Conformable Fractional Bernoulli Differential Equation with Applications

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***Abstract*—In This paper we study certain fractional forms of Abel's equation:**$y' = P(x) + Q_1(x)y + Q_2(x)y^2 + Q_3(x)y^3$. **We solve the fractional form of the equation for the cases:**
$Q_2 = 0$ **or** $Q_3 = 0$. **Such cases reduce the equation to Bernoulli fractional differential equation.**

***Index Terms*—Bernoulli Equation, Conformable Fractional Derivative, Exact Solutions, Abel's Equation**

I. Introduction

There are many definitions available in the literature for fractional derivatives. The most known are the Riemann-Liouville definition and the Caputo definition, see [1], [2] and [3], for some applications refer to [5], [6] and [7]. To mention some:

(i) Riemann-Liouville Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

(ii) Caputo Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

However, the following are the drawbacks of either of the definitions or the other:

(i) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0$, ($D_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^\alpha(fg) = f D_a^\alpha(g) + g D_a^\alpha(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^\alpha(f/g) = \frac{g D_a^\alpha(f) - f D_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^\alpha(f \circ g)(t) = f^{(\alpha)}\big(g(t)\big)\, g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$, in general

(vi) All fractional derivatives, specially Caputo definition, assumes that the function f is differentiable.

In [4], the authors gave a new definition of fractional derivative which is a natural extension to the usual first derivative as follows:

Given a function $f : [0, \infty) \longrightarrow R$. Then for all $t > 0$, $\alpha \in (0, 1)$, let

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

T_α is called the conformable fractional derivative of f of order α.

Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$. Hence

$$f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

If f is $\alpha-$differentiable in some $(0, b)$, $b > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then we let

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

The conformable derivative satisfies all the classical properties of derivative. Further, according to this derivative, the following statements are true, see[4].

1. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in R$,
2. $T_\alpha(\sin \frac{1}{\alpha} t^\alpha) = \cos \frac{1}{\alpha} t^\alpha$,
3. $T_\alpha(\cos \frac{1}{\alpha} t^\alpha) = -\sin \frac{1}{\alpha} t^\alpha$,
4. $T_\alpha(e^{\frac{1}{\alpha} t^\alpha}) = e^{\frac{1}{\alpha} t^\alpha}$.

The $\alpha-$fractional integral of a function f starting from $a \geq 0$ is:

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\, dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.For more details on conformable fractional refer to [8], [9] and [10].

In this paper we will study some forms of the Abel's equation associated with the Bernoulli equation. Some concrete examples are given. Throughout this paper we write $d^\alpha x$ for $\frac{dx}{x^{1-\alpha}}$. Ordinary differential equation is the Abel's differential equation which is of the form:

$$y' = P(x) + Q_1(x)y + Q_2(x)y^2 + Q_3(x)y^3. \tag{1}$$

II. Main Results

Clearly, if $P(x) = 0$ and $Q_3(x) = 0$, or $P(x) = Q_2(x) = 0$, then equation (1) is a Bernoulli Equation.

If $Q_3(x) = 0$, then the equation turns out to be the Riccatti Equation. This gave us the motivation to

study the fractional Bernoulli equation. The general form of the Bernoulli's Equation is:

$$y' + P(x)y = Q(x)y^n. \tag{2}$$

This can be transformed to a fractional differential equation in many ways. We will consider only two cases:

Case(i)

$$y^{(\alpha)} y^{\alpha-1} + P(x)y^\alpha = Q(x)y^{n\alpha}, n \neq 1. \tag{3}$$

Case(ii)

$$y^{(\alpha)} + P(x)y = Q(x)y^{n\alpha}, n\alpha \neq 1. \tag{4}$$

Notice that in both equations (3) and (4), if $\alpha = 1$,then both equations reduce to equation (2). The object of this section is to solve both equations (3) and (4).

