The Barycentric Bernstein Form for Control Design

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Abstract

In this paper, an algorithm for computing a polynomial control and a polynomial Lyapunov function in the simplicial Bernstein form is developed. This ensures asymptotic stability of the designed feedback system. To this end, we provide certificates of positivity for polynomials in the simplicial Bernstein form. Subsequently, the state space is partitioned into simplices. On each simplex, we simultaneously compute Lyapunov and control functions. With this control, the equilibrium is asymptotically stable.

Keywords: Bernstein polynomial, inclusion isotonicity, certificates of positivity, control design, stability.

1. Introduction

We consider the stability verification of polynomials with coefficients depending polynomially on parameters varying in a union of simplices. A fundamental problem in computer-aided design is the efficient computation of solutions of a system of nonlinear polynomial equations on bounded domains. This computation provides the theoretical foundation for analysis and design of polynomial control systems, i.e., control systems with polynomial vector fields. Bounding of the solution set of systems of polynomial equations provides general certificates of positivity over a given domain. Computing such bounds for the range (minimum and maximum) of values of a polynomial over intervals has received a good deal of attention in the past [1–4]. Sherbrooke and Patrikalakis [5] have developed a method for solving systems in high dimensions within an n-dimensional rectangular domain, relying on the representation of polynomials in the tensor product Bernstein basis. A method for solving systems within an n-dimensional simplex, which relies on the representation of polynomials in the barycentric Bernstein basis is given in [6]. The range of polynomial functions is bounded by the smallest and the largest (enclosure bound) Bernstein coefficients over a simplex. An efficient algorithm for computing Bernstein coefficients of arbitrary polynomials has been proposed in [7]. The inclusion isotonicity of the related tensorial enclosure function over boxes has been shown in [8–10]. This property, in the simplicial case, states that the enclosure bound over a subsimplex is contained within the enclosure bound over the whole simplex. In this paper, we show that the *inclusion isotonicity* in the simplicial case is also obtained if the barycentric subdivision strategy is applied for simplices. This improves the certificates of positivity, where the stability of the designed feedback system has been translated to certificates of positivity.

On the other hand, computing a Lyapunov function of polynomial vector fields has also attracted the interest of many researchers in the past [11–15]. The way to compute a Lyapunov function for a polynomial system is to provide a certificate of positivity. A certificate of positivity [16] in the Bernstein basis always exists if a polynomial is positive, and the degree of the Bernstein form is sufficiently large or the cells in the state-space partition are sufficiently small. The inclusion isotonicity property improved the certificate of positivity under subdivision. Specifically, we apply the barycentric subdivision strategy to provide a local certificate of positivity for a Lyapunov function in the Bernstein basis. Subsequently, we provide an algorithm that estimates a polynomial control and a polynomial Lyapunov function in a finite number of computations of enclosure bounds.

The organization of our paper is as follows: In the next section, we briefly recall the simplicial polynomial Bernstein form and its basic properties. In Section 3, the inclusion isotonicity of the barycentric polynomial Bernstein form is shown. The simplicial Lyapunov stability is addressed in Section 4. Finally, Section 5 comprises conclusions.

2. Bernstein Expansion

For completeness of the exposition, we introduce some notation and essential background about the simplicial Bernstein basis. Throughout the paper, $V = [\sigma_0,\ldots,\sigma_n]$ will denote a non-degenerate simplex of $\mathbb{R}^n$, i.e., the points $\sigma_0,\ldots,\sigma_n$ are affinely independent. Let $\lambda_0,\ldots,\lambda_n$ be the associated barycentric coordinates of V. In other words, they are linear polynomials of $\mathbb{R}[X] = \mathbb{R}[X_1,\ldots,X_n]$ such that $\sum_{i=0}^{n}\lambda_i(x) = 1$ and $\forall x \in \mathbb{R}^n$, $x = \lambda_0(x)\sigma_0 + \cdots + \lambda_n(x)\sigma_n$. The realization $|V|$ of the simplex V is the subset of $\mathbb{R}^n$ defined as the convex hull of the points $\sigma_0,...,\sigma_n$. Without loss of generality, we can assume that V is the standard simplex $\Delta = [e_0,e_1,\ldots,e_n]$, where $(e_1,\ldots,e_n)$ denotes the canonical basis of $\mathbb{R}^n$, and $e_0 = (0,...,0)$ the origin. This is not a restriction since any simplex V in $\mathbb{R}^n$ can be mapped affinely upon Δ.

