FOUNDATIONS OF DIFFERENTIAL GEOMETRY

VOLUME I

SHOSHICHI KOBAYASHI
University of California, Berkeley, California

and

KATSUMI NOMIZU
Brown University, Providence, Rhode Island

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PREFACE

Differential geometry has a long history as a field of mathematics and yet its rigorous foundation in the realm of contemporary mathematics is relatively new. We have written this book, the first of the two volumes of Foundations of Differential Geometry, with the intention of providing a systematic introduction to differential geometry which will also serve as a reference book.

Our primary concern was to make it as self-contained as possible and to give complete proofs of all standard results that this purpose has been achieved with the following arrangements. In Chapter I we have given a survey of differentiable manifolds, Lie groups and fibre bundles. The readers who are unfamiliar with them may learn the subjects from the books of Chevalley, Montgomery-Zippin, Pontrjagin, and Steenrod, listed in the Bibliography, which are our standard references in Chapter I. We have also included concise account of tensor-algebras and tensor fields, the central theme of which is the notion of derivation of the algebra of tensor fields. In the Appendices, we have given some results from topology, Lie group theory and others which we need in the main text. With these preparations, the main text of the book is self-contained.

Chapter II contains the connection theory of Ehresmann and its later development. Results in this chapter are applied to linear and affine connections in Chapter III and to Riemannian connections in Chapter IV. Many basic results on normal coordinates, convex neighborhoods, distance, completeness and holonomy groups are proved here completely, including the de Rham decomposition theorem for Riemannian manifolds. 14

In Chapter V, we introduce the sectional curvature of a Riemannian manifold and the spaces of constant curvature. A more complete treatment of properties of Riemannian manifolds involving sectional curvature depends on the calculus of variations and will be given in Volume II. We discuss flat affine and Riemannian connections in detail.

In Chapter VI, we first discuss transformations and infinitesimal transformations which preserve a given linear connection or a Riemannian metric. We include here various results concerning Ricci tensor, holonomy and infinitesimal isometries. We then...
treat the extension of local transformations and the so-called equivalence problem for affine and Riemannian connections. The results in this chapter are closely related to differential geometry of homogeneous spaces (in particular, symmetric spaces) which are planned for Volume II.

In all the chapters, we have tried to familiarize the readers with various techniques of computations which are currently in use in differential geometry. These are: (1) classical tensor calculus with indices; (2) exterior differential calculus of E. Cartan; (3) formalism of covariant differentiation $\nabla_X Y$, which is the newest among the three. We have also illustrated as we see fit, the methods of using a suitable bundle or working directly in the base space.

The Notes include some historical facts and supplementary results Pertinent to the main content of the present volume. The Bibliography at the end contains only those books which we quote throughout the book.

Theorems, propositions and corollaries are numbered for each section. For example, each chapter, say, Chapter 1 is in Section 1. In the rest of the same chapter, it will be referred to simply as Theorem 1.1. For quotation in subsequent chapters, it is referred to as Theorem 1.1 of Chapter II.

We originally planned to write one volume which would include all the content of the present volume as well as the following topics: submanifolds; variations of the length integral; differential geometry of complex and Kahler manifolds; differential geometry of homogeneous spaces; symmetric spaces; characteristic classes. The considerations of space and time have made it desirable to divide the book in two volumes. The topics mentioned above will therefore be included in Volume II.

In concluding the preface, we wish like to thank Professor L. Bers, who invited us to undertake this project, and Inter-science Publishers, a division of John Wiley and Sons, for their patience and kind cooperation. We are greatly indebted to Dr. A. J. Lohwater, Dr. H. C. Ozeki, Messrs. A. Howard and E. Ruh for their kind help which resulted in improvements of both the content and the presentation. We also acknowledge the grants of the National Science Foundation which supported part of the work included in this book.

SHOUSHIGI KOBAYASHI
KATSUMI NOMIZU
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CHAPTER I

Differentiable Manifolds

A pseudogroup of transformations on a topological space $S$ is a set $\Gamma$ of transformations satisfying the following axioms:

1. Each $f \in \Gamma$ is a homeomorphism of one open set (called the domain of $f$) onto another open set (called the range of $f$) of $S$.

2. If $f \in \Gamma$, then the restriction of $f$ to an arbitrary open subset of the domain of $f$ is in $\Gamma$.

3. Let $U = \bigcup U_i$, where each $U_i$ is an open set of $S$. A homeomorphism $f$ of $U$ onto an open set of $S$ belongs to $\Gamma$ if the restriction of $f$ to $U_i$ is in $\Gamma$ for every $i$.

4. For every open set $U$ of $S$, the identity transformation of $U$ is in $\Gamma$.

5. If $f \in \Gamma$, then $f^{-1} \in \Gamma$.

6. If $f, g \in \Gamma$ is a homeomorphism of $U$ onto $V$ and $f, g \in \Gamma$ is a homeomorphism of $U'$ onto $V'$ and if $V \cap U'$ is non-empty, then the homeomorphism $f \cdot g^{-1}$ of $f^{-1}(V \cap U')$ onto $f^{-1}(V \cap U')$ is in $\Gamma$.

We give a few examples of pseudogroups which are used in this book. Let $R^n$ be the space of $n$-tuples of real numbers $(x_1, x_2, \ldots, x_n)$ with the usual topology. A mapping $f$ of an open set of $R^n$ into $R^m$ is said to be of class $C^r$, $r = 1, 2, \ldots, r$, if $f$ is continuously $r$ times differentiable. By class $C^0$ we mean that $f$ is continuous. By class $C^\omega$ we mean that $f$ is real analytic. The pseudogroup $\Gamma^r(R^n)$ of transformations of class $C^r$ of $R^n$ is the set of homeomorphisms $f$ of an open set of $R^n$ onto an open set of $R^n$ such that both $f$ and $f^{-1}$ are of class $C^r$. Obviously $\Gamma^r(R^n)$ is a pseudogroup of transformations of $R^n$. If $r < s$, then $\Gamma^r(R^n)$ is a
subpseudogroup of $\Gamma^r(R^n)$. If we consider only those $f \in \Gamma^r(R^n)$ whose Jacobians are positive everywhere, we obtain a subpseudogroup of $\Gamma^r(R^n)$. This subpseudogroup, denoted by $\Gamma^r(R^n)$, is called the pseudogroup of orientation-preserving transformations of class $C^r$ of $R^n$. Let $C^r$ be the space of $n$-tuples of complex numbers with the usual topology. The pseudogroup of holomorphic transformations of $C^r$ can be similarly defined and will be denoted by $\Gamma^r(U)$. We shall identify $C^r$ with $R^{2n}$, when necessary, by mapping $(z_1, \ldots, z^n) \in C^r$ into $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, where $z^2 = x^2 + iy^2$. Under this identification, $\Gamma^r(C^r)$ is a subpseudogroup of $\Gamma^r(R^n)$ for any $r$.

An atlas of a topological space $M$ compatible with a pseudogroup $\Gamma$ is a family of pairs $(U_i, \varphi_i)$, called charts, such that

(a) Each $U_i$ is an open set of $M$ and $\bigcup U_i = M$;

(b) Each $\varphi_i$ is a homeomorphism of $U_i$ onto an open set of $S$;

(c) Whenever $U_i \cap U_j$ is non-empty, the mapping $\varphi_i \circ \varphi_j^{-1}$ of $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$ is an element of $\Gamma$.

A complete atlas of $M$ compatible with $\Gamma$ is an atlas of $M$ compatible with $\Gamma$ which is not contained in any other atlas of $M$ compatible with $\Gamma$. Every atlas of $M$ compatible with $\Gamma$ contains a unique complete atlas of $M$ compatible with $\Gamma$. In fact, given an atlas $A = \{(U_i, \varphi_i)\}$ of $M$ compatible with $\Gamma$, let $\tilde{A}$ be the family of all pairs $(U, \varphi)$ such that $\varphi$ is a homeomorphism of an open set $U$ of $M$ onto an open set of $S$ and

$$\varphi_i \circ \varphi^{-1}: \varphi(U \cap U_i) \to \varphi_i(U \cap U_j)$$

is an element of $\Gamma$ whenever $U \cap U_i$ is non-empty. Then $\tilde{A}$ is the complete atlas containing $A$.

If $\Gamma'$ is a subpseudogroup of $\Gamma$, then an atlas of $M$ compatible with $\Gamma'$ is compatible with $\Gamma$.

A differentiable manifold of class $C^r$ is a Hassdorff space with a fixed complete atlas compatible with $\Gamma^r(R^n)$. The integer $r$ is called the dimension of the manifold. Any atlas of a Hassdorff space compatible with $\Gamma^r(R^n)$ defines a differentiable structure of class $C^r$. Since $\Gamma^r(R^n) \supset \Gamma^s(R^n)$ for $r < s$, a differentiable structure of class $C^s$ defines uniquely a differentiable structure of class $C^r$. A differentiable manifold of class $C^r$ is also called a real analytic manifold. (Throughout the book we shall mostly consider differentiable manifolds of class $C^n$.)

A differentiable manifold or, simply, manifold, we shall mean a differentiable manifold of class $C^\infty$. A complex (analytic) manifold of complex dimension $n$ is a Hassdorff space with a fixed complete atlas compatible with $\Gamma_1(R^n)$. An oriented differentiable manifold of class $C^r$ is a Hassdorff space with a fixed complete atlas compatible with $\Gamma^r(R^n)$. An oriented differentiable manifold of class $C^r$ gives rise to a differentiable structure of class $C^r$ uniquely. Not every differentiable structure of class $C^r$ is thus obtained; if it is obtained from an oriented one, it is called orientable. An orientable manifold $M$ admits exactly two orientations; if it is connectd. L. any fact to the reader, we shall only indicate how to reverse the orientation of a manifold. If a family of charts $(U_i, \varphi_i)$ defines an oriented manifold, then the family of charts $(U_i, -\varphi_i)$ defines the manifold with the reverse orientation where $-\varphi_i$ is the composition of $\varphi_i$ with the transformation $(x^1, x^2, \ldots, x^n) \to (-x^1, x^2, \ldots, x^n)$ of $\R^n$. Since $\Gamma(C) \subset \Gamma^r(R^n)$, every complex manifold is oriented as a manifold of class $C^r$.

For any structure under consideration (e.g., differentiable, structure of class $C^r$), an allowable chart is a chart which belongs to the fixed complete atlas defining the structure. From now on, by a chart we shall mean an allowable chart. Given an allowable chart $(U_i, \varphi_i)$ of an $n$-dimensional manifold $M$ of class $C^r$ the system of functions $x^1 \circ \varphi_i, \ldots, x^n \circ \varphi_i$ defined on $U_i$ is called a local coordinate system in $U_i$. We say then that $U_i$ is a coordinate neighborhood. For every point $p$ of $M$, it is possible to find a chart $(U_i, \varphi_i)$ such that $\varphi_i(p)$ is the origin of $\R^n$ and $\varphi_i$ is a homeomorphism of $U_i$ onto an open set of $\R^n$ defined by $|x|^2 < a, \ldots, |x|^2 < a$ for some positive number $a$. $U_i$ is then called a cubic neighborhood of $p$.

In a natural manner $R^n$ is an oriented manifold of class $C^r$ for any $r$; a chart consists of an element of $\Gamma^r(R^n)$ and the domain off. Similarly, $C$ is a complex manifold. Any open subset $N$ of a manifold $M$ of class $C^r$ is a manifold of class $C^r$ in a natural manner; a chart of $N$ is given by $(U_i \cap N, \varphi_i)$ where $(U_i, \varphi_i)$ is a chart of $M$ and $\varphi_i$ is the restriction of $\varphi_i$ to $U_i \cap N$. Similarly, for complex manifolds.

Given two manifolds $M$ and $M'$ of class $C^r$, a mapping $f: M \to M'$ is said to be differentiable of class $C^k$, $k \leq r$, if, for every chart $(U_i, \varphi_i)$ of $M$ and every chart $(V_j, \psi_j)$ of $M'$ such that
f(U) ⊂ V', the mapping $\psi_j \circ f \circ \psi_i^{-1}$ of $\psi_i(U_i)$ into $\psi_j(V_j)$ is differentiable of class $C^k$. If $u^1, \ldots, u^n$ is a local coordinate system in $U_i$ and $v^1, \ldots, v^m$ is a local coordinate system in $V_j$, then $f$ may be expressed by a set of differentiable functions of class $C^k$:

$$u^i = f^i(u^1, \ldots, u^n), \quad v^j = f^j(u^1, \ldots, u^n).$$

By a differentiable mapping or simply, a mapping, we shall mean a mapping of class $C^k$. A differentiable function of class $C^k$ on $M$ is a mapping of class $C^k$ of $M$ into $R$. The definition of a holomorphic (or complex analytic) mapping or function is similar.

By a differentiable curve of class $C^k$ in $M$, we mean a differentiable mapping of class $C^k$ of a closed interval $[a, b]$ of $R$ into $M$; namely, the restriction of a differentiable mapping of class $C^k$ of an open interval containing $[a, b]$ into $M$. We shall now define a tangent vector (or simply a vector) at a point $p$ of $M$. Let $\mathcal{F}(p)$ be the algebra of differentiable functions of class $C^k$ defined in a neighborhood of $p$. Let $x(t)$ be a curve of class $C^k$, $a \leq t \leq b$, such that $x(t_0) = p$. The vector tangent to the curve $x(t)$ at $p$ is a mapping $X: \mathcal{F}(p) \rightarrow R$ defined by

$$Xf = \left. \frac{d}{dt} x(t) \right|_{t=t_0}.$$

In other words, $Xf$ is the derivative of $f$ in the direction of the curve $x(t)$ at $t = t_0$. The vector $X$ satisfies the following conditions:

1. $X$ is a linear mapping of $\mathcal{F}(p)$ into $R$;
2. $X(fg) = (Xf)g(p) + f(p)(Xg)$ for $f, g \in \mathcal{F}(p)$.

The set of mappings $X$ of $\mathcal{F}(p)$ into $R$ satisfying the preceding two conditions forms a real vector space. We shall show that the set of vectors at $p$ is a vector subspace of dimension $n$, where $n$ is the dimension of $M$. Let $u^1, \ldots, u^n$ be a local coordinate system in a coordinate neighborhood $U$ of $p$. For each $i$, $(\partial/\partial u^i)_p$ is a mapping of $\mathcal{F}(p)$ into $R$, which satisfies conditions (1) and (2) above. We shall show that the set of vectors at $p$ is the vector space with basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$. Given any curve $x(t)$ with $p = x(t_0)$, let $u^i = x^i(t), i = 1, \ldots, n$, be its equations in terms of the local coordinate system $u^1, \ldots, u^n$. Then

$$(dx^i(t))/dt = \sum_j (\partial x^i/\partial u^j)(\partial u^j)/dt.$$

which proves that every vector at $p$ is a linear combination of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$. Conversely, given a linear combination $\sum \xi^i(\partial/\partial u^i)_p$, consider the curve defined by

$$u^i = u^i(p) + \xi^i(t), j = 1, \ldots, 72.$$

Then the vector tangent to this curve at $t = 0$ is $\sum \xi^i(\partial/\partial u^i)_p$. To prove the linear independence of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$, assume $\sum \xi^i(\partial/\partial u^i)_p = 0$. Then

$$0 = \sum \xi^i(\partial/\partial u^i)_p = \xi^i$$

for $i = 1, \ldots, n$.

This completes the proof of our assertion. The set of tangent vectors at $p$, denoted by $T_p(M)$ or $T_p$, is called the tangent space of $M$ at $p$. The $n$-tuple of numbers $\xi^1, \ldots, \xi^n$ will be called the components of the vector $\sum \xi^i(\partial/\partial u^i)_p$ with respect to the local coordinate system $u^1, \ldots, u^n$.

Remark. It is known that if a manifold $M$ is of class $C^k$, then $T_p(M)$ coincides with the space of $X: \mathcal{F}(p) \rightarrow R$ satisfying conditions (1) and (2) above, where $\mathcal{F}(p)$ now denotes the algebra of all $C^k$ functions around $p$. From now on we shall consider mainly manifolds of class $C^k$ and mappings of class $C^k$.

A vector field $X$ on a manifold $M$ is an assignment of a vector $X_p$ to each point $p$ of $M$. If $f$ is a differentiable function on $M$, then $Xf$ is a function on $M$ defined by $(Xf)(p) = X_p f$. A vector field $X$ is called differentiable if $Xf$ is differentiable for every differentiable function $f$. In terms of a local coordinate system $u^1, \ldots, u^n$, a vector field $X$ may be expressed by $X = \sum \xi^i(\partial/\partial u^i)$, where $\xi^i$ are functions defined in the coordinate neighborhood, called the components of $X$ with respect to $u^1, \ldots, u^n$. $X$ is differentiable if and only if its components $\xi^i$ are differentiable.

Let $\mathcal{F}(M)$ be the set of all differentiable vector fields on $M$. It is a real vector space under the natural addition and scalar multiplication. If $X$ and $Y$ are in $\mathcal{F}(M)$, define the bracket $[X, Y]$ as a mapping from the multiplication of functions on $M$ into itself by

$$[X, Y]f = X(Yf) - Y(Xf).$$

We shall show that $[X, Y]$ is a vector field. In terms of a local coordinate system $u^1, \ldots, u^n$, we write

$$X = \sum \xi^i(\partial/\partial u^i), \quad Y = \sum \eta^i(\partial/\partial u^i).$$
Then 
\[
[X, Y] f = \sum_{i=1}^{n} \left( \varepsilon^i (\partial \eta^i / \partial u^k) - \eta^i (\partial \varepsilon^i / \partial u^k) \right) \partial f / \partial u^k.
\]
This means that \([X, Y] \) is a vector field whose components \(f_j\) with respect to \(u^1, \ldots, u^n\) are given by \(f_j = \sum_{i=1}^{n} \left( \varepsilon^i (\partial \eta^i / \partial u^k) - \eta^i (\partial \varepsilon^i / \partial u^k) \right) \partial f / \partial u^k\), \(j = 1, \ldots, n\). With respect to this bracket operation, \(\mathfrak{X}(M)\) is a Lie algebra over the real number field (of infinite dimensions).

In particular, we have Jacobi's identity:
\[
[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0
\]
for \(X, Y, Z \in \mathfrak{X}(M)\).

We may also regard \(\mathfrak{X}(M)\) as a module over the algebra \(\mathfrak{g}(M)\) of differentiable functions on \(M\) as follows. If \(f\) is a function and \(X\) is a vector field on \(M\), then \(fX\) is a vector field on \(M\) defined by \((fX)_p = f(p)X_p\) for \(p \in M\). Then
\[
[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.
\]

For a point \(p\) of \(M\), the dual vector space \(T^*_p(M)\) is the tangent space at \(p\). An assignment of a covector at each point \(p\) is called a \(1\)-form. (differential form of degree 1). For each function \(f\) on \(M\), the total differential \((df)_p\), of \(f\) at \(p\) is defined by
\[
((df)_p, X) = Xf
\]
for \(X \in T_p(M)\), where \((, )\) denotes the value of the first entry on the second entry as a linear functional on \(T_p(M)\). If \(u^1, \ldots, u^n\) is a local coordinate system in a neighborhood of \(p\), then the total differentials \((du^1)_p, \ldots, (du^n)_p\) form a basis for \(T^*_p(M)\). In fact, they form the dual basis of the basis \((\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p\) for \(T_p(M)\).

In a neighborhood of \(p\), each \(1\)-form \(\omega\) can be uniquely written as
\[
\omega = \sum_j f_j du^j
\]
where \(f_j\) are functions defined in the neighborhood of \(p\) and are called the components of \(\omega\) with respect to \(u^1, \ldots, u^n\). The \(1\)-form \(\omega\) is called differentiable if \(f_j\) are differentiable (this condition is independent of the choice of a local coordinate system). We shall only consider differentiable \(1\)-forms.

A \(1\)-form \(\omega\) can be defined also as an \(m\)-linear mapping of the \(S(M)\)-module \(S(M)\) into \(\mathfrak{g}(M)\). The two definitions are related by (cf. Proposition 3.1)
\[
(\omega(X))_p = \langle \omega_p, X_p \rangle, \quad X \in \mathfrak{X}(M), \quad p \in M.
\]

Let \(\Lambda T^*_p(M)\) be the exterior algebra over \(T^*_p(M)\). An \(r\)-form \(\omega\) is an assignment of an element of degree \(r\) to each point \(p\) of \(M\). In terms of a local coordinate system \(u^1, \ldots, u^n\), \(\omega\) can be expressed uniquely as
\[
\omega = \sum_i \omega_i \cdot \cdots \cdot \omega_i du^1 \wedge \cdots \wedge du^n.
\]

The \(r\)-form \(\omega\) is called differentiable if the components \(\omega_1, \ldots, \omega_r\) are \(C^\infty\)-differentiable. By an \(r\)-form we shall mean a differentiable \(r\)-form. An \(r\)-form \(\omega\) can be defined also as a skew-symmetric \(r\)-linear mapping over \(\mathfrak{g}(M)\) of \(X(M) < X(M) < \cdots < X(M)\) (\(r\) times) into \(\mathfrak{g}(M)\). The two definitions are related as follows.

If \(\omega_1, \ldots, \omega_r\) are \(1\)-forms and \(X_1, \ldots, X_r\) are vector fields, then
\[
(\omega_1 \wedge \cdots \wedge \omega_r)(X_1, \ldots, X_r) = \frac{1}{r!} \text{det}(\omega_j(X_k))_{1 \leq j, k \leq r}
\]
formed of \(\mathfrak{g}(M)\). The two definitions are related by (cf. Proposition 3.1)
\[
(\omega(X))_p = \langle \omega_p, X_p \rangle, \quad X \in \mathfrak{X}(M), \quad p \in M.
\]

We denote by \(\mathcal{D} = \text{Diff}(M)\) the total space of (differential) \(r\)-forms on \(M\) for each \(r = 0, 1, \ldots, n\). Then \(\mathcal{D}^r(M) = \mathfrak{g}(M)\). Each \(\mathcal{D}^r(M)\) is a real vector space and can be also considered as an \(\mathfrak{g}(M)\)-module; for \(f \in \mathfrak{g}(M)\) and \(w \in \mathcal{D}^r(M)\), \(f w\) is an \(r\)-form defined by \((f w)_p = f(p)w_p, p \in M\). We set \(\mathcal{D} = \mathcal{D}^r(M)\). With respect to the exterior product, \(\mathcal{D}(M)\) forms an algebra over the real \(C^\infty\) algebra. Exterior differentiation \(d\) can be characterized as follows:

1. \(d\) is an \(R\)-linear mapping of \(\mathcal{D}(M)\) into itself such that \(d(\mathcal{D}^r) \subset \mathcal{D}^{r+1}\).
2. For a function \(f \in \mathcal{D}^0, df\) is the total differential;
3. If \(\omega \in \mathcal{D}^r\) and \(\pi \in \mathcal{D}^s\), then
\[
d(\omega \wedge \pi) = dw \wedge \pi + (-1)^r \omega \wedge d\pi;
\]
4. \(d^2 = 0\).

In terms of a local coordinate system, if \(\omega = \sum_{i} \cdots \sum_{i} f_{i1} \cdots du^1 \wedge \cdots \wedge du^n\), then \(d\omega = \sum_{i} \cdots \sum_{i} df_{i1} \cdots du^1 \wedge \cdots \wedge du^n\).

It will be later necessary to consider differential forms with values in an arbitrary vector space. Let \(V\) be an \(m\)-dimensional
real vector space. A $V$-valued $r$-form $\omega$ on $M$ is an assignment to each point $p \in M$ of a skew-symmetric $r$-linear mapping of $T_p(M) \times \cdots \times T_p(M)$ ($r$ times) into $V$. If we take a basis $e_1, \ldots, e_n$, for $V$, we can write $\omega$ uniquely as

$$\omega = \sum_{i=1}^n \omega^i e_i$$

where $\omega^i$ are usual $r$-forms on $M$. $\omega$ is differentiable, by definition, if $\omega^i$ are all differentiable. The exterior derivative $d\omega$ is defined to be

$$\sum_{i=1}^n \omega^i \wedge e_i$$

where $\omega^i \wedge e_i$ is a $(r+1)$-form.

Given a mapping of a manifold $M$ into another manifold $M'$, the derivative at $p$ off is the linear mapping $f_*(T_p(M))$ into $T_{f(p)}(M')$ defined as follows. For each $X \in T_p(M)$, choose a curve $\gamma(t)$ in $M$ such that $X$ is the vector tangent to $\gamma(t)$ at $p = \gamma(t_0)$. Then $f_*(X) = f'(p)(\gamma(t))$ is the vector tangent to the curve $f(\gamma(t))$ at $f(p) = f(\gamma(t_0))$. It follows immediately that if $g$ is a function differentiable on an open neighborhood of $(f_0)$, then $(f_*(X))g = X(g \circ f)$. When it is necessary to specify the point $p$, we write $(f_*)_p$.

For any $r$-form $\omega'$ on $M'$, we define an $r$-form $f^*\omega'$ on $M$ by

$$(f^*\omega')(X_1, \ldots, X_r) = \omega'(f_*(X_1), \ldots, f_*(X_r))$$

where $X_1, \ldots, X_r \in T_p(M)$.

The exterior differentiation $d$ commutes with $f^*$: $d(f^*\omega') = f^*(d\omega')$.

A mapping $f$ of $M$ into $M'$ is said to be of rank $r$ at $p \in M$ if the dimension of $f_*(T_p(M))$ is $r$. If the rank of $f$ at $p$ is equal to $n$ the dimension of $M$, $(f_*)_p$ is injective and $\dim M' \geq \dim M$. If the rank of $f$ at $p$ is equal to $n = \dim M$, $(f_*)_p$ is surjective and $\dim M' = \dim M$. By the implicit function theorem, we have

**Proposition 1.1.** Let $f$ be a mapping defined on a manifold $M$. Let $M$ be the set of points $p \in M$ such that $f(p) = 0$. If $(df)_p \neq 0$ at every point $p$ of $M$, then it is possible to introduce the structure of a manifold $M'$ so that $M$ is a closed submanifold of $M'$, called the hypersurface defined by the equation $f = 0$. More generally, let $M$ be the set of common zeros of functions $f_1, \ldots, f_k$. If the dimension, say $k$, of the subspace of $T^*_p(M)$ spanned by $(df_1)_p, \ldots, (df_k)_p$ is independent of $p \in M$, then $M$ is a closed submanifold of $M'$ of dimension $\dim M' - k$.

A diffeomorphism of a manifold $M$ onto another manifold $M'$ is a homeomorphism $\phi$ such that both $\phi$ and $\phi^{-1}$ are differentiable. A diffeomorphism of $M'$ onto itself is called a transformation (or, simply, a transformation) of $M'$.

A transformation $\phi$ of $M$ induces an automorphism $\phi^*$ of the algebra $\mathcal{C}(M)$ of differentiable functions on $M$. In particular, an automorphism of the algebra $\mathcal{C}(M)$ of functions on $M$: $$(\phi^* f)(p) = f(q(p)), \quad f \in \mathcal{C}(M), \quad p \in M.$$
It induces also an automorphism \( \varphi_0 \) of the Lie algebra \( T(M) \) of vector fields by
\[
(\varphi_0 X)_p = (\varphi_0)_p(X_p),
\]
where
\[
\varphi(g) = \rho, \quad X \in \mathfrak{X}(M).
\]
They are related by
\[
\varphi^*(g \ast f) = X(g \ast f) \quad \text{for} \quad X \in \mathfrak{X}(M) \quad \text{and} \quad f \in \mathcal{F}(M).
\]

Although any mapping \( \varphi \) of \( M \) into \( M' \) carries a differential form \( \omega' \) on \( M' \) into a differential form \( \varphi^*(\omega) \) on \( M \), \( \varphi \) does not send a vector field on \( M \) into a vector field on \( M' \) in general. We say that a vector field \( X \) on \( M \) is \( \varphi \)-related to a vector field \( X' \) on \( M' \) if \( (\varphi_0)_pX_p = X'_p \) for all \( p \in M \). If \( X \) and \( Y \) are \( \varphi \)-related to \( X' \) and \( Y' \) respectively, then \([X, Y] \) is \( \varphi \)-related to \([X', Y'] \).

A distribution \( S \) of dimension \( r \) on a manifold \( M \) is an assignment to each point \( p \) of \( M \) of an \( r \)-dimensional subspace \( S_p \) of \( T_p(M) \).

It is called differentiable if every point \( p \) has a neighborhood \( U \) and \( r \)-differentiable vector fields on \( U \), say, \( X_1, \ldots, X_r \), which form a basis of \( S \) at every \( q \in U \). The set \( X_1, \ldots, X_r \) is called a local basis for the distribution \( S \) in \( U \). A vector field \( X \) is said to belong to \( S \) if \( X_p \in S_p \) for all \( p \in M \). Finally, \( S \) is called involutive if \([X, Y] \) belongs to \( S \) whenever two vector fields \( X \) and \( Y \) belong to \( S \). By a distribution we shall always mean a differentiable distribution.

A connected submanifold \( N \) of \( M \) is an integral manifold of the distribution \( S \) if \( f(N) \subseteq S_p \) for all \( p \in N \), where \( f \) is the imbedding of \( N \) into \( M \). If there is no other integral manifold of \( S \) which contains \( N \), \( N \) is called a maximal integral manifold of \( S \).

The classical theorem of Frobenius can be formulated as follows.

**Proposition 1.2.** Let \( S \) be an involutive distribution on a manifold \( M \). Through every point \( p \in M \), there passes a unique maximal integral manifold \( N(p) \) of \( S \). Any integral manifold through \( p \) is an open submanifold of \( N(p) \).

For the proof, see Chevalley [1, p. 94]. We also state

**Proposition 1.3.** Let \( S \) be an involutive distribution on a manifold \( M \). Let \( W \) be a submanifold of \( M \) whose connected components are all integral manifolds of \( S \). Let \( f \) be a differentiable mapping of a manifold \( N \) into \( M \) such that \( f(N) \subseteq W \). If \( W \) satisfies the second axiom of countability, then \( f \) is differentiable as a mapping of \( N \) into \( W \).

For the proof, see Chevalley [1, p. 95, Proposition 1]. Replace "analyticity there" by differentiability throughout and observe that \( W \) need not be connected since the differentiability of \( f \) is a local matter.

We now define the product of two manifolds \( M \) and \( N \) of dimension \( m \) and \( n \) respectively. If \( M \) is defined by an atlas \( A = \{(U_i, \varphi_i)\} \) and \( N \) is defined by an atlas \( B = \{(V_j, \psi_j)\} \), then the natural differentiable structure on the topologicalspace \( M \times N \) is defined by an atlas \( \{(U_i \times V_j, \varphi_i 	imes \psi_j)\} \), where \( \varphi_i \times \psi_j : U_i \times V_j \rightarrow \mathbb{R}^{m+n} \) is defined in a natural manner. Note, that this atlas is not complete even if \( A \) and \( B \) are complete. For every point \( (p, q) \) of \( M \times N \), the tangent space \( T_{(p,q)}(M \times N) \) can be identified with the direct sum \( T_p(M) \oplus T_q(N) \) in a natural manner. Namely, for \( X \in T_p(M) \) and \( Y \in T_q(N) \), choose curves \( x(t) \) and \( y(t) \) such that \( X \) is tangent to \( x(t) \) at \( p = x(0) \) and \( Y \) is tangent to \( y(t) \) at \( q = y(0) \). Then \( (X, Y) \in T_{(p,q)}(M \times N) \) is identified with the vector \( Z \in T_{(p,q)}(M \times N) \) which is tangent to the curve \( (x(t), y(t)) \) at \( (p, q) = (x(0), y(0)) \). Let \( X \in T_p(M) \) and \( Y \in T_q(N) \) be the vector tangent to the curve \( (x(t), y(t)) \) in \( M \times N \) at \( (p, q) \). Similarly, let \( Y \in T_{(p,q)}(M \times N) \) be the vector tangent to the curve \( (p, y(t)) \) in \( M \times N \) at \( (p, q) \). In other words, \( X \) is the image of \( X \) by the mapping \( M \rightarrow M \times N \) which sends \( p' \in M \) into \( (p', q) \) and \( f \) is the image of \( Y \) by the mapping \( N \rightarrow M \times N \) which sends \( q' \in N \) into \( (p, q') \). Then \( Z = X + Y \), because, for any function \( f \) on \( M \times N \), \( Zf = (dx(x(t), y(t))/dt)_{t=0} = Xf + Yf \), is, by the chain rule, equal to
\[
(dx(x(t), y(t))/dt)_{t=0} + (dy(x(t), y(t))/dt)_{t=0} = Xf + Yf.
\]

More generally,

**Proposition 1.4 (Leibniz's formula).** Let \( \varphi \) be a mapping of the product manifold \( M \times N \) into another manifold \( V \). The differential \( \varphi_0 \) at \( (p, q) \in M \times N \) can be expressed as follows. If \( Z \in T_{(p,q)}(M \times N) \) corresponds to \( (X, Y) \in T_p(M) \oplus T_q(N) \), then
\[
\varphi_0(Z) = \varphi_{10}(X) + \varphi_{20}(Y).
\]
where \( q_1: M \to V \) and \( q_2: N \to V \) are defined by
\[
q_1(p') = q(p', q) \quad \text{for} \quad p' \in M \quad \text{and} \quad q_2(q') = q(p, q') \quad \text{for} \quad q' \in N.
\]

Proof. From the definitions of \( X \), \( Y \), \( q_1 \), and \( q_2 \), it follows that \( q_1(X) = q_1(X) \) and \( q_1(Y) = g_0(Y) \). Hence, \( q_1(Z) = q_1(X) + q_2(Y) \).

Note that if \( V = M \times N \) and \( q \) is the identity transformation, then the preceding proposition reduces to the formula \( Z = X \).

Let \( X \) be a vector field on a manifold \( M \). A curve \( x(t) \) in \( M \) is called an integral curve of \( X \) if, for every parameter value \( t_0 \), the vector \( X(t_0) \) is tangent to the curve \( k(t) \) at \( k(t_0) \). For any point \( p_0 \) of \( M \), there is a unique integral curve \( x(t) \) of \( X \), defined for \( |t| < \varepsilon \) for some \( \varepsilon > 0 \), such that \( p_0 = x(0) \). In fact, let \( u^1, \ldots, u^n \) be a local coordinate system in a neighborhood \( U \) of \( p_0 \), and let \( X = \sum \xi^i \partial / \partial u^i \). Then an integral curve of \( X \) is a solution of the following system of ordinary differential equations:
\[
du^i / dt = \xi^i(u^1(t), \ldots, u^n(t)), \quad j = 1, \ldots, n.
\]

Our assertion follows from the fundamental theorem for systems of ordinary differential equations (see Appendix I).

A 1-parameter group of (differentiable) transformations of \( M \) is a mapping of \( \mathbb{R} \times M \) into \( M \), \( (t, p) \mapsto \phi_t(p) \in M \), which satisfies the following conditions:

(1) For each \( t \in \mathbb{R} \), \( \phi_t \) is a transformation of \( M \).
(2) For all \( t, s \in \mathbb{R} \) and \( p \in M \), \( \phi_{t+s}(p) = \phi_t(\phi_s(p)) \).
(3) Each 1-parameter group of transformations \( \phi_t \) induces a vector field \( X \) as follows. For every point \( p \in M \), \( X_p \) is the vector tangent to the curve \( x(t) = \phi_t(p) \), called the orbit of \( p \), at \( p = \phi_0(p) \). The orbit \( \phi_t(p) \) is an integral curve of \( X \) starting at \( p \). A local 1-parameter group of local transformations can be defined in the same way, except that \( \phi_t(p) \) is defined only for \( t \) in a neighborhood of 0 and \( p \) in an open set of \( M \). More precisely, let \( I \) be an open interval \((-\varepsilon, \varepsilon)\) and \( U \) an open set of \( M \). A local 1-parameter group of local transformations defined on \( I \times U \) is a mapping of \( I \times U \) into \( M \) which satisfies the following conditions:

(1') For each \( t \in I \), \( \phi_t: p \mapsto \phi_t(p) \) is a diffeomorphism of \( U \) onto the open set \( \phi_t(U) \) of \( M \).

As in the case of a local 1-parameter group of transformations, \( \phi_t \) induces a vector field \( X \) defined on \( U \). We now prove the converse.

**Proposition 1.5.** Let \( X \) be a vector field on a manifold \( M \). For each point \( p_0 \) of \( M \), there exist a neighborhood \( U \) of \( p_0 \), a positive number \( \varepsilon \) and a local 1-parameter group of local transformations \( \phi_t: U \to M \), \( t \in I \), which induces the given \( X \).

We shall say that \( X \) generates a local 1-parameter group of local transformations \( \phi_t \) in a neighborhood of \( p_0 \). If there exists a (global) 1-parameter group of transformations of \( M \) which induces \( X \), then we say that \( X \) is complete. If \( \phi_t(p) \) is defined on \( I \times M \) for some \( \varepsilon \), then \( X \) is complete.

Proof. Let \( u^1, \ldots, u^n \) be a local coordinate system in a neighborhood \( W \) of \( p_0 \) such that \( u^1(p_0) = \ldots = u^n(p_0) = 0 \). Let \( X = \sum \xi^i(u^1, \ldots, u^n)(\partial / \partial u^i) \) in \( W \). Consider the following system of ordinary linear differential equations:
\[
df^i / dt = \xi^i(f^1(t), \ldots, f^n(t)), \quad i = 1, \ldots, n
\]
with unknown functions \( f^1, \ldots, f^n(t) \). By the fundamental theorem for systems of ordinary differential equations (see Appendix I), there exists a unique set of functions \( f^1(t; u) \) \( \ldots, f^n(t; u) \), defined for \( u = (u^1, \ldots, u^n) \) with \( |u^i| < \delta \), such that \( \phi_t^u(p) = \phi_t^u(p_0) = 0 \) for \( |t| < \varepsilon \), which form a solution of the differential equation for each fixed \( u \) and satisfy the initial conditions:
\[
f^i(0; u) = u^i.
\]
Set \( \phi^u_t = (f^1(t; u), \ldots, f^n(t; u)) \) for \( |t| < \varepsilon \), and \( u \in U = \{u; |u^i| < \delta \} \). If \( |t|, |s| \) and \( |t| + |s| \) are all less than \( \varepsilon \) and both \( u \) and \( \psi^u_t(u) \) are in \( U \), then the functions \( \psi^u_{t+s}(u) \) are a solution of the differential equation for the initial conditions \( \psi^u_0(u) = u \). By the uniqueness of the solution, we have \( g^u(t) = \psi^u_t(u) \). This proves that \( \phi^1_t(u) = \psi^1_t(u) \), since \( \phi^1_0 \) is the identity transformation of \( U \), there exist \( \delta > 0 \) and \( \varepsilon > 0 \) such that, for \( U = \{u; |u^i| < \delta \} \), \( \psi^u(U) \subset U \) if
For each point \( p \) on \( M \), \( \varphi_t(p) \) is a local 1-parameter group of local transformations defined on \( I_t \times U \). From the construction of \( \varphi_t \), it is obvious that \( \varphi_t \) induces the given vector field \( X \) in \( U \). QED.

Remark. In the course of the preceding proof, we showed also that if two local 1-parameter groups of local transformations \( \varphi_t \) and \( \psi_t \) defined on \( I_t \times U \) induce the same vector field on \( U \), they coincide on \( U \).

**Proposition 1.6.** On a compact manifold \( M \), every vector field \( X \) is complete.

Proof. For each point \( p \in M \), let \( U(p) \) be a neighborhood of \( p \) and \( \epsilon(p) \) a positive number such that the vector field \( X \) generates a local 1-parameter group of local transformations \( \varphi_t \) on \( I_{\epsilon(p)} \times U(p) \). Since \( M \) is compact, the open covering \( \{ U(p) : p \in M \} \) has a finite subcovering \( \{ U(p_i) : i = 1, \ldots, k \} \). Let \( \epsilon = \min \{ \epsilon(p_1), \ldots, \epsilon(p_k) \} \). It is clear that \( \varphi_t(p) \) is defined on \( I_t \times M \) and, hence, on \( R \times M \). QED.

In what follows, we shall not give explicitly the domain of definition for a given vector field \( X \) and the corresponding local 1-parameter group of local transformations \( \varphi_t \). Each formula is valid whenever it makes sense, and it is easy to specify, if necessary, the domain of definition for vector fields or transformations involved.

**Proposition 1.7.** Let \( \varphi \) be a transformation of \( M \). If a vector field \( X \) generates a local 1-parameter group of local transformations \( \varphi_t \), then the vector field \( \varphi_t \circ X \) generates \( \varphi_t \circ X \).

Proof. It is clear that \( \varphi_t \circ X \) is a local 1-parameter group of local transformations. To show that it induces the vector field \( \varphi_t \circ X \), let \( p \) be an arbitrary point of \( M \) and \( q = \varphi_t^{-1}(p) \). Since \( \varphi_t \) induces \( X \), the vector \( X_q = T_t(M) \) is tangent to the curve \( x(t) = \varphi(t) \) at \( t = 0 \). It follows that the vector \( \varphi_t \circ X_q = X_{\varphi_t(q)} \) is tangent to the curve \( y(t) = \varphi_t \circ X_{\varphi_t(q)} = \varphi_t \circ X_{\varphi_t(q)} \).

**Corollary 1.8.** A vector field \( X \) is invariant by \( \varphi \), that is, \( \varphi_t \circ X = X \), if and only if \( \varphi \) commutes with \( \varphi_t \).

We now give a geometric interpretation of the bracket \([X, Y]\) of two vector fields.

**Proposition 1.9.** Let \( X \) and \( Y \) be vector fields on \( M \). If \( X \) generates a local 1-parameter group of local transformations \( \varphi_t \), then

\[
[X, Y] = \lim_{t \to 0} \frac{1}{t} [Y - (\varphi_t)_* Y].
\]

More precisely,

\[
[X, Y]_p = \lim_{t \to 0} \frac{1}{t} [Y_p - ((\varphi_t)_* Y)_p], \quad p \in M.
\]

The limit on the right-hand side is taken with respect to the natural topology of the tangent vector space \( T_p(M) \). We first prove two lemmas.

**Lemma 1.** If \( f(t, p) \) is a function on \( I \times M \), where \( I \) is an open interval \((-e, e)\), such that \( f(0, p) = 0 \) for all \( p \in M \), then there exists a function \( g(t, p) \) on \( I \times M \) such that \( f(t, p) = f(t, g(t, p), p) \). Moreover, \( g(0, p) = f(0, p) \), where \( f' = \partial f/\partial t \), for \( p \in M \).

Proof. It is sufficient to define \( g \) by

\[
g(t, p) = \int_0^t f'(s, p) \, ds.
\]

**Lemma 2.** Let \( X \) generate \( \varphi_t \). For any function \( f \) on \( M \), there exists a function \( g(t, p) = g(f(t, p)) \) such that \( f \circ \varphi_t = f + t \cdot g_t \) and \( g_0 = Xf \) on \( M \).

The function \( g(t, p) \) is defined, for each fixed \( p \in M \), in \( |t| < e \) for some \( e \).

Proof. Consider \( f(t, p) = f(\varphi_t(p)) - f(p) \) and apply Lemma 1. Then \( f = \varphi_t - f \cdot g_t \). We have

\[
\lim_{t \to 0} \frac{1}{t} [f(\varphi_t(p)) - f(p)] = \lim_{t \to 0} \frac{1}{t} f(t, p) = \lim_{t \to 0} g_t(p) = g_0(p).
\]

QED.

Proof of Proposition 1.9. Given a function \( f \) on \( M \), take a function \( g_t \) such that \( f \circ \varphi_t = f + t \cdot g_t \) and \( g_0 = Xf \) (Lemma 2).

Set \( p(t) = \varphi_t^{-1}(p) \). Then

\[
((\varphi_t)_* Y)_p = (Y(\varphi_t))_{p(t)} = (Yf)_{p(t)} + t \cdot (Yg_t)_{p(t)}
\]
and
\[ \lim_{t \to 0} \frac{1}{t} [(Y - (\varphi_t) \cdot Y)]_t = \lim_{t \to 0} \frac{1}{t} [(Yf)_t - (Yf)_0] = \lim_{t \to 0} (Yg)_t, \]
proving our assertion.

**Corollary 1.10.** With the same notation as in Proposition 1.9, we have more generally
\[ (\varphi_s)_* [X, Y] = \lim_{t \to 0} \frac{1}{t} [(\varphi_s)_* Y - (\varphi_{s+t})_* Y] \]
for any value of \( s \).

**Proof.** For a fixed value of \( s \), consider the vector field \( (\varphi_s)_* Y \) and apply Proposition 1.9. Then we have
\[ [X, (\varphi_s)_* Y] = \lim_{t \to 0} \frac{1}{t} [(\varphi_s)_* Y - (\varphi_{s+t})_* Y], \]
which since \( \varphi_s \circ \varphi_t = \varphi_{s+t} \). On the other hand, \( (\varphi_s)_* X = X \) by Corollary 1.8. Since \( \varphi_s \) preserves the bracket, we obtain
\[ (\varphi_s)_*[X, Y] = [X, (\varphi_s)_* Y]. \]
QED.

**Remark.** The conclusion of Corollary 1.10 can be written as
\[ d((\varphi_1)_* Y)/dt\big|_{t=0} = - (\varphi_1)_*[X, Y]. \]

**Corollary 1.11.** Suppose \( X \) and \( Y \) generate local 1-parameter groups \( \varphi_s \) and \( \psi_s \), respectively. Then \( \varphi_t \circ \psi_s = \psi_s \circ \varphi_t \) for every \( s \) and \( t \) if and only if \( [X, Y] = 0 \).

**Proof.** If \( \varphi_t \circ \psi_s = \psi_s \circ \varphi_t \) for every \( s \) and \( t \), then \( d((\varphi_1)_* Y)/dt\big|_{t=0} = 0 \), by Proposition 1.9. By Corollary 1.8. Conversely, if \( [X, Y] = 0 \), then \( d((\varphi_1)_* Y)/dt\big|_{t=0} = 0 \), by Corollary 1.10 (see Remark, above). Therefore, \( (\varphi_t)_* Y \) is constant at each point \( p \) so that \( Y \) is invariant by \( \varphi_t \) for every \( t \). By Corollary 1.8, \( \varphi_s \) commutes with every \( \varphi_t \).

QED.

## 2. Tensor algebras

We fix a ground field \( F \) which will be the real number field \( \mathbb{R} \) or the complex number field \( \mathbb{C} \) in our applications. All vector spaces we consider are finite dimensional over \( F \) unless otherwise stated. We define the tensor product \( U \otimes V \) of two vector spaces \( U \) and \( V \) as follows. Let \( M(U, V) \) be the vector space which has the set \( U \times V \) as a basis, i.e., the free vector space generated by the pairs \((u, v)\) where \( u \in U \) and \( v \in V \). Let \( N \) be the vector subspace of \( M(U, V) \) spanned by elements of the form
\[ (u + u', v) = (u, v) + (u', v), \]
\[ (u + u', v') - (u, v') = (u, v + v'), \]
\[ (ru, v) - r(u, v), \]
where \( u, u' \in U, v, v' \in V \) and \( r \in F \). We set \( U \otimes V = M(U, V)/\langle N \rangle \).

**Remark.** Above, \( \otimes \) denotes the tensor product, \( \langle N \rangle \) the vector subspace of \( M(U, V) \) spanned by elements of the form
\[ (u + u', v) - (u, v) - (u', v), \]
\[ (u, v + v') - (u, v) - (u, v'), \]
\[ (ru, v) - r(u, v), \]
where \( u, u' \in U, v, v' \in V \) and \( r \in F \). We set \( U \otimes V = M(U, V)/\langle N \rangle \).

**For every** pair \((u, v)\) considered as an element of \( M(U, V) \), its image by the natural projection \( M(U, V) \to U \otimes V \) will be denoted by \( u \otimes v \). Define the canonical bilinear mapping \( \varphi_u \) of \( U \otimes V \) into \( U \otimes V \) by
\[ \varphi_u(u \otimes v) = u \otimes v \quad \text{for} \quad (u, v) \in U \times V. \]

Let \( W \) be a vector space and \( \varphi : U \times V \to W \) a bilinear mapping. We say that the couple \((W, \varphi)\) has the universal factorization property for \( U \times V \) if for every vector space \( S \) and every bilinear mapping \( f : U \times V \to S \) there exists a unique linear mapping \( g : W \to S \) such that \( f = g \circ \varphi \).

**Proposition 2.1.** The couple \((U \otimes V, \varphi)\) has the universal factorization property for \( U \times V \). If a couple \((W, \varphi)\) has the universal factorization property for \( U \times V \), then \((U \otimes V, \varphi)\) and \((W, \varphi)\) are isomorphic in the sense that there exists a linear isomorphism \( \sigma : U \otimes V \to W \) such that \( \varphi = \sigma \circ \varphi \).

**Proof.** Let \( S \) be any vector space and \( f : U \times V \to S \) any bilinear mapping. Since \( U \times V \) is a basis for \( M(U, V) \), we can extend \( f \) to a unique linear mapping \( f' : M(U, V) \to S \). Since \( f \) is bilinear, \( f' \) vanishes on \( N \). Therefore, \( f' \) induces a linear mapping \( g : U \otimes V \to S \). Obviously, \( f' = g \circ \varphi \). The uniqueness of such a mapping \( g \) follows from the fact that \( \varphi(U \times V) \) spans \( U \otimes V \). Let \((W, \varphi)\) be a couple having the universal factorization property for \( U \times V \). By the universal factorization property of \( (U \otimes V, \varphi) \)
(resp. of \( W, \psi \)), there exists a unique linear mapping \( \sigma: U \otimes V \to W \) (resp. \( \tau: W \to U \otimes V \)) such that \( \psi = \sigma \circ \varphi \) (resp. \( \varphi = \tau \circ \sigma \)). Hence, \( \tau \circ \sigma \circ \varphi = \tau \circ \sigma = \varphi = \tau \circ \sigma \circ \varphi \).

Using the uniqueness of \( g \) in the definition of the universal factorization property, we conclude that \( \tau \circ \sigma \circ \varphi = \tau \circ \sigma = \varphi = \tau \circ \sigma \circ \varphi \). QED.

**Proposition 2.2.** There is a unique isomorphism of \( U \otimes V \) onto \( V \otimes V \) which sends \( u \otimes v \) into \( v \otimes u \) for all \( u \in U \) and \( v \in V \).

**Proof.** Let \( f: U \times V \to V \otimes U \) be the bilinear mapping defined by \( f(u, v) = v \otimes u \). By Proposition 1.1, there is a unique linear mapping \( g: U \otimes V \to V \otimes U \) such that \( g(u \otimes v) = v \otimes u \). Similarly, there is a unique linear mapping \( g': V \otimes U \to U \otimes V \) such that \( g'(v \otimes u) = u \otimes v \). Evidently, \( g \circ g' = g' \circ g = id \). Hence, \( g \) is the desired isomorphism. QED.

The proofs of the following two propositions are similar and hence omitted.

**Proposition 2.3.** If we regard the ground field \( F \) as a 1-dimensional vector space over \( F \), there is a unique isomorphism of \( F \otimes U \) onto \( U \) which sends \( 1 \otimes u \) into \( u \) for all \( u \in U \) and \( v \in U \).

**Proposition 2.4.** There is a unique isomorphism of \( (U \otimes V) \otimes W \) onto \( U \otimes (V \otimes W) \) which sends \( (u \otimes v) \otimes w \) into \( u \otimes (v \otimes w) \) for all \( u \in U \), \( v \in V \), and \( w \in W \).

Therefore, it is meaningful to write \( U \otimes V \otimes W \). Given vector spaces \( U_1, \ldots, U_k \), the tensor product \( U_1 \otimes \cdots \otimes U_k \) can be defined inductively. Let \( \psi: U_1 \times \cdots \times U_k \to U_1 \otimes \cdots \otimes U_k \) be the multilinear mapping which sends \( (u_1, \ldots, u_k) \) into \( u_1 \otimes \cdots \otimes u_k \). Then, as in Proposition 2.1, the couple \( (U_1 \otimes \cdots \otimes U_k) \) can be characterized by the universal factorization property for \( U_1 \times \cdots \times U_k \) if \( U_2 \) can be also generalized. For any permutation \( \pi \) of \( (1, \ldots, k) \), there is a unique isomorphism of \( U_{\pi(1)} \otimes \cdots \otimes U_{\pi(k)} \) onto \( U_{\pi(1)} \otimes \cdots \otimes U_{\pi(k)} \) which sends \( u_1 \otimes \cdots \otimes u_k \) into \( u_{\pi(1)} \otimes \cdots \otimes u_{\pi(k)} \).

**Proposition 2.4.1.** Given linear mappings \( f_j: U_i \to V \), \( j = 1, 2 \), there is a unique linear mapping \( f: U_1 \otimes U_2 \to V_1 \otimes V_2 \) such that \( f(u_1 \otimes u_2) = f_1(u_1) \otimes f_2(u_2) \) for all \( u_1 \in U_1 \) and \( u_2 \in U_2 \).

**Proof.** Consider the bilinear mapping \( U_1 \otimes U_2 \to V_1 \otimes V_2 \) which sends \( (u_1, u_2) \) into \( f_1(u_1) \otimes f_2(u_2) \) and apply Proposition 2.1. QED.

The generalization of Proposition 2.4.1 to the case with more than two mappings is obvious. The mapping \( f \) just given will be denoted by \( f_1 \otimes f_2 \).

**Proposition 2.5.** If \( U_1 + U_2 \) denotes the direct sum of \( U_1 \) and \( U_2 \), then

\[
(U_1 + U_2) \otimes V = U_1 \otimes V + U_2 \otimes V.
\]

Similarly,

\[
U \otimes (V_1 + V_2) = U \otimes V_1 + U \otimes V_2.
\]

**Proof.** Let \( i_1: U_1 \to U_1 + U_2 \) and \( i_2: U_2 \to U_1 + U_2 \) be the injections. Let \( \rho_1: U_1 + U_2 \to U_1 \) and \( \rho_2: U_1 + U_2 \to U_2 \) be the projections. Then \( \rho_1 \circ i_1 \) and \( \rho_2 \circ i_2 \) are the identity transformations of \( U_1 \) and \( U_2 \) respectively. Both \( \rho_1 \circ i_1 \) and \( \rho_2 \circ i_2 \) are the zero mappings. By Proposition 2.4.1, \( i_1 \) and the identity transformation of \( V \) induce a linear mapping \( i_1 : U_1 \otimes V \to (U_1 + U_2) \otimes V \).

Similarly, \( i_2 \), \( \rho_1 \), and \( \rho_2 \) are defined. It follows that \( \rho_1 \circ i_1 \) and \( \rho_2 \circ i_2 \) are the identity transformations of \( U_1 \otimes V \) and \( U_2 \otimes V \) respectively and \( \rho_1 \circ i_1 \) and \( \rho_2 \circ i_2 \) are the zero mappings. This proves the first isomorphism. The proof for the second is similar.

By the induction, we obtain

\[
(U_1 + \cdots + U_k) \otimes V = U_1 \otimes V + \cdots + U_k \otimes V.
\]

**Proposition 2.6.** If \( u_1, \ldots, u_n \) is a basis for \( U \) and \( v_1, \ldots, v_n \) is a basis for \( V \), then \( \{ u_i \otimes v_j \mid i = 1, \ldots, m; j = 1, \ldots, n \} \) is a basis for \( U \otimes V \). In particular, \( \dim U \otimes V = \dim U \cdot \dim V \).

**Proof.** Let \( U \) be the 1-dimensional subspace of \( U \) spanned by \( u \) and \( V \) the 1-dimensional subspace of \( V \) spanned by \( v \). By Proposition 2.5,

\[
U \otimes V = \Sigma_{ij} U_i \otimes V_j.
\]

By Proposition 2.3, each \( U_i \otimes V_j \) is a 1-dimensional vector space spanned by \( u_i \otimes v_j \). QED.
Proposition 2.7. Let \( L(U^*, V) \) be the space of linear mappings of \( U^* \) into \( V \). Then there is a unique isomorphism \( g \) of \( U \otimes V \) onto \( L(U^*, V) \) such that 
\[
(g(u \otimes v))u^* = (u, u^*)v
\]
for all \( u \in U, v \in V \) and \( u^* \in U^* \).

Proof. Consider the bilinear mapping \( f: U \times V \to L(U^*, V) \) defined by \( (f(u, v))u^* = (u, u^*)v \) and apply Proposition 2.1. Then there is a unique linear mapping \( g: U \otimes V \to L(U^*, V) \) such that \( (g(u \otimes v))u^* = (u, u^*)v \). To prove that \( g \) is an isomorphism, let \( u_1, \ldots, u_m \) be a basis for \( U \), \( u^*_1, \ldots, u^*_n \) the dual basis for \( U^* \) and \( v_1, \ldots, v_n \), a basis for \( V \). We shall show that 
\[
\{g(u_i \otimes v_j)\} ; i = 1, \ldots, m; j = 1, \ldots, n
\]
is linearly independent. If \( \Sigma a_{ij}g(u_i \otimes v_j) = 0 \) where \( a_{ij} \in F \), then 
\[
0 = (\Sigma a_{ij}(u_i \otimes v_j))u^* = \Sigma a_{ij}u_i^*v_j
\]
and, hence, all \( a_{ij} \) vanish. Since \( \dim U \otimes V = \dim L(U^*, V) \), \( g \) is an isomorphism of \( U \otimes V \) onto \( L(U^*, V) \). \( \square \)

Proposition 2.8. Given two vector spaces \( U \) and \( V \), there is a unique isomorphism \( g \) of \( U^* \otimes V^* \) onto \( (U \otimes V)^* \) such that 
\[
(g(u^* \otimes v^*))((u \otimes v)) = (u, u^*)v
\]
for all \( u \in U, u^* \in U^*, v \in V, v^* \in V^* \).

Proof. Apply Proposition 2.1 to the bilinear mapping \( f: U^* \times V^* \to (U \otimes V)^* \) defined by \( (f(u^*, v^*))((u \otimes v)) = (u, u^*)v \). To prove that \( g \) is an isomorphism, take bases for \( U, V, U^* \), and \( V^* \) and proceed as in the proof of Proposition 2.7. \( \square \)

We now define various tensor spaces over a fixed vector space \( V \). For a positive integer \( r \), we shall call \( T^r = V \otimes \cdots \otimes V \) (\( r \) times tensor product) the contravariant tensor space of degree \( r \). An element of \( T^r \) will be called a contravariant tensor of degree \( r \). If \( r = 1 \), \( T^1 \) is nothing, but \( V \). By convention, we agree that \( T^0 \) is the ground field \( F \) itself. Similarly, \( T_s = V^* \otimes \cdots \otimes V^* \) (\( s \) times tensor product) is called the covariant tensor space of degree \( s \) and its elements covariant tensors of degree \( s \). Then \( T_1 = V^* \) and, by convention, \( T_0 = F \).

We shall give the expressions for these tensors with respect to a basis of \( V \). Let \( e_1, \ldots, e_n \) be a basis for \( V \) and \( e^1, \ldots, e^n \) the dual basis for \( V^* \). By Proposition 2.6, \( \{e_i \otimes \cdots \otimes e_i; 1 \leq i_1, \ldots, i_r \leq n \} \) is a basis for \( T^r \). Every contravariant tensor \( K \) of degree \( r \) can be expressed uniquely as a linear combination
\[
K = \Sigma_{i_1, \ldots, i_r} K^{i_1, \ldots, i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}
\]
where \( K^{i_1, \ldots, i_r} \) are the components of \( K \) with respect to the basis \( e_{i_1}, \ldots, e_{i_r} \) of \( V \). Similarly, every covariant tensor \( L \) of degree \( s \) can be expressed uniquely as a linear combination
\[
L = \Sigma_{j_1, \ldots, j_s} L^{j_1, \ldots, j_s} e_{j_1} \otimes \cdots \otimes e_{j_s}
\]
where \( L^{j_1, \ldots, j_s} \) are the components of \( L \).

For a change of basis from \( U \), the components of tensors are subject to the following transformations. Let \( e_1, \ldots, e_n \) and \( \tilde{e}_1, \ldots, \tilde{e}_n \) be two bases of \( V \) related by a linear transformation
\[
\tilde{e}_i = \Sigma_j A_{ij} e_j, \quad i = 1, \ldots, n,
\]
The corresponding change of the dual bases in \( V^* \) is given by
\[
\tilde{e}^i = \Sigma_j B_{ij} e^j, \quad i = 1, \ldots, n,
\]
where \( B = (B_{ij}) \) is the inverse matrix of the matrix \( A = (A_{ij}) \) so that
\[
\Sigma_j A_{ij} B_{jk} = \delta_{ik}.
\]
If \( K \) is a contravariant tensor of degree \( r \), its components \( K^{i_1, \ldots, i_r} \), and \( \tilde{K}^{i_1, \ldots, i_r} \) with respect to \( \{e_i\} \) and \( \{\tilde{e}_i\} \) respectively are related by
\[
\tilde{K}^{i_1, \ldots, i_r} = \Sigma_{j_1, \ldots, j_r} A_{j_1 i_1} \cdots A_{j_r i_r} K^{j_1, \ldots, j_r}.
\]
Similarly, the components of a covariant tensor \( L \) of degree \( s \) are related by
\[
L^{j_1, \ldots, j_s} = \Sigma_{i_1, \ldots, i_s} B_{i_1 j_1} \cdots B_{i_s j_s} L^{i_1, \ldots, i_s}.
\]

The verification of these formulas is left to the reader.

We define the (mixed) tensor space of type \((r, s)\), or tensor space of contravariant degree \( r \) and covariant degree \( s \), as the tensor product \( T^r \otimes T_s = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^* \) (\( V \) \( r \) times \( V^* \) \( s \) times). In particular, \( T^0 = T_0 \), \( T^1 = T_1 \), and \( T_0 = F \). An element of \( T^r \) is called a tensor of type \((r, s)\), or tensor of contravariant degree \( r \) and covariant degree \( s \). In terms of a basis \( e_1, \ldots, e_n \) of \( V \),
the dual basis $e^1, \ldots, e^n$ of $V^\ast$, every tensor $K$ of type $(r, s)$ can be expressed uniquely as

$$K = \sum_{i_1, \ldots, i_r, j_1, \ldots, j_s} K_{i_1 \ldots i_r}^{j_1 \ldots j_s} e^{i_1} \otimes \cdots \otimes e^{i_r} \otimes e_{j_1} \otimes \cdots \otimes e_{j_s},$$

where $K_{i_1 \ldots i_r}^{j_1 \ldots j_s}$ are called the components of $K$ with respect to the basis $e_1, \ldots, e_n$. For a change of basis $e_i = \Sigma_j A_{ij} e_j$, we have the following transformations of components:

(2.1) \[ K_{i_1 \ldots i_r}^{j_1 \ldots j_s} = \Sigma A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_r j_r} B_{j_1 k_1} B_{j_2 k_2} \cdots B_{j_s k_s} k_{r+1} \cdots k_{s+r} \]

Set $T = \Sigma K^i_0 \otimes \cdots \otimes K_s^{j_1} \cdots j_s$, so that an element of $T$ is of the form $\Sigma \sum_i K^i_0 \otimes \cdots \otimes K_s^{j_1} \cdots j_s$, where $K^i_0 \cdot K^i_s$ are zero except for a finite number of them. We shall now make $T$ into an associative algebra over $F$ by defining the product of two tensors $K \in T^i_r$ and $L \in T^j_s$ as follows. From the universal factorization property of the tensor product, it follows that there exists a unique bilinear mapping of $T^i_r \times T^j_s$ into $T^i+j_{r+s}$ which sends $(v^1_1 \otimes \cdots \otimes v^i_r \otimes v^j_1 \otimes \cdots \otimes v^j_s)$ into $v^i_1 \otimes \cdots \otimes v^i_r \otimes v^j_1 \otimes \cdots \otimes v^j_s$. The image of $(K, L) \in T^i_r \times T^j_s$ by this bilinear mapping will be denoted by $K \otimes L$. In terms of components, if $K$ is given by $K^i_{i_1} \cdots i_r$ and $L$ is given by $L^j_{j_1} \cdots j_s$, then

$$(K \otimes L)^{i_1 \cdots i_r j_1 \cdots j_s} = K^i_{i_1} \cdots i_r L^j_{j_1} \cdots j_s.$$  

We now define the notion of contraction. Let $r, s \geq 1$. To each ordered pair of integers $(i, j)$ such that $1 \leq i \leq r$ and $1 \leq j \leq s$, we associate a linear mapping, called the contraction and denoted by $C$, of $T^i_r$ into $T^{i-1}_{r-1}$ which maps $v^i_1 \otimes \cdots \otimes v^i_r \otimes v^j_1 \otimes \cdots \otimes v^j_s$ into

$$(v^i_1, v^j_1) v^i_2 \cdots \otimes v^i_{i-1} \otimes v^i_{i+1} \cdots \otimes v^i_r \otimes v^j_2 \cdots \otimes v^j_{j-1} \otimes v^j_{j+1} \cdots \otimes v^j_s,$$

where $v^i_1, \ldots, v^i_r \in V$ and $v^j_1, \ldots, v^j_s \in V^\ast$. The uniqueness and the existence of $C$ follow from the universal factorization property of the tensor product. In terms of components, the contraction $C$ maps a tensor $K \in T^{i}_{r-1}$ whose components, are given by

$$(CK)^{i_1 \cdots i_{r-1}}_{j_1 \cdots j_s} = \Sigma_k K^i_{i_1 \cdots i_k j_1} \cdots j_s,$$

where the superscript $k$ appears at the $i$-th position and the subscript $k$ appears at the $j$-th position.

We shall now interpret tensors as multilinear mappings.

**Proposition 2.9.** $T^r$ is isomorphic, in a natural way, to the vector space of all $r$-linear mappings of $V \times \cdots \times V$ into $F$.

**Proposition 2.10.** $T^r$ is isomorphic, in a natural way, to the vector space of all $r$-linear mappings of $V^\ast \times \cdots \times V^\ast$ into $F$.

Proof. We prove only Proposition 2.9. By generalizing Proposition 2.8, we see that $T^r = V^\ast \otimes \cdots \otimes V^\ast$ is the dual vector space of $T^r = V \otimes \cdots \otimes V$. On the other hand, it follows from the universal factorization property of the tensor product that the space of linear mappings of $T^r = V \otimes \cdots \otimes V$ into $F$ is isomorphic to the space of $r$-linear mappings of $V \times \cdots x V$ into $F$.

QED.

Following the interpretation in Proposition 2.9, we consider a tensor $K \in T^r$ as an $r$-linear mapping $V \times \cdots \times V \rightarrow F$ and write $K(v_1, \ldots, v_r) \in F$ for $v_1, \ldots, v_r \in V$.

**Proposition 2.11.** $T^r$ is isomorphic, in a natural way, to the vector space of all $r$-linear mappings of $V \times \cdots \times V$ into $V$.

Proof. $T^r$ is, by definition, $V \otimes T$, which is canonically isomorphic with $T \otimes V$ by Proposition 2.2. By Proposition 2.7, $T^r \otimes V$ is isomorphic to the space of linear mappings of the dual space of $T^r$, that is $T^r_\ast$, into $V$. Again, by the universal factorization property of the tensor product, the space of linear mappings of $T^r_\ast$ into $V$ can be identified with the space of $r$-linear mappings of $V \times \cdots \times V$ into $V$.

QED.

With this interpretation, any tensor $K$ of type $(1, r)$ is an $r$-linear mapping of $V \times \cdots \times V$ into $V$ which maps $(v_1, \ldots, v_r) \rightarrow K(v_1, \ldots, v_r) \in V$. If $e_1, \ldots, e_n$ is a basis for $V$, then $K = \Sigma K^i_{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}$ corresponds to an $r$-linear mapping of $V^\ast \times \cdots \times V^\ast$ into $V$ such that $K(e_{i_1}, \ldots, e_{i_r}) = \Sigma e_{i_1} e_{i_1} \cdots e_{i_r}$.

**Similar interpretation** can be made for tensors of type $(r, s)$ in general, but we shall not go into it.

**Example 2.1.** If $v \in V$ and $v^\ast \in V^\ast$, then $v \otimes v^\ast$ is a tensor of type $(1, 1)$. The **contraction** $C: T^1 \rightarrow F$ maps $v \otimes v^\ast$ into $\langle v, v^\ast \rangle$. In general, a tensor $K$ of type $(1, 1)$ can be regarded as a linear endomorphism of $V$ and the contraction $C K$ of $K$ is then the trace of the corresponding **endomorphism**. In fact, if $\epsilon_1, \ldots, \epsilon_n$ is a
basis for $\mathcal{V}$ and $\mathcal{K}$ has components $K_i$ with respect to this basis, then the endomorphism corresponding to $\mathcal{K}$ sends $e_i$ into $\Sigma_i K_i e_i$.

Clearly, the trace of $\mathcal{K}$ and the contraction $\mathcal{K}^\mathcal{K}$ of $\mathcal{K}$ are both equal to $\Sigma_i K_i$.

Example 2.2. An inner product $g$ on a real vector space $\mathcal{V}$ is a covariant tensor of degree 2 which satisfies (1) $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$ (positive definite) and (2) $g(v, w) = g(w, v)$ (symmetric).

Let $T(U)$ and $T(V)$ be the tensor algebras over vector spaces $U$ and $V$. If $\mathcal{A}$ is a linear isomorphism of $U$ onto $V$, then its transpose $\mathcal{A}^*$ is a linear isomorphism of $V^*$ onto $U^*$ and $A^{-1}$ is a linear isomorphism of $U^*$ onto $V^*$. By Proposition 2.4, we obtain a linear isomorphism $A \otimes A^{-1} : U \otimes U^* \rightarrow V \otimes V^*$. In general, we obtain a linear isomorphism of $T(U)$ onto $T(V)$ which maps $T(E)U$ onto $T(E)V$. The isomorphism, called the extension of $A$ and denoted by the same letter $A$, is the unique algebra isomorphism $T(U) \rightarrow T(V)$ which extends $A : U \rightarrow V$; the uniqueness follows from the fact that $T(U)$ is generated by $F$, $U$ and $U^*$. It is also easy to see that the extension of $A$ commutes with every contraction $C$.

Proposition 2.1. There is a natural 1-1 correspondence between the linear isomorphisms of a vector space $U$ onto another vector space $V$ and the algebra isomorphisms of $T(U)$ onto $T(V)$ which preserve type and commute with contractions.

In particular, the group of automorphisms of $V$ is isomorphic, in a natural way, with the group of automorphisms of the tensor algebra $T(V)$ which preserve type and commute with contractions.

Proof. The only thing which has to be proved now is that every algebra isomorphism, say $f$, of $T(U)$ onto $T(V)$ is induced from an isomorphism $A$ of $U$ onto $V$, provided that $f$ preserves type and commutes with contractions. Since $f$ is type-preserving, it maps $T(E)U$ onto $T(E)V$. Denote the extension of $f$ to $T(U)$ by $A$. Since $f$ maps every element of the field $F$ into itself and commutes with every contraction $C$, we have, for all $u \in U$ and $u^* \in U^*$,

$fu, fu^* = \langle fu, fu^* \rangle = C(fu \otimes fu^*) = C(f(u \otimes u^*))$

$= f(C(u \otimes u^*)) = f(\langle u, u^* \rangle) = \langle u, u^* \rangle$.

Hence, $fu^* = A^*[u^*].$ The extension of $A$ and $f$ agrees on $F$, $U$ and $U^*$. Since the tensor algebra $T(U)$ is generated by $F$, $U$ and $U^*$, $f$ coincides with the extension of $A$. Q.E.D.

Let $T$ be the tensor algebra over a vector space $V$. A linear endomorphism $D$ of $T$ is called a derivation if it satisfies the following conditions:

(a) $D$ is type-preserving, i.e., $D$ maps $T(E)$ into itself;
(b) $D(K \otimes L) = DL \otimes K + K \otimes DL$ for all tensors $K$ and $L$;
(c) $D$ commutes with every contraction $C$.

The set of derivations of $T$ forms a vector space. It forms a Lie algebra if we set $[D, D'] = DD' - D'D$ for derivations $D$ and $D'$. Similarly, the set of linear endomorphisms of $V$ forms a Lie algebra. Since a derivation $D$ maps $T(E)U$ into itself by (a), it induces an endomorphism, say $B$, of $V$.

Proposition 2.13. The Lie algebra of derivations of $T(V)$ is isomorphic to the Lie algebra of endomorphisms of $V$. The isomorphism is given by assigning to each derivation its restriction to $V$.

Proof. It is clear that $D \mapsto B$ is a Lie algebra homomorphism. From (b) it follows easily that $D$ maps every element of $F$ into $0$. Hence, for $v \in V$ and $v^* \in V^*$, we have

$0 = D(v, v^*) = D(C(v \otimes v^*)) - C(D(v \otimes v^*))$

$= C(Dv \otimes v^* + v \otimes Dv^*) = \langle Dv, v^* \rangle + \langle v, Dv^* \rangle$.

Since $Dv = Bv$, $Dv^* = B^*v^*$ where $B^*$ is the transpose of $B$. Since $T$ is generated by $F$, $V$, and $V^*$, $D$ is uniquely determined by its restriction to $F$, $V$, and $V^*$. It follows that $D \mapsto B$ is injective. Conversely, given an endomorphism $B$ of $V$, we define $D(a) = 0$ for $a \in F$, $Dv = Bv$ for $v \in V$ and $Dv^* = -B^*v^*$ for $v^* \in V^*$ and, then, extend $D$ to a derivation of $T$ by (b). The existence of $D$ follows from the universal factorization property of the tensor product. Q.E.D.

Example 2.3. Let $K$ be a tensor of type $(1, 1)$ and consider it as an endomorphism of $V$. Then the automorphism of $T(V)$ induced by an automorphism $A$ of $V$ maps the tensor $K$ into the tensor $AKA^{-1}$. On the other hand, the derivation of $T(V)$ induced by an endomorphism $B$ of $V$ maps $K$ into $[B, K] = BK - KB$. 


Tensor fields

Let \( T_x = T^r(M) \) be the tangent space to a manifold \( M \) at a point \( x \) and \( T(x) \) the tensor algebra over \( T_x = \Sigma T^i(x) \), where \( T^i(x) \) is the tensor space of type \((r, s)\) over \( T_x \). A tensor field of type \((r, s)\) on a subset \( N \) of \( M \) is an assignment of a tensor \( K_x \in T^r(x) \) to each point \( x \) of \( N \). In a coordinate neighborhood \( U \) with a local coordinate system \( x^1, \ldots, x^n \), we take \( X_i = \partial/\partial x^i \), \( i = 1, \ldots, n \), as a basis for each tangent space \( T_x \), \( x \in U \), and \( \omega^i = dx^i, i = 1, \ldots, n \), as the dual basis, of \( T^*_x \). A tensor field \( K \) of type \((r, s)\) defined on \( U \) is then expressed by

\[
K_x = \sum K^i_{\lambda} \cdots X^i \otimes \cdots \otimes X^i \otimes \omega^j \otimes \cdots \otimes \omega^j,
\]

where \( K^i_{\lambda} \cdots \) are functions on \( U \), called the components of \( K \) with respect to the local coordinate system \( x^1, \ldots, x^n \). We say that \( K \) is of class \( C^k \) if its components \( K^i_{\lambda} \cdots \) are functions of class \( C^k \). Of course, it has to be verified that this notion is independent of a local coordinate system. This is easily done by means of the formula (2.1) where the matrix \( (A^i_j) \) is to be replaced by the Jacobian matrix between two local coordinate systems. From now on, we shall mean by a tensor field that of class \( C^k \) unless otherwise stated.

In section 5, we shall interpret a tensor field as a differentiable cross section of a certain fibre bundle over \( M \). We shall give here another interpretation of tensor fields of type \((0, r)\) (and \((1, r)\)) from the viewpoint of Propositions 2.9 and 2.11. Let \( \tilde{\xi} \) be the algebra of functions (of class \( C^k \)) on \( M \) and \( \tilde{x} \) the \( s \)-module of vector fields on \( M \).

Proposition 3.1. A tensor field \( K \) of type \((0, r)\) (resp. type \((1, r)\)) on \( M \) can be considered as an \( r \)-linear mapping of \( \tilde{x} \times \cdots \times \tilde{x} \) into \( \tilde{\xi} \) (resp. \( \tilde{x} \)) such that

\[
K(f_1X_1, \ldots, f_rX_r) = f_1 \cdots f_rK(X_1, \ldots, X_r)
\]

for \( f_i \in \tilde{\xi} \) and \( X_i \in x \).

Conversely, any such mapping can be considered as a tensor field of type \((0, r)\) (resp. type \((1, r)\)).

Proof. Given a tensor field \( K \) of type \((0, r)\) (resp. type \((1, r)\)), \( K_x \) is an \( r \)-linear mapping of \( T_x \times \cdots \times T_x \) into \( \tilde{\xi} \) (resp. \( \tilde{x} \)) by proposition 2.9 (resp. Proposition 2.11) and hence \((X_1, \ldots, X_r) \to (K(X_1, \ldots, X_r)) \) is an \( r \)-linear mapping of \( \tilde{x} \times \cdots \times \tilde{x} \) into \( \tilde{\xi} \) (resp. \( \tilde{x} \)) satisfying the preceding condition. Conversely, let \( K : \tilde{x} \times \cdots \times \tilde{x} \to \tilde{\xi} \) be an \( r \)-linear mapping over \( \tilde{\xi} \). The essential point of the proof is to show that the value of the function (resp. the \( \textit{vector field} \) \( K(X_1, \ldots, X_r) \) at a point \( x \) depends only on the values of \( X_i \) at \( x \). This will imply that \( K \) induces an \( r \)-linear mapping of \( T_x(M) \times \cdots \times T_x(M) \) into \( \tilde{\xi} \) (resp. \( T_x(M) \)) for each \( x \). We first observe that the mapping that can be \textit{localized}. Namely, we have

**Lemma.** If \( X_i = Y_i \) in a neighborhood \( U \) of \( x \) for \( i = 1, \ldots, r \), then we have

\[
K(X_1, \ldots, X_r) = K(Y_1, \ldots, Y_r)
\]

in \( U \).

Proof of Lemma. It is sufficient to show that if \( X_i \equiv 0 \) in \( U \), then \( K(X_1, \ldots, X_r) = 0 \) in \( U \). For any \( x \in U \), let \( f \in \tilde{x} \) be a differentiable function on \( M \) such that \( f(x) = 0 \) and \( f = 1 \) outside \( U \). Then \( X_i = fX_i + K(X_1, \ldots, X_r) = fK(X_1, \ldots, X_r) \), which vanishes at \( x \). This proves the lemma.

To complete the proof of Proposition 3.1, it is sufficient to show that if \( X_i \) vanishes at a point \( x_i \) so does \( K(X_1, \ldots, X_r) \). Let \( x^1, \ldots, x^n \) be a coordinate system around \( x \) so that \( X_i = \sum_j f^j(\partial/\partial x^j) \). \( W \) may take vector fields \( Y_i \) and differentiable functions \( g^i \) on \( M \) such that \( g^i = f^i \) and \( Y_i = (\partial/\partial x^j) \) for \( i = 1, \ldots, n \) in some neighborhood \( U \) of \( x \). Then \( X_i = \sum_j g^j Y_j \) in \( U \). By the lemma, \( K(X_1, \ldots, X_r) = \sum_i g^i K(Y_1, \ldots, X_r) \). In \( U \), since \( g^i(x) = f^i(x) = 0 \) for \( i = 1, \ldots, n \), \( K(X_1, \ldots, X_r) \) vanishes at \( x \).

QED.

Example 3.1. A \textit{(positive definite)} Riemannian metric on \( M \) is a covariant tensor field \( g \) of degree 2 which satisfies (1) \( g(X, X) \leq 0 \) for all \( X \in \tilde{x} \) and \( g(X, X) = 0 \) if and only if \( X = 0 \) and (2) \( g(Y, X) = g(X, Y) \) for all \( X, Y \in \tilde{x} \). In other words, \( g \) assigns an inner product in each tangent space \( T_x(M) \), \( x \in M \) (cf. Example 2.2). In terms of a local coordinate system \( x^1, \ldots, x^n \), the components of \( g \) are given by \( g_{ij} = g^j(\partial/\partial x^i, \partial/\partial x^j) \). It has been customary to write \( ds^2 = \sum g_{ij} dx^i dx^j \) for \( g \).
Example 3.2. A differential form $\omega$ of degree $r$ is nothing but a covariant tensor field of degree $r$ which is skew-symmetric:

$$\omega(X_{\alpha(1)}, \ldots, X_{\alpha(r)}) = \sum \omega(\pi) X_{\pi(1)}, \ldots, X_{\pi(r)},$$

where $\pi$ is an arbitrary permutation of $(1, 2, \ldots, r)$ and $\omega(\pi)$ is its sign. For any covariant tensor $K$ at $x$ or any covariant tensor field $K$ on $M$, we define the alternation $\Lambda$ as follows:

$$(\Lambda K)(X_1, \ldots, X_r) = \frac{1}{r!} \sum \epsilon(\pi) K(X_{\pi(1)}, \ldots, X_{\pi(r)}),$$

where the summation is taken over all permutations $\pi$ of $(1, 2, \ldots, r)$. It is easy to verify that $\Lambda K$ is skew-symmetric for any $K$ and that $K$ is skew-symmetric if and only if $\Lambda K = 0$. If $\omega$ and $\omega'$ are differential forms of degree $r$ and $s$ respectively, then $\omega \wedge \omega'$ is a covariant tensor field of degree $r + s$.

Example 3.3. The symmetrization $S$ can be defined as follows. If $K$ is a covariant tensor, or tensor field of degree $r$, then

$$(S K)(X_1, \ldots, X_r) = \frac{1}{r!} \sum \epsilon(\pi) K(X_{\pi(1)}, \ldots, X_{\pi(r)}).$$

For any $K$, $SK$ is symmetric and $SK = K$ if and only if $K$ is symmetric.

We now proceed to define the notion of Lie differentiation. Let $\mathcal{X}(M)$ be the set of tensor fields of type $(r, s)$ defined on $M$ and set $\mathcal{X}(M) = \bigoplus_{r,s} \mathcal{X}^r(M)$. Then $\mathcal{X}(M)$ is an algebra over the real number field $\mathbb{R}$, the multiplication $\odot$ being defined pointwise, i.e., if $K, L \in \mathcal{X}(M)$ then $(K \odot L)(x) = K(x) \otimes L(x)$ for all $x \in M$. If $\varphi$ is a transformation of $M$, its differential $\varphi_*$ gives a linear isomorphism of the tangent space $T_x(M)$ onto the tangent space $T_{\varphi(x)}(M)$. By Proposition 2.12, this linear isomorphism can be extended to an isomorphism of the tensor algebra $T(x)$ onto the tensor algebra $T(\varphi(x))$, which we denote by $\varphi$. Given a tensor field $K$, we define a tensor field $\tilde{\varphi}K$ by

$$(\tilde{\varphi}K)_x = \varphi(K_{\varphi^{-1}(x)}), \quad x \in M.$$ 

In this way, every transformation $\varphi$ of $M$ induces an algebra automorphism of $\mathcal{X}(M)$ which preserves type and commutes with contractions.

Let $X$ be a vector field on $M$ and $\varphi_t$ a local 1-parameter group of local transformations generated by $X$ (cf. Proposition 1.5), we shall define the Lie derivative $L_XK$ of a tensor field $K$ with respect to a vector field $X$ as follows. For the sake of simplicity, we assume that $q_t$ is a global 1-parameter group of transformations of $M$; the reader will have no difficulty in modifying the definition when $X$ is not complete. For each $t$, $\tilde{\varphi}_t$ is an automorphism of the algebra $\mathcal{X}(M)$. For any tensor field $K$ on $M$, we set

$$(L_XK)_x = \lim_{t \to 0} \frac{1}{t} [K - \tilde{\varphi}_t K]_x.$$ 

The mapping $L_X$ of $\mathcal{X}(M)$ into itself which sends $K$ into $L_X K$ is called the Lie differentiation with respect to $X$. We have

**Proposition 3.2.** Lie differentiation $L_X$ with respect to a vector field $X$ satisfies the following conditions:

(a) $L_X$ is a derivation of $\mathcal{X}(M)$, that is, it is linear and satisfies $L_X(K \odot K') = (L_X K) \odot K' + K \odot (L_X K')$ for all $K$, $K' \in \mathcal{X}(M)$;

(b) $L_X$ is type-preserving: $L_X(\mathcal{X}^r(M)) \subset \mathcal{X}^r(M)$;

(c) $L_X$ commutes with every contraction of a tensor field;

(d) $L_X f = X f$ for every function $f$;

(e) $L_X Y = [X, Y]$ for every vector field $Y$.

Proof. It is clear that $L_X$ is linear. Let $\varphi_t$ be a local 1-parameter group of local transformations generated by $X$. Then

$$L_X(K \odot K') = \lim_{t \to 0} \frac{1}{t} [K \odot K' - \tilde{\varphi}_t(K \odot K')]$$

$$= \lim_{t \to 0} \frac{1}{t} [K \odot K' - (\tilde{\varphi}_t K) \odot (\tilde{\varphi}_t K')]$$

$$= \lim_{t \to 0} \frac{1}{t} [(K \odot K' - (\tilde{\varphi}_t K) \odot (\tilde{\varphi}_t K')]$$

$$+ \lim_{t \to 0} \frac{1}{t} [L_X K - L_X (\tilde{\varphi}_t K)]$$

$$= \lim_{t \to 0} \frac{1}{t} [K - (\tilde{\varphi}_t K)] \odot K'$$

$$+ \lim_{t \to 0} \frac{1}{t} L_X K - L_X (\tilde{\varphi}_t K)$$

$$= L_X K \odot K' + K \odot (L_X K').$$
Since $\phi_i$ preserves type and commutes with contractions, so does $L_X$. If $f$ is a function on $M$, then

$$(L_Xf)(x) = \lim_{t \to 0} \frac{1}{t} [f(x) - f(q^{-1}x)] = \lim_{t \to 0} \frac{1}{t} [f(x) - f(0)] = f'(x),$$

If we observe that $q^{-1} = q_i$ is a local 1-parameter group of local transformations generated by $-X$, we see that $L_Xf = -(Xf) = Xf$. Finally (c) is a restatement of Proposition 1.9. QED.

By a derivation of $\mathfrak{X}(M)$, we shall mean a mapping of $\mathfrak{X}(M)$ to itself which satisfies conditions (a), (b) and (c) of Proposition 3.2.

Let $S$ be a tensor field of type $(1, 1)$. For each $x \in M$, $S_x$ is a linear endomorphism of the tangent space $T_x(M)$. By Proposition 2.13, $S_x$ can be uniquely extended to a derivation of the tensor algebra $T(M)$ over $T_x(M)$. For every tensor field $K$, define $SK$ by

$$(SK)_x = S_xK_x, \quad x \in M.$$ Then $S$ is a derivation of $\mathfrak{X}(M)$. We have

**Proposition 3.3.** Every derivation $D$ of $\mathfrak{X}(M)$ can be decomposed uniquely as follows:

$$D = L_X + S,$$

where $X$ is a vector field and $S$ is a tensor field of type $(1, 1)$.

Proof. Since $D$ is type-preserving, it maps $\mathfrak{X}(M)$ into itself and satisfies $D[fg] = Df \cdot g + f \cdot Dg$ for $f, g \in \mathfrak{X}(M)$. It follows that there is a vector field $X$ such that $Df = Xf$ for every $f \in \mathfrak{X}(M)$. Clearly, $D = L_X$ is a derivation of $\mathfrak{X}(M)$ which is zero on $\mathfrak{X}(M)$.

We shall show that any derivation $D$ which is zero on $\mathfrak{X}(M)$ is induced by a tensor field of type $(1, 1)$. For any vector field $Y$, $DY$ is a vector field and, for any function $f \in \mathfrak{X}(M)$, $Df = f'Y = f \cdot DY$ since $Df = 0$ by assumption. By Proposition 3.1, there is a unique tensor field $S$ of type $(1, 1)$ such that $DY = SY$ for every vector field $Y$. To show that $D$ coincides with the derivation induced by $S$: it is sufficient to prove the following

**Lemma.** Two derivations $D_1$ and $D_2$ of $\mathfrak{X}(M)$ coincide if they coincide on $\mathfrak{X}(M)$ and $\mathfrak{X}(M)$.

Proof. We first observe that a derivation $D$ can be localized, that is, if a tensor field $K$ vanishes on an open set $U$, then $DK$ vanishes on $U$. In fact, for each $x \in U$, let $f$ be a function such that $f(x) = 0$ and $f = 1$ outside $U$. Then $K = f \cdot K$ and hence $DK = Df \cdot K + f \cdot DK$. Since $K$ and $f$ vanish at $x$, so does $DK$. It follows that if two tensor fields, $K$ and $K'$ coincide on an open set $U$, then $DK$ and $DK'$ coincide on $U$.

Set $D = D_1 - D_2$. Our problem is now to prove that if a derivation $D$ is zero on $\mathfrak{X}(M)$ and $\mathfrak{X}(M)$, then it is zero on $\mathfrak{X}(M)$.

Let $K$ be a tensor field of type $(r, s)$ and $x$ an arbitrary point of $M$. To show that $DK$ vanishes at $x$, let $Y$ be a coordinate neighborhood of $x$ with a local coordinate system $x^1, \ldots, x^n$ and let

$$K = \sum K^i_1 \cdots K^n_r x^1 \cdots x^n \otimes \omega^1 \otimes \cdots \otimes \omega^n,$$

where $x_i = \partial / \partial x_i$ and $\omega^i = dx^i$. We may extend $K^i_1 \cdots K^n_r$ to $M$ to and $M$ and assume that the equality holds in a smaller neighborhood $U$ of $x$. Since $D$ can be localized, it suffices to show that

$$D(K^i_1 \cdots K^n_r x^1 \cdots x^n \otimes \omega^1 \otimes \cdots \otimes \omega^n) = 0.$$

But this will follow at once if we show that $D(\omega) = 0$ for every 1-form $\omega$ on $M$. Let $Y$ be any vector field and $C: \mathfrak{X}(M) \to \mathfrak{X}(M)$ the obvious contraction so that $C(Y \otimes \omega) = \omega(Y)$ is a function (cf. Example 2.1). Then we have

$$0 = D(C(Y \otimes \omega)) = C(D(Y \otimes \omega)) = C(DY \otimes \omega) + C(Y \otimes D\omega) = C(Y \otimes D\omega).$$

Since this holds for every vector field $Y$, we have $D\omega = 0$. QED.

The set of all derivations of $\mathfrak{X}(M)$ forms a Lie algebra over $R$ (of infinite dimensions) with respect to the natural addition and multiplication and the bracket operation defined by $[D, D']K = D(D'K) - D'(DK)$. From Proposition 2.13, it follows that the set of all tensor fields $S$ of type $(1, 1)$ forms a subalgebra of the Lie algebra of derivations of $\mathfrak{X}(M)$. In the proof of Proposition 3.3, we showed that a derivation of $\mathfrak{X}(M)$ is induced by a tensor field of type $(1, 1)$ if and only if it is zero on $\mathfrak{X}(M)$. It follows immediately that if $D$ is a derivation of $\mathfrak{X}(M)$ and $S$ is a tensor field of type $(1, 1)$, then $[D, S]$ is zero on $\mathfrak{X}(M)$ and, hence, is induced by a tensor field of type $(1, 1)$. In other words, the set of tensor fields of type $(1, 1)$ is an ideal of the Lie algebra of derivations of $\mathfrak{X}(M)$. On the other hand, the set of Lie differentiations $L_X, X \in \mathfrak{X}(M)$, forms a subalgebra of the Lie algebra of derivations of $\mathfrak{X}(M)$. This follows from the following:
PROPOSITION 3.4. For any vector fields \( X \) and \( Y \), we have
\[
L_{[X,Y]} = [L_X, L_Y].
\]

Proof. By virtue of Lemma above, it is sufficient to show that
\[
[L_{X, L_Y}] \text{ has the same effect as } L_{[X,Y]}
\]
and \( \mathfrak{x}(M) \) and \( \mathfrak{x}(M) \). For
\[
L_{X, L_Y} f = XYf - YXf = [X,Y]f = L_{[X,Y]} f.
\]
For \( Z \in \mathfrak{x}(M) \), we have
\[
[L_X, L_Y] [X, Y, Z] - [Y, [X, Z]] = [[X, Y], Z] = [[X, Y], Z]_{(2)}
\]
by the Jacobi identity. QED.

PROPOSITION 3.5. Let \( K \) be a tensor field of type \((1, r)\) which we interpret as in Proposition 3.1. For any vector field \( X \), we have then
\[
(L_X K)(Y_1, \ldots, Y_r) = [X, K(Y_1, \ldots, Y_r)]
\]
and (c) of Proposition 3.2, we have, for any derivation \( D \) of \( \mathfrak{x}(M) \),
\[
D(K(Y_1, \ldots, Y_r)) = (DK)(Y_1, \ldots, Y_r)
\]
\[+ \sum_i K(Y_1, \ldots, DY_i, \ldots, Y_r).\]

If \( D = L_X \), then (e) of Proposition 3.2 implies Proposition 3.5. QED.