Case(i). Solution Of Equation (3). First, divide both sides of the equation (3) by $(y^{n\alpha})$, to get

$$y^{\alpha-n\alpha-1}\, y^{(\alpha)} + P(x)y^{(1-n)\alpha} = Q(x),$$

$$y^{(1-n)\alpha-1}\, y^{(\alpha)} + P(x)y^{(1-n)\alpha} = Q(x),$$

Now, let $u = y^{(1-n)\alpha}$ to get:

$$u^{(\alpha)} = (1-n)\alpha y^{(1-n)\alpha-1} y^{(\alpha)},$$

Thus equation (3) becomes

$$\frac{u^{(\alpha)}}{(1-n)\alpha} + P(x)u = Q(x).$$

This can be written in the form

$$u^{(\alpha)} + (1-n)\alpha P(x)u = (1-n)\alpha Q(x). \tag{5}$$

which is a fractional linear differential equation. If we multiply equation (5) by

$$\mu(x) = Exp[\int (1-n)\alpha P(x) d^\alpha x]$$

we get:

$$\mu(x)u^{(\alpha)} + (1-n)\alpha P(x)\mu(x)u = (1-n)\alpha Q(x)\mu(x),$$

The general solution is:

$$u = \frac{1}{\mu(x)}\left[(1-n)\alpha \int \mu(x)Q(x)d^\alpha x + c\right].$$

Now replace u by $y^{(1-n)\alpha}$ to get the general solution of the equation (3)

$$y = \left[\frac{1}{\mu(x)}\left[(1-n)\alpha \int \mu(x)Q(x)d^\alpha x + c\right]\right]^{\frac{1}{(1-n)\alpha}} , n \neq 1$$

Case(ii). Solution Of Equation (4). Consider equation (4)

$$y^{(\alpha)} + P(x)y = Q(x)y^{n\alpha}.$$

Multiplying both sides of the equation by $(\frac{1}{y^{n\alpha}})$, to get

$$y^{-n\alpha}\, y^{(\alpha)} + P(x)y^{1-n\alpha} = Q(x).$$

Let $u = y^{1-n\alpha}$. Then

$$u^{(\alpha)} = (1-n\alpha)y^{-n\alpha}y^{(\alpha)}.$$

Consequently, the equation becomes

$$u^{(\alpha)} + (1-n\alpha)P(x)u = (1-n\alpha)Q(x).$$

This is a linear fractional differential equation whose solution is

$$u(x) = \frac{1}{\mu(x)}\left[(1-n\alpha)\int \mu(x)Q(x)d^\alpha x + c\right].$$

Replacing u by $y^{1-n\alpha}$ we get

$$y = \frac{1}{\mu(x)}\left[(1-n\alpha)\int \mu(x)Q(x)d^\alpha x + c\right]^{\frac{1}{1-n\alpha}} , n\alpha \neq 1.$$

A. Applications

The following are some specific examples of Bernoulli fractional differential equations.

Example 1:Consider the Bernoulli fractional differential equation

$$y^{-\frac{2}{5}}y^{\left(\frac{1}{5}\right)}+\frac{5}{3}x^{\frac{1}{5}}y^{\frac{3}{5}}=x^{\frac{1}{5}},$$

$$u=y^{\frac{3}{5}}\Rightarrow y=u^{\frac{5}{3}}$$

$$y^{\left(\frac{1}{5}\right)}=\frac{5}{3}x^{1-\frac{1}{5}}u^{\frac{2}{3}}u',$$

$$u'+x^{-\frac{3}{5}}u=\frac{3}{5}x^{-\frac{3}{5}},$$

$$\mu(x)=Exp(\int x^{-\frac{3}{5}}dx)=e^{\frac{5}{2}x^{\frac{2}{5}}},$$

$$D(e^{\frac{5}{2}x^{\frac{2}{5}}}u)=\frac{3}{5}x^{-\frac{3}{5}}e^{\frac{5}{2}x^{\frac{2}{5}}},$$

$$u=\frac{3}{5}+ce^{\frac{-5}{2}x^{\frac{2}{5}}}.$$

Now we use the condition $y(0)=1$ to get $c=\frac{2}{5}$. Hence the required solution of the equation is:

$$y=\left[\frac{3}{5}+\frac{2}{5}e^{\frac{-5}{2}x^{\frac{2}{5}}}\right]^{\frac{5}{3}}.$$

