Some Properties of Fuzzy D-Coimplication

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Abstract: **In this paper, we will introduce the definition of fuzzy D-Coimplications, then study the equivalences between D-coimplication and other fuzzy coimplication classes. Also, some examples are also discussed as well.**

Keywords: *Fuzzy implications, Fuzzy Coimplications, D-implication and D-Coimplication.*

1. INTRODUCTION

In fuzzy logic, the basic theory of connective AND (∧), OR (∨), NOT (¬) are often modeled as (t-norm T, t-conorm S and strong negations N_c). The logic connectives like conjunction ∧ is interpreted by a triangular norm, disjunction ∨ by triangular conorm and negation ¬ by a strong negation. see [1], [2] and [3].

Also, the implications are generally performed by suitable functions $I: [0,1] \times [0,1] \to [0,1]$, called implication operators, derived from t-norms, t-conorms and fuzzy negations.

For all p and q in $[0,1]$, the four most usual ways to define these implication operators are: see [4].

1. $I(p,q) = S(N(p), q)$ for a given t-conorm S and a fuzzy negation N, called S-Implications;
2. $I(p,q) = \sup\{r \in [0,1]: T(r,p) \leq q\}$ for a given left-continuous t-norm T, called R-Implications;
3. $I(p,q) = S\big(N(p), T(p,q)\big)$ for a given t-norm T, a t-conorm S and a fuzzy negation N, called QL-Implications;
4. $I(p,q) = S\big(T\big(N(p), N(q)\big), q\big)$ for a given t-norm T, a t-conorm S and a fuzzy negation N, that will call D-Implication.

2. PRELIMINARIES

In this section we will show basic definitions about t-norms, t-conorms and fuzzy negations. The conjunction ∧ in fuzzy logic, it is often modeled as follow:

Definition 2.1. [5] A function $T: [0,1] \times [0,1] \to [0,1]$ is a triangular norm (in short, t-norm), if for all p,q,r,w ∈ $[0,1]$ the following conditions are satisfied

(T.1) $T(p,q) = T(q,p)$,
(T.2) $T(p,q) \leq T(r,w)$ if $p \leq r$ and $q \leq w$,
(T.3) $T\big(p, T(q,r)\big) = T(T(p,q), r)$,
(T.4) $T(p,1) = p$.

Besides these properties, some others can be required, such as: see [6].

(T.5) Continuity: T is continuous in both arguments at the same time;
(T.6) Left-continuity: T is left-continuous in each argument;
(T.7) Idempotency: $T(p,p) = p$, for all $p \in [0,1]$;
(T.8) Positiveness: if $T(p,q) = 0$ then either $p = 0$ or $q = 0$;
(T.9) Nilpotency: T is continuous, $p \in]0,1[$ and there exists an $n \in \mathbb{N}$ such that $p_T^{[n]} = 0$ where $p_T^{[0]} = 1$ and $p_T^{[i+1]} = T\Big(p, p_T^{[i]}\Big)$.

Also, disjunction ∨ in fuzzy logic is often modeled as follows:

Definition 2.2. [5] A function $S: [0,1] \times [0,1] \to [0,1]$ is a triangular conorm (in short, t-conorm), if for all p,q,r,w ∈ $[0,1]$ the following conditions are satisfied

(S.1) $S(p,q) = S(q,p)$,
(S.2) $S(p,q) \leq S(r,w)$ if $p \leq r$ and $q \leq w$,
(S.3) $S\big(p, S(q,r)\big) = S(S(p,q), r)$,
(S.4) $S(p,0) = p$.

Additional properties: see [6].

(S.5) Continuity: S is continuous in both arguments at the same time;
(S.6) Left-continuity: S is left-continuous in each argument;
(S.7) Idempotency: $S(p,p) = p$, for all $p \in [0,1]$;
(S.8) Positiveness: if $S(p,q) = 1$ then either $p = 1$ or $q = 1$;
(S.9) Nilpotency: S is continuous, $p \in]0,1[$ and there exists an $n \in \mathbb{N}$ such that $p_S^{[n]} = 1$ where $p_S^{[0]} = 0$ and $p_S^{[i+1]} = S\Big(p, p_S^{[i]}\Big)$.

Definition 2.3. [7] A function $N: [0,1] \to [0,1]$ is a negation function, iff:

(1) $N(0) = 1$, $N(1) = 0$;
(2) $N(p) \leq N(q)$, if $p \geq q$, $\forall p, q \in [0,1]$.

A negation function is strict, iff:

(1) $N(p)$ is continuous;
(2) $N(p) < N(q)$, if $p > q$. $\forall p, q \in [0,1]$.

A strict negation function is strong or volutive, iff:

$$N\big(N(p)\big) = p, \forall p \in [0,1].$$

A negation function is weak, iff N is not strong.

