$

As this map is continuous, g is $C^{(1,0,\dots,0,1)}$.

By [2, Lemma 3.4], also $D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}g(x)$ exists and is given by

$$D_{(w_1)^*_{\beta_1}}D_{(w_n)^*_1}g(x)=d^{(\beta_1,\dots,\beta_{n-1},1)}f(x,(w_1)_1,\dots,(w_{n-1})_{\beta_{n-1}},(w_n)_1).$$

But, by definition of g, $D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}g(x)=D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}\cdots D_{(w_{n-1})^*_{n-1}}f(x)$. Hence

$$\begin{aligned}&D_{(w_n)^*_1}D_{(w_1)^*_{\beta_1}}\cdots D_{(w_{n-1})^*_{n-1}}f(x)\\&=d^{(\beta_1,\dots,\beta_{n-1},1)}f(x,(w_1)_1,\dots,(w_{n-1})_{\beta_{n-1}},(w_n)_1).\quad\square\end{aligned}$$

Proposition 3.5 (*Schwarz' Theorem for C^α-mappings*)**.** *For all $i\in\{1,\dots,n\}$, let E_i and F be locally convex spaces, $U_i\subseteq E_i$ an open subset, $x_i\in U_i$ and $\alpha_i\in\mathbb{N}_0\cup\{\infty\}$ with $\alpha:=(\alpha_1,\dots,\alpha_n)$. For $\beta_i\in\mathbb{N}_0$ such that $\beta_i\le\alpha_i$, we define $\beta:=(\beta_1,\dots,\beta_n)$, $\xi_i:=\sum_{m=1}^{i-1}\beta_m+1$, $\rho_i:=\sum_{m=1}^{i}\beta_m$, $w^*_{\xi_i},\dots,w^*_{\rho_i}\in(\{0\})^{i-1}\times E_i\times(\{0\})^{n-i}\subseteq E_1\times\cdots\times E_n$ with entries $w_{\xi_i},\dots,w_{\rho_i}$ in the E_i-coordinate. If $\sigma\in S_{\rho_n}$ is a permutation of $\{1,\dots,\rho_n\}$ and $f:U_1\times\cdots\times U_n\to F$ is a C^α-map. Then the iterated directional derivative*

$$(D_{w^*_{\sigma(1)}}\cdots D_{w^*_{\sigma(\rho_n)}}f)(x_1,\dots,x_n),$$

exists and coincides with $d^\beta f(x_1,\dots,x_n,w_1,\dots,w_{\rho_n})$.

Proof. The case $n=2$ having been settled in [2, Proposition 3.6], we may assume that $n\ge 3$ and assume that the assertion holds when n is replaced with $n-1$. We prove the nth case by induction on ρ_n. *The case* $\rho_n=0$ is trivial. *If at least one of the* $\beta_i=0$ *for* $i=1,\dots,n$ then the assertion follows from the assumption that n has been replaced with $n-1$. *The case $\beta_i\ge 1$, for all $i=1,\dots,n$.* If $\sigma(1)\in\{1,\dots,\beta_1\}$, then by induction for all $x=(x_1,\dots,x_n)\in U_1\times\cdots\times U_n$

$$\begin{aligned}&D_{w^*_{\sigma(2)}}\cdots D_{w^*_{\sigma(\rho_n)}}f(x)\\&=d^{(\beta_1-1,\beta_2,\dots,\beta_n)}f(x,w_1,\dots,w_{\sigma(1)-1},w_{\sigma(1)+1},\dots,w_{\beta_1},\dots,w_{\rho_n}).\end{aligned}$$

Because f is C^α, we can differentiate once more in the first variable:

$$\begin{aligned}&D_{w^*_{\sigma(1)}}\cdots D_{w^*_{\sigma(\rho_n)}}f(x)\\&=d^\beta f(x,w_1,\dots,w_{\sigma(1)-1},w_{\sigma(1)+1},\dots,w_{\rho_1},w_{\sigma(1)};w_{\xi_2},w_{\xi_2+1},\dots,w_{\rho_n})\\&=d^\beta f(x,w_1,w_2,\dots,w_{\rho_n}).\end{aligned}$$

For the final equality we used that, for $v_{\xi_i},\dots,v_{\rho_i}\in E_i$,

$$\begin{aligned}&d^\beta f(x,v_1,\dots,v_{\rho_n})\\&=d^{\beta_1}(d^{(\beta_2,\dots,\beta_n)}f_\bullet(x_2,\dots,x_n,v_{\xi_2},v_{\xi_2+1},\dots,v_{\rho_n}))(x_1,v_1,\dots,v_{\beta_1})\end{aligned}$$

is symmetric in $v_1,\dots,v_{\beta_1}\in E_1$, as

$$g(x_1):=d^{(\beta_2,\dots,\beta_n)}f_{x_1}(x_2,\dots,x_n,v_{\xi_2},v_{\xi_2+1},\dots,v_{\rho_n})$$

is C^{α_1} in x_1 (see Lemma 3.3).

If $\sigma(1) \in \{\xi_i, \dots, \rho_i\}$ for $i \in \{2, \dots, n\}$, then

$$D_{w^*_{\sigma(2)}} \cdots D_{w^*_{\sigma(\rho_n)}} f(x)$$
$$= d^{(\beta_1,\dots,\beta_i-1,\beta_{i+1},\dots,\beta_n)} f(x, w_1, \dots, w_{\xi_i}, \dots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \dots, w_{\rho_i}, \dots, w_{\rho_n}).$$

For fixed $w_{\xi_i}, \dots, w_{\rho_n}$, consider the function $h\colon U_1 \times \cdots \times U_n \to F$,

$$h(x) := d^{(0,\dots,0,\beta_i-1,\beta_{i+1},\dots,\beta_n)} f(x, w_{\xi_i}, \dots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \dots, w_{\rho_i}, \dots, w_{\rho_n}).$$

By Lemma 3.4, $D_{w^*_{\sigma(1)}} D_{w^*_{\rho_1}} \cdots D_{w^*_{\xi_{i-1}}} h(x)$ exists and coincides with

$$D_{w^*_{\rho_1}} \cdots D_{w^*_{\xi_{i-1}}} D_{w^*_{\sigma(1)}} h(x).$$

Now, by induction,

$$D_{w^*_{\sigma(2)}} \cdots D_{w^*_{\sigma(\rho_n)}} f(x)$$
$$= d^{(\beta_1,\dots,\beta_i-1,\beta_{i+1},\dots,\beta_n)} f(x, w_1, \dots, w_{\xi_i}, \dots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \dots, w_{\rho_i}, \dots, w_{\rho_n})$$
$$= D_{w^*_{\rho_1}} \cdots D_{w^*_{\xi_{i-1}}} h(x).$$

Let $y := (x_2, \dots, x_n)$ and let ψ denote

$$d^{(\beta_2,\dots,\beta_n)} f_\bullet(y, w_{\xi_2}, \dots, w_{\xi_i}, \dots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \dots, w_{\rho_i}, w_{\sigma(1)}, w_{\xi_{i+1}}, \dots, w_{\rho_n}).$$

By the preceding, we can apply, $D_{w^*_{\sigma(1)}}$, i.e., $D_{w^*_{\sigma(1)}} \cdots D_{w^*_{\sigma(\rho_n)}} f(x)$ exists and coincides with

$$D_{w^*_{\rho_1}} \cdots D_{w^*_{\xi_{i-1}}} D_{w^*_{\sigma(1)}} h(x)$$
$$= d^{\beta} f(x, w_1, \dots, w_{\xi_i}, \dots, w_{\sigma(1)-1}, w_{\sigma(1)+1}, \dots, w_{\rho_i}, w_{\sigma(1)}, w_{\xi_{i+1}}, \dots, w_{\rho_n})$$
$$= d^{\beta_1} \psi(x_1, w_1, \dots, w_{\rho_1})$$

where $d^{(\beta_2,\dots,\beta_n)} f_{x_1}(y, v_{\xi_2}, v_{\xi_2+1}, \dots, v_{\rho_n})$ is symmetric in $v_i, \dots, v_{\rho_i} \in E_i$ by induction on n for the $C^{(\alpha_2,\dots,\alpha_n)}$-function f_{x_1}. Hence also after differentiations in x_1:

$$d^{\beta_1} \psi(x_1, w_1, \dots, w_{\rho_1})$$

coincides with $d^{\beta} f(x, w_1, \dots, w_{\rho_n})$. $\square$

Corollary 3.6. *Under the assumptions of the preceding proposition, if*

$$f : U_1 \times \cdots \times U_n \to F$$

is a C^α-map, $\alpha_\sigma := (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$ and $\beta_\sigma := (\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)})$. Then $g : U_{\sigma(1)} \times \cdots \times U_{\sigma(n)} \to F$ is C^{α_σ} and

$$d^{\beta_\sigma} g(x_{\sigma(1)}, \dots, x_{\sigma(n)}, v_{\sigma(1)}, \dots, v_{\sigma(n)}) = d^{\beta} f(x_1, \dots, x_n, w_1, \dots, w_n),$$

where $w_i := w_{\xi_i}, \dots, w_{\rho_i}$.

Lemma 3.7. *Let $E_1, \dots, E_n$, F and H be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \dots, n\}$ and $\alpha := (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, if $\lambda : F \to H$ is a continuous linear map and $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map, then $\lambda \circ f$ is C^α and $d^\beta(\lambda \circ f) = \lambda \circ d^\beta f$ for all $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \dots, \beta_n)$.*

Proof. This follows directly from the fact that continuous linear maps and directional derivatives can be interchanged. $\square$

Lemma 3.8 (*Mappings to Products for C^α-maps*)**.** *Let $E_1, \ldots, E_n$ be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$, and $(F_j)_{j\in J}$ be a family of locally convex spaces with direct product $F := \prod_{j\in J} F_j$ and the projections $\pi_j : F \to F_j$ onto the components. Let $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and $f : U_1 \times \cdots \times U_n \to F$ be a map. Then f is C^α if and only if all of its components $f_j := \pi_j \circ f$ are C^α. In this case*

$$d^\beta f = (d^\beta f_j)_{j\in J}, \tag{3.0.9}$$

for all $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \ldots, \beta_n)$.