Generalizing Corollary 1.10, we obtain

PROPOSITION 3.6. Let \( \varphi_t \) be a local 1-parameter group of local transformations generated by a vector field \( X \). For any tensor field \( K \), we have
\[
\varphi_t(L_X K) = \frac{1}{t} [K - \varphi_t K]
\]
Proof. By definition,
\[
L_X K = \lim_{t \to 0} \frac{1}{t} [K - \varphi_t K].
\]
Replacing \( K \) by \( \varphi_t K \), we obtain
\[
L_X (\varphi_t K) = \lim_{t \to 0} \frac{1}{t} [\varphi_t K - \varphi_t K] = -(d(\varphi_t K)/dt)_t.
\]

Our problem is therefore to prove that \( \varphi_t(L_X K) = L_X (\varphi_t K) \), i.e., \( L_X K = \varphi_t^{-1} L_X \varphi_t K \) for all tensor fields \( K \). It is a straightforward verification to see that \( \varphi_t^{-1} L_X \varphi_t \) is a derivation of \( 3 (M) \). By Lemma in the proof of Proposition 3.3, it is sufficient to prove that \( L_X \) and \( \varphi_t^{-1} L_X \varphi_t \) coincide on \( \mathfrak{x}(M) \) and \( \mathfrak{x}(M) \). We already noted in the proof of Corollary 1.10 that they coincide on \( \mathfrak{x}(M) \). The fact that they coincide on \( \mathfrak{x}(M) \) follows from the following formulas (cf. §1, Chapter 1):
\[
\varphi_t((\varphi_s X) f) = X(\varphi_t f),
\]
\[
\varphi_t^{(2)} f = q_t f,
\]
which hold for any transformation \( q \) of \( M \) and from \((\varphi_s)_* X = X \) (cf. Corollary 1.8).

COROLLARY 3.7. A tensor field \( K \) is invariant by \( \varphi_t \) for every \( t \) if and only if \( L_X K = 0 \).

Let \( \mathcal{D}(M) \) be the space of differential forms of degree \( r \) defined on \( M \), i.e., skew-symmetric covariant tensor fields of degree \( r \). With respect to the exterior product, \( \mathcal{D}(M) = \Sigma^r \mathfrak{x}(M) \) forms an algebra over \( M \). A derivation (resp. skew-derivation) of \( \mathcal{D}(M) \) is a linear mapping \( 0 \) of \( \mathcal{D}(M) \) into itself which satisfies
\[
D(\omega \wedge \alpha) = D\alpha \wedge \omega - (-1)^{r+1} D\omega \wedge \alpha \wedge \alpha',
\]
for \( \omega, \alpha \in \mathcal{D}(M) \) (resp. \( D\omega \wedge \omega' = (-1)^{r+1} D\omega \wedge \alpha \wedge \alpha' \) for \( \omega, \alpha' \in \mathcal{D}(M) \)).

A derivation \( D \) or a skew-derivation \( D' \) of \( \mathcal{D}(M) \) is said to be of degree \( k \) if it maps \( \mathcal{D}(M) \) into \( \mathcal{D}^{r+k}(M) \) for every \( r \). The exterior differentiation \( d \) is a skew-derivation of degree 1. As a general result on derivations and skew-derivations of \( \mathcal{D}(M) \), we have

PROPOSITION 3.8. (a) If \( D \) and \( D' \) are derivations of degree \( k \) and \( k' \) respectively, then \([D, D'] \) is a derivation of degree \( k + k' \).
(b) If \( D \) is a derivation of degree \( k \) and \( D' \) is a skew-derivation of degree \( k' \), then \([D, D'] \) is a skew-derivation of degree \( k + k' \).
or a skew-derivation of degree $k$ and $k'$ respectively, then $dD' = D'd$ is a derivation of degree $k + k'$.

(d) A derivation or a skew-derivation is completely determined by its effect on $\mathcal{D}^0(M) = \mathcal{G}(M)$ and $\mathcal{W}(M)$.

Proof. The verification of (a), (b), and (c) is straightforward. The proof of (d) is similar to that of Lemma for Proposition 3.3.

QED.

**Proposition 3.9.** For every vector field $X$, $L_X$ is a derivation of degree 0 of $\mathcal{D}(M)$ which commutes with the exterior differentiation $d$. Conversely, every derivation of degree 0 of $\mathcal{D}(M)$ which commutes with $d$ is equal to $L_X$ for some vector field $X$.

Proof. Observe first that $L_X$ commutes with the alternation $\Lambda$ as in Example 3.2. This follows immediately from the following formula:

$$\{L_X\omega\}(Y_1, \ldots, Y_r) = X(\omega(Y_1, \ldots, Y_r)) - \sum_i \omega(Y_i, [X, Y_i], \ldots, Y_r),$$

whose proof is the same as that of Proposition 3.5. Hence, $L_X(\mathcal{D}(M)) \subset \mathcal{D}(M)$ and, for any $\omega, \omega' \in \mathcal{D}(M)$, we have

$$L_X(\omega \wedge \omega') = L_X(A(\omega \otimes \omega')) = A(L_X(\omega) \otimes \omega') = A(\omega \otimes L_X(\omega')).$$

To prove that $L_X$ commutes with $d$, we first observe that for any transformation $\varphi$ of $M$, $\varphi \omega = (\varphi^{-1})^* \omega$ and, hence, $\Lambda$ commutes with $d$. Let $\gamma$ be a local 1-parameter group of local transformations generated by $X$. From $\dot{\gamma}_t \varphi_Y = d(\varphi_Y)$ and the definition of $L_X\omega$, it follows that $L_X(\varphi_Y \omega) = d(L_X\omega)$ for every $\omega \in \mathcal{D}(M)$. Conversely, let $D$ be a derivation of degree 0 of $\mathcal{D}(M)$ which commutes with $d$. Since $D$ maps $\mathcal{D}^0(M) = \mathcal{G}(M)$ into itself, $D$ is a derivation of $\mathcal{G}(M)$ and there is a vector field $X$ such that $Df = Xf$ for every $f \in \mathcal{G}(M)$. Set $D' = D - L_X$. Then $D'$ is a derivation of $\mathcal{D}(M)$ such that $dD' = 0$ for every $\omega \in \mathcal{G}(M)$.

**Proposition 3.8**. in order to prove $D' = 0$, it is sufficient to prove $D'\omega = 0$ for every 1-form $\omega$. Just as in Lemma for Proposition 3.3, $D'$ can be localized and it is sufficient to show that $D'\omega = 0$ when $\omega$ is of the form $\sum f_i dx^i$ with respect to a local coordinate system $x^1, \ldots, x^n$. Let $\omega = \sum f_i dx^i$.

For each vector field $X$, we define a skew-derivation $\iota_X$ called the interior product with respect to $X$, of degree $-1$ of $\mathcal{D}(M)$ such that

(a) $\iota_X f = 0$ for every $f \in \mathcal{D}^0(M)$;

(b) $\iota_X \omega = \omega(X)$ for every $\omega \in \mathcal{D}^1(M)$.

By (d) of Proposition 3.8, such a skew-derivation is unique if it exists. To prove its existence, consider, for each $\iota_{r}$, the contraction $C : \mathcal{L}_{(r)}(M) \to \mathcal{D}^0(M)$ associated with the pair $(1, 1)$. Consider every $r$-form $\omega$ as an element of $\mathcal{D}^0(M)$ and define $\iota_X \omega = C(\omega, X)$, where $\omega \in \mathcal{D}(M)$ and $\omega' \in \mathcal{D}(M)$, follows easily from the following formula:

$$\langle \omega \wedge \omega' \rangle (Y_1, Y_2, \ldots, Y_{r+s}) = \frac{1}{(r + s)!} \sum_{\sigma} \epsilon(\sigma; (1, \ldots, r + s)) \omega(Y_{\sigma(1)}, \ldots, Y_{\sigma(r+s)}) \omega'(Y_{\sigma(r+s+1)}, \ldots, Y_{\sigma(r+s+k)}),$$

where the summation is taken over all possible partitions of $(1, \ldots, r + s)$ into $(j_1, \ldots, j_r)$ and $(k_1, \ldots, k_s)$ and $\epsilon(j; k)$ stands for the sign of the permutation $(1, \ldots, r + s) \to (j_1, \ldots, j_r, k_1, \ldots, k_s)$.

From $\langle \iota_X \omega \rangle (Y_1, \ldots, Y_{r-2}) = r(r - 1)\omega(X, X, Y_1, \ldots, Y_{r-2}) = 0$, we have $\iota_X (\omega(X, X, Y_1)) = 0$.

As relations among $d, L_X$, and $\iota_X$, we have

**Proposition 3.10**. (a) $L_X = d\iota_X + \iota_X d$ for every vector field $X$.

(b) $[L_X, \iota_Y] = \iota_{[X, Y]}$ for any vector fields $X$ and $Y$.

Proof. By (c) of Proposition 3.8, $d \iota_X + \iota_X d$ is a derivation.
of degree 0. It commutes with $d$ because $d^2 = 0$. By Proposition 3.9, it is equal to the Lie differentiation with respect to some vector field. To prove that it is actually equal to $L_X$, we have only to show that $L_X f = (d \circ \iota_X + \iota_X \circ df)$ for every function $f$. But this is obvious since $L_X f = Xf$ and $(d \circ \iota_X + \iota_X \circ df) = \iota_X(df) = (df)(X) = Xf$. To prove the second assertion (b), observe first that $[L_X, \iota_Y]$ is a skew-derivation of degree -1 and that both $[L_X, \iota_Y]$ and $\iota_{[X,Y]}$ are zero on $\mathcal{O}(M)$. By (d) of Proposition 3.8, it is sufficient to show that they have the same effect on every 1-form $\omega$. As we noted in the proof of Proposition 3.9, there is $(L_X \omega)(Y) = X(\omega(Y)) - \omega([X,Y])$ which can be proved in the same way as Proposition 3.5. Hence,

$$(L_X, \iota_Y) \omega = L_X(\omega(Y)) - \iota_Y(L_X \omega) = X(\omega(Y)) - (L_X \omega)(Y) = \omega([X,Y]) = \iota_{[X,Y]} \omega.$$

QED.

As an application of Proposition 3.10 we shall prove

**Proposition 3.11.** If $\omega$ is an $r$-form, then

$$(d\omega)(X_0, X_1, \ldots, X_r) = \sum_{i=0}^{r+1} \frac{(-1)^i}{r!} X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_r))$$

$$+ \frac{1}{r+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r),$$

where the symbol $\hat{X}$ means that the term is omitted. (The cases $r = 1$ and 2 are particularly useful.) If $\omega$ is a 1-form, then

$$(d\omega)(X, Y) = \frac{1}{2}(X(\omega(Y)) - Y(\omega(X))) - \omega([X, Y]),$$

$$(X, Y) \in \mathfrak{X}(M).$$

If $\omega$ is a 2-form, then

$$(d\omega)(X, Y, Z) = \frac{1}{2}(X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y))$$

$$- \omega([X, Y] Z) - \omega([Y, Z] X) - \omega([Z, X] Y)),$$

$$X, Y, Z \in \mathfrak{X}(M).$$

Proof. The proof is by induction on $r$. If $r = 0$, then $\omega$ is a function and $d\omega(X_0) = X_0 \omega$, which shows that the formula above is true for $r = 0$. Assume that the formula is true for $r - 1$. Let $\omega$ be an $r$-form and, to simplify the notation, set $X = X_0$. By (a) of Proposition 3.10,

$$(r+1)! \omega(X_0, X_1, \ldots, X_r, Y) = \frac{1}{r!} \sum_{i=j} (-1)^i \iota_X \omega(X_1, \ldots, X_i) + \iota_Y \omega(X_0, \ldots, X_{i-1}, X_i, \ldots, X_r)$$

$$+ \frac{1}{r!} \sum_{i<j} (-1)^{i+j} \iota_X \omega([X_1, X_2, \ldots, X_i], X_{i+1}, \ldots, X_r).$$

As we noted in the proof of Proposition 3.9,

$$(L_X \omega)(X_1, \ldots, X_r) - X_0(\omega(X_1, \ldots, X_r))$$

$$- \sum_{i=j} (-1)^i \iota_X \omega(X_1, \ldots, X_i) + \iota_Y \omega(X_0, \ldots, X_{i-1}, X_i, \ldots, X_r).$$

Since $\iota_X \omega$ is an $(r-1)$-form, we have, by induction assumption,

$$(d \iota_X \omega)(X_1, \ldots, X_r) = \frac{1}{r!} \sum_{i=j} (-1)^i \iota_X \omega([X_1, X_2, \ldots, X_i], X_{i+1}, \ldots, X_r)$$

$$+ \frac{1}{r!} \sum_{i<j} (-1)^{i+j} \iota_X \omega([X_1, X_2, \ldots, X_i], X_{i+1}, \ldots, X_r).$$

Our Proposition follows immediately from these three formulas. QED.

Remark. Formulas in Proposition 3.11 are valid also for vector-space valued forms.

Various derivations allow us to construct new tensor fields from a given tensor field. We shall conclude this section by giving another way of constructing new tensor fields.

**Proposition 3.12.** Let $A$ and $B$ tensor fields of type $(1, 1)$. Set

$$S(X, Y) = [A(X, B) Y] - [B(X, A) Y] - A[B(X, Y)] - B[A(X, Y)]$$


$$X, Y \in \mathfrak{X}(M).$$
Then the mapping \( S: X(M) \times X(M) \rightarrow X(M) \) is a tensor field of type \((1, 2)\) and \( S(X,Y) = -S(Y,X) \).

Proof. By a straightforward calculation, we see that \( S \) is a bilinear mapping of the \( S(M)\)-module \( X(M) \times X(M) \) into the \( B(M)\)-module \( X(M) \). By Proposition 3.1, \( S \) is a tensor field of type \((1, 2)\). The verification of \( S(X,Y) = -S(Y,X) \) is easy.

QED.

We call \( S \) the torsion of \( A \) and \( B \). The construction of \( S \) was discovered by Xijenhuis [1].

4. Lie groups

A Lie group \( G \) is a group which is at the same time a differentiable manifold such that the group operation \((a,b) \in G \times G \rightarrow ab^{-1} \in G\) is a differentiable mapping of \( G \times G \) into \( G \). Since \( G \) is locally connected, the connected component of the identity, denoted by \( G^0 \), is an open subgroup of \( G \). \( G^0 \) is generated by any neighborhood of the identity \( e \). In particular, it is the sum of at most countably many compact sets and satisfies the second axiom of countability.

It follows that \( G \) satisfies the second axiom of countability if and only if the factor group \( G/G^0 \) consists of at most countably many elements.

We denote by \( L_\alpha \) (resp. \( R_\alpha \)) the left (resp. right) translation of \( G \) by an element \( \alpha \in G \); \( L_\alpha x = \alpha x \) (resp. \( R_\alpha x = x \alpha \)) for every \( x \in G \). For \( a \in G \), \( ad \ a \) is the inner automorphism of \( G \) defined by \((\text{ad} \ a)x = \alpha x \alpha^{-1}\) for every \( x \in G \).

A vector field \( X \) on \( G \) is called left invariant (resp. right invariant) if it is invariant by all left translations \( L_\alpha \) (resp. right translations \( R_\alpha \)). A left or right invariant vector field is always differentiable. We define the Lie algebra \( \mathfrak{g} \) of \( G \) to be the set of all left invariant vector fields on \( G \) with the usual addition, scalar multiplication and bracket operation. As a vector space, \( \mathfrak{g} \) is isomorphic with the tangent space \( T_e(G) \) at the identity, the isomorphism being given by the mapping which sends \( X \in \mathfrak{g} \) into \( X_e \), the value of \( X \) at \( e \). Thus \( \mathfrak{g} \) is a Lie subalgebra of dimension \( n \) (\( n = \dim G \)) of the Lie algebra of vector fields \( X(G) \).

Every \( A \in \mathfrak{g} \) generates a (global) 1-parameter group of transformations of \( G \). Indeed, if \( q_t \) is a local 1-parameter group of local transformations generated by \( A \) and \( \varphi_\varepsilon \) is defined for \( |t| < \varepsilon \), then \( q_t \varphi_\varepsilon \) can be defined for \( |t| < \varepsilon \) for every \( \varphi \in G \) and is equal to \( L_t(q_\varepsilon) \varphi \) as \( q_\varepsilon \) commutes with every \( L_t \) by Corollary 1.8. Since \( q_t \varphi_\varepsilon \) is defined for \( |t| < \varepsilon \) for every \( \varphi \in G \), \( q_t \varphi_\varepsilon \) is defined for \( |t| < \infty \) for every \( \varphi \in G \). Set \( \alpha_\varepsilon = q_\varepsilon \varphi_\varepsilon \). Then \( \alpha_{t+s} = \alpha_t \alpha_s \) for all \( t, s \in \mathbb{R} \). We call \( \alpha_\varepsilon \), the 1-parameter subgroup of \( G \) generated by \( A \). Another characterization of \( \alpha_\varepsilon \) is that it is a unique curve in \( G \) such that its tangent vector \( \alpha_\varepsilon \) at \( \varepsilon \) is equal to \( L_{\varepsilon} \alpha_\varepsilon \) and that \( \alpha_0 = e \). In other words, it is a unique solution of the differential equation \( \alpha'_\varepsilon = A \), with initial condition \( \alpha_0 = e \). Denote \( \alpha_\varepsilon = q_\varepsilon \varphi_\varepsilon \) by \( \exp \ A \). It follows that \( \exp tA = \alpha_t \) for all \( t \). The mapping \( A \rightarrow \exp A \) of \( \mathfrak{g} \) into \( G \) is called the exponential mapping.

Example 4.1. \( GL(n; \mathbb{R}) \) and \( gl(n; \mathbb{R}) \). Let \( GL(n; \mathbb{R}) \) be the group of all real \( n \times n \) non-singular matrices \( A = (a_{ij}) \) (the matrix whose \( i\)-th row and \( j\)-th column entry is \( a_{ij} \)); the multiplication is given by

\[
(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}
\]

for \( A = (a_{ij}) \) and \( B = (b_{ij}) \).

\( GL(n; \mathbb{R}) \) can be considered as an open subset and, hence, as an open submanifold of \( \mathbb{R}^{n^2} \). With respect to this differentiable structure, \( GL(n; \mathbb{R}) \) is a Lie group. Its identity component consists of matrices of positive determinant. The set \( gl(n; \mathbb{R}) \) of all \( n \times n \) real matrices forms an \( n^2\)-dimensional Lie algebra with bracket operation defined by \([A, B] \equiv AB - BA\). It is known that the Lie algebra of \( GL(n; \mathbb{R}) \) can be identified with \( gl(n; \mathbb{R}) \) and the exponential mapping \( gl(n; \mathbb{R}) \rightarrow GL(n; \mathbb{R}) \) coincides with the usual exponential mapping \( \exp A = \sum_{k=0}^{\infty} A^k/k! \).

Example 4.2. \( O(n) \) and \( o(n) \). The group \( O(n) \) of all \( n \times n \) orthogonal matrices is a compact Lie group. Its identity component, consisting of elements of determinant 1, is denoted by \( SO(n) \). The Lie algebra \( o(n) \) of all skew-symmetric \( n \times n \) matrices can be identified with the Lie algebra of \( O(n) \) and the exponential mapping \( o(n) \rightarrow O(n) \) is the usual one. The dimension of \( O(n) \) is equal to \( n(n - 1)/2 \).

By a Lie subgroup of a Lie group \( G \), we shall mean a subgroup \( H \) which is at the same time a submanifold of \( G \) such that \( H \) itself is a Lie group with respect to this differentiable structure. A left invariant vector field on \( H \) is determined by its value at \( e \) and this tangent vector at \( e \) of \( H \) determines a left invariant vector field on
Every automorphism \( g \) of a Lie group \( G \) induces an automorphism \( g \ast \) of its Lie algebra \( g \); in fact, if \( A \in g \), \( g \ast A \) is again a left invariant vector field and \( g \ast [A, B] = [g \ast A, g \ast B] \) for \( A, B \in g \). In particular, for every \( a \in G \), \( ad \) which maps \( x \) into \( axa^{-1} \) induces an automorphism of \( g \), denoted also by \( ad \). The representation \( a \to ad \) of \( G \) is called the \textit{adjoint representation} of \( G \) in \( g \). For every \( a \in G \) and \( A \in g \), we have \( (ad a)A = (H_a) \ast A \), because \( axa^{-1} = L_{a^{-1}} R_{a} x = R_{a^{-1}} L_{a} x \) and \( L_{a} \) is left invariant. Let \( A, B \in g \) and \( \theta \), the 1-parameter group of transformations of \( G \) generated by \( A \). Set \( \theta_t = \exp tA = q_t(e) \). Then \( q_t(x) = xa_t \) for \( x \in G \). By Proposition 1.9, we have
\[
[B, A] = \lim_{t \to 0} \frac{1}{t} [(q_t) \ast B - B] = \lim_{t \to 0} \frac{1}{t} [(R_{q_t} g \ast B - B] = \lim_{t \to 0} \frac{1}{t} [\text{ad} (q_t^{-1}) B - B].
\]

It follows that if \( H \) is an invariant Lie subgroup of \( G \), its Lie algebra \( h \) is an ideal of \( g \) that is, \( A \in g \) and \( B \in h \) imply \( [B, A] \in h \). Conversely, the connected Lie subgroup \( H \) generated by an ideal \( h \) of \( g \) is an invariant subgroup of \( G \).

A differential form \( \omega \) on \( G \) is called left invariant if \( (L_a) \ast \omega = \omega \) for every \( a \in G \). The vector space \( g^* \) formed by all left invariant 1-forms is the dual space of the Lie algebra \( g \); if \( A \in g \) and \( \omega \in g^* \), then the function \( \omega(A) \) is constant on \( G \). If \( \omega \) is a left invariant form, then \( d\omega \) is \( d\omega \ast \) because the exterior differentiation commutes with \( \ast \). From Proposition 3.11 we obtain the equation of \textit{Maurer-Cartan}:
\[
d\omega(A, B) = -\frac{1}{2} \omega([A, B]) \quad \text{for \( \omega \in g^* \) and \( A, B \in g \).}
\]

The canonical 1-form \( \theta \) on \( G \) is the left invariant \( g \)-valued 1-form uniquely determined by
\[
O(A) = A \quad \text{for \( A \in g \).}
\]

Let \( E_1, \ldots, E_r \) be a basis for \( g \) and set
\[
\theta = \sum_{i=1}^r \theta_i E_i.
\]

Then \( \theta^1, \ldots, \theta^r \) form a basis for the space of left invariant real 1-forms on \( G \). We set
\[
[E_i, E_j] = \sum_{k=1}^r \epsilon_{ijk} E_k,
\]

where the \( \epsilon_{ijk} \)'s are called the \textit{structure constants} of \( g \) with respect to the basis \( E_1, \ldots, E_r \). It can be easily verified that the equation of Maurer-Cartan is given by
\[
d\theta^i = -\frac{1}{2} \sum_{k,j=1}^r \epsilon_{ijk} \theta^j \wedge \theta^k, \quad i = 1, \ldots, r.
\]

We now consider Lie transformation groups. We say that a Lie group \( G \) is a \textit{Lie transformation} group on a manifold \( M \) or that \( G \) acts (differentiably) on \( M \) if the following conditions are satisfied:

1. Every element \( a \) of \( G \) induces a \textit{transformation} of \( M \), denoted by \( x \to xa \) where \( x \in M \);
(2) \((a, x) \mapsto G \times M \to xa \in M\) is a differentiable mapping;
(3) \(x(ab) = (xa)b\) for all \(a, b \in G\) and \(x \in M\).
We also write \(R_x\) for \(xa\) and say that \(G\) acts on \(M\) on the right. If we write \(ax\) and assume \((ab)x = a(bx)\) instead of (3), we say that \(G\) acts on \(M\) on the left and we use the notation \(L_xa\) for \(ax\) also. Note that \(R_{ab} = R_b \circ R_a\) and \(L_{ab} = L_a \circ L_b\). From (3) and from the fact that each \(R_a\) or \(L_a\) is one-to-one on \(M\), it follows that \(R_t\) and \(L_t\) are the identity transformation of \(M\).

We say that \(G\) acts effectively (resp. freely) on \(M\) if \(R_x = x\) for all \(x \in M\) (resp. for some \(x \in M\)) implies that \(a = e\).

If \(G\) acts on \(M\) on the right, we assign to each element \(A \in \mathfrak{g}\) a vector field \(A^*\) on \(M\) as follows. The action of the \(t\)-parameter group of transformations \(R_{at}\) is trivial, that is, \(R_{at}\) is the identity transformation of \(M\) for every \(t\). If \(G\) is effective on \(M\), this implies that \(a = e\) for every \(t\) and hence \(A = 0\). To prove the last assertion of our proposition, assume \(\sigma A\) vanishes at some point \(x\) of \(M\). Then \(R_{a_x}\) leaves \(x\) fixed for every \(t\). If \(G\) acts freely on \(M\), this implies that \(a = e\) for every \(t\) and hence \(A = 0\).

Although we defined a Lie group as a group which is a differentiable manifold such that the group operation \((a, b) \mapsto ab^{-1}\) is differentiable, we can replace differentiability by real analyticity without loss of generality for the following reason. The exponential mapping is one-to-one near the origin of \(\mathfrak{g}\); that is, there is an open neighborhood \(N\) of 0 in \(\mathfrak{g}\) such that \(\exp\) is a diffeomorphism of \(N\) onto an open neighborhood \(U\) of \(e\) in \(G\) (cf. Chevalley [1; p. 118] or Pontrjagin [1; §39]). Consider the atlas of \(G\) which consists of charts, \((U_a, \phi_a), a \in \mathfrak{g}\), where \(\phi_a : U_a \to N\) is the inverse mapping of \(R_a \circ \exp : N \to U_a\). (Here, \(U_a\) means \(R_a(U)\) and \(N\) is considered as an open set of \(R^n\) by an identification of \(g\) with \(\mathbb{R}^n\).) With respect to this atlas, \(G\) is a real analytic manifold and the group operation \((a, b) \mapsto ab^{-1}\) is real analytic (cf. Pontrjagin [1; p. 257]). We shall need later the following

**Proposition 4.2.** Let \(G\) be a Lie group and \(H\) a closed subgroup of \(G\). Then the quotient space \(G/H\) admits a structure of real analytic manifold in such a way that the action of \(G\) on \(G/H\) is real analytic, that is, the mapping \(G \times G/H \to G/H\) which maps \((a, bH)\) into \(abH\) is real analytic. In particular, the projection \(G \to G/H\) is real analytic.

For the proof, see Chevalley [1; pp. 109-111].

There is another important class of quotient spaces. Let \(G\) be an abstract group acting on a topological space \(M\) on the right as a group of homeomorphisms. The action of \(G\) is called properly discontinuous if it satisfies the following conditions:

1. If two points \(x\) and \(x'\) of \(M\) are not congruent modulo \(G\) (i.e., \(R_x \neq x'\) for every \(a \in G\)), then \(x\) and \(x'\) have neighborhoods \(U\) and \(U'\) respectively such that \(R_a(U) \cap U'\) is empty for all \(a \in G\);
(2. For each \( x \in G \), the isotropy group \( G_x = \{ a \in G; R_a x = x \} \) is finite.

(3) Each \( x \in M \) has a neighborhood \( U_0 \) stable by \( G \), such that \( U \cap R_a(U) \) is empty for every \( a \in G \) not contained in \( G_x \).

Condition (1) implies that the quotient space \( M/G \) is Hausdorff. If the action of \( G \) is free, then condition (2) is automatically satisfied.

**Proposition 4.3.** Let \( G \) be a properly discontinuous group of differentiable (resp. real analytic) transformations acting freely on a differentiable (resp. real analytic) manifold \( M \). Then the quotient space \( M/G \) has a structure of differentiable (resp. real analytic) manifold such that the projection \( \pi: M \rightarrow M/G \) is differentiable (resp. real analytic).

Proof. Condition (3) implies that every point of \( M/G \) has a neighborhood \( V \) such that \( \pi \) is a homeomorphism of each connected component of \( \pi^{-1}(V) \) onto \( V \). Let \( U \) be a connected component of \( \pi^{-1}(V) \). Choosing \( V \) sufficiently small, we may assume that there is an admissible chart \( (U', \varphi) \), where \( \varphi: U \rightarrow \mathbb{R}^n \), for the manifold \( M \). Introduce a differentiable (resp. real analytic) structure in \( M/G \) by taking \( (U', \varphi) \), where \( \varphi \) is the composite of \( \pi^{-1} \circ V \rightarrow U \) and \( \varphi \), as an admissible chart. The verification of details is left to the reader.

**Remark.** A complex analytic analogue of Proposition 4.3 can be proved in the same way.

To give useful criteria for properly discontinuous groups, we define a weaker notion of discontinuous groups. The action of an abstract group \( G \) on a topological space \( M \) is called discontinuous if, for every \( x \in M \) and every sequence of elements \( \{ a_n \} \) of \( G \) (where \( a_n \) are all mutually distinct), the sequence \( \{ R_{a_n} x \} \) does not converge to a point in \( M \).

**Proposition 4.4.** Every discontinuous group \( G \) of isometries of a metric space \( M \) is properly discontinuous.

Proof. Observe that for each \( x \in M \), the orbit \( xG = \{ R_\alpha x; \alpha \in G \} \) is closed in \( M \). Given a point \( x' \) outside the orbit \( xG \), let \( r \) be a positive number such that \( 2r \) is less than the distance between \( x \) and \( x' \). Let \( U' \) and \( U'' \) be the open spheres of radius \( r \) and centers \( x \) and \( x' \) respectively. Then \( R_\alpha(U') \cap U'' \) is empty for all \( \alpha \in G \), thus proving condition (1). Condition (2) is always satisfied by a discontinuous action. To prove (3), for each \( x \in M \), let \( r \) be a positive number such that \( 2r \) is less than the distance between \( x \) and the closed set \( xG - \{ x \} \). It suffices to take the open sphere of radius \( r \) and center \( x \) as \( U \).

Let \( G \) be a topological group and \( H \) a closed subgroup of \( G \). Then \( G/H \), hence, any subgroup of \( G \) acts on the quotient space \( G/H \) (on the left).

**Proposition 4.5.** Let \( G \) be a topological group and \( H \) a compact subgroup of \( G \). Then the action of every discrete subgroup \( D \) of \( G \) on \( G/H \) is continuous.

Proof. Assumption that the action of \( D \) is not discontinuous, let \( x \) and \( y \) be points of \( G/H \) and \( \{ d_n \} \) a sequence of distinct elements of \( D \) such that \( d_n x \) converges to \( y \). Let \( \bar{p}: G \rightarrow G/H \) be the projection and write \( x = \bar{p}(a) \) and \( y = \bar{p}(b) \) where \( a, b \in G \). Let \( V \) be a neighborhood of the identity \( e \) of \( G \) such that \( bV \) contains \( a \). Since \( p(bV) \) is a neighborhood of \( y \), there is an integer \( N \) such that \( d_n x \in bV \) for all \( n > N \). Hence, \( d_n aH = p^{-1}(d_n x) \leq p^{-1}(bV) = bV \) for \( n \geq N \). For each \( n \geq N \), there exist \( v_n \in V \) and \( h_n \in H \) such that \( d_n a = bv_n h_n \). Since \( H \) is compact, we may assume (by taking a subsequence if necessary) that \( A_n \) converges to an element \( h \in H \) and hence \( h_n = u_n h \) for \( n > N \), where \( u_n \in V \). We have therefore \( d_n a = bv_n h a^{-1} \) for \( n > N \). Consequently, \( d_n a^{-1} \) is in \( bV V^{-1}W^{-1} \) if \( i, j > N \). This means \( d_n a^{-1} = d_i \), contradicting our assumption.

**Theorem 4.6.** Let \( G \) be a locally compact effective transformation group of a connected manifold \( M \) of class \( C^r \), \( r \geq 1 \), and let each transformation of \( G \) be of class \( C^r \). Then \( G \) is a Lie group and the mapping \( G \times M \rightarrow M \) is of class \( C^r \).

We shall now prove the following result, essentially due to van Dantzig and van der Waerden [1].
Theorem 4.7. The group $G$ of isometries of a connected, locally compact metric space $M$ is locally compact with respect to the compact-open topology.

Proof. We recall that the compact-open topology of $G$ is defined as follows. For any finite number of pairs $(K_i, U_i)$ of compact subsets $K_i$ and open subsets $U_i$ of $M$, let $W = W(K_1, \ldots, K_s; U_1, \ldots, U_s) = \{ g \in G; \varphi(K_i) \subset U_i \text{ for } i = 1, \ldots, s \}$. Then the sets $W$ of this form are taken as a base for the open sets of $G$.

Since $M$ is regular and locally compact, the group multiplication $G \times G \rightarrow G$ and the group action $G \times M \rightarrow M$ are continuous (cf. Steenrod [1; p. 19]). The continuity of the mapping $G \rightarrow G$ which sends $\varphi$ into $\varphi^{-1}$ will be proved using the assumption in Theorem 4.7, although it follows from a weaker assumption (cf. Arens [1]).

Every connected, locally compact metric space satisfies the second axiom of countability (see Appendix 2). Since $M$ is locally compact and satisfies the second axiom of countability, $G$ satisfies the second axiom of countability. This justifies the use of sequences in proving the local compactness of $G$ (cf. Kelley [1; p. 138]).

The proof is divided into several lemmas.

Lemma 1. Let $a \in M$ and let $e > 0$ be such that $U(a; e) = \{ x \in M; d(a, x) < e \}$ has compact closure (where $d$ is the distance). Denote by $V_a$ the open neighborhood $U(a; e/4)$ of $a$. Let $\varphi_n$ be a sequence of isometries such that $\varphi_n(b)$ converges for some point $b \in V_a$. Then there exists a compact set $K$ and an integer $N$ such that $\varphi_n(V_a) \subset K$ for every $n > N$.

Proof. Choose $N$ such that $n > N$ implies $d(\varphi_n(b), \varphi_m(b)) < e/4$. If $x \in V_a$ and $n > N$, then we have

$$d(\varphi_n(x), \varphi_N(x)) \leq d(\varphi_n(x), \varphi_n(b)) + d(\varphi_n(b), \varphi_N(b)) + d(\varphi_N(b), \varphi_N(x))$$

$$= d(x, b) + d(\varphi_n(b), \varphi_N(b)) + d(b, a) < e,$$

using the fact that $\varphi_n$ and $\varphi_N$ are isometries. This means that $\varphi_n(V_a)$ is contained in $U(\varphi_N(a); e)$. But $U(\varphi_N(a); e) = \varphi_N(U(a; e))$ since $\varphi_N$ is an isometry. Thus the closure $X$ of $U(\varphi_N(a); e) = \varphi_N(U(a; E))$ is compact and $\varphi_n(V_a) \subset K$ for $n > N$.

Lemma 2. In the notation of Lemma 1, assume again that $\varphi_n(b)$ converges for some $b \in V_a$. Then there is a subsequence $\varphi_{n_k}$ of $\varphi_n$ such that $\varphi_{n_k}(x)$ converges for each $x \in V_a$.

Proof. Let $\{b_i\}$ be a countable set which is dense in $V_a$. (Such a set exists since $M$ is separable.) By Lemma 1, there is an $n$ such that $\varphi_n(V_a)$ is in $K$ for $n > N$. In particular, $\varphi_n(b_i)$ is in $K$.

Choose a subsequence $\varphi_{n_i}$ such that $\varphi_{n_i}(b_i)$ converges. From this subsequence, we choose a subsequence $\varphi_{n_{i_k}}$ such that $\varphi_{n_{i_k}}(b_{i_k})$ converges, and so on. The diagonal sequence $\varphi_{n_{i_k}}(b_{i_k})$ converges for every $i = 1, 2, \ldots$. To prove that $\varphi_{n_{i_k}}(x)$ converges for every $x \in V_a$, we change the notation and may assume that $\varphi_n(b_i)$ converges for each $i = 1, 2, \ldots$. Let $x \in V_a$ and $N > 0$. Choose $b_i$ such that $d(x, b_i) < \delta/4$. There is an $N_1$ such that $d(\varphi_n(b_i), \varphi_m(b_i)) < \delta/4$ for $n, m > N_1$. Then we have

$$d(\varphi_n(x), \varphi_m(x)) \leq d(\varphi_n(x), \varphi_n(b_i)) + d(\varphi_n(b_i), \varphi_m(b_i))$$

$$+ d(\varphi_m(b_i), \varphi_m(x)) < 3\delta.$$

Thus $\varphi_n(x)$ is a Cauchy sequence. On the other hand, Lemma 1 says that $\varphi_n(x)$ is in a compact set $K$ for all $n > N$. Thus $\varphi_n(x)$ converges.

Lemma 3. Let $\varphi_n$ be a sequence of isometries such that $\varphi_n(a)$ converges for some point $a \in M$. Then there is a subsequence $\varphi_{n_k}$ such that $\varphi_{n_k}(x)$ converges for each $x \in M$. (The connectedness of $M$ is essentially used here.)

Proof. For each $x \in M$, let $V_x = U(x; e/4)$ such that $U(x; e)$ has compact closure (this $e$ may vary from point to point, but we choose one such $e$ for each $x$). We define a chain as a finite sequence of open sets $V_i$ such that (1) each $V_i$ is of the form $V_x$ for some $x$; (2) $V_i$ contains $a$; (3) $V_i$ and $V_{i+1}$ have a common point. We assert that every point $y$ of $M$ is in the last term of some chain. In fact, it is easy to see that the set of such points $y$ is open and closed. $M$ being connected, the set coincides with $M$.

This being said, choose a countable set $\{b_i\}$ which is dense in $M$. For $b_i$, let $V_{b_1}, V_{b_2}, \ldots$ be a chain with $6. \in V_{b_1}$. By assumption $\varphi_n(b_i)$ converges. By Lemma 2, we may choose a subsequence (which we may still denote by $\varphi_n$ by changing the notation) such that $\varphi_n(x)$ converges for each $x \in V_1$. Since $V_1 \cap V_2$ is non-empty, Lemma 2 allows us to choose a subsequence which converges for
each \( x \in V_2 \), and so on. Thus the original sequence \( q_n \) has a sub-sequence \( q_{1,k} \) such that \( q_{1,k}(b_1) \) converges. From this sub-sequence, we may further choose a subsequence \( q_{2,k} \) such that \( q_{2,k}(b_2) \) converges. As in the proof of Lemma 2, we obtain, the diagonal subsequence \( q_{k,k} \) such that \( q_{k,k}(b_n) \) converges for each \( n \). Denote this diagonal subsequence by \( q_n \), by changing the notation. Thus \( q_n(b) \) converges for each \( b \).

We now want to show that \( q_n(x) \) converges for each \( x \in M \). In \( V_{2^i} \), there is some \( b_i \) so that there exist an \( N \) and a compact set \( K \) such that \( q_n(V_{2^i}) \subset K \) for \( n > N \) by Lemma 1. Proceeding as in the second half of the proof for Lemma 2, we can prove that \( q_n(x) \) is a Cauchy sequence. Since \( q_n(x) \in K \) for \( n > N \), we conclude that \( q_n(x) \) converges.

**Lemma 4.** Assume that \( q_n \) is a sequence of isometries such that \( q_n(x) \) converges for each \( x \in M \). Define \( q(x) = \lim_{n \to \infty} q_n(x) \) for each \( x \). Then \( q \) is an isometry.

**Proof.** Clearly, \( d(q(x), q(y)) = d(x,y) \) for any \( x, y \in M \). For any \( a \in M \), let \( a' = q(a) \). From \( d(q_n^{-1}(a), a') = d(q(a), q_n(a)) \), it follows that \( q_n^{-1}(a') \) converges to \( a \). By Lemma 3, there is a subsequence \( q_n \) such that \( q_n^{-1}(x) \) converges for every \( x \in M \). Define a mapping \( \psi \) by \( \psi(x) = \lim_{n \to \infty} q_n^{-1}(x) \). Then \( \psi \) preserves distance, that is, \( d(\psi(x), \psi(y)) = d(x,y) \) for any \( x, y \in M \). From

\[
\begin{align*}
  d(\psi(x), x) &= d(\lim_{k \to \infty} q_n^{-1}(x), x) = \lim_{k \to \infty} d(q_n^{-1}(x), x) \\
  &= \lim_{k \to \infty} d(q(x), q_n(x)) = d(q(x), x) = 0,
\end{align*}
\]

it follows that \( \psi(x) = x \) for each \( x \in M \). This means that \( \psi \) maps \( M \) onto \( M \). Since \( \psi \) preserves distance and maps \( M \) onto \( M \), \( \psi \) exists and is obviously equal to \( q \). Thus \( q \) is an isometry.

**Lemma 5.** Let \( q_n \) be a sequence of isometries and \( q \) an isometry. If \( q_n(x) \) converges to \( q(x) \) for each \( x \in M \), then the convergence is uniform on every compact subset \( K \) of \( M \).

**Proof.** Let \( \delta > 0 \) be given. For each point \( x \in K \), choose an integer \( N_x \) such that \( n > N_x \) implies \( d(q_n(a), q(a)) < \delta/4 \). Let \( W_x = U(a; \delta/4) \). Then for any \( x \in W_a \) and \( n > N_x \), we have

\[
d(q_n(x), y(x)) \leq d(q_n(x), q_n(x)) + d(q(a), q(a)) + d(q(a), q(x)) < 2d(x, a) + \delta/4 < \delta.
\]

Now \( K \) can be covered by a finite number of \( U_x \)'s, say \( W_{i,1}, \ldots, W_{i,s} \). It follows that if \( n \) is max \( \{N_x\} \), then

\[
d(q_n(x), q(x)) < \delta \quad \text{for each } x \in K.
\]

**Lemma 6.** If \( q_n(x) \) converges to \( q(x) \) as in Lemma 5, then \( q_n^{-1}(x) \) converges to \( q^{-1}(x) \) for every \( x \in M \).

**Proof.** For any \( x \in M \), let \( y = q^{-1}(x) \). Then

\[
d(q_n^{-1}(y), q^{-1}(y)) = d(q_n^{-1}(q_n(y)), y) = d(q(y), q_n(y)) \to 0.
\]

We shall now complete the proof of Theorem 4.7. First, observe that \( q_n \to q \) with respect to the compact-open topology is equivalent to the uniform convergence of \( q_n \) to \( q \) on every compact subset of \( M \). If \( q_n \to q \) in \( G \) (with respect to the compact-open topology), then Lemma 6 implies that \( q_n^{-1}(x) \to q^{-1}(x) \) for every \( x \in M \), and the convergence is uniform on every compact subset by Lemma 5. Thus \( q_n^{-1} \to q^{-1} \) in \( G \). This means that the mapping \( G \to G \) which maps \( q \) into \( q^{-1} \) is continuous.

To prove that \( G \) is locally compact, let a \( \epsilon M \) and \( U \) an open neighborhood of \( \epsilon \) with compact closure. We shall show that the neighborhood \( W = W(\epsilon, U) = \{ g \in G, g(\epsilon) \in U \} \) of the identity element of \( G \) has compact closure. Let \( q_n \) be a sequence of elements in \( W \). Since \( q_n(\epsilon) \) is contained in the compact set \( U \), closure of \( U \), we can choose, by Lemma 3, a subsequence \( q_{n_k} \) such that \( q_{n_k}(x) \) converges for every \( x \in M \). The mapping \( \psi \) defined by \( \psi(x) = \lim q_n(x) \) is an isometry of \( M \) by Lemma 4. By Lemma 5, \( q_{n_k} \to \psi \) uniformly on every compact subset of \( M \), that is, \( q_{n_k} \to \psi \) in \( G \), proving that \( W \) has compact closure.

**Corollary 4.8.** Under the assumption of Theorem 4.7, the isotropy subgroup \( G_\epsilon = \{ g \in G, g(\epsilon) = \epsilon \} \) of \( G \) at \( \epsilon \) is compact for every \( \epsilon \in M \).

**Proof.** Let \( q_n \) be a sequence of elements of \( G \). Since \( q_n(\epsilon) = \epsilon \) for every \( n \), there is a subsequence \( q_{n_k} \) which converges to an element \( \zeta \in G_\epsilon \) by Lemmas 3, 4, and 5.

**Corollary 4.9.** If \( M \) is a locally compact metric space with a finite number of connected components, the group \( G \) of isometries \( G \) is locally compact with respect to the compact-open topology.

**Proof.** Decompose \( M \) into its connected components \( M_i \), \( M = \bigcup_{i=1}^{\infty} M_i \). Choose a point \( \epsilon_i \) in each \( M_i \) and an open
Given a principal fibre bundle \( P(M, G) \), the action of \( G \) on \( P \) induces a homomorphism \( \sigma \) of the Lie algebra \( g \) of \( G \) into the Lie algebra \( \mathfrak{X}(P) \) of vector fields on \( P \) by Proposition 4.1. For each \( A = \sigma(A) \) is called the fundamental vector field corresponding to \( A \). Since the action of \( G \) sends each fibre into itself, \( A \) is tangent to the fibre at each point \( p \in P \). As \( G \) acts freely on \( P \), \( A \) never vanishes on \( P \) if \( A \neq 0 \) (by Proposition 4.1). The dimension of each fibre being equal to that of \( g \), the mapping \( A \rightarrow \sigma(A) \), of \( g \) into \( \mathfrak{X}(P) \) is a linear isomorphism of \( g \) onto the tangent space at \( u \) of the fibre through \( u \). We prove

**Proposition 5.1.** Let \( A^* \) be the fundamental vector field corresponding to \( A \in g \). For each \( A \in g \), \( (R_a)_* A^* \) is the fundamental vector field corresponding to \( (\text{ad}(a^{-1}))A \in g \).

Proof. Since \( A \) is induced by the 1-parameter group of transformations \( R_a \), where \( a = \exp tA \), the vector field \( (R_a)_* A^* \) is induced by the 1-parameter group of transformations \( R_a R_b R_a^{-1} = R_{a \circ b} \) by Proposition 1.7. Our assertion follows from the fact that \( \sigma(a^{-1}b) \) is the 1-parameter group generated by \( (\text{ad}(a^{-1}))A \in g \).

QED.

The concept of fundamental vector fields will prove to be useful in the theory of connections.

In order to relate our intrinsic definition of a principal fibre bundle to the definition and the construction by means of an open covering, we need the concept of transition functions. By (3) for a principal fibre bundle \( P(M, G) \), it is possible to choose an open covering \( \{ U_a \} \) of \( M \) such that each \( \pi^{-1}(U_a) \) provided with a diffeomorphism \( u \rightarrow (\pi(u), \varphi_a(u)) \) of \( \pi^{-1}(U_a) \) onto \( U_a \times G \) such that \( \varphi_a(u) = (q_a(u), a) \). If \( u \in \pi^{-1}(U_a \cap U_b) \), then \( \varphi_a(u) \varphi_b(u)^{-1} = \varphi_a(u)q_a(u)^{-1}q_b(u)^{-1} \), which shows that \( \varphi_a(u) \varphi_b(u)^{-1} \) depends only on \( \pi(u) \) not on \( u \). We can define a mapping \( \varphi_{ab} : U_a \cap U_b \rightarrow G \) by \( \varphi_{ab}(x) = \varphi_a(x) \varphi_b(x)^{-1} \). The family of mappings \( \varphi_{ab} \) are called transition functions of the bundle \( P(M, G) \) corresponding to the open covering \( \{ U_a \} \) of \( M \). It is easy to verify that

\[ \varphi_{ax}(x) = \varphi_{bx}(x) : \varphi_{ab}(x) \quad \text{for} \ x \in U_a \cap U_b \cap U_v. \]

Conversely, we have
 Proposition 5.2. Let $M$ be a manifold, $\{U_\alpha\}$ an open covering of $M$ anti $G$ a Lie group. Given a mapping $\psi_\alpha: U_\alpha \cap U_\beta \to G$, for every nonempty $U_\alpha \in U_\alpha$, in such a way that the relations $(*)$ are satisfied, we can construct a (differentiable) principal fibre bundle $P(M, G)$ with transition functions $\psi_\alpha$.

Proof. We first observe that the relations $(*)$ imply $\psi_{\alpha}(x) = x$ for every $x \in U_\alpha$ and $\psi_{\alpha}(y) \psi_{\beta}(x) = \psi_{\alpha}(y)$ for every $x \in U_\alpha \cap U_\beta$. Let $X_\alpha = U_\alpha \times G$ for each index $\alpha$ and let $X = \bigcup X_\alpha$ the topological sum of $X_\alpha$; each element of $X$ is a triple $(x, a, \alpha)$ where $x$ is some index, $a \in U_\alpha$, and $\alpha \in G$. Since each $X_\alpha$ is a differentiable manifold and $X$ is a disjoint union of $X_\alpha$, $X$ is a differentiable manifold in a natural way. We introduce an equivalence relation $\rho$ in $X$ as follows. We say that $(x, a, \alpha) \in (x, a, \beta) \in X$ is equivalent to $(x, b, \beta) \in \{x\} \times X_\beta$ if and only if $x = y \in U_\alpha \cap U_\beta$ and $b = \psi_{\alpha}(x) a$. We remark that $(x, x, \alpha) \in (x, y, \beta)$ is equivalent if and only if $x = y$ and $\alpha = \beta$. Let $P$ be the quotient space of $X$ by this equivalence relation $\rho$. We first show that $G$ acts freely on $P$ on the right and that $P/G = M$. By definition, each $c \in G$ maps the $p$-equivalence class of $(c, x, a)$ into the $p$-equivalence class of $(x, x, ca)$. It is easy to see that this definition is independent of the choice of representative $(a, x, a)$ and that $G$ acts freely on $P$ on the right. The projection $\pi: P \to M$, defined by, the $p$-equivalence class of $(a, x, a)$ into $x$; the definition of $\pi$ is independent of the choice of representative $(a, x, a)$. For $u \in P$, $\pi(u) = \pi(v)$ if and only if $u = c \cdot v$ for some $c \in G$. In fact, let $(a, x, a)$ and $(b, y, b)$ be representatives for $u$ and $v$, respectively. If $u = c \cdot v$ for some $c \in G$, then $u = x = y$ and hence $\pi(u) = \pi(v)$. Conversely, if $\pi(u) = x = y = \pi(v) \in U_\alpha \cap U_\beta$, then $u = c \cdot v$ where $c = a \cdot \psi_{\alpha}(x)^{-1} \cdot b \in G$. In order to make $P$ into a differentiable manifold, we first note that, by the natural mapping $X \to P = X/\rho$, each $X_\alpha = U_\alpha \times G$ is mapped 1:1 onto $\pi^{-1}(U_\alpha)$. We introduce a differentiable structure in $P$ by requiring that $\pi^{-1}(U_\alpha)$ is an open submanifold of $P$ and that the mapping $X \to P$ induces a diffeomorphism of $X_\alpha = U_\alpha \times G$ onto $\pi^{-1}(U_\alpha)$. This is possible since every point of $P$ is contained in $\pi^{-1}(U_\alpha)$ for some $\alpha$ and the identification of $(a, x, a)$ with $(a, x, \psi_{\alpha}(x) a)$ is made by means of differentiable mappings. It is easy to check that the action of $G$ on $P$ is differentiable and $P(M, G, \pi)$ is a differentiable principal fibre bundle. Finally, the transition functions of $P$ corresponding to the covering $\{U_\alpha\}$ are precisely the given $\psi_{\alpha}$ if we define $\psi_{\alpha}: \pi^{-1}(U_\alpha) \to U_\alpha \times G$ by $y(u) = (x, a)$, where $u \in \pi^{-1}(U_\alpha)$ is the $p$-equivalence class of $(a, x, a)$. QED.

A homomorphism $f$ of a principal fibre bundle $P'(M', G')$ into another principal fibre bundle $P(M, G)$ consists of a mapping $f: P' \to P$ and a homomorphism $f': G' \to G$ such that $f'(u'a') = f'(u') f'(a')$ for all $u' \in P'$ and $a' \in G'$. For the sake of simplicity, we shall denote $f'$ and $f''$ by the same letter $f$. Every homomorphism $f: P' \to P$ maps each fibre of $P'$ into a fibre of $P$ and hence induces a mapping of $M'$ into $M$, which will be also denoted by $f$. A homomorphism $f: P'(M', G') \to P(M, G)$ is called in imbedding if $f: P' \to P$ is an imbedding and if $f': G' \to G$ is a monomorphism. If $f: P' \to P$ is an imbedding, then the induced mapping $f: M' \to M$ is also an imbedding. By identifying $P'$ with $f(P')$, $G'$ with $f(G')$ and $M'$ with $f(M')$, we say that $P'(M', G')$ is a subbundle of $P(M, G)$. If, moreover, $M' = M$ and the induced mapping $f: M' \to M$ is the identity transformation of $M$, $f: P'(M', G') \to P(M, G)$ is called a reduction of the structure group $G$ of $P(M, G)$ to $G'$. The subbundle $P'(M, G')$ is called a reduced bundle. Given $P(M, G)$ and a Lie subgroup $G'$ of $G$, we say that the structure group $G$ is reducible to $G'$ if there is a reduced bundle $P'(M, G')$. Note that we do not require in general that $G'$ is a closed subgroup of $G$. This generality is needed in the theory of connections.

Proposition 5.3. The structure group $G$ of a principal fibre bundle $P(M, G)$ is reducible to a Lie subgroup $G'$ if and only if there is an open covering $\{U_\alpha\}$ of $M$ with a set of transition functions $\psi_{\alpha}$ which take their values in $G'$.

Proof. Suppose first that the structure group $G$ is reducible to $G'$ and let $P(M', G')$ be a reduced bundle. Consider $P'$ as a submanifold of $P$. Let $\{U_\alpha\}$ be an open covering of $M$ such that each $\pi^{-1}(U_\alpha)$ (a': the projection of $P'$ onto $M$) is provided with an isomorphism $u \to (\pi'(u), \psi_{\alpha}'(u))$ of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times G'$. The corresponding transition functions take their values in $G'$. Now, for the same covering $\{U_\alpha\}$, we define an isomorphism of $\pi^{-1}(U_\alpha)$ (a': the projection of $P$ onto $M$) on $U_\alpha \times G'$ by extending $\psi_{\alpha}'$ as follows. Every $v \in \pi^{-1}(U_\alpha)$ may be represented in the form $v = u a$ for some $u \in \pi^{-1}(U_\alpha)$ and $a \in G$ and we set $\psi_{\alpha}'(v) = \psi_{\alpha}'(u) a$.
It is easy to see that \( \varphi_{\alpha}(v) \) is independent of the choice of representation \( v = u\alpha \). We see then that \( v \mapsto (\varphi(v), \varphi_{\alpha}(v)) \) is an isomorphism of \( \pi^{-1}(U_\alpha) \) onto \( U_\alpha \times G \). The corresponding transition functions \( \psi_{\alpha}(x) = \varphi_{\beta}(v) (\varphi_{\alpha}(u))^{-1} = \varphi_{\beta}(u)(\varphi_{\alpha}(u))^{-1} \) take their values in \( G' \).

Conversely, assume that there is a covering \( \{ U_\alpha \} \) of \( M \) with a set of transition functions \( \psi_{\alpha} \) all taking values in a Lie subgroup \( G' \) of \( G \). For \( U_\alpha \cap U_\beta \supseteq \emptyset \), \( \psi_{\alpha} \) is a differentiable mapping of \( U_\alpha \cap U_\beta \) into \( \text{Lie group} \) \( G' \) such that \( \psi_{\alpha}(U_\alpha \cap U_\beta) \subseteq G' \). The crucial point is that \( \psi_{\alpha} \) is a differentiable mapping of \( U_\alpha \cap U_\beta \) into \( G' \) with respect to the differentiable structure of \( G' \). This follows from Proposition 1.3; note that a Lie subgroup satisfies the second axiom of countability by definition, cf. §4. By Proposition 5.2, we can construct a principal fibre bundle \( P'(M, G') \) from \( \{ U_\alpha \} \) and \( \{ \psi_{\alpha} \} \). Finally, we imbed \( P' \) into \( P \) as follows. Let \( f_\alpha : \pi^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha) \) be the composite of the following three mappings:

\[
\pi^{-1}(U_\alpha) \to U_\alpha \times G' \to U_\alpha \times G \to \pi^{-1}(U_\alpha).
\]

It is easy to see that \( f_\alpha = f_{\alpha \beta} f_\beta \) on \( \pi^{-1}(U_\alpha \cap U_\beta) \) and that the mapping \( f : P' \to P \) thus defined by \( \{ f_\alpha \} \) is an injection. QED.

Let \( P(M, G) \) be a principal fibre bundle and \( F \) a manifold on which \( G \) acts on the left. Let \( E(M, F, G, P) \) be the fibre bundle associated with \( P \). By Proposition 5.2, we can construct a principal fibre bundle \( P'(M, G') \) from \( \{ U_\alpha \} \) and \( \{ \psi_{\alpha} \} \). Finally, we imbed \( P' \) into \( P \) as follows. Let \( f_\alpha : \pi^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha) \) be the composite of the following three mappings:

\[
\pi^{-1}(U_\alpha) \to U_\alpha \times G' \to U_\alpha \times G \to \pi^{-1}(U_\alpha).
\]

It is easy to see that \( f_\alpha = f_{\alpha \beta} f_\beta \) on \( \pi^{-1}(U_\alpha \cap U_\beta) \) and that the mapping \( f : P' \to P \) thus defined by \( \{ f_\alpha \} \) is an injection. QED.

Let \( P(M, G) \) be a principal fibre bundle and \( F \) a manifold on which \( G \) acts on the left. Let \( E(M, F, G, P) \) be the fibre bundle associated with \( P \). By an isomorphism of a fibre \( F_\alpha = \pi^{-1}(x), x \in M \), onto another fibre \( F_\beta = \pi^{-1}(y), y \in M \), we mean a diffeomorphism which can be represented in the form \( u \circ u^{-1} \), where \( u \in \pi^{-1}(x) \) and \( u \in \pi^{-1}(y) \) are considered as mappings of \( F_\alpha \) onto \( F_\beta \) and \( F_\beta \) onto \( F_\alpha \), respectively. In particular, an automorphism of the fibre \( F_\alpha \) is a mapping of the form \( u \circ u^{-1} \), where \( u \) is an arbitrarily fixed point of \( \pi^{-1}(x) \). The group of automorphisms of \( F_\alpha \) is hence isomorphic with the structure group \( G \).

Example 5.1. \( G(G/H, H) \) : Let \( G \) be a Lie group and \( H \) a closed subgroup of \( G \). We let \( H \) act on \( G \) on the right as follows: an element \( a \in G \) maps \( u \in G \) into \( u a \). The quotient space of \( G \) by this group action is denoted by \( G/H \). A differentiable structure will be introduced in \( G/H \) later and at this moment \( G/H \) is only a set. On \( G \) we have a neighbourhood \( U \) of \( \{ a \} \) such that \( \pi^{-1}(U) \) is isomorphic to \( U \times G \). Identifying \( \pi^{-1}(U) \) with \( U \times G \), we see that the action of \( G \) on \( \pi^{-1}(U) \) is a mapping \( \pi_\alpha \) of \( E \) onto \( M \). Every point \( x \) of \( M \) has a neighbourhood \( U \) such that \( \pi^{-1}(U) \) is isomorphic to \( U \times G \). It follows that the isomorphism \( \pi^{-1}(U) \approx U \times G \) induces an isomorphism \( \pi_\alpha^{-1}(U) \approx U \times F \). We can therefore introduce a differentiable structure in \( E \) by the requirement that \( \pi_\alpha^{-1}(U) \approx U \times F \) under the isomorphism \( \pi_\alpha^{-1}(U) \approx U \times F \). The projection \( \pi_\alpha \) is then a differentiable mapping of \( E \) onto \( M \). We call \( E \) or more precisely \( E(M, F, G, P) \) the fibre bundle over the base \( M \) with (standard) fibre \( F \) and (structure) group \( G \), which is associated with the principal fibre bundle \( P \).

Example 5.2. Bundle of \( \text{linear frames} \) : Let \( M \) be a manifold of dimension \( n \). A linear frame \( u \) at a point \( x \) is an ordered basis \( X_1, \ldots, X_n \) of the tangent space \( T_x(M) \). Let \( L(M) \) be the set of
all linear frames $u$ at all points of $M$ and let $\pi$ be the mapping of $L(M)$ onto $M$ which maps a linear frame $u$ at $x$ into $x$. The general linear group $GL(n; \mathbb{R})$ acts on $L(M)$ on the right as follows. If $a = (a_t^i) \in GL(n; \mathbb{R})$ and $u = (X_1, \ldots, X_s)$ is a linear frame at $x$, then $ua$ is, by definition, the linear frame $(Y_1, \ldots, Y_s)$ at $x$ defined by $Y_i = \sum_j a_j^i X_j$. It is clear that $GL(n; \mathbb{R})$ acts freely on $L(M)$ and $\pi(u) = \pi(v)$ if and only if $v = ua$ for some $a \in GL(n; \mathbb{R})$. Now, in order to introduce a differentiable structure in $L(M)$, let $(x_1, \ldots, x_s)$ be a local coordinate system in a coordinate neighborhood $U$ in $M$. Every frame $u$ at $x \in U$ can be expressed uniquely in the form $u = (X_1, \ldots, X_s)$ with $X_i = \sum_j a_j^i \partial/\partial x^j$, where $(\partial/\partial x^j)$ is a non-singular matrix. This shows that $\pi^{-1}(U)$ is in $1:1$ correspondence with $U \times GL(n; \mathbb{R})$. We can make $L(M)$ into a differentiable manifold by taking $(x^i)$ and $(X^k)$ as a local coordinate system in $\pi^{-1}(U)$. It is now easy to verify that $L(M)(M, GL(n; \mathbb{R}))$ is a principal fibre bundle. We call $L(M)$ the bundle of linear frames over $M$. In view of Proposition 5.4, a linear frame $u$ at $x \in U$ can be defined as a non-singular linear mapping of $\mathbb{R}^n$ onto $T_x(M)$. The two definitions are related to each other as follows. Let $e_1, \ldots, e_n$ be the natural basis for $\mathbb{R}^n: e_i = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$. A linear frame $u = (X_1, \ldots, X_s)$ at $x$ can be given as a linear mapping $u: \mathbb{R}^n \rightarrow T_x(M)$ such that $ue_i = X_i$ for $i = 1, \ldots, n$. The action of $GL(n; \mathbb{R})$ on $L(M)$ can be accordingly interpreted as follows. Consider $a = (a_t^i) \in GL(n; \mathbb{R})$ as a linear transformation of $\mathbb{R}^n$ which maps $e_j$ into $\sum_i a_j^i e_i$. Then $ua: \mathbb{R}^n \rightarrow T_x(M)$ is the composite of the following two mappings:

$$\mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_x(M).$$

**Example 5.3.** Tangent bundle: Let $GL(n; \mathbb{R})$ act on $\mathbb{R}^n$ as above. The tangent bundle $TM$ over $M$ is the bundle associated with $L(M)$ with standard fibre $\mathbb{R}$, It can be easily shown that the fibre of $TM$ over $x \in M$ may be considered as $T_x(M)$.

**Example 5.4.** Tensor bundles: Let $T^*_x(M)$ be the tensor space of type $(Y, s)$ over the vector space $\mathbb{R}^n$ as defined in §2. The group $GL(n; \mathbb{R})$ can be regarded as a group of linear transformations of the space $T^*_x(M)$ by Proposition 2.12. With this standard fibre $T^*_x(M)$, we obtain the tensor bundle $T^*_x(M)$ over $M$ which is associated with $L(M)$. It is easy to see that the fibre of $T^*_x(M)$ over $x \in M$ may be considered as the tensor space $T^*_x(M)$ of type $(s, s)$.

Returning to the general case, let $P(M, G)$ be a principal fibre bundle and $H$ a closed subgroup of $G$. In a natural way, $G$ acts on the quotient space $G/H$ on the left. Let $E(M, G/H, \mathcal{P}),$ be the associated bundle with standard fibre $G/H$. On the other hand, being a subgroup of $G$, $H$ acts on $E$ on the right. Let $P/E$ be the quotient space of $P$ by this action of $H$. Then we have

**Proposition 5.5:** The bundle $E = P \times G/H$ of $E$ with standard fibre $G/H$ can be identified with $P/E$ of $P$ as follows. An element of $E$ represented by $(u, a, y) \in P \times G/H$ is mapped into the element of $P/E$ represented by $(u \times a, y)$, where $a \in G$ and $y$ is the origin of $G/H$, i.e., the coset $H$.

Consequently, $P(E, H)$ is a principal fibre bundle over the base $E = P/E$ with structure group $H$. The projection $\pi: E \rightarrow P$ maps $u \times a \in P$ into $u a \in E$, where $u$ is considered as a mapping of the standard fibre $G/H$ into a fibre of $E$.

Proof. The proof is straightforward, except the local triviality of the bundle $P(E, H)$: This follows from local triviality of $E = P \times G/H$ and $G/H$ as follows. Let $U$ be an open set of $M$ such that $\pi^{-1}(U) = U \times G/H$ and let $V$ be an open set of $G/H$ such that $p^{-1}(V) = V \times H$, where $p: G \rightarrow G/H$ is the projection. Let $W$ be the open set of $p^{-1}(U) \subset E$ which corresponds to $U \times V$ under the identification $\pi^{-1}(U) = U \times G/H$. If $\mu: P \rightarrow E = P/E$ is the projection, then $\mu^{-1}(W) \approx W \times H$.

QED.

A cross section of a bundle $E(M, F, G, \mathcal{P})$ is a mapping $\sigma: M \rightarrow E$ such that $\pi_E \circ \sigma$ is the identity transformation of $M$. For $P(M, G)$ itself, a cross section $\sigma: M \rightarrow P$ exists if and only if $P$ is the trivial bundle $M \times G$ (cf. Steenrod[1, p. 36]). More generally, we have

**Proposition 5.6:** The structure group $G$ of $P(M, G)$ is reducible to a closed subgroup $H$ if and only if the associated bundle $E(M, G/H, \mathcal{P})$ admits a cross section $\sigma: M \rightarrow E = P/E$.

Proof. Suppose $G$ is reducible to a closed subgroup $H$ and let $Q(M, H)$ be a reduced bundle with injection $f: Q \rightarrow P$. Let $\mu: P \rightarrow E = P/H$ be the projection. If $u$ and $v$ are in the same fibre of $Q$, then $v = uu'$ for some $u \in H$ and hence $\mu(f(u)) = \mu(f(u)) = \mu(f(u'))$. This means that $\mu \circ f$ is constant on each fibre of $Q$ and induces a mapping $\sigma: M \rightarrow E, \sigma(x) = \mu(f(u))$.

Returning to the general case, let $P(M, G)$ be a principal fibre bundle and $H$ a closed subgroup of $G$. In a natural way, $G$ acts on the quotient space $G/H$ on the left. Let $E(M, G/H, \mathcal{P}),$ be the associated bundle with standard fibre $G/H$. On the other hand, being a subgroup of $G$, $H$ acts on $E$ on the right. Let $P/E$ be the quotient space of $P$ by this action of $H$. Then we have
where \( x = \pi(f(u)) \). It is clear that \( \sigma \) is a section of \( E \). Conversely, given a cross section \( \sigma : M \to E \), let \( Q \) be the set of points, \( u \in P \) such that \( \mu(u) = \sigma(\pi(u)) \). In other words, \( Q \) is the inverse image of \( \alpha(M) \) by the projection \( \mu : P \to E = P/H \). For every \( x \in M \), there is \( u \in Q \) such that \( \pi(u) = x \) because \( \mu^{-1}(\sigma(x)) \) is non-empty. Given \( u \in Q \) in the same fiber of \( P \), if \( u \in Q \) then \( x = \pi(u) \) where only when \( v = ua \) for some \( a \in H \). This follows from the fact that \( p(u) = \mu(v) \) if and only if \( v = ua \) for some \( a \in H \). It is now easy to verify that \( Q \) is a closed submanifold of \( P \) and that \( Q \) is a principal fibration bundle \( Q(M, H) \) imbedded in \( P(M, G) \). QED.

Remark. The correspondence between the sections \( \sigma : M \to E = P/H \) and the submanifolds \( Q \) is 1-1.

We shall now consider the question of extending a cross section defined on a subset of the base manifold. A mapping of a subset \( A \) of a manifold \( M \) into another manifold is called \textit{differentiable on} \( A \) if for each point \( x \in A \), there is a differentiable mapping \( f \) of an open neighborhood \( U_x \) of \( x \) in \( M \) such that \( f_x = f \) on \( U_x \cap A \). If \( f \) is the restriction of a differentiable mapping \( f \) of an open set \( W \) containing \( A \) into \( M' \), then \( f \) is clearly differentiable on \( A \). Given a fiber bundle \( E(M, F, G, P) \) and a subset \( A \) of \( M \), by a cross section on \( A \) we mean a differentiable mapping \( \sigma \) of \( A \) into \( E \) such that \( \pi \circ \sigma \) is the identity transformation of \( A \).

**Theorem 5.7.** Let \( E(M, F, G, P) \) be a fiber bundle such that the base manifold \( M \) is paracompact and the fiber \( F \) is diffeomorphic with a Euclidean space \( \mathbb{R}^n \). Let \( A \) be a closed subset (possibly empty) of \( M \). Then every cross section \( \sigma : A \to E \) defined on \( A \) can be extended to a cross section defined on \( M \). In the particular case where \( A \) is empty, there exists a cross section of \( E \) defined on \( M \).

Proof. By the very definition of a paracompact space, every open covering of \( M \) has a locally finite open refinement. Since \( M \) is normal, every locally finite open covering \( \{U_i\} \) of \( M \) has an open refinement \( \{V_i\} \) such that \( V_i \subset U_i \) for all \( i \) (see Appendix 3).

**Lemma 1.** A differentiable function defined on a closed set of \( \mathbb{R}^n \) can be extended to a differentiable function on \( \mathbb{R}^n \) (cf. Appendix 3).

**Lemma 2.** Every point of \( M \) has a neighborhood \( U \) such that every section of \( E \) defined on a closed subset contained in \( U \) can be extended to \( U \).

Proof. Given a point of \( M \), it suffices to take a coordinate neighborhood \( U \) such that \( \pi^{-1}(U) \) is trivial: \( \pi^{-1}(U) \approx U \times F \). Since \( F \) is diffeomorphic with \( \mathbb{R}^m \), a section on \( U \) can be identified with a set of \( m \) functions \( f_1, \ldots, f_m \) defined on \( U \). By Lemma 1, these functions can be extended to \( U \).

Using Lemma 2, we shall prove Theorem 5.7. Let \( \{U_i\}_{i \in I} \) be a locally finite open covering of \( M \) such that each \( U_i \) has the property stated in Lemma 2. Let \( \{V_i\} \) be an open refinement of \( \{U_i\} \) such that \( V_i \subset U_i \) for all \( i \in I \). For each subset \( J \) of the index set \( I \), set \( S_J = \bigcup_{i \in J} V_i \). Let \( T \) be the set of pairs \((\tau, J)\) where \( J \subset I \) and \( \tau \) is a section of \( E \) defined on \( S_J \) such that \( \tau \circ \sigma \) is a diffeomorphism of \( A \) such that \( \tau \circ \sigma \) extends \( \sigma \) on \( A \). Then \( T \) is non-empty; take \( \tau_i \) which extends \( \sigma_i \) on \( A \). On the closed set \( \bigcup_{i \in J} V_i \) contained in \( U_i \), we have a well defined section \( \sigma_i \); \( \sigma_i \circ \sigma_j \) on \( A \cap V_i \cap V_j \) and \( \sigma_i = \sigma_j \) on \( S_J \cap V_i \cap V_j \). Extend \( \sigma_i \) to a section \( \tau_i \) on \( V_i \), which is possible by the property possessed by \( U_i \). Introduce an order in \( T \) as follows: \( (\tau', J') < (\tau, J) \) if \( J' \subset J \) and \( \tau' = \tau |_{S'} \). Let \( (\tau, J) \) be a maximal element (by using Zorn's Lemma). Assume \( J \neq I \) and let \( i \in I \setminus J \). On the closed set \( \bigcup_{i \in J} V_i \) contained in \( U_i \), we have a well defined section \( \sigma_i ; \sigma_i \circ \sigma_j \) on \( A \cap V_i \cap V_j \) and \( \sigma_i |_{S_J} = \sigma_j |_{S_J} \). Extend \( \sigma_i \) to a section \( \tau_i \) on \( V_i \), which is possible by the property possessed by \( U_i \). Let \( J' = J \cup \{i\} \) and \( \tau' = \tau |_{S_J} \) and \( \tau' = \tau |_{S_J} \). Then \( (\tau, J) < (\tau', J') \), which contradicts the maximality of \( (\tau, J) \). Hence, \( I = J \) and \( \tau \) is the desired section.

The proof given here was taken from Godement [1, p. 151].

**Example 5.5.** Let \( L(M) \) be the bundle of linear frames over an \( n \)-dimensional manifold \( M \). The homogeneous space \( GL(n; \mathbb{R})/ \mathbb{R}^n \) is known to be diffeomorphic with a Euclidean space of dimension \( n(n + 1) \) by an argument similar to Chevalley [1, p. 16]. The fibre bundle \( E = L(M)/O(n) \) with fibre \( GL(n; \mathbb{R})/O(n) \), associated with \( L(M) \), admits a cross section if \( M \) is paracompact (by Theorem 5.7). By Proposition 5.6, we see that the structure group of \( L(M) \) can be reduced to the orthogonal group \( O(n) \), provided that \( M \) is paracompact.

**Example 5.6.** More generally, let \( P(M, G) \) be a principal fibre bundle over a paracompact manifold \( M \) with group \( G \) which is a connected group. It is known that \( G \) is diffeomorphic with a direct product of any of its maximal compact subgroups \( H \) and a Euclidean space (cf. Iwasawa [1]). By the same reasoning as above, the structure group \( G \) can be reduced to \( H \).
Example 5.7. Let \( L(M) \) be the bundle of linear frames over a manifold \( A4 \) of dimension \( n \). Let \((\cdot, \cdot)\) be the natural inner product in \( \mathbb{R}^n \) for which \( e_i = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1) \) are orthonormal and which is invariant by \( O(n) \) by the very definition of \( O(n) \). We shall show that each reduction of the structure group \( GL(n; \mathbb{R}) \) to \( O(n) \) gives rise to a Riemannian metric \( g \) on \( M \).

Let \( Q(N, O(n)) \) be a reduced subbundle of \( L(M) \). When \( u \in L(M) \) as a linear isomorphism of \( \mathbb{R}^n \) onto \( T_x(M) \) where \( x = \pi(u) \); each \( u \in O(n) \) defines an inner-product \( g \) in \( T_x(M) \) by

\[
g(X, Y) = (u^{-1}X, u^{-1}Y) \quad \text{for } X, Y \in T_x(M).
\]

The invariance of \((\cdot, \cdot)\) by \( O(n) \) implies that \( g(X, Y) \) is independent of the choice of \( u \in O(n) \). Conversely, if \( M \) is given a Riemannian metric \( g \), let \( Q \) be the subset of \( L(M) \) consisting of linear frames \( u = (X_1, \ldots, X_n) \) which are orthonormal with respect to \( g \). If we regard \( u \in L(M) \) as a linear isomorphism of \( \mathbb{R}^n \) onto \( T_x(M) \), then \( u \) belongs to \( Q \) if and only if \( (\xi, \xi') = g(\xi, \xi') \) for all \( \xi, \xi' \in \mathbb{R}^n \).

It is easy to verify that \( Q \) forms a reduced subbundle of \( L(M) \) over \( M \) with structure group \( O(n) \). The bundle \( Q \) will be called the orthonormal frame. The result here combined with Example 5.5 implies that every paracompact manifold \( M \) admits a Riemannian metric. We shall see later that every Riemannian manifold is a metric space and hence paracompact.

To introduce the notion of induced bundle, we prove

**Proposition 5.8.** Given a principal fibre bundle \( P(M, G) \) and a mapping \( f \) of a manifold \( N \) into \( M \), there is a unique (of course, unique up to an isomorphism) principal fibre bundle \( Q(N, G) \) with a homomorphism \( f: Q \rightarrow P \) which induces \( f: N \rightarrow M \) and which corresponds to the identity automorphism of \( G \).

The bundle \( Q(N, G) \) is called the bundle induced by \( f \) from \( P(M, G) \) or simply the induced bundle; it is sometimes denoted by \( f^{-1}P \).

Proof. In the direct product \( N \times P \), consider the subset \( Q \) consisting of \((y, u) \in N \times P \) such that \( f(y) = \pi(u) \). The group \( G \) acts on \( Q \) by \((y, u) \rightarrow (y, u a) = (y, u a) \) for \((y, u) \in Q \) and \( a \in G \).

It is easy to see that \( G \) acts freely on \( Q \) and that \( Q \) is a principal fibre bundle over \( N \) with group \( G \) and \( \pi_Q \) given by \( \pi_Q(y, u) = y \). Let \( Q' \) be another principal fibre bundle over \( N \) with group \( G \) and \( f': Q' \rightarrow P \) a homomorphism which induces \( f: N \rightarrow M \) and which corresponds to, the identity automorphism of \( G \). Then it is easy to show that the mapping of \( Q' \) onto \( Q \) defined by \( u' \rightarrow (\pi_Q(u'), f'(u')) \), \( u' \in Q' \), is an isomorphism of the bundle \( Q' \) onto \( Q \) which induces the identity transformation of \( N \) and which corresponds to the identity automorphism of \( G \).

QED.

We recall here some results on covering spaces which will be used later. Given a connected, locally arcwise connected topological space \( M \), a connected space \( E \) is called a covering space over \( M \) with projection \( p: E \rightarrow M \) if every point \( x \) of \( M \) has a connected open neighborhood. \( U \) such that each connected component of \( p^{-1}(U) \) is open in \( E \) and is mapped homeomorphically onto \( U \) by \( p \). Two covering spaces \( p: E \rightarrow M \) and \( p': E' \rightarrow M \) are isomorphic if there exists a homeomorphism \( f: E \rightarrow E' \) such that \( p' \circ f = p \). A covering space \( p: E \rightarrow M \) is a universal covering space if \( E \) is simply connected. If \( M \) is a manifold, every covering space has a (unique) structure of manifold such that \( p \) is differentiable. From now on we shall only consider, covering manifolds.

**Proposition 5.9.** (1) Given a connected manifold \( M \), there is a unique (unique up to an isomorphism) universal covering manifold, which will be denoted by \( \tilde{M} \).

(2) The universal covering manifold \( \tilde{M} \) is a principal fibre bundle over \( M \) with group \( \pi_1(M) \) and projection \( p: \tilde{M} \rightarrow M \), where \( \pi_1(M) \) is the first homotopy group of \( M \).

(3) The isomorphism classes of the covering spaces over \( M \) are in a 1:1 correspondence with the conjugate classes of the subgroups of \( \pi_1(M) \). The correspondence is given as follows. To each subgroup \( H \) of \( \pi_1(M) \), we associate \( E = \tilde{M}/H \). Then the covering manifold \( E \) corresponding to \( H \) is a fibre bundle over \( M \) with fibre \( \pi_1(M)/H \) associated with the principal fibre bundle \( M(\tilde{M}, \pi_1(M)) \). If \( H \) is a normal subgroup of \( \pi_1(M) \), \( E = \tilde{M}/H \) is a principal fibre bundle with group \( \pi_1(M)/H \) and is called a regular covering manifold of \( M \).

For the proof, see Steenrod [1, pp. 67-77] or Hu [1, pp. 89-97].

The action of \( \pi_1(M)/H \) on a regular covering manifold \( E = \tilde{M}/H \) is properly discontinuous. Conversely, if \( E \) is a connected manifold and \( G \) is a properly discontinuous group of transformations acting freely on \( E \), then \( E \) is a regular covering manifold of
Example 5.8. Consider \( \mathbb{R}^n \) as an \( n \)-dimensional vector space, and let \( \xi_1, \ldots, \xi_n \) be any basis of \( \mathbb{R}^n \). Let \( G \) be the subgroup of \( \mathbb{R}^n \) generated by \( \xi_1, \ldots, \xi_n \); \( G = \{ \Sigma m_i \xi_i : m_i \text{ integers} \} \). The action of \( G \) on \( \mathbb{R}^n \) is properly discontinuous and \( \mathbb{R}^n \) is the universal covering manifold of \( \mathbb{R}^n/G \). The quotient manifold \( \mathbb{R}^n/G \) is called an \( n \)-dimensional manifold.

Example 5.9. Let \( S^n \) be the unit sphere in \( \mathbb{R}^{n+1} \) with center at the origin: \( S^n = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \Sigma (x^i)^2 = 1\} \). Let \( G \) be the group consisting of the identity transformation of \( S^n \) and the transformation of \( S^n \) which maps \((x^1, \ldots, x^{n+1})\) into \((-x^1, \ldots, -x^{n+1})\). Then \( S^n \) is the universal covering manifold of \( \mathbb{R}^{n+1}/G \). The quotient manifold \( \mathbb{R}^{n+1}/G \) is called the \( n \)-dimensional real projective space.

CHAPTER II

Theory of Connections

1. Connections in a principal fibre bundle

Let \( P(M, G) \) be a principal fibre bundle over a manifold \( M \) with group \( G \). For each \( u \in P \), let \( T_u(P) \) be the tangent space of \( P \) at \( u \) and \( G_u \) the subspace of \( T_u(P) \) consisting of vectors tangent to the fibre through \( u \). A connection \( \Gamma \) in \( P \) is an assignment of a subspace \( Q_u \) of \( T_u(P) \) to each \( u \in P \) such that

(a) \( T_u(P) = G_u \oplus Q_u \) (direct sum);
(b) \( Q_{ua} = (R_u) Q_u \) for every \( u \in P \) and \( a \in G \), where \( R_u \) is the transformation of \( P \) induced by \( a \in G \); \( R \cdot u = u \); and
(c) \( Q_u \) depends differentiably on \( u \).

Condition (b) means that the distribution \( u \rightarrow Q_u \) is invariant by \( G \). We call \( G_u \) the vertical subspace and \( Q_u \) the horizontal subspace of \( T_u(P) \). A vector \( X \in T_u(P) \) is called vertical (resp. horizontal) if it lies in \( G_u \) (resp. \( Q_u \)). By (a), every vector \( X \in T_u(P) \) can be uniquely written as

\[
X = Y + Z \quad \text{where} \quad Y \in G_u \quad \text{and} \quad Z \in Q_u.
\]

We call \( Y \) (resp. \( Z \)) the vertical (resp. horizontal) component of \( X \) and denote it by \( \pi X \) (resp. \( hX \)). Condition (c) means, by definition, that if \( \pi X \) (resp. \( hX \)) is a differentiable vector field on \( P \) so are \( \pi X \) and \( hX \). (It can be easily verified that this is equivalent to saying that the distribution \( u \rightarrow Q_u \) is differentiable.)

Given a connection \( \Gamma \) in \( P \), we define, a 1-form \( \omega \) on \( P \) with values in the Lie algebra \( \mathfrak{g} \) of \( G \) as follows. In \( S5 \) of Chap. I, we showed that every \( a \in \mathfrak{g} \) induces a vector field \( A^* \) on \( P \), called the fundamental vector field corresponding to \( A \), and that \( A \rightarrow (A^*)_u \) is a linear isomorphism of \( \mathfrak{g} \) onto \( G_u \) for each \( u \in P \). For each \( X \in T_u(P) \), we define, \( \omega(X) \) to be the unique \( A \in \mathfrak{g} \) such that
The connection form $\omega$ of a connection satisfies the following conditions:

(a') $\delta_+(A^*) = \omega(A^*)$ for every $A \in g$;

(b') $\omega((R_\alpha)_* X) = \text{ad} \, (a^{-1}) \cdot \omega(X)$ for every $A \in G$ and every vector field $X$ on $P$, where $\text{ad}$ denotes the adjoint representation of $G$ in $\mathfrak{g}$.

Conversely, given a $G$-valued 1-form $w$ on $P$ satisfying conditions (a') and (b'), there is a unique connection $\Gamma$ in $P$ whose connection form is $w$.

Proof. Let $w$ be the connection form of a connection. The connection form $\omega$ of a connection satisfies the condition (a') for a connection. Thus, both $\omega((R_\alpha)_* X)$ and $\text{ad} \, (a^{-1}) \cdot \omega(X)$ vanish: In the case when $X$ is vertical, we may assume $G$ is given, and every horizontal vector field $X$ on $P$ where $\text{ad}$ denotes the adjoint representation of $G$ in $\mathfrak{g}$.

The verification that $\omega$ is invariant by $G$ is clear from the invariance of the horizontal component $Y$ of $X$ and hence is differentiable. The invariance of $X^*$ by $G$ is clear from the invariance of the horizontal subspaces by $G$.

Finally, let $X^*$ be a horizontal vector field on $P$ invariant by $G$. For every $x \in M$, take a point $u \in P$ such that $\pi(u) = x$ and define $X_u = \pi(X_u^*)$. The vector $X_u$ is independent of the choice of $u$ such that $\pi(u) = x$. Since if $u' = u$, then $\pi(X_u^*) = \pi(R_\alpha \cdot X_u^*) = \pi(X_{u'}^*)$. It is obvious that $X_u$ is then the lift of the vector field $X$. QED.

Proposition 1.3. Let $X^*$ and $Y^*$ be the horizontal lifts of $X$ and $Y$ respectively. Then:

1. $X^* + Y^*$ is the horizontal lifted of $X + Y$.
2. For every function $f$ on $M$, $f^* \cdot X^*$ is the horizontal lift of $fX$ where $f^*$ is the function on $P$ defined by $f^* = f \circ \pi$.
3. The horizontal component of $[X^*, Y^*]$ is the horizontal lift of $[X, Y]$.

Proof. The first two assertions are trivial: For the third, we have

$$\pi(h[X^*, Y^*]) = \pi([X^*, Y^*]) = [X, Y].$$

QED.

Let $x_1, \ldots, x^n$ be a local coordinate system in a coordinate neighborhood $U$ in $M$. Let $X^*$ be the horizontal lift in $\pi^{-1}(U)$ of the vector field $x_i = \partial/\partial x_i$ on $U$. For each $i$, then $X_i^*$ form a local basis for the distribution $\mathcal{U} \subset Q_\alpha$ in $\pi^{-1}(U)$.

We shall now express $\pi^{-1}(U)$ as a connection form $w$ on $P$ by a family of forms, each defined in an open subset of the base manifold $M$. Let $\{U_\alpha\}$ be an open covering of $M$ with a family of isomorphisms $\varphi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ and the corresponding family of transition functions $\gamma_\alpha^\beta: U_\alpha \cap U_\beta \rightarrow G$. For each $a$, let $\sigma_a: U_a \rightarrow P$ be the
cross section on $U_x$ defined by $\sigma_a(x) = \psi_{a^{-1}}(x, e)$. $x \in U_x$, where $e$ is the identity of $C$. Let $\theta$ be the (left invariant $g$-valued) canonical 1-form on $G$ defined in §4 of Chapter I of [1].

For each non-empty $U_x \cap U_y$, define a $g$-valued 1-form $\theta^{a\beta}$ on $U_x \cap U_y$ by

$$\theta^{a\beta} = \psi_{a^{-1}} \psi_{\beta^{-1}} \theta.$$

For each $a$, define a $g$-valued 1-form $\omega_a$ on $U_x$ by

$$\omega_a = \sigma_a^* \omega_a.$$

**Proposition 1.4.** The forms $\omega_a$ and $\omega_a$ are subject to the conditions:

$$\omega_a = \text{ad} (\psi_{a^{-1}}) \omega_a + \theta^{a\beta} \quad \text{on } U_x \cap U_y.$$

Conversely, for every family of $g$-valued 1-forms $\{\omega_a\}$ each defined on $U_x$ and satisfying the preceding conditions, there is a unique connection form $\omega$ on $P$ which gives rise to $\{\omega_a\}$ in the described manner.

**Proof.** If $U_x \cap U_y$ is non-empty, $\sigma_a(x) = \sigma_a(x) \psi_{a\beta}(x)$ for all $x \in U_x \cap U_y$. Denote the differentials of $\sigma_a(x)$ and $\psi_{a\beta}(x)$ by the same letters. Then for every vector $X \in T_a(U_x \cap U_y)$, the vector $\sigma_a(X) = T_a(P) \psi_{a\beta}(X)$ is the image of $(u, X), \psi_{a\beta}(X)$, where $u = \sigma_a(x)$ and $a = \psi_{a\beta}(x)$, under the mapping $P \times G \rightarrow P$. By Proposition 1.4 (Leibniz's formula) of Chapter I, we have

$$\sigma_b(X) = \sigma_a(X) \psi_{a\beta}(X) + \sigma_b(x) \psi_{b\beta}(X),$$

where $\sigma_a(x) \psi_{a\beta}(X)$ means $R_a(\psi_{a\beta}(X))$ and $\sigma_b(x) \psi_{b\beta}(X)$ is the image of $\psi_{a\beta}(X)$ by the differential of $\sigma_a(x)$, $\sigma_b(x)$ being considered as a mapping of $G$ into $P$ which maps $b \in G$ into $\sigma_a(x) \beta$. Taking the values of $\omega$ on both sides of the equality, we obtain

$$\omega_b(X) = \text{ad} (\psi_{a^{-1}}) \omega_a(X) + \theta^{a\beta}(X).$$

Indeed, if $a \in G$ is the left invariant vector field on $G$ which is equal to $\psi_a(X)$ at $a = \psi_a(x)$ so that $\theta(\psi_a(X)) = A$, then $\sigma_a(x) \psi_{a\beta}(X)$ is the value of the fundamental vector field $A^\ast$ at $u = \sigma_a(x) \psi_{a\beta}(x)$ and, hence $\omega(\sigma_a(x) \psi_{a\beta}(x)) = A$.

The converse can be verified by following back the process of obtaining $\{\omega_a\}$ from $\omega$.

QED.

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### 2. Existence and extension of connections

Let $P(M, G)$ be a principal fibre bundle and $A$ a subset of $M$.

We say that a connection is defined; over $A$ if, at every point $u \in A$ with $\pi(u) \in A$, a subspace $Q_u$ of $T_u(P)$ is given in such a way that conditions (a) and (b) for connection (see §1) are satisfied and $Q_u$ depends differentiably on $u$ in the following sense. For every point $x \in A$, there exist an open neighborhood $U$ and a connection in $P|U = \pi^{-1}(U)$ such that the horizontal subspace at every $u \in \pi^{-1}(A)$ is the given space $Q_u$.

**Theorem 2.1.** Let $P(M, G)$ be a principal fibre bundle and $A$ a closed subset of $M$ (A may be empty). If $M$ is paracompact, every connection defined over $A$ can be extended to a connection in $P$. In particular, $P$ admits a connection if $M$ is paracompact.

**Proof.** The proof is a replica of that of Theorem 5.7 in Chapter I.

**Lemma 1.** A differentiable function defined on a closed subset of $\mathbb{R}^n$ can be always extended to a differentiable function on $\mathbb{R}^n$ (cf. Appendix 3).

**Lemma 2.** Every point of $M$ has a neighborhood $U$ such that every connection defined on a closed subset contained in $U$ can be extended to a connection defined over $U$.

**Proof.** Given a point of $M$, it suffices to take a coordinate neighborhood $U$ such that $\pi^{-1}(U)$ is trivial: $\pi^{-1}(U) \approx U \times G$.

On the trivial bundle $U \times G$, a connection form $\omega$ is completely determined by its behavior at the points of $U \times \{e\} (e$ the identity of $G$) because of the property $R^\omega_\theta(\omega) = \text{ad}(\omega) \omega$. Furthermore, if $\sigma: U \rightarrow U \times G$ is the natural cross section, that is, $\sigma(x) = (x, e)$ for $x \in U$, then $\omega$ is completely determined by the $g$-valued 1-form $\sigma^\ast \omega$ on $U$. Indeed, every vector $X \in T_{\pi(x)}(U \times G)$ can be written uniquely in the form

$$X = Y + Z,$$

where $Y$ is tangent to $U \times \{e\}$ and $Z$ is vertical so that $Z = \sigma^\ast(\pi^\ast X)$. Hence we have

$$\omega(X) = \omega(\sigma^\ast(\pi^\ast X)) + \omega(Z) = (\sigma^\ast \omega)(\pi^\ast X) + A,$$

where $A$ is a unique element of $g$ such that the corresponding fundamental vector field $\sigma^\ast A$ is equal to $Z$ at $\sigma(x)$. Since $A$ depends
only on \(Z\), not on the connection, \(\omega\) is completely determined by \(\sigma^*\omega\). The equation above shows that, conversely, every \(g\)-valued 1-form on \(U\) determines uniquely a connection form on \(U \times G\). Thus Lemma 2 is reduced to the extension problem for \(g\)-valued 1-forms on \(U\). If \(\{A_i\}\) is a basis for \(g\), then \(\omega = \Sigma A_i \omega^i\), where each \(\omega^i\) is a usual 1-form. Thus it is sufficient to consider the extension problem of usual 1-forms on \(U\). Let \(x_1, \ldots, x^n\) be a local coordinate system in \(U\). Then every 1-form in \(U\) is of the form \(\sum f_i \, dx^i\) where each \(f_i\) is a function on \(U\). Thus our problem is reduced to the extension problem of functions on \(U\). Lemma 2 now follows from Lemma 1.

By means of Lemma 2, Theorem 2.1 can be proved exactly in the same way as Theorem 5.7 of Chapter I. Let \(\{U_i\}_{i \in I}\) be a locally finite open covering of \(M\) such that each \(U_i\) has the property stated in Lemma 2. Let \(\{V_i\}\) be an open refinement of \(\{U_i\}\) such that \(\overline{V_i} \subset U_i\). For each subset \(J\) of \(I\), let \(S_J = \bigcup_{i \in J} V_i\).