Specifically, if $x = (x_1,...,x_n) \in \Delta$, then $\lambda_0,...,\lambda_n = (1-\sum_{i=1}^{n} x_i, x_1,...,x_n)$. We refer to the multi-index $\alpha = (\alpha_0,\ldots,\alpha_n) \in \mathbb{N}^{n+1}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$. For $\hat{\beta} = (\beta_1,...,\beta_n)$, $\hat{\alpha} = (\alpha_1,...,\alpha_n)$ with $\hat{\beta} \le \hat{\alpha}$ (componentwise), we define

$$\binom{\hat{\alpha}}{\hat{\beta}} := \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}.$$

If k is a natural number such that $|\hat{\beta}| \le k$, we use the notation $\binom{k}{\hat{\beta}} := \frac{k!}{\beta_1!\ldots\beta_n!(k-|\hat{\beta}|)!}$. The Bernstein polynomials of degree k with respect to Δ are the polynomials $(B_\alpha^{(k)})_{|\alpha|=k}$, where

$$B_\alpha^{(k)}(\lambda) = \binom{k}{\alpha}\lambda^\alpha. \tag{1}$$

For $x \in \mathbb{R}^n$ its multi-powers are $x^{\hat{\beta}} := \prod_{i=1}^{n} x_i^{\beta_i}$. Let f be a polynomial function of degree l,

$$f(x) = \sum_{|\hat{\beta}|\le l} a_{\hat{\beta}} x^{\hat{\beta}}, \tag{2}$$

f can be uniquely expresses for $l \le k$ as

$$f(x) = \sum_{|\alpha|=k} b_\alpha(f,k,\Delta) B_\alpha^{(k)}, \tag{3}$$

where $b_\alpha(f,k,\Delta)$ are called the Bernstein coefficients of f of degree k with respect to Δ given as

$$b_\alpha(f,k,\Delta) = \sum_{\hat{\beta}\le\hat{\alpha}} \frac{\binom{\hat{\alpha}}{\hat{\beta}}}{\binom{k}{\hat{\beta}}} a_{\hat{\beta}}. \tag{4}$$

The grid points of degree k associated to Δ are the points

$$\sigma_\alpha(k,\Delta) = \frac{\alpha_0 e_0 + \cdots + \alpha_n e_n}{k} \in \mathbb{R}^n \ (|\alpha| = k), \tag{5}$$

whereas, the control points associated to f are

$$(\sigma_\alpha(k,\Delta),\ b_\alpha(f,k,\Delta)) \in \mathbb{R}^{n+1} \ (|\alpha| = k).$$

The set of control points of f forms its control net of degree k.

3. Inclusion Isotonicity

The key to finding a Lyapunov function for a polynomial system is to find a certificate of positivity, where the inclusion isotonicity, defined below, allows local certificates of positivity under subdivision. In this section, we prove that the barycentric Bernstein form is inclusion isotone, a property which is of fundamental importance in interval computations [17, Section 1.4]. We define the set of real intervals $\mathbb{I}(\mathbb{R})$. An interval function $F : \mathbb{I}(\mathbb{R})^n \longrightarrow \mathbb{I}(\mathbb{R})$ is called *inclusion isotone,* if, for all $X,Y \in \mathbb{I}(\mathbb{R})^n$, $X \subseteq Y$ implies $F(X) \subseteq F(Y)$.