Proof. π_j is continuous linear. Hence if f is C^α, then $f_j = \pi_j \circ f$ is C^α, by Lemma 3.7, with $d^\beta f_j = \pi_j \circ d^\beta f$. Hence (3.0.9) holds. Conversely, assume that each f_j is C^α. The limit in products can be formed component-wise, thus for all $(x_1, \ldots, x_n) \in U_1^0 \times \cdots \times U_n^0$, $w_i := ((w_i)_1, \ldots, (w_i)_{\beta_i})$ such that $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$,

$$d^\beta f(x_1, \ldots, x_n, w_1, \ldots, w_n) = (\check{D}_{w_1} \cdots \check{D}_{w_n} f)(x_1, \ldots, x_n)$$

exists and is given by

$$(d^\beta f_j(x_1, \ldots, x_n, w_1, \ldots, w_n))_{j\in J} \tag{3.0.10}$$

Now (3.0.10) defines a continuous function

$$U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \longrightarrow F.$$

Hence f is C^α. $\square$

Lemma 3.9. *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, $\alpha_n \geq 1$. If $f : U_1 \times \cdots \times U_n \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$, f is $C^{(0,\ldots,0,1)}$ and $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$, then f is C^α.*

Proof. Let $\beta_i \in \mathbb{N}_0$, $\beta_i \leq \alpha_i$, $w_i := ((w_i)_1, \ldots, (w_i)_{\beta_i})$ where $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$. Consider also the corresponding elements $(w_i)_1^*, \ldots, (w_i)_{\beta_i}^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$. *If* $\beta_n = 0$, for all $x := (x_1, \ldots, x_n) \in U_1^0 \times \cdots \times U_n^0$, $(\check{D}_{w_1} \cdots \check{D}_{w_{n-1}} f)(x)$ exists as f is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$, and is given by $d^{(\beta_1,\ldots,\beta_{n-1},0)} f(x, w_1, \ldots, w_{n-1})$ which extends continuously to

$$U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_{n-1}^{\beta_{n-1}}.$$

If $\beta_n > 0$, then $D_{(w_n)_1^*} f(x) = d^{(0,\ldots,0,1)} f(x, (w_n)_1)$ exists because f is $C^{(0,\ldots,0,1)}$ and because this function is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$, also the directional derivatives

$$\begin{aligned}&(\check{D}_{w_1} \cdots \check{D}_{w_n} f)(x)\\ &= (D_{((w_1)_{\beta_1}^*,0)} \cdots D_{((w_{n-1})_1^*,0)} D_{((w_n)_{\beta_n}^*,0)} \cdots D_{((w_n)_2^*,0)} d^{(0,\ldots,0,1)} f)(x, (w_n)_1)\end{aligned}$$

exist and the right hand side extends continuously to

$$(x, (w_1)_1, \ldots, (w_n)_{\beta_n}) \in U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}.$$

Hence f is C^α. $\square$

Lemma 3.10. *Let $E_1, \ldots, E_n, H_1, \ldots, H_n$ and F be locally convex spaces, $U_i \subseteq E_i$, $P_i \subseteq H_i$ be locally convex subsets with dense interior for all $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, if $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map and $\lambda_i : H_i \to E_i$ is a continuous linear map such that $\lambda_i(P_i) \subseteq U_i$, then $f \circ (\lambda_1 \times \cdots \times \lambda_n)|_{P_1 \times \cdots \times P_n} : P_1 \times \cdots \times P_n \to F$ is C^α.*

Proof. Let $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \ldots, \beta_n)$. For $(p_1, \ldots, p_n) \in P_1^0 \times \cdots \times P_n^0$, $(w_i)_1, \ldots, (w_i)_{\beta_i} \in H_i$ and corresponding elements $(w_i)_1^*, \ldots, (w_i)_{\beta_i}^* \in (\{0\})^{i-1} \times H_i \times (\{0\})^{n-i} \subseteq H_1 \times \cdots \times H_n$, we have

$$\begin{aligned}
&D_{(w_n)_1^*}(f \circ (\lambda_1 \times \cdots \times \lambda_n))(p_1, \ldots, p_n)\\
&= \lim_{t \to 0} \frac{1}{t}(f(\lambda_1(p_1), \ldots, \lambda_{n-1}(p_{n-1}), \lambda_n(p_n + t(w_n)_1)) - f(\lambda_1(p_1), \ldots, \lambda_n(p_n)))\\
&= \lim_{t \to 0} \frac{1}{t}(f(\lambda_1(p_1), \ldots, \lambda_n(p_n) + t\lambda_n((w_n)_1)) - f(\lambda_1(p_1), \ldots, \lambda_n(p_n)))\\
&= D_{(0,\ldots,0,\lambda_n((w_n)_1))} f)(\lambda_1(p_1), \ldots, \lambda_n(p_n))
\end{aligned}$$

and recursively

$$\begin{aligned}
&\breve{D}_{w_1} \cdots \breve{D}_{w_n}(f \circ (\lambda_1 \times \cdots \times \lambda_n))(p_1, \ldots, p_n)\\
&= d^\beta f(\lambda_1(p_1), \ldots, \lambda_n(p_n), \lambda_1((w_1)_1), \ldots, \lambda_n((w_n)_{\beta_n})).
\end{aligned}$$

the right hand side defines a continuous function of $(p_1, \ldots, p_n, (w_1)_1, \ldots, (w_n)_{\beta_n}) \in P_1 \times \cdots \times P_n \times H_1^{\beta_1} \times \cdots \times H_n^{\beta_n}$. Hence the assertion follows. □

Lemma 3.11. *Let $E_1, \ldots, E_n$, $H_1, \ldots, H_m$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$, $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, $H := H_1 \times \cdots \times H_m$ and $f : U_1 \times \cdots \times U_n \times H \to F$ be a map that satisfies the following conditions:*

1. *For all $x := (x_1, \ldots, x_n)$, $x_i \in U_i$, the map $f(x, \bullet) : H \to F$ is m-linear;*
2. *The map $f : U_1 \times \cdots \times U_n \times H \to F$ is $C^{(\alpha,0)}$.*

Then $f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \to F$ is C^α. Also $g : U_1 \times \cdots \times U_{i-1} \times (U_i \times H) \times U_{i+1} \times \cdots \times U_n \to F$, $(x_1, \ldots, x_{i-1}, (x_i, h), x_{i+1}, \ldots, x_n) \mapsto f(x, h)$ is C^α.

Proof. Holding $h \in H$ fixed, the map $f(\bullet, h)$ is C^α and hence, for a permutation $\sigma \in S_n$ of $\{1, \ldots, n\}$, we have $U_{\sigma(1)} \times \cdots \times U_{\sigma(n)} \to F$, $x \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, h)$ is C^{α_σ}, where $\alpha_\sigma := (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$, by Corollary 3.6. Hence $f_1 : U_{\sigma(1)} \times \cdots \times U_{\sigma(n-1)} \times (U_{\sigma(n)} \times H) \to F$, $f_1(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, h) := f(x, h)$ satisfying hypotheses analogous to those satisfied by f (with $\alpha_{\sigma(i)}$ interchanged) and will be a C^{α_σ} if the first assertion holds. Using Corollary 3.6, this implies that g is C^α. Therefore we will prove the first assertion by induction on α_n after assuming without loss of generality that $\alpha_i < \infty$.

The case $\alpha_n = 0$. Then f is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$ by the hypotheses.

Induction step. For $(w_n)_1 \in E_n$, $z = (z_1, \ldots, z_m) \in H$. By hypothesis, $D_{(0,\ldots,0,(w_n)_1,0)} f(x, h)$ exists and extends continuously on $U_1 \times \cdots \times U_n \times E_n \times H \to F$. Because $f(x, \bullet) : H \to F$ is m-linear continuous map, it is C^1 with

$$D_{(0,\ldots,0,z)} f(x, h) = \sum_{k=1}^{m} f(x, h_1, \ldots, h_{k-1}, z_k, h_{k+1}, \ldots, h_m).$$

This formula defines a continuous function $U_1 \times \cdots \times U_n \times E_n \times H \rightarrow F$. Holding $(x_1, \ldots, x_{n-1}) \in U_1 \times \cdots \times U_{n-1}$ fixed, we deduce by Proposition 2.4 (Rule on Partial Differentials) that $U_n \times H \rightarrow F, (x_n, h) \mapsto f(x, h)$ is C^1-map, with

$$D_{(0,\ldots,0,(w_n)_1,z)} f(x, h) \tag{3.0.11}$$
$$= D_{(0,\ldots,0,(w_n)_1,0)} f(x, h) + \sum_{k=1}^{m} f(x, h_1, \ldots, h_{k-1}, z_k, h_{k+1}, \ldots, h_m).$$

Because we have just seen that $d^{(0,\ldots,0,1)} f(x_1, \ldots, x_{n-1}, (x_n, h), ((w_n)_1, z))$ exists and is given by (3.0.11), which extends to a continuous map on $U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \times (E_n \times H)$, the map $f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \rightarrow F$ is $C^{(0,\ldots,0,1)}$. Also, $f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H) \rightarrow F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$ by the hypothesis.