Let \(T\) be the set of pairs \((\tau, J)\) where \(J \subset I\) and \(\tau\) is a connection over \(A \cap S_J\). Introduce an order in \(T\) as follows: \((\tau', J') < (\tau, J)\) if \(J' \subset J\) and \(\tau' = \tau\) on \(S_J\). Let \((\tau, J)\) be a maximal element of \(T\). Then \(J = I\) as in the proof of Theorem 5.7 of Chapter I and \(\tau\) is a desired connection.

Remark. It is possible to prove Theorem 2.1 using Lemma 2 and a partition of unity \(\{f_i\}\) subordinate to \(\{V_i\}\) (cf. Appendix 3).

Let \(\omega\) be a connection form on \(\pi^{-1}(U_i)\) which extends the given connection over \(A \cap V_i\). Then \(\omega = \Sigma g_i \omega_i\) is a desired connection form on \(P\), where each \(g_i\) is the function on \(P\) defined by \(g_i = f_i \circ \pi\).

3. Parallelism

Given a connection \(\Gamma\) in a principal fibre bundle \(P(M, G)\), we shall define the concept of parallel displacement offrhelons along any given curve \(\tau\) in the base manifold \(M\).

Let \(\tau = x_t\), \(a \leq t \leq b\), be a piecewise differentiable curve of class \(C^1\) in \(M\). A horizontal lift of \(\tau\) is a horizontal curve \(\tau^* = u_t\), \(a \leq t \leq b\), in \(P\) such that \(\pi(u_t) = x_t\), for \(a \leq t \leq b\).

Here a horizontal curve in \(P\) means a piecewise differentiable curve of class \(C^1\) whose tangent vectors are all horizontal.

The notion of lift of a curve corresponds to the notion of lift of a vector field. Indeed, if \(X^*\) is the lift of vector field \(X\) on \(M\), then the integral curve of \(X^*\) through a point \(u_0 \in P\) is a lift of the integral curve of \(X\) through the point \(x_0 = \pi(u_0) \in M\). We now prove

**Proposition 3.1.** Let \(\tau = x_t\), \(0 \leq t \leq 1\), be a curve of class \(C^1\) in \(M\). For an arbitrary point \(u_0\) of \(P\) with \(\pi(u_0) = x_0\), there exists an unique lift \(\tau^* = u_t\) of \(\tau\) which starts from \(u_0\).

Proof. By local triviality of the bundle, there is a curve \(v_t\) of class \(C^1\) in \(P\) such that \(v_0 = u_0\) and \(\pi(v_t) = x_t\), for \(0 \leq t \leq 1\). A lift of \(\tau\) is unique, must be of the form \(u_t = v_{t_0} a_t\), where \(a_t\) is a curve in the structure group of \(G\) such that \(a_0 = e\). We shall now look for a curve \(a\), in \(G\) which makes \(u_t = v_{t_0} a_t\), a horizontal curve. Just as in the proof of Proposition 1.4, we apply Leibniz’s formula (Proposition 1.4 of Chapter I) to the mapping \(P \times G \to P\) which maps \((v, a)\) into \(va\) and obtain

\[
u_t = v_{t_0} a_t + v_{t_0} a_t^t a_t^t,
\]

where each dotted italic letter denotes the tangent vector at that point (e.g., \(\dot{u}_t\) is the vector tangent to the curve \(\tau^* = u_t\) at the point \(u_t\)). Let \(\omega\) be the connection form of \(\Gamma\). Then, as in the proof of Proposition 1.4, we have

\[
\omega(u_t) = ad(a_t^{-1})\omega(v_{t_0}) = a_t^{-1} d_t,
\]

where \(a_t^{-1} d_t\) is now a curve in the Lie algebra \(g = T_e(G)\) of \(G\). The curve \(u_t\) is horizontal if and only if \(a_t a_t^{-1} = -\omega(v_{t_0})\) for every \(t\).

The construction of \(u_t\) is thus reduced to the following

**Lemma.** Let \(G\) be a Lie group and \(g\) its Lie algebra identified with \(\mathfrak{t}(G)\). Let \(Y_t\), \(0 \leq t \leq 1\), be a continuous curve in \(\mathfrak{t}(G)\). Then there exists in \(G\) a unique curve \(a_t\), of class \(C^1\) such that \(a_0 = e\) and \(a_t a_t^{-1} = Y_t\) for \(0 \leq t \leq 1\).

Remark. In the case where \(Y_t = A\) for all \(t\), the curve \(a_t\) is nothing but the 1-parameter subgroup of \(G\) generated by \(A\). Our differential equation \(da_t^{-1} = Y_t\) is hence a generalization of the differential equation for 1-parameter subgroups.

Proof of Lemma. We may assume that \(Y_t\) is defined and continuous for all \(t,-\infty < t < \infty\). We define a vector field \(X\) on
The value of $X$ at $(a, t) \in G \times R$ is, by definition, equal to $(Ya, (d/dz)_e)^T(T_a(G) \times T_t(R))$, where $z$ is the natural coordinate system in $R$. It is clear that the integral curve of $X$ starting from $(e, 0)$ is of the form $(a_0(t), t)$ and $a_0$ is the desired curve in $G$. The only thing we have to verify is that $a_0$ is defined for all $t$, $0 \leq t \leq 1$. Let $a_0'$ be a local 1-parameter group of local transformations of $G \times R$ generated by $X$. For each $(a, t) \in G \times R$, there is a positive number $\delta$, such that $\varphi_t(e, \gamma e)$ is defined for $|\gamma - e| < \delta$, and $|d| < \theta$ (Proposition 1.5 of Chapter 1). Since the subset $(a, t) \in [0, 1]$ of $G \times R$ is compact, we may choose $\delta > 0$ such that, for each $t \in [0, 1]$, $\varphi_t(e, e)$ is defined for $|\gamma - e| < \delta$ (cf. Proof of Proposition 1.6 of Chapter 1). Choose $s_0, s_1, \ldots, s_k$ such that $0 = s_0 < s_1 < \ldots < s_k = 1$ and $s_i - s_{i-1} < \delta$ for every $i$. Then $\varphi_t(e, s_i) = (a_{i}, t)$ is defined for $0 \leq t \leq s_i < s_i - \delta$, where $a_{i+1} = Y_{s_{i+1}}a_{i}$, and we define $a_{i+1} = Y_{s_{i+1}}a_{i}$ for $s_i \leq t \leq s_{i+1}$, ..., $\varphi_t(e, s_{k-2}) = (e, s_{k-2} + \theta) = (e, s_{k-1} + \delta)$ for $0 \leq t \leq s_k - s_{k-1}$, where $a_{k-1} = Y_{s_k}a_{k-1}$, and we define $a_{k-1} = Y_{s_k}a_{k-1}$, thus completing the construction of $a_0$, $0 \leq t \leq 1$.

Now using Proposition 3.1, we define the parallel displacement of fibres as follows. Let $x = x_0$, $0 \leq t \leq 1$, be a differentiable curve of class $C^1$ on $M$. Let $u_0$ be an arbitrary point of $P$ with $\pi(u_0) = x_0$. The unique lift $\tau$ of $\tau$ through $u_0$ has the end point $u_1$ such that $\pi(u_1) = x_1$. By varying $u_0$ in the fibre $\pi^{-1}(x_0)$, we obtain a mapping of the fibre $\pi^{-1}(x_0)$ onto the fibre $\pi^{-1}(x_1)$, which maps $u_0$ into $u_1$. We denote this mapping by the same letter $\tau$ and call it the parallel displacement along the curve $\tau$. The fact that $\tau: \pi^{-1}(x_0) \to \pi^{-1}(x_1)$ is actually an isomorphism comes from the following.

**Proposition 3.2.** The parallel displacement along any curve $\tau$ commutes with the action of $G$ on $R$, $\tau^*r = R^* \tau$ for every $a \in G$.

**Proof.** This follows from the fact that every horizontal curve is mapped into a horizontal curve by $R_e$. QED.

The parallel displacement along any piecewise differentiable curve of class $C^1$ can be defined in an obvious manner. It should be remarked that the parallel displacement along a curve $\tau$ is independent of a specific parametrization $x_t$, $0 \leq t \leq b$, and $x_0$, $0 \leq s \leq b$, in $M$. The parallel displacement along $x_t$ and the one along $\gamma$, coincide if there is a homeomorphism $\varphi$ of the interval $[0, b]$ onto $[0, c]$ such that: (1) $\varphi(a) = c$ and $\varphi(b) = d$, (2) both $\varphi$ and $\varphi^{-1}$ are differentiable of class $C^1$ except at a finite number of parameter values, and (3) $y_{s(\gamma)} = x_t$, for all $s, t \leq b$.

If $\tau$ is the curve $x_t$, $0 \leq t \leq b$, we denote by $\tau^*$ the curve $y_t$, $0 \leq t \leq b$, defined by $y_t = x_{t+\tau}$. The following proposition is evident.

**Proposition 3.3.** (a) If $\tau$ is a piecewise differentiable curve of class $C^1$ on $M$, then the parallel displacement along $\tau^{-1}$ is the inverse of the parallel displacement along $\tau$.

(b) If $\tau$ is a curve from $x$ to $y$ on $M$ and $\mu$ is a curve from $y$ to $z$ in $M$, the parallel displacement along the composite curve $\mu \circ \tau$ is the composite of the parallel displacements $\tau$ and $\mu$.

### 4. Holonomy groups

Using the notion of parallel displacement, we now define the holonomy group of a given connection $\Gamma$ in a principal fibre bundle $P(M, G)$. For the sake of simplicity we shall mean by a curve a piecewise differentiable curve of class $C^1$, $1 \leq k \leq \infty$ ($k$ will be fixed throughout $\S 4$).

For each point $x$ of $M$ we denote by $C(x)$ the loop space at $x$, that is, the set of all closed curves starting and ending at $x$. If $\tau$ and $\mu$ are elements of $C(x)$, the composite curve $\mu \circ \tau$ (followed by $\mu$) is also an element of $C(x)$. As we proved in $\S 3$, for each $x \in C(x)$, the parallel displacement along $\tau$ is an isomorphism of the fibre $\pi^{-1}(x)$ onto itself. The set of all such isomorphisms of $\pi^{-1}(x)$ onto itself forms a group by virtue of Proposition 3.3. This group is called the holonomy group of $\Gamma$ with reference point $x$. Let $C^0(x)$ be the subset of $C(x)$ consisting of loops which are homotopic to zero. The subgroup of the holonomy group consisting of the parallel displacements arising from all $\tau \in C^0(x)$ is called the restricted holonomy group of $\Gamma$ with reference point $x$. The holonomy group and the restricted holonomy group of $\Gamma$ with reference point $x$ will be denoted by $\Phi(x)$ and $\Phi^0(x)$ respectively.
It is convenient to realize these groups as subgroups of the structure group $G$ in the following way. Let $u$ be an arbitrarily fixed point of the fibre $\pi^{-1}(x)$. Each $\tau \in C(x)$ determines an element, say, $a$, of $G$ such that $\tau(u) = ua$. If a loop $\mu \in C(x)$ determines $b \in G$, then the composite $\mu \cdot \tau$ determines $ba$ because $(\mu \cdot \tau)(u) = \mu(ua) = (\mu(u))a = uba$ by virtue of Proposition 3.2. The set of elements $a \in G$ determined by all $\tau \in C(x)$ forms a subgroup of $G$ by Proposition 3.3. This subgroup, denoted by $\Phi(u)$, is called the holonomy group of $G$ with reference point $u \in P$. The restricted holonomy group $\Phi^0(u)$ of $G$ with reference point $u$ can be defined accordingly. Note that $\Phi(x)$ is a group of isomorphisms of the fibre $\pi^{-1}(x)$ onto itself and $\Phi(u)$ is a subgroup of $G$. It is clear that there is a unique isomorphism of $\Phi(x)$ onto $\Phi(u)$ which makes the following diagram commutative:

$$
\Phi(x) \rightarrow \Phi(u).
$$

Another way of defining $\Phi(u)$ is the following: When two points $u$ and $v$ of $P$ can be joined by a horizontal curve, we write $u \sim v$. This is clearly an equivalence relation. Then $\Phi(u)$ is equal to the set of $a \in G$ such that $u \sim uu$. Using the fact that $u \sim v$ implies $ua \sim vu$ for any $u, v \in P$ and $a \in G$, it is easy to verify once more that this subset of $G$ forms a subgroup of $G$.

**Proposition 4.1.** (a) If $v = ua$, $a \in G$, then $\Phi(v) = ad(a^{-1})(\Phi(u))$, that is, the holonomy group $\Phi(v)$ and $\Phi(u)$ are conjugate in $G$. Similarly, $\Phi^0(v) = ad(a^{-1})(\Phi^0(u))$.

(b) If two points $u$ and $v$ of $P$ can be joined by a horizontal curve, then $\Phi(u) = \Phi(v)$ and $\Phi^0(u) = \Phi^0(v)$.

**Proof.** (a) Let $b \in \Phi(u)$ so that $u \sim ub$. Then $ua \sim (ub)a$ so that $v \sim (vu^{-1})ba = vu^{-1}ba$. Thus $ad(a^{-1})(b) \in \Phi(u)$. It follows easily that $\Phi(v) = ad(a^{-1})(O(u))$. The proof for $\Phi^0(v)$ is analogous.

(b) The relation $u \sim v$ implies $ub \sim vb$ for every $b \in G$. Since the relation $\sim$ is transitive, $u \sim ub$ if and only if $v \sim vb$. By the factorization lemma (Appendix 7), $T$ is (equivalent to) a product of small lassos of the form $\tau_{1-1} \cdot \tau_1$, where $\tau_1$ is a piecewise differentiable curve of class $C^1$ from $x = \pi(u)$ to $y$. By the factorization lemma, it follows then that $\Phi(u)$ is a connected Lie subgroup of $G$.

Let $a \in \Phi^0(u)$ be obtained by the parallel displacement along a piecewise differentiable loop $\tau$ of class $C^1$ which is homotopic to $0$. By the factorization lemma, $\tau$ is (equivalent to) a product of small lassos of the form $\tau_{1-1} \cdot \tau_1$, where $\tau_1$ is a piecewise differentiable curve of class $C^1$ from $x = \pi(u)$ to $y$. The factorization lemma states that $\Phi(u)$ is a normal subgroup of $\Phi^0(v)$.

**Theorem 4.2.** Let $P(M, G)$ be a principal fibre bundle whose base manifold $M$ is connected and paracompact. Let $\Phi(u)$ and $\Phi^0(v)$, $u \in P$, be the holonomy group and the restricted holonomy group of a connection $\Gamma$ with reference point $u$. Then

(a) $\Phi^0(u)$ is a connected Lie subgroup of $G$;

(b) $\Phi(u)$ is a normal subgroup of $\Phi^0(u)$ and $\Phi^0(u)/\Phi(u)$ is countable.

By virtue of this theorem, $\Phi(u)$ is a Lie subgroup of $G$ whose identity component is $\Phi^0(u)$.

**Proof.** We shall show that every element of $\Phi^0(u)$ can be obtained by the parallel displacement along a piecewise differentiable curve from class $C^1$ which is homotopic to $0$. By the factorization lemma, $\tau$ is (equivalent to) a product of small lassos of the form $\tau_{1-1} \cdot \tau_1$, where $\tau_1$ is a piecewise differentiable curve of class $C^1$ from $x = \pi(u)$ to $y$. The factorization lemma states that $\Phi(u)$ is a normal subgroup of $\Phi^0(v)$.

Let $a \in \Phi^0(u)$ be obtained by the parallel displacement along a piecewise differentiable curve $\tau$ of class $C^1$ which is homotopic to $0$. By the factorization lemma, $\tau$ is (equivalent to) a product of small lassos of the form $\tau_{1-1} \cdot \tau_1$, where $\tau_1$ is a piecewise differentiable curve of class $C^1$ from $x = \pi(u)$ to $y$. The factorization lemma states that $\Phi(u)$ is a normal subgroup of $\Phi^0(v)$.

Let $x^1, \ldots, x^n$ be a local coordinate system with origin at $y$. Then the element $\Phi(u)$ can be obtained by the parallel displacement along $\tau_1$. It is therefore sufficient to show that the element $a \in \Phi^0(u)$ defined by the loop $\mu$, where $\nu$ is the point obtained by the parallel displacement of $u$ along $\tau_1$, can be sent to the identity element in $\Phi^0(u)$ by a differentiable curve $\gamma$ of $G$ which lies in $\Phi^0(u)$.
and let \( \mu \) be defined by \( x' = x'(t), i = 1, \ldots, n. \) Set \( f'(t, s) = s + (1 - s)x'(t) \) for \( i = 1, \ldots, n \) and \( 0 \leq t, s \leq 1. \) Then \( f(t, s) = (f_1(t, s), \ldots, f_n(t, s)) \) is a differentiable mapping of class \( C^k \) of \( I \times I \) into \( M \) (where \( I = [0, 1] \)) such that \( f(0) \) is the curve \( \mu \) and \( f(1) \) is the trivial curve \( \gamma. \) For each fixed \( s \), let \( b(s) \) be an element of \( \Phi(u) \) obtained from \( \gamma \) so that \( b(0) = b \) and \( b(1) = \text{identity}. \) The fact that \( b(s) \) is of class \( C^k \) in \( s \) as a mapping of \( I \) into \( G \) follows from the following

**Lemma.** Let \( f: I \times I \to M \) be a differentiable mapping of class \( C^k \) and \( u_0(s), 0 \leq s \leq 1, \) a differentiable curve of class \( C^k \) in \( P \) such that \( \pi(u_0(s)) = f_0(s) \). For each fixed \( s \), let \( u_1(s) \) be the point of \( P \) obtained by the parallel displacement of \( u_0(s) \) along the curve \( f(t, s) \), where \( 0 \leq t \leq 1 \) and \( s \) is fixed. Then the curve \( u_1(s), 0 \leq s \leq 1, \) is differentiable of class \( C^k \).

**Proof of Lemma.** Let \( F : I \times I \to P \) be a differentiable mapping of class \( C^k \) such that \( \pi(F(t, s)) = f(t, s) \) for all \( (t, s) \in I \times I \) and that \( F(0, s) = u_0(s) \). The existence of such an \( F \) follows from local triviality of the bundle \( P. \) Set \( \nu_i(s) = F(t, s). \) In the proof of Proposition 3.1, we saw that, for each fixed \( s \), there is a curve \( \alpha(s), 0 \leq t \leq 1, \) in \( G \) such that \( a_0(s) = e \) and that the curve \( \alpha(s) \) is horizontal. Set \( u_1(s) = \nu_i(s). \) To prove that \( u_1(s) \), \( 0 \leq s \leq 1, \) is a differentiable curve of class \( C^k \), it is sufficient to show that \( \nu_i(s), 0 \leq s \leq 1, \) is a differentiable curve of class \( C^k \) in \( G. \) Let \( \omega \) be the connection form of \( P. \) Set \( Y(s) = -\omega(\nu_i(s)), \) where \( \nu_i(s) \) is the vector tangent to the curve described by \( \nu_i(s), 0 \leq t \leq 1, \) when \( s \) is fixed. Then as in the proof of Proposition 3.1, \( a_i(s) \) is a solution of the equation \( \partial a_i(s)/\partial s = Y_i(s). \) As in the proof of the lemma for Proposition 3.1, we define, for each fixed \( s \), a vector field \( X(s) \) on \( G \times R \) so that \( (a_i(s), t) \) is the integral curve of the vector field \( X(s) \) through the point \( (\alpha(s), 0) \in G \times R \). The differentiability of \( a_i(s) \) follows from the fact that each solution of an ordinary linear differential equation with constant coefficients is differentiable in \( s \) as many times as the equation is (cf. Appendix 1). This completes the proof of the lemma and hence the proof of (a) of Theorem 4.2.

Let \( \pi_1(M) \) be the first homotopy group of \( M \) with reference point \( x. \) We define a homomorphism \( \pi_1(M) \to \Phi(u)/\Phi(0) \) as follows. For each element \( a \) of \( \pi_1(M) \), let \( \tau \) be a continuous loop at \( x \) which represents a. We may cover \( \tau \) by a finite number of coordinate neighborhoods, modify \( \tau \) within each neighborhood and obtain a piecewise differentiable loop \( \tau_1 \) of class \( C^k \) at \( x \) which is homotopic to \( \tau. \) If \( \tau_1 \) and \( \tau_2 \) are two such loops, then \( \tau_1 \cdot \tau_2^{-1} \) is homotopic to zero and defines an element of \( \Phi(u)/\Phi(0). \) Thus, \( \tau_1 \) and \( \tau_2 \) define the same element of \( \Phi(u)/\Phi(0) \), which is denoted by \( f(a) \). Clearly, \( f \) is a homomorphism of \( \pi_1(M) \) onto \( \Phi(u)/\Phi(0). \) Since \( M \) is connected and paracompact, it satisfies the second axiom of countability (Appendix 3). It follows easily that \( \pi_1(M) \) is countable. Hence, \( \Phi(u)/\Phi(0) \) is also countable.

**Remark:** In §3, we defined the parallel placement along any piecewise differentiable curve of class \( C^k \). In this section, we defined the holonomy group \( \Phi(u) \) using piecewise differentiable curves of class \( C^k. \) If we denote by \( \Phi_k(u) \) the holonomy group thus obtained from piecewise differentiable curves of class \( C^k \), then we have \( \Phi_k(u) \supset \Phi(u)/\Phi(0) \). We shall prove later in §7 that these holonomy groups coincide.

### 5. Curvature form and structure equation

Let \( P(M, G) \) be a principal fibre bundle and \( \rho \) a representation of \( G \) on a finite dimensional vector space \( V; \) \( p(a) \) is a linear transformation of \( V \) for each \( a \in G \) and \( p(ab) = p(a)p(b) \) for \( a, b \in G. \) A pseudotensorial form of degree \( r \) on \( P \) of type \((p, V)\) is a \( V \)-valued \( r \)-form on \( P \) such that

\[
R^*_\varphi = p(a^{-1}) \cdot \varphi \quad \text{for } a \in G.
\]

Such a form is called a tensorial form if it is horizontal in the sense that \( p(X_1, \ldots, X_r) = 0 \) whenever at least one of the tangent vectors \( X_i \) of \( P \) is vertical, i.e., tangent to a fibre.

**Example 5.1.** If \( \rho_0 \) is the trivial representation of \( G \) on \( V, \) that is, \( \rho_0(a) = I \) for each \( a \in G, \) then, as a tensorial form of degree \( r \) of type \((p_0, V)\) is nothing but a form \( \varphi \) on \( P \) which can be expressed as \( \varphi = \pi^*\varphi_M \) where \( \varphi_M \) is a \( V \)-valued \( r \)-form on the base \( M. \)
Example 5.2. Let \( p \) be a representation of \( G \) on \( V \) and \( E \) the bundle associated with \( P \) with standard fibre \( V \) on which \( G \) acts through \( p \). A tensorial form \( \varphi \) of degree \( r \) of type \((p, V)\) can be regarded as an assignment to each \( x \in M \) a multilinear skew-symmetric mapping \( \tilde{\varphi}_x: T_x(M) \times \cdots \times T_x(M) \to \pi_E^{-1}(x) \) into the vector space \( \pi_E^{-1}(x) \) which is the fibre of \( E \) over \( x \). Namely, we define

\[
\tilde{\varphi}_x(X_1, \ldots, X_r) = u(\varphi(X_1^*, \ldots, X_r^*)), \quad X_i \in T_x(M),
\]

where \( u \) is any point of \( P \) with \( \pi(u) = x \) and \( X_i^* \) is any vector at \( u \) such that \( \pi(X_i^*) = X_i \) for each \( i \). \( \varphi(X_1^*, \ldots, X_r^*) \) is then an element of the standard fibre \( V \) and \( u \) is a linear mapping of \( V \) onto \( \pi_E^{-1}(x) \) so that \( u(\varphi(X_1^*, \ldots, X_r^*)) \) is an element of \( \pi_E^{-1}(x) \). It can be easily verified that this element is, independent of the choice of \( u \) and \( X_i^* \). Conversely, given a skew-symmetric multilinear mapping \( \tilde{\varphi}_x: T_x(M) \times \cdots \times T_x(M) \to \pi_E^{-1}(x) \) for each \( x \in M, a \) tensorial form \( \varphi \) of degree \( r \) of type \((p, V)\) on \( P \) can be defined by

\[
\varphi(X_1^*, \ldots, X_r^*) = u^{-1}(\tilde{\varphi}_x(\pi(X_1^*), \ldots, \pi(X_r^*))), \quad X_i \in T_x(P),
\]

where \( x = \pi(u) \). In particular, a tensorial O-form of type \((p, V)\), that is, a function \( f: P \to V \) such that \( f(\pi(u)) = \rho(a^{-1})f(u) \), can be identified with a cross section \( M \to E \).

A few special cases of Example 5.2 will be used in Chapter III.

Let \( \Gamma \) be a connection in \( P(M, G) \). Let \( G_u \) and \( Q_u \) be the vertical and the horizontal subspaces of \( T_u(P) \), respectively. Let \( h: T_u(P) \to Q_u \) be the projection.

Proposition 5.1. If \( \varphi \) is a pseudotensorial \( r \)-form on \( P \) of type \((\rho, V)\), then

(a) The form \( q\varphi \) defined by \( (q\varphi)(X_1, \ldots, X_r) = q(hX_1, \ldots, hX_r), \) \( X_i \in T_u(P) \), is a tensorial form of type \((\rho, V)\); 

(b) \( dq \) is a pseudotensorial \((r + 1)\)-form of type \((\rho, V)\); 

(c) The \((r + 1)\)-form \( D\varphi \) defined by \( D\varphi = (dp)h \) is a tensorial form of type \((\rho, V)\).

Proof. From \( R_a \circ h = h \circ R_a, a \in G \), it follows that \( q\varphi \) is a pseudotensorial form of type \((p, V)\). It is evident that

\[
(q\varphi)(X_1, \ldots, X_r) = 0.
\]
Proposition 5.5. Let $\omega$ be a connection form and $\varphi$ a tensorial 1-form of type $\text{ad} \ G$. Then

$$D\varphi(X, Y) = d\varphi(X, Y) + \frac{1}{2}[\varphi(X), \omega(Y)] + \frac{1}{2}[\omega(X), \varphi(Y)],$$

for $X, Y \in T_u(P)$. $u \in P$.

Proof. As in the proof of Theorem 5.2, it suffices to consider the three special cases. The 'only non-trivial case' is the case where $X$ is vertical and $Y$ is horizontal. Let $X = A^e$ at $u$, where $A \in \mathfrak{g}$. We extend $Y$ to a horizontal vector field on $P$, denoted also by $Y$, which is invariant by $R_{\varphi A}$. We first extend the vector $\pi Y$ to a vector field on $P$ and then lift it to a horizontal vector field on $P$. Then $[A^e, Y] = 0$. As $A^e$ is vertical, $D\varphi(A^e, Y) = 0$. We shall show that the right hand side of the equality 'vanishes'. By Proposition 3.11 of Chapter I, we have

$$d\varphi(A^e, Y) = \frac{1}{2}(A^e(\varphi(Y)) - Y(\varphi(A^e)) - \varphi([A^e, Y])) = \frac{1}{2}A^e(\varphi(Y)),$$

so that it suffices to show $A^e(\varphi(Y)) + [\omega(A^e), \varphi(Y)] = 0$ or $A^e(\varphi(Y)) = -[A^e, \varphi(Y)]$. If $a^*$ denotes the 1-parameter subgroup of $G$ generated by $A$, then

$$A^e(\varphi(Y)) = \lim_{t \to 0} \frac{1}{t}([a^e, \varphi(Y)] - \varphi_{a^e}(Y)) = \lim_{t \to 0} \frac{1}{t}([a^e, \varphi(Y)] - \varphi(Y)) = [A, \varphi(Y)],$$

since $Y$ is invariant by $R_{\varphi A}$. QED.

6. Mappings of connections

In §5 of Chapter I, we considered certain mappings of one principal fibre bundle into another such as a homomorphism, an injection, and a bundle map. We now study the effects of these mappings on connections.

Proposition 6.1. Let $f: \mathcal{P}(M', G') \to \mathcal{P}(M, G)$ be a homomorphism with the corresponding homomorphism $f: G' \to G$ such that the induced mapping $f: M' \to M$ is a diffeomorphism of $M$ onto $M$. Let $\Gamma'$ be a connection in $\mathcal{P}$, $\omega'$ the connection form and $\Omega'$ the curvature form of $\Gamma'$. Then

(a) There is a unique connection $\Gamma$ in $\mathcal{P}$ such that the horizontal subspaces of $\Gamma'$ are mapped into horizontal subspaces of $\Gamma$ by $f$. 

Proof. Apply Proposition 1.9 of Chapter I to the left hand side of the structure equation just proved. QED.
II. THEORY OF CONNECTIONS

It is sufficient to verify the above equality in the two special cases:

(1) \( X' \) is horizontal, and (2) \( X' \) is vertical. Since \( f^* P' \rightarrow P \) maps every horizontal vector into a horizontal vector, both sides of the equality vanish if \( X' \) is horizontal. If \( X' \) is vertical, \( X' = A^*\) at \( u' \), where \( A' \in G \). Set \( A = (A')' \in G \). Since \( f(u'a') = f(u')f(a') \) for every \( a' \in G \), we have \( f(X') = A^* \) at \( u' \). Thus

\[
\omega(fX') = \omega(A^*) = A = f(A') = f(\omega(A^*)) = f(\omega(X')).
\]

From \( f*\omega = f*o' \), we obtain \( d(f*\omega) = d(f*o') \) and \( f*do = f*d'o' \). By the structure equation (Theorem 5.2):

\[
-\frac{1}{2} f*\omega + f*\Omega = -\frac{1}{2} [f*o', f*o'] + f*\Omega.
\]

This implies that \( f*\Omega = f*\Omega' \).

(c) Let \( \tau \) be a loop at \( x = \pi(u) \). Set \( \tau' = f^{-1}(\tau) \) so that \( \tau' \) is a loop at \( x' = \pi'(u') \). Let \( \tau' \) be the horizontal lift of \( \tau \) starting from \( u' \). Then \( f(\tau') \) is the horizontal lift of \( \tau \) starting from \( u \). The statement (c) is now evident.

In the situation as in Proposition 6.1, we say that \( f \) maps the connection \( \Gamma \) into the connection \( \Gamma' \). In particular, in the case where \( P'(M', G) \) is a reduced subbundle of \( P(M, G) \) with injection \( f \) so that \( M' = M \) and \( f : M' \rightarrow M \) is the identity morphism, we say that the connection \( \Gamma \) in \( P \) is reducible to the connection \( \Gamma' \) in \( P' \). An automorphism \( f \) of the bundle \( P(M, G) \) is called an automorphism of \( a \) connection \( \Gamma \) in \( P \) if it maps \( \Gamma \) into \( \Gamma' \), and in this case, \( \Gamma \) is said to be invariant by \( f \).

Proposition 6.2. Let \( f : P'(M', G) \rightarrow P(M, G) \) be a homomorphism such that the corresponding homomorphism \( f_* : G' \rightarrow G \) maps \( G' \) isomorphically onto \( G \). Let \( \Gamma' \) be a connection in \( P' \), \( \omega \) the connection form and \( \Omega \) the curvature form of \( \Gamma' \). Then

(a) There is a unique connection \( \Gamma \) in \( P \) such that the horizontal subspaces of \( \Gamma \) are mapped into horizontal subspaces of \( \Gamma' \) by \( f \).

(b) If \( \omega' \) and \( \Omega' \) are the connection form and the curvature form of \( \Gamma' \) respectively, then \( f*\omega = f*\omega' \) and \( f*\Omega = f*\Omega' \).

(c) If \( u' \in P' \) and \( u = f(u') \in P \), then the isomorphism \( f : G' \rightarrow G \) maps \( \Phi(u') \) into \( \Phi(u) \) and \( \Phi^0(u') \) into \( \Phi^0(u) \).
We define \( \Gamma' \) by defining its connection form \( \omega' \). Set \( \omega' = f^{-1} f^* \omega \), where \( f^{-1} : g \to g' \) is the inverse of the isomorphism \( f : g' \to g \) induced from \( f : G' \to G \). Let \( X' \in T_{v'}(\mathcal{P}) \) and \( a' \in G' \) and set \( X = fX' \) and \( a = f(a') \). Then we have

\[
\omega'(R_{a'}X') = f^{-1}(\omega(f(R_{a}X))) = f^{-1}(\omega(R_{a}(X))) = f^{-1}(\text{ad}(a^{-1})(\omega(X))) = \text{ad}(a^{-1})(\omega'(X)).
\]

Let \( A' \in g' \) and set \( A = f(A') \). Then \( A' \) and \( A', \) denote the fundamental vector fields corresponding to \( A \) and \( A' \), respectively. Then we have

\[
\omega'(A') = f^{-1}(\omega(A')) = f^{-1}(A') = A'.
\]

This proves that the form \( \omega' \) defines a connection (Proposition 1.1). The verification of other statements is similar to the proof of Proposition 6.1 and is left to the reader. QED.

In the situation as in Proposition 6.2, we say that \( \Gamma' \) is induced by \( f \) from \( \Gamma \). If \( f \) is a bundle map, that is, \( G' = G \) and \( f : G' \to G \) is the identity automorphism, then \( \omega' = f^* \omega \) in particular, given a bundle \( P(M, G) \) and a mapping \( f : M' \to M \), every connection in \( P \) induces a connection in the induced bundle \( f^*P \).

For any principal fibre bundles \( P(M, G) \) and \( Q(M, H) \), \( P \times Q \) is a principal fibre bundle over \( M \times M \) with group \( G \times H \). Let \( P + Q \) be the restricted \( P \times Q \) to the diagonal \( M \times M \) of \( M \times M \). Since \( M \times M \) are diffeomorphic with each other in a natural way, we consider \( P + Q \) as a principal fibre bundle over \( M \) with group \( G \times H \). The restriction of the projection \( P \times Q \to P + Q \), denoted by \( f_P \), is a homomorphism with the corresponding natural homomorphism \( f_Q : G \times H \to G \). Similarly, for \( f_Q : P + Q \to Q \) and \( f_R : G \times H \to H \).

**Proposition 6.3.** Let \( \Gamma_P \) and \( \Gamma_Q \) be connections in \( P(M, G) \) and \( Q(M, H) \) respectively. Then

(a) There is a unique connection \( \Gamma \) in \( P + Q \) such that the homomorphisms \( f_P : P + Q \to P \) and \( f_Q : P + Q \to Q \) maps \( \Gamma_P \) into \( \Gamma \) and \( \Gamma_Q \) respectively.

(b) If \( \omega \), \( \omega_P \) and \( \omega_Q \) are the connection forms and \( \Omega, \Omega_P \), and \( \Omega_Q \) are the curvature forms of \( \Gamma, \Gamma_P \), and \( \Gamma_Q \) respectively, then

\[
\omega = f_P^* \omega_P + f_Q^* \omega_Q, \quad \Omega = f_P^* \Omega_P + f_Q^* \Omega_Q.
\]
which can be joined to \( u_0 \) by a horizontal curve. Then

1. \( \Phi(u_0) \) is a reduced bundle with structure group \( \Phi(u_0) \).
2. The connection \( \Gamma \) is reducible to a connection in \( \Phi(u_0) \).

Proof. (1) We first prove

**Lemma 1.** Let \( Q \) be a subset of \( P(M, G) \) and \( H \) a Lie subgroup of \( G \). Assume (1) the projection, \( \pi: P \rightarrow M \)-maps \( Q \) onto \( M \); (2) \( Q \) is stable by \( H \); i.e., \( R_a(Q) = Q \) for each \( \alpha \in H \); (3) if \( u, v \in Q \) and \( \pi(u) = \pi(v) \), then there is an element \( \alpha \in H \) such that \( v = \alpha u \); and (4) every point \( x \) of \( M \) has a neighborhood \( U \) and a cross section \( \sigma: U \rightarrow P \) such that \( \sigma(U) \subset \Phi(u) \). Then \( Q(M, H) \) is a reduced subbundle of \( P(M, G) \).

Proof of Lemma 1. For each \( u \in \pi^{-1}(U) \), let \( x = \pi(u) \) and \( \alpha \in G \) the element determined by \( u = \alpha(x) a \). Define an isomorphism \( \psi: \pi^{-1}(U) \rightarrow U \times G \) by setting \( \psi(u) = (x, \alpha) \). It is easy to see that \( \psi \) maps \( \psi^{-1}(U) \) onto \( U \times H \). Introduce a differentiable structure in \( Q \) in such a way that \( \psi^{-1}(U) \rightarrow U \times H \) becomes \( \psi \) a diffeomorphism; using Proposition 1.3 of Chapter I as in the proof of Proposition 5.3 of Chapter I, we see that \( Q \) becomes a differentiable manifold. It is now evident that \( Q \) is a principal fibre bundle over \( M \) with \( H \) and \( Q \) is a subbundle of \( P \).

Going back to the proof of the first assertion of Theorem 7.1; we see that \( M \) being paracompact, the holonomy group \( \Phi(u_0) \) is a Lie subgroup of \( G \) (Theorem 4.2) and that the subset \( \Phi(u_0) \) and the group \( \Phi(u_0) \) satisfy conditions (1), (2), and (3) of Lemma 1 (cf. the second definition of \( \Phi(u_0) \) given before Proposition 4.1 and also Proposition 4.1 (b)). To verify condition (4) of Lemma 1, let \( x_1, \ldots, x_n \) be a local coordinate system around \( x \) such that \( x \) is the origin \((0, \ldots, 0)\) with respect to this coordinate system. Let \( U \) be cubical neighborhood of \( x \) defined by \( |x| < \delta \). Given any point \( u \in U \), let \( \tau_u \) be the segment from \( x \) to \( u \) with respect to the \( \tau_u \) coordinate system \( x_1, \ldots, x_n \). Fix a point \( u \in Q \) such that \( \pi(u) = x_0 \). Let \( \sigma(y) \) be the point of \( P \) obtained by the parallel displacement of \( u \) along \( \tau_u \). Then \( \sigma: U \rightarrow P \) is a cross section such that \( \sigma(U) \subset Q \). Now (1) of Theorem 7.1 follows from Lemma 1.

(2) This is an immediate consequence of the following

**Lemma 2.** Let \( Q(M, H) \) be subbundle of \( P(M, G) \) and \( \Gamma \) a connection in \( P \). If, for every \( u \in Q \), the horizontal subspace of \( T_u(P) \) is tangent to \( Q \), then \( \Gamma \) is reducible to a connection in \( Q \).

---

Proof of Lemma 2. We define a connection \( \Gamma' \) in \( Q \) as follows. The horizontal subspace of \( T_u(Q), u \in Q \), with respect to \( \Gamma' \) is by definition the horizontal subspace of \( T_u(P) \) with respect to \( \Gamma \). It is obvious that \( \Gamma \) is reducible to \( \Gamma' \). QED.

We shall call \( P(u) \) the holonomy bundle through \( u \). It is evident that \( P(u) = P(v) \) if and only if \( u \) and \( v \) can be joined by a horizontal curve. Since the relation introduced in \( \S 4 \) \((u \sim v \text{ if } u \text{ and } v \) can be joined by a horizontal curve) is an equivalence relation, we have, for every pair of points \( u \) and \( v \) of \( P \), either \( P(u) = P(v) \) or \( P(u) \cap P(v) = \emptyset \). In other words, \( P \) is decomposed into the disjoint union of the holonomy bundles. Since every \( \alpha \in G \) maps each horizontal curve into a horizontal curve, \( R_a(P(u)) = P(ua) \), and \( R_a: P(u) \rightarrow P(ua) \) is an isomorphism with the corresponding isomorphism \( \Phi(u) \rightarrow \Phi(ua) \) of the structure groups. It is easy to see that, given any \( u \) and \( v \), there is an element \( \alpha \in G \) such that \( P(v) = P(ua) \). Thus the holonomy bundles \( P(u), u \in P \), are all isomorphic with each other.

Using Theorem 7.1, we prove that the holonomy groups \( \Phi_k(u), 1 \leq k \leq \infty \), coincide as was pointed out in Remark of \( \S 4 \). This result is due to Nomizu and Ozeki [2].

**Theorem 7.2.** All the holonomy groups \( \Phi_k(u), 1 \leq k \leq \infty \), coincide.

Proof. It is sufficient to show that \( \Phi_k(u) = \Phi_k(v) \). We denote \( \Phi_k(u) \) by \( \Phi(u) \) and the holonomy bundle, through \( u \), by \( P(u) \). We know by Theorem 7.1 that \( P(u) \) is a subbundle of \( P \) with \( \Phi(u) \) as its structure group. Define a distribution \( S \) on \( P \) by setting

\[ S_u = T_u(P(u)) \quad \text{for } u \in P. \]

Since the holonomy bundles have the same dimension, say \( k, S \) is a \( k \)-dimensional distribution. We first prove

**Lemma 1.** (1) \( S \) is differentiable and involutive.
(2) For each \( u \in P \), \( P(u) \) is the maximal integral manifold of \( S \) through \( u \).

Proof of Lemma 1. We set

\[ S'_u = S'_u + S''_u \quad \text{for } u \in P, \]

where \( S'_u \) is horizontal and \( S''_u \) vertical. The distribution \( S' \) is differentiable by the very definition of a connection. To prove the
differentiability of $S$, it suffices to show that of $S''$. For each $u \in P$, let $U$ be a neighborhood of $x = n(u)$ with a cross section $\sigma: U \to P(u)$ such that $\sigma(x) = u$. (Such a cross section was constructed in the proof of Theorem 7.1.) Let $\mathcal{A}_1, \ldots, \mathcal{A}_r$ be a basis of the Lie algebra $g(u)$ of $\Phi(u)$. We shall define vector fields $\mathcal{A}_1, \ldots, \mathcal{A}_r$ on $\pi^{-1}(U)$ which form a basis of $S''$ at every point of $\pi^{-1}(U)$. Let $v \in \pi^{-1}(U)$. Then there is a unique $a \in G$ such that $v = a_{\pi^{-1}(U)}a$. Since $(a^{-1}) \Phi(u) = \Phi(a^{-1})$, $\mathcal{A}_i$, $i = 1, \ldots, r$, are elements of $g(v)$ and form a basis of $S''$.

We may assume that $\mathcal{A}_i$ are differentiable and form a basis of $S''$ at each point. Let $v \in P(u)$, we have $T_v(P(u)) = T_v(P(v)) = S_v$. This implies that $S$ is involutive.

(2) Let $W(u)$ be the maximal integral manifold of $S$ through $u$ (cf. Proposition 1.2 of Chapter 1). Then $P(u)$ is an open submanifold of $W(u)$. We prove that $P(u) = W(u)$. Let $v$ be an arbitrary point of $W(u)$ and let $u(t), 0 \leq t \leq 1$, be a curve in $W(u)$ such that $u(0) = u$ and $u(1) = v$. Let $t_1$ be the supremum of $t$ such that $0 \leq t \leq t_1$ implies $u(t) \in P(u)$. Since $P(u)$ is open in $W(u)$, $t_1$ is positive. We show that $u(t_1)$ lies in $P(u)$; since $P(u)$ is open in $W(u)$, this will imply that $t_1 = 1$, proving that $u(1) = v \in P(u)$. The point $u(t_1)$ lies in $P(u(t_1))$ and $P(u(t_1))$ is open in $W(u(t_1))$. There exists $\varepsilon > 0$ such that $0 < t_1 - \varepsilon < t_1 + \varepsilon$ implies $u(t) \in P(u(t_1))$. Let $\varepsilon$ be any value such that $t_1 - \varepsilon < t_1 < t_1 + \varepsilon$. By definition of $t_1$, we have $u(t) \in P(u)$. On the other hand, $u(t) \in P(u(t_1))$. This implies that $P(u) = P(u(t_1))$ so that $u(t_1) \in P(u)$, as we wanted to show. We have thereby proved that $P(u)$ is actually the maximal integral manifold of $S$ through $u$.

**Lemma 2.** Let $S$ be an involutive, $C^r$-distribution on a $C^r$-manifold. Suppose $x_0, 0 \leq t \leq 1$, is a piecewise $C^r$-curve whose tangent vectors $x_t$ belong to $S$. Let the entire curve $x_t$ lies in the maximal integral manifold $W(u)$ through the point $x_0$.

Proof of Lemma 2. We may assume that $x_t$ is a $C^r$-curve. Take a local coordinate system $x^1, \ldots, x^n$ around the point $x_0$ such that \[
\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, k = \dim S.
\] For small values of $t$, say, $0 \leq t \leq e$, $x_t$ can be expressed by $x_t = x_t(0) + t(x'(0))$, $1 \leq i \leq n$, and its tangent vectors are given by \[
\Sigma_r x_i^{(r)}(0) \frac{\partial}{\partial x^i}.
\] By assumption, we have $dx_i^{(r)}(0) = 0$ for $k + 1 \leq i \leq n$. Thus, $x_t(0) = x_t(0)$ for $1 \leq i \leq n$ so that $x_t, 0 \leq t \leq e$, lies in the slice through $x_0$ and hence in $W(u)$.

The standard continuation argument concludes the proof of Lemma 2.

We are now in position to complete the proof of Theorem 7.2. Let $a$ be any element of $\Phi_1(u)$. This means that $u$ and $ua$ can be joined by a piecewise $O$-horizontal curve $u(t), 0 \leq t \leq 1$, in $P$. The tangent vector $u(t)$ at each point obviously lies in $S_{u(t)}$.

By Lemma 2, the entire curve $u_t$ lies in the maximal integral manifold $W(u)$ of $S$ through $u$. By Lemma 1, the entire curve $u_t$ lies in $P(u)$. In particular, $ua$ is a point of $P(u)$. Since $P(u)$ is a subbundle with structure group $\Phi(u)$, $a$ belongs to $\Phi(u)$.

**Proof.**\[ \text{Corollary 7.3.}\] The restricted holonomy groups $\Phi_k(u)$, $1 \leq k \leq \infty$, coincide.

Proof. $\Phi_k(u)$ is the connected component of the identity of $\Phi_k(u)$ for every $k$ (cf. Theorem 4.2 and its proof). Now Corollary 7.3 follows from Theorem 7.2.

**QED.**

Remark. In the case where $P(M, G)$ is a real analytic principal bundle with an analytic connection, we can still define the holonomy group $\Phi(u)$ by using only piecewise analytic horizontal curves. The argument used in proving Theorem 7.2 and Corollary 7.3 shows that $\Phi(u) = \Phi_1(u)$ and $\Phi_2(u) = \Phi(u)$.

Given a connection $\Gamma$ in a principal fibre bundle $P(M, G)$, we shall define the notion of parallel displacement in the associated fibre bundle $E(M, F, G, P)$ with standard fibre $F$. For each $w \in E$, the horizontal subspace $Q_w$ and the vertical subspace $F_w$ of $T_w(E)$ are defined as follows. The vertical subspace $F_w$ is by definition the tangent space to the fibre of $E$ at $w$. To define $Q_w$, we recall that we have the natural projection $P \times F \to E = P \times F$. Choose a point $(w, \xi) \in P \times F$ which is mapped into $w$. We fix this $\xi \in F$ and consider the mapping $P \to E$ which maps $v \in P$ into $v \xi \in E$. Then the horizontal subspace $Q_w$ is, by definition, the image of the horizontal subspace $Q_w \subset T_w(P)$ by this mapping $P \to E$. We see easily that $Q_w$ is independent of the choice of...
(u, ξ) • P x F. We leave to the reader the proof that $T_u(E) = F_u + Q_u$ (direct sum). A curve in E is horizontal if its tangent vector is horizontal at each point. Given a curve $τ$ in $M$, a (horizontal) lift $\tau^*$ of $τ$ is a horizontal curve in E such that $π_E(\tau^*) = τ$. Given a curve $τ = x_t, 0 ≤ t ≤ 1$, and a point $u_0$ such that $π_E(u_0) = x_0$, there is a unique lift $\tau^* = u_1$ starting from $u_0$. To prove the existence of $\tau^*$, we choose a point $(u_0, \xi)$ in $P$ such that $u_0\xi = u_1$. Let $u_t$ be the lift of $τ = x_t$ starting from $u_0$. Then $u_t = u_0, \xi$ is a lift of $τ$ starting from $u_0$. The uniqueness of $\tau^*$ reduces to the uniqueness of a solution of a system of ordinary linear differential equations satisfying a given initial condition just as in the case of a lift in a principal fibre bundle. A cross section $σ$ of $E$ defined on an open subset $U$ of $M$ is called parallel if the image of $T(U)$ (by $π$) is horizontal for each $x ∈ U$, that is, for any curve $τ = x_t, 0 ≤ t ≤ 1$, the parallel displacement of $σ(x_t)$ along $τ$ gives $σ(x_s)$.

**Proposition 7.4.** Let $P(M, G)$ be a principal fibre bundle and $E = E(M, G, H, P)$ the associated bundle with $standard fibre G/H$, where $H$ is a closed subgroup of $G$. Let $a : M → E$ be a cross section and $Q = (M, H)$ the reduced subbundle of $P(M, G)$ corresponding to $a$ (cf. Proposition 5.6 of Chapter I). Then a connection $Γ$ in $P$ is reducible to a connection $Γ^*$ in $Q$ if and only if $σ$ is parallel with respect to $Γ$.

**Proof.** If we identify $E$ with $P/H$ (cf. Proposition 5.5 of Chapter I), then $σ(M)$ coincides with the image of $Q$ by the natural projection $μ : P → P/H$; in other words, if $u ∈ Q$ and $x = π(u)$, then $μ(u) = μ(x)$ (cf. Proposition 5.6 of Chapter I). Suppose $Γ$ is reducible to a connection $Γ^*$ in $Q$. We note that if $ξ$ is the origin (i.e., the coset $H$ of $G/H$, then $μξ = μ(u)$ for every $u ∈ P$ and hence $μ(u) = μ(x)$, so $μ(u) ∈ E$. Given a curve $x_t, 0 ≤ t ≤ 1$, in $M$, choose $u_0 ∈ P$ with $π(u_0) = x_0$ so that $σ(x_0) = μ(u_0)$. Let $u_t$ be the lift to $P$ of $x_t$ starting from $u_0$ (with respect to $Γ$), so that $μ(u_t)$ is the lift of $x_t$ to $E$ starting from $σ(x_0)$. Since $Γ$ is reducible to $Γ^*$, we have $u_t ∈ Q$ and hence $μ(u_t) = σ(x_t)$ for all $t$. Conversely, assume that $σ$ is parallel (with respect to $Γ^*$). Given any curve $x_t, 0 ≤ t ≤ 1$, in $M$ and any point $u_0$ of $Q$ with $π(u_0) = x_0$, let $u_t$ be the lift of $x_t$ to $P$ starting from $u_0$. Since $a$ is parallel, $μ(u_t) = σ(x_t)$ and hence $u_t ∈ Q$ for all $t$. This shows that every horizontal vector at $u_0 ∈ Q$ (with respect to $Γ$) is tangent to $Q$. By Proposition 7.2, $Γ$ is reducible to a connection in $Q$. QED.

### 8. Holonomy theorem

We first prove the following result of Ambrose and Singer [1] by applying Theorem 7.1.

**Theorem 8.1.** Let $P(M, G)$ be a principal fibre bundle, where $M$ is connected and paracompact. Let $Γ$ be a connection in $P$, $Ω$ the curvature form, $Φ(U)$ the holonomy group with reference point $u ∈ P$ and $P(U)$. The holonomy bundle through $u$ of $Γ$, $T$ then the Lie algebra of $Φ(U)$ is equal to the subspace of $g$, Lie algebra of $G$, spanned by all elements of the form $Ω_u(X, Y)$, where $v ∈ P(U)$ and $X$ and $Y$ are arbitrary horizontal vectors at $u$.

Proof. By virtue of Theorem 7.1, we may assume that $P(U) = P$, i.e., $Φ(U) = G$. Let $g'$ be the subspace of $g$, spanned by all elements of the form $Ω_u(X, Y)$, where $v ∈ P(U)$ and $X$ and $Y$ are arbitrary horizontal vectors at $u$. The subspace $g'$ is actually an ideal of $g$, because $Ω$ is a torsion form of type $d g$ (cf. §5), and hence $g'$ is invariant by $d g$. We shall prove that $Γ = g'$.

At each point $v ∈ P$, let $S_v$ be the subspace of $T_v(P)$ spanned by the horizontal subspace $Q_v$ of $Q_v$, and by the subspace $g' = \{A^*; A ∈ g\}',$ where $A^*$ is the fundamental vector field on $P$ corresponding to $A$. The distribution $S$ has dimension $n + r$, where $n = dim M$ and $r = dim g'$. We shall prove that $S$ is differentiable and involutive. Let $v$ be an arbitrary point of $P$ and $U$, a coordinate neighborhood of $v = π(v) ∈ M$ such that $π^{-1}(U)$ is isomorphic with $U × G$. Let $X_1, \ldots , X_n$ be differentiable vector fields on $U$ which are linearly independent everywhere on $U$ and $X_1^*, \ldots , X_n^*$ the horizontal lifts of $X_1, \ldots , X_n$. Let $A_1, \ldots , A_n$ be a basis for $g'$ and $A_1^*, \ldots , A_n^*$ the corresponding fundamental vector fields. It is clear that $X_1^*, \ldots , X_n^*, A_1^*, \ldots , A_n^*$ form a local basis for $S$. To prove that $S$ is involutive, it suffices to verify that the bracket of any two of these vector fields belongs to $S$. This is clear for $[A_i^*, A_j^*]$, since $[A_i, A_j] ∈ g'$ and $[A_i, A_j] = [A_i, A_j]^*$ by the lemma for Theorem 5.2, $[A_i^*, A_j^*]$ is horizontal; actually, $[A_i^*, X_j^*] = 0$ as $X_j^*$ is invariant by $R_v$ for each $A ∈ g$. Finally, set $A = σ([X_i^*, X_j^*]) ∈ g$ where $σ$ is the connection form of $Γ$. By Corollary 5.3, $A = σ([X_i^*, X_j^*]) = -2σ(X_i^*, X_j^*) ∈ g'$. Since the vertical component of $[X_i^*, X_j^*]$ at $v$ is equal to $A_i^* ∈ S_v$,
$[X^t, X^s]$ belongs to $S$. This proves our assertion that $S$ is involutive.

Let $P_0$ be the maximal integral manifold of $S$ through $u$. By Lemma 2 in the proof of Theorem 7.2, we have $P_0 = P$. Therefore,

$$\dim g = \dim P = n = \dim g^t.$$

This implies $g = g^t$.

Next we prove

**Theorem 8.2.** Let $P(M, G)$ be a principal fibre bundle, where $P$ is connected and $M$ is paracompact. If $\dim M \geq 2$, there exists a connection in $P$ such that all the holonomy bundles $P(u)$, $u \in P$, coincide with $P$.

Proof. Let $u_0$ be an arbitrary point of $P$ and $x^1, \ldots, x^n$ a local coordinate system with origin $x_0 = \pi(u_0)$. Let $U$ and $V$ be neighborhoods of $x_0$ defined by $|x^1| < \alpha$ and $|x^i| < \beta$ respectively, where $0 < \beta < \alpha$. Taking a sufficiently small, we may assume that $P \cap U = \pi^{-1}(U)$ is isomorphic with the trivial bundle $U \times G$. We shall construct a connection $\Gamma'$ in $P$ such that the holonomy group of the bundle $P \mid V$ coincides with the identity component of $G$. We shall then extend $\Gamma'$ to a connection $\Gamma$ in $P$ in such a way that $\Gamma'$ coincides with $\Gamma$ on $P \mid V$ (cf. Theorem 2.1).

Let $A_1, \ldots, A_n$ be a basis for the Lie algebra $g$ of $G$. Choose real numbers $\alpha_1, \ldots, \alpha_n$ such that $0 < \alpha_1 < \cdots < \alpha_n < \beta$ and let $f_i(t), t = 1, \ldots, r_i$, be differentiable functions in $-\alpha < t < \alpha + \epsilon$ such that $f_i(0) = 0$ for every $i$ and $f_i'(\alpha_i) = \delta_{ij}$ (Kronecker's symbol). On $\pi^{-1}(U) = U \times C$, we can define a connection form $\omega$ by requiring that

$$\omega_{(x,a)}(\partial/\partial x^i) = \sum_{j=1}^n f_j(x)A_j$$

and that

$$\omega_{(x,a)}(\partial/\partial x^i) = 0$$

for $i = 2, 3, \ldots, n$.

(Note that, by virtue of the property $R^*_\omega \omega = \text{ad}(a^{-1})\omega$, the preceding conditions determine the values of $\omega$ at every point $(x, a)$ of $U \times G$.)

Fixing $t, 0 < t < \beta$, and $\alpha_k$, $1 \leq k \leq r_i$, for the moment, consider the rectangle, on the $x^1x^2$-plane in $V$ formed by the line segments $\tau_1$ from $(0, 0)$ to $(0, a)$, $\tau_2$ from $(0, a)$ to $(t, \alpha_k)$, $\tau_3$ from $(t, \alpha_k)$ to $(t, 0)$, and $\tau_4$ from $(t, 0)$ to $(0, 0)$. (Here and in the following argument, the $x^1$ to $x^2$ coordinates of all the points remain 0 and are hence omitted.) In $\pi^{-1}(V) = V \times G$, we determine the horizontal lift of $\tau = \tau_1 \cdot \tau_2 \cdot \tau_3 \cdot \tau_4$ starting from the point $(0, 0; e)$. The lift $\tau^t_i$ of $\tau_i$ starting from $(0, 0; e)$ is clearly $(0, \alpha_i; e)$, $0 \leq \alpha_i \leq \alpha_n$, since its tangent vectors $\partial/\partial x^2$ are horizontal. The lift $\tau^t_i$ of $\tau_i$ starting from the end point $(0, a; e)$ of $\tau_i$ is of the form $(s, a; e)$, $0 \leq s \leq t$, where $c_s$ is a suitable curve with $c_0 = e$ in $G$. Its tangent vector is of the form $(\partial/\partial x^2)(a, s) + c_s$. By a similar computation to that for Proposition 3.1, we have

$$\omega((\partial/\partial x^2)(a, s) + c_s) = \text{ad} (c_s^{-1})\omega((\partial/\partial x^2)(a, s) + c_s + c_s^{-1} \cdot c_s$$

\[= \text{ad} (c_s^{-1}) \left( \sum_{j=1}^n f_j(a, s)A_j + c_s^{-1} \cdot c_s \right) = \text{ad} (c_s^{-1})A_k + c_s^{-1} \cdot c_s. \]

Therefore we have $c_s \cdot c_s^{-1} = -A_k$, that is, $c_s = \exp (-A_k)$. The end point of $\tau^t_i$ is hence $(t, a; \exp (-A_k))$. The lift $\tau^t_4$ of $\tau_4$ starting from $(0, a; \exp (-A_k))$, $0 \leq s \leq \alpha_n$. Finally, the lift $\tau^t_i$ of $\tau_i$ starting from the end point $(0; e)$ of $\tau_i$ is of the form $(t, s; e)$, $0 \leq s \leq t$, since $\partial/\partial x^2$ is horizontal at the points with $x^1 = 0$. This shows that the end point of the lift $\tau^t_i$ of $\tau_i$ is $(0, s; e)$, $0 \leq s \leq t$, and hence is open in $G$. Since this is the case for every $i$, we see that $A_i$ is in the Lie algebra of the holonomy group. The result being valid for any $A_i$, we see that the holonomy group of the connection in $\pi^{-1}(V)$ coincides with the identity component of $G$.

Let $\Gamma$ be a connection in $P$ which coincides with $\Gamma'$ on $\pi^{-1}(P)$. Since the holonomy group $\Phi(u_0)$ of $\Gamma$ obviously contains the identity component of $G$, the holonomy bundle $P(u_0)$ of $\Gamma$ has the same dimension as $P$ and hence is open in $P$. Since $P$ is a disjoint union of holonomy bundles each of which is open, the connectedness of $P$ implies that $P = P(u_0)$.

**Corollary 8.3.** Any connected Lie group $G$ can be realized as the holonomy group of a certain connection in a trivial bundle $P = M \times G$, where $M$ is an arbitrary differentiable manifold with $\dim M \geq 2$.

Theorem 8.2 was proved for linear connections by Hano and Ozeki [1] and then in the general case by Nomizu [5], both by making use of Theorem 8.1. The above proof which is more direct is due to E. Ruh (unpublished).
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9. Flat connections

Let \( P = M \times G \) be a trivial principal fibre bundle. For each \( a \in G \), the set \( M \times \{a\} \) is a submanifold of \( P \). In particular, \( M \times \{e\} \) is a subbundle of \( P \), where \( e \) is the identity of \( G \). The canonical flat connection in \( P \) is defined by taking the tangent space to \( M \times \{af \at \} \) \( \mathbb{R} \) as \( a \in M \times G \) as the horizontal subspace at \( u \). In other words, a connection in \( P \) is the canonical flat connection if and only if it is reducible to a unique connection in \( M \times \{e\} \).

Let \( \theta \) be the canonical 1-form on \( G \) (cf. §4 of Chapter I). Let \( f: M \times G \to G \) be the natural projection and set \( \omega = f^* \theta \).

It is easy to verify that \( \omega \) is the connection form of the canonical flat connection in \( P \). The Maurer-Cartan equation of \( \theta \) implies that the canonical flat connection has zero curvature:

\[
\omega = f^* \theta.
\]

A connection in any principal fibre bundle \( P(M, G) \) is called flat if every point \( x \) of \( M \) has a neighborhood \( U \) such that the induced connection in \( P \mid U = \pi^{-1}(U) \) is isomorphic with \( U \times G \) the canonical flat connection in \( U \times G \). More precisely, here is an isomorphism \( y: \pi^{-1}(U) \to U \times G \) which maps the horizontal subspace at each \( u \in \pi^{-1}(U) \) upon the horizontal subspace at \( \varphi(u) \) of the canonical flat connection in \( U \times G \).

Theorem 9.1. A connection in \( P(M, G) \) is flat if and only if the curvature form vanishes identically.

Proof. The necessity is obvious. Assume that the curvature form vanishes identically. For each point \( x \) of \( M \), let \( U \) be a simply connected open neighborhood of \( x \) and consider the induced connection in \( P \mid U = \pi^{-1}(U) \). By Theorems 4.2 and 8.1, the holonomy group of the induced connection in \( P \mid U \) consists of the identity only. Applying the Reduction Theorem (Theorem 7.1), we see that the induced connection in \( P \mid U \) is isomorphic with the canonical flat connection in \( U \times G \). QED.

Corollary 9.2. Let \( \Gamma \) be a connection in \( P(M, G) \) such that the curvature vanishes identically. If \( M \) is paracompact and simply connected,

then \( \Gamma \) is isomorphic with the trivial bundle \( M \times G \) and \( \Gamma \) is isomorphic with the canonical flat connection in \( M \times G \).

We shall study the case where \( M \) is not necessarily simply connected. Let \( \Gamma \) be a flat connection in \( P(M, G) \), where \( M \) is connected and paracompact. Let \( u_0 \in \Gamma \) and \( M^* = \pi(u_0) \), the holonomy bundle through \( u_0 \); \( M^* \) is a principal fibre bundle over \( M \) whose structure, group is the holonomy group \( \Phi(u_0) \). Since \( \Phi(u_0) \) is discrete by Theorems 4.2 and 8.1 and since \( M^* \) is connected, \( M^* \) is a covering space of \( M \). Set \( x_0 = \pi(u_0) \), \( x_0 \in M \).

Every closed curve of \( M \) starting from \( x_0 \) defines, by means of the parallel displacement along it, an element of \( \Phi(u_0) \). Since the restricted holonomy group is trivial by Theorems 4.2 and 8.1, any two closed curves at \( x_0 \) representing the same element of the first homotopy group \( \pi_1(M, x_0) \) give rise to the same element of \( \Phi(u_0) \). Thus we obtain a homomorphism of \( \pi_1(M, x_0) \) onto \( \Phi(u_0) \). Let \( N \) be a normal subgroup of \( \Phi(u_0) \) and set \( M' = M^*/N \). Then \( M' \) is a principal fibre bundle over \( M \) with structure group \( \Phi(u_0)/N \).

In particular, \( M' \) is a covering space of \( M \). Let \( P'(M', G) \) be the principal fibre bundle induced from \( P(M, G) \) by the covering projection \( M' \to M \). Let \( f: P' \to P \) be the natural homomorphism (cf. Proposition 5.8 of Chapter I):

\[
\text{Proposition 9.3: There exists a unique connection } \Gamma' \text{ in } P'(M', G) \text{ which is mapped into } \Gamma \text{ by the homomorphism } f: P' \to P. \text{ The connection } \Gamma' \text{ is flat. If } u_0 \text{ is a point of } P' \text{ such that } f(u_0) = x_0, \text{ then the holonomy group } \Phi(u_0) \text{ of } \Gamma' \text{ with reference point } u_0 \text{ is isomorphically mapped onto } N \text{ by } f. \]

Proof. The first statement is contained in Proposition 6.2. By the same proposition, the curvature form of \( \Gamma' \) vanishes identically and \( \Gamma' \) is flat. We recall that \( P' \) is the subset of \( M' \times P \) defined as follows (cf. Proposition 5.8 of Chapter I):

\[
P' = \{(x', u) \in M' \times P; \mu(x') = \pi(u)\},
\]

where \( \mu: M' \to M \) is the covering projection. The projection \( \pi': P' \to M' \) is given by \( \pi'(x', u) = x' \) and the homomorphism \( f: P' \to P \) is given by \( f(x', u) = u \) so that the corresponding homomorphism \( f: G \to G \) of the structure groups is the identity automorphism. To prove that \( f \) maps \( \Phi(u_0) \) isomorphically onto
N, it is therefore sufficient to prove \( \Phi(u'_0) = N \). Write
\[
u'_0 = (x'_0, u_0) \in P' \subset M' \times P.
\]
Since \( \mu(x'_0) = \pi(u_0) \), there exists an element \( a \in \Phi(u_0) \) such that
\[
x'_0 = v(u_0a),
\]
where \( v : M^* = P(u_0) \to M' = P(u_0)/N \) is the covering projection. Let \( \tau = U_t \), \( 0 \leq t \leq 1 \), be a horizontal curve in \( P' \) such that \( \pi'(u'_0) = \pi'(u'_t) \). For each \( t \), we set
\[
u_t = (x'_t, u_t) \in P' \subset M' \times P.
\]
Then the curve \( u_t, 0 \leq t \leq 1 \), is horizontal in \( P \) and hence is contained in \( M^* = P(u_0) \). Since \( \mu(x'_t) = \pi(u_t) = \mu \circ v(u_t) \) and \( x'_0 = v(u_0a) \) for \( 0 \leq t \leq 1 \), we have
\[
v(u_0a) = x'_0 = \pi'(u'_0) = \pi'(u'_t) = x'_t = v(u_0a).
\]
and consequently,
\[
v(u_t) = v(u_0),
\]
which means that \( u_1 = u_0b \) for some \( b \in N \). This shows that \( \Phi(u'_1) \subset N \). Conversely, let \( \Phi(u'_1) \subset N \). Let \( u_t, 0 \leq t \leq 1 \), be a horizontal curve in \( P \) such that \( u_1 = u_0b \). Define a horizontal curve \( u'_t, 0 \leq t \leq 1 \), in \( P' \) by
\[
u'_t = (x'_t, u_t),
\]
where \( x'_t = v(u_ta) \). Then \( u'_1 = u'_0b \), showing that \( \Phi(u'_0) = N \). QED.

10. Local and infinitesimal holonomy groups

Let \( \Gamma \) be a connection in a principal fibre bundle \( P(M, G) \), where \( M \) is connected and paracompact. For every connected open subset \( U \) of \( M \), let \( \Gamma_U \) be the connection in \( P \mid U = \pi^{-1}(U) \) induced from \( \Gamma \). For each \( u \in \pi^{-1}(U) \), we denote by \( \Phi^0(u, U) \) and \( P(u, U) \) the restricted-holonomy group with reference point \( u \) and the holonomy bundle through \( u \) of the connection \( \Gamma_U \), respectively. \( P(u, U) \) consists of points \( v \) of \( \pi^{-1}(U) \) which can be joined to \( u \) by a horizontal curve in \( \pi^{-1}(U) \).

The local holonomy group \( \Phi^0(u) \) with reference point \( u \) of \( \Gamma \) is defined to be the intersection \( \bigcap \Phi^0(u, U) \), where \( U \) runs through all connected open neighborhoods of the point \( x = \pi(u) \). If \( \{ U_k \} \) is a sequence of connected open neighborhoods of \( x \) such that \( U_k \supseteq U_{k+1} \) and \( \bigcap U_k = \{ x \} \), then we have obviously \( \Phi^0(u, U_k) \supseteq \Phi^0(u, U_{k+1}) \supseteq \cdots \supseteq \Phi^0(u, U_1) \supseteq \cdots \). Since, for every open neighborhood \( U \) of \( x \), there exists an integer \( k \) such that \( U_k \supseteq U \), we have \( \Phi^0(u) = \bigcap \Phi^0(u, U_k) \). Since each group \( \Phi^0(u, U_k) \) is a connected Lie subgroup of \( G \) (Theorem 4.2), it follows that \( \dim \Phi^0(u, U_k) \) is constant for sufficiently large \( k \) and hence that \( \Phi^0(u) = \Phi^0(u, U_k) \) for such \( k \). The following proposition is now obvious.

Proposition 10.1. The local holonomy groups have the following properties:

1. \( \Phi^0(u) \) is a connected Lie subgroup of \( G \) which is contained in the restricted holonomy group \( \Phi^0(u) \);
2. Every point \( x = \pi(u) \) has a connected open neighborhood \( U \) such that \( \Phi^0(u) = \Phi^0(u, V) \) for any connected open neighborhood \( V \) of \( x \) contained in \( U \);
3. If \( U \) is such a neighborhood of \( x = \pi(u) \), then \( \Phi^0(u) \supseteq \Phi^0(v) \) for every \( v \in P(u, U) \);
4. For every \( a \in G \), we have \( \Phi^0(ua) = \text{ad}(a^{-1}) \Phi^0(u) \);
5. For every integer \( m \), the set \( \{ n(u) \in M \mid \dim \Phi^0(u) \leq m \} \) is open.

As to (5), we remark that \( \dim \Phi^0(u) \) is constant on each fibre of \( P \) by (4) and thus can be considered as an integer valued function on \( M \). Then (5) means that this integer valued function is upper semicontinuous.

Theorem 10.2. Let \( g(u) \) and \( g^*(u) \) be the Lie algebras of \( \Phi^0(u) \) and \( \Phi^0(u) \), respectively. Then \( \Phi^0(u) \) is generated by all \( \Phi^0(v) \), \( v \in P(u) \), and \( g(u) \) is spanned by all \( g^*(v) \), \( v \in P(u) \).

Proof. If \( v \in P(u) \), then \( \Phi^0(u) \supseteq \Phi^0(v) \supseteq \Phi^0(p) \) and \( g(u) = g(v) = g^*(v) \). By Theorem 8.1, \( g(u) \) is spanned by all elements of the form \( \Omega_r(X^*, Y^*) \) where \( r \in P(u) \) and \( X^* \) and \( Y^* \) are horizontal vectors at \( v \). Since \( \Omega_r(X^*, Y^*) \) is contained in the Lie algebra of \( \Phi^0(p, V) \) for every connected open neighborhood \( V \) of \( \pi(v) \), it is contained in \( g^*(v) \). Consequently, \( g(u) \) is spanned by all \( g^*(v) \).
where \( v \in P(u) \). The first assertion now follows from the following lemma.

**Lemma.** If the Lie algebra \( g \) of a connected Lie group \( G \) is generated by a family of subspaces \( \{m_i\} \), then every element of \( G \) can be written as a product \( \exp X_1 \times \exp X_2 \times \ldots \times \exp X_k \), where each \( X_i \) is contained in some \( m_i \).

Proof of Lemma. The set \( H \) of all elements of \( G \) of the above form is clearly a subgroup which is arcwise connected; indeed, every element of \( H \) can be joined to the identity by a differentiable curve which lies in \( H \). By the theorem of Freudenthal-Kuranishi-Yamabe (proved in Appendix 4), \( H \) is a connected Lie subgroup of \( G \). Its Lie algebra contains all \( m_a \) and thus coincides with \( g \). Hence, \( H = G \).

**Theorem 10.3.** If \( \dim \Phi^*(u) \) is constant on \( P \), then \( \Phi^0(u) = \Phi^*(u) \) for every \( u \) in \( P \).

**Proof.** By (3) of Proposition 10.1, \( x = \pi(u) \) has an open neighborhood \( U \) such that \( \Phi^*(u) \supset \Phi^*(v) \) for each \( v \in P(u, U) \). Since \( \dim \Phi^*(u) = \dim \Phi^*(v) \), we have \( \Phi^*(u) = \Phi^*(v) \). By the standard continuation argument, we see that, if \( v \in P(u) \), then \( \Phi^*(u) = \Phi^*(v) \). By Theorem 10.2, we have \( \Phi^0(u) = \Phi^*(u) \).

We now define the infinitesimal holonomy group at each point \( u \) of \( P \) by means of the curvature form and study its relationship to the local holonomy group. We first define a series of subspaces \( m_i(u) \) of \( g \) by induction on \( k \). Let \( m_0(u) \) be the subspace of \( g \) spanned by all elements of the form \( \Omega(u, X, Y) \), where \( X \) and \( Y \) are horizontal vector fields at \( u \). We consider a \( g \)-valued function \( f \) on \( P \) of the form

\[
(I_k) \quad f = V_k \cdots V_1 \Omega(X, Y),
\]

where \( X, Y, V_1, \ldots, V_k \) are arbitrary horizontal vector fields on \( P \). Let \( m_i(u) \) be the subspace of \( g \) spanned by \( m_{k-1}(u) \) and, by the values at \( u \) of all functions \( f \) of the form \( (I_k) \). We then set \( \Phi^*(u) \) to be the union of all \( m_i(u) \), \( k = 0, 1, 2, \ldots \).

**Proposition 10.4.** The subspace \( g'(u) \) of \( g \) is a subalgebra of \( \Phi^*(u) \).

The connected Lie subgroup \( \Phi'(u) \) of \( G \) generated by \( g'(u) \) is called the infinitesimal holonomy group at \( u \).
which shows that $\Omega(X, Y)$ is of type ad $G$, if $X = R_uX$ and $Y = R_uY$. We have also
\[(Xf)_{ua} = X_{ua}f = (R_uX_u)f = X_u(f \circ R_u) = \text{ad} (a^{-1})(X_u)f = \text{ad} (a^{-1})(Xf)_{ua},\]
iff $i$ is of type ad $G$ and $X$ is invariant by $R_u$. This completes the proof of Lemma 1.

Let $X_i = \partial/\partial x^i$, where $x^1, \ldots, x^n$ is a local coordinate system, in a neighborhood $U$ of $x = \pi(u)$. Let $X_i^*$ be the horizontal lift of $X_i$. Consider a $g$-valued function $f$ of the form
\[
(I^*_k) \quad f = X_i^* \cdots X_j^* (\Omega(X_i^*, X_j^*)),
\]
where $i, j_1, \ldots, j_k$ are taken freely from $1, \ldots, n$.

**Lemma 2.** For each $k$, $m_k(u)$ is spanned by $m_{k-1}(u)$ and by the values at $u$ of all functions $f$ of the form $(I^*_k)$.

**Proof of Lemma 2.** The proof is by induction on $k$. The case $k = 0$ is obvious. Every horizontal vector field in $\pi^{-1}(U)$ is a linear combination of $X_i^*$, with real valued functions as coefficients. It follows that every function $f$ of the form $(I^*_k)$ is a linear combination of functions of the form (II), $s \leq k$, with real valued functions as coefficients, in a neighborhood of $u$. It is now clear that, if the assertion holds for $k - 1$, it holds for $k$.

We now prove that $g'(u)$ is a subalgebra of $g$ by establishing the relation $[m(u), m(u)] \subseteq m_{k+1}(u)$ for all pairs of integers $k$ and $s$. In view of Lemma 2, it is sufficient to prove that, for every function $f$ of the form (I) and every function $g$ of the form (II), the function $[f, g](u) = [f(u), g(u)]$ is a linear combination of functions of the form (I), $r \leq k + s + 2$, with real valued functions as coefficients. The proof is by induction on $s$.

Let $s = 0$ and let $f(u) = \Omega_u(X, Y)$, where $X$ and $Y$ are horizontal vector fields. Since $g$ is of type ad $G$, we have, by (2) of Lemma 1,
\[
2[\Omega_u(X, Y), g(u)] = v([X, Y])_u \cdot g.
\]
On the other hand, we have
\[
v([X, Y])_u \cdot g = [X, Y]_u \cdot g - h([X, Y])_u \cdot g = X_u(Yg) - Y_u(Xg) - h([X, Y])_u \cdot g,
\]
where $h([X, Y])$ denotes the horizontal component of $[X, Y]$. The functions $X(Yg)$ and $Y(Xg)$ are of the form $(I_{k+1})$ and the function $h([X, Y])g$ is of the form $(I_{k+1})$. This proves our assertion for $s = 0$ and for an arbitrary $k$.

Suppose now that our assertion holds for $s = 1$ and every $k$. Every function of the form $(I_s)$ can be written as $Xf$, where $f$ is a function of the form $(I_{s+1})$ and $X$ is a horizontal vector field. Let $g$ be an arbitrary function of the form $(II_k)$. Then
\[
[X_u f, g(u)] = X_u ([f, g]) = [f(u), X_u g].
\]
The function $[f, X_u]g$ is a linear combination of functions of the form $(I_1)$, $r \leq k + s + 2$, also by the inductive assumption. Thus, the function $Xf, g$ is a linear combination of functions of the form $(II_s)$, $r \leq s + k + 2$. QED.

**Proposition 10.5.** The infinitesimal holonomy groups have the following properties:

1. $\Phi'(u)$ is a connected Lie subgroup of the local holonomy group $\Phi^*(u)$;
2. $\Phi'(ua) = \text{ad} (a^{-1})(\Phi'(u))$ and $g'(ua) = \text{ad} (a^{-1})(g'(u))$;
3. For each integer $m$, the set $\{\pi(u) \in M : \dim \Phi'(u) \leq m\}$ is open;
4. If $\Phi'(u) = \Phi^*(u)$ at a point $u$, then there exists a connected open neighborhood $U$ of $x = \pi(u)$ such that $\Phi'(v) = \Phi^*(v) = \Phi'(u)$ for every $v \in P(u, U)$.

**Proof.** (1) is evident from Proposition 10.4. (2) follows from Lemma 1. For each $k$, we have $m_k(u)m_k(u) = \text{ad} (a^{-1})(m_k(u))$.

**Proof of Lemma 1.** The proof is by induction on $k$. The case $k = 0$ is a consequence of the fact that $\Omega$ is of type ad $G$. Suppose the assertion holds for $k - 1$. By (3) of Lemma 1 for Proposition 10.4, every function of the form $(II_k)$ is of type ad $G$. Our lemma now follows from Lemma 2 for Proposition 10.4.

(2) means that $\Phi'(u)$ can be considered as a function on $M$. (3) is a consequence of the fact that, if the values of a finite number of functions of the form $(I_k)$ are linearly independent at a point $u$, then they are linearly independent at every point of a neighborhood of $u$. Note that (3) means that $\dim \Phi'(u)$, considered as a function on $M$, is lower semicontinuous. To prove (4), assume
\[ \Phi'(u) \equiv \Phi^*(u) \text{ at a point } u. \text{ Since } \dim \Phi'(u) \text{ is lower semicontinuous and } \dim \Phi^*(u) \text{ is upper semicontinuous [cf. (5) of Proposition 10.11, the point } x = \pi(u) \text{ has a neighborhood } U \text{ such that} \\
\dim \Phi'(v) \geq \dim \Phi^*(v) \text{ for every } v \in \pi^{-1}(U). \]

On the other hand, \( \Phi^*(v) \supseteq \Phi'(v) \text{ for every } v \in \pi^{-1}(U). \) Hence,
\[ \dim \Phi^*(v) \equiv \dim \Phi'(v) = \dim \Phi^*(u) \]
and, consequently, \( \Phi^*(v) = \Phi'(v) \text{ for every } v \in \pi^{-1}(U). \) Applying Theorem 10.3 to \( P \mid U, \) we see that \( \Phi^0(u, U) = \Phi^*(u) \text{ and } \Phi^0(v, U) = \Phi^*(v). \) If \( v \in P(u, U), \) then \( \Phi^0(u, U) = \Phi^0(v, U) \) so that \( \Phi^*(u) = \Phi^*(v). \)

**Theorem 10.6.** If \( \dim \Phi'(v) \) is constant in a neighborhood of \( u \) in \( P, \) then \( \Phi'(u) = \Phi^*(u). \)

**Proof.** We first prove the existence of an open neighborhood \( U \) of \( x = \pi(u) \) such that \( g'(v) = g'(v) \text{ for every } v \in P(u, U). \) Let \( f_1, \ldots, f_s \) be a finite number of functions of the form (II) such that \( f_1(u), \ldots, f_s(u) \) form a basis of \( g'(u). \) At every point \( v \) of a small neighborhood of \( u, f_1(v), \ldots, f_s(v) \) are linearly independent and, by the assumption, they form a basis of \( g'(v). \) Since \( f_1, \ldots, f_s \) are of type \( ad G, f_1(va), \ldots, f_s(va) \) form a basis of \( g'(va) \equiv ad (a^{-1})(g'(v)). \) This means that there exists a neighborhood \( U \) of \( x = \pi(u) \) such that \( f_1(v), \ldots, f_s(v) \) form a basis of \( g'(v) \) for every point \( v \in \pi^{-1}(U). \) Now, let \( v \) be an arbitrary point of \( P(u, U) \) and let \( u_t, 0 \leq t \leq 1, \) be a horizontal curve from \( u \) to \( v \) in \( \pi^{-1}(U) \) so that \( u = u_0 \text{ and } v = u_1. \) We may assume that \( u_t \) is differentiable, the case where \( u_t \) is piecewise differentiable follows easily. Set \( g_i(t) = f_i(u_t), i = 1, \ldots, s, \) and \( X = u_t. \) Since \( X \) is horizontal, we have
\[ (dg_i/dt)_t = (Xf_i)(u_t) \in g'(u_t), \quad i = 1, \ldots, s. \]

Since \( g_1(t), \ldots, g_s(t) \) form a basis for \( g'(u_t), dg_i/dt \) can be expressed by
\[ (dg_i/dt)_t = \sum_{j=1}^s A_{ij}(t)g_j(t), \]
where \( A_{ij}(t) \) are continuous functions of \( t. \) By the lemma for Proposition 3.1, there exists a unique curve \( (a_{ij}(t))_{i,j=1, \ldots, s} \) in

\[ GL(s, R) \text{ such that} \\
d_{a_{ij}dt} = \sum_{k=1}^s A_{ik}a_{kj} \quad \text{and} \quad a_{ij}(0) = \delta_{ij}. \]

(Note that \( (A_{ij}(t)) \in gl(s, R) \) corresponds to \( Y, \in T(G) \) in the lemma for Proposition 3.1.) Let \( (b_{ij}(t)) \) be the inverse matrix of \( (a_{ij}(t)) \) so that
\[ db_{ij}/dt = -\sum_{k=1}^s A_{ik}b_{kj}. \]

Then
\[ \frac{d}{dt} \left( \sum_{i=1}^s b_{ij}g_i \right) = \sum_{i=1}^s b_{ij}\left( \frac{dg_i}{dt} \right) + \sum_{i=1}^s b_{ik}\left( \frac{dg_k}{dt} \right) = 0. \]

Since \( b_{ij}(0) = \delta_{ij}, \) we have
\[ \sum_{i=1}^s b_{ij}(t)g_i(t) = g_i(0). \]

This means that \( g'(u_t) = g'(u) \) and, in particular, \( g'(v) = g'(u). \)

Taking \( U \) sufficiently small, we may assume that
\[ g^*(v) \supseteq g^*(v) \supseteq g^*(v) \supseteq m(v) \text{ for every } v \in P(u, U). \]

By Theorem 8.1, the Lie algebra of \( \Phi^0(u, U) \) is spanned by all \( m(v), v \in P(u, U). \) A fortiori, \( g^*(v) \) is spanned by all \( g^*(v), v \in P(u, U). \) Since \( g^*(v) = g^*(v) \) for every \( v \in P(u, U) \) as we have just shown, we may conclude that \( g^*(u) = g^*(u) \) and \( \Phi^*(u) = \Phi^*(u). \)

**Corollary 10.7.** If \( \dim \Phi'(u) \) is constant on \( P, \) then \( \Phi^*(u) = \Phi^*(u). \)

**Proof.** This follows from Theorems 10.3 and 10.6. QED.

**Theorem 10.8.** For a real analytic connection in a real analytic principal fibre bundle \( P, \) we have \( \Phi^0(u) = \Phi^*(u) = \Phi^*(u) \text{ for every } u \in P. \)

**Proof.** We may assume that \( P = P(u) \) and, in particular, \( P \) is connected. It suffices to show that \( \dim \Phi'(u) \) is locally constant; it then follows that \( \dim \Phi'(u) \) is constant on \( P \) and, by Corollary 10.8, \( \Phi^0(u) = \Phi^*(u) = \Phi^*(u) \text{ for every } u \in P. \) Let \( x_1, \ldots, x_n \) be a real analytic local coordinate system with origin \( x = n(u). \)

Let \( U \) be a coordinate neighborhood of \( x \) given by \( \Sigma_i (x^i)^2 < a^2 \text{ for some } a > 0. \) We want to show that \( \dim \Phi'(u) \) is constant on \( \pi^{-1}(U). \) Let \( X_i = \partial / \partial x_i \) and let \( X^*_i \) be the horizontal lift of \( X_i. \)
For any set of numbers \((a_1, \ldots, a^n)\) with \(\Sigma_i (a'_i)^2 = 1\), consider the vector field \(X = \sum_i a'_i X_i\) on \(U\). Let \(s_i\) be the ray given by \(s_i(t) = a'_i t\) and let \(u_i\) be the horizontal lift of \(X_i\) such that \(u_i = u_0\). We prove that \(g'(u_i) = g'(u_1)\) for every \(t\) with \(|t| < a\). Consider all the functions \(f\) of the form \((II., k \geq 0)\).

\[
\int f = X^* \cdots X^*_k(\Omega(X^*_i, X^*_j))
\]
defined on \(\pi^{-1}(U)\). We set \(h(t) = f(u_t)\). Then the functions \(h(t)\) are analytic functions of \(t\). For each \(t_0\) with \(|t_0| < a\), there exists \(\delta > 0\) such that all the functions \(h(t)\) can be expanded in a common neighborhood \(|t - t_0| < \delta\) in the Taylor series:

\[
h(t) = \sum_{m=0}^{\infty} \frac{1}{m!} (t - t_0)^m h^{(m)}(t_0)
\]

If \(X^*_i\) is the horizontal lift of \(X\), then we can write \(h'(t) = X^*_u g', h''(t) = X^*_u (X^*_g)\) and so on.

The results in this section are due to Ozeki [1].

### 11. Invariant connections

Before we treat general invariant connections, we present an important special case.

**Theorem** 11.1. Let \(G\) be a connected Lie group and \(H\) a closed subgroup of \(G\). Let \(g\) and \(h\) be the Lie algebras of \(G\) and \(H\) respectively.

1. If there exists a subspace \(m\) of \(g\) such that \(g = h + m\) (direct sum) and \(\text{ad} (H) m = m\), then the \(b\)-component of the canonical 1-form \(\theta\) of \(G\) (cf. §4 of Chapter I) with respect to the decomposition \(g = h + m\) defines a connection in the bundle \(G(G/H, H)\) which is invariant by the left translations of \(G\);

2. Conversely, any connection in \(G(G/H, H)\) invariant by the left translations of \(G\) (if it exists) determines such a decomposition \(g = h + m\) and is obtainable in the manner described in (1);

3. The curvature form \(\Omega\) of the invariant connection defined by \(\omega\) in (1) is given by

\[
\Omega(X, Y) = -\frac{1}{2} [X, Y]_h \quad (b\text{-component of } -\frac{1}{2} [X, Y] \in g),
\]

where \(X\) and \(Y\) are arbitrary left invariant vector fields on \(G\) belonging to \(m\);

4. Let \(g(e)\) be the Lie algebra of the holonomy group \(\Phi(e)\) with reference point \(e\) (identity element) of the invariant connection defined in (1). Then \(g(e)\) is spanned by all elements of the form \([X, Y]_h, X, Y \in m\).

**Proof.** Since \(X_\alpha \neq 0\), we may take a local coordinate system \(x^1, \ldots, x^n\) such that \(X = \partial/\partial x_1\) and \(x_\alpha = (0, \ldots, 0, \ldots, 0)\), in a neighborhood of \(x\). The preceding expansions of \(h(t)\) are nothing but the expansions of \(f, (x)\) into power series of \(x^1\). Each \(X_i\) is of the form \(X_i = \sum_i f_{ij} \partial/\partial x_j\). Since \(f\) and \(f_{ij}\) are all analytic, they can be expanded into power series of \((x^1, \ldots, x^n)\) in a common neighborhood \(|x| < a\) for some \(a > 0\). Our lemma then follows from the fact that if \(f_1\) and \(f_2\) are real analytic functions which can be expanded into power series of \((x^1, \ldots, x^n)\) in a neighborhood \(|x| < a\), then the functions \(f_1 f_2\) and \(\partial f_1/\partial x^j\) can be expanded into power series in the same neighborhood.

The proof is straightforward and is similar to that of Proposition 6.4. Under the identification \(g \approx T_e(G)\), the subspace \(m\) corresponds to the horizontal subspace at \(e\).
(2) Let \( \omega \) be a connection form on \( G(G/H, H) \) invariant by the left translations of \( G \). Let \( m \) be the set of left invariant vector fields on \( G \) such that \( \omega(X) = 0 \). It is easy to verify that \( g = h + m \) is a desired decomposition.

(3) A left invariant vector field is horizontal if and only if \( \mathfrak{g} \) is an element of \( m \). Now, (3) follows from Corollary 5.3.

(4) Let \( g_1 \) be the subspace of \( \mathfrak{g} \) spanned by the set \( \{\Omega_\epsilon(X, Y) : X, Y \in \mathfrak{m} \} \). Let \( g_2 \) be the subspace of \( \mathfrak{g} \) spanned by the set \( \{\Omega_\epsilon(X, Y) : X, Y \in \mathfrak{m} \text{ and } u \in \mathfrak{g} \} \). By Theorem 8.1, we have \( g_1 \subseteq \mathfrak{g}(\epsilon) \subseteq \mathfrak{g} \). On the other hand, we have \( g_2 = g \) as \( \Omega_\epsilon(X, Y) = \Omega_\epsilon(X, Y) \) for any \( X, Y \in \mathfrak{m} \) and \( u \in \mathfrak{g} \). Now, (4) follows from (3).

QED.

**Remark.** (1) can be considered as a particular case of Proposition 6.4. Let \( P = (G/H) \times \mathbb{R} \) be the trivial bundle over \( G/H \) with group \( G \). We imbed the bundle \( G(G/H, H) \) into \( P \) by the mapping \( f \) defined by

\[
f(u) = (\pi(u), u), \quad u \in G.
\]

where \( \pi: G \to G/H \) is the natural projection. Let \( \varphi \) be the form defining the canonical flat connection (cf. §9) of \( P \). Its \( \mathfrak{g} \)-component, restricted to the subbundle \( G(G/H, H) \), defines a connection (Proposition 6.4) and agrees with the form \( \omega \) in (1).

Going back to the general case, we first prove the following proposition which is basic in many applications.

**Proposition 11.2.** Let \( \varphi_t \) be a 1-parameter group of automorphisms of a principal fibre bundle \( P(M, G) \) and \( X \) the vector field on \( P \) induced by \( \varphi_t \). Let \( \Gamma \) be a connection in \( P \) invariant by \( \varphi_t \). For an arbitrary point \( u_0 \) of \( P \), we define curves \( u_t, x_t, v_t \) and \( a \), as follows:

\[
u_t = \varphi_t(u_0), \quad x_t = \pi(u_t), \quad v_t = \text{the horizontal lift of } x_t \text{ such that } v_0 = u_0, \quad u_t = v_0 a_t.
\]

Then \( a_t \) is the 1-parameter subgroup of \( G \) generated by \( A = \omega_{u_0}(X) \), where \( \omega \) is the connection form of \( \Gamma \).

Proof. As in the proof of Proposition 3.1, we have

\[
\omega(a_t) = (\text{ad} (a_t^{-1})) \omega(u_t) + a_t^{-1} \dot{a}_t.
\]

Since \( v_t \) is horizontal, we have \( \omega(a_t) = a_t^{-1} \dot{a}_t \). On the other hand, we have \( \omega(u_t) = \omega(u_0) = A \), since the connection form \( \omega \) is invariant by \( \varphi_t \). Thus we obtain \( a_t^{-1} \dot{a}_t = A \).

QED.

Let \( K \) be a Lie group acting on a principal fibre bundle \( P(M, G) \) as a group of automorphisms. Let \( u_0 \) be an arbitrary point of \( P \) which we choose as a reference point. Every element of \( K \) induces a transformation of \( M \) in a natural manner. The set \( J \) of all elements of \( K \) which fix the point \( x_0 = \pi(u_0) \) of \( M \) forms a closed subgroup of \( K \), called the isotropy subgroup of \( K \) at \( x_0 \). We define a homomorphism \( \lambda : J \to G \) as follows. For each \( j \in J \), let \( j u_0 \) be a point in the same fibre as \( u_0 \) and hence is of the form \( j u_0 = u_0 a \) with some \( a \in G \). We define \( \lambda(j) = a \). If \( j \neq j' \), then

\[
u_0 \lambda(j') = (ju_0 \lambda(j')) = (ju_0 \lambda(j')) = (u_0 \lambda(j') \lambda(j')) = (u_0 \lambda(j) \lambda(j')) = (u_0 \lambda(j') \lambda(j')).
\]

Hence, \( \lambda(j') = \lambda(j) \lambda(j') \), which shows that \( \lambda : J \to G \) is a homomorphism. It is also easy to check that \( \lambda \) is differentiable. The induced Lie algebra homomorphism \( j \to \mathfrak{g} \) will be also denoted by the same \( \lambda \). Note that \( \lambda \) depends on the choice of \( u_0 \); the reference point \( u_0 \) is chosen once for all and is fixed throughout this section.