The graph of a polynomial f over Δ is contained in the *convex hull* of its associated control points. This implies the *range enclosing property* [2]

$$\min_{|\alpha|=k} b_\alpha(f,k,\Delta) \le f(x) \le \max_{|\alpha|=k} b_\alpha(f,k,\Delta),\ x \in \Delta. \tag{6}$$

It follows that the interval (enclosure bound)

$$B(f,k,\Delta) := [\min b_\alpha(f,k,\Delta), \max b_\alpha(f,k,\Delta)]$$

encloses the range of f of degree $l \le k$ over Δ. Assume that $W = (w_0,...,w_n)$ is a subsimplex extracted from Δ by the barycentric subdivision. We prove that $B(f,k,W)$ is contained in $B(f,k,\Delta)$. We apply the *barycentric subdivision strategy* [6], which is a particular way of dividing Δ at a point into subsimplices.

We aim at computing the Bernstein coefficients over W as convex combinations of the Bernstein coefficients over Δ. In order to do so, we compute the Bernstein coefficients in a particular coordinate direction, r say, since the de Casteljau algorithm [18] computes the coefficients in all coordinate directions. The barycentric subdivision strategy supposes subdivision at an edge or a non-edge point with respect to $|\Delta|$.

If we subdivide Δ at an edge point $w_r \in \mathbb{R}^n$, then Δ will be subdivided into two subsimplices constructed from Δ at $\lambda_i(w_r), \lambda_{i+1}(w_r)$, and we call them the constructed subsimplices. Otherwise (non-edge point), Δ can be subdivided into $\le n+1$ (constructed) subsimplices, Figure 1. It is sufficient to show that the inclusion isotonicity holds if we compute the coefficients in rth coordinate direction, $0 < \lambda_r(w_r) < 1$, with respect to the constructed simplices.

Let for some $r \in \{0,...,n\}$, $0 < \lambda_r(w_r) < 1$, then for all i, $i \ne r$, we have $1 > \lambda_i(w_r) \ge 0$. The following algorithm computes the Bernstein coefficients in rth coordinate direction to extract a new subsimplex W, where the barycentric subdivision is applied.

Algorithm 1: Bernstein coefficients over subsimplices

Input : Simplices $W^{[w_r]}$ and Δ with $W^{[w_r]}$ contained in Δ, and the Bernstein coefficients on Δ.

Output: The Bernstein coefficients on $W^{[w_r]}$ as convex combination of the ones on Δ .

Initialization: $\forall |\alpha| = k$, $b_\alpha^{(0)} := b_\alpha(f,k,\Delta)$.

Choose: $r \in \{0,...,n\}$, $r \neq i_0$, $1 > \lambda_r(w_r) > 0$.

for $d = 1,...,k$ ***do***

for $|\alpha| = k - d$ ***do***

if $\lambda_r + \lambda_{i_0} = 1$ ***then***

$$b_\alpha^{(d)} = \lambda_r b_{\alpha+e_r}^{(d-1)} + (1-\lambda_r) b_{\alpha+e_{r+1}}^{(d-1)}$$

else

$$b_\alpha^{(d)} = \lambda_0 b_{\alpha+e_0}^{(d-1)} + ... + \lambda_r b_{\alpha+e_r}^{(d-1)} + ... + \lambda_n b_{\alpha+e_n}^{(d-1)}$$

end ***if***

end ***for***

end ***for***

return $b_\alpha(f,k,W^{[w_r]}) = b_{\alpha^{[r]}}^{(\alpha_r)}$

$$(|\alpha| = k,\ \alpha^{[r]} := (\alpha_0, ..., \alpha_{r-1}, 0, \alpha_{r+1}, ..., \alpha_n)).$$

Theorem 3.1. *Let W be a subsimplex of Δ, which is extracted by the barycentric subdivision strategy. Then*

$$B_\alpha(f,k,W) \subseteq B_\alpha(f,k,\Delta).$$

Proof. Let $W^{[w_r]}$ be the constructed subsimplex from Δ at $\lambda_r(w_r)$, by subdivision at w_r, $r \in \{0,...,n\}$. We proceed to extract a subsimplex W into rth coordinate direction and return to Algorithm 1 at all w_r (full algorithm). Let (for simplicity) $r = 0$, then we extract at $\lambda_0(w_0)$

$$W^{[w_0]} = [w_0, e_1, ..., e_n].$$

By Algorithm 1, the Bernstein coefficients on $W^{[w_0]}$ are convex combinations of the coefficients on Δ.