Example 2.1. [7]

- The strong negation ($N_c(p) = 1 - p$),
- Strict negation but not strong ($N_k(p) = 1 - p^2$),
- Weaker negation $\left(N_{D_1}(p) = \begin{cases} 1 \text{ if } p = 0, \\ 0 \text{ if } p > 0. \end{cases}\right)$,
- Strongest negation $\left(N_{D_2}(p) = \begin{cases} 1 \text{ if } p < 1, \\ 0 \text{ if } p = 1. \end{cases}\right)$.

Definition 2.4. [8] Let T be a t-norm and S be a t-conorm. The functions N_T and N_S from [0,1] into [0,1] defined by

$$N_T(p) = \sup\{r \in [0,1] \backslash T(p,r) = 0\},\ \forall\ p \in [0,1],$$
$$N_S(p) = \inf\{r \in [0,1] \backslash S(p,r) = 1\},\ \forall p \in [0,1],$$

are called the natural negation of T and S, respectively.

The Law of Excluded Middle (LEM) is one of the well-known fundamental Boolean laws of classical theory. As the LEM, in classical logic, means that $\neg p \vee p$ is always true, so we have the following definition. see [9].

Definition 2.5. [6] Let S be a t-conorm and N be a fuzzy negation, the pair (S,N) satisfies the LEM if

$$S(N(p),p) = 1,\ p \in [0,1]. \qquad (LEM)$$

Aristotle's law of non-contradiction is defined in classical logic as "$\neg p \wedge p$ is always false". So, its fuzzy generalization is defined as follows. see [9].

Definition 2.6. [6] Let T be a t-norm and N a fuzzy negation, the pair (T,N) satisfies the law of contradiction if

$$T(N(p),p) = 0,\ p \in [0,1]. \qquad (1)$$

Proposition 2.1. [5] For a t-norm T, t-conorm S and strong negation N then S is N-dual of T iff

$$S(p,q) = N(T(N(p),N(q))),\ \forall p,q \in [0,1],$$

and T is N-dual of S iff

$$T(p,q) = N\left(S\big(N(p),N(q)\big)\right),\ \forall p,q \in [0,1]. \qquad (2)$$

The standard examples of t-norm are stated in the following:

$T_M(p,q) = \min(p,q)$, (Minimum t-norm)

$\Pi(p,q) = pq$, (Product t-norm)

$$W(p,q) = \begin{cases} p & \text{if } q = 1 \\ q & \text{if } p = 1 \\ 0 & \text{if } p,q \in [0,1), \end{cases}$$ (Drastic or weak t-norm)

$$N(p,q) = \begin{cases} \min(p,q) & \text{if } p+q \geq 1 \\ 0 & \text{if } p+q < 1. \end{cases}$$ (Nilpotent t-norm)

$L(p,q) = \max(p+q-1,0)$, (Lukasiewicz t-norm)

$$H(p,q) = \begin{cases} 0 & \text{if } p = q = 0 \\ \frac{pq}{p+q-pq} & \text{otherwise}, \end{cases}$$ (Hamacher t-norm)

$D_\alpha(p,q) = \frac{pq}{\max(p,q,\alpha)}, \alpha \in (0,1)$. Dubois-Prade t-norm)

The standard examples N-dual of T (t-conorms) are stated in the following:

$S_M(p,q) = \max(p,q)$, (Maximum t-conorm)

$S_\Pi(p,q) = p+q-pq$, (Probabilistic sum t-conorm)

$$S_W(p,q) = \begin{cases} p & \text{if } q = 1 \\ q & \text{if } p = 1 \\ 1 & \text{otherwise}, \end{cases}$$ (Drastic or largest t-conorm)

$$S_N(p,q) = \begin{cases} \max(p,q) & \text{if } p+q < 1 \\ 0 & \text{if } p+q \geq 1, \end{cases}$$ (Nilpotent t-conorm)

$S_L(p,q) = \min(p+q,1)$, (Bounded Sum t-conorm)

$$S_H(p,q) = \begin{cases} 0 & \text{if } p = q = 0 \\ \dfrac{p+q-2pq}{1-pq} & \text{otherwise}, \end{cases}$$ (Hamacher t-conorm)

$$S_{D_\alpha}(p,q) = 1 - \frac{(1-p)(1-q)}{\max(1-p,1-q,\alpha)}, \alpha \in (0,1).$$ (Dubois-Prade t-conorm)

Definition 2.7. [2]

- S distributes over T if:
  $$S\big(p,T(q,w)\big) = T\big(S(p,q),S(p,w)\big),\ \forall p,q,w \in [0,1].$$
- T distributes over S if:
  $$T\big(p,S(q,w)\big) = S\big(T(p,q),T(p,w)\big),\ \forall p,q,w \in [0,1]. \qquad (3)$$

Proposition 2.2. [2] Let T be a t-norm and S a t-conorm.

(i) S is distributive over T iff $T = T_M$;

(ii) T is distributive over S iff $S = S_M$;

(iii) (T,S) is distributive pair iff $T = T_M$ and $S = S_M$.