**Proposition 11.3.** Let \( K \) be a group of automorphisms of \( P(M, G) \) and \( \Gamma \) a 'connection in \( P \) invariant by \( K \). We define a linear mapping \( \Lambda : \mathfrak{g} \to \mathfrak{g} \) by

\[
\Lambda(X) = \omega_{u_0}(X), \quad X \in \mathfrak{g}
\]

where \( X \) is the vector field on \( P \) induced by \( X \). Then

\[
(1) \quad \Lambda(X) = \lambda(X) \quad \text{for } X \in \mathfrak{g};
\]

\[
(2) \quad \Lambda(\text{ad} (j)(X)) = \text{ad} (\text{ad} (j)(X)) = \Lambda(X) \quad \text{for } j \in J \text{ and } X \in \mathfrak{g},
\]

where \( \text{ad} (j) \) is the adjoint representation of \( j \) in \( \mathfrak{g} \) and \( \text{ad} (i(j)) \) is that of \( \text{ad} (j) \).

Note that the geometric meaning of \( \Lambda(X) \) is given by Proposition 11.2.

Proof. (1) We apply Proposition 11.2 to the 1-parameter subgroup \( \varphi_t \) of \( K \) generated by \( X \). If \( X \in \mathfrak{g} \), then the curve \( x_t = \pi(\varphi_t(u_0)) \) reduces to a single point \( x_0 = \pi(u_0) \). Hence we have \( \varphi_t(u_0) = u_0 \lambda(t) \). Comparing the tangent vectors of the orbits \( \varphi_t(u_0) \) and \( u_0 \lambda(t) \) at \( u_0 \), we obtain \( \Lambda(X) = \lambda(X) \).
(2) Let $X \in \mathfrak{t}$ and $j \in J$. We set $Y = \text{ad}(j)(X)$. Then $Y$ generates the 1-parameter subgroup $\{j^t \cdot f, \; j \in J\}$ which maps $u_0$ into $j \cdot f$. It follows that $\bar{Y} = j(\bar{R}_{Xu_0} X_{u_0})$. Since the connection form $\omega$ is invariant by $j$, we have

$$\omega_{u_0}(\bar{Y}) = \omega_{u_0}(j(\bar{R}_{Xu_0} X_{u_0})) = \omega_{j^{-1}(\bar{R}_{Xu_0} X_{u_0})} = \text{ad} \left( j^{j^{-1}}(X) \right) \left( \omega_{u_0}(X_{u_0}) \right) = \text{ad} \left( j^{j^{-1}}(X) \right) (\Lambda(X)).$$

QED.

**Proposition 11.4.** With the notation of Proposition 11.3, the curvature form $\Omega$ satisfies the following condition:

$$2\omega_{u_0}(X, Y) = \left[ A(X), A(Y) \right] - \Lambda([X, Y]) \text{ for } X, Y \in \mathfrak{t}.$$ 

Proof. From the structure equation (Theorem 5.2) and Proposition 3.11 of Chapter I, we obtain

$$2\omega_{u_0}(X, Y) = 2d\omega(X, Y) + [\omega(X), \omega(Y)] = [A(X), A(Y)] - \Lambda([X, Y]).$$

Since $\omega$ is invariant by $K$, we have by (c) of Proposition 3.2 of Chapter I (cf. also Proposition 3.5 of Chapter I)

$$X(\omega(Y)) - \omega([X, Y]) + [\omega(X), \omega(Y)].$$

On the other hand, $X \rightarrow \bar{X}$ being a Lie algebra homomorphism, we have

$$\omega_{u_0}(X, Y) = \Lambda([X, Y]).$$

Thus we obtain

$$2\omega_{u_0}(X, Y) = \Lambda([X, Y]).$$

QED.

We say that $K$ acts fibre-transitively on $P$ if, for any two fibres of $P$, there is an element of $K$ which maps one fibre onto the other, that is, if the action of $K$ on the base $M$ is transitive. If $J$ is the isotropy subgroup of $K$ at $x_0 = \pi(u_0)$ as above, then $M$ is the homogeneous space $[K]/J$.

The following result is due to Wang [1].

**Theorem 11.5.** If a connected Lie group $K$ is a fibre-transitive automorphism group of a bundle $P(M, G)$ and if $J$ is the isotropy subgroup of $K$ at $x_0 = \pi(u_0)$, then there is a $1:1$ correspondence between the set of $K$-invariant connections in $P$ and the set of linear mappings $A: \mathfrak{t} \rightarrow \mathfrak{g}$ which satisfies the following conditions in Proposition 11.3; the correspondence is given by

$$A(X) = \omega_{u_0}(X) \quad \text{for } X \in \mathfrak{t},$$

where $X$ is the vector field on $P$ induced by $X$.

Proof. In view of Proposition 11.3, it is sufficient to show that, for every $A: \mathfrak{t} \rightarrow \mathfrak{g}$ satisfying (1) and (2) of Proposition 11.3, there is a $K$-invariant connection form $\omega$ on $P$ such that $A(X) = \omega_{u_0}(X)$ for $X \in \mathfrak{t}$.

Let $X^* \in \mathcal{T}_u(P)$. Since $K$ is fibre-transitive, we can write

$$u_0 = kua = k \circ R_{u_0}^a,$$

where $k \in K$, $a \in G$, $X \in \mathfrak{k}$ and $X^*$ is the fundamental vector field corresponding to $a \circ g$. We then set

$$\omega(X^*) = \Lambda(\Lambda(X) + A).$$

We first prove that $\omega(X^*)$ is independent of the choice of $X$ and $A$. Let

$$X_{u_0} + A_{u_0} = Y_{u_0} + B_{u_0},$$

where $Y \in \mathfrak{t}$ and $B \in \mathfrak{g}$.

so that $X_{u_0} = Y_{u_0} = B_{u_0} = A_{u_0}$. From the definition of $\lambda: \mathfrak{j} \rightarrow \mathfrak{g}$, it follows that $\lambda(X - Y) = B - A$. By condition (1) of Proposition 11.3, we have $\lambda(X - Y) = \Lambda(X - Y) = \Lambda(X) - \Lambda(Y)$.

Hence, $\Lambda(X) = A + \Lambda(Y) + B$.

We next prove that $\omega(X^*)$ is independent of the choice of $k$ and $a$. Let

$$u_0 = kua = k_1ua_1 \quad (k_1 \in K \text{ and } a_1 \in G),$$

so that $k_1k^{-1}u_0 = u_0a_1^{-1}a$ and $k_1k^{-1} \in J$. We set $j = k_1k^{-1}$. Then $\lambda(j) = a_1^{-1}a$. We have

$$k_1 \circ R_{a_1} X^* = j \circ R_{a_1} \circ R_{k_1} X^* = j \circ R_{k_1} \circ R_{a_1}(X_{u_0} + A_{u_0}).$$

By Proposition 1.7 of Chapter I, we have

$$j \circ R_{k_1} \circ R_{a_1}(X_{u_0}) = j \circ R_{k_1} \circ R_{a_1} \circ R_{k_1} \circ R_{a_1}(X_{u_0}) = \Lambda(X_{u_0} + A_{u_0}).$$
By Proposition 5.1 of Chapter I, we have
\[ j \circ R_{\lambda^{-1}}(A_{u_0}^*) = R_{\lambda^{-1}}(jA_{u_0}^*) = R_{\lambda^{-1}}A_{u_0}^* = C_{u_0}^*, \]
where \( C = \text{ad} (\lambda(j))(A) \). Hence we have
\[ k_1 \circ R_{u_0} X^* = \tilde{X}_{u_0} + C_{u_0}^*, \]
\[ \text{ad} (a_1)(\lambda(Z) + C) = \text{ad} (a_1)(\lambda(j)(A(X) + A)) = \text{ad} (a_1)(\lambda(j)(\lambda(X) + A)) = \text{ad} (a_1)(\lambda(X) + A). \]

This proves our assertion that \( w(X^*) \) depends only on \( X^* \).

We now prove that \( w \) is a connection form. Let \( X^* \in T_{\epsilon}(P) \) and \( u_0 = k\mu a \) as above. Let \( b \) be an arbitrary element of \( G \). We set \( Y^* = R_{\epsilon}X^* \in T_{\epsilon}(P) \), where \( v = k\mu(b^{-1}a) \), so that \( u_0 = k\mu b^{-1}a = k\mu(b^{-1}a) \).

Then have
\[ k \circ R_{\mu^{-1}}(Y^*) = k \circ R_{\mu^{-1}}R_{\mu}X^* = k \circ R_{\mu}X^* = (\tilde{X}_{u_0} + A_{u_0}^*) \]
and hence
\[ \omega(R_{\mu}X^* \circ Y^*) = \omega(Y^*) = \text{ad} (b^{-1}a)(\lambda(X) + A) = \text{ad} (b^{-1})(\omega(X^*) \circ Y^*), \]
which shows that \( \omega \) satisfies condition (b') of Proposition 1.1.

Now, let \( A \) be any element of \( \mathfrak{g} \) and let \( u_0 = k\mu a \). Then
\[ k \circ R_{u_0} A_{u_0}^* = R_{u_0} \circ k(A_{u_0}^*) = R_{u_0} A_{u_0}^* = B_{u_0}^*, \]
where \( B = \text{ad} (a^{-1})(A) \).

Hence we have
\[ w(A_{u_0}^*) = \text{ad} (a)(B) = A, \]
which shows that \( w \) satisfies condition (a') of Proposition 1.1.

To prove that \( w \) is differentiable, let \( u_1 \) be an arbitrary point of \( P \) and let \( u_0 = k_1u_1a \). Consider the \( \text{fib}_1 \text{bund} \leq K(M, J) \), where \( M = K/J \). Let \( \gamma : U \to K \) be a local cross section of this bundle defined in a neighborhood \( U \) of \( \gamma(u_1) \) such that \( \gamma(\pi(u_1)) = k_1 \).

Then \( k = \gamma(\pi(u)) \) and \( a \in G \) by
\[ k = \gamma(\pi(u)) \text{ and } u_0 = k\mu a. \]

Then both \( k \) and \( a \) depend differentiably on \( u \). We decompose the vector space \( \mathfrak{g} \) into a direct sum of subspaces: \( \mathfrak{g} = \mathfrak{j} + \mathfrak{m} \). For an arbitrary \( X^* \cdot T_{\epsilon}(P) \), we set
\[ k \circ R_{\mu}X^* = X_{u_0} + A_{u_0}^*, \]
where \( X \in \mathfrak{m} \).

Then both \( X \) and \( A \) are uniquely determined and depend differentiably on \( X^* \). Thus \( \omega(X^*) = \text{ad} (a)(\lambda(X) + A) \) depends differentiably on \( X^* \).

Finally, we prove that \( \omega \) is invariant by \( K \). Let \( \epsilon X^* \in T_{\epsilon}(P) \) and \( u_0 = k\mu a \). Let \( k_1 \) be an arbitrary element of \( K \). Then \( k_1X^* \in T_{k_1}(P) \) and \( u_0 = k_1^{-1}(k_1X^* \circ a) \).

Hence,
\[ k_1^{-1} \circ R_{\mu}(k_1X^*) = k \circ R_{\mu}(X^*). \]

From the construction of \( \omega \), we see immediately that \( \omega(k_1X^*) = \omega(X^*). \)

In the case where \( K \) is fibre-transitive on \( P \), the curvature form \( \Omega \), which is a tensorial form of type \( \text{ad} G \) (cf. §5) and \( \Omega \) is invariant by \( K \), is completely determined by the values \( \Omega_{u_0}(X, Y) \), \( X, Y \in \mathfrak{m} \).

Proposition 11.4 expresses \( \Omega_{u_0}(X, Y) \) in terms of \( A_{u_0}^* \). As a consequence of Proposition 11.4 and Theorem 11.5, we obtain

**Corollary 11.6.** The \( K \)-invariant connection in \( P \) defined by \( A \) is flat if and only if \( \lambda : \mathfrak{m} \to \mathfrak{g} \) is a Lie algebra homomorphism.

Proof. A connection is flat if and only if its curvature form vanishes identically (Theorem 9.1). QED.

**Theorem 11.7.** Assume in Theorem 11.5 that \( \mathfrak{m} \) admits a subspace \( \mathfrak{m} \) such that \( \mathfrak{m} = \mathfrak{m} + \mathfrak{m} \) (direct sum) and \( \text{ad} (j)(m) = m \), where \( \text{ad}(j) \) is the adjoint representation of \( j \) in \( \mathfrak{m} \). Then

(1) There is a \( 1 : 1 \) correspondence between the set of \( K \)-invariant connections in \( P \) and the set of linear mappings \( A_{u_0}^* : \mathfrak{m} \to \mathfrak{g} \) such that
\[ \lambda_{u_0}(\text{ad} (j)(X)) = \lambda(j)(\lambda_{u_0}(X)) \quad \text{for } X \in \mathfrak{m} \text{ and } j \in \mathfrak{j}; \]
the correspondence is given via Theorem 11.5 by
\[ A_{u_0}^*(X) = \lambda(X) \quad \text{if } X \in \mathfrak{m}, \]
\[ A_{u_0}^*(X) = R_{\mu}X \quad \text{if } X \in \mathfrak{m}. \]

(2) The curvature form \( \Omega \) of the \( K \)-invariant connection defined by \( A_{u_0}^* \) satisfies the following condition:
\[ 2\Omega_{u_0}(X, Y) = \left[ \lambda(X), \lambda(Y) \right] - \lambda([X, Y]) \text{ for } X, Y \in \mathfrak{m}. \]
where \([X, Y]_m\) (resp. \([X, Y]_j\)) denotes the \(m\)-component (resp. \(j\)-component) of \([X, Y]_t\) in \(\mathfrak{g}\).

Proof. Let \(A: \mathfrak{g} \to \mathfrak{g}\) be a linear mapping satisfying (1) and (2) of Proposition 11.3. Let \(A_m\) be the restriction of \(A\) to \(\mathfrak{g}\). It is easy to see that \(A_m\) gives a desired correspondence. The statement (2) is a consequence of Proposition 11.4.

QED.

In Theorem 11.7, the \(K\)-invariant connection in \(\mathfrak{p}\) defined by \(A_m\) is called the horizontal vector field on \(\mathfrak{p}\) and \(8\) is the vector field on \(\mathfrak{p}\) induced by an element \(X\) of \(\mathfrak{t}\).

Remark. (1) and (3) of Theorem 11.1 follow from Theorem 11.7 if we set \(\mathfrak{p}(M, G) = G(G/H, H)\) and \(K = G\); the invariant connection in Theorem 11.1 is the canonical connection just defined.

Finally, we determine the Lie algebra of the holonomy group of a \(K\)-invariant connection.

**Theorem 11.8.** With the same assumptions and notation as in Theorem 11.5, the Lie algebra \(g(u_0)\) of the holonomy group \(\Phi(u_0)\) of the \(K\)-invariant connection defined by \(A: \mathfrak{t} \to \mathfrak{g}\) is given by

\[
m_0 + [\Lambda(t), m_0] + [\Lambda(t), [\Lambda(t), m_0]] + \cdots,
\]

where \(m_0\) is the subspace of \(\mathfrak{g}\) spanned by

\[
\{([\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]); X, Y \in \mathfrak{t})\}.
\]

Proof. Since \(K\) is fibre-transitive on \(\mathfrak{p}\), the restricted holonomy group \(\Phi'(u_0)\) coincides with the infinitesimal holonomy group \(\Phi'(u_0)\) by virtue of Corollary 10.7. We define a series of subspaces \(m_k\), \(k = 0, 1, 2, \ldots\), of \(\mathfrak{g}\) as follows:

\[
m_1 = m_0 + [\Lambda(t), m_0],
\]

\[
m_2 = m_0 + [\Lambda(t), m_0] + [\Lambda(t), [\Lambda(t), m_0]] + \cdots
\]

and so on. We defined in §10 an increasing sequence of subspaces \(m_k(u_0)\), \(k = 0, 1, 2, \ldots\), of \(\mathfrak{g}\). Since the union of these subspaces \(m_k(u_0)\) is the Lie algebra \(\mathfrak{g}(u_0)\) of the infinitesimal holonomy group \(\Phi'(u_0)\), it is sufficient to prove that \(m_k = m_k(u_0)\) for \(k = 0, 1, 2, \ldots\).

By Proposition 11.4, the subspace \(m_0\) is spanned by \(\{\Omega(w(X, Y)); X, Y \in \mathfrak{t}\}\). Since \(\Omega(w(X, Y)) = \Omega(w(hX, hY))\), where \(hX\) and \(hY\)


denote the horizontal components of \(X\) and \(Y\) respectively, \(m_0\) coincides with \(m_0(u_0)\).

We need the following lemmas.

**Lemma 1.** If \(Y\) is a horizontal vector field on \(\mathfrak{p}\) and \(8\) is the vector field on \(\mathfrak{p}\) induced by an element \(X\) of \(\mathfrak{t}\), then \([X, Y]\) is horizontal.

Proof of Lemma 1. By (c) of Proposition 3.2 of Chapter I (cf. also Proposition 3.5 of Chapter I), we have

\[
x_\omega(Y) = (L_\omega X)(Y) + \omega([X, Y]).
\]

Since \(\omega(Y) = 0\) and \(L_\omega X = 0\), we have \(\omega([X, Y]) = 0\).

**Lemma 2.** Let \(V, W, Y_1, \ldots, Y_r\) be arbitrary horizontal vector fields on \(\mathfrak{p}\) and let \(X\) be the vector field on \(\mathfrak{p}\) induced by an element \(X\) of \(\mathfrak{t}\). Then

\[
x_\omega(Y_1, \ldots, Y_r, \Omega(V, W)) \in \mathfrak{m}_r(u_0).
\]

Proof of Lemma 2. We have

\[
x_\omega(Y_1, \ldots, Y_r, \Omega(V, W))
\]

\[
= (Y_1)_\omega(Y_1, \ldots, Y_r, \Omega(V, W)) \mod \mathfrak{m}_r(u_0),
\]

since \([X, Y]\) is horizontal by Lemma 1 and \([X, Y]_m = (Y_1)_\omega(Y_1, \ldots, Y_r, \Omega(V, W))\) is in \(\mathfrak{m}_r(u_0)\). Repeating this process, we obtain

\[
x_\omega(Y_1, \ldots, Y_r, \Omega(V, W))
\]

\[
= (Y_r)_\omega(Y_1, \ldots, Y_r, \Omega(V, W)) \mod \mathfrak{m}_r(u_0).
\]

By the same argument as in the proof of Lemma 1, we have

\[
x_\omega(Y_1, \ldots, Y_r, \Omega(V, W)) = (L_\omega X)(Y_1, \Omega(V, W)) + \Omega([X, V], W) + \Omega(V, [X, W]).
\]

Since \(L_\omega \Omega = 0\), we have

\[
(Y_1)_\omega(Y_1, \ldots, Y_r, \Omega(V, W))
\]

\[
= (Y_1)_\omega(Y_1, \ldots, Y_r, \Omega([X, V], W)) + (Y_1)_\omega(Y_1, \ldots, Y_r, \Omega(V, [X, W])).
\]

The two terms on the right hand side belong to \(\mathfrak{m}_r(u_0)\) as \([X, V]\) and \([X, W]\) are horizontal by Lemma 1. This completes the proof of Lemma 2.

Let \(X_i = \partial/\partial x_i\), where \(x_1, \ldots, x^n\) is a local coordinate system in a neighborhood of \(x_0 = \pi(u_0)\). Let \(X_i^*\) be the horizontal lift of
Let $f = X_p^* \cdots X_q^*(x)$ be a function of the form (II) as defined in §10. If $\widehat{X}$ and $\widehat{x}$ denote the horizontal and the vertical components of $X$ respectively, then Lemma 1 for Proposition 10.4 implies

$$\widehat{X}_0f = -(\nu X)_0f + \nu X_0f = [\omega_{\nu X}(X), f(u_0)] + \nu X_0f.$$ 

Since $X_0f \in m_i(u_0)$ by Lemma 2 and since $\omega_{\nu X}(X) = A(X)$, we have

$$\nu X_0f = [A(X), f(u_0)] \mod m_i(u_0).$$

Assuming that $m_s = m_i(u_0)$ for all $s < r$, we shall show that $m_s = m_i(u_0)$. Since $K$ is fibre-transitive on $P$, every horizontal vector at $u_0$ is of the form $(h, X)_t$ for some $X \in \mathfrak{f}$ and $t \in \mathfrak{t}$. Hence, $m_s(u_0)$ is spanned by $m_{s-1}(u_0)$ and the set of all $(h, X)_t f$, where $X \in \mathfrak{f}$ and $t$ is a function of the form (II) on $X$. On the other hand, $m_s$ is spanned by $m_{s-1}(u_0)$ and by $[A(t), m_{s-1}(u_0)]$. In other words, $m_s$ is spanned by $m_{s-1}(u_0)$ and the set of all $[A(X), f(u_0)]$, where $X \in \mathfrak{f}$ and $t$ is a function of the form (II) on $X$. Our assertion $m_s = m_i(u_0)$ follows from the congruence $(h, X)_t f \equiv [A(X), f(u_0)] \mod m_i(u_0)$.

Remark. (4) of Theorem 11.1 is a corollary to Theorem 11.8 (cf. Remark made after the proof of Theorem 11.7).

CHAPTER III

Linear and Affine Connections

1. Connections in a vector bundle

Let $E$ be either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. $F^m$ the vector space of all $m$-tuples of elements of $F$ and $GL(m; F)$ the group of all $m \times m$ non-singular matrices with entries from $F$. The group $GL(m; F)$ acts on $F^m$ on 'the left in a natural manner; if $a = (a_j) \in GL(m; F)$ and $\xi = (\xi_1, \ldots, \xi_m) \in F^m$, then $a \xi = (\Sigma \xi_{j1}, \ldots, \Sigma \xi_{jm}) \in F^m$.

Let $P(M, G)$ be a principal fibre bundle and $\rho$ a representation of $G$ into $GL(m; F)$. Let $E(M, F^m, G, P)$ be the associated bundle with standard fibre $F^m$ on which $G$ acts through $\rho$. We call $E$ a real or complex vector bundle over $M$ according as $F = \mathbb{R}$ or $F = \mathbb{C}$.

Each fibre $\pi_E^{-1}(x), x \in M$, of $E$ has the structure of a vector space such that every $u \in \mathcal{P}$ with $\pi(u) = x$, considered as a mapping of $F^m$ onto $\pi_E^{-1}(x)$, is a linear isomorphism of $F^m$ onto $\pi_E^{-1}(x)$. Let $\mathcal{P}$ be the set of cross sections $\phi: M \to E$; it forms a vector space over $F$ (of infinite dimensions, if $m \geq 1$) with addition and scalar multiplication defined by

$$\phi + \psi, \lambda \phi \in \mathcal{P}, \quad \phi(x) + \psi(x), \quad \lambda \phi(x).$$

We may also consider $S$ as a module over the algebra of $F$-valued functions; if $\lambda$ is an $F$-valued function on $M$, then

$$(\lambda \phi)(x) = \lambda(x) \cdot \phi(x), \quad \phi \in S, \quad x \in M.$$
lift of $\tau$ to $P$, then, for each fixed $\xi \in F^m$, the curve $\tau' = u_\xi \xi$ is, by definition, a horizontal lift of $\tau$ to $E$.

Let $\varphi$ be a section of $E$ defined on $\tau = x_t$ so that $\pi_E \circ \varphi(x_t) = x_t$ for all $t$. Let $\dot{x}_t$ be the vector tangent to $\tau$ at $x_t$. Then, for each fixed $t$, the covariant derivative $\nabla_{\dot{x}_t} \varphi$ of $\varphi$ in the direction of (or with respect to) $\dot{x}_t$ is defined by

$$\nabla_{\dot{x}_t} \varphi = \lim_{h \to 0} \frac{1}{h} [\pi_{E}^{-1}(\varphi(x_{t+h})) - \varphi(x_t)],$$

where $\pi_{E}^{-1}: \pi_{E}^{-1}(x_t) \to \pi_{E}^{-1}(x_t)$ denotes the parallel displacement of the fibre $\pi_{E}^{-1}(x_t)$ along $\tau$ from $x_{t+h}$ to $x_t$. Thus, $\nabla_{\dot{x}_t} \varphi \in \pi_{E}^{-1}(x_t)$ for every $t$ and defines a cross section of $E$ along $\tau$. The cross section $\varphi$ is parallel, that is, the curve $\varphi(x_t)$ is horizontal, if and only if $\nabla_{\dot{x}_t} \varphi = 0$ for all $t$. The following formulas are evident. If $\varphi$ and $\psi$ are cross sections $\varphi$ of $E$ defined on $\tau$ at $x_t$, then

$$\nabla_{\dot{x}_t} (\varphi + \psi) = \nabla_{\dot{x}_t} \varphi + \nabla_{\dot{x}_t} \psi.$$ 

If $\lambda$ is an $F$-valued function defined on $\tau$, then

$$\nabla_{\dot{x}_t} (\lambda \varphi) = \lambda(x_t) \cdot \nabla_{\dot{x}_t} \varphi + (\dot{x}_t \lambda) \cdot \varphi(x_t).$$

The last formula follows immediately from

$$\nabla_{\dot{x}_t} (\lambda \varphi) = \nabla_{\dot{x}_t} (\lambda(x_t) \cdot \varphi(x_t)) = \lambda(x_{t+h}) \cdot \nabla_{\dot{x}_t} \varphi + (\dot{x}_t \lambda) \cdot \varphi(x_t).$$

Let $X \in T_x(M)$ and $\varphi$ a cross section of $E$ defined in a neighborhood of $x$. Then the covariant derivative $\nabla_X \varphi$ of $\varphi$ in the direction of $X$ is defined as follows. Let $\tau = x_t$, $-\varepsilon \leq t \leq \varepsilon$, be a curve such that $X = \dot{x}_0$. Then set

$$\nabla_X \varphi = \nabla_{\dot{x}_0} \varphi.$$ 

It is easy to see that $\nabla_X \varphi$ is independent of the choice of $\tau$. A cross section $\varphi$ of $E$ defined on an open subset $U$ of $A^4$ is parallel if and only if $\nabla_X \varphi = 0$ for all $X \in T_x(U)$, $x \in U$.

**Proposition 1.1.** Let $X, Y \in T_x(M)$ and let $\varphi$ and $\psi$ be cross sections $\varphi$ of $E$ defined in a neighborhood of $x$. Then

1. $\nabla_{X+Y} \varphi = \nabla_X \varphi + \nabla_Y \varphi$;
2. $\nabla_X (\varphi + \psi) = \nabla_X \varphi + \nabla_X \psi$;
3. $\nabla_{\lambda X} \varphi = \lambda \nabla_X \varphi$, where $\lambda \in F$;
4. $\nabla_X (\lambda \varphi) = \lambda(x) \cdot \nabla_X \varphi + (\dot{x}_t \lambda) \cdot \varphi(x)$, where $\lambda$ is an $F$-valued function defined in a neighborhood of $x$.

Proof. We proved (2) and (4). (3) is obvious. Finally, (1) will follow immediately from the following alternative definition of covariant differentiation.

Suppose that a cross section $\varphi$ of $E$ is defined on an open subset $U$ of $M$. As in Example 5.2 of Chapter II, we associate with $\varphi$ an $F^n$-valued function $f$ on $\pi^{-1}(U)$ as follows:

$$f(v) = v^{-1}(\varphi(v)), \quad v \in \pi^{-1}(U).$$

Given $X \in T_x(M)$, let $X^* \in T_x(P)$ be a horizontal lift of $X$. Since $f$ is an $F^n$-valued function, $X^*f$ is an element of $F^n$ and $u(X^*f)$ is an element of the fibre $\pi^{-1} E$. We have

**Lemma.** $\nabla_X \varphi = u(X^*f)$.

Proof of Lemma. Let $\tau = x_t$, $-\varepsilon \leq t \leq \varepsilon$, be a curve such that $X = \dot{x}_0$. Let $\tau^* = u_\tau$ be a horizontal lift of $\tau$ such that $u_0 = u$ so that $X^* = u_0$. Then we have

$$X^*f = \lim_{h \to 0} \frac{1}{h} [f(u_h) \cdot f(u)] = \lim_{h \to 0} \frac{1}{h} [u^{-1}(\varphi(x_h)) - u^{-1}(\varphi(x))].$$

In order to prove the lemma, it is sufficient to prove

$$\tau^0_0(\varphi(x_h)) = u \circ u^{-1}_h(\varphi(x)).$$

Set $\xi = u^{-1}_h(\varphi(x))$. Then $u_\xi$ is a horizontal curve in $E$. Since $u_\xi \xi = \varphi(x_h)$, $\varphi(x_h)$ is the element of $E$ obtained, by the parallel displacement of $u_\xi \xi$ along $\tau$ from $x_h$ to $x_t$. This implies $\tau^0_0(\varphi(x_h)) = u \circ u^{-1}_h(\varphi(x))$, thus completing the proof of the lemma.

Now, (1) of Proposition 1.1 follows from the lemma and the fact that, if $X, Y \in T_x(M)$ and $X^*, Y^* \in T_\xi(P)$ are horizontal lifts of $X$ and $Y$ respectively, then $X^* + Y^*$ is a horizontal lift of $X + Y$.

Q.E.D.

If $\varphi$ is a cross section of $E$ defined on $M$ and $X$ is a vector held on $M$, then the covariant derivative $\nabla_X \varphi$ of $\varphi$ in the direction of (or with respect to) $X$ is defined by

$$(\nabla_X \varphi)(x) = \nabla_X \varphi(x).$$
Then, as an immediate consequence of Proposition 1.1, we have

**Proposition 1.2.** Let \( X \) and \( Y \) be vector fields on \( M, p \) and \( q \) cross sections of \( E \) on \( M \) and \( \lambda \) an \( F \)-valued function on \( M \). Then

1. \( \nabla_{X+Y} \phi = \nabla_X \phi + \nabla_Y \phi; \)
2. \( \nabla_X (q + p) = \nabla_X q + \nabla_X p; \)
3. \( \nabla_{X \cdot Y} = \lambda \cdot \nabla_X \phi + (X \lambda) \phi. \)

Let \( X \) be a vector field on \( M \) and \( X^* \) the horizontal lift of \( X \) to \( P \). Then covariant differentiation \( \nabla_X \) corresponds to Lie differentiation \( L_X \), in the following sense. In Example 5.2 of Chapter II, we saw that there is a 1-1 correspondence between the set of cross sections \( \phi : M \to E \) and the set of \( F^m \)-valued functions \( f \) on \( P \) such that \( f(u \phi) = \phi^{-1}(f(u)) \), \( a \in G \) (i.e. \( \phi^{-1} \) is the inverse \( P \)-bundle). The correspondence is given by \( u \phi = \phi^{-1}(f(u)), u \in P. \) We then have

**Proposition 1.3.** If \( \phi : M \to E \) is a cross section and \( f : P \to F^m \) \( f \) is the corresponding function, then \( X^* f \) is the function corresponding to the cross section \( \nabla_X \phi. \)

Proof. This is an immediate consequence of the lemma for Proposition 1.1. QED.

A fibre metric \( g \) in a vector bundle \( E \) is an assignment, to each \( x \in M \), of an inner product \( g_x \) in the fibre \( E_x \). \( g \) is differentiable in \( x \) in the sense that, if \( \phi \) and \( \psi \) are differentiable cross sections of \( E \), then \( g_{(\phi(x), \psi(x))} \) depends differentiably on \( x \).

When \( E \) is a complex vector bundle, the inner product is understood to be hermitian:

\[ g_x((\mathbb{E}_1, \mathbb{E}_2) = g_x((\mathbb{E}_1, \mathbb{E}_2)) \]

**Proposition 1.4.** If \( M \) is paracompact, every vector bundle \( E \) over \( M \) admits a fibre metric.

Proof. This follows from Theorem 5.7 of Chapter I just as the existence of a Riemannian metric on a paracompact manifold. We shall give here another proof using a partition of unity. Let \( \{ U_i \}_{i \in I} \) be a locally finite open covering of \( M \) such that \( \pi^{-1}(U_i) \) is isomorphic with \( U_i \times F^m \) for each \( i \). Let \( \{ s_i \} \) be a partition of unity subordinate to \( \{ U_i \} \) (cf. Appendix 3). Let \( h_i \) be a fibre metric in \( E \mid U_i = \pi^{-1}(U_i) \). Set \( g = \sum_i s_i h_i \), that is,

\[ g((\mathbb{E}_1, \mathbb{E}_2)) = \sum_i s_i(x) h_i((\mathbb{E}_1, \mathbb{E}_2)) \]

for \( x \in U_i \). Since \( \{ U_i \} \) is locally finite and \( s_i \) vanishes outside \( U_i \), \( g \) is a well defined fibre metric.

**III. LINEAR AND AFFINE CONNECTIONS.**

Let \( g \) be a fibre metric in a vector bundle \( E(M, F^m, G, P) \), we construct a reduced Subbundle \( Q(M, H) \) of \( P(M, G) \) as follows. In the standard fibre \( F^m \) of \( E \), we consider the canonical inner product \( (, ) \) defined by

\[ (\xi, \eta) = \sum_i \xi_i \eta_i \]

for \( \xi, \eta \in F^m \). Then \( Q \) is a closed submanifold of \( P \) whose structure group \( H \) is given by

\[ H = \{ a \in G; p(a) \in O(m) \} \]

if \( F = \mathbb{R} \),

\[ H = \{ a \in G; p(a) \in U(m) \} \]

if \( F = \mathbb{C} \),

where \( p \) is the representation of \( G \) in \( GL(m; F) \).

Given a fibre metric \( g \) in \( E \), a connection in \( E \) is called a metric connection if the parallel displacement of fibres of \( E \) preserves the fibre metric \( g \). More precisely, for every curve \( \tau = x(t), 0 \leq t \leq 1 \), of \( M \), the parallel displacement \( \pi^{-1}(x(t)) \) along \( \tau \) is isometric.

**Proposition 1.5.** Let \( \Gamma \) be a fibre metric in a vector bundle \( E(M, F^m, G, P) \) and \( Q(M, H) \) the reduced subbundle of \( P(M, G) \) defined by \( g \). A connection \( \Gamma \) in \( P \) is reducible to a connection \( \Gamma' \) in \( Q \) if and only if \( \Gamma \) is a metric connection.

Proof. Let \( \tau = x(t), 0 \leq t \leq 1 \), be a curve in \( M \). Let \( \xi, \eta \in F^m \) and \( u_0 \in Q \) with \( \pi(u_0) = x_0 \). Let \( \tau^* = u_0 \) be the horizontal lift of \( \tau \) to \( P \) starting from \( u_0 \) so that both \( \tau^* = u_0(\xi) \) and \( \tau^* = u_0(\eta) \) are horizontal lifts of \( \tau \) to \( E \). If \( \Gamma \) is reducible to a connection \( \Gamma' \) in \( Q \), then \( u_0 \in Q \) for all \( t \). Hence,

\[ g(u_0(\xi), u_0(\eta)) = (\xi, \eta) = g(u_0(\xi), u_0(\eta)), \]

proving that \( \Gamma \) is a metric connection. Conversely, if \( \Gamma \) is a metric
connection, then
\[ g(u_i(\xi), u_i(\eta)) = g(u_0(\xi), u_0(\eta)) = (\xi, \eta). \]

Hence, \( u_i \in Q \). This means that \( \Gamma \) is reducible to a connection in \( Q \) by Proposition 7.2 of Chapter II. QED.

Proposition 1.5, together with Theorem 2.1 of Chapter II, implies that, given a fibre metric \( g \) in a vector bundle \( E \) over a paracompact manifold \( K \), there is a metric connection in \( P \).

Let \( E(M, F', \rho, P) \) be a vector bundle such that \( G = GL(m; F) \). Let \( E^i \in \mathfrak{g}(m; F) \), Lie algebra of \( GL(m; F) \), be the \( m \times m \) matrix such that the entry at the \( i \)-th column and the \( j \)-th row is 1 and other entries are all zero. Then \( \{ E^i, i = 1, \ldots, m \} \) form a basis of the Lie algebra \( \mathfrak{g}(m; F) \). Let \( \omega \) and \( \Omega \) be the connection form and the curvature form of a connection \( \Gamma \) in \( P \). Set

\[ \omega = \sum_i \omega^i E^i, \quad \Omega = \sum_i \Omega^i E^i. \]

It is easy to verify that the structure equation of the connection \( \Gamma \) (cf. §5 of Chapter II) can be expressed by

\[ d\omega^i = -\sum_k \omega^k \wedge \omega^j + \Omega^i, \quad i, j = 1, \ldots, m. \]

Let \( g \) be a fibre metric in \( E \) and \( Q \) be the reduced subbundle of \( P \) defined by \( g \). If \( \Gamma \) is a metric connection, then the restriction of \( \omega \) to \( Q \) defines a connection in \( Q \) by Proposition 6.1 of Chapter II and Proposition 1.5. In particular, both \( \omega \) and \( \Omega \), restricted to \( Q \), take their values in the Lie algebra \( \mathfrak{g}(m) \) or \( \mathfrak{u}(m) \) according as \( F = \mathbb{R} \) or \( F = \mathbb{C} \). In other words, both \( \{ \omega^i \} \) and \( \{ \Omega^i \} \), restricted to \( Q \), are skew-symmetric or skew-Hermitian according as \( F = \mathbb{R} \) or \( F = \mathbb{C} \).

2. Linear connections

Throughout this section, we shall denote the bundle of linear frames \( E(M) \) by \( P \) and the general linear group \( GL(n; \mathbb{R}) \), \( n = \dim M \), by \( G \).

The canonical form \( \theta \) of \( P \) is the \( \mathbb{R}^n \)-valued 1-form on \( P \) defined by

\[ \theta(X) = u^{-1}(\pi(X)) \quad \text{for } X \in T_u(P), \]

where \( u \) is considered as a linear mapping of \( \mathbb{R}^n \) onto \( T_{\pi(u)}(M) \) (cf. Example 5.2 of Chapter I).

**Proposition 2.1.** The canonical form \( \theta \) of \( P \) is a tensorial 1-form of type \( (GL(n; \mathbb{R}), \mathbb{R}^n) \). It corresponds to the identity transformation of the tangent space \( T_x(M) \) at each \( x \in M \) in the sense of Example 5.2 of Chapter II.

**Proof.** Let \( X \) be a vertical vector at \( u \in P \), then \( \pi(X) = 0 \) and hence \( 0(0) = 0 \). If \( X \) is any vector at \( u \in P \) and \( a \) is any element of \( G = GL(n; \mathbb{R}) \), then \( \pi(X) \) is a vector at \( a \in P \). Hence,

\[ (\pi^a X)(0) = 0 = \pi^{-1}(a^{-1} \pi(X)) = a^{-1} \pi(X), \]

thus proving our first assertion. The second assertion is clear. QED.

A connection in the bundle of linear frames \( P \) over \( M \) is called a linear connection of \( M \). Given a linear connection \( \Gamma \) of \( M \), we associate with each \( \xi \in \mathfrak{g} \) a horizontal vector field \( B(\xi) \) on \( P \) as follows. For each \( u \in P \), \( B(\xi)_u \) is the unique horizontal vector at \( u \) such that \( \pi(B(\xi)_u) = u(\xi) \). We call \( B(\xi) \) the standard horizontal vector field corresponding to \( \xi \). Unlike the fundamental vector fields, the standard horizontal vector fields depend on the choice of connections.

**Proposition 2.2.** The standard horizontal vector fields have the following properties:

1. If \( \theta \) is the canonical form of \( P \), then \( \theta(B(\xi)) = \xi \) for \( \xi \in \mathbb{R}^n \);
2. \( R_x(B(\xi)) = B(\pi(\xi)) \) for \( \pi(\xi) \in \mathfrak{g} \) and \( \xi \in \mathbb{R}^n \);
3. If \( \xi \neq 0 \), then \( B(\xi) \) never vanishes.

**Proof.** (1) is obvious. (2) follows from the fact that if \( X \) is a horizontal vector at \( u \), then \( \pi(X) \) is a horizontal vector at \( u(\xi) \) and \( \pi(R_x(X)) = \pi(X) \). To prove (3), assume that \( \pi(B(\xi))_u = 0 \) at some point \( u \in P \). Then \( u(\xi) = \pi(B(\xi)_u) = 0 \). Since \( u: \mathbb{R}^n \to T_{\pi(u)}(M) \) is a linear isomorphism, \( \xi = 0 \).

**Remark.** The conditions \( \theta(B(\xi)) = \xi \) and \( \theta(\xi) = 0 \) (where \( \omega \) is the connection form) completely determine \( B(\xi) \) for each \( \xi \in \mathbb{R}^n \).

**Proposition 2.3.** If \( A^\ast \) is the fundamental vector field corresponding to \( A \in \mathfrak{g} \) and if \( B(\xi) \) is the standard horizontal vector field corresponding
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to \( \xi \in \mathbb{R}^n \), then

\[
[A^*, B(\xi)] = B(A^* \xi),
\]

where \( A^* \xi \) denotes the image of \( \xi \) by \( A^* \in \mathfrak{gl}(n; \mathbb{R}) \) (Lie algebra of all \( n \times n \) matrices) which acts on \( \mathbb{R}^n \).

Proof. Let \( A_t \) be the 1-parameter subgroup of \( G \) generated by \( A, a_t = \exp tA \). By Proposition 1.9 of Chapter I and (2) of Proposition 2.2, we have

\[
\begin{align*}
[A^*, B(\xi)] &= \lim_{t \to 0} \frac{1}{t} [B(\xi) - R_{a_t}(B(\xi))] \\
&= \lim_{t \to 0} \frac{1}{t} [B(\xi) - B(a_t^{-1} \xi)].
\end{align*}
\]

Since \( \xi \to (B(\xi))_u \) is a linear isomorphism of \( \mathbb{R}^n \) onto the horizontal subspace \( \mathcal{Q}_u \) (cf. (3) of Proposition 2.2), we have

\[
\lim_{t \to 0} \frac{1}{t} [B(\xi) - B(a_t^{-1} \xi)] = B(A^* \xi).
\]

We define the torsion form \( \Theta \) of a linear connection \( \Gamma \) by

\[
\Theta = D\theta \quad \text{(exterior, covariant differential of \( \theta \))}.
\]

By Proposition 5.1 of Chapter II and Proposition 2.1, \( \Theta \) is a tensorial \( P \)-form on \( P \) of type \((GL(n; \mathbb{R}), \mathbb{R}^*)\).

**Theorem 2.4 (Structure equations).** Let \( \omega, \Theta, \) and \( \Omega \) be the connection form, the torsion form and the curvature form of a linear connection \( \Gamma \) of \( M \). Then

1st Structure equation:

\[
d\omega(X, Y) = -\frac{1}{2} (\omega(X) \cdot \theta(Y) - \omega(Y) \cdot \theta(X)) + \Theta(X, Y),
\]

2nd structure equation:

\[
d\omega(X, Y) = -\frac{1}{2} [\omega(X), \omega(Y)] + \Omega(X, Y),
\]

where \( X, Y \in T_p M \) and \( u \in P \).

Proof. The second structure equation was proved in Theorem 5.2 of Chapter II (see also §1). The proof of the first structure equation is similar to that of Theorem 5.2 of Chapter II. There are three cases which have to be verified and the only non-trivial case is the one where \( X \) is vertical and \( Y \) is horizontal. Choose \( A \in \mathfrak{g} \) and \( \xi \in \mathbb{R}^n \) such that \( X = A^*_\xi \) and \( Y = B(\xi)_u \).

Then \( \Theta(X, Y) = 0 \), \( \omega(Y) \cdot \theta(X) = 0 \) and \( \omega(X) \cdot \theta(Y) = \omega(A^*_\xi) \cdot \theta(B(\xi)) = A^* \xi \cdot \theta(\xi) \), since \( \omega(A^*_\xi) = A \) and \( \theta(B(\xi)) = \xi \). On the other hand, \( 2d\theta(X, Y) = A^*_\xi \theta(\xi) - B(\xi)_u \theta(A^*_\xi) = \theta[A^*_\xi, B(\xi)_u] = -\theta(A^*_\xi, B(\xi)_u) = -\theta(B(\xi)_u, A^*_\xi) = -A^*_\xi \) by Proposition 2.3. This proves the first structure equation. QED.

We define the torsion form \( \theta \) of a linear connection \( \Gamma \) by

\[
\theta = \sum_i \theta_\epsilon^i, \quad \Theta = \sum_i \Theta_\epsilon^i.
\]

As in §1, with respect to the basis \( E_i^j \) of \( \mathfrak{gl}(n; \mathbb{R}) \), we write

\[
v = \sum_i \omega_\epsilon^i E_i^j, \quad \Omega = \sum_i \Omega_\epsilon^i E_i^j.
\]

Then the structure equations can be written as

\[
\begin{align*}
(1) \quad &d\omega^j_i = -\sum_k \omega_k^i \land \omega^j_k + \Theta^j_i, \quad i = 1, \ldots, n, \\
(2) \quad &d\omega_j^i = -\sum_k \omega_k^j \land \omega_j^k + \Omega_j^i, \quad i, j = 1, \ldots, A.
\end{align*}
\]

Considering \( \theta \) as a vector valued form and \( \omega \) as a matrix valued form, we also write the structure equations in the following simplified form:

\[
\begin{align*}
(1') \quad &de = -\omega \land \theta + \Theta, \\
(2') \quad &d\omega = -\omega \land \omega + \Omega.
\end{align*}
\]

In the next section, we shall give an interpretation of the torsion form and the first structure equation from the viewpoint of affine connections.

**Theorem 2.5 (Bianchi's identities).** For a linear connection, we have

1st identity: \( D\Theta = \Omega \land \Theta \), that is,

\[
3D\Theta(X, Y, Z) = \Omega(X, Y) \theta(Z) + \Omega(Y, Z) \theta(X) + \Omega(Z, X) \theta(Y),
\]

where \( X, Y, Z \in T_p M \).

2nd identity: \( D\Omega = 0 \).

Proof. The second identity was proved, in Theorem 5.4 of Chapter II. The proof of the first identity is similar to that of Theorem 5.4. If we apply the exterior differentiation \( d \) to the first structure equation \( d\theta = -\omega \land \Theta + \Theta \), then we obtain

\[
0 = -d\omega \land \Theta + w \land d\theta + d\Theta.
\]
Denote by \( hX \) the horizontal component of \( X \). Then \( \omega(hX) = 0 \), \( \theta(hX) = c(X) \) and \( d\omega(hX, hY) = \Omega(X, Y) \). Hence,

\[
D\Omega(X, Y, Z) = (d\omega \wedge \theta)(hX, hY, hZ) = (\Omega \wedge 0)(X, Y, Z).
\]

QED.

Let \( B_1, \ldots, B_n \) be the standard horizontal vector fields corresponding to the natural basis \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \) and \( \{E_i^*\} \) the fundamental vector fields corresponding to the basis \( \{E_i\} \) of \( g(\mathbb{R}; \mathbb{R}) \). It is easy to verify that \( \{B_i, E_i^*\} \) and \( \{\theta^i, \omega^i_j\} \) are dual to each other in the sense that

\[
\theta^k(B_i) = \delta^k_i; \quad \theta^k(E_i^*) = 0,
\]

\[
\omega^k_j(B_i) = 0; \quad \omega^k_j(E_i^*) = \delta^k_j.
\]

**Proposition 2.6.** The \( n^2 + n \) vector fields \( \{B_i, E_i^*; i, j, k = 1, \ldots, n\} \) define an absolute parallelism in \( I' \), that is, the \( n^2 + n \) vectors \( \{(B_k)_v, (E_j^*)_v\} \) form a basis of \( T(v) \) for every \( v \in P \).

**Proof.** Since the dimension of \( P \) is \( n^2 + n \), it is sufficient to prove that the above \( n^2 + n \) vectors are linearly independent. Since \( A \to A^* \) is a linear isomorphism of \( g \) onto the vertical subspace of \( T_u(P) \) (cf. §5 of Chapter I), \( \{E_i^*\} \) are linearly independent at every point of \( P \). By (3) of Proposition 2.2, \( \{B_k\} \) are linearly independent at every point of \( P \). Since \( \{B_k\} \) are horizontal and \( \{E_i^*\} \) are vertical, \( \{B_k, E_i^*\} \) are linearly independent at every point of \( P \).

QED.

Let \( T^*_x(M) \) be the tensor bundle \( T^*_x(M) \) of \( M \) of type \((r, s)\) (cf. Example 5.4 of Chapter I). It is a vector bundle with standard fiber \( T^*_x(M) \) (tensor space over \( \mathbb{R}^n \) of type \((r, s)\)) associated with the bundle \( P \) of linear frames. A tensor field \( K \) of type \((r, s)\) is a cross section of the tensor bundle \( T^*_x(M) \). In §1, we defined covariant derivatives of a cross section in a vector bundle in general. As in §1, we can define covariant derivatives of \( K \) in the following three cases:

1. \( \nabla_{\xi} K \), when \( K \) is defined along a curve \( \tau = x_t \) of \( M \);
2. \( \nabla_X K \), when \( X \in T_x(M) \) and \( K \) is defined in a neighborhood of \( x_0 \);
3. \( \nabla_X K \), when \( X \) is a vector field on \( M \) and \( K \) is a tensor field on \( M \).

For the sake of simplicity, we state the following proposition in case (3) only, although it is valid in cases (1) and (2) with obvious changes.

**Proposition 2.7.** Let \( Z(M) \) be the algebra of tensor fields on \( M \). Let \( X \) and \( Y \) be vector fields on \( M \). Then the covariant differentiation has the following properties:

1. \( \nabla_X f \) is a type preserving derivation;
2. \( \nabla_X \) commutes with every contraction;
3. \( \nabla_X f = Xf \) for every function \( f \) on \( M \);
4. \( \nabla_X(Y + f) = \nabla_X Y + f \nabla_X \); for every function \( f \) on \( M \) and \( K \in Z(M) \);
5. \( \nabla_{X^+} K = f \cdot \nabla_X K \) for every function \( f \) on \( M \) and \( K \in Z(M) \).

**Proof.** Let \( \tau = x_t \), \( 0 \leq t \leq 1 \), be a curve in \( M \). Let \( T(x_t) \) be the tensor algebra over \( T_x(M) \), \( T(x_t) = \sum T^*_x(M) \) (cf. §3 of Chapter I). The parallel displacement along \( \tau \) gives an isomorphism of the algebra \( T(x_0) \) onto the algebra \( T(x_1) \) which preserves type and commutes with every contraction. From the definition of covariant differentiation given in §1, we obtain (1) and (2) by an argument similar to the proof of Proposition 3.2 of Chapter I. (3), (4) and (5) were proved in Proposition 1.2, QED.

By the lemma for Proposition 3.3 of Chapter I, the operation of \( \nabla_X \) on \( Z(M) \) is completely determined by its operation on the algebra of functions \( Z(M) \) and the module of vector fields \( X(M) \). Since \( \nabla_X f = Xf \) for every function \( f \in Z(M) \), the operation of \( \nabla_X \) on \( Z(M) \) is determined by its operation on \( Z(M) \). As an immediate corollary to Proposition 1.2, we have

**Proposition 2.8.** If \( X, Y \) and \( Z \) are vector fields on \( M \), then

1. \( \nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z \);
2. \( \nabla_X Y Z = \nabla_X Z Y + \nabla_Y Z \);
3. \( \nabla_X(f Y) = f \cdot \nabla_X Y \) for every function \( f \in Z(M) \);
4. \( \nabla_X(f Y) = f \cdot \nabla_X Y + (f Y) \nabla_X Y \) for every function \( f \in Z(M) \).

We shall prove later in §7 that any mapping \( \mathfrak{X}(M) \times 3(M) \to X(M) \), denoted by \( (X, Y) \to \nabla_X Y \), satisfying the four conditions above is actually the covariant derivative with respect to a certain linear connection.

The proof of the following proposition, due to Kostant [1], is similar to that of Proposition 3.3 of Chapter I and hence is left to the reader.
**Proposition 2.9.** Let $M$ be a manifold with a linear connection. Every derivation $D$ (preserving type and commuting with contractions) of the algebra $Z(M)$ of tensor fields into the tensor algebra $T(x)$ at $x \in M$ can be uniquely decomposed as follows:

$$D = \nabla_x + S,$$

where $X \in T_x(M)$ and $S$ is a linear endomorphism of $T_x(M)$.

Observe that, in contrast to Lie differentiation $L_x$ with respect to a vector field, covariant differentiation $\nabla_x$ makes sense when $X$ is a vector at a point of $M$.

Given a tensor field $K$ of type $(r, s)$, the covariant differential $\nabla_x K$ of $K$ is a tensor field of type $(r, s + 1)$ defined as follows. As in Proposition 2.11 of Chapter I, we consider a tensor of type $(r, s)$ at a point $x \in M$ as a multilinear mapping of $T_x(M) \times \cdots \times T_x(M)$ (s times product) into $T_y(x)$ (space of contravariant tensors of degree $r$ at $y$). We set

$$(W, V, \ldots, X, Y, X, X, \ldots, X) = (\nabla_x K)(X_1, \ldots, X_s), \quad X, X, \in T_x(M).$$

**Proposition 2.10.** If $K$ is a tensor field of type $(r, s)$, then

$$(\nabla K)(X_1, \ldots, X_s; X) = \nabla_x(K(X_1, \ldots, X_s)) = \Sigma_{i=1}^s K(X_1, \ldots, \nabla_x X_i, \ldots, X_s),$$

where $X_1, X_s \in \pi(M)$.

Proof. This follows from the fact that $\nabla_x$ is a derivation commuting with every contraction. The proof is similar to that of Proposition 3.5 of Chapter I and is left to the reader. QED.

A tensor field $K$ on $M$, considered as a cross section of a tensor bundle, is parallel if and only if $\nabla_x K = 0$ for all $X \in T_x(M)$ and $x \in M$ (cf. §1). Hence we have

**Proposition 2.11.** A tensor field $K$ on $M$ is parallel if and only if $\nabla K = 0$.

The second covariant differential $\nabla^2 K$ of a tensor field $K$ of type $(r, s)$ is defined to be $\nabla(\nabla K)$, which is a tensor field of type $(r, s + 2)$. We set

$$(\nabla^2 K)(X; Y) = (\nabla_x(\nabla K))(X),$$

where $X, Y \in T_x(M)$.

that is, if we regard $K$ as a multilinear mapping of $T_x(M) \times \cdots \times T_x(M)$ (s times product) into $T_y(x)$, then

$$(\nabla^2 K)(X_1, \ldots, X_s; Y, X; Y) = (\nabla_x(\nabla K))(X_1, \ldots, X_s; X, Y).$$

Similarly to Proposition 2.10, we have

**Proposition 2.12.** For any tensor field $K$ and for any vector fields $X$ and $Y$, we have

$$(\nabla^2 K)(X; Y) = \nabla_x(\nabla K) - \nabla_y K.$$
where $\alpha$ is an isomorphism of the vector group $\mathbb{R}^n$ into $A(n; \mathbb{R})$ which maps $\xi \in \mathbb{R}^n$ into \( \begin{pmatrix} I_n & \xi \\ 0 & 1 \end{pmatrix} \in A(n; \mathbb{R}) \) ($I_n$ = identity of \( GL(n; \mathbb{R}) \)) and $\beta$ is a homomorphism of $A(n; \mathbb{R})$ onto $GL(n; \mathbb{R})$ which maps \( \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix} \in A(n; \mathbb{R}) \) into $\xi \in GL(n; \mathbb{R})$. The sequence is exact in the sense that the kernel of each homomorphism is equal to the image of the preceding one. It is a splitting-exact sequence in the sense that there is a homomorphism \( y: GL(n; \mathbb{R}) \to A(n; \mathbb{R}) \) such that $\beta \circ y$ is the identity automorphism of $GL(n; \mathbb{R})$; indeed, we define $y$ by $y(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in A(n; \mathbb{R})$, a $\xi \in GL(n; \mathbb{R})$.

The group $A(n; \mathbb{R})$ is a semidirect product of $\mathbb{R}^n$ and $GL(n; \mathbb{R})$, that is, for every $a \in A(n; \mathbb{R})$, there is a unique pair $(a, \xi) \in GL(n; \mathbb{R}) \times \mathbb{R}^n$ such that $a = a(\xi) \cdot y(a)$.

An affine frame of a manifold $M$ at $x$ consists of a point $p \in A(x)$ and a linear frame $(x_1, \ldots, x_n)$ at $x$; it will be denoted by $(p; x_1, \ldots, x_n)$. Let $a$ be the origin of $\mathbb{R}^n$ and $(e_1, \ldots, e_n)$ the natural basis for $\mathbb{R}^n$. We shall call $(p; e_1, \ldots, e_n)$ the canonical frame of $A^n$. Every affine frame $(p; x_1, \ldots, x_n)$ at $x$ can be identified with an affine transformation $\tilde{u}: A^n \to A(x)$ which maps $(0; e_1, \ldots, e_n)$ into $(p; x_1, \ldots, x_n)$, because $(p; x_1, \ldots, x_n)$ gives a 1:1 correspondence between the set of affine frames at $x$ and the set of affine transformations of $A^n$ onto $A(x)$. We denote by $A(M)$ the set of all affine frames of $M$ and define the projection $\pi: A(M) \to M$ by setting $\pi(\tilde{u}) = x$ for every affine frame $\tilde{u}$ at $x$. We shall show that $A(M)$ is a principal fibre bundle over $M$ with group $A(n; \mathbb{R})$ and shall call $A(M)$ the bundle of affine frames over $M$. We define an action of $A(n; \mathbb{R})$ on $A(M)$ by $(\tilde{u}, a) \to \tilde{u}a$, $a \in A(n; \mathbb{R})$, where $\tilde{u}a$ is the composite of the affine transformations $a^{-1} A^n \to A^n$ and $\tilde{u}: A^n \to A(x)$. It can be proved easily (cf. Example 5.2 of Chapter I) that $A(n; \mathbb{R})$ acts freely on $A(M)$ on the right and that $A(M)$ is a principal fibre bundle over $M$ with group $A(n; \mathbb{R})$.

Let $L(M)$ be the bundle of linear frames over $M$. Corresponding to the homomorphisms $\beta: A(n; \mathbb{R}) \to GL(n; \mathbb{R})$ and $\gamma: GL(n; \mathbb{R}) \to A(n; \mathbb{R})$, we have homomorphisms $\beta: A(M) \to L(M)$ and $\gamma: L(M) \to A(M)$. Namely, $\beta: A(M) \to L(M)$ maps $(p; x_1, \ldots, x_n)$ into $(X_1, \ldots, X_n)$ and $\gamma: L(M) \to A(M)$ maps $(X_1, \ldots, X_n)$ into $(a_1 x_1, \ldots, a_n x_n)$, where $a_i \in A(M)$ is the point corresponding to the origin of $T_{x_i}(M)$. In particular, $L(M)$ can be considered as a subbundle of $A(M)$. Evidently, $\beta \circ \gamma$ is the identity transformation of $L(M)$.

A generalized affine connection of $M$ is a connection in the bundle $A(M)$ of affine frames over $M$. We shall study the relationship between generalizd affine connections and linear connections. We denote by $R^n$ the Lie algebra of the vector group $\mathbb{R}^n$. Corresponding to the splitting exact sequence $0 \to R^n \to A(n; \mathbb{R}) \to GL(n; \mathbb{R}) \to 1$ of groups, we have the following splitting exact sequence of Lie algebras:

$$0 \to R^n \to a(n; \mathbb{R}) \to gl(n; \mathbb{R}) \to 0.$$ 

Therefore,

$$a(n; \mathbb{R}) = gl(n; \mathbb{R}) + R^n \text{ (semidirect sum)}.$$ 

Let $\bar{\omega}$ be the connection form of a generalized affine connection of $M$. Then $\gamma \ast \bar{\omega}$ is an $a(n; \mathbb{R})$-valued 1-form on $L(M)$. Let

$$\gamma \ast \bar{\omega} = \omega + \varphi,$$

be the decomposition corresponding to $a(n; \mathbb{R}) = gl(n; \mathbb{R}) + R^n$, so that $\omega$ is a $gl(n; \mathbb{R})$-valued 1-form on $L(M)$ and $\varphi$ is an $R^n$-valued 1-form on $L(M)$. By Proposition 6.4 of Chapter I, $\omega$ defines a connection in $L(M)$. On the other hand, we see easily that $\varphi$ is a tensorial 1-form on $L(M)$ of type $(gl(n; \mathbb{R}), R^n)$ (cf. §5 of Chapter II) and hence corresponds to a tensor, field of type $(1, 1)$ of $M$ as explained in Example 5.2 of Chapter II.

**Proposition 3.1.** Let $\bar{\omega}$ be the connection form of a generalized affine connection $\Gamma$ of $M$ and let

$$\gamma \ast \bar{\omega} = \omega + \varphi,$$

where $\omega$ is $gl(n; \mathbb{R})$-valued and $\varphi$ is $R^n$-valued. Let $\Gamma$ be the linear connection of $M$ defined by $\omega$ and let $K$ be the tensor field of type $(1, 1)$ of $M$ defined by $\varphi$. Then

1. The correspondence between the set of generalized affine connections of $M$ and the set of pairs consisting of a linear connection of $M$ and a tensor field of type $(1, 1)$ of $M$ given by $\Gamma \to (\Gamma, K)$ is 1:1.

2. The homomorphism $\beta: A(M) \to L(M)$ maps $\Gamma$ into $\pi$ (cf. §6 of Chapter II).
Proof. (1) It is sufficient to prove that, given a pair \((\Gamma', K)\), there is \(\bar{\Gamma}\) which gives rise to \((\Gamma', K)\). Let \(\omega\) be the connection form of \(\Gamma\) and \(\varphi\) the tensorial 1-form on \(L(M)\) of type \((GL(n; \mathbb{R}), \mathbb{R}^n)\) corresponding to \(K\). Given a vector \(X \in T_a(A(M))\), choose \(X = T_a(A(M))\) and \(\tilde{a} \in A(n; \mathbb{R})\) such that \(\tilde{a} = u\tilde{a}\) and \(\tilde{a} - R_2(X)\) is vertical. There is an element \(A \in A(n; \mathbb{R})\) such that
\[
X = R_2(X) + A^*;
\]
where \(A^*\) is the fundamental vector corresponding to \(A\). We define \(\tilde{\omega}\) by
\[
\tilde{\omega}(X) = \text{ad} (\tilde{a}^{-1})(\omega(X) + \varphi(X)) + A.
\]
It is straightforward to verify that \(\tilde{\omega}\) defines the desired connection \(\bar{\Gamma}\).

(2) Let \(X \in T_a(A(M))\). We set \(u = \beta(\tilde{a})\) and \(X = \beta(X)\) so that \(X \in T_a(A(M))\), since \(\beta: A(M) \to t(M)\), is the homomorphism associated to \(\beta\) the homomorphism \(\beta: A(n; \mathbb{R}) \to GL(n; \mathbb{R}) = A(n; \mathbb{R})^* \to L(M)\). L(M) can be identified with \(A(M)/\mathbb{R}^n\) and \(\beta: A(M) - L(M)\) can be considered as the natural projection \(A(M) \to A(M)/\mathbb{R}^n\). Since \(X = \beta(X) = X\), there exist \(\tilde{a} \in \mathbb{R}^n \subset A(n; \mathbb{R})\) and \(A \in \mathbb{R}^n \subset A(n; \mathbb{R})\) such that \(\tilde{a} = u\tilde{a}\) and \(X = R_2(X) + A\). Assume that \(X\) is horizontal with respect to \(\Gamma\), so that \(0 = \tilde{\omega}(X) = \tilde{\omega}(R_2(X)) + \tilde{\omega}(A^*) = \text{ad} (\tilde{a}^{-1})(\omega(X)) + \varphi(A^*)\). Hence, \(\tilde{\omega}(X) = \text{ad} (\tilde{a})\) and \(\omega(X) = \varphi(X) = \text{ad} (\tilde{a})(\varphi(A^*))\). Since both \(\varphi(X)\) and \(\text{ad} (\tilde{a})\) are in \(\mathbb{R}^n\) and \(\omega(X)\) is in \(A(n; \mathbb{R})\), we have \(\omega(X) = 0\). This proves that if \(X\) is horizontal with respect to \(\Gamma\), then \(\beta(X)\) is horizontal with respect to \(\bar{\Gamma}\). QED.

**Proposition 3.2.** In Proposition 3.1, let \(\Omega\) and \(\bar{\Omega}\) be the curvature forms of \(\Gamma\) and \(\bar{\Gamma}\) respectively. Then
\[
\gamma^*\bar{\Omega} = \Omega + D\varphi,
\]
where \(D\) is the exterior covariant differentiation with respect to \(\Gamma\).

**Proof.** Let \(X, Y \in T_a(L(M))\). To prove that \((\gamma^*\bar{\Omega})(X, Y) = \Omega(X, Y) + D\varphi(X, Y)\), it is sufficient to consider the following two cases: (1) at least one of \(X\) and \(Y\) is vertical, (2) both \(X\) and \(Y\) are horizontal with respect to \(\Gamma\). In the case (1), both sides vanish. In the case (2), \(\omega(X) = \omega(Y) = 0\) and hence \(\tilde{\omega}(X) = \varphi(X)\) and \(\tilde{\omega}(Y) = \varphi(Y)\). From the structure equation of \(\bar{\Gamma}\), we have
\[
\begin{align*}
\tilde{\omega}(X, Y) &= -\frac{1}{2} \tilde{\omega}([X, Y], J) + \bar{\Omega}(X, Y) \\
&= -\frac{1}{2} \varphi([X, Y]) + \bar{\Omega}(X, Y),
\end{align*}
\]
(Here, considering \(L(M)\) as a subbundle of \(A(M)\), we identified \(y(X)\) with \(X\).) On the other hand, \(\gamma^*\tilde{\omega} = \tilde{\omega} + \varphi\) and hence \(\tilde{\omega}(X, Y) = \omega(X, Y) + \varphi(X, Y)\). Since \(\mathbb{R}^n\) is abelian, \([\varphi(X), \varphi(Y)] = 0\). Hence, \(\omega(X, Y) + \varphi(X, Y) = \Omega(X, Y)\). Since both \(X\) and \(Y\) are horizontal, \(\omega(X, Y) + \varphi(X, Y) = \Omega(X, Y)\). QED.

Consider again the structure equation of a generalized affine connection \(\bar{\Gamma}\):
\[
d\tilde{\omega} = -\frac{1}{2}[\tilde{\omega}, \tilde{\omega}] + \bar{\Omega}.
\]
By restricting both sides of the equation to \(L(M)\) and by comparing the \(gl(n; \mathbb{R})\)-components and the \(\mathbb{R}^n\)-components we obtain
\[
\begin{align*}
d\varphi(X, Y) &= -\frac{1}{2}([\omega(X), \varphi(Y)] - [\omega(Y), \varphi(X)]) + D\varphi(X, Y), \\
d\omega(X, Y) &= -\frac{1}{2}([\omega(X), \omega(Y)] + \Omega(X, Y),
\end{align*}
\]
Just as in §2, we write
\[
\begin{align*}
d\varphi &= -\omega \wedge \varphi + D\varphi, \\
d\omega &= -\omega \wedge \omega + \Omega.
\end{align*}
\]
A generalized affine connection \(\bar{\Gamma}\) is called an affine connection if, with the notation of Proposition 3.1, the \(\mathbb{R}^n\)-valued 1-form \(\varphi\) is the canonical form \(\beta\) defined in §2. In other words, \(\bar{\Gamma}\) is an affine connection if the tensor field \(K\) corresponding to \(\varphi\) is the field of identity transformations. of tangent spaces of \(M\). As an immediate consequence of Proposition 3.1, we have

**Theorem 3.3.** The homomorphism \(\beta: A(M) \to L(M)\) maps every affine connection \(\bar{\Gamma}\) of \(M\) into a linear connection \(\Gamma\) of \(M\). Moreover, \(\bar{\Gamma} \to \Gamma\) gives a 1:1 correspondence between the set of affine connections \(\bar{\Gamma}\) of \(M\) and the set of linear connections \(\Gamma\) of \(M\).

Traditionally, the words “linear connection” and “affine connection” have been used interchangeably. This is justified by Theorem 3.3. Although we shall not break with this tradition, we
shall make a logical distinction between a linear connection and an affine connection whenever necessary; a linear connection of \( M \) is a connection in \( L(M) \) and an affine connection is a connection in \( A(M) \).

From Proposition 3.2, we obtain

**Proposition 3.4.** Let \( \Theta \) and \( \Omega \) be the torsion form and the curvature form of a linear connection \( \Gamma \) of \( M \). Let \( \Omega \) be the curvature form of the corresponding affine connection. Then

\[
y^*\Omega = \Theta + \Omega,
\]

where \( y: L(M) \to A(M) \) is the natural injection.

Replacing \( \varphi \) by the canonical form \( \theta \) in the formulas:

\[
d\varphi = -\omega \wedge \varphi + D\varphi, \quad d\omega = -\omega \wedge \omega + \Omega,
\]

we rediscover the structure equations of a linear connection proved in Theorem 2.4.

Let \( \Phi(\tilde{u}) \) be the holonomy group of an affine connection \( \Gamma \) of \( M \) with reference point \( \tilde{u} \in A(M) \). Let \( \Psi(\tilde{u}) \) be the holonomy group of the corresponding linear connection \( \Gamma \) of \( M \) with reference point \( u = \beta(\tilde{u}) \in L(M) \). We shall call \( \Phi(\tilde{u}) \) the affine holonomy group of \( \tilde{\Gamma} \) or \( \Gamma \) and \( \Psi(u) \) the linear holonomy group (or homogeneous holonomy group) of \( \tilde{\Gamma} \) or \( \Gamma \). The restricted affine and linear holonomy groups \( \Phi(\tilde{u}) \) and \( \Psi(u) \) are defined accordingly. From Proposition 6.1 of Chapter II, we obtain

**Proposition 3.5.** The homomorphism \( \beta: A(n; \mathbb{R}) \to GL(n; \mathbb{R}) \) maps \( \Phi(\tilde{u}) \) onto \( Y(u) \) and \( \Phi(\tilde{u}) \) onto \( Y'(u) \).

### 4. Developments

We shall study in this section the parallel displacement arising from an affine connection of a manifold \( M \). Let \( \tau = x_t, 0 \leq t \leq 1, \) be a curve in \( M \). The affine parallel displacement along \( \tau \) is an affine transformation of the affine tangent space at \( x_0 \) onto the affine tangent space at \( x_1 \) defined by the given connection in \( A(M) \). It is a special case of the parallelism in an associated bundle which is, in our case, the affine tangent bundle whose fibres are \( A(M), x \in M \). We shall denote this affine parallelism by \( \tau \).

The total space (i.e., the bundle space) of the affine tangent bundle over \( M \) is naturally homeomorphic with that of the tangent (vector) bundle over \( M \); the distinction between the two is that the affine tangent bundle is associated with \( A(M) \) whereas the tangent (vector) bundle is associated with \( L(M) \). A cross section of the affine tangent bundle is called a point field. There is a natural 1:1 correspondence between the set of point fields and the set of vector fields.

Let \( \tau_t \) be the affine parallel displacement along the curve \( \tau \) from \( x_t \) to \( x_0 \). In particular, \( \tau_0 \) is the parallel displacement \( A_0(M) \to A(M) \) along \( \tau \) (in the reversed direction) from \( x_1 \) to \( x_0 \). Let \( p \) be a point field defined along \( \tau \) so that \( \tau_t(p) \) is an element of \( A_0(M) \) for each \( t \). Then \( \tau_0(p) \) describes a curve in \( A_0(M) \). We identify the curve \( \tau = x_t \) with the trivial point field along \( \tau \), that is, the point field corresponding to the zero vector field along \( \tau \). Then the development of the curve \( \tau \) in \( M \) into the affine tangent space \( A_0(M) \) is the curve \( \tau_0(x_t) \) in \( A_0(M) \).

The following proposition allows us to obtain the development of a curve by means of the linear parallel displacement, that is, the parallel displacement defined by the corresponding linear connection.

**Proposition 4.1.** Given a curve \( \tau = x_t, 0 \leq t \leq 1, in \ M, \) set \( Y_t = \tau_t(x_t), \) where \( \tau_t \) is the linear parallel displacement along \( \tau \) from \( x_1 \) to \( x_0 \) and \( x_t \) is the vector tangent to \( \tau \) at \( x_t \). Let \( C_t, 0 \leq t \leq 1, \) be the curve in \( A_0(M) \) starting from the origin (that is, \( C_0 = x_0 \)) such that \( C_t \) is parallel (in the affine space \( A(M) \) in the usual sense) to \( Y_t \) for every \( t \). Then \( C_t \) is the development of \( \tau \) into \( A_0(M) \).

**Proof.** Let \( u_0 \) be any point in \( L(M) \) such that \( \pi(u_0) = x_0 \) and \( u_t \) the horizontal lift of \( x_t \) in \( L(M) \) with respect to the linear connection. Let \( \tilde{u}_t \) be the horizontal lift of \( x_t \) in \( A(M) \) with respect to the affine connection such that \( \tilde{u}_0 = u_0 \). Since the homomorphism \( \beta: A(M) \to L(M) = A(M)/\mathbb{R}^n \) (cf. \S 3) maps \( \tilde{u}_t \) into \( u_t \), there is a curve \( \tilde{u}_t \) in \( A(M)/\mathbb{R}^n \) such that \( \tilde{u}_0 = u_0 \), and that \( \tilde{u}_0 \) is the identity. As in the proof of Proposition 3.1 of Chapter II, we shall find a necessary and sufficient condition for \( \tilde{u}_t \) in order that \( \tilde{u}_t \) be horizontal with respect to the affine connection. From

\[
\tilde{u}_t = \tilde{u}_0 + u_t \cdot \tilde{a}_t,
\]
5. Curvature and torsion tensors.

We have already defined the torsion form $\Theta$ and the curvature form $\Omega$ of a linear connection. We now define the torsion tensor field (or simply, torsion) $T$ and the curvature tensor field (or simply, curvature) $R$. We set

$$T(X, Y) = u(2\Theta(X*, Y*))$$

for $X, Y \in \mathcal{T}_a(M)$, where $u$ is any point of $\mathcal{L}(M)$ with $n(u) = x$ and $X*$ and $Y*$ are vectors of $\mathcal{L}(M)$ at $u$ with $\pi(X*) = X$ and $\pi(Y*) = Y$. We already know that $T(X, Y)$ is independent of the choice of $u$, $X*$, and $Y*$ (cf. Example 5.2 of Chapter II); this fact can be easily verified directly also. Thus, at every point $x$ of $M$, $T$ defines a skew symmetric bilinear mapping $T(M) \times T(M) \to \mathcal{T}(M)$. In other words, $T$ is a tensor field of type $(1, 2)$ such that $T(X, Y) = -T(Y, X)$. We shall call $T(X, Y)$ the torsion translation in $T(M)$ determined by $X$ and $Y$. Similarly, we set

$$R(X, Y, Z) = u((2\Omega(X*, Y*)) (u^{-1}Z)$$

for $X, Y, Z \in \mathcal{T}_a(M)$, where $u$, $X*$ and $Y*$ are chosen as above. Then $R(X, Y, Z)$ depends only on $X, Y$ and $Z$, not on $u$, $X*$ and $Y*$. In the above definition, $((2\Omega(X*, Y*)) (u^{-1}Z)$ denotes the image of $u^{-1}Z \in \mathcal{R}^n$ by the linear endomorphism $2\Omega(X*, Y*) \in \mathcal{E}(\mathcal{D}; \mathcal{R})$ of $\mathcal{R}^n$. Thus, $R(X, Y)$ is an endomorphism of $\mathcal{T}_a(M)$ and is called the curvature transformation of $\mathcal{T}_a(M)$ determined by $X$ and $Y$. It follows that $R$ is a tensor field of type $(1, 3)$ such that $R(X, Y) = -R(Y, X)$.

**Theorem 5.1.** In terms of covariant differentiation, the torsion $T$ and the curvature $R$ can be expressed as follows:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

and,

$$R(X, Y, Z) = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$$

where $X, Y$ and $Z$ are vector fields on $M$.

**Proof.** Let $X*$, $Y*$ and $Z*$ be the horizontal lifts of $X$, $Y$ and $Z$, respectively. We first prove

**Lemma** $(\nabla_X Y)_x = u(X* (\theta(Y*)))$, where $\sigma(u) = x$.

**Proof of Lemma.** In the lemma for Proposition 1.1, we proved that $(\nabla_X Y)_x = u(X* f)$, where $f$ is an $\mathcal{R}^n$-valued function defined by $f(u) = u^{-1}(Y*)$. Hence, $f(u) = \theta(Y*)$ for $u \in \mathcal{L}(M)$. This completes the proof of the lemma.

We have therefore

$$T(X*, Y*) = u(2\Theta(X*, Y*))$$

$$= u(X* (\theta(Y*)) - Y* (\theta(X*)) - \theta([X*, Y*])_u)$$

$$= (\nabla_X Y)_x - (\nabla_Y X)_x - [X, Y]$$

since $\pi([X*, Y*]) = [X, Y]$.

To prove the second equality, we set $f = \theta(Z*)$ so that $f$ is an $\mathcal{R}^n$-valued function on $\mathcal{L}(M)$ of type $(GL(n; \mathcal{R}), \mathcal{R}^n)$. We have
then 
\[
\{(\nabla_X, \nabla_Y)Z - \nabla_{[X,Y]}Z\} = u(X^*_f(Y^*f) - Y^*_f(X^*f) - \langle h[X^*, Y^*]\rangle u_f) = u(\langle [X^*, Y^*]\rangle u_f),
\]
where \(h\) (resp. \(v\)) denotes the horizontal (resp. vertical) component. 

Let \(A^*\) be an element of \(gl(n; \mathbb{R})\) such that \(A^* = (\langle v[X^*, Y^*]\rangle u)\), where \(A^*\) is the fundamental vector field corresponding to \(A\). Then by Corollary 5.3 of Chapter II, we have 
\[
2\Omega(X^*_f, Y^*_f) = -\omega([X^*, Y^*]_u) = -\dot{A}.
\]

On the other hand, if \(a^*_t\) is the \(1\)-parameter subgroup of \(GL(n; \mathbb{R})\) generated by \(A\), then 
\[
A^*_f = \lim_{t \to 0} \frac{1}{t} [f(a^*_t) - f(u)]
\]
where \(A(f(u))\) denotes the result of the linear transformation 
\[A: \mathbb{R}^n \to \mathbb{R}^n\] applied to \(f(u) \in \mathbb{R}^n\). Therefore, we have 
\[
\{(\nabla_X, \nabla_Y)Z - \nabla_{[X,Y]}Z\} = u(\langle [X^*, Y^*]\rangle u_f) = u(-A(f(u)))
\]
\[
= u(2\Omega(X^*_f, Y^*_f)(f(u))) = u(2\Omega(X^*_f, Y^*_f)(u^{-1}Z)) = R(X, Y)Z.
\]

QED.

**Proposition 5.2.** Let \(X, Y, Z, W \in T^*_\infty(M)\) and \(u \in L(M)\) with \(\pi(u) = x\). Let \(X^*, Y^*, Z^*, W^*\) be the standard horizontal vector fields on \(L(M)\) corresponding to \(u^{-1}X\), \(u^{-1}Y\), \(u^{-1}Z\) and \(u^{-1}W\) respectively, so that \(\pi(X^*_f) = X\), \(\pi(Y^*_f) = Y\), \(\pi(Z^*_f) = Z\) and \(\pi(W^*_f) = W\). Then 
\[
(\nabla_X T)(Y, Z) = u(X^*_f(2\Theta(Y^*, Z^*)))
\]
and 
\[
((\nabla_X R)(Y, Z))W = u((X^*_f(2\Omega(Y^*, Z^*)))(u^{-1}W)).
\]

Proof. We shall prove only the first formula. The proof of the second formula is similar to that of the first. Consider the torsion \(T\) as a cross section of the tensor bundle \(T^1(M)\) whose standard fibre is the tensor space \(T^1\) of type \((1, 2)\) over \(\mathbb{R}^n\). Let \(f\) be the \(T\&\text{valued}\) function on \(L(M)\) corresponding to the torsion \(T\) as in Example 5.2 of Chapter II so that, if we set 
\[
\eta = u^{-1}Y\text{ and } \zeta = u^{-1}Z,
\]
then 
\[
f_\eta(\eta, \zeta) = u^{-1}(T(X, Z)) = \Theta(Y^*_f, Z^*_f).
\]

By Proposition 1.3, \(X^*_f\) corresponds to \(\nabla_X T\). Hence, 
\[
u^{-1}((\nabla_X T)(Y, Z)) = (X^*_f(\eta, \zeta))
\] 
\[
= X^*_f(f(\eta, \zeta)) = X^*_f(2\Theta(Y^*, Z^*))
\]
thus proving our assertion. QED.

Using Proposition 5.2, we shall express the Bianchi's identities (Theorem 2.5) in terms of \(T, R\) and their covariant derivatives.

**Theorem 5.3.** Let \(T\) and \(R\) be the torsion and the curvature of a linear connection of \(M\). Then, for \(X, Y, Z \in T^*_\infty(M)\), we have

- **Bianchi's 1st identity:**
  
  \[G(R(X, Y)Z) = G(T(T(X, Y), Z) + (\nabla_X T)(Y, Z));\]

- **Bianchi's 2nd identity:**
  
  \[G((\nabla_X R)(Y, Z))W = G((\nabla_X R)(Y, Z));\]

where \(G\) denotes the cyclic sum with respect to \(X, Y\) and \(Z\).

In particular, if \(T = 0\), then

- **Bianchi's 1st identity:**
  
  \[G(R(X, Y)Z) = 0;\]

- **Bianchi's 2nd identity:**
  
  \[G((\nabla_X R)(Y, Z)) = 0.\]

Proof. Let \(u\) be any point of \(L(M)\) such that \(\pi(u) = x\). We lift \(X\) to a horizontal vector at \(u\) and then extend it to a standard horizontal vector field \(X^*_f\) on \(L(M)\) as in Proposition 5.2. Similarly, we define \(Y^*_f\) and \(Z^*_f\). We shall derive the first identity from 
\[
D\Theta = \Omega \wedge \theta \text{ (Theorem 2.5)}
\]

We have 
\[
6(\Omega \wedge \theta)(X^*_f, Y^*_f, Z^*_f) = G(2\Omega(X^*_f, Y^*_f)\Theta(Z^*_f))
\]
\[
= G(u^{-1}(R(X, Y)Z)).
\]

On the other hand, by Proposition 3.11 of Chapter I, we have 
\[
6D\Theta(X^*_f, Y^*_f, Z^*_f) = 6D\Theta(X^*_f, Y^*_f, Z^*_f)
\]
\[
= G(X^*_f(2\Theta(Y^*, Z^*)) - 2\Theta([X^*, Y^*]_u, Z^*_u)).
\]
By Proposition 5.2, \(X_u^*(2\theta(Y^*, Z^*)) = u^{-1}((\nabla_X T)(Y, Z))\). It is therefore sufficient to prove that

\[-2\theta([X^*, Y^*]_u, Z^*_u) = u^{-1}(T(T(X, Y), Z)).\]

We observe first that

\[\pi([X^*, Y^*]_u) = u(\theta([X^*, Y^*]_u)) = -u(2d\theta(X^*_u, Y^*_u))\]

Hence we have

\[-2\theta([X^*, Y^*]_u, Z^*_u) = -u^{-1}(T(\pi[X^*, Y^*]_u, Z)).\]

We have

\[-2\theta([X^*, Y^*]_u, Z^*_u) = -u^{-1}(T(X, Y), Z)).\]

We shall derive the second identity from

\[D\Omega = 0 \quad \text{(Theorem 2.5)}.
\]

We have

\[0 = 3\nabla(X^*_u, Y^*_u, Z^*_u) = \nabla\{X^*_u(\Omega(Y^*, Z^*)) - \Omega([X^*, Y^*]_u, Z^*_u)\}.
\]

On the other hand, by Proposition 5.2, we have

\[X^*_u(\Omega(Y^*, Z^*)) = \frac{1}{2}u^{-1}((\nabla_X R)(Y, Z)).\]

As in the proof of the first identity, we have

\[-\Omega([X^*, Y^*]_u, Z^*_u) = \frac{1}{2}u^{-1}(T(X, Y), Z)).\]

The second identity follows from these three formulas. QED.

Remark. Theorem 5.3 can be proved from the formulas in Theorem 5.1 (see, for instance; Nomizu [7, p. 611]).

**Proposition 5.4.** Let \(B, B'\) be arbitrary standard horizontal vector fields on \(L(M)\). Then we have

1. If \(T = 0\), then \([B, B']\) is vertical; (1)
2. If \(R = 0\), then \([B, B']\) is horizontal. (2)

Proof. (1) \(\theta([B, B']) = -2d\theta(B, B') = -2\theta(B, B') = 0\). Hence, \([B, B']\) is vertical. (2) \(\omega([B, B']) = -2\omega(B, B') = -2\Omega(B, B') = 0\). Hence, \([B, B']\) is horizontal. QED.

Let \(P(u_0)\) be the holonomy subbundle of \(L(M)\) through \(u_0\) and \(\Psi(u_0)\) the linear holonomy group with reference point \(u_0\). Let \(A_1, \ldots, A_n\) be a basis of the Lie algebra of \(\Psi(u_0)\) and \(A^*_1, \ldots, A^*_n\) the corresponding fundamental vector fields. Let \(B_1, \ldots, B_n\) be the standard horizontal vector fields corresponding to the basis \(e_1, \ldots, e_n\) of \(\mathbb{R}^n\). These vector fields \(A^*_1, \ldots, A^*_n, B_1, \ldots, B_n\) (originally defined on \(L(M)\)), restricted to \(P(u_0)\), define vector fields on \(P(u_0)\). Just as in Proposition 2.6, they define an absolute parallelism on \(P(u_0)\). We know that \([A^*_1, A^*_n]\) is the fundamental vector field corresponding to \([A_1, A_n]\) and hence is a linear combination of \(A^*_1, \ldots, A^*_n\) with constant coefficients. By Proposition 2.3, \([A^*_1, B_1]\) is the standard horizontal vector field corresponding to \(A_1 e_1 \in \mathbb{R}^n\). The following proposition gives some information about \([B_1, B_1]\).

**Proposition 5.5.** Let \(P(u_0)\) be the holonomy subbundle of \(L(M)\) through \(u_0\). Let \(B, B'\) be arbitrary standard horizontal vector fields. Then we have,

1. If \(\nabla T = 0\), then the horizontal component of \([B, B']\) coincides with a standard horizontal vector field on \(P(u_0)\).
2. If \(\nabla R = 0\), then the vertical component of \([B, B']\), coincides with the fundamental vector field \(A^\ast\) on \(P(u_0)\), which corresponds to an element \(A\) of the Lie algebra of the linear holonomy group \(\Psi(u_0)\).

Proof. (1) Let \(X^*\) be any horizontal vector at \(u \in L(M)\). Set \(X = \pi(X^*), Y = \pi(B_u), Z = \pi(B'_u)\). By Proposition 5.2, we have

\[X^*(2\theta(B, B')) = u^{-1}((\nabla_X T)(Y, Z)) = 0;\]

This means that \(\theta(B, B')\) is a constant function (with values in \(\mathbb{R}^n\)) on \(P(u_0)\). Since \(\theta([B, B']) = -2\theta(B, B')\), the horizontal component of \([B, B']\) coincides on \(P(u_0)\) with the standard horizontal vector field corresponding to the element \(-2\theta(B, B')\) of \(\mathbb{R}^n\).

2. Again, by Proposition 5.2, \(VR = 0\) implies

\[X^*(\Omega(B, B')) = 0;\]

This means that \(\Omega(B, B')\) is a constant function on \(P(u_0)\) (with values in the Lie algebra of \(\Psi(u_0)\)). Since \(\omega([B, B']) = -2\Omega(B, B')\), the vertical component of \([B, B']\) coincides on \(P(u_0)\) with the fundamental vector field corresponding to the element \(-2\Omega(B, B')\) of the Lie algebra of \(\Psi(u_0)\). QED.

It follows that, if \(VT = 0\) and \(VR = 0\), then the restriction of \([B, B']\) to \(P(u_0)\) coincides with a linear combination of \(A^*_1, \ldots, A^*_n\).
Hence we have such that 

\[ \text{Connection is } \] 

is of class \( \mathcal{C}^\infty \) and for any integral curve \( \xi(t) \) of \( \mathcal{B} \) corresponding to an element \( \xi \in \mathbb{R}^n \). Let \( \xi(t) \) be an integral curve of \( \mathcal{B} \). Let \( \xi(t) \) is a horizontal lift of \( \xi \) and \( \xi(t) \) is independent of \( t, \xi \). \( \xi(t) \) is parallel along the curve \( \xi(t) \) (see §7, Chapter II, in particular, before Proposition 7.4).

Conversely, let \( \xi(t) \) 'be a geodesic' in \( \mathcal{M} \) defined in some open interval containing 0. Let \( \xi(t) \) be any point of \( \mathcal{B} \) such that \( \pi(\xi(t)) = \mathcal{M} \). We set \( \xi(t) = u_0^{-1}x_0 \in \mathbb{R}^n \). Let \( \xi(t) \) be the horizontal lift of \( \xi(t) \) through \( \xi(t) \). Since \( \xi(t) \) is a geodesic, we have \( \xi(t) = u_0^{-1}\xi(t) = \mathcal{M} \). \( \xi(t) \) is an integral curve of the standard horizontal vector field \( \mathcal{B} \) corresponding to \( \xi(t) \).

As an application of Proposition 6.3, we obtain the following

**Theorem 6.4.** For any point \( x \in \mathcal{M} \) and for any vector \( X \in T_x(\mathcal{M}) \), there is a unique geodesic with the initial condition \( (x, X) \), that is, a unique geodesic \( \xi(t) \) such that \( x_0 = x \) and \( x_0 = X \).

Another consequence of Proposition 6.3 is that a geodesic, which is a curve of class \( \mathcal{C}^\infty \), is automatically, of class \( \mathcal{C}^\infty \) (provided that the linear connection is of class \( \mathcal{C}^\infty \)). In fact, every standard horizontal vector field is of class \( \mathcal{C}^\infty \) and hence its integral curves are all of class \( \mathcal{C}^\infty \). The projection onto \( \mathcal{M} \) of a curve of class \( \mathcal{C}^\infty \) in \( \mathcal{B} \) is a curve of class \( \mathcal{C}^\infty \) in \( \mathcal{M} \).

A linear connection of \( \mathcal{M} \) is said to be complete if every geodesic can be extended to a geodesic \( \xi(t) \) defined for \( -\infty < t < \infty \), where \( t \) is an affine parameter. In other words, for any \( x \in \mathcal{M} \) and \( X \in T_x(\mathcal{M}) \), the geodesic curve \( \xi(t) \) in Theorem 6.4 with the initial condition \( x \) is defined for all values of \( t \) in the open interval \( (-\infty, \infty) \).
Immediate from Proposition 6.3 is the following

**Proposition 6.5.** A linear connection is complete if and only if every standard horizontal vector field on $\mathcal{L}(M)$ is complete.

We recall that a vector field on a manifold is said to be complete if it generates a global 1-parameter group of transformations of the manifold.

When the linear connection is complete, we can define the exponential map at each point $x \in M$ as follows. For each $X \in T_x(M)$, let $\gamma = x_t$ be the geodesic with the initial condition $(x, X)$ as in Theorem 6.4. We set

$$\exp X = x_1.$$ 

Thus we have a mapping of $T_x(M)$ into $M$ for each $x$. We shall later (in $\S 8$) define the exponential map in the case where the linear connection is not necessarily complete and discuss its differentiability and other properties.

### 7. Expressions in local coordinate systems

In this section, we shall express a linear connection and related concepts in terms of local coordinate systems.