Repeatedly splitting at the remaining w_r, $r = 1,...,n$, with respect to the constructed simplices, then we will have finally at w_n, the Bernstein coefficients on

$$W^{[w_n]} = [w_0, ..., w_n]$$

as convex combinations of the coefficients on $W^{[w_{n-1}]}$, which completes the proof. □

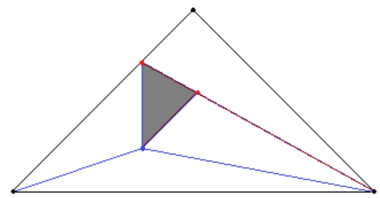

Figure 1. Subsimplices are constructed by subdivision steps at edge points (colored red) and a non edge point (colored blue).

Corollary 1. *The union of enclosure bounds of f over $W^{[i]}$, $i = 0,...,n$, is contained in $B(f,k,\Delta)$.*

Example 3.1. The polynomial $f = 5x^2 - 2x + 1$ is positive on the simplex $\Delta = [-1,1]$ but $b(f,2,\Delta) = (8,-4,4)$. However, by the first binary splitting of Δ, the certificate of positivity of f follows since $b(f,2,[-1,0]) = (8,2,1)$ and $b(f,2,[0,1]) = (1,0,4)$.

The following lemma computes the (Complexity) number of Bernstein coefficients and the computation steps needed to perform one call to Algorithm 1, see [6].

Lemma 3.2. *Let η denotes the number of barycentric coordinates associated to Δ, $2 \leq \eta \leq n+1$. The number of multiplication steps needed to perform one call to Algorithm 1 at $w \in \mathbb{R}^n$ with respect to Δ is*

$$\frac{\eta(k+n)!}{(k-1)!(n+1)!}.$$

Proof. The number of Bernstein coefficients describing an n-dimensions simplicial Bernstein polynomial of degree k is:

$$S(k,n) := \binom{k+n}{n} = \frac{(k+n)!}{k!n!}.$$

Therefore the total number of the calculated Bernstein coefficients in one call to Algorithm 1 is

$$D(t,n) := \sum_{t=0}^{k-1} S(t,n) = \binom{k+n}{k-1} = \frac{(k+n)!}{(k-1)!(n+1)!},$$

from which the statement follows. □

4. Lyapunov Stability Analysis

In this section, we devise an algorithm for control synthesis. We suppose that all vector fields are polynomials defined on a union of simplices, $\Delta = W^{[0]} \cup ... \cup$

$W^{[n]}$. To this end, we use the representation of the given control system in the (barycentric) Bernstein basis. The affine control system is given by

$$\dot{x} = F_u(x) := p(x) + g(x)u(x), \tag{7}$$

where the vector field $F_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by the drift $p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the control with the input matrix function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$.

We will represent all polynomials in the Bernstein form. We follow the definition of asymptotic and stability and call v satisfying the conditions of Definition 4.1 a Lyapunov function for F_u.

Definition 4.1. Let x_0 be an equilibrium point for (7) and let $A \subseteq \mathbb{R}^n$ be a collection of simplices containing the interior point x_0. Let $v : A \longrightarrow \mathbb{R}$ be a continuously differentiable function such that $v(x_0) = 0$,

$$v(x) > 0,\ \forall x \in A \setminus \{x_0\},$$

$$\mathscr{L}_{F_u}(v)(x) = \frac{\partial v}{\partial x}(x)F_u(x) < 0,\ \forall x \in A \setminus \{x_0\},$$

where $\mathscr{L}$ denotes the Lie derivative. Then v will be called a Lyapunov function for F_u.

Specifically, if there exist a Lyapunov function for F_u, then x_0 is an asymptoticly stable equilibrium of the system F_u.