3. Fuzzy Implications

Fuzzy implications are play a significant role in many fields, being crucial in fuzzy control and approximate reasoning see [10].

In the following they are four ways to define an implication in the Boolean lattice $(L,\wedge,\vee,\neg)$: see [3], [11], [12] and [13].

(1) $p \Rightarrow q \equiv \neg p \vee q$, (S-Implication)

(2) $p \Rightarrow q \equiv \max\{t \in L: r \wedge t \leq q\}$, (R-Implication)

(3) $p \Rightarrow q \equiv \neg p \vee (p \wedge q)$, (Quantum logic)

(4) $p \Rightarrow q \equiv q \vee (\neg p \wedge \neg q)$, (D-Implication)

where $p,q \in L$.

Definition 3.1. [7] A function $I: [0,1] \times [0,1] \to [0,1]$ is fuzzy implication if, $\forall p,q,r \in [0,1]$, the following conditions are satisfied:

($I1$) $I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$.

($I2$) $I(p,q) \geq I(r,q)$ if $p \leq r$.

($I3$) $I(p,q) \leq I(p,r)$ if $q \leq r$.

The set of all fuzzy implications is denoted by FI.

There is also a property that relates fuzzy implications and negations: see [6].

($I4$) Contraposition: $I(p,q) = I\big(N(q),N(p)\big)$, for all $p,q \in [0,1]$.

The following properties are generalization of fuzzy implication from classical logic.

Definition 3.2. [12] A fuzzy implications I is said to satisfy the following most important properties, $\forall p,q,r \in [0,1]$.

$I(1,q) = q$; (NP) $\quad I\big(p,I(q,r)\big) = I\big(q,I(p,r)\big)$; (EP)

$I(p,p) = 1$; (IP) $\quad I(p,q) = 1 \Leftrightarrow p \leq q$. (OP)

There are four best-known classes of fuzzy implications, (S,N), R, QL and D-implication.

Definition 3.3. [12] A function $I: [0,1] \times [0,1] \to [0,1]$ is called an (S,N) implication if there exist a fuzzy negation N and a t-conorm S such that

$$I_{S,N}(p,q) = S(N(p),q),\ \forall p,q \in [0,1].$$

Definition 3.4. [12] Let T a left-continuous t-norm. Then, the residual implication or R-implication derived from T is given by:

$$I_T(p,q) = \sup\{r \in [0,1]: T(r,p) \leq q\},\ \forall p,q \in [0,1].$$

Definition 3.5. [10] Let T be a t-norm, S be a t-conorm and N be a fuzzy negation. A QL-implication is defined by:

$$I_{T,S,N}(p,q) = S\big(N(p),T(p,q)\big),\ \forall p,q \in [0,1].$$

The fourth type of implication, diskant-implication (D-implication, for short) is the contraposition ($I4$) of the QL-implication if N is a strong negation [6].

A D-implication is generated from a fuzzy negation, a t-conorm and a t-norm, getting idea from the equivalency in classical binary logic:

$$p \Rightarrow q \equiv q \vee (\neg p \wedge \neg q), \text{ for all } p,q \in [0,1].$$

Definition 3.6. [9] A function $I: [0,1] \times [0,1] \to [0,1]$ is called a D-implication if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that

$$I^{T,S,N}(p,q) = S\big(T\big(N(p),N(q)\big),q\big),$$

for all $p, q \in [0,1]$.

A D-implication I generated from a t-conorm S, a fuzzy negation N and a t-norm T satisfies:

(1) $\forall p, q, w \in [0,1],\ I(p,q) \geq I(w,q)$ if $p \leq w$;
(2) $\forall q \in [0,1],\ I(0,q) = 1$;
(3) $I(1,0) = 0$;
(4) $\forall q \in [0,1],\ I(1,q) = q$;
(5) $\forall p \in [0,1],\ N(p) = I(p,0)$, is a strong fuzzy negation N_c.

Lemma 3.1. [6] Every D-implication satisfies ($I1$), ($I2$) and (NP).

Lemma 3.2. [9] If a D-implication $I^{T,S,N} \in FI$ or (QL-implication $I_{T,S,N} \in FI$), then the pair (S, N) satisfies (LEM).

4. COIMPLICATION

Fuzzy coimplications, one of the new connectives used in fuzzy logic and fuzzy inference also they are a generalization of binary coimplications presented in classical logic [14]. In [15], [16] and [17], the concept of fuzzy coimplication was introduced as a new approach to approximate reasoning of expert systems using the equivalence relation for modus ponens of the inference in fuzzy expert systems instead of fuzzy implication. The algebraic properties of fuzzy (T, N) co-implications and residual coimplications are studied in [17], in order to provide a theoretical background for approximate reasoning applications. see [18].