Let $\mathcal{M}$ be a manifold and $U$ a coordinate neighborhood in $\mathcal{M}$ with a local coordinate system $x^1, \ldots, x^n$. We denote by $X_i$ the vector field $\partial / \partial x^i$, $i = 1, \ldots, n$, defined in $U$. Every linear frame at a point $x$ of $U$ can be uniquely expressed by

$$(\Sigma_i X_i^1(x), \ldots, \Sigma_i X_i^n(x)),$$

where $\det (X_i^j) \not= 0$. We take $(x^1, X_1^j)$ as a local coordinate system in $\mathcal{M}$, (cf. Example 5.2 of Chapter I). Let $(Y_1^j)$ be the inverse matrix of $(X_i^j)$ so that

$$\Sigma_i X_i^j Y^k = \Sigma_i Y_i^j X_k^i = \delta^k_i.$$

We shall express first the canonical form $\theta$ in terms of the local coordinate system $(x^i, X_i^j)$. Let $e_i, \ldots, e_n$ be the natural bases for $\mathbb{R}^n$ and set

$$\theta = \Sigma_i \theta_i e_i.$$

**Proposition 7.1.** In terms of the local coordinate system $(x^i, X_i^j)$, the canonical form $\theta = \Sigma_i \theta_i e_i$ can be expressed as follows:

$$\theta^i = \Sigma_j Y_j^i dx^j.$$
Let $\varphi$ be the left invariant $\mathfrak{gl}(n; \mathbb{R})$-valued canonical 1-form on $GL(n; \mathbb{R})$ defined in §4 of Chapter I; this form was denoted by $\theta$ in §4 of Chapter I and in §1 of Chapter II. If $(s_i^j)$ is the natural coordinate system in $GL(n; \mathbb{R})$ and if $(t_i^j)$ denotes the inverse matrix of $(s_i^j)$, then

$$\varphi = \sum_{i,j,k} t_i^j \frac{ds_i}{dx^k} E_k^i,$$

the proof being similar to that of Proposition 7.1. It is easy to verify that

$$\varphi_{UV} = (\partial x^i/\partial x')$$

and hence

$$\varphi_{UV} \varphi = \sum_{i,j} \left( \sum_{a,b,d} \frac{\partial x^a}{\partial x^i} d \left( \frac{\partial x^d}{\partial x'} \right) \right) \frac{\partial x^d}{\partial x'} = \sum_{a,b} \left( \sum_{i,j} \frac{\partial x^a}{\partial x^i} \frac{\partial x^b}{\partial x^j} d \frac{\partial x^i}{\partial x'} \frac{\partial x^j}{\partial x'} \right) E_k^i E_l^j.$$

With our notation, the formula in Proposition 1.4 of Chapter II can be expressed as follows:

$$\omega_U = (\text{ad} (\varphi_{UV})) \omega_U + \varphi_{UV} \varphi.$$

By a simple calculation, we see that this formula is equivalent to the transformation rule of our proposition. QED.

From the components $\Gamma_{jk}^i$ we can reconstruct the connection form $\omega$.

**Proposition 7.3.** Assume that, for each local coordinate system $x_1, \ldots, x_n$, there is given a set of functions $\Gamma_{jk}^i, i, j, k = 1, \ldots, n$, in such a way that they satisfy the transformation rule of Proposition 7.2. Then there is a unique linear connection $\Gamma'$ whose components with respect to $x_1, \ldots, x_n$ are exactly the given functions $\Gamma_{jk}^i$. Moreover, the connection form $\omega = \sum_i \omega_i^j E_j^i$ is given in terms of the local coordinate system $(x_1, X_k^j)$ by

$$\omega_{ij}^k = \sum_i X_i^j (dX_k^j + \sum_{m} \Gamma_{mk}^i X_j^m \ dx^m), \quad i, j = 1; \ldots, n.$$

Proof. It is easy to verify that the form $\omega$ defined by the above formula defines a connection in $L(M)$, that is, $\omega$ satisfies the conditions (a') and (b') of Proposition 1.1 of Chapter II. The fact that $\omega$ is independent of the local coordinate system used

follows from the transformation rule of $\Gamma_{jk}^i$. This can be proved by reversing the process in the proof of Proposition 7.2. The cross section $\sigma: U \rightarrow \mathfrak{gl}(n)$ used above to define $\omega_U$ is given in terms of the local coordinate systems $(x^i)$ and $(x^j)$ by $(x^i) \rightarrow (x^j, \delta^i_j)$. Hence, $\varphi_{ij} = \sum_k \Gamma_{mk}^i dX^m_j$. This shows that the components of the connection $\Gamma$ defined by $\omega$ are exactly the functions $\Gamma_{jk}^i$. QED.

The components of a linear connection can be expressed also in terms of covariant derivatives.

**Proposition 7.4.** Let $x_1, \ldots, x_n$ be a local coordinate system in $M$ with a linear connection $\Gamma$. Set $X_i = \partial/\partial x^i$, $i = 1, \ldots, n$. Then the components $\Gamma_{jk}^i$ of $\Gamma$ with respect to $x_1, \ldots, x_n$ are given by

$$\nabla_x X_i = \sum_k \Gamma_{jk}^i X_k.$$

Proof. Let $X_i^j$ be the horizontal lift of $X_i$. From Proposition 7.3, it follows that, in terms of the coordinate system $(x^i, X_k^j)$, $X_n$ is given by

$$X_n^j = (\partial/\partial x^j) = -\sum_{k} \Gamma_{jk}^i X_k^j (\partial/\partial X_i^j).$$

To apply Proposition 1.3, let $f$ be the $\mathbb{R}$-valued function on $\pi^{-1}(U) \subseteq L(M)$ which corresponds to $X_i$. Then

$$f = \sum_k \gamma_k f e_k.$$

A simple calculation shows that

$$X_i^j f = \sum_k \gamma_k f Y_k^j e_k.$$

By Proposition 1.3, $X_i^j f$ is the function corresponding to $\nabla_x X_i$ and hence

$$\nabla_x X_i = \sum_k \gamma_k \Gamma_{jk}^i X_k.$$

QED.
\( \text{Proposition 7.6.} \) We have

\[
T_{jk}^l = \Gamma_{jk}^l - \Gamma_{kj}^l;
\]

\[
R_{jkl}^i = \left( \partial \Gamma_{jk}^i / \partial x^s - \partial \Gamma_{kj}^i / \partial x^j \right) + \Sigma_m (\Gamma_{jm}^r \Gamma_{kl}^r - \Gamma_{kj}^r \Gamma_{lm}^r).
\]

Proof. These formulas follow immediately from Theorem 5.1 and Proposition 7.4.

The proof of the following proposition is a straightforward calculation.

\( \text{Proposition 7.7.} \) (1) If \( f \) is a function defined on \( M \), then

\[
f_{jk}^l = \Gamma_{jk}^l f_i^j = \Sigma_i \Gamma_{jk}^l \eta^i_j.
\]

(2) If \( X \) is a vector field on \( M \) with components \( \xi^i \), then

\[
\xi_{jk}^l = \xi_{jkl}^l = \Sigma_i \Gamma_{jk}^l \xi^i_j.
\]

Since \( n^2 \) \( n \) 1-forms \( \theta^i, \omega^i, \xi^i, \eta^i \) \( k = 1, \ldots, n \) define an absolute parallelism (Proposition 2.6), every differential form on \( L(M) \) can be expressed in terms of these 1-forms and functions. Since the torsion form \( \Theta \) and the curvature form \( \Omega \) are tensorial forms, they can be expressed in terms of \( n \) 1-forms \( \theta^i \) and functions.

We define a set of functions \( T_{jk}^l \) and \( R_{jkl}^i \) on \( L(M) \) by

\[
\Theta^i = \Sigma_i \xi_j^k \Gamma^i_{jk}, \quad \Gamma_{jk}^i = - \Gamma_{kj}^i,
\]

\[
\Omega_j^i = \Sigma_i \xi_j^k \Gamma^i_{jk} \theta^k, \quad R_{jkl}^i = - R_{kjl}^i.
\]

These functions are related to the components of the torsion \( T \) and the curvature \( R \) as follows. Let \( \sigma : U \to L(M) \) be the cross section over \( U \) defined at the beginning of this section. Then

\[
\sigma^* T_{jk}^i = T_{jk}^i, \quad \sigma^* R_{jkl}^i = R_{jkl}^i.
\]

These formulas follow immediately from Proposition 7.6 and from

\[
\sigma^* \omega^i_j = - \Sigma_j \sigma^* \theta^i + \sigma^* \Theta^i, \quad \sigma^* \theta^i_j = \Sigma_k \sigma^* \omega^i_k + \sigma^* \Omega_k^i.
\]

**Proposition 7.8.** Let \( x^i = x^i(t) \) be the equations of a curve
\[ \tau = \chi_i \text{ of class } C^2. \text{ Then } \tau \text{ is a geodesic if and only if} \]
\[ \frac{d^2 x_i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad i = 1, \ldots, n. \]

Proof. The components of the vector field \( \dot{x} \) along \( \tau \) are given by \( dx^i/dt \). From the formula for the components of \( \nabla_X Y \) given above, we see that, if we set \( X = \dot{x} \), then \( \nabla_X X = 0 \) is equivalent to the above equations.

We shall compare two or more linear connections by their components.

**Proposition 7.9.** Let \( \Gamma \) be a linear connection of \( M \) with components \( \Gamma^i_{jk} \). For each fixed \( t \), \( 0 \leq t \leq 1 \), the set of functions \( t \Gamma^i_{jk} + (1 - t) \Gamma'^i_{jk} \) defines a linear connection \( \Gamma^* \) which has the same geodesics as \( \Gamma \). In particular, \( \Gamma^* i_{jk} = \frac{1}{2} (\Gamma^i_{jk} + \Gamma'^i_{jk}) \) define a linear connection with vanishing torsion.

Proof. Our proposition follows immediately from Propositions 7.3, 7.5 and 7.8.

Q E D.

In general, given two linear connections \( \Gamma \) and \( \Gamma' \) respectively, then \( S_{jk} = \Gamma^i_{jk} - \Gamma'^i_{jk} \) are the components of a tensor \( S \) of type \((1,2)\). Conversely, if \( S_{jk} \) are the components of a linear connection \( \Gamma \) and \( S'_{jk} \) are the components of a tensor \( S \) of type \((1,2)\), then \( \Gamma^i_{jk} = \Gamma^i_{jk} + S'_{jk} \) define a linear connection \( \Gamma' \). In terms of covariant derivatives, they are related to each other as follows:

\[ \nabla'_X Y = \nabla_X Y + S(X,Y) \]

for any vector fields \( X \) and \( Y \) on \( M \), \( \nabla \) and \( \nabla' \) are the covariant differentiations with respect to \( \Gamma \) and \( \Gamma' \) respectively.

### 8. Normal coordinates

In this section we shall prove the existence of normal coordinate systems and convex coordinate neighborhoods as well as the differentiability of the exponential map.

Let \( M \) be a manifold with a linear connection \( \Gamma \). Given \( X \in T_x(M) \), let \( \tau = x_t \) be the geodesic with the initial condition \( (x, X) \) (cf. Theorem 6.5). We set

\[ \exp tX = x_t \]

As we have seen already in \( \S6 \), \( \exp tX \) is defined in some open interval \(-\epsilon < t < \epsilon \), where \( \epsilon \) and \( \epsilon_2 \) are positive. If the connection is complete, the exponential map \( \exp \) is defined on the whole of \( T(M) \) for each \( x \in M \). In general, \( \exp \) is defined only on a subset of \( T_x(M) \) for each \( x \in M \).

**Proposition 8.1.** Identifying each \( x \in M \) with the zero vector at \( x \), we consider \( M \) as a submanifold of \( T(M) = \bigcup_{x \in M} T_x(M) \). Then there is a neighborhood \( N \) of \( M \) in \( T(M) \) such that the exponential map \( \exp \) is defined on \( N \). The exponential map \( N \to M \) is differentiable of class \( C^2 \), provided that the connection is of class \( C^2 \).

Proof. Let \( x_0 \) be any point of \( M \) and \( u_0 \) a point of \( L(M) \) such that \( \pi(u_0) = x_0 \). For each \( \xi \in \mathbb{R}^n \), we denote by \( B(\xi) \) the corresponding standard horizontal vector field on \( L(M) \) (cf. \( \S2 \)). By Proposition 1.5 of Chapter I, there exist a neighborhood \( U^* \) of \( u_0 \) and a positive number \( \delta \) such that the local L-parameter group of local transformations \( tB(\xi) : U^* \to L(M) \) is defined for \( |t| < \delta \). Given a compact set \( K \) of \( \mathbb{R}^n \), we can choose \( U^* \) for all \( \xi \in K \) simultaneously, because \( B(\xi) \) depends differentiably on \( \xi \). Therefore, there exist a neighborhood \( U^* \) of \( u_0 \) and a neighborhood \( V \) of \( 0 \) in \( \mathbb{R}^n \) such that \( \exp tB(\xi) : U^* \to L(M) \) is defined for \( \xi \in V \) and \( |t| \leq 1 \). Let \( U \) be a neighborhood of \( x_0 \) in \( M \) and \( \sigma \) a cross section of \( L(M) \) over \( U \) such that \( \sigma(x_0) = u_0 \) and \( \sigma(U) \subset U^* \). Given \( \xi \in U \), let \( N_{\xi} \) be the set of \( x \in T(M) \) such that \( \sigma(x)^{-1}X \in V \) and set \( N(x_0) = \bigcup_{x \in U} N_{\xi} \). Given \( x \in N \), set \( \xi = \sigma(x)^{-1}X \). Then \( \pi(\exp tB(\xi)) \cdot \sigma(x) \) is the geodesic with the initial condition \( (x, X) \) and hence

\[ \exp x = \pi(\exp B(\xi)) \cdot \sigma(x) \]

It is now clear that \( \exp : N(x_0) \to M \) is differentiable of class \( C^2 \). Finally, we set \( N = \bigcup_{x_0 \in M} N(x_0) \).

Q E D.

**Proposition 8.2.** For every point \( x \in M \), there is a neighborhood \( N_x \) of \( x \) (more precisely, the zero vector at \( x \)) in \( T(M) \) which is mapped diffeomorphically onto a neighborhood \( U_x \) of \( x \) in \( M \) by the exponential map.
PROPOSITION 8.3. Let \( x_1, \ldots, x^n \) be the normal coordinate system determined by a linear frame \( u = (X_1, \ldots, X_n) \) at \( x \in M \). Then the geodesic \( \gamma = x_t \) with the initial condition \( (x, X) \), where \( X = \sum_i a^i X_i \), is expressed by

\[
x^i = a^i t \quad i = 1, \ldots, n.
\]

Conversely, a local coordinate system \( x_1, \ldots, x^n \) with the above property is necessarily the normal coordinate system determined by \( u = (X_1, \ldots, X_n) \).

Proof. The first assertion is an immediate consequence of the definition of a normal coordinate system. The second assertion follows from the fact that a geodesic is uniquely determined by the initial condition \((x, X)\). QED.

Remark. In the above definition of a normal coordinate system, we did not specify the neighborhood in which the coordinate system is valid. This is because if \( x_1, \ldots, x^n \) is the normal coordinate system valid in a neighborhood \( U \) of \( x \) and \( y^1, \ldots, y^n \) is the normal coordinate system valid in a neighborhood \( V \) of \( x \), then they coincide in a neighborhood of \( x \).

PROPOSITION 8.4. Given a linear connection \( \Gamma \) on \( M \), let \( \Gamma^i_{jk} \) be its components with respect to a normal coordinate system with origin \( x_0 \). Then

\[
\Gamma^i_{jk} + \Gamma^i_{kj} = 0 \quad \text{at } x_0.
\]

Consequently, if the torsion of \( \Gamma \) vanishes, then \( \Gamma^i_{jk} = 0 \) at \( x_0 \).

Proof. Let \( x_1, \ldots, x^n \) be a normal coordinate system with origin \( x_0 \). For any \( (a^1, \ldots, a^n) \in \mathbb{R}^n \), the curve defined by \( x^i = a^i t \), \( i = 1, \ldots, n \), is a geodesic and, hence, by Proposition 7.8,

\[
\sum_{j<k} \Gamma^i_{jk}(a^1, \ldots, a^n) a^i a^k = 0.
\]

In particular,

\[
\sum_{j<k} \Gamma^i_{jk}(x_0) a^i a^k = 0.
\]

Since this holds for every \( (a^1, \ldots, a^n) \), \( \Gamma^i_{jk} + \Gamma^i_{kj} = 0 \) at \( x_0 \). If the torsion vanishes, then \( \Gamma^i_{jk} = 0 \) at \( x_0 \) by Proposition 7.6. QED.

COROLLARY 8.5. Let \( K \) be a tensor field on \( M \) with components \( K^i_{jk} \) with respect to a normal coordinate system \( x_1, \ldots, x^n \) with origin \( x_0 \). If the torsion vanishes, then the covariant derivative \( K^i_{jk} \) coincides with the partial derivative \( \partial K^i_{jk}/\partial x^k \) at \( x_0 \).

Proof. This is immediate from Proposition 8.4 and the formula for the covariant differential of \( K \) in terms of \( \Gamma^i_{jk} \) given in \$7 \).

COROLLARY 8.6. Let \( \omega \) be any differential form on \( M \). If the torsion vanishes, then

\[
\omega = A(\nabla \omega),
\]

where \( \nabla \omega \) is the covariant differential of \( \omega \) and \( A \) is the alternation defined in Example 3.2 of Chapter I.

Proof. Let \( x_0 \) be an arbitrary point of \( M \) and \( x_1, \ldots, x^n \) a normal coordinate system with origin \( x_0 \). By Corollary 8.5,

\[
d\omega = A(\nabla \omega) \quad \text{at } x_0.
\]

THEOREM 8.7. Let \( x_1, \ldots, x^n \) be a normal coordinate system with origin \( x_0 \). Let \( U(x_0; \rho) \) be the neighborhood of \( x_0 \) defined by \( \Sigma_i (x^i)^2 < \rho^2 \).

Then there is a positive number \( \rho \) such that if \( 0 < \rho < a_1 \) then

(1) \( U(x_0; \rho) \) is convex in the sense that any two points of \( U(x_0; \rho) \) can be joined by a geodesic which lies in \( U(x_0; \rho) \).

(2) Each point of \( U(x_0; \rho) \) has a normal coordinate neighborhood containing \( U(x_0; \rho) \).

Proof. By Proposition 7.9, we may assume that the linear connection has no torsion.

LEMMA 1. Let \( S(x_0; \rho) \) denote the sphere defined by \( \Sigma_i (x^i)^2 = \rho^2 \). Then there exists a positive number \( \rho \) such that, if \( 0 < \rho < c \), then any geodesic which is tangent to \( S(x_0; \rho) \) at a point, say \( y \), of \( S(x_0; \rho) \) lies outside \( S(x_0; \rho) \) in a neighborhood of \( y \).

Proof of Lemma 1. Since the torsion vanishes by our assumption, the components \( \Gamma^i_{jk} \) of the linear connection vanish at \( x_0 \) by Proposition 8.4. Let \( x'(t) = x'(t) \) be the equations of a geodesic
which is tangent to $S(x_0; \rho)$ at a point $y = (x^1(0), \ldots, x^n(0))$ ($\rho$ will be restricted later). Set

$$F(t) = \Sigma_i (x^i(t))^2.$$ 

Then

$$(dF/dt)_{t=0} = 2\Sigma_i x^i(0) (dx^i/dt)_{t=0} = 0,$$

$$d^2F/dt^2 = 2\Sigma_i (d^2x^i/dt^2 + x^i(t) d^2x^i/dt^2).$$

Because of the equations of a geodesic given in Proposition 7.8, we have

$$(d^2F/dt^2)_{t=0} = \Sigma_{j,k} \left( \delta_{jk} - \Sigma_i \Gamma^i_{jk} x^i \right) (dx^j/dt)(dx^k/dt).$$

Since $\Gamma^i_{jk}$ vanish at $x_0$, there exists a positive number $c$ such that the quadratic form with coefficients $\left( \delta_{jk} - \Sigma_i \Gamma^i_{jk} x^i \right)$ is definite in $U(x_0; c).$ If $0 < \rho < c$, then $(d^2F/dt^2)_{t=0} > 0$ and hence $F(t) > \rho^2$ when $t \neq 0$ is in a neighborhood of 0. This completes the proof of the lemma.

**Lemma 2.** Choose a positive number $c$ as in Lemma 1. Then there exists a positive number $a, c$ such that

1. Any two points of $U(x_0; a)$ can be joined by a geodesic which lies in $U(x_0; c)$.
2. Each point of $U(x_0; a)$ has a normal coordinate neighborhood containing $U(x_0; a)$.

Proof of Lemma 2. **We consider** $A^4$ as a submanifold of $T(M)$ in a natural manner. Set

$$\varphi(X) = (x, \exp X) \quad \text{for } X \in T_x(M).$$

If the connection is complete, $\varphi$ is a mapping of $T(M)$ into $M \times M.$ In general, $\varphi$ is defined only in a neighborhood of $M$ in $T(M).$ Since the differential of $\varphi$ at $x_0$ is nonsingular; there exist a neighborhood $V$ of $x_0$ in $T(M)$ and a positive number $a < c$ such that $\varphi: V \rightarrow U(x_0; a) \times U(x_0; a)$ is a diffeomorphism. Taking $V$ and a small, we may assume that $\exp tX \in U(x_0; \epsilon)$ for all $X \in V$ and $|t| \leq 1.$ To verify condition (1), let $x$ and $y$ be points of $U(x_0; a).$ Let $X = \varphi^{-1}(x, y), X \in V.$ Then the geodesic with the initial condition $(x, y)$ joins $x$ and $y$ in $U(x_0; a).$ To verify (2), let $V = V \cap T(M).$ Since $exp: V \rightarrow U(x_0; a)$ is a diffeomorphism, condition (2) is satisfied.

To complete the proof of Theorem 8.7, let $0 < \rho < a.$ Let $x = x(t), 0 \leq t \leq 1,$ be the equations of a geodesic from $x$ to $y$ in $U(x_0; c)$ (see Lemma 2). We shall show that this geodesic lies in $U(x_0; \rho).$ Set

$$F(t) = \Sigma_i (x^i(t))^2 \quad \text{for } 0 \leq t \leq 1.$$ 

Assume that $F(t) \geq \rho^2$ for some $t$ (that is, $x^i(t)$ lies outside $U(x_0; \rho)$ for some $t$). Let $t_0, 0 < t_0 < 1,$ be the value for which $F(t)$ attains the maximum. Then

$$0 = \left( \frac{dF}{dt} \right)_{t=t_0} = 2\Sigma_i (x^i(t_0) \frac{dx^i}{dt})_{t=t_0}.$$ 

This means that the geodesic $x^i(t)$ is tangent to the sphere $S(x_0; \rho_0)$, where $\rho_0 = F(t_0)$, at the point $x(t_0).$ By the choice of $t_0$, the geodesic $x^i(t)$ lies inside the sphere $S(x_0; \rho_0)$, contradicting Lemma 1. This proves (1). (2) follows from (2) of Lemma 2.

Q.E.D.

The existence of convex neighborhoods is due to J. H. C. Whitehead [1].

**9. Linear infinitesimal holonomy groups**

Let $\Gamma$ be a linear *connection* on a manifold $M$. For each point $u$ of $L(M)$, the holonomy group $Y(u)$, the local holonomy group $\Psi^*(u)$ and the infinitesimal holonomy group $\Psi'(u)$ are defined as in §10 of Chapter II. These groups can be realized as groups of linear transformations of $T_x(M), x = w(u)$, denoted by $\Psi(x), \Psi^*(x)$ and $Y(x)$ respectively (cf. §8 of Chapter II).

**Theorem 9.1.** The Lie algebra $\mathfrak{g}(x)$, of the holonomy group $\Psi(x)$ is equal to the subspace of linear endomorphisms of $T_x(M)$ spanned by all elements of the form $(\tau R)(X; Y) = \tau^{-1} \circ R(\tau X, \tau Y) \circ \tau,$ where $X, Y \in T_x(M)$ and $\tau$ is the parallel displacement along an arbitrary piecewise-differentiable curve $\tau$ starting from $x$.
Proof. This follows immediately from Theorem 8.1 of Chapter II and from the relationship between the curvature form $\Omega$ on $L(M)$ and the curvature tensor field $R$ (cf. §5 of Chapter III). QED.

It is easy to reformulate Proposition 10.1, Theorems 10.2 and 10.3 of Chapter II in terms of $Y(x)$ and $Y^{\prime}(x)$. We shall therefore proceed to the determination of the Lie algebra of $\Psi'(x)$.

**Theorem 9.2.** The Lie algebra $\mathfrak{g}'(x)$ of the infinitesimal holonomy group $\Psi'(x)$ is spanned by all linear endomorphisms of $T^1(M)$ of the form $(\nabla^k R)(X, Y; V, \ldots; V_k)$, where $X, Y, V, \ldots, V_k \in T^1(M)$ and $0 \leq k < \infty$.

Proof. The proof is achieved by the following two lemmas.

**Lemma 1.** If a tensor field of type $A_k$ (resp. $B_k$), we mean a tensor field of type $(1, 1)$ of the form $\nabla^{i_1} \cdots \nabla^{i_k}(R(X, Y))$ (resp. $(\nabla^k R)(X, Y; V, \ldots; V_k)$), where $X, Y, V, \ldots, V_k$ are arbitrary vector fields on $M$. Then every tensor field of type $A_k$ (resp. $B_k$) is a linear combination (with differentiable functions as coefficients) of a finite number of tensor fields of type $B_j$ (resp. $A_j$), $0 \leq j \leq k$.

Proof of Lemma 1. The proof is by induction on $k$. The case $k = 0$ is trivial. Assume that $\nabla_{V_{k+1}} \cdots \nabla_{V_1}(R(X, Y))$ is a sum of terms like $f(\nabla^j R)(U, V; W_1; \ldots; W_j)$, $0 \leq j \leq k - 1$, where $f$ is a function. Then we have

$$\nabla_{V_k} (f(\nabla^j R)(U, V; W_1; \ldots; W_j))$$

$$(V_k f)(\nabla^j R)(U, V; W_1; \ldots; W_j) + (V_k R)(\nabla^j_r U, V; W_1; \ldots; W_j) + (V_k R)(U, \nabla^j_r V; W_1; \ldots; W_j) + (V_k R)(U, V, \nabla^j_r W_1; \ldots; W_j) + \sum_{i=1}^j (\nabla^i R)(U, V; W_1; \ldots; \nabla_{V_k} W_i; \ldots; W_j).$$

This shows that every tensor field of type $A_k$ is a linear combination of tensor fields of type $B_j$, $0 \leq j \leq k$.

Assume now that every tensor field of type $B_{k-1}$ is a linear combination of tensor fields of type $A_j$, $0 \leq j \leq k - 1$. We have

$$(\nabla^k R)(X, Y; V, \ldots; V_k) = \nabla_{V_k} ((\nabla^{k-1} R)(X, Y; V, \ldots; V_{k-1}))$$

$$(\nabla^{k-1} R)(\nabla_{V_k} X, Y; V_1; \ldots; V_{k-1})$$

$$(\nabla^{k-1} R)(X, \nabla_{V_k} Y; V_1; \ldots; V_{k-1})$$

$$\nabla^{k-1} R)(X, Y; \nabla_{V_k} V_1; \ldots; V_{k-1})$$

The first term on the right hand side is a linear combination of tensor fields of type $B_j$, $0 \leq j \leq k$. The remaining terms on the right hand side are linear combinations of tensor fields of type $A_j$, $0 \leq j < k$. This completes the proof of Lemma 1.

By definition, $\mathfrak{g}'(u)$ is spanned by the values at $u$ of all $\mathfrak{gl}(n; \mathbb{R})$-valued functions of the form $(1, k)$, $k = 0, 1, 2, \ldots$ (cf. §5 of Chapter II). Theorem 9.2 will follow from Lemma 1 and the following lemma.

**Lemma 2.** If $X, Y, V, \ldots, V_k$ are vector fields on $M$ and if $X^*, Y^*, V^*, \ldots, V_k^*$ are their horizontal lifts to $L(M)$, then we have

$$(\nabla_{V_k} \cdots \nabla_{V_1}(R(X, Y)))_{\ast} Z = u \ast (V_k^* \cdots V_1^* (2\Omega(X^*, Y^*)) |_{\ast})_{\ast} u^{-1}(Z)$$

for $Z \in T^1_z(M)$.

Proof of Lemma 2. This follows immediately from Proposition 1.3 of Chapter III; we take $R(X, Y)$ and $2\Omega(X^*, Y^*)$ as $q$ and $f$ in Proposition 1.3 of Chapter III. QED.

By Theorem 10.8 of Chapter II and Theorem 9.2 the restricted holonomy group $\Psi^0(x)$ of a real analytic linear connection is completely determined by the values of all successive covariant differentials $\nabla^k R$, $k = 0, 1, 2, \ldots$, at the point $x$.

The results in this section were obtained by Nijenhuis [2].
IV. RIEMANNIAN CONNECTIONS

1. Riemannian metrics

Let $\mathcal{M}$ be an $n$-dimensional paracompact manifold. We know (cf. Examples 5.5, 5.7 of Chapter I and Proposition 1.4 of Chapter III), that $\mathcal{M}$ admits a Riemannian metric and that there is a 1:1 correspondence between the set of Riemannian metrics' on $\mathcal{M}$ and the set of reductions of the bundle $L(\mathcal{M})$ of linear frames to a bundle $O(\mathcal{M})$ of orthonormal frames. Every Riemannian metric $g$ defines a positive definite inner product in each tangent space $T_x(\mathcal{M})$; we write, $g_x(X, Y) = g(X, Y)$ for $X, Y \in T_x(\mathcal{M})$ (cf. Example 3.1 of Chapter I).

**Example 1.1.** The Euclidean metric $g$ on $\mathbb{R}^n$ with the natural coordinate system $x_1, \ldots, x_n$ is defined by

$$g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij} \quad \text{(Kronecker's symbol).}$$

**Example 1.2.** Let $f: N \rightarrow A^4$ be an immersion of a manifold $N$ into a Riemannian manifold $A^4$ with metric $g$. The induced Riemannian metric $h$ on $N$ is defined by $h(X, Y) = g(f_*X, f_*Y)$, $X, Y \in T_x(N)$.

**Example 1.3.** A homogeneous space $G/H$, where $G$ is a Lie group and $H$ is a compact subgroup, admits an invariant metric. Let $\tilde{H}$ be the linear isotropy group at the origin $o$ (i.e., the point represented by the coset $H$) of $G/H$. $\tilde{H}$ is a group of linear transformations of the tangent space $T_o(G/H)$, each induced by an element of $H$ which leaves the point $o$ fixed. Since $H$ is compact, so is $\tilde{H}$ and there is a positive definite inner product, say $g_o$, in $T_o(G/H)$ which is invariant by $\tilde{H}$. For each $x \in G/H$, we take an element $a \in G$ such that $a(o) = x$ and define an inner product $g_x$ in $T_x(G/H)$ by $g_x(X, Y) = g_o(a^{-1}X, a^{-1}Y)$, $X, Y \in T_x(G/H)$. It is easy to verify that $g_x$ is independent of the choice of $a \in G$ such that $a(o) = x$ and that the Riemannian metric $g$ thus obtained is invariant by $G$. The homogeneous space $G/H$ provided with an invariant Riemannian metric is called a Riemannian homogeneous space.

**Example 1.4.** Every compact Lie group $G$ admits a Riemannian metric which is invariant by both right and left translations. In fact, the group $G \times G$ acts transitively on $G$ by $(a, b) \cdot x = ab^{-1}$, for $(a, b) \in G \times G$ and $x \in G$. The isotropy subgroup of $G \times G$ at the identity $e$ of $G$ is the diagonal $D = \{(a, a); a \in G\}$, so that $G = (G \times G)/D$. By Example 1.3, $G$ admits a Riemannian metric invariant by $G \times G$, thus proving our assertion. If $G$ is compact and semisimple, then $G$ admits the following canonical invariant Riemannian metric. In the Lie algebra $\mathfrak{g}$, identified with the tangent space $T_e(G)$, we have the Killing-Cartan form

$$\varphi(X, Y) = \text{trace}(\text{ad}X \circ \text{ad}Y),$$

where $X,Y \in \mathfrak{g} = Z'(G)$. The form $\varphi$ is bilinear, symmetric and invariant by ad $G$. When $G$ is compact and semisimple, $\varphi$ is negative definite. We define a positive definite inner product $g$, in $T_e(G)$ by

$$g(x, y) = -\varphi(x, y),$$

where $-\varphi$ is invariant by ad $G$. Since $\varphi$ is invariant by ad $G$, $g_x$ is invariant by the diagonal $D$. By Example 1.3, we obtain a Riemannian metric on $G$ invariant by $G \times G$. We discuss this metric in detail in Volume II.

By a Riemannian metric, we shall always mean a positive definite symmetric covariant tensor field of degree 2. By an indefinite Riemannian metric, we shall mean a symmetric covariant tensor field $g$ of degree 2 which is nondegenerate at each $x \in M$, that is, $g(X, Y) = 0$ for all $Y \in T_x(M)$ implies $X = 0$.

**Example 1.5.** An indefinite Riemannian metric on $\mathbb{R}^n$ with the coordinate system $x_1, \ldots, x_n$ can be given by

$$\Sigma_{p-1} (dx^i)^2 - \Sigma_{n-p+1} (dx^i)^2,$$

where $0 \leq p \leq n - 1$. Another example of an indefinite Riemannian metric is the canonical metric on a noncompact, semisimple Lie group $G$ defined as follows. It is known that for such a group the Killing-Cartan form $\varphi$ is indefinite and nondegenerate. The construction in Example 1.4 gives an indefinite Riemannian metric on $G$ invariant by both right and left translations.
Let $M$ be a manifold with a Riemannian metric or an indefinite Riemannian metric $g$. For each $x$, the inner product $g_x$ defines a linear isomorphism $\mathcal{L}$ of $T_x(M)$ onto its dual $T^*_x(M)$ (space of covectors at $x$) as follows: To each $X \in T_x(M)$, we assign the covector $a \in T^*_x(M)$ defined by
\[
\langle Y, a \rangle = g_x(X, Y)
\]
for all $Y \in T_x(M)$.

The inner product $g_x$ in $T_x(M)$ defines an inner product, denoted also by $g_x$, in the dual space $T^*_x(M)$ by means of the isomorphism $\mathcal{L}$:
\[
\langle a, b \rangle = g_x(Y', \mathcal{L}(a)) \quad \text{for } a, b \in T^*_x(M).
\]

Let $x^1, \ldots, x^n$ be a local coordinate system in $M$. The components $g_{ij}$ of $g$ with respect to $x^1, \ldots, x^n$ are given by
\[
g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), \quad i, j = 1, \ldots, n.
\]

The contravariant components $g^{ij}$ of $g$ are defined by
\[
g^{ij} = g(\partial^i, \partial^j), \quad i, j = 1, \ldots, n.
\]
We have then
\[
\sum g_{ij} e^j = \delta^i.
\]

In fact, define $\psi_{ij}$ by $\psi(\partial/\partial x^i, \partial/\partial x^j) = \sum \psi_{ij} dx^l$. Then we have
\[
g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j) = (\partial/\partial x^i, \psi(\partial/\partial x^j)) = \psi_{ij}.
\]

On the other hand, we have
\[
\delta^i = (\partial/\partial x^i, dx^k) = g(dx^k, \psi(\partial/\partial x^i)) = g(dx^k, \sum \psi_{ij} dx^l) = g_{ij} e^j,
\]
thus proving our assertion.

If $x^i$ are the components of a vector or a vector field $X$ with respect to $x^1, \ldots, x^n$, that is, $X = \sum x^i \partial/\partial x^i$, then the components $\xi^i$ of the corresponding covector or the corresponding 1-form $a = \gamma(X)$ are related to $x^i$ by
\[
\xi^i = \sum g_{ij} \xi^j, \quad \xi^i = \sum g_{ij} x^j.
\]

The inner product, $g$ in $T_x(M)$ and in $T^*_x(M)$ can be extended to an inner product, denoted also by $g$, in the tensor space $T^r_s(M)$ at $x$ for each type $(r, s)$. If $K$ and $L$ are tensors at $x$ of type $(r, s)$, with components $K_{i_1}^{i_2} \cdots {i_r}$ and $L_{i_1}^{i_2} \cdots {i_s}$ (with respect to $x^1, \ldots, x^n$), then
\[
\langle K, L \rangle = \sum g_{i_1}^{k_1} \cdots g_{i_r}^{k_r} g_{j_1}^{j_1} \cdots g_{j_s}^{j_s} K_{i_1}^{i_2} \cdots {i_r} L_{j_1}^{j_2} \cdots {j_s}.
\]

The isomorphism $\mathcal{L}$ of $T_x(M)$ can be extended to tensors. Given a tensor $K \in T^r_s(M)$, we obtain a tensor $K^{r_1} \cdots {r_s}$ with components
\[
K_{i_1}^{i_2} \cdots {i_r} L_{j_1}^{j_2} \cdots {j_s} = \sum g_{i_1}^{k_1} \cdots g_{i_r}^{k_r} g_{j_1}^{l_1} \cdots g_{j_s}^{l_s} K_{i_1}^{i_2} \cdots {i_r} L_{j_1}^{j_2} \cdots {j_s},
\]
or $K' \in T^r_s(M)$ with components
\[
K'_{i_1}^{i_2} \cdots {i_r} L'_{j_1}^{j_2} \cdots {j_s} = \sum g_{i_1}^{k_1} \cdots g_{i_r}^{k_r} g_{j_1}^{l_1} \cdots g_{j_s}^{l_s} K_{i_1}^{i_2} \cdots {i_r} L_{j_1}^{j_2} \cdots {j_s}.
\]

Example 1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be skew-symmetric endomorphisms of the tangent space $T_x(M)$, that is, tensors at $x$ of type $(1, 1)$ such that
\[
g(\mathcal{A} X, Y) = -g(\mathcal{A} Y, X)
\]
and
\[
g(\mathcal{B} X, Y) = -g(\mathcal{B} Y, X)
\]
for $X, Y \in T_x(M)$. Then the inner product $g(\mathcal{A}, \mathcal{B})$ is equal to $-\text{trace } (\mathcal{A} \mathcal{B})$. In fact, take a local coordinate system $x^1, \ldots, x^n$ such that $g_{ij} = \delta_{ij}$ at $x$ and let $a^i$ and $b^i$ be the components of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then
\[
g(\mathcal{A}, \mathcal{B}) = \sum (a^i b^j + b^j a^i) = \sum a^i b^j = -\text{trace } (\mathcal{A} \mathcal{B}),
\]
since $\mathcal{B}$ is skew-symmetric, i.e., $b^j = -b^j$.

On a Riemannian manifold $M$, the arc length of a differentiable curve $\gamma = x^t$, $a \leq t \leq b$, of class $C^1$ is defined by
\[
L = \int_a^b g(\dot{x}, \dot{x}) \, dt.
\]
In terms of a local coordinate system $x^1, \ldots, x^n$, $L$ is given by
\[
L = \int_a^b \left( \sum g_{ij} \dot{x}^i \dot{x}^j \right) \, dt.
\]
This definition can be generalized to a piecewise differentiable curve of class $C^1$ in an obvious manner.

Given a Riemannian metric $g$ on a-connected manifold $M$, we define the distance function $d(x, y)$ on $M$ as follows. The distance $d(x, y)$ between two points $x$ and $y$ is, by definition, the infimum of the lengths of all piecewise differentiable curves of class $C^1$. 

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IV. RIEMANNIAN CONNECTIONS
2. Riemannian connections

Although the results in this section are valid for manifolds with indefinite Riemannian metrics, we shall consider (positive definite) Riemannian metrics only for the sake of simplicity.

Let \( M \) be an \( n \)-dimensional Riemannian manifold with metric \( g \) and \( O(M) \) the bundle of orthonormal frames over \( M \). Every connection in \( O(M) \) determines a connection in the bundle \( L(M) \) of linear frames, that is, a linear connection of \( M \) by virtue of Proposition 6.1 of Chapter II. A linear connection of \( M \) is called a metric connection if it is thus determined by a connection in \( O(M) \).

**Proposition 2.1.** A linear connection \( \Gamma \) of a Riemannian manifold \( M \) with metric \( g \) is a metric connection iff and only if \( g \) is parallel with respect to \( \Gamma \).

Proof. Since \( g \) is a fibre metric (cf. §1 of Chapter III) in the tangent bundle \( T(M) \), our proposition follows immediately from Proposition 1.5 of Chapter III. QED.

Among all possible metric connections, the most important is the Riemannian connection (sometimes called the Levi-Civita connection) which is given by the following theorem.

**Theorem 2.2.** Every Riemannian manifold admits a unique metric connection with vanishing torsion.

We shall present here two proofs, one using the bundle \( O(M) \), and the other using the formalism of covariant differentiation.

Proof (A). **Uniqueness.** Let \( \theta \) be the canonical form of \( L(M) \) restricted to \( O(M) \). Let \( \omega \) be the connection form on \( O(M) \) defining a metric connection of \( M \). With respect to the basis \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \) and the basis \( E_{ij}^k, i < j, i, j = 1, \ldots, n \), of the Lie algebra of \( \mathfrak{o}(n) \), we represent \( \theta \) and \( \omega \) by \( n \) forms \( \theta^i, i = 1, \ldots, n \), and a skew-symmetric matrix of differential forms \( \omega^i \) respectively.

The proof of the following lemma is similar to that of Proposition 2.6 of Chapter III and hence is left to the reader.

**Lemma.** The \( n \) forms \( \theta^i, i = 1, \ldots, n \), and the \( \frac{1}{2} n(n - 1) \) forms \( \omega^i_{jk}, 1 \leq j < k \leq n \), define an absolute parallelism on \( O(M) \).

Let \( \varphi \) be the connection form defining another metric connection of \( M \). Then \( \varphi - \omega \) can be expressed in terms of \( \theta^i \) and \( \omega^i_{jk} \) by the lemma. Since \( \varphi - \omega \) annihilates the vertical vectors, we have

\[
\varphi^i_{jk} = \omega^i_{jk} = \sum_k F^i_{jk} \theta^k,
\]

where the \( F^i_{jk} \)s are functions on \( O(M) \). Assume that the connections defined by \( \omega \) and \( \varphi \) have no torsion. Then, from the first structure equation of Theorem 2.4 of Chapter III, we obtain

\[
0 = \sum_i (\varphi^i_{jk} - \omega^i_{jk}) \wedge \theta^i = \sum_{j,k} F^i_{jk} \theta^k \wedge \theta^i.
\]

This implies that \( F^i_{jk} = F^i_{kj} \). On the other hand, \( F^i_{jk} = -F^i_{kj} \) since \( (\omega^i_{jk}) \) and \( (\varphi^i_{jk}) \) are skew-symmetric. It follows that \( F^i_{jk} = 0 \), proving the uniqueness.

**Existence.** Let \( \varphi \) be an arbitrary metric connection form on \( O(M) \) and \( \theta \) its torsion form on \( O(M) \). We write

\[
\Theta^i = \sum_{j,k} T^i_{jk} \theta^j \wedge \theta^k, \quad T^i_{jk} = -T^i_{kj},
\]

and set

\[
\tau^i_{jk} = \sum_k \frac{1}{2} (T^i_{jk} + T^i_{kj}) \theta^k \wedge \theta^i
\]

and

\[
\omega^i_{jk} = \varphi^i_{jk} + \tau^i_{jk}.
\]

We shall show that \( \omega = (\omega^i_{jk}) \) defines the desired connection. Since both \( (T^i_{jk} + T^i_{kj}) \) and \( T^i_{kj} \) are skew-symmetric in \( i \) and \( j \), so is \( \tau^i_{jk} \). Hence \( \omega \) is \( \mathfrak{o}(n) \)-valued. Since \( \Theta \) annihilates the vertical vectors, so does \( \tau = (\tau^i_{jk}) \). It is easy to show that \( R^i \sigma = \mathfrak{ad}(a^{-1}) (\sigma) \) for every \( a \in \mathfrak{o}(n) \). Hence, \( \omega \) is a connection form. Finally, we verify that the metric connection defined by \( \omega \) has zero torsion. Since \( (T^i_{jk} + T^i_{kj}) \) is symmetric in \( j \) and \( k \), we have

\[
\sum_j \tau^i_{jk} \wedge \theta^j = -\Theta^i,
\]

and hence

\[
d\theta^i = -\sum_j \varphi^i_{jk} \wedge \theta^j + \Theta^i = -\sum_j \omega^i_{jk} \wedge \theta^j,
\]

proving our assertion.
Let $M$ and $M'$ be Riemannian manifolds with Riemannian metrics $g$ and $g'$ respectively. A mapping $f: M \to M'$ is called isometric at a point $x$ of $M$ if $g(X, Y) = g'(f_*(X), f_*(Y))$ for all $X, Y \in T_x(M)$. In this case, $f_*$ is injective at $x$, because $f_*X = 0$ implies that $g(X, Y) = 0$ for all $Y$ and hence $X = 0$. A mapping $f$ which is isometric at every point of $M$ is thus an immersion which we call an isometric immersion. If, moreover, $f$ is 1:1, then it is called an isometric embedding of $M$ into $M'$. If $f$ maps $M$ 1:1 onto $M'$, then $f$ is called an isometry of $M$ onto $M'$.

**Proposition 2.5.** If $f$ is an isometry of a Riemannian manifold $M$ onto another Riemannian manifold $M'$, then the differential $df$ commutes with the parallel displacement. More precisely, if $\gamma$ is a curve from $x$ to $y$ in $M$, then the following diagram is commutative:

$$
\begin{array}{ccc}
T_x(M) & \xrightarrow{T_y(M)} & T_y(M') \\
\downarrow & & \downarrow \\
T_{x'}(M') & \xrightarrow{T_{y'}(M')} & T_{y'}(M')
\end{array}
$$

where $x' = f(x)$, $y' = f(y)$ and $\gamma' = f(\gamma)$.

Proof. This is a consequence of the uniqueness of the Riemannian connection in Theorem 2.2. Being a diffeomorphism between $M$ and $M'$, $f$ defines a 1:1 correspondence between the set of vector fields on $M$ and the set of vector fields on $M'$. From the Riemannian connection $\Gamma^i_x$ on $M$, we obtain a linear connection $\Gamma^i_x$ on $M'$ by $\nabla_XY = f^{-1}(\nabla_{fX}(fY))$, where $X$ and $Y$ are vector fields on $M$. It is easy to verify that $\Gamma$ has no torsion and is metric with respect to $g$. Thus, $\Gamma$ is the Riemannian connection of $M$. This means that $f_*(\nabla_XY) = \nabla_{fX}(fY)$ with respect to the Riemannian connection of $M$. This implies immediately our proposition.

**Proposition 2.6.** If $f$ is an isometric immersion of a Riemannian manifold $M$ into another Riemannian manifold $M'$ and if $f(M)$ is open in $M'$, then the differential $df$ commutes with the parallel displacement.

Proof. Since $f(M)$ is open in $M'$, $\dim M = \dim M'$. Since $f$ is an immersion, every point $x$ of $M$ has an open neighborhood $U$ such that $f(U)$ is open in $M'$ and $f: U \to f(U)$ is a diffeomorphism. Thus, $f$ is an isometry of $U$ onto $f(U)$. By Proposition 2.5, the differential $df$ commutes with the parallel displacement.
along any curve in \( U \). Given an arbitrary curve \( \tau \) from \( x \) to \( y \) in \( M \), we can find a finite number of open neighborhoods in \( M \) with the above property which cover \( \tau \). It follows that the differential off commutes with the parallel displacement along \( \tau \). QED.

Remark. It follows immediately that, under the assumption of Proposition 2.6, every geodesic of \( M \) is mapped by \( f \) into a geodesic of \( M' \).

Example 2.1. Let \( M \) be a Riemannian manifold with metric \( g \). Let \( M^\ast \) be a covering manifold of \( M \) with projection \( \rho \). We can introduce a Riemannian metric \( g^\ast \) on \( M^\ast \) in such a way that \( \rho : M^\ast \to M \) is an isometric ‘immersion’. Every geodesic of \( M^\ast \) projects on a geodesic of \( M \). Conversely, given a geodesic \( \tau \) from \( x \) to \( y \) in \( M \) and a point \( x^\ast \) of \( M^\ast \) with \( \rho(x^\ast) = x \), there is a unique curve \( \tau^\ast \) in \( M^\ast \) starting from \( x^\ast \) such that \( \rho(\tau^\ast) = \tau \).

Since \( \rho \) is a local isometry, \( \tau^\ast \) is a geodesic of \( M^\ast \). A similar argument, together with Proposition 2.6, shows that if \( \rho(x^\ast) = x \), then the restricted linear holonomy group of \( M^\ast \) with reference point \( x^\ast \) is isomorphic by \( \rho \) to the restricted linear holonomy group of \( M \) with reference point \( x \).

Proposition 2.5 and 2.6 were stated with respect to Riemannian connections which are special linear connections. Similar statements hold with respect to the corresponding affine connections.

The statement concerning linear holonomy groups in Example 2.1 holds also for affine holonomy groups.

3. Normal coordinates and convex neighborhoods

Let \( M \) be a Riemannian manifold with metric \( g \). The length of a vector \( X \), i.e., \( g(X, X)_x \), will be denoted by \( \| X \| \).

Let \( \tau = x_t \) be a geodesic in \( M \). Since the tangent vectors \( \dot{x}_t \) are parallel along \( \tau \) and since the parallel displacement is isometric, the length of \( \dot{x}_t \) is constant along \( \tau \). If \( \| \dot{x}_t \| = 1 \), then \( t \) is called the canonical parameter of the geodesic \( \tau \).

By a normal coordinate system at \( x \) of a Riemannian manifold \( M \), we always mean a normal coordinate system \( x_1, \ldots, x^n \) at \( x \) such that \( \partial / \partial x^1, \ldots, \partial / \partial x^n \) form an orthonormal frame at \( x \). However, \( \partial / \partial x^1, \ldots, \partial / \partial x^n \) may not be orthonormal at other points.

Let \( U \) be a normal coordinate neighborhood of \( x \) with a normal coordinate system \( x_1, \ldots, x^n \) at \( x \). We define a cross section \( \sigma \) of \( O(M) \) over \( U \) as follows: Let \( u \) be the orthonormal frame at \( x \) given by \( (\partial / \partial x^1)_x, \ldots, (\partial / \partial x^n)_x \). By the parallel displacement of \( u \) along the geodesics through \( x \), we attach an orthonormal frame to every point of \( U \). For the study of Riemannian manifolds, the cross section \( \sigma : U \to O(M) \) thus defined is more useful than the cross section \( U \to O(M) \) given by \( \partial / \partial x^1, \ldots, \partial / \partial x^n \). Let \( \theta = (\theta^i) \) and \( \omega = (\omega^k) \) be the canonical form and the Riemannian connection form of \( O(M) \) respectively. We set

\[
\tilde{\theta} = \sigma^* \theta = (\tilde{\theta}^i) \quad \text{and} \quad \tilde{\omega} = \sigma^* \omega = (\tilde{\omega}^k),
\]

where \( \tilde{\theta}^i \) and \( \tilde{\omega}^k \) are 1-forms on \( U \). To compute these forms explicitly, we introduce the polar coordinate system \( (p^1, \ldots, p^n; t) \) by

\[
x^i = p^i t, \quad i = 1, \ldots, n; \quad \Sigma_i (p^i)^2 = 1.
\]

Then, \( \tilde{\theta}^i \) and \( \tilde{\omega}^k \) are linear combinations of \( dp^1, \ldots, dp^n \) and \( dt \) with functions of \( p^1, \ldots, p^n, t \) as coefficients.

**Proposition 3.1.** (1) \( \tilde{\theta}^i = p^i dt + q^i \), where \( q^i, i = 1, \ldots, n \), do not involve \( dt \); (2) \( \tilde{\omega}^k \) do not involve \( dt \); (3) \( q^i = 0 \) and \( \tilde{\omega}^k = 0 \) at \( t = 0 \) (i.e., at the origin \( x \)); (4)

\[
dp^i = -dp^i + \Sigma_j \tilde{\omega}^j dp^j \wedge dt + \cdots,
\]

\[
dt^k = -\Sigma_j R^k_{ij} p^j \wedge dt + \cdots,
\]

where the dots . . . indicate terms not involving \( dt \) and \( R^k_{ij} \) are the components of the curvature tensor field with respect to the frame field \( \sigma \).

**Proof.** (1) For a fixed direction \( (p^1, \ldots, p^n) \), let \( \tau = x_t \) be the geodesic defined by \( x^i = p^i t, i = 1, \ldots, n \). Set \( u_t = \sigma(x_t) \). To prove that \( \tilde{\theta} = p^i dt \) do not involve \( dt \), it is sufficient to prove that \( \tilde{\theta}^i(x_t) = p^i \). From the definition of the canonical form \( \theta \), we have

\[
\theta(u_t) = \tilde{\theta}^i(x_t) = u^{-1}(x_t),
\]

Since both \( u_t \) and \( x_t \) are parallel along \( \tau \), \( \tilde{\theta}^i(x_t) \) is independent of \( t \). On the other hand, we have \( \tilde{\theta}^i(x_t) = p^i \), and hence \( \tilde{\theta}^i(x_t) = p^i \) for all \( t \).

(2) Since \( u_t \) is horizontal by the construction of \( \sigma \), we have

\[
\tilde{\omega}^k(x_t) = \omega^k(u_t) = 0.
\]

This means that \( \tilde{\omega}^k \) do not involve \( dt \).
(3) Given any unit vector $X$ at $x$ (i.e., the point where $t = 0$), let $\tau = x$, be the geodesic with the initial condition $(x, X)$ so that $X = \dot{x}_t$. By (1) and (2), we have $q^i(x_0) = 0$ and $\ddot{w}_k(x_0) = 0$.

(4) From the structure equations, we obtain

$$d(p^i dt + q^i) = -\Sigma_i \dot{q}^i \wedge (p^i dt + q^i)$$

$$d\ddot{w}_k = -\Sigma_k \dot{w}_k \wedge \dot{\omega}_k + \ddots,$$

where

$$\ddots = \Sigma_{k, l} \frac{1}{2} R_{ij}^{kl}\dot{\omega}^i \wedge \dot{\omega}^j = \Sigma_{k, i} \frac{1}{2} R_{ijkl}(p^k dt + q^k) \wedge (p^i dt + q^i)$$

(cf. §7 of Chapter III)

and hence (4).

In terms of $dt$ and $q^i$, we can express the metric tensor $g$ as follows (cf. the classical expression $ds^2 = \Sigma g_{ij} dx^i dx^j$ for $g$ as explained in Example 3.1 of Chapter I).

PROPOSITION 3.2. The metric tensor $g$ can be expressed by

$$ds^2 = (dt)^2 + \Sigma_i (q^i)^2.$$  

Proof. Since $\tilde{b}(X) = (\sigma(y))^{-1}(X)$ for every $X \in T_u(M)$, $y \in U$, and since $\sigma(y)$ is an isometric mapping of $\mathbb{R}^n$ onto $T_u(M)$, we have

$$g(X, Y) = \Sigma_i \tilde{b}(X)\tilde{b}(Y)$$

for $X, Y \in T_u(M)$ and $y \in U$.

In other words,

$$ds^2 = \Sigma_i (\tilde{b})^2.$$  

By Proposition 3.1, we have

$$ds^2 = (dt)^2 + \Sigma_i (q^i)^2 + \Sigma_i p^i \dot{p}^i dt.$$  

Since $q^i = 0$ at $t = 0$, we shall prove that $\Sigma_i p^i \dot{q}^i = 0$ by showing that $\Sigma_i p^i \dot{q}^i$ is independent of $t$. Since $\Sigma_i p^i \dot{q}^i$ does not involve $dt$ by Proposition 3.1, it is sufficient to show that $d(\Sigma_i p^i \dot{q}^i)$ does not involve $dt$. We have, by Proposition 3.1,

$$d(\Sigma_i p^i \dot{q}^i) = -\Sigma_i \dot{p}^i (dp^i + \Sigma_i \dot{q}^i) \wedge dt + \cdots,$$

where the dots indicate terms not involving $dt$.

From $\Sigma_i (p^i)^2 = 1$, we obtain

$$0 = d(\Sigma_i (p^i)^2) = 2 \Sigma_i p^i \dot{p}^i.$$
We shall now prove (3). If y is in \( U(x; r) \), then, clearly, \( d(x, y) < r \). Conversely, let \( d(x, y) < r \) and let \( \tau \) be a curve from \( x \) to \( y \) such that \( L(\tau) < r \). Suppose \( \tau \) does not lie in \( U(x; r) \). Let \( y' \) be the first point on \( \tau \) which belongs to the closure of \( U(x; r) \) but not to \( U(x; r) \). Then, \( d(x, y') = r \) by (1) and (2). The length of \( \tau \) from \( x \) to \( y' \) is at least \( r \). Hence, \( L(\tau) \geq r \), which is a contradiction. Thus \( \tau \) lies entirely in \( U(x; r) \) and hence \( y \) is in \( U(x; r) \). QED.

**Proposition 3.5.** \( d(x, y) \) is a distance function (i.e., metric) on \( A^4 \) and defines the same topology as the manifold topology of \( M \).

Proof. As we remarked earlier (cf. the end of §1), we have

\[
d(x, y) \geq 0, d(x, y) = d(y, x), d(x, y) + d(y, z) \geq d(x, z).
\]

From Proposition 3.4, it follows that if \( x \neq y \), then \( d(x, y) > 0 \). Thus \( d \) is a metric. The second assertion follows from (3) of Proposition 3.4. QED.

A geodesic joining two points \( x \) and \( y \) of a Riemannian manifold \( M \) is called minimizing if its length is equal to the distance \( d(x, y) \). We now proceed to prove the existence of a convex neighborhood around each point of a Riemannian manifold in the following form.

**Theorem 3.6.** Let \( x^1, \ldots, x^n \) be a normal coordinate system at \( x \) of a Riemannian manifold \( M \). There exists a positive number \( a \) such that, if \( 0 < p < a \), then

1. Any two points of \( U(x; p) \) can be joined by a unique minimizing geodesic; and it is the unique geodesic joining the two points and lying in \( U(x; p) \);
2. In \( U(x; p) \), the square of the distance \( d(y, z) \) is a differentiable function of \( y \) and \( z \).

Proof. (1) Let \( a \) be the positive number given in Theorem 8.7 of Chapter III and let \( 0 < p < a \). If \( y \) and \( z \) are points of \( U(x; p) \), they can be joined by a geodesic \( \tau \) lying in \( U(x; p) \) by the same theorem. Since \( U(x; p) \) is contained in a normal coordinate neighborhood of \( y \) (cf. Theorem 8.7 of Chapter III), we see from Proposition 3.4 that \( \tau \) is a unique geodesic joining \( y \) and \( z \) and lying in \( U(x; p) \) and that the length of \( \tau \) is equal to the distance, that is, \( \tau \) is minimizing. It is clear that \( \tau \) is the unique minimizing geodesic joining \( y \) and \( z \) in \( M \).

(2) Identifying every point of \( M \) with the zero vector at \( y \), we consider \( y \) as a point of \( T(M) \). For each \( y \) in \( U(x; p) \), let \( N_y \) be the neighborhood of \( y \) in \( T(M) \) such that \( \exp: N_y \rightarrow U(x; p) \) is a diffeomorphism (cf. (2) of Theorem 8.7 of Chapter III). Set

\[
N = \bigcup_{y \in U(x; p)} N_y.
\]

Then the mapping \( \exp: N \rightarrow U(x; p) \times U(x; p) \) which sends \( y \in N_y \) into \( (y, \exp y) \) is a diffeomorphism (cf. Proposition 8.1 of Chapter III). If \( z = \exp y \), then \( d(y, z) = \|Y\| \). In other words, \( \|Y\| \) is the function on \( V \) which corresponds to the distance function \( d(y, z) \) under the diffeomorphism \( \exp: N \rightarrow U(x; p) \times U(x; p) \). Since \( \|Y\|^2 \) is a differentiable function on \( V \), \( d(y, z)^2 \) is a differentiable function on \( U(x; p) \times U(x; p) \). QED.

As an application of Theorem 3.6, we obtain the following

**Theorem 3.7.** Let \( M \) be a paracompact differentiable manifold. Then every open covering \( \{U_i\} \) of \( M \) has an open refinement \( \{V_i\} \) such that

1. each \( V_i \) has compact closure;
2. \( \{V_i\} \) is locally finite in the sense that every point of \( M \) has a neighborhood which meets only a finite number of \( V_i \)'s;
3. any nonempty finite intersection of \( V_i \)'s is diffeomorphic with an open cell of \( R^n \).

Proof. By taking an open refinement if necessary, we may assume that \( \{U_i\} \) is locally finite and that each \( U_i \) has compact closure. Let \( \{U_{i1}, \ldots, U_{ia}\} \) be an open refinement of \( \{U_i\} \) (with the same index set) such that \( U_{ij} \subset U_i \) for all \( a \) (cf. Appendix 3). Take any Riemannian metric on \( M \). For each \( x \in M \), let \( V_x \) be a convex neighborhood of \( x \) in the sense of Theorem 3.6 which is contained in some \( U_{i1} \). For each \( x \), let

\[
\mathcal{B}_x = \{W_x; W_x \cap U_{ij} \neq \text{non-empty}\}.
\]

Since \( U_{ij} \) is compact, there is a finite subfamily \( \mathcal{B}_x \) of \( \mathcal{B}_x \) which covers \( U_{ij} \). Then the family \( \mathcal{B} = \bigcup \mathcal{B}_x \) is a desired open refinement of \( \{U_i\} \). In fact, it is clear from the construction that \( \mathcal{B} \) satisfies (1) and (2). If \( V_1, \ldots, V_k \) are members of \( \mathcal{B} \) and if \( x \) and \( y \) are points of the intersection \( \bigcap V_i \), then there is a unique minimizing geodesic joining \( x \) and \( y \) in \( M \). Since the geodesic lies in each \( V_i \), it lies in the intersection \( \bigcap V_i \). It follows that the intersection is diffeomorphic with an open cell of \( R^n \). QED.
Remark. A covering \( \{ V_i \} \) satisfying (1), (2) and (3) is called a simple covering. Its usefulness lies in the fact that the Čech cohomology of \( M \) can be computed by means of a simple covering of \( A^4 \) (cf. Weil [1]).

In any metric space \( M \), a segment is defined to be a continuous image \( x(t) \) of a closed interval \( a \leq t \leq b \) such that

\[
d(x(t_1), x(t_2)) + d(x(t_2), x(t_3)) = d(x(t_1), x(t_3))
\]

for \( a \leq t_1 \leq t_2 \leq t_3 \leq b \),

where \( d \) is the distance function. As an application of Theorem 3.6, we have

**Proposition 3.8.** Let \( M \) be a Riemannian manifold with metric \( g \) and \( d \) the distance function defined by \( g \). Then every segment is a geodesic (as a point set).

The parametrization of a segment may not be affine.

**Proof.** Let \( x(t), a \leq t \leq b \), be a segment in \( M \). We first show that \( x(t) \) is a geodesic for \( a \leq t \leq a + \varepsilon \) for some positive \( \varepsilon \). Let \( U \) be a convex neighborhood of \( x(a) \) in the sense of Theorem 3.6. There exists \( \varepsilon > 0 \) such that \( x(t) \in U \) for \( a \leq t \leq a + \varepsilon \). Let \( \tau \) be the minimizing geodesic from \( x(a) \) to \( x(a + \varepsilon) \). We shall show that \( \tau \) and \( x(t) \), \( a \leq t \leq a + \varepsilon \), coincide as a point set. Suppose there is a number \( c \), \( a < c < a + \varepsilon \), such that \( x(c) \) is not on \( \tau \). Then

\[
d(x(a), x(a + \varepsilon)) < d(x(a), x(c)) + d(x(c), x(a + \varepsilon)),
\]

contradicting the fact that \( x(t) \), \( a \leq t \leq a + \varepsilon \), is a segment. This shows that \( x(t) \) is a geodesic for \( a \leq t \leq a + \varepsilon \). By continuing this argument, we see that \( x(t) \) is a geodesic for \( a \leq t \leq b \). QED.

Remark. If \( x_t \) is a continuous curve such that \( d(x_{t_1}, x_{t_2}) = |t_1 - t_2| \) for all \( t_1 \) and \( t_2 \), then \( x_t \) is a geodesic with arc length \( t \) as parameter.

**Corollary 3.9.** Let \( \tau = x_{t, a \leq t \leq b} \) be a piecwise differentiable curve of class \( C^1 \) from \( x \) to \( y \) such that its length \( L(\tau) \) is equal to \( d(x, y) \). Then \( \tau \) is a geodesic as a point set. If, moreover, \( ||x_t|| \) is constant along \( \tau \), then \( \tau \) is a geodesic including the parametrization.

**Proof.** It suffices to show that \( \tau \) is a segment. Let \( a \leq t_1 \leq t_2 \leq t_3 \leq b \). Denoting the points \( x_i \) by \( x_i \), \( i = 1, 2, 3 \), and the arcs into which \( \tau \) is divided by these points by \( \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \) respectively, we have

\[
d(x, x_1) L(\tau_1), d(x_1, x_2) L(\tau_2), d(x_2, x_3) L(\tau_3),
\]

\[
d(x_3, y) L(\tau_4).
\]

If we did not have the equality everywhere, we would have

\[
d(x, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, y)
\]

\[
\leq L(\tau_1) + L(\tau_2) + L(\tau_3) + L(\tau_4) = L(\tau) = d(x, y),
\]

which is a contradiction. Thus we have

\[
d(x_1, x_2) = L(\tau_2), d(x_2, x_3) = L(\tau_3).
\]

Similarly, we see that

\[
d(x_1, x_2) = L(\tau_2 + \tau_3).
\]

Finally, we obtain

\[
d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3).
\]

QED.

Using Proposition 3.8, we shall show that the distance function determines the Riemannian metric.

**Theorem 3.10.** Let \( M \) and \( M' \) be Riemannian manifolds with Riemannian metrics \( g \) and \( g' \), respectively. Let \( \tilde{d} \) and \( d' \) be the distance functions of \( M \) and \( M' \) respectively. If \( \tilde{f} \) is a mapping (which is not assumed to be continuous or differentiable) of \( M \) onto \( M' \) such that \( d(x, y) = d'(f(x), f(y)) \) for all \( x, y \in M \), then \( \tilde{f} \) is a diffeomorphism of \( M \) onto \( M' \) which maps the tensor field \( g \) into the tensor field \( g' \).

In particular, every mapping \( \tilde{f} \) of \( M \) onto itself which preserves \( d \) is an isometry, that is, preserves \( g \).

**Proof.** Clearly, \( \tilde{f} \) is a homeomorphism. Let \( x \) be an arbitrary point of \( M \) and set \( x' = f(x) \). For a normal coordinate neighborhood \( U' \) of \( x' \), let \( U \) be a normal coordinate neighborhood of \( x \) such that \( f(U) \subseteq U' \). For any unit tangent vector \( x \) at \( x' \) let \( \tau \) be a geodesic in \( U' \) with the initial condition \( (x, X) \). Since \( \tau \) is a segment with respect to \( \tilde{d} \), \( f(\tau) \) is a segment with respect to \( d' \) and hence is a geodesic in \( U' \) with origin \( x' \). Since \( \tau = x_{t, a \leq t \leq b} \) is parametrized by the arc length \( s \) and since \( d'(f(x_1), f(x_2)) = d(x_1, x_2) = |s_2 - s_1| \), \( f(\tau) = f(x) \) is parametrized by the arc
length $\xi$ also. Let $F(X)$ be the unit vector tangent to $f(\tau)$ at $x'$. Thus, $F$ is a mapping of the set of unit tangent vectors at $x$ into the set of unit tangent vectors at $x'$. It can be extended to a mapping, denoted by the same $F$, of $T_x(M)$ into $T_{x'}(M')$ by proportion. Since $f$ has an inverse which also preserves the distance functions, it is clear that $F$ is a 1:1 mapping of $T_x(M)$ into $T_{x'}(M')$. It is also clear that

$$f \circ \exp_x = \exp_{x'} \circ F$$

and $\|F[X]\| = \|X\|$ for $X \in T_x(M)$, where $\exp_x$ (resp. $\exp_{x'}$) is the exponential map of a neighborhood of 0 in $T_x(M)$ (resp. $T_{x'}(M')$) onto $U$ (resp. $U'$). Both $\exp_x$ and $\exp_{x'}$ are diffeomorphisms. To prove that $f$ is a diffeomorphism of $M$ onto $M'$, which maps $g$ into $g'$, it is therefore sufficient to show that $F$ is a linear isometric mapping of $T_x(M)$ onto $T_{x'}(M')$.

We first prove that $g(X, Y) = g'(F(X), F(Y))$ for all $X, Y \in T_x(M)$. Since $F(cX) = cF(X)$ for any $X \in T_x(M)$ and any constant $c$, we may assume that both $X$ and $Y$ are unit vectors. Then both $F(X)$ and $F(Y)$ are unit vectors at $x'$. Set $\cos \alpha = g(X, Y)$ and $\cos \alpha' = g'(F(X), F(Y))$.

We shall now prove that $F$ is linear. We already observed that $F(cX) = cF(X)$ for any $X \in T_x(M)$ and for any constant $c$.

**Lemma.** $\sin \frac{1}{2} \alpha = \lim_{s \to 0} \frac{1}{2s} d(x_{s}, y_{s})$ and $\sin \frac{1}{2} \alpha' = \lim_{s \to 0} \frac{1}{2s} d(x'_{s}, y'_{s})$.

We shall give the proof of the lemma shortly. Assuming the lemma for the moment, we shall complete the proof of our theorem.

Since $f$ preserves distance, the lemma implies that

$$\sin \frac{1}{2} \alpha = \sin \frac{1}{2} \alpha'$$

and hence

$$g(X, Y) = \cos \alpha = 1 - 2 \sin^2 \frac{1}{2} \alpha = 1 - 2 \sin^2 \frac{1}{2} \alpha' = \cos \alpha' = g'(F(X), F(Y)).$$

We shall now prove that $F$ is linear. We already observed that $F(cX) = cF(X)$ for any $X \in T_x(M)$ and for any constant $c$. Let $X_1, \ldots, X_n$ be an orthonormal basis for $T_x(M)$. Then $X'_i = F(X_i)$, $i = 1, \ldots, n$, form an orthonormal basis for $T_{x'}(M')$ as we have just proved. Given $X$ and $Y$ in $T_x(M)$, we have

$$g(F(X + Y), X'_i) = g(X + Y, X'_i) = g(X, X'_i) + g(Y, X'_i)$$

$$= g'(F(X), X'_i) + g'(F(Y), X'_i) = g'(F(X) + F(Y), X'_i)$$

for every $i$, and hence

$$F(X + Y) = F(X) + F(Y).$$

**Proof of Lemma.** It is sufficient to prove the first formula. Let $U$ be a coordinate neighborhood with a normal coordinate system $x^1, \ldots, x^n$ at $x$. Let $\check{h}$ be the Riemannian metric in $U$ given by $\sum_i (dx^i)^2$ and let $\delta(y, z)$ be the distance between $y$ and $z$ with respect to $\check{h}$. Supposing that

$$\lim_{s \to 0} \frac{1}{2s} d(x^s, y^s) > \sin \frac{1}{2} \alpha,$$

we shall obtain a contradiction. (The case where the inequality is reversed can be treated in a similar manner.) Choose $c > 1$ such that

$$\lim_{s \to 0} \frac{1}{2s} d(x^s, y^s) < c \sin \frac{1}{2} \alpha.$$

Taking $U$ small, we may assume that $\frac{1}{c} \check{h} < \delta < c \check{h}$ on $U$ in the sense that

$$\frac{1}{c} h(Z, Z) < \delta(Z, Z) < c h(Z, Z)$$

for $Z \in T_x(M)$ and $z \in U$.

From the definition of the distances $d$ and $\delta$, we obtain

$$\frac{1}{c} \delta(y, z) < d(y, z) < c \delta(y, z).$$

Hence we have

$$\frac{c}{2s} \delta(x_{s}, y_{s}) > \frac{1}{2s} d(x_{s}, y_{s}) > c \sin \frac{1}{2} \alpha$$

for small $s$.

On the other hand, $\check{h}$ is a Euclidean metric and hence

$$\frac{1}{2s} \delta(x_{s}, y_{s}) = \sin \frac{1}{2} \alpha.$$
This is a contradiction. Hence,
\[
\lim_{s \to 0} \frac{1}{2s} d(x_s, y_s) = \sin \frac{1}{2}x.
\]
Similarly, we obtain
\[
\lim_{s \to 0} \frac{1}{2s} d(x_s, y_s) = \sin \frac{1}{2}x.
\]

Theorem 3.10 is due to Myers and Steenrod [11]; the proof is adopted from Palais [2].

4. Completeness

A Riemannian manifold \( M \) or a Riemannian metric \( g \) on \( M \) is said to be complete if the Riemannian connection is complete, that is, if every geodesic of \( M \) can be extended for arbitrarily large values of its canonical parameter (cf. §6 of Chapter III). We shall prove the following two important theorems.

**Theorem 4.1.** For a connected Riemannian manifold \( M \), the following conditions are mutually equivalent:

1. \( M \) is a complete Riemannian manifold;
2. \( M \) is a complete metric space with respect to the distance function \( d \);
3. Every bounded subset of \( M \) (with respect to \( d \)) is relatively compact;
4. For an arbitrary point \( x \) of \( M \) and for an arbitrary curve \( C \) in the tangent space \( T_x(M) \) (or more precisely, the affine tangent space \( A_x(M) \)) skirting from the origin, there is a curve \( \gamma \) in \( M \) starting from \( x \) which is developed upon the given curve \( C \).

**Theorem 4.2.** If \( M \) is a connected complete Riemannian manifold, then any two points \( x \) and \( y \) of \( A^4 \) can be joined by a minimizing geodesic.

Proof. We divide the proofs of these theorems into several steps.

(i) The implication (i) \( \Rightarrow \) (1). Let \( x_s \to 0 \) be a geodesic, where \( s \) is the canonical parameter. We show that this geodesic can be extended beyond \( L \). Let \( \{x_s\} \) be an infinite sequence such that \( x_{s_n} \to 0 \). Then
\[
d(x_{s_n}, x_{s_m}) < |s_n - s_m|,
\]
so that \( \{x_{s_n}\} \) is a Cauchy sequence in \( M \) with respect to \( d \) and hence converges to a point, say \( x \). The limit point \( x \) is independent of the choice of a sequence \( \{s_n\} \) converging to \( L \). We set \( x_L = x \).

By using a normal coordinate system at \( x \), we can extend the geodesic for the values of \( s \) such that \( L \leq s \leq L + \varepsilon \) for some \( \varepsilon > 0 \).

(ii) Proof of Theorem 4.2. Let \( x \) be any point of \( M \). For each \( \varepsilon > 0 \), we set
\[
S(\varepsilon) = \{y \in M; d(x, y) \leq \varepsilon\}
\]
and
\[
E(\varepsilon) = \{y \in S(\varepsilon); y \text{ can be joined to } x \text{ by a minimizing geodesic}\}.
\]

We are going to prove that \( E(\varepsilon) \) is compact and coincides with \( S(\varepsilon) \) for every \( \varepsilon > 0 \). To prove the compactness of \( E(\varepsilon) \), let \( y_i \) be a sequence of points of \( E(\varepsilon) \) and, for each \( i \), let \( \gamma_i \) be a minimizing geodesic from \( x \) to \( y_i \). Let \( X_i \) be the unit vector tangent to \( \gamma_i \) at \( x \). By taking a subsequence if necessary, we may assume that \( \{X_i\} \) converges to a unit vector \( X_0 \) in \( T_x(M) \). Since \( d(x, y_i) \leq \varepsilon \) for all \( i \), we may assume, again by taking a subsequence if necessary, that \( d(x, y_i) \) converges to a non-negative number \( r_0 \) since \( \gamma_i \) is minimizing, we have
\[
y_i = \exp (d(x, y_i)X_i).
\]

Since \( M \) is a complete Riemannian manifold, \( \exp r_0X_0 \) is defined. We set
\[
x_0 = \exp r_0X_0.
\]

It follows that \( \{y_i\} \) converges to \( x_0 \) and hence that \( d(x, x_0) = r_0 \). This implies that the geodesic \( \exp sX_0 \) for \( 0 \leq s \leq r_0 \) is minimizing and that \( x_0 \) is in \( E(\varepsilon) \). This proves the compactness of \( E(\varepsilon) \).

Now we shall prove that \( E(\varepsilon) = S(\varepsilon) \) for all \( \varepsilon > 0 \). By the existence of a normal coordinate system and a convex neighborhood around \( x \) (cf. Theorem 3.6), we know that \( E(\varepsilon) = S(\varepsilon) \) for \( 0 < \varepsilon < \varepsilon' \) for some \( \varepsilon' > 0 \). Let \( r^* \) be the supremum of \( r_0 \) such that \( E(\varepsilon) = S(\varepsilon) \) for \( \varepsilon < r_0 \). To show that \( r^* = \infty \), assume that \( r^* \) is finite. We first prove that \( E(r^*) = S(r^*) \). Let \( y \) be a point of \( S(r^*) \) and let \( \{y_i\} \) be a sequence of points with \( d(x, y_i) < r^* \) which converges to \( y \). The existence of such a sequence \( \{y_i\} \) follows from the fact that \( x \) and \( y \) can be joined by a curve whose length is as close to \( d(x, y) \) as we wish. Then each \( y_i \) belongs to some \( E(y_i) \), where \( r < r^* \), and hence each \( y_i \) belongs to \( E(r^*) \).
Since $E(r^* + \delta)$ is compact, $y$ belongs to $E(r^*)$. Hence $S(r^* + \delta) = E(r^*)$. Next we shall show that $S(r) \subseteq E(r)$ for $r < r^* + \delta$ for some $\delta > 0$, which contradicts the definition of $r^*$. We need the following

**Lemma.** On a Riemannian manifold $M$, there exists a positive continuous function $r(z), z \in M$, such that any two points of $S_r(r(z)) = \{ y \in M; d(z,y) \leq r(z) \}$ can be joined by a minimizing geodesic.

**Proof of Lemma.** For each $z \in M$, let $r(z)$ be the supremum of $r$ such that any two points $y$ and $y'$ with $d(z,y) \leq r$ and $d(z,y') \leq r$ can be joined by a minimizing geodesic. The existence of a convex neighborhood (cf. Theorem 3.6) implies that $r(z) > 0$. If $r(z) = \infty$ for some point $z$, then $r(y) = \infty$ for every point $y$ of $M$ and any positive continuous function on $M$ satisfies the condition of the lemma. Assume that $r(z) < \infty$ for every $z \in M$.

We shall prove the continuity of $r(z)$ by showing that $|r(z) - r(y)| \leq d(z,y)$. Without any loss of generality, we may assume that $r(z) > r(y)$. If $d(z,y) \geq r(z)$, then obviously $|r(z) - r(y)| < d(z,y)$. If $d(z,y) < r(z)$, then $S(y) = \{ y'; d(y,y') \leq r \}$ is contained in $S_r(r(z))$, where $r = r(z) - d(z,y)$. Hence $r(y) \geq r(z) - d(z,y)$, that is, $|r(z) - r(y)| \leq d(z,y)$, completing the proof of the lemma.

Going back to the proof of Theorem 4.2, let $r(z)$ be the continuous function given in the lemma and let $\delta$ be the minimum of $r(z)$ on the compact set $E(r^*)$. To complete the proof of Theorem 4.2, we shall show that $S(r^* + \delta) = E(r^* + S)$. Let $y \in S(r^* + \delta)$ but $\not\in S(r^*)$. We show first that there exists a point $y'$ in $S(r^*)$ such that $d(y', y') = r^*$ and that $d(x,y) = d(x,y') + d(y,y')$. To this end, for every positive integer $k$, choose a curve $\gamma_k$ from $x$ to $y$ such that $L(\gamma_k) = d(x,y) + \frac{1}{k}$, where $L(\gamma_k)$ is the length of $\gamma_k$. Let $y_k$ be the last point on $\gamma_k$ which belongs to $E(r^*)$. Then $d(x,y_k) = r^*$ and $d(y_k,y_k) \leq L(\gamma_k) = d(x,y) + \frac{1}{k}$.

Since $E(r^*)$ is compact, we may assume, by taking a subsequence if necessary, that $\{ y_k \}$ converges to a point, say $y'$, of $E(r^*)$. We have $d(x,y') = r^*$ and $d(y', y') = d(x,y)$. Let $\tau'$ be a minimizing geodesic from $x$ to $y'$. Since $d(y', y') \leq \delta \leq r(y')$, there is a minimizing geodesic $\tau''$ from $y'$ to $y$. Let $\tau$ be the join of $\tau'$ and $\tau''$. Then $L(\tau') = L(\tau'') + L(\tau) = d(x,y') + d(y', y) = d(x,y)$. By Corollary 3.9, $\tau$ is a geodesic, in fact, a minimizing geodesic from $x$ to $y$. Hence $y \in E(r^* + \delta)$, completing the proof of Theorem 4.2.

Remark. To prove that $E(r) = S(r)$ is compact for every $r$, it is sufficient to assume that every geodesic issuing from the particular point $x$ can be extended infinitely.

(iii) The implication $(1) \rightarrow (3)$ in Theorem 4.1. In (ii) we proved that $(1)$ implies that $E(r) = S(r)$ is compact for every $r$. Every bounded subset of $M$ is contained in $S(r')$ for some $r'$, regardless of the point $x$ we choose in the proof of (ii).

(iv) The implication $(3) \rightarrow (2)$ is evident.

(v) The implication $(4) \rightarrow (1)$. Since a geodesic is a curve in $M$ which is developed upon a straight line (or a segment) in the tangent space, it is obvious that every geodesic can be extended infinitely.

(vi) The implication $(1) \rightarrow (4)$. Let $C_0, 0 \leq t \leq a$, be an arbitrary curve in $T_x(M)$ starting from the origin. We know that there is $\varepsilon > 0$ such that $C_0, 0 \leq t \leq \varepsilon$, is the development of a curve $x_\varepsilon, 0 \leq t \leq \varepsilon$, in $M$. Let $b$ be the supremum of such $\varepsilon > 0$. We want to show that $b = a$. Assume that $b < a$.

First we show that if $x_\varepsilon$ exists in $M$. Let $t_n \uparrow b$. Since the development preserves the arc length, the length of $x_{t_n}, t_n \leq t \leq t_n$, is equal to the length of $C_0, t_n \leq t \leq t_n$. On the other hand, the distance $d(x_{t_n}, x_b)$ is less than or equal to the length of $x_{t_n}, t_n \leq t \leq t_n$. This implies that $\{ x_{t_n} \}$ is a Cauchy sequence in $M$. Since we know the implication $(1) \rightarrow (3)$ (by (i) and (iv)), we see that $\{ x_{t_n} \}$ converges to a point, say $y$. It is easy to see that $\lim_{t \rightarrow b} x_t = y$. Let $C_t$ be the curve in $T_y(M)$ (or more precisely, in $A_x(M)$) obtained by the affine (not linear!) parallel displacement of the curve $C_t$ along the curve $x_{t_n}, 0 \leq t \leq b$. Then $C_t$ is the origin of $T_y(M)$. There exist $\delta > 0$ and a curve $x_\varepsilon, b \leq t \leq b + \delta$, which is developed upon $C_t, b \leq t \leq b + \delta$. Then the curve $x_{t_n}, 0 \leq t \leq b + \delta$, is developed upon $C_0, 0 \leq t \leq b + \delta$. This contradicts the definition of $b$.

**Corollary 4.3.** If all geodesics starting from any particular point $x$ of a connected Riemannian manifold $M$ are infinitely extendable, then $M$ is complete.
Proof. As we remarked at the end of (ii) in the proof of Theorem 4.2, $E(r) = S(r)$ is compact for every $r$. Every bounded subset of $M$ is contained in $S(r)$ for some $r$ and hence is relatively compact.

**QED.**

**Corollary 4.4.** Every compact Riemannian manifold is complete.

Proof. This follows from the implication $(3) \Rightarrow (1)$ in Theorem 4.1.

**QED.**

A Riemannian manifold $M$ is said to be homogeneous if the group of isometries, i.e., transformations preserving the metric tensor $g$ of $M$ is transitive on $M$. (Cf. Example 1.3 and Theorem 3.4, Chapter VI.)

**Theorem 4.5.** Every homogeneous Riemannian manifold is complete.

Proof. Let $x$ be a point of a homogeneous Riemannian manifold $M$. There exists $r > 0$ such that, for every unit vector $X$ at $x$, the geodesic $\exp sX$ is defined for $|s| \leq r$ (cf. Proposition 8.1 of Chapter III). Let $\mathbf{r} = x_0$, $0 \leq s \leq a$, be any geodesic with canonical parameter $s$ in $M$. We shall show that $\mathbf{r} = x_0$ can be extended to a geodesic defined for $0 \leq s \leq a + r$. Let $\varphi$ be an isometry of $M$ which maps $x$ into $x_0$. Then $\varphi^{-1}$ maps the unit vector $x_0$ at $x_0$ into a unit vector $X$ at $x$: $X = \varphi^{-1}(x_0)$. Since $\exp sX$ is a geodesic through $x$, $\varphi(\exp sX)$ is a geodesic through $x_0$. We set

$$x_{ats} = \varphi(\exp sX) \quad \text{for } 0 \leq s \leq r.$$

Then $\mathbf{r} = x_0$, $0 \leq s \leq a + r$, is a geodesic.

**QED.**

Theorem 4.5 follows also from the general fact that every locally compact homogeneous metric space is complete.

**Theorem 4.6.** Let $M$ and $M^*$ be connected Riemannian manifolds of the same dimension. Let $p: M^* \rightarrow M$ be an isometric immersion.

1. If $M^*$ is complete, then $M^*$ is a covering space of $M$ with projection $p$ and $M$ is also complete.

2. Conversely, if $p: M^* \rightarrow M$ is a covering projection and if $M$ is complete, then $M^*$ is complete.

Proof. The proof is divided into several steps.

(i) If $M^*$ is complete so is $M$. Let $x^* \in M^*$ and set $x = p(x^*)$. Let $X$ be any unit vector of $M$ at $x$ and choose a unit vector $X^*$ at $x^*$ such that $\varphi(X) = X$. Then $\exp sX = \varphi(\exp sX^*)$ is the geodesic in $M$ with the initial condition $(x, X)$. Since $\exp sX^*$ is defined for all $s$, $-\infty < s < \infty$, so is $\exp sX$. By Corollary 4.3, $M$ is complete.

(ii) If $M^*$ is complete, $p$ maps $M^*$ onto $M$. Let $x^* \in M^*$ and $x = p(x^*)$. Given a point $y$ of $M$, let $\exp sx^*, 0 \leq s \leq a$, be a geodesic from $x$ to $y$, where $X$ is a unit vector at $x$. Such a geodesic exists by Theorem 4.2 since $M$ is complete by (i). Let $X^*$ be the unit vector of $M^*$ at $X^*$ such that $\varphi(X^*) = X$. Set $y^* = \exp aX^*$. Then $p(y^*) = \exp ax^* = y$.

(iii) If $M^*$ is complete, then $p: M^* \rightarrow M$ is a covering projection. For a given $x \in M$ and for each positive number $r$, we set

$$U(x; r) = \{y \in M : d(x, y) < r\}, N(x; r) = \{X \in T_x(M) : \|X\| < r\}.$$

Similarly, we set, for $x^* \in M^*$,

$$U(x^*; r) = \{y^* \in M^* : d(x^*, y^*) < r\}, N(x^*; r) = \{X^* \in T_{x^*}(M^*) : \|X^*\| < r\}.$$

Choose $r > 0$ such that $\exp (N(x^*; 2r)) = U(x^*; 2r)$ is a diffeomorphism. Let $\{x^*_1, x^*_2, \ldots\}$ be the set $p^{-1}(x)$. For each $x^*_i$, we have the following commutative diagram:

$$\begin{array}{c}
N(x^*_i; 2r) \\
\exp \downarrow \quad \downarrow \exp
\end{array}$$

$$\begin{array}{c}
U(x^*_i; 2r)
\end{array}$$

It is sufficient to prove the following three statements:

(a) $p: U(x^*_i; r) \rightarrow U(x; r)$ is a diffeomorphism for every $i$;

(b) $p^{-1}(U(x^*_i; r)) = \bigcup U(x^*_i; r)$;

(c) $U(x^*_i; r) \cap U(x^*_j; r)$ is empty if $x^*_i \neq x^*_j$.

Now, (a) follows from the fact that both $p: N(x^*_i; 2r) \rightarrow N(x; 2r)$ and $\exp: N(x^*_i; 2r) \rightarrow U(x^*_i; 2r)$ are diffeomorphisms in the above diagram. To prove (b), let $y^* \in p^{-1}(U(x; r))$ and set $y = p(y^*)$. Let $\exp sx^*, 0 \leq s \leq a$, be a minimizing geodesic from $y$ to $x$, where $Y$ is a unit vector at $y$. Let $Y^*$ be the unit vector at $y^*$ such that $\varphi(Y) = Y$. Then $\exp sy^*, 0 \leq s \leq a$, is a geodesic in $M^*$ starting from $y^*$ such that $\varphi(\exp sy^*) = \exp sY$. In particular, $\exp ax^* = x$ and hence $\exp aX^* = x^*$ for some $x^*_i$. Evidently, $y^* \in U(x^*_i; r)$, proving that $p^{-1}(U(x; r)) = \bigcup U(x^*_i; r)$. On
the other hand, it is obvious that $p(U(x^*_t; s)) \subset U(x; r)$ for every $i$, and hence $p^{-1}(U(x; r)) = \bigcup U(x^*_t; r)$. To prove (c), suppose $y^* \in U(x^*_t; s) \cap U(x^*_r; r)$. Then $x^*_t \in U(x^*_r; 2r)$. Using the above diagram, we have shown that $p(x^*_t) = p(x^*_r)$, we must have $x^*_t = x^*_r$.

(iv) Proof of (2). Assume that $\tilde{\rho} : M^* \to M$ is a covering projection and that $M$ is complete. Observe first that, given a curve $x_\alpha, 0 \leq t \leq \alpha$, in $M$ and a point $x_0^* \in M^*$ such that $\rho(x_0^*) = x_0^*$, there is a unique curve $x^*_t$, $0 \leq t \leq \alpha$, in $M^*$ such that $\rho(x^*_t) = x_0^*$ for $0 \leq t \leq \alpha$. Let $x^*_t \in M^*$ and let $X^*$ be any unit vector at $X^*$. Set $X^* = \rho(X^*)$. Since $M$ is complete, the geodesic $sX^*$ is defined for $-\infty < s < \infty$. From the above observation, we see that there is a unique curve $X^*$, $-\infty < s < \infty$, in $M^*$ such that $x^*_0 = x^*$ and that $\rho(x^*_s) = \exp sX$. Evidently, $x^*_s = \exp sX$. This shows that $M^*$ is complete. QED.

**Corollary 4.7.** Let $M$ and $M^*$ be connected manifolds of the same dimension and let $\rho : M^* \to M$ be an immersion. If $M^*$ is compact, so is $M$, and $\rho$ is a covering projection.

**Proof.** Take any Riemannian metric $g$ on $M$. It is easy to see that there is a unique Riemannian metric $g^*$ on $M^*$ such that $\rho$ is an isometric immersion. Since $M^*$ is complete by Corollary 4.4, $\rho$ is a covering projection by Theorem 4.6 and hence $M$ is compact. QED.

**Example 4.1.** A Riemannian manifold is said to be non-prolongable if it cannot be isometrically imbedded into another Riemannian manifold as a proper open submanifold. Theorem 5.6 shows that every complete Riemannian manifold is non-prolongable. The converse is not true. For example, let $M$ be the Euclidean plane with origin removed and $M^*$ the universal covering space of $M$. As an open submanifold of the Euclidean plane, $M$ has a natural Riemannian metric which is obviously not complete. With respect to the natural Riemannian metric on $M^*$ (cf. Example 2.1), $M^*$ is not complete by Theorem 4.6. It can be shown that $M^*$ is non-prolongable.

**Corollary 4.8.** Let $G$ be a group of isometries of a connected Riemannian manifold $M$. If the orbit $G(x)$ of a point $x$ of $M$ contains an open set of $M$, then the orbit $G(x)$ coincides with $M$, that is, $M$ is homogeneous.

**Proof.** It is easy to see that $G(x)$ is open in $M$. Let $M^*$ be a connected component of $G(x)$. For any two points $x^*$ and $y^*$ of $M^*$, there is an element $f$ of $G$ such that $f(x^*) = y^*$. Since $f$ maps every connected component of $G(x)$ onto a connected component of $G(y)$, $f(M^*) = M^*$. Hence $M^*$ is a homogeneous Riemannian manifold isometrically imbedded into $M$ as an open submanifold. Hence, $M^* = M$. QED.

**Proposition 4.9.** Let $M$ be a Riemannian manifold and $M^*$ a submanifold of $M$ which is locally closed in the sense that every point $x$ of $M$ has a neighborhood $U$ such that every connected component of $U \cap M^*$ (with respect to the topology of $M^*$) is closed in $U$. If $M$ is complete, so is $M^*$ with respect to the induced metric.

**Proof.** Let $d$ be the distance function defined by the Riemannian metric of $M$ and $d^*$ the distance function defined by the induced Riemannian metric of $M^*$. Let $x^* \in M$ and let $a$ be the supremum of $s$ such that $x^*_s$ is defined. To show that $a = \infty$, assume $a < \infty$. Let $s_a \uparrow a$. Since $d(x^*_s, x^*_a) \leq d^*(x^*_s, x^*_a) \leq |s_a - s|$, the sequence $\{x^*_s \}$ is a Cauchy sequence in $M$ and hence converges to a point, say $y$, of $M$. Then $y = \lim x^*_s$. Let $U$ be a neighborhood of $x$ in $M$ with the property stated in Proposition 4.4. Then $x^*_s, b \leq s < a$, lies in $U$ for some $b$. Since the connected component of $M^* \cap U$ containing $x^*_a, b \leq s < a$, is closed in $U$, the point $x_a$ belongs to $M^*$. Set $x_a = x$. Then $x^*_a, 0 \leq s \leq a$, is a geodesic in $M^*$. Using a normal coordinate system at $x^*_a$, we see that this geodesic can be extended to a geodesic $x^*_a, 0 \leq s \leq a + 6$, for some $\delta > 0$. QED.