We claim that $d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_{n-1} \times (U_n \times H \times E_n \times H) \rightarrow F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$. If this is true, then f is C^α by Lemma 3.9. To prove the claim, for fixed $k \in \{1, \ldots, m\}$, consider

$$\phi : U_1 \times \cdots \times U_{n-1} \times (U_n \times H \times E_n \times H) \rightarrow F,$$
$$(x, h, (w_n)_1, z) \mapsto f(x, h_1, \ldots, h_{k-1}, z_k, h_{k+1}, \ldots, h_m).$$

The map

$$\psi : U_1 \times \cdots \times U_n \times H_1 \times \cdots \times H_{m-1} \times (H_m \times E_n \times H) \rightarrow F,$$
$$(x, h_1, \ldots, h_{m-1}, (h_m, (w_n)_1, z)) \mapsto f(x, h_1, \ldots, h_m)$$

is m-linear in $(h_1, \ldots, h_{n-1}, (h_n, (w_n)_1, z))$. By induction, ψ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$ as a map on $U_1 \times \cdots \times U_{n-1} \times (U_n \times H_1 \times \cdots \times H_m \times E_n \times H)$. By Lemma 3.10, also ϕ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$. Hence each of the final k summands in (3.0.11) is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$ in $(x, h_1, \ldots, h_{m-1}, (h_m, (w_n)_1, z))$. To take care of the first summands in (3.0.11), observe that $\theta \colon U_1 \times \cdots \times U_n \times (H \times E_n) \rightarrow F$, $(x, h, (w_n)_1) \mapsto D_{(0,\ldots,0,(w_n)_1,0)} f(x, h)$ is $(m+1)$-linear in the final argument and satisfies hypotheses analogous to those of f, with $(\alpha_1, \ldots, \alpha_n)$ replaced by $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1)$. Hence by induction, θ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$. Consequently, $d^{(0,\ldots,0,1)} f$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$ (like each of the summands in (3.0.11)). □

The following lemma illustrates the relation between $C^{(r,s)}$-maps and C^α-maps.

Lemma 3.12. *For all $i \in \{1, \ldots, n\}$, let E_i, H_1, H_2 and F be locally convex spaces, $U_i \subseteq E_i$, $V \subseteq H_1$ and $W \subseteq H_2$ be locally convex subsets with dense interior and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Assume that $U_n = V \times W \subseteq H_1 \times H_2 = E_n$. If for all $k, l \in \mathbb{N}_0$ with $k + l \leq \alpha_n$ the map $f : U_1 \times \cdots \times U_{n-1} \times V \times W \rightarrow F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},k,l)}$, then $f : U_1 \times \cdots \times U_n \rightarrow F$ is C^α.*

Proof. We may assume that $\alpha_n < \infty$. The proof is by induction on α_n. For the case $\alpha_n = 0$, the assertion follows by the definition of the C^α-map. For the case $\alpha_n \geq 0$, let $x := (x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n$ and $(h_1, h_2), (h'_1, h'_2) \in H_1 \times H_2$. By the Rule on Partial Differentials (Proposition 2.4),

$$d^{(0,\ldots,0,1)} f(x, (h_1, h_2)) = d^{(0,\ldots,0,1,0)} f(x, h_1) + d^{(0,\ldots,0,1)} f(x, h_2).$$

Also, for fixed (h_1, h_2) and differentiations in the x-variables,

$$D_{(0,\ldots,0,(h'_1,h'_2))}(d^{(0,\ldots,0,1)} f(x, (h_1, h_2))) \tag{3.0.12}$$
$$= D_{(0,\ldots,0,(h'_1,h'_2))}(d^{(0,\ldots,0,1,0)} f(x, h_1) + d^{(0,\ldots,0,1)} f(x, h_2)). \tag{3.0.13}$$

Using Lemma 3.13, we can show that (3.0.13) is $C^{(\alpha_1,\ldots,\alpha_{n-1},k,l)}$ as a map on $U_1 \times \cdots \times U_{n-1} \times (V \times H_1^2) \times (W \times H_2^2)$ for $k+l \leq \alpha_{n-2}$, hence by induction and again by Lemma 3.13, (3.0.12) is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-2)}$ on $U_1 \times \cdots \times (U_n \times E_n r)$. Thus, $d^{(0,\ldots,0,1)}f : U_1 \times \cdots \times U_n \times E_n \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$ and by induction $f : U_1 \times \cdots \times U_n \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$. Hence, it is C^α, by Lemma 3.13. □

Lemma 3.13. *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$, $\alpha_n \geq 1$. Then $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map if and only if f is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$, f is $C^{(0,\ldots,0,1)}$ and $d^{(0,\ldots,0,1)}f : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to F$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$.*

Proof. If f is C^α, then f is $C^{(\alpha_1,\ldots,\alpha_{n-1},0)}$ and f is $C^{(0,\ldots,0,1)}$. Also $d^{(0,\ldots,0,1)}f : U_1 \times \cdots \times U_n \times E_n \to F$ is linear in the E_n-variable and for all $\beta_i \in \mathbb{N}_0$, $\beta_i \leq \alpha_i$, $\beta_n \leq \alpha_n - 1$, $(x_1, \ldots, x_n) \in U_1^0 \times \cdots \times U_n^0$, $(w_i)_1, \ldots, (w_i)_{\beta_i} \in E_i$ and corresponding elements $(w_i)_1^*, \ldots, (w_i)_{\beta_i}^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$,

$$\begin{aligned} &D_{((w_1)_{\beta_1}^*,0)} \cdots D_{((w_n)_1^*,0)}(d^{(0,\ldots,0,1)}f)(x_1, \ldots, x_n, z) \\ &= d^{(\beta_1,\ldots,\beta_{n-1},\beta_n+1)}f(x_1, \ldots, x_n, (w_1)_1, \ldots, (w_{n-1})_{\beta_{n-1}}, z, (w_n)_1, \ldots, (w_n)_{\beta_n}) \end{aligned}$$

exists and extends continuously in $U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$. Hence by Lemma 3.11, $d^{(0,\ldots,0,1)}f$ is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$. The converse has already been established in Lemma 3.9. □

Lemma 3.14. *Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i be a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$ and $\alpha_0 \in \mathbb{N}_0$, if the map $f : U_1 \times \cdots \times U_n \to F$ is $C^{(\alpha_0,\ldots,\alpha_0)}$, then f is C^{α_0}.*

Proof. The proof is by induction on α_0. *The case* $\alpha_0 = 0$. If f is $C^{(0,\ldots,0)}$, then f is continuous and hence C^0. *The case* $\alpha_0 \geq 1$. Assume that $U_1, \ldots, U_n$ are open subset. Then $D_{(w_i)^*}f(x_1, \ldots, x_n)$ exists and is continuous in

$(x_1, \ldots, x_n, w_i)$ for all $x_i \in U_i$ and all $i \in \{1, \ldots, n\}$, where $w_i \in E_i$, $(w_i)^* \in (\{0\})^{i-1} \times E_i \times (\{0\})^{n-i} \subseteq E_1 \times \cdots \times E_n$. Hence by Proposition 2.4 f is C^1 and

$$df((x_1, \ldots, x_n)(w_1, \ldots, w_n)) = D_{(w_1)^*}f(x_1, \ldots, x_n) + \cdots + D_{(w_n)^*}f(x_1, \ldots, x_n), \tag{3.0.14}$$

which is continuous in $(x_1, \ldots, x_n, w_1, \ldots, w_n)$. Thus f is C^1. In the general case, the right hand side of (3.0.14) is continuous in $(x_1, \ldots, x_n, w_1, \ldots, w_n) \in U_1 \times \cdots \times U_n \times E_1 \times \cdots \times E_n$ and extends $d(f|_{U_1^0 \times \cdots \times U_n^0})$. Hence f is C^1. Next note that $D_{(w_i)^*}f(x_1, \ldots, x_n)$ is $C^{(\alpha_0-1,\ldots,\alpha_0-1)}$-mappings, by Lemma 3.9 and Corollary 3.6. Hence df is C^{α_0-1}, by induction. Since f is C^1 and df is C^{α_0-1}, then f is C^{α_0}. □

As an immediate consequence of Lemma 3.14.

Remark 3.15. The map $f : U_1 \times \cdots \times U_n \to F$ is smooth if and only if it is $C^{(\infty,\ldots,\infty)}$.

Lemma 3.16 (*Chain Rule for C^α-mappings*)**.** *For all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m_i\}$, let $E_i, X_{i,j}$ and F be locally convex spaces, $U_i \subseteq E_i$, $P_{i,j} \subseteq X_{i,j}$ be locally convex subsets with dense interior,*

$\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0 \cup \{\infty\})^n$, $f : U_1 \times \cdots \times U_n \to F$ is a C^α-map and $g_i : P_{i,1} \times P_{i,2} \times \cdots \times P_{i,m_i} \to U_i$ is a C^{γ_i}-map, where $\gamma_i := (\gamma_{i,1}, \ldots, \gamma_{i,m_i}) \in (\mathbb{N}_0 \cup \{\infty\})^{m_i}$, $|\gamma_i| := \gamma_{i,1} + \cdots + \gamma_{i,m_i} \leq \alpha_i$. Then the composition

$$f \circ (g_1 \times \cdots \times g_n) : (P_{1,1} \times \cdots \times P_{1,m_1}) \times \cdots \times (P_{n,1} \times \cdots \times P_{n,m_n}) \to F,$$
$$(p_{1,1}, \ldots, p_{n,m_n}) \mapsto f(g_1(p_{1,1}), \ldots, g_n(p_{n,m_n}))$$

is a $C^{(\gamma_1,\ldots,\gamma_n)}$-map.