Remark 4.1. By application of the first interior subdivision step of Δ at x_0, we will have $\Delta = W^{[0]} \cup \ldots \cup W^{[n]}$, such that from Section 3, for all $i \in \{0,\ldots,n\}$,

$$\underline{c}^f := \min_{|\alpha|=k} b_\alpha(f,k,\Delta) \le \min_{|\alpha|=k} b_\alpha(f,k,W^{[i]}) \le f(x)$$

$$\le \max_{|\alpha|=k} b_\alpha(f,k,W^{[i]}) \le \max_{|\alpha|=k} b_\alpha(f,k,\Delta) =: \overline{c}^f,\ \forall x \in W^{[i]}.$$

Suppose the candidate Lyapunov function v is a positive polynomial expressed in the power form (2) of degree l, where $v(x_0) = 0$. Without loss of generality, we assume that $x_0 = e_0$.

Remark 4.2. By the Bernstein theorem (cf. Remark 4.1), there exists $k \ge l$ such that $b_\alpha(v,k,W^{[i]})$ are positive for all $|\alpha| = k$ if and only if v of degree l is positive.

4.1. Controller Bounding Functions

We provide a subdivision strategy of a simplex to compute a pair (u,v) of polynomial functions such that v is a Lyapunov function for F_u. If there exists such a pair (u,v), we will briefly say that there exists a stabilizing control. Specifically, the enclosure bounds of initial control and a candidate Lyapunov function are recursively shrinked until the bounds for Bernstein coefficients of u and v are computed.

The number of Bernstein coefficients of any $n-$dimensional polynomial is $N := \binom{k+n}{k}$. The Bernstein form of v of degree $l \le k$ over $W \in \{W^{[0]},\ldots,W^{[n]}\}$ is given as

$$v = \sum_{|\alpha|=k} b_\alpha(v,k,W)B_\alpha^{(k)},$$

where $b_\alpha(v,k,\Delta) = 0$ if $|\hat{\alpha}| = 0$, $\alpha_0 = k$. Therefore, as given by Farouki and Rajan [19]

$$v_i' := \frac{\partial v}{\partial x_i}(x) = \sum_{|\alpha|=k} b_\alpha(v,k,W)\frac{\partial B_\alpha^{(k)}}{\partial x_i}(x)$$

$$= \sum_{|\alpha|=k-1} k(b_{\alpha+e_i} - b_\alpha)B_\alpha^{(k-1)}(x), \tag{8}$$

from which the Bernstein coefficients of v_i' are linear combinations of the coefficients of v.

Define a sub-bound of any $[\underline{b},\overline{b}]$ by $[\underline{b}_{\varepsilon_1},\overline{b}_{\varepsilon_2}]$, where

$$\underline{b}_{\varepsilon_1} = \underline{b} + (\overline{b} - \underline{b})\varepsilon_1,\ 0 \le \varepsilon_1 \le 1, \tag{9}$$

and

$$\overline{b}_{\varepsilon_2} = \overline{b} - (\overline{b} - \underline{b})\varepsilon_2,\ 0 \le \varepsilon_2 \le 1. \tag{10}$$

Define $L[\underline{b},\overline{b}] = [\underline{b}_{\varepsilon_1},\overline{b}]$ and $R[\underline{b},\overline{b}] = [\underline{b},\overline{b}_{\varepsilon_2}]$. Denote by H^* the set of all finite strings of compositions of elements of $H = \{L,R\}$. For example, the string $LR \in H^*$ is the composition $L \circ R[\underline{b},\overline{b}] = [\underline{b}_{\varepsilon_1},\overline{b}_{\varepsilon_2}]$.

Theorem 4.1. *Let $\{\overline{b}_\alpha,\overset{N}{\cdots},\overline{b}_\alpha\} \subset \mathbb{R}^+$ and $\underline{b}_j,\overline{b}_j \in \mathbb{R}$, $j = 1,\ldots,m$, be real numbers. Suppose there exist a stabilizing polynomial control function. Then, for any $0 \le \varepsilon_1,\varepsilon_2 \le 0$, there exit strings $S_\alpha,T_j \in H^*$, $|\alpha| = k$, $j \in \{1,\ldots,m\}$, and a pair (u,v) such that v is a Lyapunov function for F_u results Bernstein coefficients of v and u bounded as*

$$b_\alpha(v,k,W) \in S_\alpha[0,\overline{b}_\alpha] \text{ and } b_\alpha(u_j,k,W) \in T_j[\underline{b}_j,\overline{b}_j].$$