### 5. Holonomy groups

Throughout this section, let $M$ be a connected Riemannian manifold with metric $g$ and $\Psi(x)$ the linear or homogeneous holonomy group of the Riemannian connection with reference point $x \in M$ (cf. §4 of Chapter II and §3 of Chapter III). Then $M$ is said to be reducible or irreducible according as $\Psi(x)$ is reducible or irreducible as a linear group acting on $T(M)$. In this section, we shall study $\Psi(x)$ and local structures of a reducible Riemannian manifold.
Assuming that $M$ is reducible, let $T'_0$ be a non-trivial subspace of $T(M)$ which is invariant by $\Psi(x)$. Given a point $x \in M$, let $\tau$ be a curve from $x$ to $y$ and $T'_y$ be the image of $T'_0$ by the (linear) parallel displacement along $\tau$. The subspace $T'_y$ of $T(M)$ is independent of the choice of $\tau$. In fact, if $\mu$ is any other curve from $x$ to $y$, then $\mu^{-1} \cdot \tau$ is a closed curve at $x$ and the subspace $T'_y$ is invariant by the parallel displacement along $\mu^{-1} \cdot \tau$, that is $\mu^{-1} \cdot \tau(T'_y) = T'_y$, and hence $\tau(T'_y) = \mu(T'_y)$. We thus obtain a distribution $T'$ which assigns to each point $y$ of $M$ the subspace $T'_y$ of $T(M)$.

A submanifold $N$ of a Riemannian manifold (or more generally, a manifold with a linear connection) $M$ is said to be totally geodesic at a point $x$ of $N$ if, for every $x \in T_x(N)$, the geodesic $\tau = x_t$ of $M$ determined by $(x, X)$ lies in $N$ for small values of the parameter $t$. If $N$ is totally geodesic at every point of $N$, it is called a totally geodesic submanifold of $M$.

**Proposition 5.1.** (1) The distribution $T'$ is differentiable and involutive.

(2) Let $M'$ be the maximal integral manifold of $T'$ through a point of $M$. Then $M'$ is a totally geodesic submanifold of $M$. If $M$ is complete, so is $M'$.

Proof. (1) To prove that $T'$ is differentiable, let $y$ be any point of $M$ and $x_1, \ldots, x^n$ a normal coordinate system at $y$, valid in a neighborhood $U$ of $y$. Let $X_1, \ldots, X_n$ be a basis for $T'_y$. Then for each $i, 1 \leq i \leq n$, we define a vector field $X^*_i$ in $U$ by

$$(X^*_i)_z = \tau X_i$$

for $z \in U$,

where $\tau$ is the geodesic from $y$ to $z$ given by $x^i = a^i t$, $j = 1, \ldots, n$, $(a^1, \ldots, a^n)$ being the coordinates of $z$. Since the parallel displacement $\tau$ depends differentiably on $(a^1, \ldots, a^n)$, we obtain a differentiable vector field $X^*_i$ in $U$. It is clear that $X^*_1, \ldots, X^*_n$ form a basis of $T'_y$ for every point $z$ of $U$.

To prove that $T'$ is involutive, it is sufficient to prove that if $X$ and $Y$ are vector fields belonging to $T'$, so are $\nabla_X Y$ and $\nabla_Y X$, because the Riemannian connection has no torsion and $[X, Y] = \nabla_X Y - \nabla_Y X$ (cf. Theorem 5.1 of Chapter III). Let $x_t$ be the integral curve of $X$ starting from an arbitrary point $x$. Let $\tau_t$ be the parallel displacement along the curve from the point $x_t$ to the point $y = x_0$. Since $Y_{x_0}$ and $Y_{x_t}$ belong to $T'$ for every $t$, $(\nabla_X Y)_t = \left(\frac{1}{t} \tau_t Y_{x_0} - Y_{x_t}\right)$ belongs to $T'_{x_t}$.

(2) Let $M'$ be a maximal integral manifold of $T'$. Let $\tau = x_t$ be a geodesic of $M$ with the initial condition $(y, X)$, where $\tau \in M$ and $x \in T_x(M') = T'_y$. Since the tangent vectors $\dot{x}_t$ are parallel along $\tau$, we see that $\dot{x}_0$ belongs to $T'_x$ for every $t$ and hence $\tau$ lies in $M'$ (cf. Lemma 2 for Theorem 7.2 of Chapter II). This proves that $M'$ is a totally geodesic submanifold of $M$. From the following lemma, we may conclude that, if $M$ is complete, so is $M'$.

**Lemma.** Let $N$ be a totally geodesic submanifold of a Riemannian manifold $M$. Every geodesic of $N$ with respect to the induced Riemannian metric of $N$ is a geodesic in $M$.

Proof of Lemma. Let $x \in N$ and $X \in T_x(N)$. Let $\tau = x_t$, $0 \leq t \leq a$, be the geodesic of $M$ with the initial condition $(x, X)$. Since $N$ is totally geodesic, $\tau$ lies in $N$. It now suffices to show that $\tau$ is a geodesic of $N$ with respect to the induced Riemannian metric of $N$. Let $d$ and $d'$ be the distance functions of $M$ and $N$ respectively. Considering only small values of $t$, we may assume that $\tau$ is a minimizing geodesic from $x = x_0$ to $x_a$ so that $d(x, x_a) = L(\tau)$, where $L(\tau)$ is the arc length of $\tau$. The arc length of $\tau$ measured by the metric of $M$ is the same as the one measured with respect to the induced metric of $N$. From the definition of the distance functions $d$ and $d'$, we obtain

$$d'(x, x) = L(\tau).$$

Hence, $d'(x, x) = L(\tau)$. By Corollary 3.9, $\tau$ is a geodesic with respect to the induced metric of $N$.

**Remark.** The lemma is a consequence of the following two facts which will be proved in Volume II. (1) If $M$ is a manifold with a linear connection whose torsion vanishes and if $N$ is a totally geodesic submanifold of $M$, then $N$ has a naturally induced linear connection such that every geodesic of $N$ is a geodesic of $M$; (2) If $N$ is a totally geodesic submanifold of a Riemannian manifold $M$, then the naturally induced linear connection of $N$ is the Riemannian connection with respect to
Let $y$ be a point of $M$. Let $M'$ and $M''$ be the maximal integral manifolds of the distributions $T'$ and $T''$ defined above. Then $y$ has an open neighborhood $V$ such that $V = V' \times V''$, where $V'$ (resp. $V''$) is an open neighborhood of $y$ in $M'$ (resp. $M''$), and that the Riemannian metric in $V$ is the direct product of the metrics in $V'$ and $V''$.

Proof. We first prove the following

**Lemma.** If $T'$ and $T''$ are two involutive distributions on a manifold $M$, there exists an integral manifold of $T'$ and $T''$ respectively. In other words, for any set of constants $(c_1, \ldots, c_k, c^{k+1}, \ldots, c^n)$, the equations $x^i = c^i, 1 \leq i \leq k, (\text{resp. } x^i = c^i, k + 1 \leq j \leq n)$ define an integral manifold of $T'$ (resp. $T''$).

Proof of Lemma. Since $T'$ is involutive, there exists a local coordinate system $x^1, \ldots, x^n$ with $y$ such that $(\partial/\partial x^1, \ldots, \partial/\partial x^k)$ form a local basis for $T'$. In other words, the equations $x^i = c^i, 1 \leq i \leq k, \text{define an integral manifold of } T'$. Similarly, there exists a local coordinate system $x^1, \ldots, x^k, x^{k+1}, \ldots, x^n$ with $y$ such that $(\partial/\partial x^{k+1}, \ldots, \partial/\partial x^n)$ form a local basis for $T''$. In other words, the equations $x^i = c^i, k + 1 \leq j \leq n, \text{define an integral manifold of } T''$. It is easy to see that $x^1, \ldots, x_k, x^{k+1}, \ldots, x^n$ is a local coordinate system with the desired property.

Making use of the local coordinate system $x^1, \ldots, x^n$ thus obtained, we shall prove Proposition 5.2. Let $V$ be the neighborhood of $y$ defined by $|x^i| < c, 1 \leq i \leq n$, where $c$ is a sufficiently small positive number so that the coordinate system $x^1, \ldots, x^n$ gives a homeomorphism of $V$ onto the cube $|x^i| < c$ in $\mathbb{R}^n$. Let $V'$ (resp. $V''$) be the set of points in $V$ defined by $|x^i| < c, 1 \leq i \leq k, \text{and } x^i = 0, k + 1 \leq j \leq n$ (resp. $x^i = 0, 1 \leq i \leq k, \text{and } |x^i| < c$). It is clear that $V'$ (resp. $V''$) is an integral manifold of $T'$ (resp. $T''$) through $y$ and is a neighborhood of $y$ in $M'$ (resp. $M''$) and that $V = V' \times V''$. We set $X_i = \partial/\partial x^i, 1 \leq i \leq n$. To prove that the Riemannian metric of $V$ is the direct product of those in $V'$ and $V''$, we show that $g_{ij} = g(X_i, X_j)$ are independent of $x^{k+1}, \ldots, x^n$ for $1 \leq i, j \leq k$, that $g_{ij} = g(X_i, X_j)$ are independent of $x^{k+1}, \ldots, x^n, 1 \leq i, j \leq n$ and that $g_{ij} = g(X_i, X_j) = 0$ for $0 \leq i, j \leq k$ and $k + 1 \leq i, j \leq n$. The last assertion is obvious since $X_1, 1 \leq i \leq n, \text{belong to } T'$ and $X_{k+1}, k + 1 \leq j \leq n, \text{belong to } T''$ and $T'$ and $T''$ are orthogonal to each other at every point. We now prove the first assertion, and the proof of the second assertion is similar. Let $1 \leq i \leq k$ and $k + 1 \leq i \leq n$. As in the proof of (1) of Proposition 5.1, we see that $X_i$ belongs to $T'$ and that $X_i$ belongs to $T''$. Since the torsion is zero and since $[X_i, X_m] = 0$, we have

$$
\nabla_{X_i} X_m - \nabla_{X_m} X_i = \nabla_{X_m} X_i - [X_i, X_m] = 0.
$$

Hence, $\nabla_{X_i} X_m = \nabla_{X_m} X_i = 0$. Since $g$ is parallel, we have

$$
X_m(g_{ij}) = g(\nabla_{X_i} X_j) = 0, 1 \leq i, j \leq k,
$$

thus proving our assertion.

**QED.**

**Proposition 5.3.** Let $T'$ and $T''$ be the distributions on $M$ used in Proposition 5.2. If $M$ is simply connected, then the homogeneous holonomy group $\Psi(x)$ is decomposed into the direct product of two normal subgroups $\Psi^{(1)}(x)$ and $\Psi^{(2)}(x)$ such that $\Psi^{(1)}(x)$ is trivial on $T'$ and that $\Psi^{(2)}(x)$ is trivial on $T''$.

Proof. Given an element $a \in \Psi(x)$, let $a_1$ (resp. $a_2$) be the restriction of $a$ to $T'_s$ (resp. $T''_s$). Let $a$ (resp. $a''$) be the orthogonal transformation of $T'_s(M)$ which coincides with $a$ on $T'_s$ (resp. with $a_2$ on $T''_s$) and which is trivial on $T''_s$ (resp. $T''_s$). If we take an orthonormal basis for $T'_s(M)$ such that the first $k$ vectors lie in $T'_s$ and the remaining $n-k$ vectors lie in $T''_s$, then these linear
transformations can be expressed by matrices as follows:
\[ a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad a' = \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a'' = \begin{pmatrix} 1 & 0 \\ 0 & -a_2 \end{pmatrix}. \]

We shall show that both \( a' \) and \( a'' \) are elements of \( \Psi(x) \). Let \( \tau \) be a closed curve at \( x \) such that the parallel displacement along \( \tau \) is the given element \( \epsilon \in \Psi(x) \). First we consider the special case where \( \tau \) is a small lasso in the following sense. A closed curve \( \tau \) at \( X \) is called a small lasso if it can be decomposed into three curves as follows: \( \tau = \mu^{-1} \circ \sigma \circ \mu \), where \( \mu \) is a curve from \( X \) to a point \( y \) (so that \( \mu^{-1} \) is a curve from \( Y \) to \( x \) going backward) and \( \sigma \) is a closed curve at \( y \) which is small enough to be contained in a neighborhood \( V = V' \times V'' \) of \( x \) and \( y \) given in Proposition 5.2. In this special case, we denote by \( \sigma' \) (resp. \( \sigma'' \)) the image of \( \sigma \) by the natural projection \( V \to V' \) (resp. \( V \to V'' \)). We set

\[ \tau' = \mu^{-1} \circ \sigma' \circ \mu, \quad \tau'' = \mu^{-1} \circ \sigma'' \circ \mu. \]

The parallel displacement along \( \tau' \) (resp. \( \tau'' \)) is trivial on \( T_y' \) (resp. \( T_y'' \)). The parallel displacement along \( \tau \) is the product of those along \( \tau' \) and \( \tau'' \). Hence the parallel displacement along \( \tau \) is the product of those along \( \tau' \) and \( \tau'' \). On the other hand, \( \tau' \) (resp. \( \tau'' \)) is trivial on \( T_x' \) (resp. \( T_x'' \)). It follows that \( \tau' \) (resp. \( \tau'' \)) is the parallel displacement -along \( \tau' \) (resp. \( \tau'' \)), thus proving our assertion in the case where \( \tau \) is a small lasso.

In the general case, we decompose \( \tau \) into a product of small las as follows.

**Lemma.** If \( M \) is simply connected, then the parallel displacement along \( \tau \) is the product of all parallel displacements along a finite number of small lassos at \( x \).

Proof of Lemma. This follows from the factorization lemma (cf. Appendix 7).

It is now clear that both \( a' \) and \( a'' \) belong to \( \Psi(x) \) in the general case. We set

\[ \Psi'(x) = \{ a' ; a \in \Psi(x) \}, \quad \Psi''(x) = \{ a'' ; a \in \Psi(x) \}. \]

Then \( \Psi(x) = \Psi'(x) \times \Psi''(x) \).

We now proceed to define a most natural decomposition of \( T_x(M) \) and derive its consequences. Let \( T_x(M) \) be the set of elements in \( T_x(M) \) which are left fixed by \( \Psi(x) \). It is the maximal linear subspace of \( T_x(M) \) on which \( \Psi(x) \) acts trivially. Let \( T_x' \) be the orthogonal complement of \( T_x'' \) in \( T_x(M) \). It is invariant by \( \Psi(x) \) and can be decomposed into a direct sum \( T_x = T_x' + T_x'' \) of mutually orthogonal, invariant and irreducible subspaces. We shall call \( T_x(M) = \Sigma_{i=0}^n T_x(i) \) a canonical decomposition (or de Rham decomposition) of \( T_x(M) \).

**Theorem 5.4.** Let \( M \) be a Riemannian manifold, \( T_x(M) = \Sigma_{i=0}^n T_x(i) \) a canonical decomposition of \( T_x(M) \) and \( T_x(i) \) the involutive distribution on \( M \) obtained by parallel displacement of \( T_x(i) \) for each \( i = 0, 1, \ldots, n-1 \). Let \( x \) be a point of \( M \) and let, for each \( i = 0, 1, \ldots, k \), \( M_i \) be the maximal integral manifold of \( T_x(i) \) through \( x \).

(1) The point \( x \) has an open neighborhood \( V \) such that \( V = V_0 \times V_1 \times \cdots \times V_k \) where \( V_i \) is an open neighborhood of \( x \) in \( M_i \), and that the Riemannian metric in \( V \) is the direct product of \( M \) the Riemannian metrics in the \( V_i \).

(2) The maximal integral manifold \( M_0 \) is locally Euclidean in the sense that every point of \( M_0 \) has a neighborhood which is isometric with an open set of an \( n \)-dimensional Euclidean space, where \( n = \dim M_0 \).

(3) If \( M \) is simply connected, then the homogeneous holonomy group \( Y(x) \) is the direct product \( \Psi(x) \times \Psi(x) \times \cdots \times \Psi(x) \) of normal subgroups, where \( Y(x) \) is trivial on \( T_x(i) \) if \( i \neq j \) and is irreducible on \( T_x(i) \) for each \( i = 1, \ldots, k \), and \( \Psi(x) \) consists of the identity only.

(4) If \( M \) is simply connected, then a canonical decomposition \( T_x(M) = \Sigma_{i=0}^n T_x(i) \) is unique up to an order.

Proof. (1) This is a generalization of Proposition 5.2.

(2) Since \( x \) is an arbitrary point of \( M \), it is sufficient to prove that \( V_0 \) is isometric to an open subset of an \( n \)-dimensional Euclidean space. Since the homogeneous holonomy group of \( V_0 \) consists of the identity only, \( T_x'' \) is the direct sum of \( n \) 1-dimensional subspaces. From the proof of Proposition 5.2, it follows that \( V_0 \) is a direct product of 1-dimensional submanifolds and that the Riemannian metric on \( V_0 \) is the direct product of the Riemannian metrics on these 1-dimensional submanifolds. On the other hand, on any 1-dimensional manifold with a local coordinate system \( x \) every Riemannian metric is of the form \( dx_1 \, dx_2 \). If \( x \) is a normal coordinate system, then the metric is of the form \( dx_1 \, dx_2 \). Hence \( V_0 \) is isometric to an open set of a Euclidean space.
(3) This is clear from the definition of a canonical decomposition of T(M) and from the proof of Proposition 5.3.

(4) First we prove

**Lemma.** Let $S_z$ be any subspace of $T(M)$ invariant by $Y(x)$. Then, for each $i = 1, \ldots, k$, either $S_z$ is orthogonal to $T_z^i$ or $S_z$ contains $T_z^i$.

Proof of Lemma. (i) Assume that all vectors of $S_z$ are left fixed by $\Psi_t(x)$. Then $S_z$ is orthogonal to $T_z^i$. In fact, let $X = \sum_{j=0}^k X_j$ be any element of $S_z$, where $X_j \in T_z^j$. For an arbitrary element $a$, of $\Psi_t(x)$, we have

$$a_i(X) = X_0 + X_1 + \cdots + a_i(X_k) + \cdots + X_k$$

since $a_i$ acts trivially on $T_z^i$ for $j \neq i$. If $a_i(X) = X$, then $a_i(X_j) = X_j$. Since this holds for every $a$, $\in \Psi_t(x)$ and since $Y(x)$ is irreducible in $T_z^i$, we must have $X_0 = 0$. This shows that $X$ is orthogonal to $T_z^i$.

(ii) Assume that $a_i(X) \neq X$ for some $a_i \in \Psi_t(x)$ and for some $X \in S_z$. Let $X = \sum_{j=0}^k X_j$ where $X_j \in T_z^j$. Since each $X_j$, $j \neq i$, is left fixed by every element of $\Psi_t(x)$, $X - a_i(X) = X_i - a_i(X_i) \neq 0$ is a vector in $T_z^i$ as well as in $S_z$. The subset $\{b_1(X - a_i(X_1)) : b_i \in \Psi_t(x)\}$ is in $T_z^i \cap S_z$ and spans $T_z^i$, since $\Psi_t(x)$ is irreducible in $T_z^i$. This implies that $T_z^i$ is contained in $S_z$, thus proving the lemma.

Going back to the proof of (4), let $T(M) = E_{\xi}$ be any other canonical decomposition. First it is clear that $T_z^0 = S_z^0$.

It is therefore sufficient to prove, that each $S_z^j$, $1 \leq j \leq i$, coincides with some $T_z^j$. Consider, for example, $S_z^j$. By the lemma, either it is orthogonal, to $T_z^j$ for every $t \geq 1$ or it contains $T_z^j$ for some $i \geq 1$. In the first case, it must be contained in the orthogonal complement $T_z^{i-1}$ of $\sum_{j=1}^{i-1} T_z^j$ in $T(M)$. This is obviously a contradiction. In the second case, the irreducibility of $S_z^j$ implies that $S_z^j$ actually coincides with $T_z^j$.

The following result is due to Borel and Lichnerowicz [1].

**Theorem 5.5.** The restricted homogeneous holonomy group of a Riemannian manifold $M$ is a closed subgroup of $SO(n)$, where $n = \dim M$.

Proof. Since the homogeneous holonomy group of the universal covering space of $M$ is isomorphic with the restricted homogeneous holonomy group of $M$ (cf. Example 2.1), we may assume, without loss of generality, that $M$ is simply connected. In view of (3) of Theorem 5.4, our assertion follows from the following result in the theory of Lie groups:

Let $G$ be a connected Lie subgroup of $SO(n)$ which acts irreducibly on the $n$-dimensional vector space $\mathbb{R}^n$. Then $G$ is closed in $SO(n)$.

The proof of this result is given in Appendix 5.

6. The decomposition theorem of de Rham

Let $M$ be a connected, simply connected and complete Riemannian manifold. Assuming that $M$ is reducible, let $T(M) = T_1 + T_2$ be a decomposition into subspaces invariant by the linear holonomy group $\Psi_t(x)$ and let $T_1$ and $T_2$ be the parallel distributions defined by $T_1$ and $T_2$, respectively, as in the beginning of §5. We fix a point $0 \in M$ and let $M_1$ and $M_2$ be the maximal integral manifolds of $T_1$ and $T_2$ through $0$, respectively. By Proposition 5.1, both $M_1$ and $M_2$ are complete, totally geodesic submanifolds of $M$.

The purpose of this section is to prove

**Theorem 6.1.** $M$ is isometric to the direct product $M_1 \times M_2$.

Proof. For any curve $z_t$, $0 \leq t \leq 1$, in $M$ with $z_0 = 0$, we shall define its projection on $M_1$ to be the curve $x_t(\varphi_0, t \geq 1$ with $x_0 = 0$ which is obtained as follows. Let $C_t$ be the development of $z_t$ in the affine tangent space $T_z(M)$. (For the sake of simplicity we identify the affine tangent space with the tangent (vector) space.) Since $T_z(M)$ is the direct product of the two Euclidean spaces $T_z^1$ and $T_z^2$, $C_t$ may be represented by a pair $(A_t, B_t)$, where $A_t$ and $B_t$ are curves in $T_z^1$ and $T_z^2$, respectively. By applying (4) of Theorem 4.1 to $M$, we see that there exists a unique curve $x_t$ in $M$ which is the image of $C_t$. In view of Proposition 4.1 of Chapter III we may define the curve $x_t$ as follows. For each $t$, let $X_t$ be the result of the parallel displacement of the $T$-component of $z_t$ from $z_t$ to $t = 0$ along the curve $x_t$. The curve $x_t$ is a curve in $M$ with $x_0 = 0$ such that the result of the parallel displacement of $x_t$ along itself to $0$ is equal to $X_t$ for each $t$.

Before proceeding further, we shall indicate the main idea of the proof. We show that the end point $x_t$ of the projection $x_t$ depends only on the end point $z_t$ of the curve $z_t$ if $M$ is simply connected.
Thus we obtain a projection $p': M \to M'$ and, similarly, a projection $p'' : M \to M''$. The mapping $p = (p', p'')$ of $M$ into $M' \times M''$ will be shown to be isometric at every point. Theorem 4.6 then implies that $p$ is a covering projection of $M$ onto $M' \times M''$. If $h$ is a homotopy in $M$ from a curve of $M'$ to another curve of $M'$, then $P'(h)$ is a homotopy between the two curves in $M'$. Thus, $M'$ is simply connected. Similarly, $M''$ is simply connected. Thus $p$ is an isometry of $M$ onto $M' \times M''$. The detail now follows.

**Lemma 1.** Let $T = z(t)$, $0 \leq t \leq 1$, be a curve in $M'$ with $z(0) = 0$ and let $a$ be any number with $0 \leq a \leq 1$. Let $T_a = z(t)$, $0 \leq t \leq 1$, and let $T_a$ be the curve $z(a)$. Thus, $T_a$ is the projection of $T$ in the maximal integral manifold $M'(z_a)$ of $T'$ through $z$. Then the projection of $T_a$ is the parallel displacement of $M'(z_a)$ coincides with the projection of $T'$.

**Proof of Lemma 1.** This is obvious from the second definition of the projection by means of the (linear) parallel displacement of tangent vectors.

**Lemma 2.** Let $z \in M$ and let $V = V' \times V''$ be an open neighborhood of $z$ in $M$, where $V'$ and $V''$ are open neighborhoods of $z$ in $M'(z)$ and $M''(z)$, respectively. For any curve $z(t)$ with $z(t) \in V$, the projection of $z(t)$ in $M'(z)$ is given by the natural projection of $V' \times V''$.

**Proof of Lemma 2.** For the existence of a neighborhood $V = V' \times V''$, see Proposition 5.2. Let $z(t)$ be given by the pair $(x(t), y(t))$ where $x(t)$ (resp. $y(t)$) is a curve in $V'$ (resp. $V''$) with $x(0) = z$ (resp. $y(0) = z$). Since $V = V' \times V''$, the parallel displacement of the $T$-component of $z(t)$ to $z(t) = 0$ along the curve $z(t)$ is the same as the parallel displacement of $x(t)$ from $x(0) = z$ along the curve $x(t)$. Thus $x(t)$ is the projection of the curve $z(t)$ in $M'(z)$.

We introduce the following terminologies. A (piecewise differentiable) curve $z(t)$ is called a $T$-curve (resp. $T'$-curve) if it belongs to $T_t$ (resp. $T'_t$) for every $t$. Given a (piecewise differentiable) homotopy $z : [0, 1] \times [0, s] \to M$ which is $\delta$-d&Led by $z(t, s)$, we shall denote by $z(t, s)$ the curve with parameter $t$ for the fixed value of $s$ (resp. the curve with parameter $s$ for the fixed value of $t$). Their tangent vectors will be denoted by $z'(t, s)$ and $z'(s, t)$, respectively. For any point $x \in M$, let $d'$ (resp. $d''$) denote the distance function on the maximal integral manifold $M'(z)$ of $T'$ (resp. $M''(z)$) through $z$. Let $U''(z'; r)$ (resp. $U''(z; r)$) denote the set of points $w \in M'(z)$ (resp. $w \in M''(z)$) such that $d'(z; w) < r$ (resp. $d''(z; w) < r$).

**Lemma 3.** Let $\tau = x_t$, $0 \leq t \leq 1$, be a $T$-curve. Then there exist a number $r > 0$ and a family of isometries $f_t$, $0 < \tau < 1$, of $U''(x_t; r)$ onto $U''(x_t; r)$ with the following properties:

1. The differential of $f_t$ at $x_t$ coincides with the parallel displacement along the curve $\tau$ from $x_0$ to $x_t$.
2. For any curve $\tau' = x'_t$, $0 < s < s_0$, in $U''(x_t; r)$ with $x'_t = x_t$, set $x'_t = f_t(x'_t)$. Then
   a. For any $0 < t_1 < 1$ and $0 < t_2 < s_0$, the parallel displacement along the “parallelogram” formed by the curve $x_t$, $0 \leq t \leq t_1$, and the curve $x'_t$, $0 \leq t \leq t_1$, is the curve $z(t_1, t_2)$, $0 \leq s < s_1$, the inverse of the curve $z(t_1, t_2)$, $0 \leq t \leq t_1$, and the inverse of the curve $y_0$, $0 \leq s < s_1$, is trivial;
   b. For any $s$ and $t$, $z(t, s)$ is parallel to $x_t$ along the curve $z(t, s)$;
   c. For any $s$ and $t$, $z(t, s)$ is parallel to $x_t$ along the curve $z(t, s)$.

**Proof of Lemma 3.** Let $V$ be a neighborhood of $x_0$ of the form $V = V' \times V''$ as in Proposition 5.2. Choose a number $r > 0$ sufficiently small so that $x_t \in V$ and $U''(x_t; r) \subset V'$ for $0 < t < r$. We define $f_t$ by $f_t(x_0, y) = (x_t, y)$ for any $y \in U''(x_t; r)$. It is clear that the family of isometries $f_t$, $0 < t < r$, has all the properties (1) and (2). The family $f_t$ can be extended easily for $0 < t < 1$ and for a suitable $r > 0$ by covering the curve $\tau = x_t$, by a finite number of neighborhoods of the form $V = V' \times V''$ and using the above argument for each neighborhood.

**Lemma 4.** Let $\tau = x_t$, $0 \leq t \leq 1$, be a $T$-curve and let $\tau' = y_t$, $0 < s < s_0$, be a P-geodesic with $y_s = x_t$, where $s$ is the arc length. Then there exists a homotopy $z(t, s)$, $0 < t < 1$, $0 < s < s_0$, with the following properties:

1. $z(0, s) = x_t$ and $z(1, s) = y_t$;
2. $z(t, s)$ has properties (a), (b), and (c) of Lemma 3.

The homotopy $z(t, s)$ is uniquely determined. In fact, if $Y_t$ is the result of parallel displacement of the initial tangent vector $Y_0 = \gamma_0$ of the geodesic $\tau$ along the curve $\tau$, then $z(t, s) = \exp s Y_t$.

**Proof of Lemma 4.** We first prove the uniqueness. By (a) and (b) and the fact that $\tau'$ is a geodesic, it follows that $z(t, s)$ is parallel to $Y_t$ along the curve $z(t, s)$. This means that $z(t, s)$ is a
geodesic with initial tangent vector $Y$. Thus, $z'_t = \exp sY$, proving the uniqueness.

It remains therefore to prove that $z'_t = \exp sY$, actually satisfies conditions (1) and (2). Condition (1) is obvious. To prove (2), we may assume that $\tau'$ is a differentiable curve so that $z'_t$ is differentiable in $t, s$. Let $f_t$ be the family of isometries as in Lemma 3. It is obvious that there exists a number $\delta > 0$ such that $z'_t = f_t(\tau')$ for $0 \leq t \leq 1$ and $0 \leq s \leq \delta$. Thus, $z'_t$ satisfies condition (2) for $0 \leq t \leq 1$ and $0 \leq s \leq \delta$. Let $a$ be the supremum of such $\delta$. In order to prove $a = s_0$, assume $a < s_0$.

First we show that $z'_t$ satisfies (2) for $0 \leq t \leq 1$ and $0 \leq s \leq a$. Since $z'_t$ is differentiable in $a, s$, the parallel displacement along the curve $z'(a)$ is the limit of the parallel displacement along the curve $z'(s)$ as $s \uparrow a$ (cf. Lemma for Theorem 4.2 of Chapter II). Hence condition (a) is satisfied. We have also $z'_{t(0)} = \lim s_{t(a)}$ and $z'_{t(0)} = \lim z'_{t(a)}$. Combined with the above limit argument, this gives conditions (b) and (c) for $0 \leq t \leq 1$ and $s = a$.

In order to show that $z'_t$ has property (2) beyond the value $a$, we apply Lemma 3 to the $T'$-curve $z'(a) = z'(0)$ and the $Y$-geodesic $y'$, where $u = s = a$. We see then that there exist a number $\tau > 0$ and a homotopy $w_t$, $0 \leq t \leq 1, -\tau \leq u \leq \tau$, satisfying a condition similar to (2), such that $w_t(0) = z'(a)$ and $w_t(\tau) = y'$. Since $w_t(0)$ is parallel to $y'$ along the curve $w_t(0) = z'(0)$, it follows that $z'_t = w_t$ for $0 \leq t \leq 1$ and $a - \tau \leq s \leq a + \tau$. This proves that $z'_t$ satisfies condition (2) for $0 \leq t \leq 1$ and $0 \leq s \leq a + \tau$, contradicting the assumption that $a < s_0$.

**Lemma 5.** Keeping the notation of Lemma 4, the projection of the curve $\tau^*$, $\tau^* - 1$ in $M'(y')$ coincides with $\tau^*(a) = z'_t(0)$, $0 \leq t \leq 1$.

Proof of Lemma 5. Since $\tau^* - 1$ is a $T'$-curve, its projection in $M'(y')$ is trivial, that is, reduces to the point $y'$. Conditions (a) and (b) imply that, for each $t$, the parallel displacement of $t$ along $\tau^* - 1$ to $y'$ is the same as the parallel displacement of $z'(a)$ along $z'(a)$ to $y'$. This means that $\tau^* - 1$ projects on $z'(a)$.

We now come to the main step for the proof of Theorem 6.1.

**Lemma 6.** If two curves $\tau_1$ and $\tau_2$ from $a$ to $b$ in $M$ are homotopic to each other, then their projections in $M' = M'(o)$ have the same end point.

Proof of Lemma 6. We first remark that $\tau_1$ is obtained from $\tau_1$ by a finite succession of small deformations. Here a small deformation of a curve $z$, means that for a certain small neighborhood $V$, we replace a portion $z_1, t_1 \leq t \leq t_2$, of the curve lying in $V$ by a curve $w_1, t_1 \leq t \leq t_2$, with $w_1 = z_1$ and $w_{t_2} = z_{t_1}$ lying in $V$. As a neighborhood $V$, we shall always take a neighborhood of the form $V^* = V$ as in Lemma 2.

It suffices therefore to prove the following assertion. Let $\tau$ be a curve from $0$ to $z_1$, a curve from $z_1$ to $\tau_1$, which lies in a small neighborhood $V = V^* = V^*$. By Lemma 2, the projection $\mu_1'$ of $\mu_1$ in $M'(z_1)$ is $\mu'$. Let $\mu_2$ be a $T'$-geodesic in $V$ joining $\tau_1$ and the end point of $\mu$. The parallel displacement of $\mu_1$ at $z_1$ along $\mu' - 1$, $\mu'$, because $\mu'$ and $\mu^*$ have the same parallel displacement for $V$-vectors. By Lemma 5, we see that the projection of $\mu_1$ in $M'(z_1)$ is the curve $\mu'$ obtained by using the homotopy $z'_t$ constructed from the Y-geodesic $\mu^*$ and the $T'$-curve $\mu$. The homotopy $z'_t$ depends only on $\mu^*$ and $\mu$ and not on $\mu$. Thus if we replace $\mu$ by $\nu$ in the above argument, we see that the projection of $\nu$ is equal to $\nu'$, followed by $\nu''$, where $\tau = (\nu', \nu'')$ in $V = V^* = V^*$.

We now divide $\tau$ into a finite number of arcs, say, $\tau_1, \tau_2, \ldots, \tau_k$, such that each $\tau_i$ lies in a small neighborhood $V_i$ of the form $V'_i \times V'_i$. We show that the projections of the curves $\mu_1', \mu_2'$, and $\mu_2'$ are the same end point in the maximal integral manifold of $V^*$ through the initial point of $\tau_1$. Again, let $\sigma_k$ be the geodesic in $V_k$ joining the end point of $\tau_k$ to the end point of $\tau_k$. As before, the projection of $\mu_1', \mu_2', \tau_i$ is the curve $\mu'_k$ followed by the curve obtained by the homotopy, which is constructed from the $T'$-geodesic $\tau'_k$ and the $T'$-curve $\mu_1'$. Similarly, for the projection of $\mu_2'$, $\nu''$, $\tau_i$. Each homotopy was constructed by the parallel displacement of the initial tangent vector of the geodesic $\tau'_k$ along $\mu_1'$.
\[ k' \cdot v' \]. Since \( v'^{-1} \cdot \mu' \) is a curve in \( V' \), the parallel displacement along \( v'^{-1} \). \( \mu' \) is trivial for \( T'-\text{vectors} \). This means that the parallel displacements of the initial tangent vector of \( \tau_k \) along \( \mu' \) and \( v' \) are the same so that the two homotopies produce the curves \( \mu_k \) and \( v_k \) starting at the end point of \( \tau_k \) and ending at the same point, where a curve \( K_k \) starts in such a way that \( K_k \cdot \mu_k \cdot \tau_k \) and \( K_k \cdot v_k \cdot \tau_k \) are the projections of \( k \cdot \mu \cdot \tau \) and \( k \cdot v \cdot \tau \) respectively. We also remark that the parallel displacements of every \( T'-\text{vector} \) along \( \mu_k \) and \( v_k \) are the same; this indeed follows from property (a) of the homotopy in Lemma 4.

We continue to the next stage of projecting the curves \( K_k \cdot \mu_k \cdot \tau_k \cdot \tau_{k-1} \) and \( K_k \cdot v_k \cdot \tau_k \cdot \tau_{k-1} \) by the same method. As a result of the above remark, we have two curves ending at the same point. Now it is obvious that this process can be continued, thus completing the proof of Lemma 6.

Now we are in position to complete the proof of Theorem 6.1.

Lemma 6 allows us to define a mapping \( \phi \) of \( M \) into \( M' \). Similarly, we define a mapping \( \psi \) of \( M \) into \( M'' \). These mappings are differentiable. As we indicated before Lemma 1, we have only to show that the mapping \( \phi = (\phi', \psi') \) of \( M \) into \( M' \times M'' \) is isometric at each point. Let \( z \) be any point of \( M \) and let \( t \) be a curve from \( 0 \) to \( z \). For any tangent vector \( Z \in T_z(M) \), let \( Z = X + Y \), where \( X \in T'_z(M) \) and \( Y \in T''_z(M) \). By definition of the projection, it is clear that \( \phi'(Z) \) is the same as the vector obtained by the parallel displacement of \( X \) from \( Z \) to \( 0 \) along \( t \) and then from \( 0 \) to \( \phi'(t) \) along \( \phi'(t) \). Therefore, \( \phi'(Z) \) and \( X \) have the same length. Similarly, \( \psi'(Z) \) and \( Y \) have the same length. It follows that \( Z \) and \( \phi(Z) \) are isometric at \( z \).

Combing Theorem 5.4 and Theorem 6.1, we obtain the decomposition theorem of de Rham.

**Theorem 6.2.** A connected, simply connected and complete Riemannian manifold \( M \) is isometric to the direct product \( M_0 \times M_1 \times \cdots \times M_k \), where \( M_0 \) is a Euclidean space (possibly of dimension 0) and \( M_1, \ldots, M_k \) are all simply connected, complete, irreducible Riemannian manifolds. Such a decomposition is unique up to an order.

Theorems 6.1 and 6.2 are due to de Rham [1]. The proof of Theorem 6.1 is new; it was inspired by the work of Reinhart [1].

### 7. Affine holonomy groups

Let \( M \) be a connected Riemannian manifold. Fixing a point \( x \) of \( M \), we denote the affine holonomy group \( \Phi(x) \) and the linear holonomy group \( \Psi(x) \) simply by \( \Phi \) and \( \Psi \), respectively. We know (cf. Theorem 5.5) that the restricted linear holonomy group \( \Psi_0 \) is a closed subgroup of \( \text{SO}(n) \), where \( n = \dim M \). \( \Phi \) is a group of Euclidean motions of the affine (or rather Euclidean) tangent space \( T_x(M) \).

We first prove the following result.

**Theorem 7.1.** If \( \Phi \) is irreducible, then either

1. \( \Phi \) contains all translations of \( T_x(M) \).
2. \( \Psi_0 \) fixes a point of \( T_x(M) \).

**Proof.** Let \( K \) be the kernel of the homomorphism of \( \Phi_0 \) onto \( \Psi_0 \) (cf. Proposition 3.5 of Chapter III). Since \( K \) is a normal subgroup of \( \Phi_0 \) and since every element \( a \in \Phi_0 \) of the form \( a = \xi \cdot a \) where \( \xi \in \Psi_0 \) and \( \xi \) is a pure translation, \( \Psi_0 \) normalizes \( K \), that is, \( \xi^{-1}K \xi = K \) for every \( \xi \in \Psi_0 \). Consider first the case where \( K \) is not discrete. Since \( \Psi_0 \) is connected, it normalizes the identity component \( K_0 \) of \( K \). Let \( v \) be the orbit of the origin of \( T_x(M) \) by \( K_0 \). It is a non-trivial linear subspace of \( T_x(M) \) invariant by \( \Psi_0 \); the invariance by \( \Psi_0 \) is a consequence of the fact that \( K_0 \) normalizes \( K_0 \). Since \( \Psi_0 \) is irreducible by assumption, we have \( v = T_x(M) \). This means that \( \Phi \) contains all translations of \( T_x(M) \). Consider next the case where \( K \) is discrete. Since \( \Psi_0 \) is connected, \( \Psi_0 \) commutes with \( K \) elementwise. Hence, for every \( \xi \in K \), \( \xi(0) \) is invariant by \( \Psi_0 \) (where 0 denotes the origin of \( T_x(M) \)). Since \( \Psi_0 \) is irreducible, \( \xi(0) = 0 \) for every \( \xi \in K \). This means that \( K \) consists of the identity element only and hence that \( \Phi_0 \) is isomorphic to \( \Psi_0 \) in a natural manner. In particular, \( \Phi_0 \) is compact. On the other hand, any compact group of affine transformations of \( T_x(M) \) has a fixed point. Although we shall prove a more general statement in Volume II, we shall give here a direct proof of this fact. Let \( f \) be the mapping from \( \Phi_0 \) into \( T_x(M) \) defined by

\[ f(a) = a(0) \quad \text{for } a \in \Phi_0. \]
Let $da$ be a bi-invariant Haar measure on $\Phi^0$ and define

$$X_0 = \int f(u) \, da.$$ 

It is easy to verify that $X_0$ is a fixed point of $\Phi^0$. QED.

We now investigate the second case of Theorem 7.1 (without assuming the irreducibility of $M$).

**Theorem 7.2.** Let $M$ be a connected, simply connected and complete Riemannian manifold. If the (restricted) affine holonomy group $\Phi^0$ at a point $x$ fixes a point of the Euclidean tangent space $T_x(M)$, then $M$ is isometric to a Euclidean space.

Proof. Assuming that $X_0 \in T_x(M)$ is a point fixed by $\Phi^0$, let $\tau$ be the geodesic from $x$ to a point $y$ which is developed upon the line segment $tX_0$, $0 \leq t \leq 1$. We observe that the affine holonomy group $\Phi^0(y)$ acts trivially. In fact, for any closed curve $\mu$ at $y$, the affine parallel displacement along $\tau^{-1} \cdot \mu \cdot \tau$ maps $X_0$ into itself. Hence the origin of $T_x(M)$ given by $\tau(X_0)$ is left fixed by $\mu$. This shows that we may assume that $\Phi^0$ fixes the origin of $T_x(M)$. Since $M$ is complete, the exponential mapping $T_x(M) \to M$ is surjective.

We show that it is 1:1. Assume that two geodesics $\tau$ and $\mu$ issuing from $x$ meet at a point $y$. The affine parallel displacement $\mu^{-1} \cdot \tau$ maps the origin of $T_x(M)$ into itself and hence we have

$$\tau^{-1}(0) = \mu^{-1}(0),$$

where $0$, denotes the origin of $T_0(M)$. Since $\tau^{-1}(0)$ and $\mu^{-1}(0)$ are the end points of the developments of $\tau$ and $\mu$ in $T_x(M)$ respectively, these developments which are line segments coincide with each other. Thus $\tau = \mu$, contradicting the assumption that $x \neq y$. This proves that the exponential mapping $T_x(M) \to M$ is 1:1.

Assume that $x$ is a normal coordinate system on $M$. Since $f(x) \leq M$. Let $x^1, \ldots, x^n$ be a normal coordinate system on $U(x; \epsilon)$. We set $X = -\sum_{i=1}^n \partial_i \sqrt{g(x)}$ and let $\phi$ be the corresponding point field (cf. §4 of Chapter II). We show that $\phi$ is a parallel point field. Since $\Phi^0$ fixes the origin of $T_x(M)$, it is sufficient to prove that $\phi$ is parallel along each geodesic through $x$. Our assertion follows therefore from

**Lemma 1.** Let $\tau = x_t, 0 \leq t \leq 1$, be a curve in a Riemannian manifold $M$ and let $\tau_t^*$ (resp. $\tau_t$) denote the affine (resp. linear) parallel displacement along $\tau$ from $x_t$ to $x$. Then

$$\tau_t^*(Y) = \tau_t^*(Y) + \tau_t C_t, \quad Y \in T_{x_t}(M),$$

where $C_t, 0 \leq t \leq 1$, is the development of $\tau = x_t$ into $T_{x_t}(M)$.

Proof of Lemma 1. Given $Y \in T_x(M)$, let $\phi$ (resp. $q$) be the point field along $\tau$ defined by the affine parallel displacement of $Y$ (resp. the origin of $T_x(M)$) and let $Y^*$ be the vector field along $\tau$ defined by the linear parallel displacement of $Y$. Then $\phi = q + Y^*$ at each point of $\tau$, that is, $Y^*$ is the vector with initial point $q$ and end point $p$ at each point of $\tau$. At the point $x_0$, this means precisely $\tau_0^*(Y) = \tau_0^*(Y) + C$.

Going back to the proof of Theorem 7.2, we assert that

$$\nabla_Y X + V = 0 \quad \text{for any vector field } V.$$

This follows from

**Lemma 2.** Let $\phi$ be a point field along a curve $\tau = x_t, 0 \leq t \leq 1$, in a Riemannian manifold $M$ and let $X$ be the corresponding vector field along $\tau$. Then $\phi$ is a parallel point field if and only if

$$\nabla_X X + \dot{x}_t = 0 \quad \text{for } 0 \leq t \leq 1.$$

Proof of Lemma 2. From Lemma 1, we obtain

$$\tau_t^* \phi(p_{x_t}) = \tau_t^* (X_{x_t}^*) + C_t,$$

where $C_t$ (for a fixed $\tau$ and with parameter $h$) is the development of $\tau$ into $T_{x_t}(M)$. Since $\tau_t^* (p_{x_t})$ is independent of $h$ (and depends only on $\tau$) if and only if $\phi$ is parallel, we have

$$0 = \lim_{h \to 0} \frac{1}{h} [\tau_t^* h (X_{x_t}) - X_{x_t}] + \lim_{h \to 0} \frac{1}{h} C_t,$$

for $0 \leq t \leq 1$ if and only if $\phi$ is parallel, completing the proof of Lemma 2.
Let $X$ and $Z$ be arbitrary vector fields on $M$. From $\nabla_Y X + Y = 0$ and $\nabla_Z X + Z = 0$, we obtain (cf. Theorem 5.1 of Chapter III)

$$\nabla_X Y = \nabla_Y X + [X, Y] = -Y + [X, Y]$$
and

$$\nabla_X Z = \nabla_Z X + [X, Z] = -Z + [X, Z].$$

Hence,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = -2g(Y, Z) + g([X, Y], Z) + g(Y, [X, Z]).$$

Let $Y = \frac{\partial}{\partial x^j}$ and $Z = \frac{\partial}{\partial x^k}$ for any fixed $j$ and $k$. Then we have

$$X \cdot g_{jk} = -2g_{jk} + g_{jk} + g_{jk} = 0.$$ 

This means that the functions $g_{jk}$ are invariant by the local 1-parameter group of transformations $q_t$ generated by $X$. But $q_t$ is of the form

$$q_t(x^1, \ldots, x^n) = (e^{-t}x^1, \ldots, e^{-t}x^n).$$

Thus the functions $g_{jk}$ are constant along each geodesic through $x$. Hence,

$$g_{jk} = g_{jk}(x) = (I., \ldots, x^n).$$

This shows that $\exp_x$ is an isometric mapping of $N(x; r)$ with Euclidean metric onto $U(x; r)$. Let $r_0$ be the supremum of $r > 0$ such that $\exp_x$ is a diffeomorphism of $N(x; r)$ onto $U(x; r)$. Since the differential $d(\exp_x)_y$ is non-singular at every point of $N(x; r_0)$, $\exp_x$ is a diffeomorphism, hence an isometry, by the argument above, of $N(x; r_0)$ onto $U(x; r_0)$. If $r_0 < \infty$, it follows that $\exp_x$ is isometric at every point $y$ of the boundary of $N(x; r_0)$ and hence non-singular in a neighborhood of such $y$. Since the boundary of $N(x; r_0)$ is compact, we see that there exists $\varepsilon > 0$ such that $\exp_x$ is a diffeomorphism of $N(x; r_0 + \varepsilon)$ onto $U(x; r_0 + \varepsilon)$, contradicting the definition of $r_0$. This shows that $\exp_x$ is a diffeomorphism of $T_x(M)$ onto $M$. By choosing a normal coordinate system $x^1, \ldots, x^n$ on the whole $M$, we conclude that $g_{jk} = \delta_{jk}$ at every point of $M$, that is, $M$ is an Euclidean space. QED.

As a consequence we obtain the following corollary due to Goto and Sasaki [1].

**Corollary 7.3.** Let $M$ be a connected and complete Riemannian manifold. If the restricted affine holonomy group $\Phi^0(x)$ fixes a point of the Euclidean tangent space $T_x(M)$ for some $x \in M$, then $M$ is locally Euclidean (that is, every point of $M$ has a neighborhood which is isometric to an open subset of a Euclidean space).

Proof. Apply Theorem 7.2 to the universal covering space of $M$. QED.

**Corollary 7.4.** If $M$ is a complete Riemannian manifold of dimension $> 1$ and if the restricted linear holonomy group $\Psi^0(x)$ is irreducible, then the restricted affine holonomy group $\Phi^0(x)$ contains all translations of $T_x(M)$.

Proof. Since $\Psi^0(x)$ is irreducible, $M$ is not locally Euclidean. Our assertion now follows from Theorem 7.1 and Corollary 7.3. QED.
CHAPTER V

Curvature and Space Forms

1. Algebraic preliminaries

Let $V$ be an $n$-dimensional real vector space and $R: V \times V \times V \times V \rightarrow \mathbb{R}$ a quadrilinear mapping with the following three properties:

(a) $R(v_1, v_2, v_3, v_4) = -R(v_2, v_1, v_3, v_4)$
(b) $R(v_1, v_2, v_3, v_4) = -R(v_1, v_2, v_4, v_3)$
(c) $R(v_1, v_2, v_3, v_4) + R(v_1, v_3, v_4, v_2) + R(v_1, v_3, v_2, v_4) = 0$.

PROPOSITION 1.1. If $R$ possesses the above three properties, then it possesses also the following fourth property:

(d) $R(v_1, v_2, v_3, v_4) = R(v_4, v_3, v_2, v_1)$.

Proof. We denote by $S(v_1, v_2, v_3, v_4)$ the left hand side of (c). By a straightforward computation, we obtain

\[ 0 = S(v_1, v_2, v_3, v_4) - S(v_2, v_3, v_4, v_1) = S(v_3, v_4, v_1, v_2) + S(v_4, v_1, v_2, v_3) \]
\[ - R(v_1, v_2, v_3, v_4) - R(v_2, v_1, v_3, v_4) = R(v_2, v_3, v_4, v_1) + R(v_3, v_4, v_1, v_2) \]
\[ + R(v_4, v_3, v_1, v_2). \]

By applying (a) and (b), we see that

\[ 2R(v_1, v_2, v_3, v_4) - 2R(v_3, v_4, v_1, v_2) = 0. \]

QED.

PROPOSITION 1.2. Let $R$ and $T$ be two quadrilinear mappings with the above properties (a), (b) and (c). If

\[ R(v_1, v_2, v_3, v_4) = T(v_1, v_2, v_3, v_4) \]

for all $v_1, v_2 \in V$,

then $R = T$.

Proof. We may assume that $T \equiv 0$; consider $R - T$ and 0 instead of $R$ and $T$. We assume therefore that $R(v_1, v_2, v_3, v_4) = 0$ for all $v_1, v_2 \in V$. We have

\[ 0 = R(v_1, v_2 + v_3, v_4) = R(v_1, v_2, v_3) + R(v_1, v_4, v_2) + R(v_1, v_2, v_4). \]

Hence,

(1) $R(v_1, v_2, v_3, v_4) = 0$ for all $v_1, v_2, v_3, v_4 \in V$.

From (1) we obtain

\[ 0 = R(v_1 + v_3, v_2, v_3, v_4) = R(v_1, v_2, v_3) + R(v_3, v_2, v_1, v_4). \]

Now, by applying (d) and then (b), we obtain,

\[ 0 = R(v_1, v_2, v_3, v_4) + R(v_1, v_3, v_2, v_4) = R(v_1, v_2, v_3, v_4). \]

Hence,

(2) $R(v_1, v_2, v_3, v_4) = R(v_1, v_2, v_3, v_4)$ for all $v_1, v_2, v_3, v_4 \in V$.

Replacing $v_2, v_3, v_4$ by $v_3, v_4, v_2$, respectively, we obtain

(3) $R(v_1, v_2, v_3, v_4) = R(v_1, v_3, v_2, v_4)$ for all $v_1, v_2, v_3, v_4 \in V$.

From (2) and (3), we obtain

\[ 3R(v_1, v_2, v_3, v_4) = R(v_1, v_2, v_3, v_4) + R(v_1, v_3, v_2, v_4) + R(v_1, v_4, v_2, v_3), \]

where the right hand side vanishes by (c). Hence,

\[ R(v_1, v_2, v_3, v_4) = 0 \]

for all $v_1, v_2, v_3, v_4 \in V$.

QED.

Besides a quadrilinear mapping $R$, we consider an inner product (i.e., a positive definite symmetric bilinear form) on $V$, which will be denoted by $(\cdot, \cdot)$. Let $p$ be a plane, that is, a 2-dimensional subspace, in $V$ and let $v_1$ and $v_2$ be an orthonormal basis for $p$. We set

\[ K(p) = \mu R(v_1, v_2, v_1, v_2). \]
As the notation suggests, \( K(p) \) is independent of the choice of an orthonormal basis for \( p \). In fact, if \( w_1 \) and \( w_2 \) form another orthonormal basis of \( \mathbb{R}^2 \), then

\[
    w_1 = aw_1 + bw_2, \quad w_2 = -bw_1 + aw_2
\]

where \( a \) and \( b \) are real numbers such that \( a^2 + b^2 = 1 \). Using (a) and (b), we easily obtain \( R(v_1, v_2, v_1, v_2) = R(w_1, w_2, w_1, w_2) \).

**Proposition 1.3.** If \( v_1, v_2 \) is a basis (not necessarily orthonormal) of a plane \( p \) in \( V \), then

\[
    K(p) = \frac{R(v_1, v_2, v_1, v_2)}{(v_1, v_1)(v_2, v_2) - (v_1, v_2)^2}.
\]

Proof. We obtain the formula making use of the following orthonormal basis for \( p \):

\[
    \begin{align*}
        \frac{v_1}{(v_1, v_1)}, \quad \frac{v_2}{(v_2, v_2)}
    \end{align*}
\]

where \( a = [(v_1, v_1)((v_1, v_1)(v_2, v_2) - (v_1, v_2)^2)]^{1/2} \).

We set

\[
    R_1(v_1, v_2, v_3, v_4) = (v_1, v_3)(v_2, v_4) - (v_2, v_3)(v_1, v_4)
\]

for \( v_1, v_2, v_3, v_4 \in V \).

It is a trivial matter to verify that \( R_1 \) is a quadrilinear mapping having the properties (a), (b) and (c) and that, for any plane \( p \) in \( V \), we have

\[
    K_1(p) = R_1(v_1, v_2, v_1, v_2) = 1,
\]

where \( v_1, v_2 \) is an orthonormal basis for \( p \).

**Proposition 1.4.** Let \( R \) be a quadrilinear mapping with properties (a), (b) and (c). If \( K(p) = c \) for all planes \( p \), then \( R = cR_1 \).

Proof. By Proposition 1.3, we have

\[
    R(v_1, v_2, v_1, v_2) = cR_1(v_1, v_2, v_1, v_2)
\]

for all \( v_1, v_2 \in V \).

Applying Proposition 1.2 to \( R \) and \( cR_1 \), we conclude \( R = cR_1 \).

QED.

Let \( e_1, \ldots, e_n \) be an orthonormal basis for \( V \) with respect to the inner product \( (, ,) \). To each quadrilinear mapping \( R \) having properties (a), (b) and (c), we associate a symmetric bilinear form \( S \) on \( V \) as follows:

\[
    S(v, w) = R(e_1, v, e_1, w) + R(e_2, v, e_2, w) + \cdots
\]

It can be easily verified that \( S \) is independent of the choice of an orthonormal basis \( e_1, \ldots, e_n \). From the definition of \( S \), we obtain

\[
    S(v, v) = K(p) + \cdots + K(p_n),
\]

where \( p \) is the plane spanned by \( v \) and \( e_i \).

2. **Sectional curvature**

Let \( M \) be an \( n \)-dimensional Riemannian manifold with metric tensor \( g \). Let \( R(X, Y) \) denote the curvature transformation of \( T(M) \) determined by \( X, Y \in T(M) \) (cf. §5 of Chapter III). The Riemannian curvature tensor (field) of \( M \), denoted also by \( R \), is the tensor field of covariant degree 4 defined by

\[
    R(X_1, X_2, X_3, X_4) = g(R(X_3, X_4)X_2, X_1),
\]

\[
    X_i \in T(M), i = 1, \ldots, 4.
\]

**Proposition 2.1.** The Riemannian curvature tensor, considered as a quadrilinear mapping \( T(M) \times T(M) \times T(M) \times T(M) \to \mathbb{R} \) at each \( x \in M \), possesses properties (a), (b), (c) and hence (d) of §1.

Proof. Let \( u \) be any point of the bundle \( O(M) \) of orthonormal frames such that \( \pi(u) = x \). Let \( X^*_3, X^*_4 \in T^*(M) \) with \( \pi(X^*_3) = X_3 \) and \( \pi(X^*_4) = X_4 \). From the definition of the curvature transformation \( R(X_3, X_4) \) given in §5 of Chapter III, we obtain

\[
    g(R(X_3, X_4)X_2, X_1) = g(2\Omega(X^*_3, X^*_4)(u^{-1}X_2), X_1)
\]

where ( , ) is the natural inner product in \( R^n \). Now we see that property (a) is a consequence of the fact that \( \Omega(X^*_3, X^*_4) \epsilon O(n) \) is a skew-symmetric matrix. (b) follows from \( R(X_3, X_4) = -R(X_4, X_3) \). Finally, (c) is a consequence of Rianchi's first identity given in Theorem 5.3 of Chapter III.

QED.
For each plane \( p \) in the tangent space \( T_x(M) \), the sectional curvature \( K(p) \) for \( p \) is defined by

\[
K(p) = R(X_1, X_2, X_1, X_2) = g(R(X_1, X_2)X_2, X_1),
\]

where \( X_1, X_2 \) is an orthonormal basis for \( p \). As we saw in §1, \( K(p) \) is independent of the choice of an orthonormal basis \( X_1, X_2 \). Proposition 1.2 implies that the set of values of \( K(p) \) for all planes \( p \) in \( T_x(M) \) determines the Riemannian curvature tensor at \( x \).

If \( K(p) \) is a constant for all planes \( p \) in \( T_x(M) \) and for all points \( x \in M \), then \( M \) is called a space of constant curvature.

The following theorem is due to F. Schur [1].

**Theorem 2.2.** Let \( M \) be a connected Riemannian manifold of dimension \( \geq 3 \). If the sectional curvature \( K(p) \), where \( p \) is a plane in \( T_x(M) \), depends only on \( x \), then \( M \) is a space of constant curvature.

**Proof.** follows:

We define a covariant tensor field \( R \) of degree 4 as

\[
R(U, X, Y) = g(\nabla U R(W, Z, X, Y) - g(W, X)g(Z, Y) - g(Z, X)g(Y, W),
\]

where \( U, X, Y, Z \) are mutually orthogonal and that \( g(U, Z) = 1 \). Then we obtain

\[
(Xk)Y = (Yk)X = 0.
\]

Since \( X \) and \( Y \) are linearly independent, we have \( Xk = Yk = 0 \). This shows that \( k \) is a constant.

**Corollary 2.3.** For a space of constant curvature \( k \), we have

\[
R(X, Y)Z = k(g(Z, Y)X - g(Z, X)Y).
\]

This was established in the course of proof for Theorem 2.2.

If \( k \) is a positive (resp. negative) constant, \( M \) is called a space of positive (resp. negative) curvature.

As in §7 of Chapter III, we define a set of function \( \xi_{ij} \) on \( L(M) \) where \( B = (\xi_{ij}) \) is the curvature form of the Riemannian connection. For an arbitrary point \( u \) of \( O(M) \), we choose a local coordinate system \( x_1, \ldots, x^n \) with origin \( x = \pi(u) \) such that \( u \) is the frame given by \( (\partial/\partial x_1), \ldots, (\partial/\partial x^n) \). With respect to this coordinate system, we have

\[
\xi_{ij} = \delta_{ij} \text{ at } x,
\]

and hence

\[
R^i_{jkl} = R^i_{jkl} = k(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}) \text{ at } x.
\]

Let \( \sigma \) be the local cross section of \( L(M) \) given by the field of linear frames \( \partial/\partial x^1, \ldots, \partial/\partial x^n \). As we have shown in §7 of
Chapter III, we have $\sigma^* R_{jkl} = R_{jkl}$. Hence, 
\[
R_{jkl} = k(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}) \quad \text{at} \quad u,
\]
\[
\Omega_{j'k} = k\theta^j A \theta^k \quad \text{at} \quad u.
\]
Since $u$ is an arbitrary point of $O(M)$, we have

**Proposition 2.4.** If $M$ is a space of constant curvature with sectional curvature $k$, then the curvature form $\Omega = (\Omega_j^i)$ is given by
\[
\Omega_j^i = k\delta^i A \theta^j \circ n \quad O(M),
\]
where $\theta = (\theta^j)$ is the canonical form on $O(M)$.

### 3. Spaces of constant curvature

In this section, we shall construct, for each constant $k$, a simply connected, complete space of constant curvature with sectional curvature $k$. Namely, we prove

**Theorem 3.1.** Let $(x^1, \ldots, x^n, t)$ be the coordinate system of $\mathbb{R}^{n+1}$ and $M$ the hypersurface of $\mathbb{R}^{n+1}$ defined by
\[
(x^1)^2 + \cdots + (x^n)^2 + rt^2 = r \quad (r: \text{a nonzero constant}).
\]
Let $g$ be the Riemannian metric of $M$ obtained by restricting the following form to $M$:
\[
(dx^1)^2 + \cdots + (dx^n)^2 + rd t^2.
\]
Then
1. $M$ is a space of constant curvature $1/r$.
2. The group $G$ of linear transformations of $\mathbb{R}^{n+1}$ leaving the quadratic form $(x^1)^2 + \cdots + (x^n)^2 + rt^2$ invariant acts transitively on $M$ as a group of isometries of $M$.
3. If $r > 0$, then $M$ is isometric to a sphere of a radius $r$. If $r < 0$, then $M$ consists of two mutually isometric connected manifolds each of which is diffeomorphic with $\mathbb{R}^n$.

**Proof.** First we observe that $M$ is a closed submanifold of $\mathbb{R}^{n+1}$ (cf. Example 1.1 of Chapter I); we leave the verification to the reader.

We begin with the proof of (3). If $r > 0$, then we set $x^{n+1} = rt$. Then $M$ is given by
\[
(x^1)^2 + \cdots + (x^n)^2 = 1,
\]
and the metric $g$ is the restriction of $(dx^1)^2 + \cdots + (dx^n)^2$ to $M$. This means that $M$ is isometric with a sphere of radius $1/r$. If $r < 0$, then $t^2 \geq 1$ at every point of $M$. Let $M^*$ be the set of points of $M$ with $t \geq 1$ (resp. $-1 
 leq 1$). The mapping $(x^1, \ldots, x^n, t) \to (y^1, \ldots, y^n)$ defined by
\[
y^i = x^i/t, \quad i = 1, \ldots, n,
\]
is a diffeomorphism of $M^*$ onto the open subset of $\mathbb{R}^n$ given by
\[
\Sigma_{i=1}^n (y^i)^2 + r < 0.
\]
In fact, the inverse mapping is given by
\[
x^i = y^i t, \quad i = 1, \ldots, n,
\]
\[
t = \pm \left(1 + \Sigma_{i=1}^n (y^i)^2 r \right)^{1/2}
\]
A straightforward computation shows that the metric $g$ is expressed in terms of $y^1, \ldots, y^n$ as follows:
\[
\frac{r[(r + \Sigma_{i=1}^n (y^i)^2)(\Sigma_{i=1}^n (dy^i)^2) - (\Sigma_{i=1}^n y^i dy^i)^2]}{(r + \Sigma_{i=1}^n (y^i)^2)^2}
\]

To prove (2), we first consider $G$ as a group acting on $\mathbb{R}^{n+1}$. Since $G$ is a linear group leaving $(x^1)^2 + \cdots + (x^n)^2 + rt^2$ invariant, it leaves the form $(dx^1)^2 + \cdots + (dx^n)^2 + rd t^2$ invariant. Thus, considered as a group acting on $M$, $G$ is a group of isometries of the Riemannian manifold $M$. The transitivity of $G$ on $M$ is a consequence of Witt's theorem, which may be stated as follows.

Let $Q$ be a nondegenerate quadratic form on a vector space $V$. If $f$ is a linear mapping of a subspace $U$ of $V$ into $V$ such that $Q(f(x)) = Q(x)$ for all $x \in U$, then $f$ can be extended to a linear isomorphism of $V$ onto itself such that $Q(f(x)) = Q(x)$ for all $x \in V$. In particular, if $x_0$ and $x_1$ are elements of $V$ with $Q(x_0) = Q(x_1)$, there is a linear isomorphism $f$ of $V$ onto itself which leaves $Q$ invariant and which maps $x_0$ into $x_1$.

For the proof of Witt's theorem, see, for example, Artin [1, p. 121].

Finally, we shall prove (1). Let $H$ be the subgroup of $G$ which consists of transformations leaving the point $o$ with coordinates
(0, ..., 0, 1) fixed. We define a mapping $f: G \to O(M)$ as follows. Let $u_0 \in O(M)$ be the frame at the point $o = (0, \ldots, 0, 1)$ of $M$ given by $(\partial/\partial x^1)_o, \ldots, (\partial/\partial x^n)_o$. Every element $a \in G$, being an isometric transformation of $M$, maps each orthonormal frame of $M$ into an orthonormal frame. In particular, $a(u_0)$ is an orthonormal frame of $M$ at the point $a(o)$. We define

$$f(a) = a(u_0), \quad a \in G.$$  

**Lemma 1.** The mapping $f: G \to O(M)$ is an isomorphism of the principal fibre bundle $G(G/H, H)$ onto the bundle $O(M^1; M, O(n))$.

Proof of Lemma 1. If we consider $G$ as a group of $(n + 1)$-matrices in a natural manner, then $H$ is naturally isomorphic with $O(n)$. Let $v = (0(n), 0)$. It is easy to verify that $f: G \to O(M)$ commutes with the right translation $R$, for every $a \in H = O(n)$. The transitivity of $G$ on $M$ implies that the induced mapping $f: G/H \to M$ is a diffeomorphism and hence that $f: G \to O(M)$ is a bundle isomorphism.

The quadratic form defining $M$ is given by the following $(n + 1)$ x $(n + 1)$-matrix:

$$Q = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}.$$  

An $(n + 1)$ x $(n + 1)$-matrix $a$ is an element of $G$ if and only if

$$\begin{pmatrix} a \cdot X \\ a \cdot Y \end{pmatrix} = Q,$$  

where $X$ is an $n$ x $n$-matrix and $z$ are elements of $R^n$ and $w$ is a real number. Then the condition for $a$ to be in $G$ is expressed by

$$X + r \cdot z = 1, \quad Y + r \cdot w = 0, \quad Y + r \cdot w^2 = r.$$  

It follows that the Lie algebra of $G$ is formed by the matrices of the form

$$\begin{pmatrix} A & b \\ c & 0 \end{pmatrix},$$  

where $A$ is an $n \times n$-matrix with $iA + A = 0$ and $b$ and $c$ are elements of $R^n$ satisfying $b + c = 0$. Let

$$\begin{pmatrix} x_1 \ldots x_n & \beta^1 \\ \vdots & \vdots \\ x_1^* \ldots x_n^* & \beta^a \\ \gamma_1 \ldots \gamma_n & 0 \end{pmatrix}$$  

be the (left invariant) canonical 1-form on $G$ (cf. §4 of Chapter I). We have $\alpha^i + \alpha^i = 0, \quad \beta^i + r\gamma_i = 0, \quad i, j = 1, \ldots, n$.

The Maurer-Cartan equation of $G$ is expressed by

$$d\beta^i = -\Sigma_k \alpha^k \wedge \beta^i,$$

$$d\alpha^i = -\Sigma_k \alpha^k \wedge \alpha^j = \beta^i \wedge \gamma_j, \quad i, j = 1, \ldots, n.$$  

**Lemma 2.** Let $\theta = (\theta^i)$ and $\omega = (\omega^i)$ be the canonical form and the connection form on $O(M)$. Then

$$f \ast \theta^i = \beta^i, \quad f \ast \omega^i = \alpha^i, \quad i, j = 1, \ldots, n.$$  

Proof of Lemma 2. As we remarked earlier, every element $a \in G$ induces a transformation of $O(M)$; this transformation corresponds to the left translation by $a$ in $G$ under the isomorphism $f: G \to O(M)$. From the definition of $\theta$, we see easily that $\theta = (\theta^i)$ is invariant by the transformation induced by each $a \in G$. On the other hand, $\beta$ is invariant by the left translation by each $a \in G$. To prove $f \ast \theta^i = \beta^i$, it is therefore sufficient to show that $(f \ast \theta^i)(X^*) = \beta^i(X^*)$ for all $X^* \in T_a(G)$. Set $X_i = (\partial/\partial x^i)_o$, so that the frame $u_0$ is given by $(X^*, \ldots, X_n)$. The composite mapping $\pi \circ f: G \to O(M)$ maps an element of $T_e(G)$ identified with the Lie algebra of $G$ of the form

$$\begin{pmatrix} A & b \\ c & 0 \end{pmatrix}$$  

onto
into the vector $\sum_i b_i X_i$, $b_i \in T_e(M)$, where $b_1, \ldots, b_n$ are the components of $b$. Therefore, if $X_1,\ldots, X_n$ are the components of $X$, then $\pi \circ f(X_i) = \sum_i \beta^i(X_i) X_i$, and hence

$$(f \circ \pi(X_i), \ldots, f \circ \pi(X_n)) = u_0^{-1}(\pi \circ f(X_i)) = (\beta^1(X_i), \ldots, \beta^n(X_i)),$$

which proves the first assertion of the lemma. Let $g$ and $H$ be the Lie algebras of $G$ and $H$, respectively. Let $m$ be the linear subspace of $g$ consisting of matrices of the form

$$
\begin{pmatrix}
0 & b \\
-\gamma & 0
\end{pmatrix}.
$$

It is easy to verify that $m$ is stable under $ad H$, that is $ad(a)(m) = m$ for every $a \in H$. Applying Theorem 11.1 of Chapter II, we see that $\{a_j\}$ defines a connection in the bundle $G(G/H, H)$. Now the second assertion of Lemma 2 follows from the following three facts: (1) $\{\beta^i\}$ corresponds to $\{\theta^i\}$ under the isomorphism $G \to O(M)$; (2) the Riemannian connection form $\omega^j_i$ is characterized by the property that the torsion is zero (Theorem 2.2 of Chapter IV), that is, $d\theta^j = -\Sigma_k \alpha_k^i \wedge \theta^i$; (3) the connection form $\omega^j_i$ satisfies the equality: $d\beta^i = -\Sigma_k \alpha_k^i \wedge \beta^i$.

We shall now complete the proof of Theorem 3.1, Lemma 2, together with

$$dx_j^i = -\Sigma_k \alpha_k^i \wedge \alpha_j^k - \beta^i \wedge \gamma_j^i$$

and

$$\beta^i + \gamma_j^i = 0,$$

implies

$$dx_j^i = -\Sigma_k \alpha_k^i \wedge \alpha_j^k + \frac{1}{r} \beta^i \wedge \beta^j,$$

showing that the curvature form of the Riemannian connection is given by $\frac{1}{r} \theta^i \wedge \theta^j$. By Proposition 2.4, $M$ is a space of constant curvature with sectional curvature $1/r$. QED.

Remark. The group $G$ is actually the group of all isometries of $M$. To see this, let $\mathcal{O}(M)$ be the group of isometries of $M$ and define a mapping $f: \mathcal{O}(M) \to O(M)$ in the same way as we defined $f: G \to O(M)$. Then $G \subset \mathcal{O}(M)$ and $f: \mathcal{O}(M) \to O(M)$ is an extension of $f: G \to O(M)$. Since $f$ maps $3(M)$ 1: 1 into $O(M)$ and since $f(G) = O(M)$, we must have $G = \mathcal{O}(M)$.

In the course of the proof of Theorem 3.1, we obtained

**Theorem 3.2.** (1) Let $M$ be the sphere in $\mathbb{R}^{n+1}$ defined by

$$(x_1)^2 + \cdots + (x_n)^2 + (x_{n+1})^2 = a^2.$$ 

Let $g$ be the restriction of $(dx_1)^2 + \cdots + (dx_n)^2 + (dx_{n+1})^2$ to $M$. Then, with respect to this Riemannian metric $g$, $M$ is a space of constant curvature $1/a^2$ with sectional curvature $1/a^2$.

(2) Let $M$ be the open set in $\mathbb{R}^n$ defined by

$$(x_1)^2 + \cdots + (x_n)^2 < a^2.$$ 

Then, with respect to the Riemannian metric given by

$$a^2 \left[ \frac{(a^2 - \Sigma_{i,j} \gamma_{ij}^k)(\Sigma_k (df_k)^2) - (\Sigma_i (d\gamma_i)^2)}{(a^2 - \Sigma_{i,j} \gamma_{ij}^k)(a^2 - \Sigma_{k,l} \gamma_{k,l}^n)} \right],$$

$M$ is a space of constant curvature $1/a^2$.

The spaces $M$ constructed in Theorem 3.2 are all simply connected, homogeneous and hence complete by Theorem 4.5 of Chapter IV. The space $\mathbb{R}^n$ with the Euclidean metric $(dx)^2 + \cdots + (dx_n)^2$ gives a simply connected, complete space of zero curvature.

A Riemannian manifold of constant curvature is said to be elliptic, hyperbolic or flat (or locally Euclidean) according as the sectional curvature is positive, negative or zero. These spaces are also called space forms (cf. Theorem 7.10 of Chapter V).

**4. Flat affine and Riemannian connections**

Throughout this section, $M$ will be a connected, paracompact manifold of dimension $n$.

Let $A(M)$ be the bundle of affine frames over $M$; it is a principal fiber bundle with structure group $G = A(n; \mathbb{R})$ (cf. §3 of Chapter III). An affine connection of $M$ is said to be flat it every point of $M$ has an open neighborhood $U$ and an isomorphism $\psi: A(M) \to U \times G$ which maps the horizontal space at each $u \in A(U)$ into the horizontal space at $\psi(u)$ of the canonical flat connection of $U \times G$. A manifold with a flat affine connection is
said to be \textit{locally affine}, a Riemannian manifold is \textit{flat} (or locally Euclidean) if the Riemannian connection is a flat affine connection.

\textbf{Theorem 4.1.} For an affine connection of $M$, the following conditions are mutually equivalent:

1. $M$ is flat;
2. The torsion and the curvature of the corresponding linear connection vanish identically;
3. The affine holonomy group is discrete.

\textbf{Proof.} By Theorem 9.1 of Chapter II, an affine connection is flat if and only if its curvature form $\Omega$ on $A(M)$ vanishes identically. The equivalence of (1) and (2) follows from Proposition 3.4 of Chapter III. The equivalence of (2) and (3) follows from Theorems 4.2 and 8.1 of Chapter II.

\textbf{QED.}

\textbf{Remark.} Similarly, for a linear connection of $M$, the following conditions are mutually equivalent:

1. It is flat, i.e., the connection in $L(M)$ is flat; (2) Its curvature vanishes identically; (3) The linear (or homogeneous) holonomy group is discrete.

When we say that the affine holonomy group and the linear holonomy group are discrete, we mean that they are $O$-dimensional Lie groups. Later (cf. Theorem 4.2) we shall see that the affine holonomy group of a complete flat affine connection is discrete in the affine group $A(n; R)$. But the linear holonomy group need not be discrete in $GL(n; R)$ (cf. Example 4.3). It will be shown that the linear holonomy group of a compact flat Riemannian manifold is discrete in $O(n)$ (cf. the proof of $\Theta$ of Theorem 4.2 and the remark following Theorem 4.2).

\textbf{Example 4.1.} Let $\xi_1, \ldots, \xi_k$ be linearly independent elements of $R^n$, $k \leq n$. Let $G$ be a subgroup of $R^n$ generated by $\xi_1, \ldots, \xi_k$:

$$G = \{ \Sigma m_i \xi_i \mid m_i \text{ integers} \}.$$ 

The action of $G$ on $R^n$ is properly discontinuous, and $R^n$ is the universal covering manifold of $R^n/G$. The Euclidean metric $(dx^1)^2 + \cdots + (dx^n)^2$ of $R^n$ is invariant by $G$ and hence induces a flat Riemannian metric on $R^n/G$. The manifold $R^n/G$ with the Riemannian metric thus defined will be called a Euclidean cylinder.

If it is called a Euclidean torus if $\xi_1, \ldots, \xi_k$ form a basis of $R^n$, i.e., $k = n$. Every connected abelian Lie group with an invariant Riemannian metric is a Euclidean cylinder, and if it is, moreover, compact, then it is a Euclidean torus. In fact, the universal covering group of such a Lie group is isomorphic with a vector group $R^n$ and its invariant Riemannian metric is given by $(dx^1)^2 + \cdots + (dx^n)^2$ by a proper choice of basis in $R^n$. Our assertion is now clear.

The following example shows that a torus can admit a flat affine connection which is not Riemannian. This was taken from Kuiper [1].

\textbf{Example 4.2.} The set $G$ of transformations

$$\begin{align*}
(x, y) &\rightarrow (x + ny + m, y + n), \\
n, m &\in \mathbb{Z}, \pm \pm \pm \pm, \ldots,
\end{align*}$$

of $R^2$ with coordinate system $(x, y)$ forms a discrete subgroup of the group of affine transformations; it acts properly discontinuously on $R^2$ and the quotient space $R^2/G$ is diffeomorphic with a torus. The flat affine connection of $R^2$ induces a flat affine connection on $R^2/G$. This flat affine connection of $R^2/G$ is not Riemannian. In fact, if it is Riemannian, the induced Riemannian metric on the universal covering space $R^2$ must be of the form:

$$dx^2 + \sum (dx^i + c dy^k)^2,$$

where $a, b, c$ are constants, since the metric must be parallel. On the other hand, $G$ is not a group of isometries of $R^2$ with respect to this metric, thus proving our assertion.

Let $M$ be locally affine and choose a linear frame $u_0 \in L(M) \subset A(M)$. Let $M^*$ be the holonomy bundle through $u_0$ of the flat affine connection and $M'$ the holonomy bundle through $u_0$ of the corresponding flat linear connection. Then $M^*$ (resp. $M'$) is a principal fibre bundle over $M$ whose structure group is the affine holonomy group $\Phi(u_0)$ (resp. the linear holonomy group $\Psi$). Since $\Phi(u_0)$ and $\Psi(u_0)$ are discrete, both $M^*$ and $M'$ are covering manifolds of $M$. The homomorphism $\beta: A(M) \rightarrow L(M)$ defined in $\S$ of Chapter III maps $M^*$ onto $M'$ (cf. Proposition 3.5 of Chapter III). Hence $M^*$ is a covering manifold of $M'$.

\textbf{Theorem 4.2.} Let $M$ be a manifold with a complete flat affine connection. Let $u_0 \in L(M) \subset A(M)$. Let $M^*$ be the holonomy bundle through $u_0$ of the flat affine connection and $M'$ the holonomy bundle through $u_0$ of the corresponding flat linear connection. Then
We identify the tangent space $\mathcal{A}$.

(1) $M^*$ is the universal covering space of $M$ and, with respect to the flat affine connection induced on $M^*$, it is isomorphic to the ordinary affine space $\mathbb{A}$.

(2) With respect to the flat affine connection induced on $M'$, $M'$ is a Euclidean cylinder, and the first homotopy group of $M'$ is isomorphic to the kernel of the homomorphism $\Phi(u_0) \to \Psi(u_0)$.

(3) If $M^*$ is a Euclidean cylinder and is a covering space of $M$, then it is a covering space of $M'$.

(4) $M'$ is a Euclidean torus if and only if $M$ is a compact flat Riemannian manifold.

Proof. Let $X$ be a vector and only

Since $X^*$ is complete (cf. Proposition 6.5 of Chapter III), so is $X$. Note that (b) implies that the group is abelian.

It is clear that $\bar{\theta}^i \partial \bar{\theta}^i + \cdots + \bar{\theta}^n \partial \bar{\theta}^n$ is an invariant Riemannian metric on the abelian Lie group $M'$. As we have seen in Example 4.1, $M'$ is a Euclidean cylinder.

**Lemma 1.** Let $R^n/G = \{\sum_{i=1}^n m_i \xi_i ; m_i \text{ integers}\}$ be a Euclidean cylinder as defined in Example 4.1. Then the affine holonomy group of $\mathbb{R}^n/G$ is a group of translations isomorphic with $G$.

Proof of Lemma 1. We identify the tangent space $T_a(R^n)$ at each point $a \in \mathbb{R}^n$ with $R^n$ by the following correspondence:

$T_a(R^n) \to \sum_{i=1}^n \lambda^i (\partial/\partial x^i)_a \to (\lambda^1, \ldots, \lambda^n) \in \mathbb{R}^n$.

The linear parallel displacement from $0 \in \mathbb{R}^n$ to $a \in \mathbb{R}^n$ sends $(\lambda^1, \ldots, \lambda^n) \in T_a(R^n)$ into the vector with the same components $(\lambda^1 + a^1, \ldots, \lambda^n + a^n)$, considered as an element of the tangent affine space $A_a(R^n)$, into $A_a(R^n)$. Let $\tau = x^i$, $0 \leq t \leq 1$, be a line from $0$ to $\sum_{i=1}^n m_i \xi_i \in G$ and let $\tau = x^i$, $0 \leq t \leq 1$, be the image of $\tau$ by the projection $R^n \to \mathbb{R}^n/G$. Then $\tau$ is a closed curve in $R^n/G$. Let $\sum_{i=1}^n m_i \xi_i = (a^1, \ldots, a^n) \in \mathbb{R}^n$.

Then the affine parallel displacement along $\tau$ yields the translation $(\lambda^1, \ldots, \lambda^n) \to (\lambda^1 + a^1, \ldots, \lambda^n + a^n)$.

This completes the proof of Lemma 1.

Being a covering space of $M'$, $M^*$ is also a Euclidean cylinder. By Proposition 9.3 of Chapter II, the affine holonomy group of $M^*$ is trivial. By Lemma 1, $M^*$ must be the ordinary affine space $\mathbb{A}$, proving (1).

Since $M' = M^*/N$, the first homotopy group of $M'$ is isomorphic with $N$. This completes the proof of (2).

Let $M^*$ be a covering space of $M$. Since $M^*$ is the universal covering space of $M$, we can write $M^* = M^*/H$, where $H$ is a subgroup of $\Phi(u_0)$. The affine holonomy group of $M^*$ is $H$ by Proposition 9.3 of Chapter II. If $M^*$ is a Euclidean cylinder, the affine holonomy group $H$ consists of translations only (cf. Lemma 1) and hence is contained in the kernel $N$ of the homomorphism
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Since $M' = M^*/N$, we may conclude that $M^*$ is a covering space of $M'$, thus proving (3).

Suppose $M'$ is a Euclidean torus. It follows that $M$ is compact and the linear holonomy group $\Phi(u_0)$ of $M$ is a finite group. This implies that the flat affine connection of $M$ is Riemannian. In fact, we choose an inner product in $T_x(M)$, $x_0 = \pi(u_0)$, invariant by the linear holonomy group with reference point $x_0$, and then we extend it to a Riemannian metric by parallel displacement. The flat affine connection of $M$ is the Riemannian connection with respect to the Riemannian metric thus constructed.

Conversely, suppose $M$ is a compact, connected, flat Riemannian manifold. By virtue of (1), identifying $M^*$ with $R^n$, we may write $M = R^n/G$, where $G$ is a discrete subgroup of the group of Euclidean motions acting on $R^n$. Let $N$ be the subgroup of $G$ consisting of pure translations. In view of (2) and (3) our problem is to prove that $R^n/N$ is a Euclidean torus. We first prove several lemmas.

**Lemma 2.** Let $A$ and $B$ be unitary matrices of degree $n$ such that $A$ commutes with $ABA^{-1}B^{-1}$. If the characteristic roots of $B$ have positive real parts, then $A$ commutes with $B$.

Proof of Lemma 2. Since $AABA^{-1}B^{-1} = ABA^{-1}B^{-1}A$, we have $ABABA^{-1}B^{-1} = BABA^{-1}B^{-1}A$. Without loss of generality, we may assume that $B$ is diagonal with diagonal elements $b_k = \cos \beta_k + J-1 \sin \beta_k, k = 1, \ldots, n$. Since $A^{-1} = tA$ and $B^{-1} = tB = B$, we have

$$ABABA^{-1}B^{-1} = BABA^{-1}B^{-1}A = BABA^{-1}B^{-1}A = BABA^{-1}B^{-1}A.$$  

Comparing the $(i,i)$-th entries, we have

$$\Sigma_{j=1}^n a_j b_j a_j^* = \Sigma_{j=1}^n b_j a_j b_j a_j^*,$$  

where $A = (a_j^*)$.

Comparing the imaginary parts, we obtain

$$\Sigma_{j=1}^n |a_j|^2 + |a_j|^2 \cdot \sin (\beta_j - \beta_j) = 0 \text{ for } i = 1, \ldots, n.$$  

We may also assume that $\beta_1 = \beta_2 = \cdots = \beta_{p_1} < \beta_{p_1 + 1} = \cdots = \beta_{p_1 + p_2} < \cdots \leq \beta_n < \beta_1 + \pi$. Since all the $b_k$'s have positive real parts, we have

$$\sin (\beta_j - \beta_i) > 0 \text{ for } i \leq p_1 \text{ and } j > p_1.$$  

Hence we must have

$$a_j^2 = a_i^2 = 0 \text{ for } i \leq p_1 \text{ and } j > p_1.$$  

Similarly, we have

$$a_i^2 = a_j^2 = 0 \text{ for } i \leq p_1 + p_2 \text{ and } j > p_1 + p_2.$$  

Continuing this argument we have

$$A = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \\ 0 & \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \\ 0 & \end{pmatrix},$$  

where $A_1, A_2, \ldots$ are unitary matrices of degree $p_1, p_2, \ldots$, and $I_{n1}, I_{n2}, \ldots$ are the identity matrices of degree $p_1, p_2, \ldots$. This shows clearly that $A$ and $B$ commute.

For any matrix $A = (a_j)$ of type $(r, s)$ we set

$$\varphi(A) = (\Sigma_{i,j} |a_{ij}|^2)^{\frac{1}{2}}.$$  

In other words, $\varphi(A)$ is the length of $A$ when $A$ is considered as a vector with $rs$ components. We have

$$\varphi(A + B) \leq \varphi(A) + \varphi(B),$$  

$$\varphi(AB) \leq \varphi(A) \cdot \varphi(B).$$  

The latter follows from the inequality of Schwarz. If $A$ is an orthogonal matrix, we have

$$\varphi(AB) = \varphi(B), \varphi(CA) = \varphi(C).$$  

Every Euclidean motion of $R^n$ is given by

$$x \to Ax + p, \quad x \in R^n,$$  

where $A$ is an orthogonal matrix (called the rotation part of the motion) and $p$ is an element of $R^n$ (called the translation part of the motion). This motion will be denoted by $(A, p)$.

**Lemma 3.** Given any two Euclidean motions $(A, p)$ and $(B, q)$, set

$$\begin{pmatrix} A_1 & p_1 \end{pmatrix} = (A, p) (B, q) (A, p)^{-1}(B, q)^{-1}.$$
Let I be the identity matrix of degree n. If \( q(A - I) < a \) and \( q(B - I) < b \), then we have

1. \( q(A - I) < 2ab \);
2. \( q(p_1) < b \cdot q(p) + a \cdot q(q) \).

**Proof of Lemma 3.** We have

\[
A - Z = ABA^{-1}B^{-1} - Z = (AB - BA)A^{-1}B^{-1} = ((A - Z)(B - I) - (B - I)(A - I))A^{-1}B^{-1}.
\]

Since \( A^{-1}B^{-1} \) is an orthogonal matrix, we have

\[
q(A - I) \leq q(A - I) \cdot q(B - I) + q(B - I) \cdot q(A - I) < 2ab.
\]

By a simple calculation, we obtain

\[
p_1 = A(I - B)A^{-1}p + AB(Z - A^{-1})B^{-1}q.
\]

By the same reasoning as above, we obtain

\[
q(p_1) \leq q(I - B) \cdot q(p) + q(I - A) \cdot q(B) < b \cdot q(p) + a \cdot q(q).
\]

**Lemma 4.** With the same notation as in Lemma 3, set

\[
(A_k, p_k) = (A, p)(A_{k-1}, p_{k-1})(A, p)^{-1}(A_{k-1}, p_{k-1})^{-2}; k = 2, 3, \ldots
\]

Then, for \( k = 1, 2, 3, \ldots \), we have

1. \( q(A_k - I) < 2^k a^k b; \)
2. \( q(p_k) < (2^k - 1)a^k b \cdot q(p) + a^k \cdot q(q) \).

**Proof of Lemma 4.** A simple induction using Lemma 3 establishes the inequalities.

**Lemma 5.** Let G be a discrete subgroup of the group of Euclidean motions of \( \mathbb{R}^n \). Let \( a < \frac{1}{2} \) and

\[
G(a) = \{(A, p) \cdot G; q(A - I) < a\}.
\]

Then any two elements \((A, p)\) and \((B, q)\) of \( G(a) \) commute.

**Proof of Lemma 5.** By Lemma 4, \( q(A_k - I) \) and \( q(p_k) \) approach zero as \( k \) tends to infinity. Since G is discrete in \( \mathbb{R}^n \), there exists an integer \( k \) such that \( A_k = \text{Z} \) and \( p_k = 0 \). We show that the characteristic roots \( a_1, \ldots, a_n \) of an orthogonal matrix \( A \) with \( q(A - I) < \frac{1}{2} \) have positive real parts. If \( U \) is a unitary matrix such that \( UAU^{-1} \) is diagonal, then

\[
q(A - I) = q(U(A - I)U^{-1}) = q(UBU^{-1} - I)
\]

\[
= |a_1 - 1|^2 + \cdots + |a_n - 1|^2 < \frac{1}{2},
\]

which proves our assertion. By applying Lemma 2 to \( A_k^{-1}A_{k-1}^{-1}(I - B)p - (I - A)q \), we see that \( A_k^{-1} = I \). Continuing this argument, we obtain \( A_1 = I \). Thus \( A \) and \( B \) commute.

Hence,

\[
p_1 = (I - B)p - (I - A)q, p_2 = (A - I)p_1,
\]

\[
p_3 = (A - I)p_2 = (A - I)^2p_1,
\]

\[
\vdots
\]

\[
p_k = (A - I)p_{k-1} = (A - I)^{k-1}p_1.
\]

Since \( p_k = 0 \), we have

\[
(A - I)^{k-1}p_1 = 0.
\]

Changing the roles of \((A, p)\) and \((B, q)\) and noting that

\[
(B, q)(A, p)(B, q)^{-1}(A, p)^{-1} = (I, -p_1),
\]

we obtain

\[
(B - I)^{m-1}p_1 = 0 \quad \text{for some integer } m.
\]

Since \( A \) and \( B \) commute, there exists a unitary matrix \( U \) such that \( UAU^{-1} \) and \( UBU^{-1} \) are both diagonal. Set

\[
UAU^{-1} = \begin{pmatrix} a_1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & a_n \end{pmatrix}, \quad UBU^{-1} = \begin{pmatrix} b_1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & b_n \end{pmatrix},
\]

Then, from \( (A - I)^k p_1 = (A - I)^{k-1}((I - B)p - (I - A)q) = 0 \), we obtain

\[
(a_i - 1)^{k-1}(1 - b_i)r_i - (1 - a_i)s_i = 0, \quad i = 1, \ldots, n.
\]

Similarly, from

\[
(B - I)^m p_1 = (B - I)^{m-1}((I - B)p - (I - A)q) = 0,
\]

\[
(a_i - 1)^{m-1}(1 - b_i)r_i - (1 - a_i)s_i = 0, \quad i = 1, \ldots, n.
\]
we obtain
\[(b_i - 1)^{n-1}(1 - b_i)r_i - (1 - a_i)s_i = 0, \quad a = 1, \ldots, n.\]
Hence we have
\[(1 - b_i)r_i - (1 - a_i)s_i = 0, \quad i = 1, \ldots, n.\]
In other words, we have
\[(1 - b_i)r_i - (1 - a_i)s_i = 0, \quad i = 1, \ldots, n.\]
In other words, we have
\[\rho_1 = (I - B)p - (I - A)q = 0,\]
which completes the proof of Lemma 5.

If \((A, p) \in G(a)\) and \((B, q) \in G\), then \((B, q) (A, p) (B, q)^{-1} \in G(a)\).
Indeed,
\[\varphi(BAB^{-1} - I) = \varphi(B(A - I)B^{-1}) = \varphi(A - I) < a.\]
This shows that the group generated by \(G(a)\) is an invariant subgroup of \(G\). By Lemma 5, it is moreover abelian if \(a < \frac{1}{2}\).
A subset \(V\) of \(\mathbb{R}^n\) is called a Euclidean subspace if there exist an element \(x_0 \in \mathbb{R}^n\) and a vector subspace \(S\) of \(\mathbb{R}^n\) such that \(V = \{x + x_0; x \in S\}\). We say that a group \(G\) of Euclidean motions of \(\mathbb{R}^n\) is irreducible if \(\mathbb{R}^n\) is the only Euclidean subspace invariant by \(G\).

**Lemma 6.** If \(H\) is an abelian normal subgroup of an irreducible group \(G\) of Euclidean motions of \(\mathbb{R}^n\), then \(H\) contains pure translations only.