Definition 4.1. [19] A function $J: [0,1] \times [0,1] \to [0,1]$ is a fuzzy co-implication if, $\forall p, q, r \in [0,1]$, the following conditions are satisfied:

($J1$) $J(1,1) = J(1,0) = J(0,0) = 0$ and $J(0,1) = 1$.
($J2$) $J(p,q) \geq J(r,q)$ if $p \leq r$.
($J3$) $J(p,q) \leq J(p,r)$ if $q \leq r$.

The set of all fuzzy co-implication is denoted by $Co-FI$. From the previous definition we can deduce that for each fuzzy co-implication $J(1,q) = J(p,0) = 0,\ \forall p, q \in [0,1]$. Moreover, J satisfies also the normality condition $J(p,p) = 0$.

There is also a property that relates fuzzy co-implications and negations:

($J4$) Contraposition: $J(p,q) = J(N(q), N(p))$, for all $p, q \in [0,1]$.

Remark 4.1. [19] Directly from Definition 4.1. we see that each fuzzy co-implication J satisfies the following left and right boundary conditions, respectively:

$J(1,q) = 0, q \in [0,1],$ (Co-LB)
$J(p,0) = 0,\ p \in [0,1].$ (Co-RB)

Therefore, J satisfies also the normality condition:

$J(1,0) = 0.$ (Co-NC)

Lemma 4.1. [17] If a function $J: [0,1] \times [0,1] \to [0,1]$ satisfies ($J1$) and ($J2$) then the function $N_J: [0,1] \to [0,1]$ defined by

$$N_J(p) = J(p,1),\ \forall p \in [0,1],$$

is a fuzzy negation.

The following properties are generalization of fuzzy co-implication from classical logic.

Definition 4.2. [12] A fuzzy co-implications J is said to satisfy the following most important properties, $\forall p, q, r \in [0,1]$.

$J(0,q) = q;$ (Co-NP)
$J(p,p) = 0;$ (Co-IP)
$J(p, J(q,r)) = J(q, J(p,r));$ (Co-EP)
$J(p,q) = 0 \Leftrightarrow p \geq q.$ (Co-OP)

In 2016, Jebril, I. used the idea of the function of fuzzy co-implication by introducing the definition of (T,N) and residual co-implication.

Definition 4.3. [17] A function $J: [0,1] \times [0,1] \to [0,1]$ is called a (T,N) co-implication if there exists a t-norm T and a fuzzy negation N such that

$$J_{T,N}(p,q) = T(q, N(p)),\ \forall p, q \in [0,1]. \quad (4)$$

Proposition below states how an (S,N) implication gives rise to a fuzzy (T,N) co-implication and vice-versa.

Proposition 4.1. [17] A function $J_{T,N}$ from $[0,1]^2$ into $[0,1]$ is a (T,N) co-implication with strong negation iff

$$J_{T,N}(p,q) = N\left(I_{S,N}(q,p)\right), \quad (5)$$

for some $I_{S,N}$ and fuzzy (strong) negation N.

Conversely, $I_{S,N}$ from $[0,1]^2$ into $[0,1]$ is an (S,N) implication with strong negation iff

$$I_{S,N}(p,q) = N\left(J_{T,N}(q,p)\right),$$

for some $J_{T,N}$ and fuzzy (strong) negation N.

Theorem 4.1. [17] All (T,N) co-implication are fuzzy implications satisfy (Co-NP) and (Co-EP).

Definition 4.4. [17] Let S is the t-conorm of right continuous T. Then, the residual co-implication (R^*-implication) derived from S, is

$$J_S(p,q) = \inf\{r \in [0,1]: S(r,p) \geq q\},\ \forall p, q \in [0,1].$$

Then in 2017, Ghoneim, H. and Jebril, I. studied the class of QL-coimplications under certain conditions in crisp logic and fuzzy logic in [20].

Definition 4.5. [20] A function $J: [0,1] \times [0,1] \to [0,1]$ is called a QL-coimplication if there exists a t-norm T, t-conorm S and a fuzzy negation N such that:

$$J_{T,S,N}(p,q) = T(S(p,q), N(p)),\ \forall p, q \in [0,1].$$

5. Main Results

In this paper we will study the fourth type of fuzzy co-implications, which is called D-coimplication. Fuzzy co-implications are extensions of the Boolean co-implication ($p \nLeftarrow q$) meaning that p is not necessary for q [21]. In classical logic, the operator `$\nRightarrow$' is generated by Boolean negation '$\neg$', conjunction '$\wedge$' and disjunction '$\vee$',

$$q \nRightarrow p \equiv (\neg p \vee \neg q) \wedge q.$$

In the following table 1, we can see the truth table for the classical co-implication.

Table 1

p	q	$p \Rightarrow q$	$\neg p \vee \neg q$	$(\neg p \vee \neg q) \wedge q$	$q \nRightarrow p$
0	0	1	1	0	0
0	1	1	1	1	1
1	0	0	1	0	0
1	1	1	0	0	0

5.1 D-coimplication

In this section we will introduce the definition of D-coimplication and some characteristics of D-coimplication.