Proof. Without loss of generality, we may assume that $\gamma_i < \infty$ for all $i \in \{1, \ldots, n\}$. The proof is by induction on $|\gamma| := |\gamma_1| + \cdots + |\gamma_n|$. If $|\gamma| = 0$, then $f \circ (g_1 \times \cdots \times g_n)$ is just a composition of continuous maps, which is continuous, hence $C^{(0,\ldots,0)}$. Now if $|\gamma| > 0$, by Corollary 3.6, we may assume that $\gamma_n > 0$. Again by Corollary 3.6, we may assume that $\gamma_{n,m_n} > 0$. For $p := (p_1, \ldots, p_n) \in P_1 \times \cdots \times P_n$ and $z \in X_{n,m_n}$, the map $d^{(0,\ldots,0,1)} g_n(p_n, z)$ is $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$, by Lemma 3.13. Also, the function

$$P_{n,1} \times \cdots \times P_{n,m_n-1} \times P_{n,m_n} \times X_{n,m_n} \to U_n,$$
$$(p_{n,1}, \ldots, p_{n,m_n-1}, (p_{n,m_n}, z)) \mapsto g_n(p_{n,1}, \ldots, p_{n,m_n-1}, p_{n,m_n})$$

is C^{γ_n}, by Lemma 3.11. In particular, the latter is $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$. Thus both components of

$$\varphi : P_{n,1} \times \cdots \times P_{n,m_n} \times X_{n,m_n} \to U_n \times E_n,\ (p_n, h) \mapsto (g_n(p_n), d^{(0,\ldots,0,1)} g_n(p_n, z))$$

are $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$, so φ is $C^{(\gamma_{n,1},\ldots,\gamma_{n,m_i-1},\gamma_{n,m}-1)}$. By Lemma 3.13,

$$d^{(0,\ldots,0,1)} f : U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to F$$

is $C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$, whence $\alpha_n \geq 1$. Thus, by the preceding, the map

$$d^{(0,\ldots,0,1)}(f \circ (g_1 \times \cdots \times g_{n-1} \times \varphi))(p_1, \ldots, p_{n-1}, (p_n, z))$$

is $C^{(\gamma_1,\ldots,\gamma_{n-1},\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$. Hence,

$$d^{(0,\ldots,0,1)}(f \circ (g_1 \times \cdots \times g_n))(p, z) = (d^{(0,\ldots,0,1)} f)((g_1 \times \cdots \times g_n), d^{(0,\ldots,0,1)} g_n(p_n, z))$$

is $C^{(\gamma_1,\ldots,\gamma_{n-1},\gamma_{n,1},\ldots,\gamma_{n,m_n-1},\gamma_{n,m_n}-1)}$ and by induction, $f \circ (g_1 \times \cdots \times g_n) : (P_{1,1} \times \cdots \times P_{1,m_1}) \times \cdots \times (P_{n,1} \times \cdots \times P_{n,m_n}) \to F$ is $C^{(\gamma_1,\ldots,\gamma_{n-1},\gamma_{n,1},\ldots,\gamma_{n,m_n-1},0)}$. Hence, by Lemma 3.13, $f \circ (g_1 \times \cdots \times g_n)$ is a $C^{(\gamma_1,\ldots,\gamma_n)}$-map. □

Proposition 3.17. *Let $E_1, \ldots, E_n$ be finite-dimensional vector spaces and F be a locally convex space. For all $i \in \{1, \ldots, n\}$, let U_i be a locally compact and locally compact subset with dense interior of E_i and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Then the evaluation map*

$$\varepsilon : C^\alpha(U_1 \times \cdots \times U_n, F) \times U_1 \times \cdots \times U_n \to F,\ \varepsilon(\gamma, x_1, \ldots, x_n) := \gamma(x_1, \ldots, x_n)$$

is $C^{(\infty,\alpha)}$.

Proof. Without loss of generality, we may assume that $\alpha_i < \infty$ for all $i \in \{1, \ldots, n\}$. The proof is by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\alpha = 0$, then ε is continuous because U_i is locally compact ([7, Theorem 3.4.3]). Also, in the first argument ε is linear. Hence it is $C^{(\infty,0,\ldots,0)}$-map, by Lemma 3.11 and Corollary 3.6. If $\alpha \neq 0$, we may assume that $\alpha_n \geq 1$, Corollary 3.6. For $x_i \in U_i^0$, $w \in E_n$, $\gamma \in C^\alpha(U_1 \times \cdots \times U_n, F)$ and small $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}&\frac{1}{t}(\varepsilon(\gamma, x_1, \ldots, x_{n-1}, x_n + tw) - \varepsilon(\gamma, x_1, \ldots, x_n))\\ &= \frac{1}{t}(\gamma(x_1, \ldots, x_{n-1}, x_n + tw) - \gamma(x_1, \ldots, x_n)) \to d^{(0,\ldots,0,1)}\gamma(x_1, \ldots, x_n, w)\\ &\text{as } t \to 0.\end{aligned}$$

Hence $d^{(0,\ldots,0,1)}\varepsilon(\gamma, x_1, \ldots, x_n, w)$ exists and is given by

$$d^{(0,\ldots,0,1)}\varepsilon(\gamma, x_1, \ldots, x_n, w) \tag{3.0.15}$$
$$= d^{(0,\ldots,0,1)}\gamma(x_1, \ldots, x_n, w) \tag{3.0.16}$$
$$= \varepsilon_1(d^{(0,\ldots,0,1)}\gamma, (x_1, \ldots, x_n, w)), \tag{3.0.17}$$

where $\varepsilon_1 \colon C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}(U_1 \times \cdots \times U_{n-1} \times U_n \times E_n, F) \times (U_1 \times \cdots \times U_{n-1} \times U_n \times E_n) \to F$, $(\zeta, x_1, \ldots, x_{n-1}, z) \mapsto \zeta(x_1, \ldots, x_{n-1}, z)$ is $C^{(\infty,\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$, by induction.

As (3.0.17) defines a continuous map (in fact a $C^{(\infty,\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}$-map) by induction and Lemma 3.10, using that

$$C^\alpha(U_1 \times \cdots \times U_n, F) \to C^{(\alpha_1,\ldots,\alpha_{n-1},\alpha_n-1)}(U_1 \times \cdots \times U_{n-1} \times U_n \times E, F),$$
$$\gamma \mapsto d^{(0,\ldots,0,1)}\gamma$$

is continuous linear. Thus, by Lemma 3.13, ε is $C^{(\infty,\alpha)}$. □

Definition 3.18. Let $E_1, \ldots, E_n$ and F be locally convex spaces, U_i is a locally convex subset with dense interior of E_i for all $i \in \{1, \ldots, n\}$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Given $C^\alpha(U_1 \times \cdots \times U_n, F)$ the initial topology with respect to the mappings

$$d^\beta \colon C^\alpha(U_1 \times \cdots \times U_n, F) \to C(U_1 \times \cdots \times U_n \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F),\ \gamma \mapsto d^\beta\gamma$$

for $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$ and $\beta := (\beta_1, \ldots, \beta_n)$, where the space on the right hand side is endowed with the compact-open topology.

Lemma 3.19. *For all $i \in \{1, \ldots, n\}$. Let E_i and F be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and $\beta_i \in \mathbb{N}_0$ with $\beta_i \leq \alpha_i$. Define $U := U_1 \times \cdots \times U_n$ and $\varsigma_m := (\tau_1, \tau_2, \ldots, \tau_n)$ such that $m \in \mathbb{N}$, $\tau_i = 1, 2, \ldots, \beta_i$. Then the sets of the form*

$$W = \{f \in C^\alpha(U, F) \colon d^{\varsigma_m} f(K_{\varsigma_m}) \subseteq P_{\varsigma_m}\}$$

form a basis of 0-neighbourhoods for $C^\alpha(U, F)$. where $P_{\varsigma_m} := P_{\tau_1,\tau_2,\ldots,\tau_n} \subseteq F$ are 0-neighbourhoods and $K_\tau := K_{\tau_1,\tau_2,\ldots,\tau_n} \subseteq U \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n}$ is compact.

Proof. The space $C^\alpha(U, F)$ endowed with the initial topology with respect to the maps

$$d^{\varsigma_m} \colon C^\alpha(U, F) \to C(U \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n}, F)_{c.o},\ f \mapsto d^{\varsigma_m} f.$$

Therefore the map

$$\Psi \colon C^\alpha(U, F) \to \prod_{\mathbb{N}_0 \ni \tau_i \leq \alpha_i} C(U \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n}, F),\ f \mapsto (d^{\varsigma_m} f)_{\mathbb{N}_0 \ni \tau_i \leq \alpha_i}$$

is a topological embedding. Sets of the form

$$B := \{(g_{\varsigma m})_{\mathbb{N}_0\ni\tau_i\leq\alpha_i} \in \prod_{\mathbb{N}_0\ni\tau_i\leq\alpha_i} C(U \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n}, F) : \ g_{\varsigma m}(K_{\varsigma m}) \subseteq Q_{\varsigma m}\}$$

with compacts sets $K_{\varsigma m} \subseteq U \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n}$ and 0-neighbourhoods $Q_{\varsigma m} := (Q_{\tau_1,\tau_2,\ldots,\tau_n} \subseteq F)$, form a basis of 0-neighbourhoods in $\prod_{\mathbb{N}_0\ni\tau_i\leq\alpha_i} C(U \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n}, F)$. Hence the sets $\Phi^{-1}(W)$ form a basis of 0-neighbourhoods in $C^\alpha(U, F)$. $\square$

Theorem 3.20. *For all $i \in \{1, \ldots, n\}$. Let E_i and F be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. For $j \in \mathbb{N}$, $2 \leq j \leq n$, let $x := (x_1, \ldots, x_{j-1}) \in U := U_1 \times \cdots \times U_{j-1}$, $V := U_j \times \cdots \times U_n$, $\gamma := (\alpha_1, \ldots, \alpha_{j-1})$ and $\eta := (\alpha_j, \ldots, \alpha_n)$. If $f : U \times V \to F$ is $C^{(\gamma,\eta)}$, then*

1. *The map $f_x : V \to F$ is C^η.*
2. *The map $f^\vee : U \to C^\eta(V, F)$, $x \mapsto f_x$ is C^γ.*
3. *The mapping $\Phi : C^{(\gamma,\eta)}(U \times V, F) \to C^\gamma(U, C^\eta(V, F))$, $f \mapsto f^\vee$ is a linear topological embedding.*

Proof.
(1) By Lemma 3.3, $f_x : V \to F$ is C^η for all $x \in U$.
(2) As $C^\infty(U_i, F) = \varprojlim_{\alpha_i\in\mathbb{N}_0} C^{\alpha_i}(U_i, F)$ (see [14]), we have

$$C^\gamma(U, C^\infty(V, F)) = \varprojlim_{\eta\in(\mathbb{N}_0)^{n-j+1}} C^\gamma(U, C^\eta(V, F)).$$

it suffices to prove the assertion for $\eta \in (\mathbb{N}_0)^{n-j+1}$ (cf. [4, Lemma 10.3]). Without loss of generality, we may assume that $\gamma \in (\mathbb{N}_0)^{n-j+1}$. The proof is by induction on γ.