Proof of Lemma 6. Since \(H\) is abelian, we may assume, by applying an orthogonal change of basis of \(\mathbb{R}^n\) if necessary, that the elements \((A, p)\) of \(H\) are simultaneously reduced to the following form:
\[
A = \begin{pmatrix}
A_1 & 0 \\
. & . \\
. & . \\
A_k & I_{n-2k}
\end{pmatrix}, 
\rho = \begin{pmatrix}
\rho_1 \\
. \\
. \\
\rho_k
\end{pmatrix}, 
A_i = \begin{pmatrix}
\cos \alpha_i & \sin \alpha_i \\
-\sin \alpha_i & \cos \alpha_i
\end{pmatrix}, 
(i = 1, \ldots, k)
\]
where \(I_{n-2k}\) is the identity matrix of degree \(n - 2k\), each \(\rho_i\) is a vector with 2 components and \(\rho^*\) is a vector with \(n - 2k\) components. Moreover, for each \(i\), there exists an element \((A, p)\) of \(H\) such that \(A_i\) is different from the identity matrix \(I_2\) so that \(A_i - I_2\) is non-singular.

Our task is now to prove that \(k = 0\), i.e., \(A \equiv 1\), for all \((A, p) \in H\). Assuming \(k \geq 1\), we shall derive a contradiction.
For each \(i\), choose \((A, p) \in H\) such that \(A_i - I_2\) is non-singular and define a vector \(t_i\) with 2 components by
\[(A_i - I_2)t_i = \rho_i.\]
We shall show that
\[(B_i - I_2)t_i = \rho_i \quad \text{for all} \quad (B, q) \in H.\]
Since \((A, p)\) and \((B, q)\) commute, we have
\[A_iq_i + p_i = B_i\rho_i + q_i\]
or
\[(A_i - I_2)q_i = (B_i - I_2)p_i.\]
Hence we have
\[(B_i - I_2)t_i = (B_i - I_2)(A_i - I_2)^{-1}p_i = (A_i - I_2)^{-1}(B_i - I_2)p_i\]
\[= (A_i - I_2)^{-1}(A_i - I_2)q_i = q_i,\]
thus proving our assertion. We define a vector \(t \in \mathbb{R}^n\) by
\[t = \begin{pmatrix}
t_1 \\
. \\
. \\
t_k \\
0
\end{pmatrix}.
\]
We have now
\[(I_n, t)(A, p)(I_n, t)^{-1} = (I_n, t)(A, p)(I_n, -t)\]
where
\[q = (A, p - (A - I_n)t), \quad (A, p) \in H,\]
\[p - (A - I_n)t = \begin{pmatrix}
\rho_1 \\
. \\
. \\
\rho_k \\
\rho^*
\end{pmatrix}, \quad p^* = \begin{pmatrix}
0 \\
. \\
. \\
0 \\
\rho^*
\end{pmatrix}.\]
By translating the origin of $\mathbb{R}^n$ to $t$, we may now assume that the elements $(A, p)$ of $H$ are of the form

$$A = \begin{pmatrix} A_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k \\ 0 & \ldots & I_{n-2k} \end{pmatrix}, \quad p = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $V$ be the vector subspace of $\mathbb{R}^n$ consisting of all vectors whose first $2k$ components are zero. Then $V$ is invariant by all elements $(A, p)$ of $H$. We shall show that $V$ is also invariant by $G$. First we observe that $V'$ is precisely the set of all vectors which are left fixed by all $A$ where $(A, p) \in H$. Let $(C, r) \in G$. Since $H$ is a normal subgroup of $G$, for each $(A, p) \in H$, there exists an element $(B, q) \in H$ such that

$$(A, p) (C, r) = (C, r) (B, q).$$

If $v \in V$, then $ACv = CBv = Cv$. Since $Cv$ is left fixed by all $A$, it lies in $V$. Hence $C$ is of the form

$$C = \begin{pmatrix} C' & 0 \\ 0 & C'' \end{pmatrix},$$

where $C'$ and $C''$ are of degree $2k$ and $n - 2k$, respectively. To prove that the first $2k$ components of $r$ are zero, write

$$r = \begin{pmatrix} r_1 \\ \vdots \\ r_k \\ r' \end{pmatrix}.$$

For each $i$, let $(A, p)$ be an element of $H$ such that $A_i - I_2$ is non-singular. Applying the equality $(A, p) (C, r) = (C, r) (B, q)$ to the zero vector of $\mathbb{R}^n$ and comparing the $(2i - 1)$-th and $i$-th components of the both sides, we have

$$A_i r_i = r_i.$$

Since $\{A_i - I_2\}$ is non-singular, we obtain $r_i = 0$. Thus every element $(C, r)$ of $G$ is of the form

$$C = \begin{pmatrix} C' & 0 \\ 0 & C'' \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This shows that $V$ is invariant by $G$, thus contradicting the irreducibility of $G$. This completes the proof of Lemma 6.

**Lemma 7.** A group $G$ of Euclidean motions of $\mathbb{R}^n$ is irreducible if $\mathbb{R}^n/G$ is compact.

Proof of Lemma 7. Assuming that $G$ is not irreducible, let $V$ be a proper Euclidean subspace of $\mathbb{R}^n$ which is invariant by $G$. Let $x_0$ be any point of $V$ and let $L$ be a line through $x_0$ perpendicular to $V$. Let $x_1, x_2, \ldots, x_m, \ldots$ be a sequence of points on $L$ such that the distance between $x_0$ and $x_n$ is equal to $m$. Let $G(x_0)$ denote the orbit of $G$ through $x_0$. Since $G(x_0)$ is in $V$, the distance between $G(x_0)$ and $x_m$ is at least $m$ and, hence, is equal to $m$. Therefore the distance between the images of $x_0$ and $x_m$ in $\mathbb{R}^n/G$ by the projection $\mathbb{R}^n \to \mathbb{R}^n/G$ is equal to $m$. This means that $\mathbb{R}^n/G$ is not compact.

We are now in position to complete the proof of (4). Let $G$ and $G(a)$ be as in Lemma 5 and assume $\mathcal{N} < \frac{1}{2}$. Let $H$ be the group generated by $G(a)$; it is an abelian normal subgroup of $G$. Assume that $\mathbb{R}^n/G$ is compact. Lemmas 6 and 7 imply that $H$ contains nothing but pure, translations. On the other hand, since $G$ is discrete, $G/H$ is finite by construction of $G(a)$. Hence $\mathbb{R}^n/H$ is also compact and hence is a Euclidean torus. Let $N$ be the subgroup of $G$ consisting of all pure translations of $G$. Since $G(a)$ contains $N$, we have $N = H$. This proves that $\mathbb{R}^n/N$ is a Euclidean torus.

**Remark.** (4) means that the linear holonomy group of a compact flat Riemannian manifold $M = \mathbb{R}^n/G$ is isomorphic to $G/N$ and hence is finite.

Although (1), (2) and (3) are essentially in Auslander-Markus [1], we lay emphasis on affine holonomy groups. (4) was originally
proved by Bieberbach [1]. The proof given here was taken from Frobenius [1] and Zassenhaus [1].

Example 4.3. The linear holonomy group of a non-compact flat Riemannian manifold may not be finite. Indeed, fix an arbitrary irrational real number \( \lambda \). For each integer \( m \), we set

\[
A(m) = \begin{pmatrix} \cos \lambda m & \sin \lambda m & 0 \\ -\sin \lambda m & \cos \lambda m & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad p(m) = \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}.
\]

Then we set \( G = \{ (A(m), p(m)); m = 0, \pm 1, \pm 2, \ldots \} \). It is easy to see that \( G \) is a discrete subgroup of the group of Euclidean motions of \( \mathbb{R}^3 \) and acts freely on \( \mathbb{R}^3 \). The linear holonomy group of \( \mathbb{R}^3/G \) is isomorphic to the group \( \{ A(m); m = 0, \pm 1, \pm 2, \ldots \} \).

**COROLLARY 4.3.** A manifold \( M \) with a flat affine connection admits a Euclidean torus as a covering space if and only if \( M \) is a compact flat Riemannian manifold.

Proof. Let \( \hat{M} \) be a Euclidean torus which is a covering space of \( M \). By (3) of Theorem 4.2, \( \hat{M} \) is a covering space of \( M' \). Thus, \( M' \) is a compact, Euclidean cylinder and hence is a Euclidean torus. By (3) of Theorem 4.2, \( M \) is a compact flat Riemannian manifold. The converse is contained in (4) of Theorem 4.2. QED.

Example 4.4. In Example 4.2, set \( M = \mathbb{R}^2/G \). Let \( \hat{N} \) be the subgroup of \( G \) consisting of translations:

\[
(x, y) \rightarrow (x + m, y), \quad m = 0, \pm 1, \pm 2, \ldots
\]

Then the covering space \( \hat{M} \) defined in Theorem 4.2 is given by \( \mathbb{R}^2/\hat{N} \) in this case. Clearly, \( \hat{M} \) is an ordinary cylinder, that is, the direct product of a circle with a line.

The determination of the \( P \)-dimensional complete flat Riemannian manifolds is due to Killing [1,2], Klein [1,2] and H. Hopf [1]. We shall present here their results with an indication of the proof.

There are four types of two-dimensional complete flat Riemannian manifolds other than the Euclidean plane. We give the fundamental group (the first homotopy group) for each type, describing its action on the universal covering space \( \mathbb{R}^2 \) in terms of the Cartesian coordinate system \((x, y)\).

1. **Ordinary cylinder** (orientable)
   
   \[
   (x, y) \rightarrow (x + n, y),
   \]
   
   \[
   n = 0, \pm 1, \pm 2, \ldots
   \]

2. **Ordinary torus** (orientable)
   
   \[
   (x, y) \rightarrow (x + ma, y + mb),
   \]
   
   \[
   m, n = 0, \pm 1, \pm 2, \ldots
   \]
   
   \[
   a, b: \text{real numbers, } a \neq 0.
   \]

3. **M"uller-Schneider torus** (non-orientable)
   
   \[
   (x, y) \rightarrow (x + n, (-1)^m y),
   \]
   
   \[
   n = 0, \pm 1, \pm 2, \ldots
   \]
   
   \[
   b: \text{non-zero real number.}
   \]

Any two-dimensional complete flat Riemannian manifold \( M \) is isometric, up to a constant factor, to one of the above four types of surfaces.

The proof goes roughly as follows. By Theorem 4.2, the problem reduces to the determination of the discrete groups of motions acting freely on \( \mathbb{R}^2 \). Let \( G \) be such a discrete group. We first prove that every element of \( G \) which preserves the orientation of \( \mathbb{R}^2 \) is necessarily a translation. Set \( z = x + iy \). Then every orientation preserving motion of \( \mathbb{R}^2 \) is of the form

\[
z \rightarrow \varepsilon z + w,
\]

where \( \varepsilon \) is a complex number of absolute value 1 and \( w \) is a complex number. If we iterate the transformation \( z \rightarrow \varepsilon z + w \) \( r \) times, then we obtain the transformation

\[
Z \rightarrow \varepsilon^r z + (\varepsilon^{r-1} + \varepsilon^{r-2} + \cdots + 1)w.
\]

We see easily that, if \( \varepsilon \neq 1 \), then the point \( w/(1 - \varepsilon) \) is left fixed by the transformation \( Z \rightarrow \varepsilon Z + w \); in contradiction to the assumption that \( G \) acts freely on \( \mathbb{R}^2 \). Hence \( \varepsilon = 1 \), which proves our assertion. If \( f \) is an element of \( G \) which reverses the orientation of \( \mathbb{R}^2 \), then \( f^2 \) is an orientation preserving transformation and hence is a translation; We thus proved that every element of \( G \) is a transformation of the type
It is now easy to conclude that \( M \) must be one of the above four types of surfaces. The detail is left to the reader.
(1) $\nabla_X Y$ is $f$-related to $\nabla_{X'} Y'$, where $V$ denotes covariant differentiation both in $M$ and $M'$;

(2) $T(X, Y)$ is $f$-related to $T'(X', Y')$, where $T$ and $T'$ are the torsion tensor fields of $M$ and $M'$, respectively;

(3) $R(X, Y)Z$ is $f$-related to $R'(X', Y')Z'$, where $R$ and $R'$ are the curvature tensor fields of $M$ and $M'$, respectively.

Proof. (1) Let $x_0$ be an integral curve of $X$ such that $x = x_0$ at $x = 0$. Then (cf. §1 of Chapter III)

$$\lim_{t \to 0} \frac{1}{t} (\tau^v_t Y_z - Y_z) = (\nabla_X Y)_z.$$ 

Set $x'_0 = f(x_0)$ and let $\tau^v_0$ be the parallel displacement along this curve from $x_0$ to $x = x_0$. Then (cf. §1 of Chapter III)

$$\lim_{t \to 0} \frac{1}{t} (\tau^v_t Y_z - Y_z) = \lim_{t \to 0} \frac{1}{t} (\tau^v_t Y_z - Y_z) = (\nabla_X Y)_z.$$ 

(2) and (3) follow from (1) and Theorem 5.1 of Chapter III. QED.

A diffeomorphism $f$ of $M$ onto itself is called an affine transformation of $M$ if it is an affine mapping. Any transformation $f$ of $M$ induces a natural automorphism $\tilde{f}$ of the bundle $P(M, G)$; $\tilde{f}$ maps a frame $u = (X_1, \ldots, X_n)$ at $x \in M$ into the frame $\tilde{f}(u) = (fX_1, \ldots, fX_n)$ at $f(x) \in M$. Since $\tilde{f}$ is an automorphism of the bundle $P$, it leaves every fundamental vector field of $P$ invariant.

**Proposition 1.3.** (1) For every transformation $f$ of $M$, the induced automorphism $\tilde{f}$ of the bundle $P$ of linear frames leaves the canonical form $\theta$ invariant. Conversely, every fibre-preserving transformation of $P$ leaving $\theta$ invariant is induced by a transformation of $M$.

(2) If $f$ is an affine transformation of $M$, then the induced automorphism $\tilde{f}$ of $P$ leaves both the canonical form $\theta$ and the connection form $\omega$ invariant. Conversely, every fibre-preserving transformation of $P$ leaving both $\theta$ and $\omega$ invariant is induced by an affine transformation of $M$.

Proof. (1) Let $X^* \in T_0(P)$ and set $X = \pi(X^*)$ so that $X \in T_0(M)$, where $x = n(u)$. Then (cf. §2 of Chapter III)

$$\theta(X^*) = u^{-1}(X) \quad \text{and} \quad \theta(fX^*) = \tilde{f}(u)^{-1}(fX),$$

where the frames $u$ and $\tilde{f}(u)$ are considered as linear mappings of $\mathbb{R}^n$ onto $T_x(M)$ and $T_{\tilde{f}(x)}(M)$, respectively. It follows from the definition of $\tilde{f}$ that the following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\tilde{f}(u)} & \mathbb{R}^n \\
\downarrow_{u} & & \downarrow_{\tilde{f}(u)} \\
T_x(M) & \xrightarrow{T_{\tilde{f}(x)}} & T_{\tilde{f}(x)}(M).
\end{array}$$

Hence, $u^{-1}(X) = \tilde{f}(u)^{-1}(fX)$, thus proving that $\theta$ is invariant by $\tilde{f}$.

Conversely, let $F$ be a fibre-preserving transformation of $P$ leaving $\theta$ invariant. Let $f$ be the transformation of the base $M$ induced by $F$. We prove that $\tilde{f} = F$. We set $J = f^{-1} \circ F$. Then $J$ is a fibre-preserving transformation of $P$ leaving $\theta$ invariant. Moreover, $J$ induces the identity transformation on the base $M$.

Therefore, we have

$$u^{-1}(X) = \theta(X^*) = \theta(fX^*) = J(u)^{-1}(X) \quad \text{for} \quad X^* \in T_0(P).$$

This implies that $J(u) = u$, that is, $\tilde{f}(u) = f(u)$.

(2) Let $f$ be an affine transformation of $M$. The automorphism $\tilde{f}$ of $P$ maps the connection $\Gamma$ into a connection, say, $\tilde{f}(\Gamma)$, and the form $\tilde{f}^{*}\omega$ is the connection form of $\tilde{f}(\Gamma)$ (cf. Proposition 6.1 of Chapter II). From the definition of an affine transformation, we see that $f$-maps, for each $u \in P$, the horizontal subspace of $T_0(P)$ onto the horizontal subspace of $T_{\tilde{f}(u)}(P)$. This means that $	ilde{f}(\Gamma) = \Gamma$ and hence $\tilde{f}^{*}\omega = \omega$.

Conversely, let $F$ be a fibre-preserving transformation of $P$ leaving $\theta$ and $\omega$ invariant. By (1), there exists a transformation $f$ of $M$ such that $F = \tilde{f}$. Since $\tilde{f}$ maps every horizontal curve of $P$ into a horizontal curve of $P$, the transformation $f: T(M) \to T(M)$ maps every horizontal curve of $T(M)$ into a horizontal curve of $T(M)$, This means that $f: M \to M$ is an affine mapping thus completing the proof. QED.

Remark. Assume that $M$ is orientable. Then the bundle $P'$ consists of two principal fibre bundles, say $P'(M, G^0)$ and
Proposition 1.4. Let $\Gamma$ be a linear connection on $M$. For a transformation $f$ of $M$, the following conditions are mutually equivalent:

1. $f$ is an affine transformation of $M$;
2. $f^*\omega = \omega$, where $\omega$ is the connection form of $I'$ and $f$-is the transformation of $P$ induced by $f$;
3. $f$-leaves every standard horizontal vector field $B(\xi)$ invariant;
4. $f^*(\nabla_Y Z) = \nabla_{f^*Y}(f^*Z)$ for any vector fields $Y$ and $Z$ on $M$.

Proof. (i) The equivalence of (1) and (2) is contained in Proposition 1.3.

(ii) (2) $\implies$ (3). By Proposition 1.3, we have

$$\omega(B(\xi)) = 0,$$

(2) implies

$$0 = \omega(B(\xi)) = (f^*\omega)(B(\xi)) = \omega(f^*B(\xi)).$$

This means that $f^*\omega = \omega$.

(iii) (3) $\implies$ (2). The horizontal subspace at $u$ is given by the set of $B(\xi)_u$. Hence (3) implies that $f$-maps every horizontal subspace into a horizontal subspace. This means that $f^*\Gamma = \Gamma$ and hence $f^*\omega = \omega$.

(iv) (1) $\implies$ (4). This follows from Proposition 1.2.

(v) (4) $\implies$ (1). Let $Z$ be a parallel vector field along a curve $\tau = x_\tau$. Let $Y$ be the vector field along $\tau$ tangent to $\tau$, that is, $Y_{\dot{\tau}} = x_\tau$. We extend $Y$ and $Z$ to vector fields defined on $M$, which will be denoted by the same letters $Y$ and $Z$ respectively. (4) implies that $fZ$ is parallel along $f(\tau)$. This means that $f$ is an affine transformation.

The set of affine transformations of $M$, denoted by $\mathfrak{A}(M)$ or $\mathfrak{A}^{\Gamma}(\Gamma)$, forms a group. The set of all fibre-preserving transformations of $P$ leaving $\theta$ and $\omega$ invariant, denoted by $\mathfrak{A}(P)$, forms a group which is canonically isomorphic with $\mathfrak{A}(M)$. We prove that $\mathfrak{A}(M)$ is a Lie group by establishing that $\mathfrak{A}(P)$ is a Lie group with respect to the compact-open topology in $P$.

Theorem 1.5. Let $\Gamma$ be a linear connection on a manifold $M$ with a finite number of connected components. Then the group $\mathfrak{A}(M)$ of affine transformations of $M$ is a Lie transformation group with respect to the compact-open topology in $P$.

Proof. Let $\theta = (\theta^i)$ and $\omega = (\omega_{ik})$ be the canonical form and the connection form on $P$. We set

$$g(X^*, Y^*) = \sum_i \theta^i(X^*)\theta_i(Y^*) + \sum_{j,k} \omega_{jk}(X^*)\omega_{jk}(Y^*),$$

$$X^*, Y^* \in T^\pi(P).$$

Since the $n^2 + n$ 1-forms $\theta^i, \omega_{ik}, i, j, k = 1, \ldots, n$, form a basis of the space of covectors at every point $u$ of $P$ (cf. Proposition 2.6 of Chapter III), $g$ is a Riemannian metric on $P$ which is invariant by $\mathfrak{A}(P)$ by Proposition 1.3. The group of isometries of $P$ is a Lie transformation group of $P$ with respect to the compact-open topology by Theorem 4.6 and Corollary 4.9 of Chapter I (cf., also Theorem, 3.10 of Chapter IV). Since $41(P)$ is clearly a closed subgroup of the group of isometries of $P$, $\mathfrak{A}(P)$ is also a Lie transformation group of $P$.

QED.

2. Infinitesimal affine transformations

Throughout this section, $P(M, G)$ denotes the bundle of linear frames over a manifold $M$ so that $G = GL(n; \mathbb{R})$, where $n = \dim M$.

Every transformation $g$ of $M$ induces a transformation of $P$ in a natural manner. Correspondingly, every vector field $X$ on $M$ induces a vector field $\hat{X}$ on $P$, in a natural manner. More precisely, we prove:

Proposition 2.1. For each vector field $X$ on $M$, there exists a unique vector field $\hat{X}$ on $P$ such that

1. $\hat{X}$ is invariant by $R_u$ for every $u \in G$;
2. $L_{\hat{X}}\theta = 0$;
3. $\hat{X}$ is $n$-related to $X$, that is, $\pi(\hat{X}_u) = X_{\pi(u)}$ for every $u \in P$. 

QED.
Conversely, given a vector field \( X \) on \( P \) satisfying (1) and (2), there exists a unique vector field \( X \) on \( M \) satisfying (3).

We shall call \( \bar{X} \) the natural lift of \( X \).

Proof. Given a vector field \( X \) on \( M \) and a point \( x \in M \), let \( \varphi_i \) be a local 1-parameter group of local transformations generated by \( X \) in a neighborhood \( U \) of \( x \). For each \( i \), \( \varphi_i \) induces a transformation \( \tilde{\varphi}_i : \pi^{-1}(U) \rightarrow P \) and hence the induced vector field on \( P \), which will be denoted by \( \bar{X} \). Since \( \tilde{\varphi}_i \) commutes with \( R_x \) for every \( x \in G \), \( \bar{X} \) satisfies (1) (cf. Corollary 1.8 of Chapter I). Since \( \tilde{\varphi}_i \) preserves the form \( \theta \), \( \bar{X} \) satisfies (2). Finally, \( \pi \circ \tilde{\varphi}_i = \tilde{\varphi}_i \circ \pi \) implies (3).

To prove the uniqueness of \( \bar{X} \), let \( \bar{X}' \) be another vector field on \( P \) satisfying (1), (2) and (3). Let \( \bar{\varphi}_i \) be a local 1-parameter group of local transformations generated by \( \bar{X}' \). Then \( \bar{\varphi}_i \) commutes with every \( R_x \), \( x \in G \), and preserves the canonical form \( \theta \). By (1) of Proposition 1.3, it follows that \( \bar{\varphi}_i \) is induced by a local 1-parameter group of local transformations \( \varphi_i \) of \( M \). Because of (3), \( \varphi_i \) induces the vector field \( X \) on \( M \). Thus \( \varphi_i = \tilde{\varphi}_i \) and hence \( \bar{\varphi}_i = \tilde{\varphi}_i \), which implies that \( \bar{X}' = \bar{X} \).

Conversely, let \( \bar{X} \) be a vector field on \( P \) satisfying (1) and (2). For each \( x \in M \), choose a point \( u \in P \) such that \( \pi(u) = x \). We then set \( X_u = \pi(\bar{X}_u) \). Since \( \bar{X} \) satisfies (1), \( X_u \) is independent of the choice of \( u \) and thus we obtain a vector field \( X \) which satisfies (3).

The uniqueness of \( \bar{X} \) is evident.

Let \( \Gamma \) be a linear connection on \( M \). A vector field \( \tilde{X} \) on \( M \) is called an infinitesimal affine transformation of \( M \) if, for each \( x \in M \), a local 1-parameter group of local transformations \( \varphi_i \) of a neighborhood \( U \) of \( x \) into \( M \) preserves the connection \( \Gamma \), more precisely, if each \( \varphi_i : U \rightarrow M \) is an affine mapping, where \( U \) is provided with the affine connection \( \Gamma \) which is the restriction of \( \Gamma \) to \( U \).

**PROPOSITION 2.2.** Let \( \Gamma \) be a linear connection on \( M \). For a vector field \( \tilde{X} \) on \( M \), the following conditions are mutually equivalent:

1. \( \tilde{X} \) is an infinitesimal affine transformation of \( M \).
2. \( L_{\tilde{X}} \omega = 0 \), where \( \omega \) is the connection form of \( \Gamma \) and \( \tilde{X} \) is the natural lift of \( X \);
3. \( [\tilde{X}, B(\xi)] = 0 \) for every \( \xi \in \mathbb{R} \), where \( B(\xi) \) is the standard horizontal vector field corresponding to \( \xi \);
4. \( L_x \circ \nabla_Y - \nabla_Y \circ L_x = [\nabla_Y, X] \) for every vector field \( Y \) on \( M \).

Proof. Let \( \varphi_i \) be a local 1-parameter group of local transformations of \( M \) generated by \( X \) and let, for each \( i \), \( \tilde{\varphi}_i \) be a local transformation of \( P \) induced by \( \varphi_i \).

(i) \( (1) \rightarrow (2) \). By Proposition 1.4, \( \tilde{\varphi}_i \) preserves \( \omega \). Hence we have (2).

(ii) \( (2) \rightarrow (3) \). For every vector field \( X \), we have (Proposition 2.1)

\[
0 = \tilde{X}(t(\theta(x))) = (L_X \theta)(x) + \theta([\tilde{X}, B(x)]) = \theta([\tilde{X}, B(x)]),
\]

which means that \( [\tilde{X}, B(x)] \) is vertical. If \( L_X \omega = 0 \), then

\[
0 = \tilde{X}(t(\omega(x))) = (L_X \omega)(x) + \omega([\tilde{X}, B(x)]) = \omega([\tilde{X}, B(x)]),
\]

which means that \( [\tilde{X}, B(x)] \) is horizontal. Hence, \( [\tilde{X}, B(x)] = 0 \).

(iii) \( (3) \rightarrow (1) \). If \( [\tilde{X}, B(x)] = 0 \), then \( \tilde{\varphi}_i \) leaves \( B(\xi) \) invariant and thus maps the horizontal subspace \( \tilde{H}(\varphi_i(u)) \) at \( u \) into the horizontal subspace at \( \varphi_i(u) \), whenever \( \tilde{\varphi}_i(u) \) is defined. Therefore, \( \tilde{\varphi}_i \) preserves the connection \( \Gamma \) and \( \tilde{X} \) is an infinitesimal affine transformation of \( M \).

(iv) \( (1) \rightarrow (4) \). By Proposition 1.4, we have

\[
\varphi_i(\nabla_Y Z) = \nabla_{\varphi_i(Y)}(\varphi_i(Z))
\]

for any vector fields \( Y \) and \( Z \) on \( M \).

From the definition of Lie differentiation given in §3 of Chapter I, we obtain

\[
L_X \circ \nabla_Y Z = \lim_{t \to 0} \frac{1}{t} [\nabla_Y Z - \varphi_i(\nabla_Y Z)]
\]

\[
= \lim_{t \to 0} \frac{1}{t} [\nabla_Y Z - \nabla_{\varphi_i(Y)}Z] + \lim_{t \to 0} \frac{1}{t} [\nabla_{\varphi_i(Y)}Z - \nabla_{\varphi_i(Y)}(\varphi_i(Z))]
\]

\[
= -\nabla_{\varphi_i(Y)}Z + \nabla_Y \circ L_X Z = \nabla_{[X,Y]}Z + \nabla_Y \circ L_X Z.
\]

We thus verified the formula:

\[
L_X \circ \nabla_Y K - \nabla_Y \circ L_X K = \nabla_{[X,Y]}K,
\]

when \( K \) is a vector field. If \( K \) is a function, the above formula is evidently true. By the lemma for Proposition 3.3 of Chapter I, the formula holds for any tensor field \( K \).

(v) \( (4) \rightarrow (1) \). We fix a point \( x \in M \). We set

\[
V(t) = (\varphi_i(\nabla_Y Z))_t \quad \text{and} \quad W(t) = (\nabla_{\varphi_i(Y)}(\varphi_i(Z)))_t.
\]
For each $t$, both $V(t)$ and $W(t)$ are elements of $T_v(M)$. In view of Proposition 1.4, it is sufficient to prove that $V(t) = W(t)$. As in (iv), we obtain

$$dV(t)/dt = \varphi_1((L_X \circ \nabla_Y Z)_{\varphi^{-1}(u)}),$$
$$dW(t)/dt = \varphi_1((\nabla_{L_Y X} Z + L_X Z)_{\varphi^{-1}(u)}).$$

From our assumption we obtain $dV(t)/dt = dW(t)/dt$. On the other hand, we have evidently $V(0) = W(0)$. Hence, $V(t) = W(t)$. QED.

Let $a(M)$ be the set of infinitesimal affine transformations of $M$. Then $a(M)$ forms a subalgebra of the Lie algebra $X(M)$ of all vector fields on $M$. In fact, the correspondence $X \rightarrow \hat{X}$ defined in Proposition 2.1 is an isomorphism of the Lie algebra $X(M)$ of vector fields on $M$ into the Lie algebra $X(P)$ of vector fields on $P$. Let $a(P)$ be the set of vector fields $X$ on $P$ satisfying (1) and (2) of Proposition 2.1 and also (2) of Proposition 2.2. Since $L_{[X,X]} = L_X \circ L_X - L_X \circ L_X$ (cf. Proposition 3.4 of Chapter II), $a(P)$ forms a subalgebra of the Lie algebra $X(P)$. It follows that $a(M)$ is a subalgebra of $X(M)$ isomorphic with $a(P)$, under the correspondence $X \rightarrow \hat{X}$ defined in Proposition 2.1.

**Theorem 2.3.** If $M$ is a connected manifold with an affine connection, the Lie algebra $a(M)$ of infinitesimal affine transformations of $M$ is of dimension at most $n^2 + n$, where $n \equiv \dim M$. If $\dim a(M) = n^2 + n$, then $\Gamma$ is flat, that is, both the torsion and the curvature of $\Gamma$ vanish identically.

Proof. To prove the first statement it is sufficient to show that $a(P)$ is of dimension at most $n^2 + n$, since $a(M)$ is isomorphic with $a(P)$. Let $u$ be an arbitrary point of $P$. The following lemma implies that the linear mapping $f: a(P) \rightarrow T_u(P)$ defined by $f(X) = X_u$ is injective so that $\dim a(P) \leq \dim T_u(P) = n^2 + n$.

**Lemma.** If an element $X_u$ of $a(P)$ vanishes at some point of $P$, then it vanishes identically on $P$.

Proof of Lemma. If $X_u = 0$, then $X_u = 0$ for every $a \in G$ as 8 is invariant by $R_u$ (cf. Proposition 2.1). Let $F$ be the set of points $x = n(u) \in M$ such that $X_u = 0$. Then $F$ is closed in $M$. Since $M$ is connected, it suffices to show that $F$ is open. Assume $X_u = 0$. Let $b_t$ be a local 1-parameter group of local transformations generated by a standard horizontal vector field $B(\xi)$ in a neighborhood of $u$. Since $[X, B(\xi)] = 0$ by Proposition 2.2, $X$ is invariant by $b_t$ and hence $X_{b_t u} = 0$. In the definition of a normal coordinate system (cf. §8 of Chapter III), we saw that the points of the form $\pi(b u)$ cover a neighborhood of $x = \pi(u)$ when $\xi$ and $t$ vary. This proves that $F$ is open.

To prove the second statement, we assume that $\dim a(M) = n^2$. Let $u$ be an arbitrary point of $P$. Then the linear mapping $f: a(P) \rightarrow T_u(P)$ is an isomorphism of the Lie algebra $X(M)$ of vector fields on $M$ into the Lie algebra $X(P)$ of vector fields on $P$. Thus, we have evidently $V(0) = W(0)$. Hence, $V(t) = W(t)$.

$\varphi_1((L_X \circ \nabla_Y Z)_{\varphi^{-1}(u)}) = (L_X \circ \nabla_Y Z)_{\varphi^{-1}(u)} - \varphi_1((\nabla_{L_Y X} Z + L_X Z)_{\varphi^{-1}(u)})$, and hence $V(t) = W(t)$. QED.

**Theorem 2.4.** If $M$ is a connected manifold with an affine connection $\Gamma$, the Lie algebra $a(M)$ of infinitesimal affine transformations of $M$ is of dimension $n^2 + n$, where $n \equiv \dim M$. If $\dim a(M) = n^2 + n$, then $\Gamma$ is flat, that is, both the torsion and the curvature of $\Gamma$ vanish identically.

Proof. To prove the first statement it is sufficient to show that $a(P)$ is of dimension at most $n^2 + n$, since $a(M)$ is isomorphic with $a(P)$. Let $u$ be an arbitrary point of $P$. The following lemma implies that the linear mapping $f: a(P) \rightarrow T_u(P)$ defined by $f(X) = X_u$ is injective so that $\dim a(P) \leq \dim T_u(P) = n^2 + n$.

**Lemma.** If an element $X_u$ of $a(P)$ vanishes at some point of $P$, then it vanishes identically on $P$.

Proof of Lemma. If $X_u = 0$, then $X_u = 0$ for every $a \in G$ as 8 is invariant by $R_u$ (cf. Proposition 2.1). Let $F$ be the set of points $x = n(u) \in M$ such that $X_u = 0$. Then $F$ is closed in $M$. Since $M$ is connected, it suffices to show that $F$ is open. Assume $X_u = 0$. Let $b_t$ be a local 1-parameter group of local transformations generated by a standard horizontal vector field $B(\xi)$ in a neighborhood of $u$. Since $[X, B(\xi)] = 0$ by Proposition 2.2, $X$ is invariant by $b_t$ and hence $X_{b_t u} = 0$. In the definition of a normal coordinate system (cf. §8 of Chapter III), we saw that the points of the form $\pi(b u)$ cover a neighborhood of $x = \pi(u)$ when $\xi$ and $t$ vary. This proves that $F$ is open.

To prove the second statement, we assume that $\dim a(M) = n^2$. Let $u$ be an arbitrary point of $P$. Then the linear mapping $f: a(P) \rightarrow T_u(P)$ is an isomorphism of the Lie algebra $X(M)$ of vector fields on $M$ into the Lie algebra $X(P)$ of vector fields on $P$. Thus, we have evidently $V(0) = W(0)$. Hence, $V(t) = W(t)$.

$\varphi_1((L_X \circ \nabla_Y Z)_{\varphi^{-1}(u)}) = (L_X \circ \nabla_Y Z)_{\varphi^{-1}(u)} - \varphi_1((\nabla_{L_Y X} Z + L_X Z)_{\varphi^{-1}(u)})$, and hence $V(t) = W(t)$. QED.
Thus we have

\[ 0 = X_u(\Theta(B, B')) = A_u^*(\Theta(B, B')) \]

\[ = -\Theta_u(B, B') + \Theta_u(B, B') = \Theta_u(B, B') \]

showing that the torsion form vanishes.

Similarly, comparing the both sides of the equality:

\[ X_u(\Omega(B, B')) = A_u^*(\Omega(B', B')) \]

and letting A equal the identity matrix of g = \text{gl}(n; R), we see that the curvature form vanishes identically.

We now prove the following result due to Kobayashi [2].

**Theorem 2.4.** Let 17 be a complete linear connection on M. Then every infinitesimal affine transformation X of M is complete, that is, generates a global 1-parameter group of affine transformations of M.

Proof. It suffices to show that every element \( \overline{X} \) of \( a(P) \) is complete under the assumption that M is connected. Let \( u_0 \) be an arbitrary point of P and let \( \phi : U \to P, |t| < \delta \), be a local 1-parameter group of local transformations generated by \( \overline{X} \) (cf. Theorem 1.5 of Chapter I). We shall prove that \( \phi_t(u) \) is defined for every \( u \in P \) and \( |t| < \delta \). Then it follows that \( \overline{X} \) is complete.

By Proposition 6.5 of Chapter III, every standard horizontal vector field \( B(\xi) \) is complete since the connection is complete. Given any point \( u \in P \), there exist standard horizontal vector fields \( B(\xi_1), \ldots, B(\xi_k) \) and an element \( a \cdot G \) such that

\[ u = (b_1^1 \circ b_2^2 \circ \cdots \circ b_k^k u_0) a, \]

where each \( b_i^i \) is the 1-parameter group of transformations of P generated by \( B(\xi_i) \). In fact, the existence of normal coordinate neighborhoods (cf. Proposition 8.2 of Chapter III) and the connectedness of M imply that the point \( x = \pi(u) \) can be joined to the point \( x_0 = \pi(u_0) \) by a finite succession of geodesics. By Proposition 6.3 of Chapter III, every geodesic is the projection of an integral curve of a certain standard horizontal vector field. This means that by taking suitable \( B(\xi_1), \ldots, B(\xi_k) \), we obtain a point \( v = (b_1^1 \circ b_2^2 \circ \cdots \circ b_k^k u_0) a \) which lies in the same fibre as \( u \). Then \( u = v a \) for a suitable \( a \in G \), thus proving our assertion. We then define \( \phi_t(u) \) by:

\[ \phi_t(u) = (b_1^1 \circ b_2^2 \circ \cdots \circ b_k^k(\phi_t(u_0))) a, \]

where \( |t| < \delta \).

The fact that \( \phi_t(u) \) is independent of the choice of \( b_1^1, \ldots, b_k^k \) and that \( \phi_t \) is generated by \( \overline{X} \) follows from (1) of Proposition 2.1 and (3) of Proposition 2.2; note that (3) of Proposition 2.2 implies that \( b_i \circ \phi_t(u) = \phi_t \circ b_i(u) \) whenever they are both defined.

QED.

In general, every element of the Lie algebra of the group \( \mathfrak{A}(M) \) of affine transformations of M gives rise to an element of \( a(M) \) which is complete, and conversely. In other words, the Lie algebra of \( \mathfrak{A}(M) \) can be identified with the subalgebra of \( a(M) \) consisting of complete vector fields. Theorem 2.4 means that if the connection is complete, then \( a(M) \) can be considered as the Lie algebra of \( \mathfrak{A}(M) \).

For any vector field X on M, the derivation \( A_X = L_X - \nabla_X \) is induced by a tensor field of type (1, 1) because it is zero on the function algebra \( \mathfrak{g}(M) \) (cf. the proof of Proposition 3.3 of Chapter I). This fact may be derived also from the following

**Proposition 2.5.** For any vector fields X and Y on M, we have

\[ A_X Y = -\nabla_Y X - T(X, Y), \]

where T is the torsion.

Proof. By Theorem 5.1 of Chapter III, we have

\[ A_X Y = L_X Y - \nabla_Y X = [X, Y] - (\nabla_Y X + [X, Y] + T(X, Y)) \]

\[ = -\nabla_Y X - T(X, Y). \]

QED.

We conclude this section by

**Proposition 2.6.** (1) A vector field X on M is an infinitesimal affine transformation if and only if

\[ \nabla_Y A_X = R(X, Y) \]

for every vector field Y on M.

(2) If both X and Y are infinitesimal affine transformations of M, then

\[ A_{[X,Y]} = [A_X, A_Y] + R(X, Y), \]

where R denotes the curvature.

Proof. (1) By Theorem 5.1 of Chapter III, we have

\[ R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = [L_X - A_X, \nabla_Y] - \nabla_{[X,Y]} \]

\[ = [CL_X, \nabla_Y] - \nabla_{[X,Y]} = [A_X, \nabla_Y]. \]
By Proposition 2.2, $X$ is an infinitesimal affine transformation if and only if $R(X, Y) = -[A_x, \nabla_Y]$ for every $Y$, that is, if and only if

$$R(X, Y)Z = \nabla_Y(A_XZ) - A_X(\nabla_YZ) = (\nabla_Y(A_X))Z$$

for all $Y$ and $Z$.

(2) By Theorem 5.1 of Chapter III and Proposition 2.2, we have

$$[A_X, A_Y] = [-L_x, L_y] = [L_x - \nabla_x, L_y - \nabla_y] = (L(x, y) - \nabla(x, y))$$

$$= [L_x, L_y] - [\nabla_x, L_y] + [\nabla_x, L_y] - [L_x, \nabla_y]$$

$$= R(X, Y).$$

QED.

3. Isometries and infinitesimal isometries

Let $M$ be a manifold with a Riemannian metric $g$ and the corresponding Riemannian connection $\Gamma$. An isometry of $M$ is a transformation of $M$ which leaves the metric $g$ invariant. We know from Proposition 2.5 of Chapter IV that an isometry of $M$ is necessarily an affine transformation of $M$ with respect to $\Gamma$.

Consider the bundle $O(M)$ of orthonormal frames over $M$ which is a subbundle of the bundle $L(M)$ of linear frames over $M$. We have

**Proposition 3.1.** (1) A transformation $f$ of $M$ is an isometry if and only if the induced transformation $\tilde{f}$ of $L(M)$ maps $O(M)$ into itself;

(2) A fibre-preserving transformation $F$ of $O(M)$ which leaves the canonical form on $O(M)$ invariant is induced by an isometry of $M$.

Proof. (1) This follows from the fact that a transformation $f$ of $M$ is an isometry if and only if it maps each orthonormal frame at an arbitrary point $x$ into an orthonormal frame at $f(x)$.

(2) Let $f$ be the transformation of the base $M$ induced by $F$. We set $J = f^{-1} \circ F$. Then $J$ is a fibre-preserving mapping of $O(M)$ into $L(M)$ which preserves $\theta$. Moreover, $J$ induces the identity transformation on the base $M$, Therefore we have

$$A_x(JX) = 0(X) = 0(JX) = J(u)(X) = u^*(X)$$

for $X \in T_u(O(M))$, $X = \pi(X^*)$.

By Corollary 3.7, moreover, $\Delta_x 1$.

Moreover, $\Delta_x$ from Corollary 3.7 with respect to $g$ everywhere on $M$, that is, $g(X, Y) = -[X, Y]$ for all $X$ and $Y$.

**Proposition 3.2.** For a vector field $X$ on a Riemannian manifold $M$, the following conditions are mutually equivalent:

(1) $X$ is an infinitesimal isometry;

(2) The natural lift $\tilde{X}$ of $X$ to $O(M)$ is tangent to $O(M)$ at every point of $O(M)$;

(3) $\Delta X = 0$, where $g$ is the metric tensor field of $M$;

(4) The tensor field $A_X = L_X - V^X$ of type $(1, 1)$ is skew-symmetric with respect to $g$ everywhere on $M$, that is, $g(A(X, Y)) = -g(A(X, Y))$ for arbitrary vector fields $X$ and $Y$.

Proof. (i) To prove the equivalence of (1) and (2), let $q_t$ and $q_t'$ be the local $I$-parameter groups of local transformations generated by $X$ and $Y$ respectively. If $X$ is an infinitesimal isometry, then $q_t$ are local isometries and hence $q_t$ maps $O(M)$ into itself. Thus $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$.

Conversely, if $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$, the integral curve of $\tilde{X}$ through each point of $O(M)$ is contained in $O(M)$ and hence each $q_t$ maps $O(M)$ into itself. This means, by Proposition 3.1; that each $q_t$ is a local isometry and hence $X$ is an infinitesimal isometry.

(ii) The equivalence of (1) and (3) follows from Corollary 3.7 of Chapter I.

(iii) Since $\nabla X g = 0$ for any vector field $X$, $L_X g = 0$ is equivalent to $A_X g = 0$. Since $A_X$ is a derivation of the algebra of tensor fields, we have

$$A_X(g(Y, Z)) = (A_X g)(Y, Z) + g(A_X Y, Z) + g(Y, A_X Z)$$

for $Y, Z \in \mathfrak{X}(M)$.

This implies that $f(u) = u$, that is, $f(u) = f(u)$. By (1), $f$ is an isometry of $M$.

A vector field $X$ on $M$ is called an infinitesimal isometry (or, a Killing vector field) if the local $I$-parameter group of local transformations generated by $X$ in a neighborhood of each point of $M$ consists of local isometries. An infinitesimal isometry is necessarily an infinitesimal affine transformation.

**Proposition 3.3.** For a vector field $X$ on a Riemannian manifold $M$, the following conditions are mutually equivalent:

(1) $X$ is an infinitesimal isometry;

(2) The natural lift $\tilde{X}$ of $X$ to $O(M)$ is tangent to $O(M)$ at every point of $O(M)$;

(3) $\Delta X = 0$, where $g$ is the metric tensor field of $M$;

(4) The tensor field $A_X = L_X - V^X$ of type $(1, 1)$ is skew-symmetric with respect to $g$ everywhere on $M$, that is, $g(A(X, Y)) = -g(A(X, Y))$ for arbitrary vector fields $X$ and $Y$.

Proof. (i) To prove the equivalence of (1) and (2), let $q_t$ and $q_t'$ be the local $I$-parameter groups of local transformations generated by $X$ and $Y$ respectively. If $X$ is an infinitesimal isometry, then $q_t$ are local isometries and hence $q_t$ maps $O(M)$ into itself. Thus $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$.

Conversely, if $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$, the integral curve of $\tilde{X}$ through each point of $O(M)$ is contained in $O(M)$ and hence each $q_t$ maps $O(M)$ into itself. This means, by Proposition 3.1; that each $q_t$ is a local isometry and hence $X$ is an infinitesimal isometry.

(ii) The equivalence of (1) and (3) follows from Corollary 3.7 of Chapter I.

(iii) Since $\nabla X g = 0$ for any vector field $X$, $L_X g = 0$ is equivalent to $A_X g = 0$. Since $A_X$ is a derivation of the algebra of tensor fields, we have

$$A_X(g(Y, Z)) = (A_X g)(Y, Z) + g(A_X Y, Z) + g(Y, A_X Z)$$

for $Y, Z \in \mathfrak{X}(M)$.

This implies that $f(u) = u$, that is, $f(u) = f(u)$. By (1), $f$ is an isometry of $M$.

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**Proposition 3.4.** For a vector field $X$ on a Riemannian manifold $M$, the following conditions are mutually equivalent:

(1) $X$ is an infinitesimal isometry;

(2) The natural lift $\tilde{X}$ of $X$ to $O(M)$ is tangent to $O(M)$ at every point of $O(M)$;

(3) $\Delta X = 0$, where $g$ is the metric tensor field of $M$;

(4) The tensor field $A_X = L_X - V^X$ of type $(1, 1)$ is skew-symmetric with respect to $g$ everywhere on $M$, that is, $g(A(X, Y)) = -g(A(X, Y))$ for arbitrary vector fields $X$ and $Y$.

Proof. (i) To prove the equivalence of (1) and (2), let $q_t$ and $q_t'$ be the local $I$-parameter groups of local transformations generated by $X$ and $Y$ respectively. If $X$ is an infinitesimal isometry, then $q_t$ are local isometries and hence $q_t$ maps $O(M)$ into itself. Thus $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$.

Conversely, if $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$, the integral curve of $\tilde{X}$ through each point of $O(M)$ is contained in $O(M)$ and hence each $q_t$ maps $O(M)$ into itself. This means, by Proposition 3.1; that each $q_t$ is a local isometry and hence $X$ is an infinitesimal isometry.

(ii) The equivalence of (1) and (3) follows from Corollary 3.7 of Chapter I.

(iii) Since $\nabla X g = 0$ for any vector field $X$, $L_X g = 0$ is equivalent to $A_X g = 0$. Since $A_X$ is a derivation of the algebra of tensor fields, we have

$$A_X(g(Y, Z)) = (A_X g)(Y, Z) + g(A_X Y, Z) + g(Y, A_X Z)$$

for $Y, Z \in \mathfrak{X}(M)$.
The set of all infinitesimal isometries of $M$, denoted by $\mathfrak{I}(M)$, forms a Lie algebra. In fact, if $X$ and $Y$ are infinitesimal isometries of $M$, then

$$L_{[X,Y]}g = L_X \circ L_Y g - L_Y \circ L_X g = 0$$

by Proposition 3.2. By the same proposition, $[X, Y]$ is an infinitesimal isometry of $M$.

**Theorem 3.3.** The Lie algebra $\mathfrak{I}(M)$ of infinitesimal isometries of a connected Riemannian manifold $M$ is of dimension at most $\frac{1}{2}n(n+1)$, where $n \approx \dim M$. If $\dim \mathfrak{I}(M) = \frac{1}{2}n(n+1)$, then $M$ is a space of constant curvature.

Proof. To prove the first assertion, it is sufficient to show that, for any point $y$ of $O(M)$, the linear mapping $X \mapsto X_y$ maps $\mathfrak{I}(M)$ 1:1 into $T_y(O(M))$. By Proposition 3.2, $X_y$ is certainly an element of $T_y(O(M))$. If $X_y = 0$, then the proof of Theorem 23 shows that $X = 0$. We now prove the second assertion.

Let $X$, $X'$ be an orthonormal basis of a plane $p$ in $T_x(M)$ and let $u$ be a point of $O(M)$ such that $p(u) = x$. We set $\xi = u^{-1}(X)$, $\xi' = u^{-1}(X')$, $B = B(\xi)$ and $B' = B(\xi')$, where $B(5)$ and $B(5')$ are the restrictions to $O(M)$ of the standard horizontal vector fields corresponding to $\xi$ and $\xi'$, respectively. From the definition of the curvature transformation given in §5 of Chapter III we see that the sectional curvature $K(j)$ (cf. 92 of Chapter V) is given by

$$K(p) = (\Omega(B_{(u, B')})a, e),$$

where $(\ , \ )$ denotes the natural inner product in $\mathbb{R}^n$. To prove that $K(p)$ is independent of $p$, let $Y$, $Y'$ be an orthonormal basis of another plane $q$ in $T_x(M)$ and set $\eta = u^{-1}(Y)$ and $\eta' = u^{-1}(Y')$. Let $a$ be an element of $SO(n)$ such that $a\xi = -\eta$ and $a\xi' = -\eta'$. By Proposition 2.2 of Chapter III, we have

$$\Omega(B(\eta, \xi), B(\eta', \xi')) = \Omega(B_{(u, B')}, B_{(u', B')}) = \text{ad}(a) \Omega(B_{(u, B')}, B_{(u', B')}) = \Omega(B_{(u, B')}, B_{(u', B')}).$$

Hence the sectional curvature $K(q)$ is given by

$$K(q) = ((2\Omega(B(\eta, \xi), B(\eta', \xi'))a, \eta)) = ((2\Omega(B_{(u, B')}, B_{(u', B')}a^{-1})a\xi', a\xi) = ((2\Omega(B_{(u, B')}, B_{(u', B')})a^{-1})a\xi', a\xi).$$

To prove that $K(p) = K(q)$, it is sufficient to show that $\Omega(B_{(u, B')}, B_{(u', B')}) = \Omega(B_{(u, B')}, B_{(u', B')})$ for every $a \in SO(n)$. We thus proved that $K(p)$ depends only on the point $x$. We prove that $K(p)$ does not depend even on $x$. Given any vector $Y \in T_x(O(M))$, let $Y$ be an element of $i(M)$ such that $Y_u = Y^*$. We have again $\Omega(B(\eta, \xi')) = 0$. Hence, for fixed $\xi$ and $\xi'$, the function $(\Omega(B(\eta, \xi')))$ is constant in a neighborhood of $u$. This means that $K(p)$, considered as a function on $M$, is locally constant.

Since it is continuous and $M$ is connected, it must be constant on $M$. (If $\dim M \geq 3$, the fact that $K(p)$ is independent of $x$ follows also from Theorem 2.2 of Chapter V.)

QED.

**Theorem 3.4.** (1) For a Riemannian manifold $M$ with a finite number of connected components, the group $\mathfrak{S}(M)$ of isometries of $M$ is a Lie transformation group with respect to the compact-open topology in $M$.

(2) The Lie algebra of $\mathfrak{S}(M)$ naturally isomorphic with the Lie algebra of all complete infinitesimal isometries.

(3) The isotropy subgroup $\mathfrak{S}(M)$ of $\mathfrak{S}(M)$ at an arbitrary point $x$ is compact.

(4) If $M$ is complete, the Lie algebra of $\mathfrak{S}(M)$ is naturally isomorphic with the Lie algebra $\mathfrak{I}(M)$ of all infinitesimal isometries of $M$.

(5) If $M$ is compact, then the group $\mathfrak{S}(M)$ is compact.

Proof. (1) As we indicated in, the proof of Theorem 1.5, this follows from Theorem 4.6 and Corollary 4.9 of Chapter I and Theorem 3.10 of Chapter IV.

(2) Every 1-parameter subgroup of $\mathfrak{S}(M)$ induces an infinitesimal isometry $X$ which is complete on $M$ and, conversely, every complete infinitesimal isometry $X$ generates a 1-parameter subgroup of $\mathfrak{S}(M)$.

(3) This follows from Corollary 4.8 of Chapter I.

(4) This follows from (2) and Theorem 2.4.

(5) This follows from Corollary 4.10 of Chapter I.

QED.

Clearly, $\mathfrak{S}(M)$ is a closed subgroup of $\mathfrak{I}(M)$. We shall see that,
in many instances, the identity component \( \mathfrak{g}_0(M) \) of \( 3(M) \) coincides with the identity component \( \mathfrak{a}_0(M) \) of \( \mathfrak{a}(M) \). We first prove a result by Hano [1].

**Theorem 3.5.** If \( M = M_0 \times M_1 \times \cdots \times M_k \) is the de Rham decomposition of a complete, simply connected Riemannian manifold \( M \), then

\[
\mathfrak{a}_0(M) \cong \mathfrak{a}_0(M_0) \times \mathfrak{a}_0(M_1) \times \cdots \times \mathfrak{a}_0(M_k),
\]

\[
\mathfrak{g}_0(M) \cong \mathfrak{g}_0(M_0) \times \mathfrak{g}_0(M_1) \times \cdots \times \mathfrak{g}_0(M_k).
\]

Proof. We need the following two lemmas.

**Lemma 1.** Let \( T(M) = \sum_{i=0}^{k} T_i^{(0)} \) be the canonical decomposition:

\( \mathfrak{a}(M) \cong \mathfrak{a}_0(M) \times \mathfrak{r}(M) \) and for each \( i, 1 \leq i \leq k \),

\( T_i^{(0)}(x) = T_i^{(0)}(y) \) for some \( i \).

(2) If \( \phi \in \mathfrak{g}(M) \), then \( \phi(T_i^{(0)}(x)) = T_i^{(0)}(y) \) for every \( i, 0 \leq i \leq k \).

Proof of Lemma 1. Let \( \tau \) be any loop at \( x \) and set \( \tau = \phi(\tau) \) so that \( \tau \) is a loop at \( \phi(x) \). If we denote by the same letter \( \tau \) and \( \tau \) the parallel displacements along \( \tau \) and \( \tau \) respectively, then

\[
\phi \circ \tau(X) = \tau' \circ \phi(X) \quad \text{for} \quad X \in T_x(M).
\]

It follows easily that \( \phi(T_i^{(0)}) \) is invariant elementwise and every \( \phi(T_i^{(0)}) \), \( 1 \leq i \leq k \), is irreducible by the linear holonomy group \( \mathfrak{g}(\phi(x)) \). Hence, \( \phi(T_i^{(0)}(x)) \subseteq T_i^{(0)}(y) \) and, their dimensions being the same, \( \phi(T_i^{(0)}(x)) = T_i^{(0)}(y) \) for \( 1 \leq i \leq k \).

Thus we obtain the canonical decomposition \( T_i^{(0)}(x) = \sum_{j=0}^{k} \phi(T_j^{(0)}(x)) \) which should coincide with the canonical decomposition \( T_i^{(0)}(x) = \sum_{j=0}^{k} \phi(T_j^{(0)}(x)) \) up to an \( \phi \)-order by (4) of Chapter IV. This means precisely the statement (1). Let \( \phi \) be a \( 1 \)-parameter subgroup of \( \mathfrak{a}_0(M) \) and let \( X \) be a non-zero element of \( T_i^{(0)} \). Let \( \tau = \phi_{\delta} \) be the parallel displacement from \( x_0 \) to \( x_t \) along \( \tau \). This means that

\[
\phi_{\delta}(T_i^{(0)}(x_0)) = T_i^{(0)}(x_t) \quad \text{for} \quad |t| < \delta
\]

for some positive number \( \delta \); in fact, if \( X_1, \ldots, X_r \) is a basis for \( T_i^{(0)} \), then

\[
\phi_{\delta}(X_j) = \delta X_j \neq 0 \quad \text{for} \quad 1 \leq j \leq r \quad \text{and} \quad |t| < \delta
\]

for some positive number \( \delta \) and hence \( \phi_{\delta}(X_j) \in T_i^{(0)}(x_t) \) for \( |t| < \delta \), which implies \( \phi_{\delta}(T_i^{(0)}(x_0)) = T_i^{(0)}(x_t) \) for \( |t| < \delta \) because of the linearity of \( \phi_{\delta} \). This concludes the proof of the statement (2), since \( \mathfrak{a}_0(M) \) is generated by \( 1 \)-parameter subgroups.

Lemma 1 is due to Nomizu [3].

**Lemma 2.** Let \( \phi \) be an arbitrary transformation of \( M_i \) for every \( i, 0 \leq i \leq k \). Let \( \phi \) be the transformation of \( M = M_0 \times M_1 \times \cdots \times M_k \) defined by

\[
\phi(x) = (\phi_0(x_0), \phi_1(x_1), \ldots, \phi_k(x_k)) \quad \text{for} \quad x = (x_0, x_1, \ldots, x_k) \in M.
\]

Then

(1) \( \phi \) is an affine transformation of \( M \) if and only if every \( \phi_i \) is an affine transformation of \( M_i \).

(2) \( \phi \) is an isometry of \( M \) if and only if every \( \phi_i \) is an isometry of \( M_i \).

The proof is trivial.

The correspondence \( (\phi_0, \phi_1, \ldots, \phi_k) \rightarrow \phi \) defined in Lemma 2 maps \( \mathfrak{a}(M) \) isomorphically into \( \mathfrak{a}(M) \). To complete the proof of Theorem 3.5, it suffices to show that, for every \( \phi \in \mathfrak{g}(M) \), there exist transformations \( \phi_i : M_i \rightarrow M_i, 0 \leq i \leq k \), such that

\[
\phi(x) = (\phi_0(x_0), \phi_1(x_1), \ldots, \phi_k(x_k)) \quad \text{for} \quad x = (x_0, x_1, \ldots, x_k) \in M.
\]

We prove that, if \( \phi_i : M_i \rightarrow M_i \) denotes the natural projection, then \( \phi_i \) depends only on \( x_i \). Given any Point. \( y = (y_0, y_1, \ldots, y_{k-1}, y_k) \), let, for each \( j = 1, \ldots, k \), \( y_j = y_j \), \( \phi_i = \phi_i \), \( 0 \leq t \leq 1 \).

For each \( t \), the tangent vector \( \phi(y) \) of \( \tau \) at \( y \) is in the distribution \( T^{(0)} + \cdots + T^{(i-1)} + T^{(i+1)} + \cdots + T^{(k)} \). By Lemma 1, \( \phi(y) \) lies also in the same distribution. Hence \( \phi_i \) is independent of \( \phi_i \) (cf. Lemma 2 for Theorem 7.2 of Chapter II). In particular, \( \phi_i \) is an isometry of \( M \) if and only if \( \phi_i \) is an isometry of \( M_i \).

Hence we can define a transformation \( \phi_* : M_i \rightarrow M_i \) by

\[
\phi_i(x_i) = \phi_i(x_i).
\]

Clearly, we have

\[
\phi(x) = (\phi_0(x_0), \phi_1(x_1), \ldots, \phi_k(x_k)).
\]

QED.

It is therefore important to study \( \mathfrak{a}(M) \) when \( M \) is irreducible. The following result is due to Kobayashi [4].
A transformation \( \varphi \) of a Riemannian manifold is said to be homothetic if there is a positive constant \( c \) such that \( g(\varphi(X), \varphi(Y)) = c g(X, Y) \) for all \( X, Y \in T_x(M) \) and \( x \in M \). Consider the Riemannian metric \( g^* \) defined by \( g^*(X, Y) = g(\varphi(X), \varphi(Y)) \).

From the proof (B) of Theorem 2.2 of Chapter III, we see that the Riemannian connection defined by \( g^* \) coincides with the one defined by \( g \). This means that every homothetic transformation of a Riemannian manifold \( M \) is an affine transformation of \( M \).

**Lemma 1.** If \( M \) is an irreducible Riemannian manifold, then every affine transformation \( \varphi \) of \( M \) is homothetic.

Proof of Lemma 1. Since \( \varphi \) is an affine transformation, the two Riemannian metrics \( g \) and \( g^* \) (defined above) determine the same Riemannian connection, say \( \Gamma \). Let \( \Psi(x) \) be the linear holonomy group of \( \Gamma \) with reference point \( x \). Since it is irreducible and leaves both \( g \) and \( g^* \) invariant, there exists a positive constant \( c_2 \) such that \( g^*(X, Y) = c_2^2 g(X, Y) \) for all \( X, Y \in T_x(M) \), that is, \( g^* = c_2^2 g \) (cf. Theorem 1 of Appendix 5). Since both \( g^* \) and \( g \) are parallel tensor fields with respect to \( \Gamma \), \( c_2 \) is constant.

**Lemma 2.** If \( A \) is a complete Riemannian manifold which is not locally Euclidean, then every homothetic transformation \( \varphi \) of \( M \) is an isometry.

Proof of Lemma 2. Assume that \( \varphi \) is a non-isometric homothetic transformation of \( M \). Considering the inverse transformation if necessary, we may assume that the constant \( c \) associated with \( \varphi \) is less than 1. Take, an arbitrary point \( x \) of \( M \). If the distance between \( x \) and \( \varphi(x) \) is less than \( d \), then the distance between \( \varphi^m(x) \) and \( \varphi^{m+1}(x) \) is less than \( c^m d \). It follows that \( \{ \varphi^m(x) : m = 1, 2, \ldots \} \) is a Cauchy sequence and hence converges to some point, say \( x^* \), since \( M \) is complete. It is easy to see that the point \( x^* \) is left fixed by \( \varphi \).

Let \( U \) be a neighborhood of \( x \) such that \( \varphi \) is compact. Let \( K^* \) be a positive number such that \( |g(R(Y_1, Y_2, Y_3, Y_4))| < K^* \) for any unit vectors \( Y_1 \) and \( Y_2 \) at \( y \in U \), where \( R \) denotes the curvature tensor field. Let \( z \in M \) and \( q \) any plane in \( T_z(M) \). Let \( X, Y \) be an orthonormal basis for \( q \). Since \( \varphi \) is an affine transformation, (3) of Proposition 1.2 implies that

\[
R(q^m X, q^m Y) (q^m Y) = q^m (R(X, Y) Y). 
\]

Hence we have

\[
g(R(q^m X, q^m Y) (q^m Y), q^m X) = g(q^m (R(X, Y) Y), q^m X) = c^m g(R(X, Y) Y, X) = c^m K(q). 
\]

On the other hand, the distance between \( q^m(x^*) \) and \( q^m(z) \) approaches 0 as \( m \) tends to infinity. In other words, there exists an integer \( m_0 \) such that \( q^m(z) \in U \) for every \( m \geq m_0 \). Since the lengths of the vectors \( q^m X \) and \( q^m Y \) is equal to \( c^m \), we have

\[
2^m K^* \geq |g(R(q^m X, q^m Y)(q^m Y), q^m X)| \quad \text{for } m \geq m_0. 
\]

Thus we obtain

\[
2^m K^* \geq |K(q)| \quad \text{for } m \geq m_0. 
\]

Letting \( m \) tend to infinity, we have \( K(q) = 0 \). This shows that \( M \) is locally Euclidean.

Let \( X \) be an infinitesimal affine transformation on a complete Riemannian manifold \( M \). Using Theorems 3.5 and 3.6, we shall find a number of sufficient conditions for \( X \) to be an infinitesimal isometry. Assuming that \( M \) is connected, let \( \tilde{M} \) be the universal covering manifold with the naturally induced Riemannian metric \( \tilde{g} = p^*(g) \), where \( p : \tilde{M} \rightarrow M \) is the natural projection. Let \( \tilde{X} \) be the vector field on \( \tilde{M} \) induced by \( X \); \( \tilde{X} \) is \( p \)-related to \( X \). Then \( \tilde{X} \) is an infinitesimal affine transformation of \( \tilde{M} \). Clearly, \( \tilde{X} \) is an infinitesimal isometry of \( \tilde{M} \) if and only if \( X \) is an infinitesimal isometry of \( M \). Let \( \tilde{M} = M_0 \times M_1 \times \cdots \times M_k \) be the de Rham decomposition of the complete simply connected Riemannian manifold \( \tilde{M} \). By Theorem 3.5, the Lie algebra \( \tilde{a}(\tilde{M}) \) is isomorphic with \( a(M_0) + a(M_1) + \cdots + a(M_k) \). Let \( (X_0, X_1, \ldots, X_k) \) be the element of \( a(M_0) + a(M_1) + \cdots + a(M_k) \) corresponding to \( \tilde{X} \in \tilde{a}(\tilde{M}) \). Since \( X_1, \ldots, X_k \) are all infinitesimal isometries by Theorem 3.6, \( X \) is an infinitesimal isometry if and only if \( X_0 \) is.

**Corollary 3.7.** If \( M \) is a connected, complete Riemannian manifold whose linear holonomy group \( \Psi(x) \) leaves no non-zero vector at \( x \) fixed, then \( \Psi(M) = \mathcal{S}(M) \).

Proof. The linear holonomy group of \( M \) is naturally isomorphic
Let $x$ be an arbitrary point of $M$ and let $q_t$ be a local 1-parameter group of affine transformations generated by $X$ in a neighborhood of $x$. Let $\tau$ be the orbit $x_t = q_t(x)$ of $x$. We denote by $\tau_t$ the parallel displacement along the curve $\tau$ from $x_0$ to $x_t$. For each $t$, we consider a linear transformation $C_t = \tau_0^{-1} \circ (q_t)_* \circ \tau_0$ of $T_x(M)$.

**Proposition 4.1.** $C_t$ is a local 1-parameter group of linear transformations of $T(M)$: $C_{t+s} = C_t \circ C_s$, and $C_t = \exp (-t(A_x))$.

Proof. Since $q_t$ maps the portion of $\tau$ from $x_0$ to $x_t$ into the portion of $\tau$ from $x_1$ to $x_{1+t}$ and since $q_t$ is compatible with parallel displacement, we have

$$q_t \circ \tau_0 = \tau_0 \circ q_t \circ \tau_0.$$

Hence

$$C_t = \tau_0 ^{-1} \circ q_t \circ \tau_0 ^{-1} \circ \tau_0 ^{-1} = \tau_0 ^{-1} \circ q_t \circ q_0 = \tau_0 ^{-1} \circ q_t \circ \tau_0 = C_{t+1}.$$

This proves the first assertion: Thus there is a linear endomorphism, say, $A$ of $T(M)$ such that $C_t = \exp tA$. The second assertion says that $A = -(A_x)_x$. To prove this, we show that

$$\lim_{t \to 0} \frac{1}{t} (C_t Y_x - Y_x) = -(A_x)_x Y_x \quad \text{for } Y_x \in T_x(M).$$

First, consider the case where $X_x \neq 0$. Then $x$ has a coordinate neighborhood with local coordinate system $x^1, \ldots, x^r$ such that the curve $\tau = x_t$ is given by $x^1 = t, x^2 = \cdots = x^r = 0$ for small values of $t$. We may therefore extend $Y_x$ to a vector field $Y$ on $M$ in such a way that $q_t(Y_x) = Y_{x_t}$ for small values of $t$. Evidently, $(L_X Y)_x = 0$. We have

$$-(A_x)_x Y_x = (\nabla_X Y)_x = (L_X Y)_x = (\nabla_X Y)_x = \lim_{t \to 0} \frac{1}{t} (\tau_0 Y_x - Y_x) \neq \lim_{t \to 0} \frac{1}{t} (t_0 q_t Y_x - Y_x) = \lim_{t \to 0} \frac{1}{t} (C_t Y_x - Y_x).$$

Second, consider the case where $X_x = 0$. In this case, $q_t$ is a local 1-parameter group of local transformations leaving $x$ fixed and the
parallel displacement $\tau_0^*$ reduces to the identity transformation of $T, (M)$. Thus $(\nabla_X Y)_x = 0$. We have
\[-(A_X)_x Y_x = (\nabla_X Y)_x - (L_X Y)_x = -(L_X Y)_x = -\lim_{t \to \infty} \frac{1}{t} (Y_x - \tau_t Y_x) = \lim_{t \to \infty} \frac{1}{t} (C_t Y_x - Y_x).
\]
This completes the proof of the second assertion. QED.

Remark. Proposition 4.1 is indeed a special case of Proposition 11.2 of Chapter II and can be derived from it.

**Proposition 4.2.** Let $N(\Psi(x))$ and $N(\Psi^\theta(x))$ be the normalizers of the Linear holonomy group $\Psi(x)$ and the restricted linear holonomy group $\Psi^\theta(x)$ in the group of linear transformations of $T, (M)$. Then $C_t$ is contained in $N(\Psi(x))$ as well as in $N(\Psi^\theta(x))$.

Proof. Let $\tau_t$ and $\tau_t^+$ be as before. For any loop $\mu$ at $x$, we set $\mu_t' = \exp \tau_t \mu$ so that $\mu_t'$ is a loop at $x_t = \tau_t (x)$. We denote by $\mu_t$ and $\mu_t'$ the parallel displacements along $\mu$ and $\mu'_t$, respectively. Then $\tau_t \circ \mu = \mu_t' \circ \tau_t$. We have
\[C_t \circ \mu \circ C_t^{-1} = \tau_t \circ \mu \circ \tau_t^{-1} = \tau_t \circ \mu_t' \circ \tau_t^{-1} = \tau_t \circ \tau_t \circ \mu_t' \circ \tau_t^{-1} = \tau_t \circ \mu_t \circ \tau_t^{-1} = \tau_t \circ \mu \circ \tau_t^{-1} = \tau_t \circ \mu \circ C_t^{-1}.
\]
This shows that $C_t \circ \mu \circ C_t^{-1}$ is an element of $\Psi(x)$. It is in $\Psi^\theta(x)$ if and only if $\mu$ is in $\Psi^\theta(x)$. (Note that $N(\Psi(x)) \subset N(\Psi^\theta(x))$ since $\Psi^\theta(x)$ is the identity component of $N(\Psi(x))$.)

QED.

**Corollary 4.3.** If $X$ is an infinitesimal affine transformation of $M$, then at each point $x \in M$, $(A,)$ belongs to the normalizer $N(g(x))$ of the Lie algebra $g(x)$ of $\Psi(x)$ in the Lie algebra of endomorphisms of $T_x (M)$.

We recall that $N(g(x))$ is defined by the set of linear endomorphisms $A$ of $T_x (M)$ such that $[A, B] \in g(x)$ for every $B \in g(x)$.

If $X$ is an infinitesimal isometry of a Riemannian manifold $M$, then $A$ is skew-symmetric (cf. Proposition 3.2) and, for each $t$, $C_t$ is an orthogonal transformation of $T_x (M)$. We have then

**Theorem 4.4.** Let $M$ be a Riemannian manifold and $g(x)$ the Lie algebra of $Y(x)$. If $X$ is an infinitesimal isometry of $M$, then for each $x \in M$, $(A,)$ belongs to the normalizer $N(g(x))$ of $g(x)$ in the Lie algebra $E(x)$ of skew-symmetric linear endomorphisms of $T_x (M)$.

The following theorem is due to Kostant [1].

**Theorem 4.5.** If $X$ is an infinitesimal isometry of a compact Riemannian manifold $M$, then for each $x \in M$, $(A,)$ belongs to the Lie algebra $g(x)$ of the linear holonomy group $Y(x)$.

Proof. In the Lie algebra $E(x)$ of skew-symmetric endomorphisms of $T_x (M)$, we introduce a positive definite inner product $(,)$ by setting
\[(A, B) = -\text{trace}(AB).
\]
Let $B(x)$ be the orthogonal complement of $g(x)$ in $E(x)$ with respect to this inner product. For the given infinitesimal isometry $X$ of $M$, we set $A_X = S_X + B_X$, where $S_X \in g(x)$, $B_X \in B(x)$, $x \in M$.

**Lemma.** The tensor field $B_X$ of type $(1,1)$ is parallel.

Proof. Let $\tau$ be an arbitrary curve from a point $x$ to another point $y$. The parallel displacement, $\tau$ gives an isomorphism of $E(x)$ onto $E(y)$ which maps $g(x)$ onto $g(y)$. Since the inner products in $E(x)$ and in $E(y)$ are preserved by $\tau$, $\tau$ maps $B(x)$ onto $B(y)$. This means that, for any vector field $Y$ on $M$, $\nabla_Y (S_X)$ is in $g(x)$ whereas $\nabla_Y (B_X)$ is in $B(x)$ at each point $x \in M$.

On the other hand, the formula $\nabla_Y (A_X) = R(Y, X)$ (cf. Proposition 2.6) implies that $\nabla_Y (A_X)$ belongs to $g(x)$ at each $x \in M$ (cf. Theorem 9.1 of Chapter III). By comparing the $g(x)$-component and the $B(x)$-component of the equality $\nabla_Y (A_X) = \nabla_Y (B_X) + \nabla_Y (S_X)$, we see that $\nabla_Y (B_X)$ belongs to $g(x)$ also. Hence $\nabla_Y (B_X) = 0$, concluding the proof of the lemma.

We shall show that $B_X = 0$. We set $Y = B_X X$. By Green's theorem (cf. Appendix 6), we have (assuming that $M$ is orientable for the moment)
\[
\int_M \text{div} \ Y \, dx = 0 \quad (\text{div: the volume element}).
\]
Since $\text{div} \ Y$ is equal to the trace of the linear mapping $V \rightarrow \nabla_Y V$ at each point $x$, we have (Lemma and Proposition 2.5)
\[
\text{div} \ Y = \text{trace} (V \rightarrow \nabla_Y (B_X X)) = \text{trace} (V \rightarrow B_X X (Y)) = \text{trace} (B_X X) = \text{trace} (B_X S_X) = \text{trace} (B_X B_X) = 0.
\]
Thus
\[ \int_M \text{trace } (B_X B_X) \, dv = 0, \]
which implies \( B_X B_X = 0 \) and hence \( B_X = 0 \). If \( M \) is not orientable, we lift \( X \) to an infinitesimal isometry \( X^* \) of the two-fold orientable covering space \( M^* \) of \( M \). Then \( B_X = 0 \) implies \( B_X = 0 \).

As an application of Theorem 4.5, we prove a result of H. C. Wang [1].

**Theorem 4.6.** If \( M \) is a compact Riemannian manifold, then

1. Every parallel tensor field \( K \) on \( M \) is invariant by the identity component \( \mathfrak{K}(M) \) of the group of isometries of \( M \);
2. At each point \( x \), the linear isotropy group of \( \mathfrak{K}(M) \) is contained in the linear holonomy group \( \mathfrak{H}(x) \).

Proof. (1) Let \( X \) be an arbitrary infinitesimal isometry of \( M \). By Proposition 4.1 and Theorem 4.5, the \( 1 \)-parameter group of \( \mathfrak{K}(M) \) is contained in \( \Psi(x) \). Since \( \mathfrak{K}(M) \) is contained in \( \mathfrak{P}(M) \), it leaves \( K \) invariant. Thus \( \phi_t(K_x) = \tau_t \tau_0 K_x = \tau_t \tau_0 K_x \) for every \( t \), where \( \phi_t \) is the \( 1 \)-parameter group of isometries generated by \( X \). Since \( \mathfrak{K}(M) \) is connected, it leaves \( K \) invariant.

(2) Let \( \phi \) be any element of \( \mathfrak{K}(M) \) such that \( \phi(x) = x \). Since \( \mathfrak{K}(M) \) is a compact connected Lie group, there exists a \( 1 \)-parameter subgroup \( \tau_t \) such that \( \phi = \tau_{t_0} \) for some \( t_0 \). In the proof of (1), we saw that \( \tau_{t_0}(x) \) belongs to \( \Psi(x) \).

5. Ricci tensor and infinitesimal isometries

Let \( M \) be a manifold with a linear connection \( \Gamma \). The Ricci tensor field \( S \) is the covariant tensor field of degree 2 defined as follows:

\[ S(X, Y) = \text{trace of the map } V \rightarrow R(V, X) Y \text{ of } T_x(M), \]
where \( X, Y, V \in T_x(M) \). If \( M \) is a Riemannian manifold and if \( V_1, \ldots, V_n \) is an orthonormal basis of \( T_x(M) \), then

\[ S(X, Y) = \sum_{i=1}^n g(R(V_i, X) Y, V_i) \]

where \( R \) in the last equation denotes the Riemannian curvature tensor (cf. §2 of Chapter V). Property (d) of the Riemannian curvature tensor (cf. §1 of Chapter V) implies \( S(X, Y) = S(Y, X) \), that is, \( S \) is symmetric.

**Proposition 5.4.** If \( f \) is an infinitesimal affine transformation of a Riemannian manifold \( M \), then

\[ \text{div } (A_X Y) = -S(X, Y) - \text{trace } (A_X A_Y) \]

for every vector field \( Y \) on \( M \). In particular,

\[ \text{div } (A_X X) = -S(X, X) - \text{trace } (A_X A_X). \]

Proof. By Proposition 2.6, we have \( g(V, X) = -g(R(V, X) Y, X) = -\nabla_V(A_X Y) \) for any vector field \( V \) on \( M \). Hence

\[ \text{div } (A_X Y) = -S(X, Y) - \text{trace } (A_X A_Y) \]

Our proposition follows from the fact that \( S(X, Y) \) is the trace of \( V \rightarrow R(V, X) Y \) and that \( \text{div } (A_X Y) \) is the trace of \( V \rightarrow \nabla_V(A_X Y) \).

**Proposition 5.5.** For an infinitesimal isometry \( X \) of a Riemannian manifold \( M \), consider the function \( f = \frac{1}{2} g(X, X) \) on \( M \). Then

1. \( \text{div } V = g(\nabla_V A_X X) \) for every tangent vector \( V \);
2. \( \nabla_V V = 0 \);
3. \( \text{div } (A_X X) \leq 0 \) at any point where \( f \) attains a relative minimum; \( \text{div } (A_X X) \geq 0 \) at any point where \( f \) attains a relative maximum.

Proof. Since \( g \) is parallel, we have

\[ Z(g(X, Y)) = \nabla_Z(g(X, Y)) = g(\nabla_Z X, Y) + g(\nabla_Z Y, X) \]

for arbitrary vector fields \( X, Y \) and \( Z \) on \( M \). Applying this formula to the case where \( X = Y \) and \( Z = V \), we obtain

\[ \text{div } V = g(\nabla_V X, X) = -g(A_X X) \]

where \( \text{div } V = -g(A_X X) \).
by virtue of Proposition 2.5 and the skew-symmetry of A. (cf. Proposition 3.2). This proves (1). If V is a vector field such that \( \nabla_V V = 0 \) everywhere on \( \text{Ricci} \), then every two-form \( \omega \) satisfies
\[
\text{div} \omega = \text{trace} \omega,
\]
proving (2). To prove (3), let \( V_1, \ldots, V_n \) be an orthonormal basis for \( T_x \). For each i, let \( \tau_i = x_i(t) \) be the geodesic with the initial condition \( (x, V_i) \) so that \( V_i = \dot{x}_i(0) \). We extend each \( V_i \) to a vector field which coincides with \( \dot{x}_i(t) \) at \( x_i(t) \) for small values of t. Then we have
\[
d^2f(x_i(t)) dt^2 = V_i^2 f = g(\nabla V_i, A x X) + g(V_i, \nabla (A x X))
\]
\[
= g(V_i, \nabla V_i, (A x X)).
\]
Since \( \text{div} (A x X) \) is the trace of the linear mapping \( V \rightarrow \nabla V (A x X) \), we have
\[
\text{div} (A x X)_x = \sum_{i=1}^n (V_i^2 f)_x.
\]
Now (3) follows from the fact that, if \( f \) attains a relative minimum (resp. maximum) at \( x \), then \( (V_i^2 f)_x \geq 0 \) (resp. \( \leq 0 \)). QED.

As an application of these two propositions, we prove the following, result of Bochner [1].

**Theorem 5.3.** Let \( M \) be a connected Riemannian manifold whose Ricci tensor field \( S \) is negative definite everywhere on \( M \). If the length of an infinitesimal isometry \( \hat{X} \) attains a relative maximum at some point of \( M \), then \( \hat{X} \) vanishes identically on \( M \).

Proof. Assume the length of \( \hat{X} \) attains a relative maximum at \( x \). By Proposition 5.2, we have \( \text{div} (A x X) \leq 0 \) at \( x \). By Proposition 5.1, we obtain \(-S(X, X) = \text{trace} (A x A x X) \leq 0 \). But \( S(X, X) \leq 0 \) by assumption, and \( \text{trace} (A x A x X) \leq 0 \) since \( A x \) is skew-symmetric. Thus we have \( S(X, X) = 0 \) and \( A x = 0 \) at \( x \). Since \( S \) is negative definite, \( X = 0 \) at \( x \). Since the length of \( X \) attains a relative maximum at \( x \), \( X \) vanishes in a neighborhood \( x \). If \( u \) is any point of \( O(M) \) such that \( \pi(u) = x \), then the natural lift \( \hat{X} \) of \( X \) vanishes in a neighborhood of \( u \). As we have seen in the proof of Theorem 3.3, \( X \) vanishes identically on \( M \). QED.

**Corollary 5.4.** If \( M \) is a compact Riemannian manifold with negative definite Ricci tensor field, then the group \( \mathfrak{g}(M) \) of isometries of \( M \) is finite.

Proof. By Theorem 5.3, \( \mathfrak{g}(M) \) reduces to the identity. Since \( \mathfrak{g}(M) \) is compact (cf. Theorem 3.4), it is finite. QED.

Remark. Corollary 5.1 can be derived from Proposition 5.1 by means of Green's theorem in the following way.

We may assume that \( M \) is orientable; otherwise, we have only to consider the orientable twofold covering space of \( M \). From Proposition 5.1 and Green's theorem, we obtain
\[
\int_M [S(X, X) + \text{trace} (A x A x X)] \, dv = 0.
\]
Since \( S(X, X) \leq 0 \). and \( \text{trace} (A x A x X) \leq 0 \), we must have \( S(X, X) = 0 \) and \( \text{trace} (A x A x X) = 0 \) everywhere on \( M \). Since \( S \) is negative definite, \( \hat{X} = 0 \) everywhere on \( M \). This proof gives also

**Corollary 5.5.** If \( M \) is a compact Riemannian manifold with vanishing Ricci tensor field, then every infinitesimal isometry of \( M \) is a parallel vector field.

Proof. By Proposition 2.5, we have \( 0 = A x V = -\nabla V X \) for every vector field \( V \) on \( M \). QED.

From Corollary 5.5, we obtain the following result of Lichnerowicz [1].

**Corollary 5.6.** If a connected compact homogeneous Riemannian manifold \( M \) has zero Ricci tensor, then \( M \) is a Euclidean torus.

Proof. By Theorem 5.1 of Chapter III and Corollary 5.5, we have
\[
[X, Y] = \nabla_X Y - \nabla_Y X = 0
\]
for any infinitesimal isometries \( X, Y \). Thus \( \mathfrak{g}(M) \) is a compact abelian group. Since \( \mathfrak{g}(M) \) acts effectively on \( M \), the isotropy subgroup of \( \mathfrak{g}(M) \) at every point \( M \) reduces to the identity element. As we have seen in Example 4.1 of Chapter V, \( M \) is a Euclidean torus. QED.

As another application of Proposition 5.2, we prove...
Proposition 5.7. Let $q_t$ be the 1-parameter group of isometries generated by an infinitesimal isometry $X$ of a Riemannian manifold $M$. If $x$ is a critical point of the length function $g(X,X)$, then the orbit $q_t(x)$ is a geodesic.

Proof. If $x$ is a critical point of $g(X,X)$, it is a critical point of the function $f_1 = \frac{1}{2} g(X,X)$ also. By (1) of Proposition 5.2, we have $g(V,x,X) = 0$ for every vector $V$ at $x$. Hence $A_x X = 0$ at $x$, that is, $\nabla X = 0$ at $x$. Since $q_t(x) = x$ of (1) of Proposition 1.2, we have $V_x X = 0$ along the orbit $q_t(x)$. This shows that the orbit $q_t(x)$ is a geodesic. QED.

6. Extension of local isomorphisms

Let $M$ be a real analytic manifold with an analytic linear connection $\Gamma$. The bundle $L(M)$ of linear frames is an analytic manifold and the connection form $\omega$ is analytic. The distribution $Q$ which assigns the horizontal subspace $Q_u$ to each point $u \in L(M)$ is analytic in the sense that each point $u$ has a neighborhood and a local basis for the distribution $Q$ consisting of analytic vector fields. The same is true for the distribution on the tangent bundle $T(M)$ which defines the notion of parallel displacement in the bundle $L(M)$ (for the notion of horizontal subspaces in an associated fibre bundle, see §7 of Chapter II).

The main object of this section is to prove the following theorem.

Theorem 6.1. Let $M$ be a connected, simply connected analytic manifold with an analytic linear connection. Let $M'$ be an analytic manifold with a complete analytic linear connection. Then every affine mapping $f_U$ of a connected open subset $U$ of $M$ into $M'$ can be uniquely extended to an affine mapping $f$ of $M$ into $M'$.

The proof is preceded by several lemmas.

Lemma 1. Let $f$ and $g$ be analytic mappings of a connected analytic manifold $M$ into an analytic manifold $M'$. If $f$ and $g$ coincide on a non-empty open subset of $M$, then they coincide on $M$.

Proof of Lemma 1. Let $x$ be any point of $M$ and let $x^1, \ldots, x^n$ be an analytic local coordinate system in a neighborhood of $x$. Let $y^1, \ldots, y^m$ be an analytic local coordinate system in a neighborhood of the point $f(x)$. The mapping $f$ can be expressed by a set of analytic functions

$$y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, m.$$  

These functions can be expanded at $x$ into convergent power series of $x^1, \ldots, x^n$. Similarly for the mapping $g$. Let $U$ be the set of points $x \in M$ such that $f(x) = g(x)$ and $f(x)$ and $g(x)$ coincide. Then $N$ is a closed subset of $M$. From the well known properties of power series, it follows that $N$ is open in $M$. Since $M$ is connected, $N = M$.

Lemma 2. Let $S$ and $S'$ be analytic distributions on analytic manifolds $M$ and $M'$. Let $f$ be an analytic mapping of $M$ into $M'$ such that $f(S) = S'$ at every point $x$ of $M$. If $M$ is connected, then $(*)$ is satisfied at every point $x$ of $M$.

Proof of Lemma 2. Let $N$ be the set of all points $x \in M$ such that $(*)$ is satisfied in a neighborhood of $x$. Then $N$ is clearly a non-empty open subset of $M$. Since $M$ is connected, it suffices to show that $N$ is closed. Let $x_0 \in N$ and $x_1 \rightarrow x_0$. Let $y_1, \ldots, y_m$ be an analytic local coordinate system in a neighborhood $V$ of $(x_0)$. Let $Z_1, \ldots, Z_n$ be a local basis for the distribution $S'$ in $V$. From $\partial / \partial y_i, \partial / \partial y_j$, choose $m$ (for the notion of horizontal subspaces, see §7 of Chapter II).

Then $Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_m$ are linearly independent at $f(x_0)$ and hence in a neighborhood $V' \cap f(V)$ of $(x_0)$. Let $U$ be a connected neighborhood of $x_0$ with an analytic local coordinate system $x^1, \ldots, x^n$ such that $f(U) \subseteq V'$ and that $S$ has a local basis $X_1, \ldots, X_n$ consisting of analytic vector fields defined on $U$. Since $f$ is analytic, we have

$$f(U) = \Sigma_{i=1}^n f^i(x) \cdot Z_i, \quad i = 1, \ldots, k,$$

where $f(x)$ are analytic functions of $x^1, \ldots, x^n$. Since $x_0 \in N$ and $x_0 \rightarrow x_0$, there exists a neighborhood $U_1$ of some $x_0$ such that $U_1 \subseteq U$ and that $(*)$ is satisfied at every point $x$ of $U_1$. In other words, $f^i(x) = 0$ on $U_1$ for $1 \leq i \leq k$ and $h + 1 \leq j \leq m$. It follows that $f^i = 0$ on $U$ for the same $i$ and $j$. This proves that $(*)$ is satisfied at every point $x$ of $U$.

Lemma 3. Let $M$ and $M'$ be analytic manifolds with analytic linear connections and $f$ an analytic mapping of $M$ into $M'$. If the
restriction of \( f \) to an open subset \( U \) of \( M \) is an affine mapping and if \( M \) is connected, then \( f \) is an affine mapping of \( M \) into \( M' \).

Proof of Lemma 3. Let \( f \) be the analytic mapping of the tangent bundle \( T(M) \) into \( T(M') \) induced by \( f \). By assumption, \( f \) maps the horizontal subspace at each point of \( \pi^{-1}(U) \) into a horizontal subspace in \( T(M') \). Applying Lemma 2 to the mapping \( f \), we see that \( f \) is an affine mapping of \( M \) into \( M' \).

**Lemma 4.** Let \( M \) and \( M' \) be differentiable manifolds with linear connections and let \( f \) and \( g \) be affine mappings of \( M \) into \( M' \). If \( f(X) = g(X) \) for every \( X \in \mathcal{T}_x(M) \) at some point \( x \in M \) and if \( M \) is connected, then \( f \) and \( g \) coincide on \( M \).

Proof of Lemma 4. Let \( N \) be the set of all points \( x \in M \) such that \( f(X) = g(X) \) for \( X \in \mathcal{T}_x(M) \). Then \( N \) is clearly a non-empty closed subset of \( M \). Since \( f \) and \( g \) commute with the exponential mappings (Proposition 1), \( x \in N \) implies that a normal coordinate neighborhood of \( x \) is in \( N \). Thus \( N \) is open. Since \( M \) is connected, we have \( N = M \).

We are now in position to prove Theorem 6.1. Under the assumptions in Theorem 6.1, let \( x(t) \), \( 0 \leq t \leq 1 \), be a curve in \( M \) such that \( x(0) \in U \). An analytic continuation of \( x(t) \) along the curve \( x(t) \) is, by definition, a family of affine mappings \( f_t \) of \( M \) into \( M' \), satisfying the following conditions:
1. For each \( t \), \( f_t \) is an affine mapping of \( M \) into \( M' \);
2. For each \( t \), there exists a positive number \( \delta \) such that if \( |s - t| < \delta \), then \( f(s) \in U \), and \( f_t \) coincides with \( f_t \) in a neighborhood of \( x(s) \);
3. \( f_0 = f \).

It follows easily from Lemma 4 that an analytic continuation of \( f_t \) along the curve \( x(t) \) is unique if it exists. We now show that it exists.

Let \( t_0 \) be the supremum of \( t \) such that an analytic continuation \( f_t \) exists for \( 0 \leq t \leq t_0 \). Let \( W \) be a convex neighborhood of the point \( x(t_0) \) as in Theorem 6.1 such that every point of \( W \) has a normal coordinate neighborhood containing \( W \). Take \( t_1 \) such that \( t_1 < t_0 \) and \( x(t_1) \in W \). Let \( V \) be a normal coordinate neighborhood of \( x(t_1) \) which contains \( W \). Since there exists an analytic continuation \( f_{t_1} \) of \( f_t \) for \( 0 \leq t \leq t_1 \),

we have the affine mapping \( f_{t_1} \) of a neighborhood \( x(t_1) \) into \( M' \). We extend \( f_{t_1} \) to an analytic mapping, say \( g \), of \( V \) into \( M' \) as follows. Since the exponential mapping gives a diffeomorphism of an open neighborhood \( V \) of the origin in \( T_{x(t_1)}(M) \) onto \( T_{x(t_1)}(M) \), each point \( y \in V \) determines a unique element. \( X \in V \) is a vector at \( f_t(x(t_1)) \). Since \( M' \) is complete, \( \exp X \) is well defined and we set \( g(y) := \exp X \). The extension of \( f_{t_1} \) thus defined commutes with the exponential mappings. Since the exponential mapping is analytic, \( g \) is also analytic. By Lemma 3, \( g \) is an analytic mapping of \( V \) into \( M' \). We can easily define the continuation \( f_t \) beyond \( t_1 \) by using this affine mapping \( g \). We have thus proved the existence of an analytic continuation \( f_t \) along the whole curve \( x(t) \), \( 0 \leq t \leq 1 \).

To complete the proof of Theorem 6.1, let \( x \) be an arbitrary fixed point of \( U \). For each point \( y \) of \( M \), let \( x(t) \), \( 0 \leq t \leq 1 \), be a curve from \( x \) to \( y \). The affine mapping \( f_t \) can be uniquely continued along the curve \( x(t) \) and gives rise to an affine mapping \( g \) of a neighborhood of \( y \) into \( M' \). We show that \( g(y) \) is independent of the choice of a curve from \( x \) to \( y \). For this, it is sufficient to observe that if \( x(t) \) is a closed curve, then the analytic continuation \( f_t \) of \( f_t \) along \( x(t) \) gives rise to the affine mapping \( f_t \), which coincides with \( f_t \) in a neighborhood \( x \). Since \( M \) is simply connected, the curve \( x(t) \) is homotopic to zero and our assertion follows readily from the factorization lemma (cf. Appendix 7.) and from the uniqueness of the analytic continuation we have already proved. It follows that the given mapping \( f_t \) can be extended to an affine mapping \( f_t \) of \( M' \) into \( M' \). The uniqueness of \( f_t \) follows from Lemma 4.