Definition 5.1. A function $J: [0,1] \times [0,1] \to [0,1]$ is called a D-coimplication if there exists a t-norm T, a t-conorm S and fuzzy negation N such that

$$J^{T,S,N}(p,q) = T(S(N(p), N(q)), q), \text{ for all } p, q \in [0,1].$$

Lemma 5.1. Every D-coimplication satisfies ($J1$), ($J2$), (Co-RB) and (Co-NP).

Proof. Let $J^{T,S,N}$ be a D-coimplication, so $J^{T,S,N}$ satisfies ($J1$) since:

$$J^{T,S,N}(1,1) = J^{T,S,N}(1,0) = J^{T,S,N}(0,0) = 0,$$

and $J^{T,S,N}(0,1) = T(S(1,0),1) = T(1,1) = 1$.

Now, assume that $p_1, p_2, q \in [0,1]$ and $p_1 \le p_2$. Then, $N(p_1) \ge N(p_2)$.

$$\Rightarrow S(N(p_1), N(q)) \ge S(N(p_2), N(q))$$
$$\Rightarrow T(S(N(p_1), N(q)), q) \ge T(S(N(p_2), N(q)), q)$$
$$\Rightarrow J^{T,S,N}(p_1, q) \ge J^{T,S,N}(p_2, q).$$

Hence $J^{T,S,N}$ satisfies ($J2$).

Let $J^{T,S,N}$ be a D-coimplication, for all $p \in [0,1]$.

$$J^{T,S,N}(p,0) = T(S(N(p), N(0)), 0) = 0, \text{ by } (T.8).$$

Then $J^{T,S,N}$ satisfies (Co-RB).

For any $q \in [0,1]$,

$$J^{T,S,N}(0,q) = T(S(N(0), N(q)), q) = T(S(1, N(q)), q),$$

and $S(1, N(q)) = 1$ (by the ordering on all t-conorms). Since $T(1,q) = q$, so $J^{T,S,N}(0,q) = q$. Hence $J^{T,S,N}$ satisfies (Co-NP).

Remark 5.1. Given any D-coimplication $J^{T,S,N}$ it is trivially satisfied that $J^{T,S,N}(p,1) = N(p)$. Thus, if any D - coimplication satisfies the (Co-EP) then it also satisfies ($J4$), since

$$J^{T,S,N}(N(q), N(p)) = J^{T,S,N}(N(q), J^{T,S,N}(p,1))$$
$$= J^{T,S,N}(p, J^{T,S,N}(N(q),1))$$
$$= J^{T,S,N}(p,q).$$

Theorem 5.1. For t-norm T, t-conorm S and a fuzzy negation N, Then

$$J^{T,S,N}(p,p) = 0, \forall p \in [0,1],$$

if and only if

$$T(S(N(p), N(p)), p) = 0, \forall p \in [0,1].$$

Proof. If $J^{T,S,N}(p,p) = 0, \forall p \in [0,1]$, then

$$T(S(N(p), N(p)), p) = J^{T,S,N}(p,p) = 0, \forall p \in [0,1].$$

Conversely, if $T(S(N(p), N(p)), p) = 0, \forall p \in [0,1]$, then

$$J^{T,S,N}(p,p) = T(S(N(p), N(p)), p) = 0, \forall p \in [0,1].$$

Theorem 5.2. For a left continuous t-norm T, continuous t-conorm S and a continuous fuzzy negation N, then

$$J^{T,S,N}(p,p) = 0, \forall p \in [0,1],$$

if and only if

$$N(p) \le N_T(p), \forall p \in [0,1].$$

Proof. Let T is a left continuous t-norm, S continuous t-conorm and a N continuous fuzzy negation, by [[17], proposition 5.2.]

$$N_T(p) = \max\{r \in [0,1]: T(p,r) = 0\}, \forall p \in [0,1].$$

Then

$$J^{T,S,N_T}(p,p) = T(S(N_T(p), N_T(p)), p) = 0.$$

By increasing of T, S and if $N(p) \le N_T(p)$, then

$$T(S(N(p), N(p)), p) = 0.$$

Conversely, let $J^{T,S,N}(p,p) = 0, \forall p \in [0,1]$, then

$$T(S(N(p), N(p)), p) = 0, \forall p \in [0,1],$$

and if $S(N(p), N(p)) = N(p) \in \{r \in [0,1]: T(p,r) = 0\}$, then $N(p) \le \max\{r \in [0,1]: T(p,r) = 0\} = N_T(p)$.

A relation between fuzzy negations and D-coimplication is given in the next proposition.

Proposition 5.1. Let $J^{T,S,N}$ be a D-coimplication, then $N_{J^{T,S,N}} = N$.

Proof. For any $p \in [0,1]$ we have

$$N_{J^{T,S,N}}(p) = J^{T,S,N}(p,1) = T(S(N(p), N(1)), 1)$$
$$= T(S(N(p), 0), 1) = T(N(p), 1) = N(p).$$

5.2 Equivalences between D-coimplication and fuzzy coimplication classes

Now we will study the equivalences between D-coimplication and fuzzy coimplication classes.