The case $\gamma = 0$. If $\eta = 0$, the assertion follows from [7, Theorem 3.4.1]. If $\eta \neq 0$, The space $C^\eta(V, F)$ endowed with the initial topology with respect to the maps

$$d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} : C^\eta(V, F) \to C(V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}, F)_{c.o}, \ g \mapsto d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}g,$$

for $\beta_i \in \mathbb{N}_0$ such that $\beta_j \leq \alpha_i$. Hence, we only need that $d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ f^\vee : U \to C(V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}, F)_{c.o}$ is continuous for $\beta_i \in \{0, 1, \ldots, \alpha_i\}$. Now

$$\begin{aligned} &d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}(f^\vee(x)) = d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}(f(x, \bullet)) \\ &= d^{(0,\ldots,0,\beta_j,\beta_{j+1},\ldots,\beta_n)} f(x, \bullet) = (d^{(0,\ldots,0,\beta_j,\beta_{j+1},\ldots,\beta_n)} f)^\vee(x). \end{aligned}$$

By induction $d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ f^\vee = (d^{(0,\ldots,0,\beta_j,\beta_{j+1},\ldots,\beta_n)} f)^\vee : U \to C(V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}, F)_{c.o}$ is continuous. Consequently, $g^\vee : U \to C^\eta(V, F)$ is continuous.

The case $\gamma \neq 0$, using Corollary 3.6, we may assume that, $\alpha_{j-1} \neq 0$. Let $\eta = 0$ then $f^\vee : U \to C(V, F)$. Let $x \in U^0 := U_1^0 \times \cdots \times U_{j-1}^0$, $z \in (\{0\})^{j-2} \times E_{j-1}$ then $x + tz \in U^0$, for small $t \in \mathbb{R} \cup \{\infty\}$; we show that

$$\frac{1}{t}(f^\vee(x + tz) - f^\vee(x)) \to d^{(0,\ldots,0,1,0\ldots,0)} f(x, \bullet, z)$$

in $C(V, F)$ as $t \to 0$. Therefore, for a compact subset $K \subseteq V$, we need to show that

$$(\frac{1}{t}(f^\vee(x + tz) - f^\vee(x)))|_K \to (d^{(0,\ldots,0,1,0\ldots,0)} f(x, \bullet, z))|_K$$

uniformly as $t \to 0$. Consider a 0-neighbourhood W in F. Without loss of generality, we can assume that W is closed and absolutely convex. There is $\varepsilon \geq 0$ such that $x + B_\varepsilon^{\mathbb{R}}(0)z \subseteq U^0$. For $y \in K$ and $t \in \mathbb{R} \setminus \{0\}$ such that $|t| < \varepsilon$, we have

$$\begin{aligned}\Delta(t,y) :&= \frac{1}{t}(f^\vee(x+tz) - f^\vee(x))(y) - d^{(0,\dots,0,1,0\dots,0)}f(x,y,z)\\ &= \frac{1}{t}(f(x+tz,y) - f(x,y)) - d^{(0,\dots,0,1,0\dots,0)}f(x,y,z)\\ &= \int_0^1 d^{(0,\dots,0,1,0\dots,0)}f(x+\sigma tz,y,z)\,d\sigma - d^{(0,\dots,0,1,0\dots,0)}f(x,y,z)\\ &= \int_0^1 (d^{(0,\dots,0,1,0\dots,0)}f(x+\sigma tz,y,z) - d^{(0,\dots,0,1,0\dots,0)}f(x,y,z))\,d\sigma.\end{aligned}$$

The function $\psi\colon B_\varepsilon^{\mathbb{R}}(0) \times K \times [0,1] \to F, (t,y,\sigma) \mapsto d^{(0,\dots,0,1,0\dots,0)}f(x+\sigma tz,y,z) - d^{(0,\dots,0,1,0\dots,0)}f(x,y,z)$ is continuous and $\psi(0,y,\sigma) = 0$ for all $(y,\sigma) \in K \times [0,1]$. Because $K \times [0,1]$ is compact, by the Wallace Lemma (see [7, 3.2.10]), there exists $\delta \in (0,\varepsilon]$ such that $\psi(B_\delta^{\mathbb{R}}(0) \times K \times [0,1]) \subseteq W$. Hence $\Delta(t,y) = \int_0^1 g(t,y,\sigma)\,d\sigma \in W$ for all $y \in K$ and all $t \in B_\delta^{\mathbb{R}}(0) \setminus \{0\}$. Because this holds for all $y \in K$, we see that $\Delta(t,\bullet) \to 0$ uniformly, as required. Thus $d^{(0,\dots,0,1)}f^\vee(x,z)$ exists for all $x \in U^0$, $z \in E_1 \times (\{0\})^{j-2}$ and is given by $d^{(0,\dots,0,1)}f^\vee(x,z) = d^{(0,\dots,0,1,0\dots,0)}f(x,\bullet,z)$. Now since $\gamma = 0$, the function $U \to C(V,F), x \mapsto d^{(0,\dots,0,1,0\dots,0)}f(x,\bullet,z)$ is continuous in all of U; so $f^\vee$ is $C^{(0,\dots,0,1)}$ on U, and $d^{(0,\dots,0,1)}f^\vee(x,z) = d^{(0,\dots,0,1,0\dots,0)}f(x,\bullet,z)$. Because

$$h\colon (U \times (\{0\})^{j-2} \times E_{j-1}) \times V \to F,\ ((x,z),y) \mapsto d^{(0,\dots,0,1,0\dots,0)}f(x,y,z)$$

is $C^{(\alpha_1,\dots,\alpha_{j-2},\alpha_{j-1}-1,\eta)}$ (see Lemma 3.13 and Corollary 3.6), by induction

$$d^{(0,\dots,0,1)}(f^\vee) = h^\vee\colon U \times E_{j-1} \to C(V,F)$$

is $C^{(\alpha_1,\dots,\alpha_{j-2},\alpha_{j-1}-1)}$. Hence f is C^γ.

Let $\eta \neq 0$, again by Corollary 3.6, we may assume that, $\alpha_n \neq 0$. Because

$$C^\eta(V,F) \to C(V,F) \times C^{(\alpha_j,\dots,\alpha_{n-1},\alpha_n-1)}(V \times E_n,F),\ \varphi \mapsto (\varphi, d^{(0,\dots,0,1)}\varphi)$$

is a linear topological embedding with closed image, $f^\vee\colon U \to C^\eta(V,F)$ will be C^γ if $f^\vee\colon U \to C(V,F)$ is C^γ (which holds by induction) and the map

$$h\colon U \to C^{(\alpha_j,\dots,\alpha_{n-1},\alpha_n-1)}(V \times E_n,F),\ x \mapsto d^{(0,\dots,0,1)}(f^\vee(x))$$

is C^γ (see [14]; cf. [4, Lemma 10.1]). For $x \in U$, $y \in V$ and $z \in (\{0\})^{n-j} \times E_n$, we have $h(x)(y,z) = d^{(0,\dots,0,1)}(f^\vee(x))(y,z) = d^{(0,\dots,0,1)}(f(x,\bullet))(y,z) = d^{(0,\dots,0,1)}f(x,y,z)$, thus $h = (d^{(0,\dots,0,1)}f)^\vee$ for $d^{(0,\dots,0,1)}f\colon U \times (V \times E_n) \to F$. By Lemma 3.13 this function is $C^{(\gamma,\alpha_j,\dots,\alpha_{n-1},\alpha_n-1)}$. Hence h is C^γ by induction.

(3) The linearity of Φ is clear. For $y \in V$, the point evaluation $\lambda\colon C^\eta(V,F) \to F$, $\psi \mapsto \psi(y)$ is continuous linear. Hence, for $\beta_i \in \mathbb{N}_0$, $\beta_i \leq \alpha_i$, $x \in U$ and $w \in E_1^{\beta_1} \times \cdots \times E_{j-1}^{\beta_{j-1}}$,

$$\begin{aligned}(d^{(\beta_1,\beta_2,\dots,\beta_{j-1})}f^\vee)(x,w)(y) &= \lambda((d^{(\beta_1,\beta_2,\dots,\beta_{j-1})}f^\vee)(x,w))\\ &= d^{(\beta_1,\beta_2,\dots,\beta_{j-1})}(\lambda \circ f^\vee)(x,w)\\ &= d^{(\beta_1,\beta_2,\dots,\beta_{j-1})}(f(\bullet,y)(x,w))\\ &= d^{(\beta_1,\beta_2,\dots,\beta_{j-1})}f(x,y,w),\end{aligned}$$

using that $(\lambda \circ f^\vee)(x) = \lambda(f^\vee(x)) = f^\vee(x)(y) = f(x, y)$. Hence

$$(d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})} f^\vee)(x, w) = (d^{(\beta_1,\beta_2,\ldots,\beta_{j-1},0,\ldots,0)} f)(x, \bullet, w).$$

Hence by Schwarz' Theorem (Proposition 3.5), for $v \in E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}$,

$$d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)}((d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})} f^\vee)(x, w))(y, v) = d^{(\beta_1,\beta_2,\ldots,\beta_n)} f(x, y, w, v).$$