Proof. Let $\{\overline{b}_\alpha,\overset{N}{\cdots},\overline{b}_\alpha\} \subset \mathbb{R}^+$ and $\underline{b}_j,\overline{b}_j \in \mathbb{R}$, $j = 1,\ldots,m$. Suppose $b_\alpha(v,k,W) = \overline{b}_\alpha$ for all $|\alpha| = k$, $\alpha_0 \neq k$, and $b_\alpha(v,k,W) = 0$ for $\alpha_0 = k$. Let (case 1) $b_\alpha(u_j,k,W) = \underline{b}_j$ for all $|\alpha| = k$, and we denote this control by $\underline{u}_j(x)$. The polynomial $\mathscr{L}_{F_{\underline{u}}}(v)(x)$ can be rearranged as

$$\mathscr{L}_{F_{\underline{u}}}(v)(x) = \sum_{i=1}^{n} v_i'(x)p_i(x) + \underline{u}_1(x)\sum_{i=1}^{n} v_i'(x)g_{i1}(x) + \ldots$$

$$+\underline{u}_m(x)\sum_{i=1}^{n} v_i'(x)g_{im}(x). \tag{11}$$

Hence, we can compute (by Remark 4.1)

$$\overline{E}_1 := \sum_{i=1}^{n} \overline{c}^{v_i' p_i} + \sum_{i=1}^{n} \overline{c}^{v_i' g_{i1} \underline{u}_1} + \ldots + \sum_{i=1}^{m} \overline{c}^{v_i' g_{im} \underline{u}_m}. \quad (12)$$

Now, we let (case 2) $b_\alpha(u_j, k, W) = \overline{b}_j$ for all $|\alpha| = k$, and we denote this control by $\overline{u}(x)$. Then compute the upper bound

$$\overline{E}_2 := \sum_{i=1}^{n} \overline{c}^{v_i' p_i} + \sum_{i=1}^{n} \overline{c}^{v_i' g_{i1} \underline{u}_1} + \ldots + \sum_{i=1}^{m} \overline{c}^{v_i' g_{im} \overline{u}_m}. \quad (13)$$

Let $\overline{E}_1$ and $\overline{E}_2$ are negative, then by Remark 4.1, we can estimate (in case 1)

$$\mathscr{L}_{F_{\underline{u}}}(v)(x) \leq \overline{E}_1, \ \forall x \in W - \{e_0\}, \quad (14)$$

and (in case 2) $\mathscr{L}_{F_{\overline{u}}}(v)(x) \leq \overline{E}_2$, and (u, v) have Bernstein coefficients within $[0, \overline{b}_\alpha]^N$ and $[\underline{b}_j, \overline{b}_j]$, $j = 1, \ldots, m$, $\varepsilon_1 = \varepsilon_2 = 0$. Otherwise, suppose $\overline{E}_2$ is non-negative. Here, we let H^* be the set of all finite strings of compositions of elements of $H = \{L, R\}$. We give the proof in case $H = R$, the proof of the other cases analogous. So, we apply the shrinking method (10) to $[0, \overline{b}_\alpha]$ and $[\underline{b}_j, \overline{b}_j]$, for a fixed ε_2, and compute $R_\alpha[0, \overline{b}_\alpha]$ and $R_j[\underline{b}_j, \overline{b}_j]$, see Figure 2. This ensures the negativity of a new $\overline{E}_2^*$ with $R_\alpha[0, \overline{b}_\alpha]$ and $R_j[\underline{b}_j, \overline{b}_j]$. Finally, by (14) and Remark 4.1, the Bernstein coefficients of (u, v) are contained within the shrinked bounds. $\square$

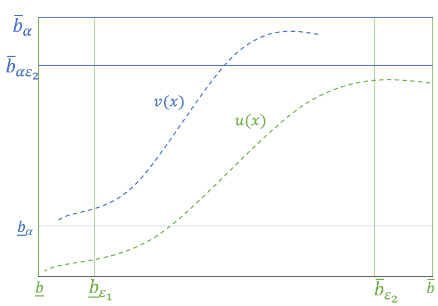

Figure 2. Srinking Bounds of u (colored green) and v (colored blue).