Proposition 5.2. Let T be a t-norm, S a t-conorm and N a fuzzy negation. If $J^{T,S,N}$ is a fuzzy co-implication, then the $J_{T,N}$ satisfies (1).

Proof. If $J^{T,S,N}$ is a fuzzy coimplication, then by Remark 4.1. it satisfies (Co-LB). Thus $J^{T,S,N}(1,q) = 0$ if and only if $T(S(N(1), N(q)), q) = 0$, i.e., $T(N(q), q) = 0$, for every $q \in [0,1]$ as well as for a $J^{T,S,N}$, to be a co-implication. That means the $J_{T,N}$ satisfies (1).

Theorem 5.3. Let T be a t-norm, S a t-conorm and N a strong negation. A D-coimplication $J^{T,S,N}$ satisfies (Co-EP) if and only if it is also a (T,N) co-implication.

Proof. $\Longrightarrow$: Let $J^{T,S,N}$ satisfies (Co-EP), it also satisfies ($J4$) by Remark 5.1. and then by Definition 2.7. and Proposition 2.2. for all $p, q \in [0,1]$.

$$J^{T,S,N}(p,q) = J^{T,S,N}(N(q), N(p))$$
$$= T(S(p,q), N(p))$$
$$= S(T(N(p), p), T(N(p), q))$$
$$= S(0, T(N(p), q))$$
$$= T(q, N(p))$$
$$= J_{T,N}(p,q).$$

$\Longleftarrow$: Let D-coimplication is a (T,N) co-implication $\forall p, q, r \in [0,1]$. i.e., $J^{T,S,N}(p,q) = J_{T,N}(p,q)$.

$$J^{T,S,N}(p, J^{T,S,N}(q,r)) = J_{T,N}(p, J_{T,N}(q,r))$$
$$= J_{T,N}(q, J_{T,N}(p,r)),$$

from Theorem 4.1. $J_{T,N}$ satisfy (Co-EP). Then

$$J^{T,S,N}(p, J^{T,S,N}(q,r)) = J^{T,S,N}(q, J^{T,S,N}(p,r)).$$

Proposition 5.3. Let T be a t-norm, S a t-conorm and N a strong negation such that the corresponding $J^{T,S,N}$ (equivalently the $J_{T,S,N}$) is a co-implication. The following statements are equivalent:

(i) $J^{T,S,N}$ satisfies the (Co-EP).
(ii) $J_{T,S,N}$ satisfies the (Co-EP).
(iii) $J^{T,S,N}$ is a (T,N) co-implication.
(iv) $J_{T,S,N}$ is a (T,N) co-implication.
(v) There exists a t-norm T_1 such that

$$T(S(N(p), N(q)), q) = T_1(q, N(p)),$$

for all $p, q \in [0,1]$. (6)

Proof. Let us prove only the equivalence among (i), (iii) and (v) since the equivalence among (ii), (iv) and (v) follows similarly.

- (i) $\Longrightarrow$ (iii) If $J^{T,S,N}$ satisfies (Co-EP), it also satisfies ($J4$) by Remark 5.1 and then, by Theorem 5.3., $J^{T,S,N}$ is a (T,N) co-implication.
- (iii) $\Longrightarrow$ (v) If $J^{T,S,N}$ is a (T,N) co-implication, there exist a t-norm T_1 and strong negation N_1 such that

$$J^{T,S,N}(p,q) = T(S(N(p), N(q)), q) = T_1(q, N_1(p)),$$

for all $p, q \in [0,1]$.

But taking $q = 1$ in the above equation we obtain $N(p) = N_1(p)$ for all $p \in [0,1]$ and consequently equation (6) follows.

- (v) $\Longrightarrow$ (i) If equation (6) is satisfied, $J^{T,S,N}$ is in fact a (T,N) co-implication and so it satisfies (Co-EP).

Lemma 5.2. Given a D-coimplication $J^{T,S,N}$ and a (T,N) co-implication $J_{T,N}$. If $S = S_M$ and $J_{T,N}$ satisfies (1), then $J^{T,S,N} = J_{T,N}$.

Proof. By proposition 2.2 (ii), $S = S_M$ iff (T,S) satisfy (3). So, for all $p, q \in [0,1]$:

$$\begin{aligned} J^{T,S,N}(p,q) &= T\big(S(N(p),N(q)),q\big) \\ &= T\Big(q, S\big(N(p),N(q)\big)\Big) && \text{by } (T.1) \\ &= S\Big(T\big(q,N(p)\big),T\big(q,N(q)\big)\Big) && \text{by (3)} \\ &= S\big(T(q,N(p)),0\big) && \text{by } (S.4) \text{ and (4)} \\ &= T\big(q,N(p)\big) = J_{T,N}(p,q). \end{aligned}$$

Theorem 5.4. Given a QL-coimplication $J_{T,S,N}$, a D-coimplication $J^{T,S,N}$ and a (T,N) co-implication $J_{T,N}$. If $S = S_M$ and $J_{T,N}$ satisfies (1), then $J^{T,S,N} = J_{T,N} = J_{T,S,N}$.