Φ is continuous at 0. Let W be a 0-neighbourhood in $C^\gamma(U, C^\eta(V, F))$. After shrinking W, without loss of generality

$$W = \{f \in C^\gamma(U, C^\eta(V, F)) \colon d^{\varsigma_m} f(K_{\varsigma_m}) \subseteq P_{\varsigma_m}\}$$

where $\varsigma_m := (\tau_1, \tau_2, \ldots, \tau_{j-1})$, $m \in \mathbb{N}$, $\tau_i = 1, 2, \ldots, \beta_i$ such that $\beta_i \in \mathbb{N}_0$, $\beta_i \le \alpha_i$, $K_{\varsigma_m} := K_{\tau_1,\tau_2,\ldots,\tau_{j-1}} \subseteq U \times E_1^{\tau_1} \times \cdots \times E_{j-1}^{\tau_{j-1}}$ is compact and $P_{\varsigma_m} := P_{\tau_1,\tau_2,\ldots,\tau_{j-1}} \subseteq C^\eta(V, F)$ is 0-neighbourhood (see Lemma 3.19). Using Lemma 3.19 again, after shrinking P_τ, we may assume that,

$$P_{\varsigma_m} = \{g \in C^\eta(V, F) \colon d^{\rho_k} g(K_{\varsigma_m,\rho_k}) \subseteq P_{\varsigma_m,\rho_k}\}$$

where $\rho_k := (\tau_j, \tau_{j+1}, \ldots, \tau_n)$, $k \in \mathbb{N}$, $K_{\varsigma_m,\rho_k} := K_{\tau_1,\tau_2,\ldots,\tau_n} \subseteq V \times E_j^{\tau_j} \times \cdots \times E_n^{\tau_n}$ is compact and $P_{\varsigma_m,\rho_k} := P_{\tau_1,\tau_2,\ldots,\tau_n} \subseteq F$ is 0-neighbourhood shrinking P_{ς_m} further. Then W is the set of all $f \in C^\gamma(U, C^\eta(V, F))$ such that $d^{\rho_k}(d^{\varsigma_m} f(x, w))(y, v) \in P_{\varsigma_m,\rho_k}$ for all $(x, w) \in K_{\varsigma_m} \subseteq U \times E_1^{\tau_1} \times \cdots \times E_{j-1}^{\tau_{j-1}}$ and $(y, v) \in K_{\varsigma_m,\rho_k} \subseteq V \times E_j^{\tau_j} \times \cdots \times E_n^{\tau_n}$. The projections of $U \times E_1^{\tau_1} \times \cdots \times E_{j-1}^{\tau_{j-1}}$ onto the factors $U, E_1^{\tau_1}, \ldots, E_{j-1}^{\tau_{j-1}}$ are continuous, hence the images $K^1_{\varsigma_m}, K^2_{\varsigma_m}, \ldots, K^j_{\varsigma_m}$ of K_{ς_m} under these projections are compact. After replacing K_{ς_m} by $K^1_{\varsigma_m} \times K^2_{\varsigma_m} \times \cdots \times K^j_{\varsigma_m}$, without loss of generality $K_{\varsigma_m} = K^1_{\varsigma_m} \times K^2_{\varsigma_m} \times \cdots \times K^j_{\varsigma_m}$. Likewise, without loss of generality $K_{\varsigma_m,\rho_k} = K^1_{\varsigma_m,\rho_k} \times K^2_{\varsigma_m,\rho_k} \times \cdots \times K^{n-j+2}_{\varsigma_m,\rho_k}$ with compact sets $K^1_{\varsigma_m,\rho_k} \subseteq V$ and $K^2_{\varsigma_m,\rho_k} \subseteq E_j^{\tau_j}, \ldots, K^{n-j+2}_{\varsigma_m,\rho_k} \subseteq E_n^{\tau_n}$.

Now if $f \in C^{(\gamma,\eta)}(U \times V, F)$ then $d^{\rho_k}(d^{\varsigma_m} f^\vee(x, w))(y, v) = d^{(\varsigma_m,\rho_k)} f(x, y, w, v)$. Hence $f^\vee \in W$ if and only if $d^{(\varsigma_m,\rho_k)} f(K^1_{\varsigma_m} \times K^1_{\varsigma_m,\rho_k} \times K^2_{\varsigma_m} \times \cdots \times K^j_{\varsigma_m} \times K^2_{\varsigma_m,\rho_k} \times \cdots \times K^{n-j+2}_{\varsigma_m,\rho_k}) \subseteq P_{\varsigma_m,\rho_k}$. This is a basis neighbourhood in $C^{(\gamma,\eta)}(U \times V, F)$ (see Lemma 3.19). Thus $\Phi^{-1}(W)$ is a 0-neighbourhood, whence Φ is continuous at 0, and hence Φ is continuous.

It is clear that Φ is injective. To see that Φ is an embedding, it remains to show that $\Phi(W)$ is a 0-neighbourhood in $\mathrm{im}(\Phi)$ for each W in a basis of 0-neighbourhoods in $C^{(\gamma,\eta)}(U \times V, F)$.

Let $W := \{f \in C^{(\gamma,\eta)}(U \times V) : d^{(\varsigma_m,\rho_k)}(K_{\varsigma_m,\rho_k}) \subseteq P_{\varsigma_m,\rho_k}\}$, without loss of generality, after increasing K_{ς_m,ρ_k}, we may assume $K_{\varsigma_m,\rho_k} = L^1_{\varsigma_m,\rho_k} \times K^1_{\varsigma_m,\rho_k} \times L^2_{\varsigma_m,\rho_k} \times \cdots \times L^j_{\varsigma_m,\rho_k} \times K^2_{\varsigma_m,\rho_k} \times \cdots \times K^{n-j+2}_{\varsigma_m,\rho_k}$ with compact sets $L^1_{\varsigma_m,\rho_k} \subseteq U$, $K^1_{\varsigma_m,\rho_k} \subseteq V$, $L^2_{\varsigma_m,\rho_k} \times \cdots \times L^j_{\varsigma_m,\rho_k} \subseteq E_1^{\tau_1} \times \cdots \times E_{j-1}^{\tau_{j-1}}$ and $K^2_{\varsigma_m,\rho_k} \times \cdots \times K^{n-j+2}_{\varsigma_m,\rho_k} \subseteq E_j^{\tau_j} \times \cdots \times E_n^{\tau_n}$. Then $\Phi(W) := \{\varphi \in \mathrm{im}(\Phi) \colon d^\rho(d^\eta \varphi(x, w))(y, v) \in P_{\varsigma_m,\rho_k}\}$ for all $x \in L^1_{\varsigma_m,\rho_k}$, $y \in K^1_{\varsigma_m,\rho_k}$, $w \in L^2_{\varsigma_m,\rho_k} \times \cdots \times L^j_{\varsigma_m,\rho_k}$ and $v \in K^2_{\varsigma_m,\rho_k} \times \cdots \times K^{n-j+2}_{\varsigma_m,\rho_k}$, which is a 0-neighbourhood in $\mathrm{im}(\Phi)$, by Lemma 3.19. $\square$

Lemma 3.21. *Let Q be a topological space, for all $i \in \{1, \ldots, n\}$, let E_i, F be locally convex spaces, $\tau_i \in \mathbb{N}$ and $f \colon Q \times E_1^{\tau_1} \times \cdots \times E_n^{\tau_n} \to F$ be a map such that $f(x, w_1, \ldots, w_{i-1}, \bullet, w_{i+1}, \ldots, w_n) \colon E_i^{\tau_i} \to F$ is symmetric (τ_i)-linear for all $x \in Q$ and $w_i \in E_i^{\tau_i}$. Then f is continuous if and only if for $v_i \in E_i$, $g \colon Q \times E_1 \times \cdots \times E_n \to F$, $g(x, v_1, v_2, \ldots, v_n) := f(x, \underbrace{v_1, \ldots, v_1}_{\tau_1-times}, \underbrace{v_2, \ldots, v_2}_{\tau_2-times}, \ldots, \underbrace{v_n, \ldots, v_n}_{\tau_n-times})$ is continuous.*

Proof. The continuity of g follows directly from the continuity of f. If, conversely, g is continuous, then the assertion follows by n applications of the Polarization Identity [6, Theorem A]. $\square$

Theorem 3.22 (*Exponential Law for C^α-mappings*)**.** *For all $i \in \{1, \dots, n\}$. Let E_i and F be locally convex spaces, $U_i \subseteq E_i$ be a locally convex subset with dense interior, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and let $X_i := \{0\}$ if $\alpha_i = 0$, otherwise $X_i := E_i$. For $j \in \{2, \dots, n\}$ define $U := U_1 \times \cdots \times U_{j-1}$, $V := U_j \times \cdots \times U_n$, $\gamma := (\alpha_1, \dots, \alpha_{j-1})$, $\eta := (\alpha_j, \dots, \alpha_n)$. Assume that V is locally compact or $U \times V \times X_1 \times X_2 \times \cdots \times X_n$ is a k-space. Then*

$$\Phi\colon C^{(\gamma,\eta)}(U \times V, F) \to C^\gamma(U, C^\eta(V, F)),\ \ f \mapsto f^\vee$$

is an isomorphism of topological vector spaces. Moreover, if $g\colon U \to C^\eta(V, F)$ is C^γ, then

$$g^\wedge\colon U \times V \to F,\ \ g^\wedge(x, y) := g(x)(y)$$

is $C^{(\gamma,\eta)}$.

Proof. It suffices to prove the final assertion. In fact, given g in the space $C^\gamma(U, C^\eta(V, F))$, the map $g^\wedge$ will be $C^{(\gamma,\eta)}$ hence $g = (g^\wedge)^\vee = \Phi(g^\wedge)$. Thus Φ is surjective and by Theorem 3.20 it is an isomorphism of topological vector spaces.