Example 4.1. Let

$$\dot{x}_1 = -x_1^3, \ \dot{x}_2 = -x_1 - x_2^2 + u$$

be of degree $l = 3$ over $V = W^{[0]} \cup W^{[1]} \cup W^{[2]} \cup W^{[3]}$, where $W^{[0]} = (e_0, (-1,-1), (1,-1))$, $W^{[1]} = (e_0, (1,-1), (1,1))$, $W^{[2]} = (e_0, (1,1), (-1,1))$ and $W^{[3]} = (e_0, (-1,1), (-1,-1))$. In order to compute $v(x)$ and $u(x)$, we let by an arbitrary way $[0, \overline{b}_\alpha]^N = [0,4] \times [0,6]^8 \times [0,3]$, and $[\underline{b}, \overline{b}] = [-8, 50]$. It is sufficient to suppose $b_\alpha(v, l, V) = \overline{b}_\alpha, \forall |\alpha| = l$, $\alpha_0 \neq l$. Furthermore, we suppose $b_\alpha(v, l, V) = 0$ if $\alpha_0 = l$. Let $\overline{u}'(x)$ be in the Bernstein form with coefficients equal 50, and note that

$$\overline{E}_2 = \overline{c}^{v_1' \dot{x}_1} + \overline{c}^{-v_2' x_1} + \overline{c}^{-v_2' x_2^2} + \overline{c}^{-v_2' \overline{u}'}$$

is positive with the given bounds ($\overline{E}_1$ is negative). We shrink the bounds $[-8, 50]$ from the right endpoint (using (10)) to be $[-8, \overline{b}_{\varepsilon_2}]$, $\varepsilon_2 = 1/10$. As above, we let $\overline{u}(x)$ be in the Bernstein form with coefficients equal $\overline{b}_{\varepsilon_2}$. Therefore, we have

$$\overline{E}_2^* = \overline{c}^{v_1' \dot{x}_1} + \overline{c}^{-v_2' x_1} + \overline{c}^{-v_2' x_2^2} + \overline{c}^{-v_2' \overline{u}} < 0.$$

Eventually, we have $\frac{\partial v}{\partial x}(x) F_{\overline{u}}(v)(x) \leq \overline{E}_2^*, \forall x \in V \setminus \{e_0\}$, from which the Bernstein coefficients of $(u(x), v(x))$ are belong to the shrinked bounds.

4.2. Algorithm for Controller Synthesis

In this section, we address the question if we can compute a pair (u, v) within enclosure bounds so that $\mathscr{L}_{F_u}(v)$ is negative (cf. Theorem 4.1). Otherwise, there is no feasible solution in these bounds. To this end, we provide an algorithm that decides and finds bounds that estimate $(u(x), v(x))$ over $\Delta = W^{[0]} \cup \ldots \cup W^{[n]}$. Let

$$\mathscr{L}_{F_u}(v)(x) = \frac{\partial v}{\partial x}(x)(p(x) + g(x)u(x))$$
$$= v'(x)p(x) + v'(x)g(x)u(x),$$

where $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the control polynomial function. Then for $k \leq l$ and $W \in \{W^{[0]}, \ldots, W^{[n]}\}$, we have

Algorithm 2: Computing of controller

Input : $\underline{b}_j, \overline{b}_j \in \mathbb{R}$, $j = 1, \ldots, m$ ($\underline{b} < \overline{b}$), $\{\overline{b}_\alpha, \overset{N}{\cdots}, \overline{b}_\alpha\} \subset \mathbb{R}^+$, and strings $S_\alpha, T_j \in H^*$, with $0 \leq \varepsilon_1, \varepsilon_2 \leq 1$.
Output: $T_j[\underline{b}_j, \overline{b}_j]$ and $S_\alpha[0, \overline{b}_\alpha]$.