Proof. By proposition 2.2 (ii), $S = S_M$ iff (T,S) satisfies (3). So, for all $p, q \in [0,1]$:

$$\begin{aligned} J_{T,S,N}(p,q) &= T\big(S(p,q),N(p)\big) \\ &= T\big(N(p),S(p,q)\big) && \text{by } (T.1) \\ &= S\big(T(N(p),p),T(N(p),q)\big) && \text{by (3)} \\ &= S\big(0,T(N(p),q)\big) && \text{by } (S.4) \text{ and } (S.1) \\ &= T\big(N(p),q\big) \\ &= T\big(q,N(p)\big) && \text{by (4)} \\ &= J_{T,N}(p,q). && (7) \end{aligned}$$

Straightforward from Lemmas 5.2. and relationship (7).

If we assume that N is a fuzzy negation, then we get another result relating (T,N)-, QL- and D-coimplications.

Proposition 5.4. Let T be a t-norm, S be the t-conorm maximum and N be a fuzzy negation and given a QL-coimplication $J_{T,S,N}$, a D-coimplication $J^{T,S,N}$ and a (T,N) co-implication $J_{T,N}$. If $J^{T,S,N}$ and $J_{T,N}$ satisfy $Co-FI$, then the corresponding QL- and D-coimplication coincide and are given by:

$$J_{T,S,N}(p,q) = J^{T,S,N}(p,q) = \begin{cases} 0, & \text{if } p \geq q \\ T\big(q,N(p)\big), & \text{otherwise.} \end{cases}$$

Proof. Let $J^{T,S,N}$, $J_{T,S,N}$ be a D-coimplication and QL-coimplication, respectively.

Not that when $p \geq q \Rightarrow N(p) \leq N(q)$, we have

$$\begin{aligned} J^{T,S,N}(p,q) &= T\big(\max(N(p),N(q)),q\big) \\ &= T(N(q),q) = 0, \text{ by (Proposition 5.2).} \end{aligned}$$

and

$$J_{T,S,N}(p,q) = T(\max(p,q),N(p)) = T\big(p,N(p)\big) = 0.$$

Similarly, when $p < q$,

$$\begin{aligned} J^{T,S,N}(p,q) &= T\big(\max(N(p),N(q)),q\big) = T(N(p),q) \\ &= T(\max(p,q),N(p)) = J_{T,S,N}(p,q). \end{aligned}$$

Remark 5.2. If N is strong negation and S is a t-conorm, then (T,N) co-implication be come (N,S) co-implication such that:

$$J_{N,S}(p,q) = N\big(S(N(q),p)\big), \forall p,q \in [0,1]. \quad (8)$$

The (N,S) co-implication generated by a t-conorm S and a fuzzy negation N is denoted by $J_{N,S}$.

Lemma 5.3. For t-conorm S and a fuzzy strong negation N, then $J_{N,S} \in Co-FI$.

Proof. Let $p, q, r \in [0,1]$.

$J1$: $J_{N,S}(1,1) = J_{N,S}(1,0) = J_{N,S}(0,0) = 0$, and $J_{N,S}(0,1) = N\big(S(N(1),0)\big) = N\big(S(0,0)\big) = N(0) = 1$.

$J2$: Now, assume that $p \leq r \Rightarrow S(N(q),p) \leq S(N(q),r)$

$$\begin{aligned} &\Rightarrow N\big(S(N(q),p)\big) \geq N\big(S(N(q),r)\big) \\ &\Rightarrow J_{N,S}(p,q) \geq J_{N,S}(r,q). \end{aligned}$$

$J3$: Assume that $q \leq r \Rightarrow N(q) \geq N(r)$

$$\begin{aligned} &\Rightarrow S(N(q),p) \geq S(N(r),p) \\ &\Rightarrow N\big(S(N(q),p)\big) \leq N\big(S(N(r),p)\big) \\ &\Rightarrow J_{N,S}(p,q) \leq J_{N,S}(p,r). \end{aligned}$$

Lemma 5.4. A (N,S) co-implication $J_{N,S}$ satisfies (Co-NP), if N is a strong negation.

Proof. Since N is a strong negation,

$$J_{N,S}(0,q) = N\big(S(N(q),0)\big) = N\big(N(q)\big) = q.$$

Theorem 5.5. Given a D-coimplicatiom $J^{T,S,N}$, a (T,N) co-implication $J_{T,N}$ and a QL-coimplication $J_{T,S,N}$. If $S = S_M$, N is strong, T is N-dual of S and $J_{T,N}$ satisfies (1). Then $J^{T,S,N} = J_{T,N} = J_{T,S,N} = J_{N,S}$.