Locally compact condition. Let $x := (x_1, \dots, x_{j-1}) \in U$, $y := (y_j, \dots, y_n) \in V$ and $\varepsilon\colon C^\eta(V, F) \times V \to F$, $(\psi, y) \mapsto \psi(y)$. By Proposition 3.17, $g^\wedge(x, y) = g(x)(y) = \varepsilon(g(x), y)$ is $C^{(\infty,\eta)}$. Hence $g^\wedge$ is $C^{(\gamma,\eta)}$ by Chain Rule for C^α-mappings (Lemma 3.16).

k-space condition. If $g\colon U \to C^\eta(V, F)$ is C^γ, define $g^\wedge\colon U \times V \to F$, $g^\wedge(x, y) = g(x)(y)$. For fixed $x \in U$, we have $g^\wedge(x, \bullet) = g(x)$ which is C^η, hence

$$\begin{aligned}(\breve{D}_{w_j} \cdots \breve{D}_{w_n} g^\wedge)(x, y) &= d^{(\beta_j, \beta j+1, \dots, \beta_n)}(g(x))(y, w_j, \dots, w_n)\\ &= (d^{(\beta_j, \beta j+1, \dots, \beta_n)} \circ g)(x)(y, w_j, \dots, w_n)\end{aligned}$$

exists for $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$, $y \in V^0 := U_j^0 \times \cdots \times U_n^0$ and $w_i \in E_i^{\beta_i}$. Also, for $\varepsilon_{(y, w_j, \dots, w_n)}\colon C^{(\alpha_j - \beta_j, \dots, \alpha_n - \beta_n)}(V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}, F) \to F$, $f \mapsto f(y, w_j, \dots, w_n)$,

$$(\breve{D}_{w_j} \cdots \breve{D}_{w_n} g^\wedge)(x, y) = (\varepsilon_{(y, w_j, \dots, w_n)} \circ d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)} \circ g)(x).$$

For fixed $(y, w_j, \dots, w_n)$, this is the function $\varepsilon_{(y, w_j, \dots, w_n)} \circ d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)} \circ g$ of x, which is C^γ. By linearity of $\varepsilon_{(y, w_j, \dots, w_n)}$ and

$$d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)}\colon C^\eta(V, F) \to C^{(\alpha_j - \beta_j, \dots, \alpha_n - \beta_n)}(V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n}, F)$$

we obtain the directional derivative

$$\begin{aligned}&(\breve{D}_{w_1} \cdots \breve{D}_{w_n} g)(x, y)\\ &= \varepsilon_{(y, w_j, \dots, w_n)}(d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)}(d^{(\beta_1, \beta_2, \dots, \beta_{j-1})} g(x, w_1, \dots, w_{j-1})))\\ &= d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)}(d^{(\beta_1, \beta_2, \dots, \beta_{j-1})} g(x, w_1, \dots, w_{j-1}))(y, w_j, \dots, w_n)\\ &= (d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)} \circ (d^{(\beta_1, \beta_2, \dots, \beta_{j-1})} g))(x, w_1, \dots, w_{j-1})(y, w_j, \dots, w_n)\\ &= (d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)} \circ (d^{(\beta_1, \beta_2, \dots, \beta_{j-1})} g))^\wedge((x, w_1, \dots, w_{j-1}), (y, w_j, \dots, w_n))\end{aligned}$$

for $x \in U^0 := U_1^0 \times \cdots \times U_n^0$. To see that $g^\wedge$ is $C^{(\gamma,\eta)}$, it therefore suffices to show that the map $h\colon U \times E_1^{\beta_1} \times \cdots \times E_{j-1}^{\beta_{j-1}} \times V \times E_j^{\beta_j} \times \cdots \times E_n^{\beta_n} \to F$, $h := (d^{(\beta_j, \beta_{j+1}, \dots, \beta_n)} \circ (d^{(\beta_1, \beta_2, \dots, \beta_{j-1})} g))^\wedge$

is continuous for all $\beta_i \in \mathbb{N}_0$ such that $\beta_i \leq \alpha_i$. By Lemma 3.21, this follows if we can show that

$$f : U \times X_1 \times \cdots \times X_{j-1} \times V \times X_j \times \cdots \times X_n \to F,$$

$$\begin{aligned}&(x, w_1, \ldots, w_{j-1}, y, w_j, \ldots, w_n)\\ &\mapsto h(x, \underbrace{w_1, \ldots, w_1}_{\beta_1-\text{times}}, \ldots, \underbrace{w_{j-1}, \ldots, w_{j-1}}_{\beta_{j-1}-\text{times}}, y, \underbrace{w_j, \ldots, w_j}_{\beta_j-\text{times}}, \ldots, \underbrace{w_n, \ldots, w_n}_{\beta_n-\text{times}})\end{aligned}$$

is continuous. Now

$$\psi : U \times X_1 \times \cdots \times X_{j-1} \to U \times E_1^{\beta_1} \times \cdots \times E_{j-1}^{\beta_{j-1}},$$
$$(x, w_1, \ldots, w_{j-1}) \mapsto (x, \underbrace{w_1, \ldots, w_1}_{\beta_1-\text{times}}, \underbrace{w_2, \ldots, w_2}_{\beta_2-\text{times}}, \ldots, \underbrace{w_{j-1}, \ldots, w_{j-1}}_{\beta_{j-1}-\text{times}})$$

is continuous and the map $\theta : U \times X_1 \times \cdots \times X_{j-1} \to C^0(V \times Y, F)$, $\theta := C^0(\varphi, F) \circ d^{(\beta_j,\beta_{j+1},\ldots,\beta_n)} \circ d^{(\beta_1,\beta_2,\ldots,\beta_{j-1})} g \circ \psi$, is continuous. By hypothesis $U \times X_1 \times \cdots \times X_{j-1} \times V \times X_j \times \cdots \times X_n$ is a k-space, it follows that $\theta^\wedge : U \times X_1 \times \cdots \times X_{j-1} \times V \times X_j \times \cdots \times X_n \to F$ is continuous (see [11, Proposition B.15]). Since $\theta^\wedge = f$, this implies the continuity of f. □

4. The exponential law for C^α-mappings on manifolds

Definition 4.1. For all $i \in \{1, \ldots, n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on a locally convex space, $\alpha := (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and F be a locally convex space. A map $f : M_1 \times \cdots \times M_n \to F$ is called C^α (in particular, continuous) if $f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1}) : V_{\varphi_1} \times \cdots \times V_{\varphi_n} \to F$ is C^α for all charts $\varphi_i : U_{\varphi_i} \to V_{\varphi_i}$ of M_i. Then in particular f is continuous.

Definition 4.2. In the situation of Definition 4.1, let $C^\alpha(M_1 \times \cdots \times M_n, F)$ be the space of all C^α-maps $f : M_1 \times \cdots \times M_n \to F$. Endow $C^\alpha(M_1 \times \cdots \times M_n, F)$ with the initial topology with respect to the maps $C^\alpha(M_1 \times \cdots \times M_n, F) \to C^\alpha(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, F)$, $f \mapsto f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$, for φ_i in the maximal smooth atlas of M_i.

Proposition 4.3. *For all $i \in \{1, \ldots, n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on locally convex space, $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$ and F be a locally convex space. Let $j \in \{2, \ldots, n\}$. Define $M := M_1 \times \cdots \times M_{j-1}$, $N := M_j \times \cdots \times M_n$, $\gamma := (\alpha_1, \ldots, \alpha_{j-1})$ and $\eta := (\alpha_j, \ldots, \alpha_n)$. Then*

1. *$f^\vee \in C^\gamma(M, C^\eta(N, F))$ for all $f \in C^{(\gamma,\eta)}(M \times N, F)$.*
2. *The map*

$$\Phi : C^{(\gamma,\eta)}(M \times N, F) \to C^\gamma(M, C^\eta(N, F)),\ f \mapsto f^\vee$$

is a linear topological embedding.

Proof. (1) It is clear that $f^\vee(x) = f(x, \bullet)$ for $x \in M$ is a C^η-map $N \to F$. It suffices to show that $f \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1}) : U_{\varphi_1} \times \cdots \times U_{\varphi_{j-1}} \to C^\eta(N, F)$ is C^γ for each chart $\varphi_k : U_{\varphi_k} \to V_{\varphi_k}$ of M, where $k \in \{1, \ldots, j-1\}$. For all $l \in \{j, \ldots, n\}$, let $\mathcal{A}_l$ be the maximal smooth atlas for M_l. Since the map

$$\Psi : C^\eta(N, F) \to \prod_{\substack{\varphi_l \in \mathcal{A}_l,\\ j \leq l \leq n}} C^\eta(U_{\varphi_j} \times \cdots \times U_{\varphi_n}, F),\ h \mapsto (h \circ (\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}))_{\substack{\varphi_l \in \mathcal{A}_l,\\ j \leq l \leq n}}$$

is a linear topological embedding with closed image (see [14]; cf. [10, 4.7 and Proposition 4.19(d)]), $f^\vee \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1})$ is C^γ if and only if $\Psi \circ f \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1})$ is C^γ (see [14]; cf. [4, Lemma 10.2]), which holds if all components are C^γ. From this the assertion follows if we can show that

$$\theta\colon V_{\varphi_1} \times \cdots \times V_{\varphi_{j-1}} \to C^\eta(V_{\varphi_j} \times \cdots \times V_{\varphi_n}, F),$$

$$x \mapsto f^\vee((\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1})(x)) \circ (\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}) = (f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})^\vee)(x)$$

is C^γ. But $\theta = (f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1}))^\vee$ where $f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})\colon V_{\varphi_1} \times \cdots \times V_{\varphi_n} \to F$ is $C^{(\gamma,\eta)}$, hence θ is C^γ by Theorem 3.20.