Compute: (By application of (12) and (13) to (11) with $\underline{u}$ and $\overline{u}$) $\overline{E}_1$ and $\overline{E}_2$.
if $\overline{E}_1$ and $\overline{E}_2 < 0$
  then $\mathscr{L}_{F_u}(v)(x) < 0$
else if $\overline{E}_1 \geq 0$
  then compute $L_j[\underline{b}_j, \overline{b}_j]$ and $L_\alpha[0, \overline{b}_\alpha]$
  else if $\overline{E}_2 \geq 0$
  then compute $R_j[\underline{b}_j, \overline{b}_j]$ and $R_\alpha[0, \overline{b}_\alpha]$
otherwise ($\overline{E}_1$ and $\overline{E}_2 \geq 0$) there is no solution in $[\underline{b}_j, \overline{b}_j]$ and $[0, \overline{b}_\alpha]^N$
  end if
  end if
end if

$$\mathscr{L}_{F_u}(v) = \sum_{|\alpha|=k} b_\alpha(\mathscr{L}_{F_u}(v),k,W)B_\alpha^{(k)},$$

where $b_\alpha(\mathscr{L}_{F_u}(v),k,W) : \mathbb{R}^n \longrightarrow \mathbb{R}$. In Algorithm 2, we test if $[0,\overline{b}_\alpha]$, $[\underline{b},\overline{b}]$ are the enclosure bounds of the pair (u,v). Finally, we guarantee that $\mathscr{L}_{F_u}(v)(x) < 0$ for all $x \in W$ by degree elevation or subdivision of Δ, see [16].

4.3. Subdivision of bounds

In this section, we provide a subdivision strategy for the picked bounds of the Lyapunov and control functions in Theorem 4.1. The polynomial $\mathscr{L}_{F_{\overline{u}}}(v)(x)$ (similarly $\mathscr{L}_{F_{\underline{u}}}(v)(x)$) can be rearranged as (11). Let $G_i(x) := v_i'(x)p_i(x)$ and $L_j(x) := \overline{u}_j(x)\sum_{i=1}^n v_i'(x)g_{ij}(x)$, $j \in \{1,...,m\}$. Assume the bounds $[0,\overline{b}_\alpha]^N$ and $[\underline{b}_j,\overline{b}_j]$ in Algorithm 2 contain (u,v) but $\overline{E}_1 \geq 0$, $\sum_{j=1}^m \overline{c}^{L_j} \geq -\sum_{i=0}^n \overline{c}^{G_i}$. Then, we will pick the set of bounds $\{[\underline{b}_1,\overline{b}_1],...,[\underline{b}_s,\overline{b}_s]\}$ from $\{[\underline{b}_1,\overline{b}_1],...,[\underline{b}_m,\overline{b}_m]\}$ that satisfy

$$\sum_{r=1}^s \overline{c}^{L_r} < -\sum_{i=0}^n \overline{c}^{G_i}. \tag{15}$$

For some $[\underline{b}_{j_0},\overline{b}_{j_0}]$, $j_0 \in \{1,...,m\}$, $j_0 \neq r$, we have $\overline{c}^{L_{j_0}} + \sum_{r=1}^s \overline{c}^{L_r} \geq -\sum_{i=0}^n \overline{c}^{G_i}$. To this end, we need to shrink some of $\{[\underline{b}_{j_0},\overline{b}_{j_0}],[\underline{b}_1,\overline{b}_1],...,[\underline{b}_s,\overline{b}_s]\}$. Apply the procedure for all $j = 1,...,m$, we conclude that $\sum_{j=1}^m \overline{c}^{L_j} < -\sum_{i=0}^n \overline{c}^{G_i}$.

5. Conclusions

We investigated certificates of positivity in the simplicial Bernstein basis by degree elevation and subdivision. By the Barycentric subdivision strategy, we proved that the barycentric Bernstein form is inclusion isotone. This property improved the local certificate of positivity under subdivision. Finally, the control design algorithm was developed that computes Lyapunov and control functions in the Bernstein form.

Acknowledgements
This work is supported by the Danish Council for Independent Research under grant number DFF- 4005-00452 in the project CodeMe.

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