Example 2: Consider the Bernoulli fractional differential equation:

$$y^{-\frac{1}{4}}y^{\left(\frac{3}{4}\right)}-\frac{4}{3}y^{\frac{3}{4}}=y^{\frac{3}{2}},\alpha=\frac{3}{4},n=2,$$

$$y(0)=1.$$

Solution:

$$y^{-\frac{7}{4}}y^{\left(\frac{3}{4}\right)}-\frac{4}{3}y^{-\frac{3}{4}}=1,$$

Using $y(0)=1$ to get $c=\frac{7}{4}$. thus the resulting solution is

$$y=\left[-\frac{3}{4}+\frac{7}{4}e^{-\frac{4}{3}x^{\frac{3}{4}}}\right]^{-\frac{4}{3}}$$

.

Example 3: Consider the Bernoulli fractional differential equation

$$y^{\left(\frac{1}{2}\right)}-\frac{2}{5}\sqrt{x}y=-\frac{2}{5}\sqrt{x}\sin(x)\,y^{\frac{7}{2}},\alpha=\frac{1}{2},n=7,$$

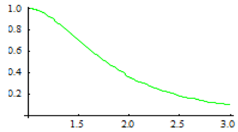
$$y(0)=0.$$

Solution:

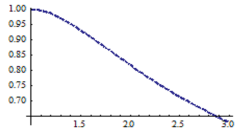
$$y^{-\frac{7}{2}}y^{\left(\frac{1}{2}\right)}-\frac{2}{5}\sqrt{x}y^{-\frac{5}{2}}=-\frac{2}{5}\sqrt{x}\sin(x),$$

Where $y(0)=0$, $c=\frac{1}{2}$. The general solution of the equation

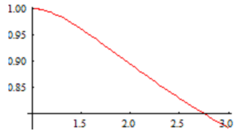
$$y(x)=\left(\frac{1}{2}(\sin(x)-\cos(x))+\frac{1}{2}e^{-x}\right)^{-\frac{2}{5}}.$$



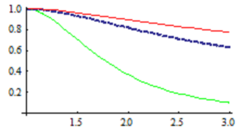
(a) Figure 1



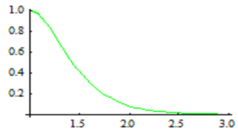
(b) Figure 2



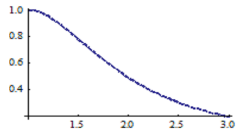
(c) Figure 3



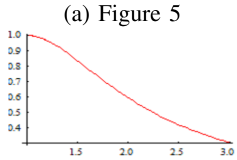
(d) Figure 4



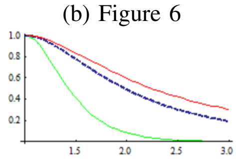
(a) Figure 5



(b) Figure 6



(c) Figure 7



(d) Figure 8

B. Graphs

The sketches bellow illustrate the graphs of the solutions of the following equations

$$y^{(\alpha)}y^{\alpha-1}+\frac{x^{-\alpha}}{\alpha(1-n)}y^{\alpha}=\frac{x}{\alpha(1-n)}y^{n\alpha}, \qquad (6)$$

$$y(1)=1, n\neq 1.$$

$$y^{(\alpha)}+\frac{1}{(1-n\alpha)}y=\frac{1}{(1-n\alpha)}y^{n\alpha}, \qquad (7)$$

$$y(0)=2^{\frac{1}{1-n\alpha}}, n\alpha\neq 1.$$

Solution: the general solution of the equation (6) is

$$y(x)=\left[\frac{x^{\alpha+1}}{\alpha+2}+\frac{\alpha+1}{(\alpha+2)\,x}\right]^{\frac{1}{\alpha(1-n)}}$$

(a) where $\alpha=0.3$ with $n=2$, $n=6$ and $n=10$.respectively.

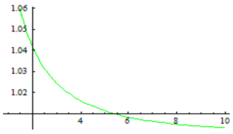
(b) where $n=2$, $\alpha=0.1$, $\alpha=0.5$ and $\alpha=0.95$, respectively.