(2) It is clear that Φ is an injective linear map. Because Ψ is a linear topological embedding, also for $P := \prod_{\substack{\varphi_l \in \mathcal{A}_l, \\ j \le l \le n}} C^\eta(V_{\varphi_j} \times \cdots \times V_{\varphi_n}, F)$ the map

$$C^\gamma(M, \Psi)\colon C^\gamma(M, C^\eta(N, F)) \to C^\gamma(M, P), \;\; f \mapsto \Psi \circ f$$

is a topological embedding [14]. Let $\mathcal{A}_k$ be the maximal smooth atlas for M_k where $k \in \{1, \ldots, j-1\}$. The map

$$\Xi\colon C^\gamma(M, P) \to \prod_{\substack{\varphi_k \in \mathcal{A}_k, \\ 1 \le k \le j-1}} C^\gamma(V_{\varphi_1} \times \cdots \times V_{\varphi_{j-1}}, P),$$

$$f \mapsto (f \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1}))_{\substack{\varphi_k \in \mathcal{A}_k, \\ 1 \le k \le j-1}}$$

is a linear topological embedding. Let

$$Q := \prod_{\substack{\varphi_k \in \mathcal{A}_k, \\ 1 \le k \le j-1}} \prod_{\substack{\varphi_l \in \mathcal{A}_l, \\ j \le l \le n}} C^\gamma(V_{\varphi_1} \times \cdots \times V_{\varphi_{j-1}}, C^\eta(V_{\varphi_j} \times \cdots \times V_{\varphi_n}, F)).$$

Using the isomorphism $\prod_{\substack{\varphi_k \in \mathcal{A}_k, \\ 1 \le k \le j-1}} C^\gamma(V_{\varphi_1} \times \cdots \times V_{\varphi_{j-1}}, P) \cong Q$ we obtain a linear topological embedding

$$\Gamma := \Xi \circ C^\gamma(M, \Psi)\colon C^\gamma(M, C^\eta(N, F)) \to Q,$$

$$f \mapsto (C^\eta(\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}, F) \circ f \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1}))_{\substack{\varphi_i \in \mathcal{A}_i, \\ 1 \le i \le n}}$$

where $C^\eta(\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}, F)\colon C^\eta(N, F) \to C^\eta(V_{\varphi_j} \times \cdots \times V_{\varphi_n}, F)$, $f \mapsto f \circ (\varphi_j^{-1} \times \cdots \times \varphi_n^{-1})$. Also the map

$$\omega\colon C^{(\gamma,\eta)}(M \times N, F) \to \prod_{\substack{\varphi_i \in \mathcal{A}_i, \\ 1 \le i \le n}} C^{(\gamma,\eta)}(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, F),$$

$$f \mapsto (f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1}))_{\substack{\varphi_i \in \mathcal{A}_i, \\ 1 \le i \le n}}$$

is a topological embedding, by Definition 4.2. Thus we obtain the following commutative diagram.

$$\begin{tikzcd}
C^{(\gamma,\eta)}(M \times N, F) \arrow[r, "\Phi"] \arrow[d, "\omega"] & C^\gamma(M, C^\eta(N, F)) \arrow[d, "\Gamma"] \\
\prod_{\substack{\varphi_i \in \mathcal{A}_i, \\ 1 \le i \le n}} C^{(\gamma,\eta)}(V_{\varphi_1} \times \cdots \times V_{\varphi_n}, F) \arrow[r, "\zeta"] & Q
\end{tikzcd}$$

where ζ is the map $(f_{\varphi_1,\dots,\varphi_n})_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\le i\le n}} \mapsto (f^{\vee}_{\varphi_1,\dots,\varphi_n})_{\substack{\varphi_i\in\mathcal{A}_i,\\ 1\le i\le n}}$. By using that open subsets of k-spaces are k-spaces and because the vertical maps and also the horizontal arrow at the bottom (by [2, Lemma 4.4] and Theorem 3.20) are topological embeddings, hence the map Φ is a topological embedding. $\square$

Theorem 4.4. *For all $i \in \{1,\dots,n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on a locally convex space E_i, F be a locally convex space and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. Let $X_i := \{0\}$ if $\alpha_i = 0$, otherwise $X_i := E_i$. For $j \in \{2,\dots,n\}$ define $M := M_1 \times \cdots \times M_{j-1}$, $N := M_j \times \cdots \times M_n$, $\gamma := (\alpha_1,\dots,\alpha_{j-1})$ and $\eta := (\alpha_j,\dots,\alpha_n)$. Assume that N is locally compact or $M \times N \times X_1 \times X_2 \times \cdots \times X_n$ is a k-space. Then*

$$\Phi\colon C^{(\gamma,\eta)}(M \times N, F) \to C^{\gamma}(M, C^{\eta}(N,F)),\ f \mapsto f^{\vee} \tag{4.0.1}$$

is an isomorphism of topological vector spaces. Moreover, a map $g\colon M \to C^{\eta}(N,F)$ is C^{γ} if and only if

$$g^{\wedge}\colon M \times N \to F,\ g^{\wedge}(x,y) := g(x)(y)$$

is $C^{(\gamma,\eta)}$.

Proof. By Proposition 4.3, it suffices to show that Φ is surjective.

Let $g \in C^{\gamma}(M, C^{\eta}(N,F))$ and define $f := g^{\wedge}\colon M \times N \to F,\ f(x,y) := g(x)(y)$. For all $i \in \{1,\dots,n\}$, let $\varphi_i : U_{\varphi_i} \to V_{\varphi_i}$ be charts for M_i. Then

$$f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})\colon V_{\varphi_1} \times \cdots \times V_{\varphi_n} \to F,$$
$$(x_1,\dots,x_n) \mapsto (C^{\eta}(\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}, F) \circ g \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1}))^{\wedge}(x_1,\dots,x_n)$$

with $C^{\eta}(\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}, F)\colon C^{\eta}(N,F) \to C^{\eta}(V_{\varphi_j} \times \cdots \times V_{\varphi_n}, F), h \mapsto h \circ (\varphi_j^{-1} \times \cdots \times \varphi_n^{-1})$ continuous linear. Hence $C^{\eta}(\varphi_j^{-1} \times \cdots \times \varphi_n^{-1}, F) \circ g \circ (\varphi_1^{-1} \times \cdots \times \varphi_{j-1}^{-1})\colon V_{\varphi_1} \times \cdots \times V_{\varphi_{j-1}} \to C^{\eta}(V_{\varphi_j} \times \cdots \times V_{\varphi_n}, F)$ is C^{γ}. Therefore, by Theorem 3.22, the map $f \circ (\varphi_1^{-1} \times \cdots \times \varphi_n^{-1})$ is $C^{(\gamma,\eta)}$.

Locally compact condition. For all $l \in \{j,\dots,n\}$, from the locally compactness of N it follows then that the open subset U_{φ_l} is locally compact and hence also the V_{φ_l}. Therefore, Theorem 3.22 applies.

k-space condition. $V_{\varphi_1} \times \cdots \times V_{\varphi_n} \times X_1 \times X_2 \times \cdots \times X_n$ is homeomorphic to the open subset $U_{\varphi_1} \times \cdots \times U_{\varphi_n} \times X_1 \times X_2 \times \cdots \times X_n$ of the k-space $M \times N \times X_1 \times X_2 \times \cdots \times X_n$ and hence a k-space. Again, Theorem 3.22 applies. $\square$

Corollary 4.5. *For all $i \in I := \{1,\dots,n\}$, let M_i be a smooth manifold (possibly with rough boundary) modelled on locally convex space E_i, F be a locally convex space and $\alpha_i \in \mathbb{N}_0 \cup \{\infty\}$. For $j \in \{2,\dots,n\}$ define $M := M_1 \times \cdots \times M_{j-1}$, $N := M_j \times \cdots \times M_n$, $\gamma := (\alpha_1,\dots,\alpha_{j-1})$ and $\eta := (\alpha_j,\dots,\alpha_n)$. Assume that (1), (2) or (3) is satisfied:*

1. *For all $i \in I$, E_i is a metrizable.*
2. *For all $i \in I$, M_i is manifold with corners and E_i is a hemicompact k-space.*
3. *For all $i \in \{j,\dots,n\}$, M_i is a finite-dimensional manifold with corners.*

Then

$$\Phi\colon C^{(\gamma,\eta)}(M \times N, F) \to C^{\gamma}(M, C^{\eta}(N,F)),\ f \mapsto f^{\vee}$$

is an isomorphism of topological vector spaces. Moreover, a map $g: M \to C^{\eta}(N, F)$ *is* C^{γ} *if and only if*

$$g^{\wedge}: M \times N \to F,\ g^{\wedge}(x, y) := g(x)(y)$$

is $C^{(\gamma,\eta)}$.

Proof. *Case* $M_j \dots, M_n$ *are finite-dimensional manifolds with corners.* Let M_l be of dimension m_l for all $l \in \{j, \dots, n\}$. Then each point of M_l has an open neighbourhood homeomorphic to an open subset V_l of $[0, \infty[^{\,m_l}$. Hence V_l is locally compact, thus M_l is locally compact. Thus Theorem 4.4 applies.

Case E_i *is a metrizable.* Then for all $i \in I$, all point $x_i \in M_i$ has an open neighbourhood $U_i \subseteq M_i$ homeomorphic to subset $V_i \subseteq E_i$. Since $V_1 \times \cdots \times V_n$ is metrizable, it follows that $U_1 \times \cdots \times U_n \times E_1 \times \cdots \times E_n$ is metrizable and hence a k-space. Hence by [2, Lemma 4.7] $M_1 \times \cdots \times M_n \times E_1 \times \cdots \times E_n$ is a k-space and Theorem 4.4 applies.

Case $E_1, \dots, E_n$ *are* k_{ω}*-spaces,* M_i *is a manifold with corners.* For all $x_i \in M_i$ there is an open neighbourhood $U_i \subseteq M_i$ homeomorphic to an open subset V_i of finite intersections of closed half-space in E_i. Hence $V_1 \times \cdots \times V_n \times E_1 \times \cdots \times E_n$ is (relatively) open subset of a closed subset of $(E_1 \times \cdots \times E_n)^2$. Since the product of hemicompact k-spaces is a hemicompact k-space (see [13, Proposition 4.2(i)]), and hence a k-space, $(E_1 \times \cdots \times E_n)^2$ is a k-space and since open subset (and also closed subset) of k-spaces is k-space, it follows that $V_1 \times \cdots \times V_n \times E_1 \times \cdots \times E_n$ is a k-space. Now [2, Lemma 4.7] shows that $M_1 \times \cdots \times M_n \times E_1 \times \cdots \times E_n$ is a k-space, and thus Theorem 4.4 applies. □

Remark 4.6. The case of manifold with corners and $(E_1 \times \cdots \times E_n)^2$ is a k-space is proved along the same lines as (2) in Corollary 4.5.

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