Proof. Assume that N is a strong negation, for all $p, q \in [0,1]$

$$\begin{aligned} &\Leftrightarrow J_{T,N}(p,q) = J_{N,S}(p,q) \\ &\Leftrightarrow T\big(q,N(p)\big) = N\big(S(N(q),p)\big) && \text{by (5) and (8)} \\ &\Leftrightarrow T\big(q,N(p)\big) = N\Big(S\Big(N(q),N\big(N(p)\big)\Big)\Big) && \text{by (2)} \end{aligned}$$

Then $J_{T,N} = J_{N,S}$ iff T is N-dual of S. (9)

Straightforward from (9) and Theorem 5.4. the proof has been completed.

Proposition 5.5. Let N be a strong negation and, given a D-coimplication $J^{T,S,N}$ and a QL-coimplication $J_{T,S,N}$. If $J^{T,S,N}$ or $J_{T,S,N}$ satisfies the contraposition (J4), then $J^{T,S,N} = J_{T,S,N}$.

Proof. Assume that N is a strong negation, $\forall p, q \in [0,1]$. Let $J^{T,S,N}$ satisfies the contraposition (J4), then

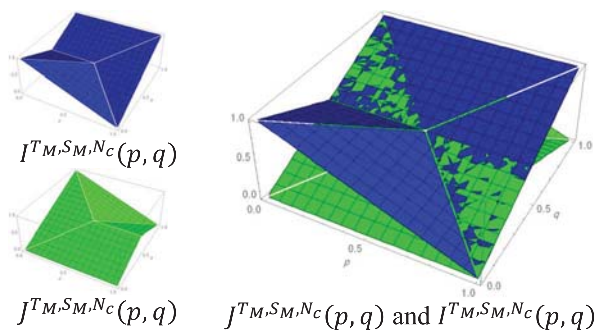
$$\begin{aligned} J^{T,S,N}(p,q) &= J^{T,S,N}\big(N(q),N(p)\big) = T\big(S(q,p),N(p)\big) \\ &= T\big(S(p,q),N(p)\big) = J_{T,S,N}(p,q). \end{aligned}$$

5.3 Examples

To confirm what we have studied in this paper, we will mention some examples of D-coimplications.

Example 5.1. For a t-norm T_M and t-conorm S_M and a strong fuzzy negation N_c then D-coimplication J^{T_M,S_M,N_c} given by

$$\begin{aligned} J^{T_M,S_M,N_c}(p,q) &= T_M\big(S_M(N_c(p),N_c(q)),q\big) \\ &= \min(\max(1-p,1-q),q). \end{aligned}$$



$I^{T_M,S_M,N_c}(p,q)$ — $J^{T_M,S_M,N_c}(p,q)$ — $J^{T_M,S_M,N_c}(p,q)$ and $I^{T_M,S_M,N_c}(p,q)$

Fig. 5.1.

Example 5.2. For a t-norm Π and t-conorm S_Π and a strong fuzzy negation N_c then D-coimplication J^{Π,S_Π,N_c} given by

$$\begin{aligned} J^{\Pi,S_\Pi,N_c}(p,q) &= \Pi\big(S_\Pi(N_c(p),N_c(q)),q\big) \\ &= \Pi(-1-pq,q) \\ &= q(1-pq). \end{aligned}$$

Example 5.3. For a t-norm L and t-conorm S_L and a strong fuzzy negation N_c then D-coimplication J^{L,S_L,N_c} given by

$$J^{L,S_L,N_c}(p,q) = L\big(S_L\big(N_c(p), N_c(q)\big), q\big) = \max(\min(2-p-q,1)+q-1,0).$$

Example 5.4. For a t-norm T_M and t-conorm S_N and a strong fuzzy negation N_c then D-coimplication J^{T_M,S_N,N_c} given by

$$J^{T_M,S_N,N_c}(p,q) = T_M\big(S_N\big(N_c(p), N_c(q)\big), q\big)$$
$$= T_M\left(\begin{cases}\max(1-p,1-q) & \text{if } 1-p+1-q<1 \\ 0 & \text{if } 1-p+1-q \geq 1\end{cases}, q\right)$$
$$= \min\left(\begin{cases}\max(1-p,1-q) & \text{if } p+q>1 \\ 0 & \text{if } p+q \leq 1\end{cases}, q\right).$$

Example 5.5. For a t-norm T_M and t-conorm S_M and a strict negation but not strong N_k then D-coimplication J^{T_M,S_M,N_k} given by

$$J^{T_M,S_M,N_k}(p,q) = T_M\big(S_M\big(N_k(p), N_k(q)\big), q\big) = \min(\max(1-p^2, 1-q^2), q).$$

Conclusion

In this paper we have introduced the fourth usual model of fuzzy co-implication that is D-coimplication then we studied the equivalences between D-coimplication and fuzzy coimplications classes. Through this paper we noted it generally, D-coimplication is defined from non strong negations, that means it is not contraposition of QL-coimplication.

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