Figures (1), (2) and (3) represent the graphs of the solutions of equation (6) where α is fixed as $\alpha=0.3$ and n was given three random values to investigate the effect of n on the solution of the equation when α is fixed. Figure (4) is the consists of the graphs 1,2, and 3 to compare between equation (6) behaviors when α is fixed and n changes.

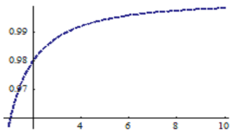
Solution: the general solution of the equation (7) is

$$y(x)=\left[1+e^{-\frac{1}{\alpha}x^{\alpha}}\right]^{\frac{1}{1-n\alpha}},$$

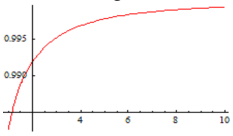
(a) where $\alpha=0.3, n=2$, $n=6$ and $n=10$,respectively.



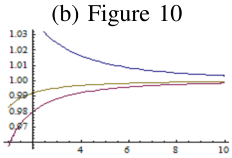
(a) Figure 9



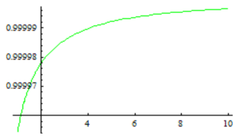
(b) Figure 10



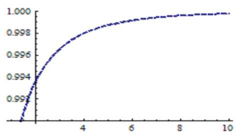
(c) Figure 11



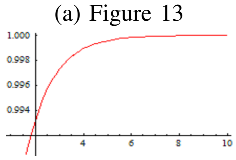
(d) Figure 12



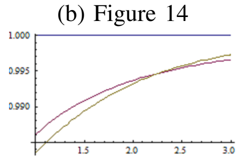
(a) Figure 13



(b) Figure 14



(c) Figure 15



(d) Figure 16

Figures (5), (6) and (7) represent the graphs of the solutions of equation (6) where n is fixed as $n = 2$ and α was given three random values to investigate the effect of α on the solution of the equation when n is fixed. Figure (8) is the consists of the graphs 5,6, and 7 to compare between equation (6) behaviors when α is fixed and n changes.

Figures (9), (10) and (11) represent the graphs of the solutions of equation (7) where α is fixed as $\alpha =$ 0.3 and n was given three random values to investigate the effect of $n$on the solution of the equation when $\alpha$is fixed.

Figure (12) is the consists of the graphs 9,10, and 11 to compare between equation (7) behaviors when α is fixed and n changes.

(b) where $n = 20$, $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.95$ respectively.

Figures (13), (14) and (15) represent the graphs of the solutions of equation (7) where n is fixed as $n = 20$ and α was given three random values to investigate the effect of α on the solution of the equation when n is fixed. Figure (16) is the consists of the graphs 13, 14 , and 15 to compare between equation (7) behaviors when α is fixed and n changes.

III. Conclusions

This paper produced an exact solution of the fractional Bernoulli differential equation for certain cases in a simpler and more efficient method than common methods, by using conformable derivative.

Acknowledgment

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IV. References

[1]. R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000

[2] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006

[3] K. S. Miller, B. Ross, An introduction to fractional calculus and fractional differential equations, John Wiley and Sons, New York, (1993)

[4] R. Khalil , M. Al Horani , A. Yousef , M. Sababheh. Anew definition of fractional derivative,
Journal of Computational and Applied Mathematics 264 (2014) 65–70

[5].K.S. Miller, and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley-Interscience Publication, 1993.

[6] K. Oldham, and J. Spanier. The fractional calculus, theory and applications of differentiation and integration of arbitrary order. Academic Press, U.S.A. 1974

[7] I. Podlubny. Fractional differential equations. Academic Press, U.S.A. 1999.

[8] Abu-Hammad, M., and Khalil, R. Conformable Fractional Heat Differential Equation. International Journal of Pure and Applied Mathematics, 94 (2014) 215-2217.

[9] Abu-Hammad, M., and Khalil, R. Abel's Formula And Wronskian For Conformable Fractional Differential Equations. I.J. Differential Equations and Applications, 13 (2014) 177-183.

[10] Abu-Hammad, M., and Khalil, R. Legendre fractional differential equation and Legender fractional polynomials. I.J of Applied Mathematical Research, 3 (3) (2014) 214-219.