QED.

**Corollary 6.2.** Let \( M \) and \( M' \) be connected and simply connected analytic manifolds with complete analytic linear connections. Then every affine isomorphism between connected open subsets of \( M \) and \( M' \) can be uniquely extended to an affine isomorphism between \( M \) and \( M' \).

We have the corresponding results for analytic Riemannian manifolds. The Riemannian connection of an analytic Riemannian metric is analytic; this follows from Corollary 2.1 of Chapter IV.

**Theorem 6.3.** Let \( M \) and \( M' \) be analytic Riemannian manifolds. If \( M \) is connected and simply connected and if \( M' \) is complete, then every
isometric immersion $f: U \to M'$ can be uniquely extended to an isometric immersion $f$ of $M$ into $M'$.

Proof. The proof is quite similar to that of Theorem 6.1. We indicate only the necessary changes. Lemma 1 can be used without any change. Lemma 2 was necessary only to derive Lemma 3. In the present case, we prove the following Lemma 3' directly.

**Lemma 3'.** Let $M$ and $M'$ be analytic manifolds with analytic Riemannian metrics $g$ and $g'$, respectively, and let $f$ be an analytic mapping of $M$ into $M'$. If the restriction of $f$ off an open subset $U$ of $M$ is an isometric immersion and if $M$ is connected, then $f$ is an isometric immersion of $M$ into $M'$.

Proof of Lemma 3'. Compare $g$ and $f(g)$.) Since they coincide on $U$, the argument similar to the one used in the proof of Lemma 1 shows that they coincide on the whole of $M$.

In Lemma 4, we replace "affine mappings" by "isometric immersions." Since an isometric immersion maps every geodesic into a geodesic and hence commutes with the exponential mappings, the proof of Lemma 4 is still valid.

In the rest of the proof of Theorem 6.1, we replace "affine mapping" by "isometric immersion." Then the proof goes through without any other change.

Remark. Since an isometric immersion $f: M \to M'$ is not necessarily an affine mapping, Theorem 6.3 does not follow from Theorem 6.1. If $\dim M = \dim M'$, then every isometric immersion $f: M \to M'$ is an affine mapping (cf. Proposition 2.6 of Chapter IV). Hence the following corollary follows from Corollary 6.2 as well as from Theorem 6.3.

**Corollary 6.4.** Let $M$ and $M'$ be connected and simply connected, complete analytic Riemannian manifolds. Then every isometry between connected open subsets of $M$ and $M'$ can be uniquely extended to an isometry between $M$ and $M'$.

7. Equivalence problem

Let $M$ be a manifold with a linear connection. Let $x^1, \ldots, x^n$ be a normal coordinate system at a point $x_0$ and let $U$ be a neighborhood of $x_0$ given by $|x^i| < 6, i = 1, \ldots, n$. Let $u_0$ be the linear frame at the origin $x_0$ given by $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. We define a cross section $\sigma: U \to L(M)$ as follows. If $x$ is a point of $U$ with coordinates $(a^1, \ldots, a^n)$, then $\sigma(x)$ is the frame obtained by the parallel displacement of $u_0$ along the geodesic given by $x^t = ta^i, 0 \leq t \leq 1$. We call $\sigma$ the cross section adapted to the normal coordinate system $x^1, \ldots, x^n$.

The first objective of this section: is to prove the following theorem.

**Theorem 7.1.** Let $M$ and $M'$ be manifolds with linear connections. Let $U$ (resp. $V$) be a normal coordinate neighborhood of a point $x_0 \in M$ (resp. $y_0 \in M'$) with a normal coordinate system $x^1, \ldots, x^n$ (resp. $y^1, \ldots, y^n$) and let $f: U \to L(M)$ (resp. $\sigma: V \to L(M')$) be the cross section adapted to $x^1, \ldots, x^n$ (resp. $y^1, \ldots, y^n$). A diffeomorphism $f$ of $U$ onto $V$ is an affine isomorphism if it satisfies the following two conditions:

(1) $f$ maps the frame $a(x)$ into the frame $f^*(a(x))$ for each point $x \in U$;
(2) $f$ preserves the torsion and curvature tensor fields.

Proof. Let $\theta = (\theta^i)$ and $\omega = (\omega^{ij})$ be the canonical form and the connection form on $L(M)$ respectively. We set

$$
(1) \quad \theta^i = \sigma^* \theta^i = \Sigma J^i_j dx^j, \quad i = 1, \ldots, n,
$$

$$
(2) \quad \omega^{ij} = \sigma^* \omega^{ij} = \Sigma B^{ij}_k dx^k, \quad i, j = 1, \ldots, n.
$$

**Lemma 1.** For any $(a^1, \ldots, a^n) \neq (0, \ldots, 0)$ with $|a^i| < \delta$, we have

$$
(3) \quad \Sigma J^i_j(ta^i) a^j = a^i, \quad 0 \leq t \leq 1, \quad i = 1, \ldots, n,
$$

$$
(4) \quad \Sigma B^{ij}_k(ta^i) a^j a^k = 0, \quad 0 \leq t \leq 1, \quad i, j = 1, \ldots, n,
$$

where $t$ stands for $(ta^1, \ldots, tan)$.

Proof of Lemma 1. For a fixed $a = (a^i)$, consider the geodesic $x$, given by $x^t = ta^i, 0 \leq t \leq 1, i = 1, \ldots, n$. Let $u_t = \sigma(x_t)$, which is the horizontal lift of $x_t$ starting from $u_0$. Since the frames $u_t$ are parallel along $x_t$, we have

$$
\theta_t^i(x_t) = \theta^i(u_t) = a^i.
$$

On the other hand, we have

$$
x_t = \Sigma J^i_j(x_t) \partial/\partial x^j \quad \text{and} \quad \theta_t^i(x_t) = \Sigma J^i_j(ta^i) a^j.
$$

This proves (3). Similarly, (4) follows from the fact that $\omega(u_t) = 0$. 
We consider the

\[ \tilde{\Theta}^i = \sigma^* \Theta^i = \sum_{j,k} \frac{1}{2} \tilde{T}^i_{jk} \tilde{\Theta}^j \wedge \tilde{\Theta}^k \quad \text{(10)} \]

\[ \tilde{\Omega}^j = \sigma^* \Omega^j = \sum_{k,i} \frac{1}{2} R^j_{ik} \tilde{\Theta}^i \wedge \tilde{\Theta}^k \quad \text{(11)} \]

**Lemma 2.** For an arbitrarily fixed \((a', \ldots, a^n)\), we set

\[
\tilde{A}_j(t) = t A_j(ta), \quad \tilde{B}_i(t) = t B_i(ta), \quad \tilde{T}^i_{jk}(t) = t T^i_{jk}(ta), \quad \tilde{R}^i_{jk}(t) = t R^i_{jk}(ta).
\]

Then the functions \(\tilde{A}_j(t)\) and \(\tilde{B}_i(t)\) satisfy the following system of ordinary linear differential equations:

\[ d\tilde{A}_j(t)/dt = \delta^j_i + \tilde{S}_i \tilde{B}_i(t) + \sum_{l,m} \tilde{T}^l_{im}(t) \tilde{A}_m \cdot a^l \quad \text{(7)} \]

\[ d\tilde{B}_i(t)/dt = \sum_{l,m} \tilde{R}^l_{im}(t) \tilde{A}_m \cdot a^l \quad \text{(8)} \]

with the initial conditions:

\[ \tilde{A}_j(0) = 0, \quad \tilde{B}_i(0) = 0 \quad \text{(9)} \]

Proof of Lemma 2. We consider the open set \(Q\) of \(\mathbb{R}^{n+1}\) defined by \(Q = \{t, a', \ldots, a^n \mid |ta'| < \delta \text{ for } i = 1, \ldots, n\} \). Let \(\rho\) be the mapping of \(Q\) into \(U\) defined by

\[ \rho(t, a', \ldots, a^n) = (ta', \ldots, ta^n). \]

We set

\[ \tilde{\Theta}^i = \rho^* \Theta^i, \quad \tilde{\Omega}^j = \rho^* \Omega^j, \quad \tilde{\Theta}^i = \rho^* \Theta^i, \quad \tilde{\Omega}^j = \rho^* \Omega^j. \]

From Lemma 1, we obtain

\[ \tilde{A}_j(t) = \sum_{l,m} \tilde{T}^l_{im}(ta) \tilde{A}_m \cdot a^l \quad \text{(10)} \]

\[ \tilde{B}_i(t) = \sum_{l,m} \tilde{R}^l_{im}(ta) \tilde{A}_m \cdot a^l \quad \text{(11)} \]

From (5) and (6), we obtain

\[ \tilde{\Theta}^i = \sum_{j,k} \frac{1}{2} \tilde{T}^i_{jk}(ta) \tilde{\Theta}^j \wedge \tilde{\Theta}^k \quad \text{(12)} \]

\[ \tilde{\Omega}^j = \sum_{k,i} \frac{1}{2} R^j_{ik}(ta) \tilde{\Theta}^i \wedge \tilde{\Theta}^k \quad \text{(13)} \]

From (10) and (11), we obtain

\[ \tilde{A}_j(t) = \sum_{l,m} \frac{1}{2} \tilde{T}^l_{im}(ta) \tilde{A}_m \cdot da^l \quad \text{(14)} \]

\[ \tilde{B}_i(t) = \sum_{l,m} \frac{1}{2} \tilde{R}^l_{im}(ta) \tilde{A}_m \cdot da^l \quad \text{(15)} \]

where the dots denote the terms not involving \(dt\).
system exists and is unique. Let $f$ be the analytic homeomorphism of $U$ onto $V$ defined by

\[ y^i \circ f = x^i, \quad i = 1, \ldots, n. \]

Clearly the differential of $f$ at $x_0$ coincides with $F$. We shall show that $f$ is an affine isomorphism of $U$ onto $V$.

We use the same notation as in the proof of Theorem 7.1. It suffices to prove the following five statements. If the normal coordinate system $x_1, \ldots, x^n$ is fixed, then

(i) The tensors $\nabla^m T_{x_1}, m = 0, 1, 2, \ldots$, determine the functions $\tilde{T}_{ik}(t), 0 \leq t \leq 1$;

(ii) The tensors $\nabla^m R_{x_1}, m = 0, 1, 2, \ldots$, determine the functions $\tilde{R}_{ik}(t), 0 \leq t \leq 1$;

(iii) The functions $\tilde{T}_{ik}(t)$ and $\tilde{R}_{ik}(t)$ determine the forms $\tilde{\sigma}^i$ and $\tilde{\sigma}^j$;

(iv) The forms $\tilde{\sigma}^i$ determine the cross section $\sigma$; and

(v) The cross section $\sigma$ and the forms $\tilde{\sigma}^j$ determine the connection form $\omega$.

To prove (i) and (ii) we need the following lemma.

**Lemma 1.** Let $u_t$, $0 \leq t \leq 1$, be a horizontal lift of a curve $x_1, \ldots, x^n$, $0 \leq t \leq 1$, to $L(M)$. Let $T_x$ be the tensor space of type $(r, s)$ over $\mathbb{R}^n$. Given a tensor field $K$ of type $(r, s)$ along $x_1, \ldots, x^n$, let $K$ be the $T_x$-valued function defined along $u_t$ by

\[ K(u_t) = u_t^{-1}(K_{x_1}), \quad 0 \leq t \leq 1, \]

where $u_t$ is considered as a linear mapping of $T_x$ onto the tensor space $T_x(x_1, \ldots, x_n)$ of type $(r, s)$. Then we have

\[ \frac{dK(u_t)}{dt} = u_t^{-1}(\nabla_x K), \quad 0 \leq t \leq 1. \]

**Proof of Lemma 1.** This is a special case of Proposition 1.3 of Chapter III. The tensor field $K$ and the function $K$ here correspond to the cross section $\sigma$ and the function $\sigma$ there. Although $\sigma$ in Proposition 1.3 of Chapter III is defined on the whole of $M$, the proof goes through when $\sigma$ is defined on a curve in $M$ (cf. the lemma for Proposition 1.1 of Chapter III).

To prove (i), we apply Lemma 1 to the torsion $T$, the geodesic $x$, given by $x^i = t a^i$, $i = 1, \ldots, n$, and the horizontal lift $u_t$ of $x_t$ with $u_t = ((\partial/\partial x^1)_t, \ldots, (\partial/\partial x^n)_t)$. Then Lemma 1 (applied $m$ times) implies that, for each $t$, $u_t^{-1}((\nabla_x)^m T)$ is the element of the tensor space $T^1$ with components $d^m T_{ij}(t)/dt^m$. In particular, setting $t = 0$, we see that, once the coordinate system $x_1, \ldots, x^n$ and $(a^1, \ldots, a^n)$ are fixed, $(d^m T_{ij}(t))_{t=0}, m = 0, 1, 2, \ldots$, are all determined by $(\nabla_x T)_{x_1}$. (Actually, it is not hard to see that

\[ (d^m T_{ij}(t))_{t=0} = \sigma^1_{ij}, \ldots, \sigma^n_{ij}(x_0) d^1 \cdots d^n, \]

where $T_{ij}(x_0), \ldots, \sigma^i_j(x_0)$ are the components of $\nabla_x T$ with respect to $x_1, \ldots, x^n$). Since each $T_{ij}(t)$ is an analytic function of $t$, it is determined by $(\nabla_x T)_{x_1}$, $m = 0, 1, 2, \ldots$. This proves (i). The proof of (ii) is similar.

**Lemma 2.** Let $\sigma$ and $\sigma'$ be two cross sections of $L(M)$ over an open subset $U$ of $M$. If $\sigma \circ \theta = \sigma' \circ \theta$ on $U$, then $\sigma = \sigma'$.

**Proof of Lemma 2.** For each $X \in T_x(M)$, where $x \in U$, we have

\[ (\sigma \circ \theta)(X) = \theta(\sigma X) = \sigma(\theta^{-1}(\pi(\sigma X))) = \sigma(X)^{-1} X, \]

where $\sigma(x) \in L(M)$ is considered as a linear isomorphism of $\mathbb{R}^n$ onto $T_x(M)$. Using the same equation for $\sigma'$, we obtain

\[ a(x) \cdot X = a'(x) \cdot X. \]

Since this holds for every $X$ in $T_x(M)$, we obtain $\sigma(x) = \sigma'(x)$.

Finally, (v) is evident from the definition of $\tilde{\sigma}^j$.

**QED.**

**Corollary 7.3.** In Theorem 7.2, if $M$ and $M'$ are, moreover, connected, simply connected analytic manifolds with complete analytic linear connections, then there exists a unique affine morphism $f$ of $M$ onto $M'$ whose differential at $x_0$ coincides with $F$.

Proof. This is an immediate consequence of Corollary 6.2 and Theorem 7.2.

**Theorem 7.4.** Let $M$ and $M'$ be differentiable manifolds with linear connections. Let $T$, $R$ and $V$ (resp. $T'$, $R'$ and $C'$) be the torsion, the curvature and the covariant differentiation of $M$ (resp. $M'$). Assume
VT = 0, \nabla R = 0, V' T' = 0 and V'R' = 0. If F is a linear isomorphism of \( T_x(M) \) onto \( T_y(M) \) and maps the tensors \( T_0 \) and \( R_0 \) at \( x_0 \) into the tensors \( T_y \) and \( R_y \) at \( y \), respectively, then there is an affine isomorphism of a neighborhood \( U \) of \( x_0 \) onto a neighborhood \( V \) of \( y \) such that \( f(x_0) = y_0 \) and that the differential off at \( x_0 \) coincides with \( F \).

Proof. We follow the notation and the argument in the proof of Theorem 7.2. By Lemma 1 in the proof of Theorem 7.2, the functions \( \tilde{t}_0(t) \) and \( \tilde{r}_0(t) \) are constant functions and hence are determined by \( T_0 \) and \( R_0 \) (and the coordinate system \( x_1, \ldots, x_n \)). Our theorem now follows from (iii), (iv) and (v) in the proof of Theorem 7.2.

COROLLARY 7.5. Let \( M \) be a differentiable manifold with a linear connection such that \( VT = 0 \) and \( VR = 0 \). Then, for any two points \( x \) and \( y \) of \( M \), there exists an affine isomorphism of a neighborhood of \( x \) onto a neighborhood of \( y \).

Proof. Let \( \tau \) be an arbitrary curve from \( x \) to \( y \). Since \( VT = C \) and \( VR = 0 \), the parallel displacement \( \tau: T_x(M) \rightarrow T_y(M) \) maps the tensors \( T_0 \) and \( R_0 \) at \( x \) into the tensors \( T_y \) and \( R_y \) at \( y \). By Theorem 7.4, there exists a local affine isomorphism \( f \) such that \( f(x) = y \) and that the differential off at \( x \) coincides with \( \tau \).

Let \( M \) be a manifold with a linear connection \( \Gamma \). The connection \( \Gamma \) is said to be invariant by parallelism if, for arbitrary points \( x \) and \( y \) of \( M \) and for an arbitrary curve \( \tau \) from \( x \) to \( y \), there exists a (unique) local affine isomorphism \( f \) such that \( f(x) = y \) and that the differential off at \( x \) coincides with the parallel displacement \( \tau: T_x(M) \rightarrow T_y(M) \). In the proof of Corollary 7.5, we saw that if \( VT = 0 \) and \( VR = 0 \), then the connection is invariant by parallelism. The converse is also true. Namely, we have

COROLLARY 7.6. A linear connection is invariant by parallelism if and only if \( VT = 0 \) and \( VR = 0 \).

Proof. Assuming that the connection is invariant by parallelism, let \( \tau \) be an arbitrary curve from \( x \) to \( y \). Let be a local affine isomorphism such that \( f(x) = y \) and that the differential off at \( x \) coincides with the parallel displacement \( \tau \). Then \( f \) maps \( T_0 \) and \( R_0 \) into \( T_y \) and \( R_y \) respectively. Hence the parallel displacement \( \tau \) maps \( T_0 \) and \( R_0 \) into \( T_y \) and \( R_y \) respectively. This means that \( T \) and \( R \) are parallel tensor fields.

QED.

VI. -TRANSFORMATIONS

THEOREM 7.7. Let \( M \) be a differentiable manifold with a linear connection such that \( VT = 0 \) and \( VR = 0 \). With respect to the atlas consisting of normal coordinate systems, \( M \) is an analytic manifold and the connection is analytic.

Proof. Let \( x_1, \ldots, x^n \) be a normal coordinate system in an open set \( U \). We introduce a coordinate system \( (x^i, X^j) \), \( i, j = 1, \ldots, n \) in \( \pi^{-1}(U) \subset L(M) \) in a natural way as in \( S7 \) of Chapter III (cf. Example 5.2 of Chapter I). If we denote by \( (U^j) \) the inverse matrix of \( (X^i) \), then the canonical form and the connection form can be expressed as follows (cf. Propositions 7.1 and 7.2 of Chapter III):

\[
\begin{align*}
&18. \quad \theta^j = \sum_i U^j_i dx^i, \quad i = 1, \ldots, n; \\
&19. \quad \omega^j = \sum_k U^j_k (dX^k + \sum_l \Gamma^k_{ml} X^l dx^m), \quad i, j = 1, \ldots, n.
\end{align*}
\]

The forms \( \theta^j \) are analytic with respect to \( (x^i, X^j) \). We show that the forms \( \omega^j \) are also analytic with respect to \( (x^i, X^j) \). Clearly it is sufficient to show that the components \( \Gamma^k_{ij} \) of the connection are analytic in \( x^1, \ldots, x^n \). We use the same notation as in the proof of Theorem 7.1. Since the functions \( \hat{t}_k(t) \) and \( \hat{r}_k(t) \) are constants which do not depend on \( (a^1, \ldots, a^n) \) by virtue of the assumption that \( VT = 0 \) and \( VR = 0 \), Lemma 2 in the proof of Theorem 7.1 implies (cf. Appendix 1) that the functions \( \hat{A}_j(t) \) and \( \hat{B}_j(t) \) are analytic in \( t \) and depend analytically on \( (a^1, \ldots, a^n) \). Hence the functions \( A_j \) and \( B_j \) are analytic in \( x^1, \ldots, x^n \). From (1) in the proof of Theorem 7.1, we see that the cross section \( \sigma: U \rightarrow L(M) \) is given by

\[
(20) \quad U^j_i = A^j_i, \quad i, j = 1, \ldots, n.
\]

Let \( (C^j_i) \) be the inverse matrix of \( (A^j_i) \). From (19) and (20), we obtain

\[
(21) \quad \sigma^*\omega^j = \tilde{\omega}^j = \sum_k A^j_k (dC_k^j + \sum_l \Gamma^k_{ml} C^j_l dx^m).
\]

By comparing (21) with (2) in the proof of Theorem 7.1, we obtain

\[
(22) \quad B^j_{im} = \sum_k A^j_k (\partial C_k^j / \partial x^m + \sum_l \Gamma^k_{ml} C^j_l).
\]

Transforming (22) we obtain

\[
(23) \quad \Gamma^j_{ml} = \sum_i (\partial C_i^j/\partial x^m) A^j_i.
\]
which shows that the components $\Gamma_{jk}^i$ are analytic functions of $x^1, \ldots, x^n$.

Since the $n^2 + n$ 1-forms $\theta^i$ and $\omega^i_j$ are analytic with respect to $(x^1, X^j_1)$ and define an absolute parallelism (Proposition 2.5 of Chapter III), the following lemma implies that $L(M)$ is an analytic manifold with respect to the atlas consisting of the coordinate system $(x^1, X^j_1)$ induced from the normal coordinate systems $(x^1, \ldots, x^n)$ of $M$.

**Lemma.** Let $\omega^1, \ldots, \omega^m$ be 1-forms defining an absolute parallelism on a manifold $P$ of dimension $m$. Let $u^1, \ldots, u^m$ (resp. $v^1, \ldots, v^m$) be a local coordinate system valid in an open set $U$ (resp. $V$) if the form $\omega^1, \ldots, \omega^m$ are analytic with respect to both $u^1, \ldots, u^m$ and $v^1, \ldots, v^m$, then the functions

$$v^i = f^i(u^1, \ldots, u^m), \quad i = 1, \ldots, m,$$

which define the coordinate change are analytic.

**Proof of Lemma.** We write

$$\omega^i = \sum_j a^i_j(u) \, du^j = \sum_j b^i_j(v) \, dv^j,$$

where the functions $a^i_j(u)$ (resp. $b^i_j(v)$) are analytic in $u^1, \ldots, u^m$ (resp. $v^1, \ldots, v^m$). Let $(e^i(v))$ be the inverse matrix of $(b^i_j(v))$. Then the system of functions $v^1 = f^1(u^1, \ldots, u^m), \ldots, v^m = f^m(u^1, \ldots, u^m)$ is a solution of the following system of linear partial differential equations:

$$\frac{\partial v^i}{\partial u^j} = \sum_k c^i_k(v) \, a^k_j(u), \quad i, j = 1, \ldots, m.$$

Since the functions $c^i_k(v)$ and $a^k_j(u)$ are analytic in $v^1, \ldots, v^m$ and $u^1, \ldots, u^m$ respectively, the functions $f^i(u^1, \ldots, u^m)$ are analytic in $v^1, \ldots, v^m$ (cf. Appendix 1). This proves the lemma.

Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$ be two normal coordinate systems in $M$. Let $(x^1, X^j_1)$ and $(y^1, Y^j_1)$ be the local coordinate systems in $L(M)$ induced by these normal coordinate systems. By the lemma just proved, $y^1, \ldots, y^n$ are analytic functions of $x^1, \ldots, x^n$. Since $y^1, \ldots, y^n$ are clearly independent of $X^j_1$, they are analytic functions of $x^1, \ldots, x^n$. This proves the first assertion of Theorem 7.7. Since we have already proved that the forms $\omega^i_j$ are analytic with respect to $(x^1, X^j_1)$, the connection is analytic. QED.

As an application of Theorem 7.7 we have...
APPENDIX 1

Ordinary linear differential equations

The purpose of this appendix is to state the fundamental theorem on ordinary linear differential equations in the form needed in the text. The proof will be found in various text books on differential equations.

For the sake of simplicity, we use the following abbreviated notation:

\[ 3' = (y^1, \ldots, y^n), \quad \eta = (\eta^1, \ldots, \eta^n), \quad f = (f^1, \ldots, f^n), \]
\[ q = (q^1, \ldots, q^n), \quad s = (s^1, \ldots, s^m), \quad x = (x^1, \ldots, x^m). \]

Then we have

**Theorem.** Let \( f(t, y, s) \) be a family of \( n \) functions defined in \(|t| < \delta\) and \((y, s) \in D\), where \( D \) is an open set in \( \mathbb{R}^{n+m} \). If \( f(t, y, s) \) is continuous in \( t \) ad differentiable of class \( C^1 \) in \( y \), then there exists a unique family \( q'(t, y, s) \) of \( n \) functions defined in \(|t| < \delta'\) and \((y, s) \in D'\), where \( 0 < \delta' < \delta \) and \( D' \) is an open subset of \( D \), such that

1. \( q'(t, y, s) \) is differentiable of class \( C^1 \) in \( t \) and \( y \);
2. \( \frac{\partial q(t, y, s)}{\partial t} = f(t, q(t, y, s), s) \);
3. \( q(0, y, s) = \eta \).

If \( f(t, y, s) \) is differentiable of class \( C^p, 0 \preceq p \leq \omega_1 \) in \( t \) and of class \( C^r, 1 \leq r \leq \omega_1 \) in \( y \) and \( s \), then \( q(t, y, s) \) is differentiable of class \( C^{p+1} \) in \( t \) and of class \( C^r \) in \( y \) and \( s \).

Consider the system of differential equations:

\[ dy/dt = f(t, y, s) \]

which depend on the parameters \( s \). Then \( y = q(t, \eta, s) \) is called the solution satisfying the initial condition:

\[ y = \eta \text{ when } t = 0. \]
Consider now a system of partial differential equations:
\[ \frac{\partial y_i}{\partial x^j} = f_{ij}(x,y), \quad i = 1, \ldots, n; \ j = 1, \ldots, m. \]

It follows from the theorem that if the functions \( f_{ij}(x,y) \) are differentiable of class \( C^r \), \( 0 \leq r \leq \omega \), then every solution \( y = \psi(x) \) is differentiable of class \( C^{r+1} \). This fact is used in the proof of Theorem 7.7 of Chapter VI.

**APPENDIX 2**

A connected, locally compact metric space is separable

We recall that a topological space \( M \) is separable if there exists a dense subset \( D \) which contains at most countably many points. It is called **locally separable** if every point of \( M \) has a neighborhood which is separable. Note that, for a metric space, the separability is equivalent to the second axiom of countability (cf. Kelley [1, p. 120]). The proof of the statement in the title is divided into the following three lemmas.

**Lemma 1.** A compact metric space is separable.

For the proof, see Kelley [1, p. 138].

**Lemma 2.** A locally compact metric space is **locally** separable.

This is a trivial consequence of Lemma 1.

The following lemma is due to Sierpinski [1].

**Lemma 3.** A connected, locally separable metric space is separable.

Proof of **Lemma 3.** Let \( d \) be the metric of a connected, locally separable metric space \( M \). For every point \( x \in M \) and every positive number \( r \), let \( U(x; r) \) be the interior of the sphere of center \( x \) and radius \( r \), that is, \( U(x; r) = \{ y \in M ; d(x,y) < r \} \). We say that two points \( x \) and \( y \) of \( M \) are \( R \)-related and write \( xRy \), if there exist a separable \( U(x; r) \) containing \( y \) and a separable \( U(y; r') \) containing \( x \). Evidently, \( xRx \) for every \( x \in M \). We have also \( xRy \) if and only if \( yRx \).

For every subset \( A \) of \( M \), we denote by \( SA \) the set of points which are \( R \)-related to a point of \( A \); \( SA = \{ y \in M ; yRx \text{ for some } x \in A \} \). Set \( S^nA = SS^{n-1}A \), \( n = 2, 3, \ldots \). If \( \{x\} \) is the set consisting of a single point \( x \), we write \( Sx \) for \( S\{x\} \). We see easily that \( y \in S^n x \) if
and only if \( x \in S^n_y \). We prove the following three statements:

(a) \( S_x \) is open for every \( x \in M \);

(b) If \( A \) is separable, so is \( S_A \);

(c) Set \( U(x) = \bigcup_{n=1}^\infty S^n_x \) for each \( x \in M \). Then, for any \( x, y \in M \), either \( U(x) \cap U(y) \) is empty or \( U(x) = U(y) \).

Proof of (a). Let \( y \) be a point of \( S_x \). Since \( xR_y \), there exist positive numbers \( r \) and \( r' \) such that \( U(x; r) \) and \( U(y; r') \) are separable and that \( y \in U(x; r) \) and \( x \in U(y; r') \). Since \( d(x, y) < r' \), there is a positive number \( r_1 \) such that

\[
d(x, y) < r_1 < r'.
\]

Let \( r_0 \) be any positive number such that

\[
r_0 < r' - r_1,
\]

\[
r_0 < r - d(x, y),
\]

\[
r_0 < r_1 - d(x, y).
\]

It suffices to show that \( U(y, r_0) \) is contained in \( S_x \). If \( z \in U(y; r_0) \), then

\[
d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r_0 < m \in \{r, r_1\},
\]

Hence \( z \) is in \( U(x; r) \) which is separable and \( x \) is in \( U(z; r_1) \). To prove that \( U(z; r_1) \) is separable, we shall show that \( U(z; r_1) \) is contained in \( U(y; r') \) which is separable. Let \( w \in U(z; r_1) \) so that \( d(z, w) < r_1 \). Then

\[
d(y, w) \leq d(y, z) + d(z, w) < d(y, z) + r_1 < r_0 + r_1 < r',
\]

Hence \( w \) is in \( U(y; r') \). This proves that \( xR_y \) for every \( z \in U(y; r_0) \), that is, \( U(y; r_0) \subseteq S_x \).

Proof of (b). Let \( A \) be a separable subset of \( M \) and \( D \) a countable dense subset of \( A \). It suffices to prove that every \( x \in S_A \) is contained in a separable sphere whose center is a point of \( D \) and whose radius is a rational number, because there are only countably many such spheres and the union of these spheres is separable. Let \( x \in S_A \). Then there is \( y \in A \) such that \( xR_y \) and there is a separable sphere \( U(y; r) \) containing \( x \). Let \( r_0 \) be a positive rational number such that \( d(x, y) < r_0 < r \). Since \( D \) is dense in \( A \), there is \( z \in D \) such that

\[
d(z, y) < m \in \{r_0, d(x, y), r - r_0\}.
\]

It suffices to show that \( U(z; r_0) \) contains \( x \) and is separable. From

\[
d(x, z) = d(x, y) + d(y, z) < r_0,
\]

it follows that \( x \in U(z; r_0) \). To prove that \( U(z; r_0) \) is separable, we show that \( U(z; r_0) \) is contained in \( U(y; r) \) which is separable. If \( w \in U(z; r_0) \), then

\[
d(w, y) \leq d(w, z) + d(z, y) < r_0 + d(z, y) < r,
\]

and hence \( w \in U(y; r) \).

Proof of (c). Assume that \( U(x) \neq U(y) \). From this, we obtain \( x \in S^n_x \) and \( z \in S^n_y \) for some \( m \) and \( n \). Hence \( x \in S^n_x \subset S^n_z \), and \( x \in S^n_y \). This implies \( S^n_x \subset S^k+m+n \) for every \( k \) and hence \( U(x) \subseteq U(y) \). Similarly, we have \( U(y) \subseteq U(x) \), thus proving \( (z) \).

By (a), \( S_A = \bigcup_{x \in A} S_x \) is open for any subset \( A \) of \( M \). Hence \( U(x) \) is open for every \( x \in M \). By (b), \( S^n_x \) is separable for every \( n \). Hence \( U(x) \) is separable. Since \( A \) is connected and since each \( U(x) \) is open, (c) implies \( M = \bigcup_{x \in M} U(x) \). Hence \( M \) is separable, thus completing the proof of the statement in the title.

We are now in position to prove

**Theorem.** For a connected differentiable manifold \( M \), the following conditions are mutually equivalent:

1. There exists a Riemannian metric on \( M \);
2. \( M \) is metrizable;
3. \( M \) satisfies the second axiom of countability;
4. \( M \) is paracompact.

**Proof.** The implication (1) \( \rightarrow \) (2) was proved in Proposition 3.5 of Chapter IV. As we stated at the beginning, for a metric space, the second axiom of countability is equivalent to the separability. The implication (2) \( \rightarrow \) (3) is therefore a consequence of the statement in the title. If (3) holds, then \( M \) is metrizable by Urysohn’s metrization theorem (cf. Kelley [1, p. 125]) and, hence, \( M \) is paracompact (cf. Kelley [1, p. 156]). This shows that (3) implies (4). The implication (4) \( \rightarrow \) (1) follows from Proposition 1.4 of Chapter III. QED.
APPENDIX 3

Partition of unity

Let \{U_i\}_{i=1} be a locally finite open covering of a differentiable manifold \(M\), i.e., every point of \(M\) has a neighborhood which intersects only finitely many \(U_i\)'s. A family of differentiable functions \(\{f_i\}\) on \(M\) is called a partition of unity subordinate to the covering \(\{U_i\}\) if the following conditions are satisfied:

1. \(0 \leq f_i \leq 1\) on \(M\) for every \(i \in I\);
2. The support of each \(f_i\), i.e., the closure of the set \(\{x \in M; f_i(x) \neq 0\}\), is contained in the corresponding \(U_i\);
3. \(\sum f_i(x) = 1\).

Note that in (3), for each point \(x \in M\), \(f_i(x) = 0\) except for a finite number of \(i\)'s so that \(\sum f_i(x)\) is a finite sum for each \(x\).

We first prove

**Theorem 1.** Let \(\{U_i\}\) be a locally finite open covering of a paracompact manifold \(M\) such that each \(U_i\) has compact closure \(\overline{U_i}\). Then there exists a partition of unity \(\{f_i\}\) subordinate to \(\{U_i\}\).

Proof. We first prove the following three lemmas. The first two are valid without the assumption that \(M\) is paracompact whereas the third holds for any paracompact topological space.

**Lemma 1.** For each point \(x \in M\) and for each neighborhood \(U\) of \(x\), there exists a differentiable function \(f\) (of class \(C^\infty\)) on \(M\) such that (1) \(0 \leq f \leq 1\) on \(M\); (2) \(f(x) = 1\); and (3) \(f = 0\) outside \(U\).

Proof of Lemma 1. This can be easily reduced to the case where \(M = \mathbb{R}^n\), \(x = 0\) and \(U = \{x_1, \ldots, x_n\}; |x| < a\}. Then, for each \(j, j, j, \ldots, n\), we let \(f_j(x)\) be a differentiable function such that \(f_j(0) = 1\) and \(f_j(x) = 0\) for \(|x| \geq a\}. We set \(f(x) = \prod f_j(x)\). This proves Lemma 1.

**Lemma 2.** For every compact subset \(K\) of \(M\) and for every neighborhood \(U\) of \(K\), there exists a differentiable function \(f\) on \(M\) such that (1) \(f \geq 0\) on \(M\); (2) \(f > 0\) on \(K\); and (3) \(f = 0\) outside \(U\).

**Lemma 3.** Let \(\{U_i\}\) be a locally finite open covering of \(M\). Then there exists a locally finite open refinement \(\{V_i\}\) such that \(P_i \subset U_i\) for every \(i\).

Proof of Lemma 3. For each point \(x \in M\), let \(W_x\) be an open neighborhood of \(x\) such that \(W_x\) is contained in some \(U_i\). Let \(W'_x\) be a locally finite refinement of \(\{W_x; x \in M\}\). For each \(i\), let \(V_i\) be the union of all \(W'_x\) whose closures are contained in \(U_i\). Since \(\{W'_x\}\) is locally finite, we have \(\bigcup V_i = U_i\), where the union is taken over all such a \(W'_x\). Thus we obtained 'an open covering \(\{V_i\}\) with the required property.

We are now in position to complete the proof of Theorem 1. Let \(\{V_i\}\) be as in Lemma 3. For each \(i\), let \(V_i\) be an open set such that \(\overline{V}_i \subset \overline{W}_i \subset U_i\). By Lemma 2, there exists, for each \(i\), a differentiable function \(g_i\) on \(M\) such that (1) \(g_i \equiv 0\) on \(M\); (2) \(g_i > 0\) on \(V_i\); and (3) \(g_i = 0\) outside \(W_i\). Since the support of each \(g_i\) contains \(\overline{V}_i\) and is contained in \(U_i\) and since \(\{U_i\}\) is locally finite, the sum \(g = \sum g_i\) is defined and differentiable on \(M\). Since \(\{V_i\}\) is an open covering of \(M\), \(g > 0\) on \(M\). We set, for each \(i\),

\[ f_i = g_i g. \]

Then \(\{f_i\}\) is a partition of unity subordinate to \(\{U_i\}\). QED.

Proof of Lemma 2. For each point \(x\) of \(K\), let \(f_x\) be a differentiable function on \(M\) with the property off in Lemma 1. Let \(V_x\) be the neighborhood of \(x\) defined by \(f_x > \frac{1}{2}\). Since \(K\) is compact, there exist a finite number of points \(x_1, \ldots, x_k\) of \(K\) such that \(V_{x_1} \cup \cdots \cup V_{x_k} \supseteq K\). Then we set

\[ f = f_{x_1} + \cdots + f_{x_k}. \]

This completes the proof of Lemma 2.

**Theorem 2.** Let \(F\) be a closed subset of a paracompact manifold \(M\). Then every differentiable function \(f\) defined on \(F\) can be extended to a differentiable function on \(M\).
Proof. For each \( x \in F \), let \( f_x \) be a differentiable function on an open neighborhood \( V_x \) of \( x \) such that \( f_x = f \) on \( F \cap V_x \). Let \( U_i \) be a locally finite open refinement of the covering of \( M \) consisting of \( M \setminus F \) and \( V_x \) for \( x \in F \). For each \( i \), we define a differentiable function \( g_i \) on \( U_i \) as follows. If \( U_i \) is contained in some \( V_x \), we choose such a \( V_x \) and set
\[
g_i = \text{restriction of } f_x \text{ to } U_i.
\]
If there is no \( V_x \) which contains \( U_i \), then we set \( g_i = 0 \).

Let \( \{ f_i \} \) be a partition of unity subordinate to \( \{ U_i \} \). We define
\[
g = \sum_i f_i g_i.
\]
Since \( \{ U_i \} \) is locally finite, every point of \( M \) has a neighborhood in which \( \sum_i f_i g_i \) is really a finite sum. Thus \( g \) is differentiable on \( M \). It is easy to see that \( g \) is an extension of \( f \). QED.

In the terminologies of the sheaf theory, Theorem 2 means that the sheaf of germs of differentiable functions on a paracompact manifold \( M \) is soft ("mou" in Codement [1]).

**APPENDIX 4**

On arcwise connected subgroups of a Lie group

Kuranishi and Yamabe proved that every arcwise connected subgroup of a Lie group is a Lie subgroup (see Yamabe [1]). We shall prove here the following weaker theorem, which is sufficient for our purpose (cf. Theorem 4.2 of Chapter II). This result is essentially due to Freudenthal [1].

**Theorem.** Let \( G \) be a Lie group and \( H \) a subgroup of \( G \) such that every element of \( H \) can be joined to the identity \( e \) by a piecewise differentiable curve of class \( C^1 \) which is contained in \( H \). Then \( H \) is a Lie subgroup of \( G \).

**Proof.** Let \( S \) be the set of vectors \( X \in T_e(G) \) which are tangent to differentiable curves of class \( C^1 \) contained in \( H \). We identify \( T_e(G) \) with the Lie algebra \( \mathfrak{g} \) of \( G \). Then \( S \) is a subalgebra of \( \mathfrak{g} \).

**Lemma.** \( S \) is a subalgebra of \( \mathfrak{g} \).

Proof of Lemma. Given a curve \( x_t \) in \( G \), we denote by \( \dot{x}_t \) the vector tangent to the curve at the point \( x_t \). Let \( \tau \) be any real number and set \( z_\tau = x_{x_\tau} \). Then \( \dot{z}_\tau = \tau \dot{x}_0 \). This shows that if \( X \in S \), then \( \tau X \in S \). Let \( x_t \) and \( y_t \) be curves in \( G \) such that \( x_0 = y_0 = e \). If we set \( v_t = x_t y_{x_0} \), then \( \dot{v}_t = \dot{x}_t + \dot{y}_0 \) (cf. Chevalley [1, pp. 120–122]). This shows that if \( X, Y \in S \), then \( X + Y e S \).

There exists a curve \( w_t \) such that \( w_\tau = x_t y_0 \dot{y}_0^{-1} \) and we have \( w_0 = [x_0, y_0] \) (cf. Chevalley [1, pp. 120–122] or Pontrjagin [1, p. 238]). This shows that if \( X, Y \in S \), then \( [X, Y] \in S \), thus completing the proof of the lemma.

Since \( S \subseteq T_e(G) = \mathfrak{g} \) is a subalgebra of \( \mathfrak{g} \), the distribution \( x \mapsto L_x S, x \in G \), is involutive (where \( L_x \) is the left translation by \( x \)) and its maximal integral manifold through \( e \), denoted by \( K \), is the Lie subgroup of \( G \) corresponding to the subalgebra \( S \). We shall show that \( H = K \).

We first prove that \( K \supseteq H \). Let \( a \) be any point of \( H \) and \( \tau = x^t \), \( 0 \leq t \leq 1 \), a curve from \( e \) to \( \mathfrak{g} \) so that \( e = x_0 \) and \( a = x_1 \). We claim...
that the vector $\dot{x}_t$ is in $L_2 S$ for all $t$. In fact, for each fixed $t$, $L_x^{-1}(\dot{x}_t)$ is the vector tangent to the curve $L_x^{-1}(\tau)$ at $e$ and hence lies in $S$, thus proving our assertion. Since $\dot{x}_t \in L_2 S$ for all $t$ and $x_0 = e$, the curve $x_t$ lies in the maximal integral manifold $K$ of the distribution $x \to L_x S$ (cf. Lemma 2 for Theorem 7.2 of Chapter II). Hence $a \in K$, showing that $K \subseteq H$.

To prove that $H \supseteq K$, let $e_{t_1}, \ldots, e_{t_k}$ be a basis for $S$ and $x_0, \ldots, x_k$ be curves in $H$ such that $x_0 = e$ and $x_0 = e_i$ for $i = 1, \ldots, k$. Consider the mapping $f$ of a neighborhood $U$ of the origin in $R^n$ into $K$ defined by $f(t_1, \ldots, t_k) = x_0 + x_1 t_1 + \cdots + x_k t_k$, $t_j \in R$, $j = 1, \ldots, k$. Since $x_0, \ldots, x_k$ form a basis for $S$, the differential of $f$ at the origin is non-singular. Taking $U$ sufficiently small, we may assume that $f$ is a diffeomorphism of $U$ onto an open subset $f(U)$ of $K$. From the definition of $f$, we have $f(U) \subseteq H$. This shows that a neighborhood of $e$ in $K$ is contained in $H$. Since $K$ is connected, $K \subseteq H$.

APPENDIX 5

Irreducible subgroups of $O(n)$

We prove the following two theorems.

**Theorem 1.** Let $G$ be a subgroup of $O(n)$ which acts irreducibly on the $n$-dimensional real vector space $R^n$. Then every symmetric bilinear form on $R^n$ which is invariant by $G$ is a multiple of the standard inner product $(x, y) = \sum_{i=1}^n x_i y_i$.

**Theorem 2.** Let $G$ be a connected Lie subgroup of $SO(n)$ which acts irreducibly on $R^n$. Then $G$ is closed in $SO(n)$.

We begin with the following lemmas.

**Lemma 1.** Let $G$ be a subgroup of $GL(n; R)$ which acts irreducibly on $R^n$. Let $A$ be a linear transformation of $R^n$ which commutes with every element of $G$. Then

1. If $A$ is nilpotent, then $A = 0$.
2. The minimal polynomial of $A$ is irreducible over $R$.
3. Either $A = a I_n$ (a real number, $I_n$ the identity transformation of $R^n$), or $A = a I_n + b J$, where $a$ and $b$ are real numbers, $b \neq 0$, $J$ is a linear transformation such that $J^2 = -I_n$, and $n$ is even.

Proof. (1) Let $k$ be the smallest integer such that $A^k = 0$. Assuming that $k \geq 2$, we derive a contradiction. Let $W = \{x \in R^n : A^k x = 0\}$. Since $W$ is invariant by $G$, we have either $W = R^n$ or $W = \{0\}$. In the first case, $A = 0$. In the second case, $A$ is non-singular and $A^{k-1} = A^{-1} \cdot A^k = 0$.

(2) If the minimal polynomial $f(x)$ of $A$ is a product $f_1(x) \cdot f_2(x)$ with $(f_1, f_2) = 1$, then $R^n = W_1 + W_2$ (direct sum), where $W_i = \{x \in R^n : f_i(A) x = 0\}$. Since every element of $G$ commutes with $A$ and hence with $f_i(A)$, it follows that $W_i$ are both invariant by $G$, contradicting the assumption of irreducibility. Thus
f(x) = g(x)x, where g(x) is irreducible. Applying (1) to g(A), we see that f(A) = g(A)x = 0 implies g(A) = 0. Thus f = g.

(3) By (2), the minimal polynomial f(x) of A is either (x - a) or (x - a)^2 + b^2 with b ≠ 0. In the first case, A = aI_n. In the second case, let J = (A - aI_n)/b. Then J^2 = -I, and A = aI_n + bJ. We have (-1)^n = det J^n = (det J)^2 > 0 so that n is even.

**Lemma 2.** Let G be a subgroup of O(n) which acts irreducibly on R^n.

Let A, B, . . . be linear transformations of R^n which commute with G.

(1) If A is symmetric, i.e., (Ax, y) = (x, Ay), then A = aI_n.

(2) If A is skew-symmetric, i.e., (Ax, y) + (y, x) = 0, then A = 0 or A = bJ, where J^2 = -I, and n = 2m.

(3) If A ≠ 0 and B are skew-symmetric and AB = BA, then B = cA.

**Proof.** (1) By (3) of Lemma 1, A = aI_n + bJ, possibly with b = 0. If A is symmetric, so is bJ. If b ≠ 0, J is symmetric so that (Jx, Jy) = (x, Jy) = -(x, y) is a contradiction for x ≠ 0.

(2) Since the eigenvalues of skew-symmetric A are 0 or purely imaginary, the minimal polynomial of A is either x or x^2 + b^2, b ≠ 0. In the first case, A = 0. In the second case, A = -bJ with J^2 = -I.

(3) Let A = bJ and B = b'K, where J^2 = K^2 = -I,. We have JK = KJ. We show that R^n = W_1 + W_2 (direct sum), where W_1 = {x ∈ R^n : Jx = Kx} and W_2 = {x ∈ R^n : Jx = -Kx}. Clearly, W_1 ∩ W_2 = {0}. Every x ∈ R^n is of the form y + z with y ∈ W_1 and z ∈ W_2, as we see by setting y = (x - Jy)/2 and z = (x + Jy)/2. W_1 and W_2 are invariant by G, because J and K commute with every element of G. Since G is irreducible, we have either W_1 = R^n or W_2 = R^n, that is, either K = J or K = -J. This means that tB = cA for some c.

**Proof of Theorem 1.** For any symmetric bilinear form f(x, y), there is a symmetric linear transformation A such that f(x, y) = (Ax, y). If f is invariant by G, then A commutes with every element of G. By (1) of Lemma 2, A = aI_n and hence f(x, y) = a(x, y).

**Proof of Theorem 2.** We first show that the center 3 of the Lie algebra g of G is at most 1-dimensional. Let A ≠ 0 and B ∈ 3. Since A, B are skew-symmetric linear transformations, which commute with every element of G, (3) of Lemma 2 implies that B = cA for some c. Thus dim 3 ≤ 1. If dim 3 = 1, then 3 = {cJ : c real}, where J is a certain skew-symmetric linear transformation with J^2 = -I. Now J is representable by a matrix which is a block form, each block being

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

with respect to a certain orthonormal basis of R^n. The l-parameter subgroup exp tJ consists of matrices of a block form, each block being

\[
\begin{pmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{pmatrix}
\]

and hence is isomorphic with the circle group.

Since g is the subalgebra of the Lie algebra of all skew-symmetric matrices, g has a positive definite inner product (A, B) = -trace(AB) which is invariant under G. It follows that the orthogonal complement s of the center 3 in g with respect to this inner product is an ideal of g and g = 3 + s is the direct sum. If s contains a proper ideal, say, s_1, then the orthogonal complement s' of s_1 in s is an ideal of s (in fact of g) and s = s_1 + s'. Thus we see that s is a direct sum of simple ideals: s = s_1 + · · · + s_k. We have already seen that the connected Lie subgroup generated by 3 is closed in SO(n), with respect to the inner product.

We now show that the connected Lie subgroup generated by s is closed in SO(n). This will finish the proof of Theorem 2.

We first remark that Yosida [1] proved the following result. Every connected semisimple Lie subgroup G of GL(n; C) is closed in GL(n; C). His proof, based on a theorem of Weyl that any representation of a semisimple Lie algebra is completely reducible, also works when we replace GL(n; C) by GL(n; R). In the case of a subgroup G of SO(n), we need not use the Weyl theorem. We now prove the following result by the same method as Yosida's.

A connected semisimple Lie subgroup G of SO(n) is closed in SO(n). Proof. Since g is a direct sum of simple ideals g_1, . . . . g_k of dimension > 1 and since g_i = g_i, for each i, it follows that g = [g, g]. Now consider SO(n) and hence its subgroup G as acting on the complex vector space C^n with standard hermitian inner product which is left invariant by SO(n). Then C^n is the direct sum of complex subspaces V_1, . . . , V_r which are invariant and irreducible by G. Assuming that G is not closed in SO(n), let G be its closure. Since G is a connected closed subgroup of SO(n), it is a Lie subgroup. Let G be its Lie algebra. Obviously, g ⊂ G, g ⊂ g. Since ad (G)g ⊂ g, we have ad (G)g ⊂ g. which implies that g is
an ideal of \( \mathfrak{g} \). Since the Lie algebra of \( SO(n) \) has a positive definite inner product invariant by \( \text{ad} \) (\( SO(n) \)) as we already noted, it follows that \( \mathfrak{g} \) is the direct sum of \( \mathfrak{g} \) and the orthogonal complement \( \mathfrak{u} \) of \( \mathfrak{g} \) in \( \mathfrak{g} \). Each summand \( V_i \) of \( \mathbb{C}^n \) is also invariant by \( \mathfrak{g} \) and hence by \( \mathfrak{g} \) acting on \( \mathbb{C}^n \). For any \( A \in \mathfrak{g} \), denote by \( A_i \) its action on \( V_i \) for each \( i \). For any \( A, B \in \mathfrak{g} \), we have obviously trace \([A, B_i]\) = 0. Since \( A \rightarrow A_i \) is a representation of \( \mathfrak{g} \) on \( V_i \), and since \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \), we have trace \( A_i = 0 \) for every \( A \in \mathfrak{g} \). Thus the restriction of \( A \in \mathfrak{g} \) on each \( V_i \) has determinant 1. (cf. Corollary 1 of Chevalley [I, p. 5]). By continuity, the restriction of \( A \in \mathfrak{g} \) on each \( V_i \) has determinant 1. This means that trace \( A_i = 0 \) for every \( A \in \mathfrak{g} \) and for each \( i \). Now let \( B \in \mathfrak{g} \). Its action \( B_i \) on \( V_i \) commutes with the actions of \( \{A_i; A \in \mathfrak{g}\} \). By Schur's Lemma (which is an obvious consequence of Lemma 1, (2), which is valid for any field instead of \( \mathbb{R} \)), we have \( B_i = b_i I \), where \( I \) is the identity transformation of \( V_i \). Since trace \( B_i = 0 \), it follows that \( b_i = 0 \), that is, \( B = 0 \). This being the case for each \( i \), we have \( B = 0 \). This means that \( u = (0) \) and \( \overline{\mathfrak{g}} = \mathfrak{g} \). This proves that \( \mathfrak{g} = G \), that is, \( G \) is closed in \( SO(n) \).

### Appendix 6

**Green's theorem**

Let \( M \) be an oriented \( n \)-dimensional differentiable manifold. An \( n \)-form \( \omega \) on \( M \) is called a **volume element**, if \( \omega(\partial/\partial x^1, \ldots, \partial/\partial x^n) > 0 \) for each oriented local coordinate system \( x^1, \ldots, x^n \). With a fixed volume element \( \omega \) (which will be also denoted by a more intuitive notation \( dv \)), the integral \( \int_M f \, dv \) of any continuous function \( f \) with compact support can be defined (cf. Chevalley [I, pp. 161-167]).

For each vector field \( X \) on \( M \) with a fixed volume element \( \omega \), the **divergence** of \( X \), denoted by \( \text{div} \, X \), is a function on \( M \) defined by

\[
(\text{div} \, X) \omega = L_X \omega,
\]

where \( L_X \) is the Lie differentiation in the direction of \( X \).

**Green's theorem.** Let \( M \) be an oriented compact manifold with a fixed volume element \( \omega = dv \). For every vector field \( X \) on \( M \), we have

\[
\int_M \text{div} \, X \, dv = 0.
\]

Proof. Let \( \varphi_t \) be the \( t \)-parameter group of transformations generated by \( X \) (cf. Proposition 1.6 of Chapter I). Since we have (cf. Chevalley [I, p. 165])

\[
\int_M \varphi_t^{-1} * \omega = \int_M \omega,
\]

\[
\int_M \varphi_t^{-1} * \omega, \text{ considered as a function of } t, \text{ is a constant. By definition}
\]

(\( \text{div} \, X \) \( \omega = L_X \omega \),

where \( L_X \) is the Lie differentiation in the direction of \( X \).
of \( L_X \), we have \( \left[ \frac{d}{dt} (q_{t}^{-1} \omega) \right]_{t=0} = -L_X \omega \). Hence
\[
0 = \left[ \frac{d}{dt} \int_M q_{t}^{-1} \omega \right]_{t=0} = \int_M \left[ \frac{d}{dt} (q_{t}^{-1} \omega) \right]_{t=0} = -\int_M L_X \omega = -\int_M \text{div} X \, dv.
\]
Hence we have \( \omega = -\text{div} X \, dv \). QED.

Remark 1. The above formula is valid for a non-compact manifold \( M \) as long as \( X \) has a compact support.

Remark 2. The above formula follows also from Stokes' formula. In fact, since \( ds = 0 \), we have \( \omega = d \circ \sigma (\omega) \),
\[
\int_M L_X \omega = \int_M \text{div} X \, dv.
\]
We then have \( \omega = -\text{div} X \, dv \).

Proposition. Let \( M \) be an oriented manifold with a fixed volume element \( \omega = dv \). If \( \Gamma \) is an affine connection with no torsion on \( M \), then, for every vector field \( X \) on \( M \), we have
\[
\text{div} X = -\text{trace} (A_X).
\]
Proof. Let \( A_X \) be the tensor field of type \((1,1) \) defined by \( A_X = L_X - \nabla_X \) as in §2 of Chapter VI. Let \( X_1, \ldots, X_n \) be a basis of \( T, (M) \). Since \( \nabla_X \omega = 0 \) and since \( A_X \), as a derivation, maps every function into zero, we have
\[
(L_X \omega)(X_1, \ldots, X_n) = (A_X \omega)(X_1, \ldots, X_n) = A_X(\omega(X_1, \ldots, X_n)) - \omega(X_1, \ldots, A_X X_1, \ldots, X_n) = -\omega(X_1, \ldots, A_X X_1, \ldots, X_n) = -\omega(\text{trace} A_X, X_1, \ldots, X_n).
\]
This shows that \( \text{div} X = -\text{trace} A_X \).

Our assertion follows from the formula (cf. Proposition 2.5 of Chapter VI):
\[
A_X Y = -\nabla_Y X - T(X, Y)
\]
and from the assumption that \( T = 0 \). QED.

Remark 3. The formula \( \text{div} X = -\text{trace} A_X \) holds without the assumption \( T = 0 \).

Let \( M \) be an oriented Riemannian manifold. We define a natural volume element \( dv \) on \( M \). At an arbitrary point \( x \) of \( M \), let \( X_1, \ldots, X_n \) be an orthonormal basis of \( T, (M) \) compatible with the orientation of \( M \). We define an n-form \( dv \) by
\[
dv(X_1, \ldots, X_n) = 1.
\]
It is easy to verify that \( dv \) is defined independently of the frame \( X_1, \ldots, X_n \) chosen. In terms of an allowable local coordinate system \( x^1, \ldots, x^n \) and the components \( g_{ij} \) of the metric tensor \( g \), we have
\[
dv = \sqrt{G} \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,
\]
where \( G = \det (g_{ij}) \).

In fact, let \( \left( \frac{\partial}{\partial x^i} \right)_x = \sum_k C^k_i X_k \) so that \( g_{ij} = \sum_k C^k_i C^j_k \) and \( G = \det (C^k_i) \) at \( x \). Since \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \) and \( X_1, \ldots, X_n \) have the same orientation, we have \( \det (C^k_i) = \sqrt{G} > 0 \). Hence, at \( x \), we have
\[
dv(\partial/\partial x^1, \ldots, \partial/\partial x^n) = \sum_i \ldots \sum_i \epsilon C^i_1 \ldots C^n_i \, dv(X_i, \ldots, X_n) = \det (C^k_i) = \sqrt{G},
\]
where \( \epsilon \) is 1 or -1 according as \((i_1, \ldots, i_n)\) is an even or odd permutation of \((1, \ldots, n)\).

Since the parallel displacement along any curve \( \tau \) of \( M \) maps every orthonormal frame into an orthonormal frame and preserves the orientation, the volume element \( dv \) is parallel. Thus the proposition as well as Green's theorem is valid for the volume element \( dv \) of a Riemannian manifold.
APPENDIX 7

Factorization lemma

Let $\mathcal{A}$ be a differentiable manifold. Two continuous curves $x(t)$ and $y(t)$ defined on the unit interval $Z = [0, 1]$ with $x(0) = y(0)$ and $x(1) = y(1)$ are said to be homotopic to each other if there exists a continuous mapping $f: (t, s) \in Z \times Z \rightarrow f(t, s) \in \mathcal{A}$ such that $f(0, s) = x(0), f(1, s) = y(0)$ and $f(t, 0) = x(t), f(t, 1) = y(t)$ for every $t$ and $s$ in $I$. When $x(t)$ and $y(t)$ are piecewise differentiable curves of class $C^k$ (briefly, piecewise $C^k$-curves), they are piecewise $C^k$-homotopic, if the mapping $f$ can be chosen in such a way that it is piecewise $C^k$ on $Z \times Z$, that is, for a certain subdivision $Z = \sum_i I_i$, $f$ is a differentiable mapping of class $C^k$ of $I_i \times Z$, into $M$ for each $(i, j)$.

**LEMMA.** If two piecewise $C^k$-curves $x(t)$ and $y(t)$ are homotopic to each other, then they are piecewise $C^k$-homotopic.

Proof. We can take a suitable subdivision $Z = \sum_i I_i$ so that $f(I_i \times Z_i)$ is contained in some coordinate neighborhood for each pair $(i, j)$. By modifying the mapping $f$ in the small squares $I_i \times Z_i$, we can obtain a piecewise $C^k$-homotopy between $x(t)$ and $y(t)$.

Now let $\mathcal{U}$ be an arbitrary open covering. We shall say that a closed curve $\tau$ at a point $x$ is a $\mathcal{U}$-lasso if it can be decomposed into three curves $\tau = \mu^{-1} \cdot \sigma \cdot \mu$, where $\mu$ is a curve from $x$ to a point $y$ and $\sigma$ is a closed curve at $y$ which is contained in an open set of $\mathcal{U}$. Two curves $\tau$ and $\tau'$ are said to be equivalent, if $\tau'$ can be obtained from $\tau$ by replacing a finite number of times a portion of the curve of the form $\mu^{-1} \cdot \mu$ by a trivial curve consisting of a single point or vice versa. With these definitions, we prove

Factorization LEMMA. Let $\mathcal{U}$ be an arbitrary open covering of $M$.

(a) Any closed curve which is homotopic to zero is equivalent to a product of a finite number of $\mathcal{U}$-lassos.

(b) If the curve is moreover piecewise $C^k$, then each $\mathcal{U}$-lasso in the product can be chosen to be of the form $\mu^{-1} \cdot \sigma \cdot \mu$, where $\mu$ is a piecewise $C^k$-curve and $\sigma$ is a $C^k$-curve.

Proof. (a) Let $\tau = x(t), 0 \leq t \leq 1$, so that $x = x(0) = x(1)$. Let $f$ be a homotopy $Z \times I \rightarrow M$ such that $f(t, 0) = x(t), f(t, 1) = x_s, f(0, s) = f(1, s) = x$ for every $t$ and $s$ in $I$. We divide the square $Z \times Z$ into $m^2$ equal squares so that the image of each small square by $f$ falls in some open set of the covering $\mathcal{U}$. For each pair of integers $(i, j), 1 \leq i, j \leq m$, let $\lambda(i, j)$ be the closed curve in the square $Z \times Z$ consisting of line segments joining lattice points in the following order:

$$(0, 0) \rightarrow (0, j|m) \rightarrow ((i - 1)|m, j|m) \rightarrow ((i - 1)|m, j|m) \rightarrow ((i - 1)|m, j|m) \rightarrow (0, 0).$$

Geometrically, $\lambda(i, j)$ looks like a lasso. Let $\tau(i, j)$ be the image of $\lambda(i, j)$ by the mapping $f$. Then $\tau$ is equivalent to the product of $\mathcal{U}$-lassos

$$\tau(m, m) \cdots \tau(1, 1).$$

(b) By the preceding lemma, we may assume that the homotopy mapping $f$ is piecewise $C^k$. By choosing $m$ larger if necessary, we may also assume that $f$ is $C^k$ on each of the $m^2$ small squares. Then each lasso $\tau(i, j)$ has the required property. QED.

The factorization lemma is taken from Lichnerowicz [2, p. 513.}

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NOTES

1. **Connections and holonomy groups**

   Although differential geometry of surfaces in the 3-dimensional Euclidean space goes back to Gauss, the notion of a Riemannian space originates with Riemann’s Habilitationsschrift [1] in 1854. The Christoffel symbols were introduced by Christoffel [1] in 1869. Tensor calculus, founded and developed in a series of papers by Ricci, was given a systematic account in Levi-Civita and Ricci [1] in 1901. Covariant differentiation which was formally introduced in this tensor calculus was given a geometric interpretation by Levi-Civita [1] who introduced in 1917 the notion of parallel displacement for the surfaces. This discovery led Weyl [1, 2] and E. Cartan [1, 2, 4, 5, 8, 9] to the introduction of affine, projective and conformal connections. Although the approach of Cartan is the most natural, one and reveals best the geometric nature of the connections. It was not until 1950 that Ehresmann [2] clarified the general notion of connections from the point of view of contemporary mathematics. His paper was followed by Chern [1, 2], Ambrose-Singer [1], Kobayashi [6], Nomizu [7], Lichnerowicz [2] and others.

   Ehresmann [2] defined, for the first time, a connection in an arbitrary fibre bundle as a field of horizontal subspaces and proved the existence of connections in any bundle. He introduced also a connection form $\omega$ and defined the curvature form $\Omega$ by means of the structure equation. The definition of $\Omega$ given in this book is due to Ambrose and Singer [1] who proved also the structure equation (Theorem 5.2 of Chapter II). Chern [1, 2] defined a connection by means of a set of differential forms $\omega_a$ on $U_a$ with values in the Lie algebra of the structure group, where $\{U_a\}$ is an open covering of the base manifold (see Proposition 1.4 of Chapter II).

   Ehresmann [2] also defined the notion of a Cartan connection, whose examples include affine, projective and conformal connections. See also Kobayashi [6] and Takizawa [1]. We have
The notion of holonomy group is due to E. Cartan [1, 6]. The fact that the holonomy group is a Lie group was taken for granted even for a Riemannian connection until Borel and Lichnerowicz [1] proved it explicitly. The holonomy theorem (Theorem 8.1 of Chapter II) of E. Cartan was rigorously proved first by Ambrose-Singer [1]. The proof was simplified by Nomizu [7] and Kobayashi [6] by first proving the reduction theorem (Theorem 7.1 of Chapter II), which is essentially due to, Cartan and Ehresmann. Kobayashi [6] showed that Theorem 8.1 is essentially equivalent to the following fact. For a principal fibre bundle $P(M, G)$, consider the principal fibre bundle $T(P)$ over $T(M)$ with group $T(G)$, where $T(\cdot)$ denotes the tangent bundle. For any connection $\Gamma$ in $P$, there is a naturally induced connection $T(\Gamma)$ in $T(P)$ whose holonomy group is $T(\Phi)$, where $\Phi$ is the holonomy group of $\Gamma$.

The result of Hano and Ozeki [1] and Nomizu [5] (Theorem 8.2 of Chapter II) to the effect that the structure group $G$ of $P(M, G)$ can be reduced to a subgroup $H$ if and only if there exists a connection in $P$ whose holonomy group is exactly $H$ means that the holonomy group by itself does not give any information other than those obtainable by topological methods. However, combined with other circumstances (such as a "torsion-free linear connection"), the holonomy group is of considerable interest.

3. Chern [3] defined the notion of a $G$-structure on a differentiable manifold $M$, where $G$ is a certain Lie subgroup of $GL(n; \mathbb{R})$ with $n = \dim M$. In our terminologies, a $G$-structure on $M$ is a reduction of the bundle of linear frames $t(M)$ to the subgroup $G$. For $G \leq O(n)$, a $G$-structure is nothing but a Riemannian metric given on $M$ (see Example 5.7 of Chapter I). For a general theory of $G$-structures, see Chern [3], Bernard [1] and Fujimoto [1]. We mention some other special cases.

Weyl [3] and E. Cartan [9] proved the following. For a closed subgroup $G$ of $GL(n; \mathbb{R})$, $n \geq 3$, the following two conditions are equivalent:

1. $G$ is the group of all matrices which preserve a certain non-degenerate quadratic form of any signature.

(2) For every $n$-dimensional manifold $M$ and for every reduced subbundle $P$ of $L(M)$ with group $G$, there is a unique torsion-free connection in $P$.

The implication (1) $\Rightarrow$ (2) is clear from Theorem 2.2 of Chapter IV (in which $g$ can be an indefinite Riemannian metric); in fact, if $G$ is such a group, any $G$-structure on $M$ corresponds to an indefinite Riemannian metric on $M$ in a similar way to Example 5.7 of Chapter I. The implication (2) $\Rightarrow$ (1) is non-trivial. See also Klingenberg [1].

Let $G$ be the subgroup of $GL(n; \mathbb{C})$ consisting of all matrices which leave the $r$-dimensional subspace $\mathbb{R}^r$ of $\mathbb{R}^n$ invariant. A $G$-structure on an $n$-dimensional manifold $M$ is nothing but an $r$-dimensional distribution. Walker [3] proved that an $r$-dimensional distribution is parallel with respect to a certain torsion-free linear connection if and only if the distribution is integrable. See also Willmore [1, 2].

Let $G$ be $GL(n; \mathbb{C})$ regarded as a subgroup of $GL(2n; \mathbb{R})$ in a natural manner. A $G$-structure on a $2n$-dimensional manifold $M$ is nothing but an almost complex structure on $M$. This structure will be treated in Volume II.

4. The notions of local and infinitesimal holonomy groups were introduced systematically by Nijenhuis [2]. The results in §10 of Chapter II were obtained by him in the case of a linear connection (§9 of Chapter III). Nijenhuis' results were generalized by Ozeki [1] to the general case as presented in §10 of Chapter II. See also Nijenhuis [3], Chevalley also obtained Corollary 10.7 of Chapter II in the case of a linear connection (unpublished) and his result was used by Nomizu [2] who discussed invariant linear connections on homogeneous spaces. His results were generalized by Wang [1] as in §11 of Chapter II.

5. By making use of a connection, one can define characteristic classes of any principal fibre bundle. This will be treated in Volume II. See Chern [2], H. Cartan [2, 3]. We shall here state a result of Narasimhan and Ramanan [1] which is closely related to the notion of a universal bundle (cf. Steenrod [1, p. 101]).

Theorem. Given a compact Lie group $G$ and a positive integer $n$, there exists a principal bundle $E(N, G)$ and a connection $\Gamma_0$ on $E$ such that any connection $\Gamma$ in any principal bundle $P(M, G)$, dim $M \leq n$, can be obtained as the inverse image of $\Gamma_0$ by a certain homomorphism of $P$ into $E$. 

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(that is, \( \omega = f \cdot \omega_0 \), where \( \omega_0 \) and \( \omega \) are the connection forms of \( \Gamma \) and \( \Gamma_0 \), respectively, see Proposition 6.2 of Chapter II).

The connection \( \Gamma_0 \) is therefore called a universal connection for \( G \) (and \( n \)). For example, the canonical connection in a Stiefel manifold with structure group \( O(k) \) is universal for \( O(k) \). For the canonical connections in the Stiefel manifolds, see also Kobayashi [5] who gave an interpretation of the Riemannian connections of manifolds imbedded in Euclidean spaces (see Volume II).

6. The holonomy groups of linear and Riemannian connections were studied in detail by Berger [1]. By a careful examination of the curvature tensor, he obtained a list of groups which can be restricted linear holonomy groups of irreducible Riemannian manifolds with non-parallel curvature tensor. His list coincides with the list of connected orthogonal groups acting transitively on spheres. Simons [1] proved directly that the linear holonomy group of an irreducible Riemannian manifold with non-parallel curvature tensor is transitive on the unit sphere in the tangent space. See Note 7 (symmetric spaces).

7. The local decomposition of a Riemannian manifold (Proposition 5.2 of Chapter IV) has been treated by a number of authors. The global decomposition (Theorem 6.2 of Chapter IV) was proved by de Rham [1]; the same problem was also treated by Walker [2]. A more general situation than the direct product has been studied by Reinhart [1], Nagano [2] and Hermann [1].

It is worthwhile noting that even the local decomposition is a strongly metric property. Ozeki gave an example of a torsion-free linear connection with the following property. The linear holonomy group is completely reducible (that is, the tangent space is the direct sum of invariant irreducible subspaces) but the linear connection is not a direct product even locally. His example is as follows: On \( \mathbb{R}^2 \) with coordinates \((x^1, x^2)\), take the linear connection given by the Christoffel symbols \( \Gamma^i_{jk} = x^j \) and other \( \Gamma^i_{jk} = 0 \). The holonomy group is \( \{(a, 0) \mid a > 0 \} \).

8. The restricted linear holonomy group of an arbitrary Riemannian manifold is a closed subgroup of the orthogonal group. Hano and Ozeki [1] gave an example of a torsion-free linear connection whose restricted holonomy group is not closed in the general linear group. The linear holonomy group of an arbitrary Riemannian manifold is not in general compact, as Example 4.3 of Chapter V shows. For a compact flat Riemannian manifold, it is compact (Theorem 4.2 of Chapter V). Recently, Wolf [6] proved that this is also the case for a compact locally symmetric Riemannian manifold.

dote 2. Complete affine and Riemannian connections

Hoff and Rinow [1] proved Theorem 4.1 (the equivalence of (1), (2) and (3)), Theorem 4.2 and Theorem 4.4 of Chapter IV. Theorem 4.2 goes back to Hilbert [1]; his proof can be also found in E. Cartan's book [8]. In §4 of Chapter IV, we followed the appendix of de Rham [1]. Condition (4) of Theorem 4.1 of Chapter IV was given as the definition of completeness by Ehresmann [1, 2].

For a complete affine connection, it does not hold in general that every pair of points can be joined by a geodesic. To construct counterexamples, consider an affine connection on a connected Lie group \( G \) such that the geodesics emanating from the identity are precisely the l-parameter groups of \( G \). Such connections will be studied in Volume II. For our present purpose, it suffices to consider the affine connection which makes every left invariant vector field parallel; the existence and the uniqueness of such a connection is easy to see. Then the question is whether every element of \( G \) is on a l-parameter subgroup. The answer is yes, if \( G \) is compact (well known) or if \( G \) is nilpotent (cf. Matsushima [1]). For a solvable group \( G \), this is no longer true in general; Saito [1] gave a necessary and sufficient condition in terms of the Lie algebra of \( G \) for the answer to be affirmative when \( G \) is a simply connected solvable group. For some linear real algebraic groups, this question was studied by Sibuya [1]. Even for a simple group, the answer is not affirmative in general. For instance, a direct computation shows that an element

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\]

of \( SL(2; \mathbb{R}) \) lies on some l-parameter subgroup if and only if
either \( a + d > -2 \) or \( a = d = -1 \) and \( b = c = 0 \). This means that, for every element \( A \) of \( SL(2; \mathbb{R}) \), either \( A \) or \(-A \) (possibly both) lies on a 1-parameter subgroup. Thus the answer to our question is negative for \( SL(2; \mathbb{R}) \) and is affirmative for \( SL(2; \mathbb{R}') \) modulo its center. Smith \([1]\) also constructed a Lorentz metric, i.e., an indefinite Riemannian metric, on a 2-dimensional manifold such that the (Riemannian) connection is complete, and that not every pair of points can be connected by a geodesic. It is not known whether every pair of points of a compact, connected manifold with a complete affine connection can be joined by a geodesic.

An affine connection on a compact manifold is not necessarily complete as the following example of Auslander and Markus \([1]\) shows. Consider the Riemannian connection on the real line \( \mathbb{R}^1 \) defined by the metric \( ds^2 = c^2 \, dx^2 \), where \( x \) is the natural coordinate system in \( \mathbb{R}^1 \); it is flat. It is not complete as the length of the geodesic from \( x = 0 \) to \( x = -\infty \) is equal to 2. The translation \( x \to x + 1 \) is an affine transformation as it sends the metric \( c^2 \, dx^2 \) into \( c^2 \, dx^2 \). Thus the real line modulo 1, i.e., a circle, has a non-complete flat affine connection. This furnishes a non-complete, compact, homogeneous affine connected manifold. An example of a non-complete affine connection on a simply connected compact manifold is obtained by defining the above affine connection on the equator of a sphere and extending it on the whole sphere so that the equator is a geodesic.

It is known that every metrizable space admits a complete uniform structure (compatible with the topology) \( (\text{Dieudonné}\ [1]) \). Nomizu and Ozeki \([1]\) proved that, given a Riemannian metric \( g \) on a manifold \( M \), there exists a positive function \( f \) such that \( f^{-1}g \) is a complete Riemannian metric.

**Note 3. Ricci tensor and scalar curvature**

Analogous to the theorem of Schur (Theorem 2.2 of Chapter V), we have the following classical result.

**Theorem 1.** Let \( M \) be a connected Riemannian manifold with metric tensor \( g \) and Ricci tensor \( S \). If \( S = \lambda g \), where \( \lambda \) is a function on \( M \), then \( \lambda \) is necessarily a constant provided that \( n = \dim M \geq 3 \).

Proof. The simplest proof is probably by means of the classical tensor calculus. Let \( g_{ij}R_{ijkl} \) and \( R_{ij} \) be the components of the metric tensor \( g \), the Riemannian curvature tensor \( R \) and the Ricci tensor \( S \), respectively, with respect to a local coordinate system \( x^1, \ldots, x^n \). Then Bianchi's second identity (Theorem 5.3 of Chapter III) is expressed by

\[
R_{ijkl} = R_{ijkm} + R_{ijml} - R_{ijlm} - R_{ijmk} = 0.
\]

Multiplying by \( g^{ik} \) and \( g^{jl} \), summing with respect to \( i, j, k \) and \( l \) and finally using the following formulas

\[
R_{ijkl} = -R_{ijkl} - R_{ijlk} + R_{iklj},
\]

we obtain

\[
(n - 2) \lambda_{,km} = 0.
\]

Hence \( \lambda \) is a constant.

A Riemannian manifold is called an **Einstein manifold** if \( S = \lambda g \), where \( \lambda \) is a constant.

The following proposition is due to Schouten and Struik \([1]\).

**Proposition 2.** If \( M \) is a 3-dimensional Einstein manifold, then it is a space of constant curvature.

Proof. Let \( p \) be any plane in \( T_p(M) \) and let \( X_1, X_2, X_3 \) be an orthonormal basis for \( T_p(M) \) such that \( p \) is spanned by \( X_1, X_2 \). Let \( p_{ij} \) be the plane spanned by \( X_i \) and \( X_j \) \((i \neq j)\) so that \( p_{ii} = p_{ij} = p_{ji} \).

Then

\[
S(X_1, X_1) = K(p_{11}) + K(p_{12}),
\]

\[
S(X_2, X_2) = K(p_{22}) + K(p_{23}),
\]

\[
S(X_3, X_3) = K(p_{33}) + K(p_{34}),
\]

where \( K(p_{ij}) \) denotes the sectional curvature determined by the plane \( p_{ij} \). Hence we have

\[
S(X_1, X_1) + S(X_2, X_2) - S(X_3, X_3) = 2K(p_{11}) = 2\lambda.
\]

Since \( S(X_1, X_1) = 1 \), we have \( K(p) = \lambda \). Q.E.D.

**Remark.** The above formula implies also that, if \( 0 < c < S(X, X) < 2\epsilon \) for all unit vectors \( X \in T_p(M) \), then \( K(p) > 0 \) for all planes \( p \) in \( T_p(M) \). Similarly, if \( c \leq S(X, X) < c \) for all unit vectors \( X \in T_p(M) \), then \( K(p) < 0 \) for all planes \( p \) in \( T_p(M) \).
Going back to the general case where \( n = \dim M \) is arbitrary, let \( X_1, \ldots, X_n \) be an orthonormal basis for \( T_x(M) \). Then \( S(X_1, X_1) + \cdots + S(X_n, X_n) \) is independent of the choice of orthonormal basis and is called the scalar curvature at \( x \). In terms of the components \( R_{ij} \) and \( g_{ij} \) of \( S \) and \( g \), respectively, the scalar curvature is given by \( \Sigma_{j=1}^{n} g^{ij}R_{ij} \).

**Note 4. Spaces of constant positive curvature**

Let \( M \) be an \( n \)-dimensional, connected, complete Riemannian manifold of constant curvature \( 1/a^2 \). Then, by Theorem 3.2 of Chapter V and Theorem 7.10 of Chapter VI, the universal covering manifold of \( M \) is isometric to the sphere \( S^n \) of radius \( a \) in \( \mathbb{R}^{n+1} \) given by \((x^1)^2 + \cdots + (x^{n+1})^2 = a^2 \), that is, \( M = S^n/G \), where \( G \) is a finite subgroup of \( O(n + 1) \) which acts freely on \( S^n \).

In the case where \( n \) is even, the determination of these groups \( G \) is extremely simple. Let \( \chi(M) \) denote the Euler number of \( M \). Then we have (cf. Hu [1, p. 277])

\[
\chi(S^n) = \chi(M) \times \text{order of } G \quad (\text{if } n \text{ is even}).
\]

Hence, \( G \) consists of either the identity 1 only or \( Z \) and another element \( A \) of \( O(n + 1) \) such that \( A^2 = 1 \). Clearly, the eigen-values of \( A \) are \( \pm 1 \). Since \( A \) cannot have any fixed point on \( S^n \), the eigen-values of \( A \) are all equal to \( -1 \). Hence, \( A = -I \). We thus obtained

**Theorem 1.** Every connected, complete Riemannian manifold \( M \) of even dimension \( n \) with constant curvature \( 1/a^2 \) is isometric either to the sphere \( S^n \) of radius \( a \) or to the real projective space \( S^n/(\pm I) \).

The case where \( n \) is odd has not been solved completely. The most general result in this direction is due to Zassenhaus [2].

**Theorem 2.** Let \( G \) be a finite subgroup of \( O(n + 1) \) which acts freely on \( S^n \). Then, any subgroup of \( G \) of order \( pq \) (where \( p \) and \( q \) are prime numbers, not necessarily distinct) is cyclic.

Proof. It suffices to prove that if \( G \) is order \( pq \), then \( G \) is cyclic.

First, consider the case \( G \) is of order \( p^2 \). Then, \( G \) is either cyclic or a direct product of two cyclic groups \( G_1 \) and \( G_2 \) of order \( p \) (cf. Hall [1, p. 49]). Assuming the latter, let \( A \) and \( B \) be generators of \( G_1 \) and \( G_2 \), respectively. Since every element \( T \neq 1 \) of \( G \) is of order \( p \), we have \( T(S^{n-1}T') = S^{n-1}T'y \) for each \( y \in \mathbb{R}^{n+1} \). Since \( T \) has no fixed point on \( S^n \), we have

\[
S^{n-1}T'y = 0 \quad \text{for } y \in \mathbb{R}^{n+1}.
\]

By setting \( T = A'B \) and \( y = x \), we obtain

\[
\Sigma_{j=0}^{n-1} (A'B)^j x = 0 \quad \text{for } x \in \mathbb{R}^{n+1} \text{ and } i = 0, 1, \ldots, q - 1,
\]

and hence

\[
0 = \Sigma_{j=0}^{n-1} \Sigma_{s=0}^{p-1} A'zB^s x = \Sigma_{s=0}^{p-1} \Sigma_{t=0}^{p-1} A'^sB^t x \quad \text{for } x \in \mathbb{R}^{n+1}.
\]

On the other hand, by setting \( T = A \) and \( y = B^s x \), we obtain

\[
\Sigma_{j=0}^{n-1} A'^j B^s x = 0 \quad \text{for } x \in \mathbb{R}^{n+1} \text{ and } j = 1, 2, \ldots, p - 1.
\]

Hence, we have

\[
0 = \Sigma_{j=0}^{n-1} \Sigma_{s=0}^{p-1} A'^j B^s x = \Sigma_{s=0}^{p-1} A'^pB^s x = px \quad \text{for } x \in \mathbb{R}^{n+1},
\]

which is obviously a contradiction. Thus, \( G \) must be cyclic.

Second, consider the case where \( p < q \). Then \( G \) is either cyclic or non-abelian. Assuming that \( G \) is non-abelian, let \( A \) and \( B \) be elements of order \( p \) and \( q \), respectively. Then, we have (cf. Hall [1, p. 511])

\[
SAS^{-1} = A',
\]

where \( 1 < t < q \) and \( t^p \equiv 1 \mod q \), and every element of \( G \) can be written uniquely as \( A'S^t \), where \( 0 \leq t \leq q - 1 \) and \( 0 \leq k \leq p - 1 \). For each integer \( k \), define an integer \( f(k) \) by \( f(k) = 1 + t + p + \cdots + t^{k-1} \). We then have

\[
\begin{align*}
\text{(a)} & \quad f(p) \equiv 0 \mod q; \\
\text{(b)} & \quad f(k) \equiv 1 \mod q, \text{ if } k \equiv 1 \mod p; \\
\text{(c)} & \quad (A'S)^k = A'^{f(k)}S^k.
\end{align*}
\]

Indeed, (a) follows from \( t^p \equiv 1 \mod q \), and (c) follows from

\[
SAS^{-1} = A'.
\]

For each \( i, 0 \leq i \leq q - 1 \), let \( G_i \) be the cyclic subgroup of \( G \) generated by \( A'S^i \). Since \( (A'S)^* = A'^{f(p)}S^k = I \), \( G_i \) is of order \( p \). Hence, we have either \( G_i \cap G_j = \{I\} \) or \( G_i \equiv G_j \) for \( 0 \leq i, j \leq q - 1 \). We prove that \( G_i \equiv G_j \) if \( i \neq j \). If \( G_i = G_j \), there exists an integer \( k \) such that \( f(k) = f(k') \). By (c), we have \( A'^{f(k)}S^k = A'S^{k'} \), hence, \( S^k = S \). This implies \( k \equiv 1 \mod p \) and \( f(k) \equiv 1 \mod q \), hence, we have \( A'^kS^k = A'S \), which
implies $i = j$. Let $N$ be the normal subgroup of $G$ generated by $A$. Since $N$ is of order $q$ and since $G_i$ is of order $q$, we have $G_i \cap N = \{1\}$ for each $i$, $0 \leq i \leq q - 1$. By counting the orders of $N$, $G_0$, $G_1$, \ldots, $G_q$, we see that $G$ is a disjoint union of $N$, $G_0 = \{1\}$, $G_1 = \{1\}$, \ldots, $G_q = \{1\}$. Therefore we have

$$
\Sigma_{T \in N} Tx + \Sigma_{T \in G_0} Tx + \cdots + \Sigma_{T \in G_{q-1}} Tx = \Sigma_{T \in G} Tx + qx
$$

for $x \in \mathbb{R}^{n+1}$.

On the other hand, for every $T_0 \in N$, we have

$$
T_0(\Sigma_{T \in N} Tx) = \Sigma_{T \in N} T_0Tx = \Sigma_{T \in N} Tx
$$

for $x \in \mathbb{R}^{n+1}$. Since $G$ acts freely on $\mathbb{S}^n$, we have $\Sigma_{T \in N} Tx = 0$. By the same reasoning, we have $\Sigma_{T \in G_i} Tx = 0$ for $i = 0, 1, \ldots, q - 1$ and $\Sigma_{T \in G} Tx = 0$. Hence, we have $qx = 0$ for each $x \in \mathbb{R}^{n+1}$, which is obviously a contradiction. QED.

Recently, Wolf [1] classified the homogeneous Riemannian manifolds of constant curvature $1/a^2$. His result may be stated as follows.

**Theorem 3.** Let $M = S^n/G$ be a homogeneous Riemannian manifold of constant curvature $1/a^2$.

1. $Z \mathbb{F} + 1 = 2m$ (but not divisible by 4), then

   $$
   S^n = \{(z_1, \ldots, z_m) \in \mathbb{C}^m; |z_1|^2 + \cdots + |z_m|^2 = a^2\},
   $$

   and $G$ is a finite group of matrices of the form $AZ'$, where $A \in \mathbb{C}$ with $|A| = 1$ and $I_m$ is the $m \times m$ identity matrix;

2. If $n + 1 = 4m$, then

   $$
   S^n = \{q', \ldots, q^m; |q'|^2 + \cdots + |q^m|^2 = a^2\}
   $$

   (where $Q$ is the field of quaternions), and $G$ is a finite group of matrices of the form $pI_m$, where $p \in Q$ with $|p| = 1$.

Conversely, if $G$ is a finite group of the type described in (1) or (2), then $M = S^n/G$ is homogeneous.

In view of Theorem 1, we do not have to consider the case where $n$ is even.

The reader interested in the classification problem of elliptic spaces, i.e., spaces of constant positive curvature, is referred to the following papers: Vincent [1], Wolf [5]; for $n = 3$, H. Hopf [1] and Seifert and Threlfall [1]. Milnor [1] partially, generalized Theorem 2 to the case where $G$ is a group of homeomorphisms acting freely on $\mathbb{S}^n$, Calabi and Markus [1] and Wolf [3, 4] studied Lorentz manifolds of constant positive curvature. See also Helgason [1]. For the study of spaces covered by a homogeneous Riemannian manifold, see Wolf [2].

**Note 5. Flat Riemannian manifolds**

Let $M = R^n/G$ be a compact flat Riemannian manifold, where $G$ is a discrete subgroup of the group of Euclidean motions of $\mathbb{R}^n$. Let $N$ be the subgroup of $G$ consisting of pure translations. Then

1. $N$ is an abelian normal subgroup of $G$ and is free on $n$ generators;

2. $N$ is a maximal abelian subgroup of $G$;

3. $G/N$ is finite;

4. $G$ has no finite subgroup.

Indeed, (1) and (3) have been proved in (4) of Theorem 4.2 of Chapter V. To prove (2), let $K$ be any abelian subgroup of $G$ containing $N$. Since $G/K$ is also finite by (2), $\mathbb{R}^n/K$ is a compact flat Riemannian manifold. Since $K$ is an abelian normal subgroup of $K$, $K$ contains nothing but translations by Lemma 6 for Theorem 4.2, of Chapter V. Hence $K = N$. Finally, (4) follows from the fact that $G$ acts freely on $\mathbb{R}^n$. In fact, any finite group of Euclidean motions has a fixed point (cf. the proof of Theorem 7.1 of Chapter IV) and hence $G$ has no finite subgroup.

Auslander and Kuranishi [1] proved the converse:

Let $G$ be a group with a subgroup $N$ satisfying the above conditions (1), (2), (3) and (4). Then $G$ can be realized as a group of Euclidean motions of $\mathbb{R}^n$ such that $\mathbb{R}^n/G$ is a compact flat Riemannian manifold.

Let $\mathbb{R}^n/G$ and $\mathbb{R}^n/G'$ be two compact flat Riemannian manifolds. We say that they are equivalent, if there exists an affine transformation $Q$ such that $QGQ^{-1} = G'$, that is, if $G$ and $G'$ are conjugate in the group of affine transformations of $\mathbb{R}^n$. In addition to (4) of Theorem 4.2 of Chapter V, Bieberbach [1] obtained the following results:

(a) If $G$ and $G'$ are isomorphic as abstract group, then $\mathbb{R}^n/G$ and $\mathbb{R}^n/G'$ are equivalent.

(b) For each $n$, there are only a finite number of equivalence classes of compact flat Riemannian manifolds $\mathbb{R}^n/G$. 


We shall sketch here an outline of the proof. We denote by 
\((A, \rho)\) an affine transformation of \(\mathbb{R}^n\) with linear part \(A\) and translation part \(\rho\). Let \(N\) be the subgroup of \(G\) consisting of pure translations and let \((Z, t_1), \ldots, (I, t_n)\) be a basis of \(N\), where \(Z\) is the identity matrix and \(t_j \in \mathbb{R}^n\). Since \((A, \rho)(I, t_i)(A, -\rho)^{-1} = (I, A t_i)\) \(\in N\) for any \((A, \rho) \in G\), we can write 
\[ A t_i = \sum_{j=1}^n a_{ij} t_j, \]
where each \(a_{ij}\) is an integer. Let \(T\) be an \((n \times n)\)-matrix whose i-th column is given by \(t_{i0}\) that is, \(T = (t_1, \ldots, t_n)\). Then \(A t_i = T^{-1} A T\) is unimodular. (A matrix is called unimodular if it is non-singular and integral together with its inverse.)

To prove (a), let \((A', \rho') \in G'\) be the element corresponding to \((A, \rho) \in G\) by the isomorphism \(G' \approx G\). Let \(N'\) be the subgroup of \(G'\) corresponding to \(N\) by the isomorphism \(G' \approx G\). Then \(N'\) is normal and maximal abelian in \(G'\). Hence \(N'\) is the subgroup of \(G'\) consisting of pure translations. Let \((Z, t_i')\) correspond to \((Z, t_i)\). Since \((A', \rho')(Z, t_i')(A', -\rho')^{-1} = (Z, A' t_i')\), \((Z, A' t_i')\) corresponds to \((Z, A t_i)\). Hence we have \(A' t_i = \sum_{j=1}^n a_{ij} t_j\). In other words, if we set \(T' = (t_1', \ldots, t_n')\), then \(T'^{-1} A T' = T^{-1} A T\). Set 
\[ G^* = \{ (T^{-1} A T, T^{-1} \rho T - T^{-1} \rho') : (A, \rho) \in G \}. \]
Then \(G^*\) is a group which contains no pure translations and hence is finite. Let \(u \cdot \mathbb{R}^n\) be a point left fixed by \(G^*\). Then we have 
\[ (T, Tu)^{-1}(A, \rho)(T, Tu) = (T', 0)^{-1}(A', \rho')(T, 0) \]
for all \((A, \rho) \in G\).

This completes the proof of (a).

To prove (b), it suffices to show that there are only a finite number of mutually non-isomorphic groups \(G\) such that \(\mathbb{R}^n / G\) are compact flat Riemannian manifolds. Each \(G\) determines a group extension 
\[ 0 \to N \to G \to K \to 1, \]
where the finite group \(K = G/N\) acts linearly on \(N\) when \(N\) is considered as a subgroup of \(\mathbb{R}^n\) via the affine transformation 
\[ (x, v) \mapsto (A x, A v + \rho). \]

Given such a finite group \(K\), the set of group extensions 
\[ 0 \to N \to G \to K \to 1 \]
is given by \(H^2(K, N)\). Since \(K\) is finite and \(N\) is finitely generated, \(H^2(K, N)\) is finite. As we have seen in the proof of (a), if we identify \(N\) with the integral lattice points of \(\mathbb{R}^n\), then \(K = G/N\) is given by unimodular matrices. Let \(K\) and \(K'\) be two finite groups of unimodular matrices of degree \(n\) which are conjugate in the group \(GL(n; \mathbb{Z})\) of all unimodular matrices so that 
\[ SKS^{-1} = K' \]
for some \(S \in GL(n; \mathbb{Z})\). The mapping which sends \(I \in \mathbb{N}\) into \(St \in \mathbb{N}\) is an automorphism of \(\mathbb{N}\). Hence \(S\) induces an isomorphism, 
\[ H^2(K, N) \approx H^2(K', N), \]
and if \(0 \to N \to G \to K \to 1\) is the element of \(H^2(K', N)\) corresponding to an element \(0 \to N \to G \to K \to 1\) of \(H^2(K, N)\), then \(G\) and \(G'\) are isomorphic. Thus our problem is reduced to the following theorem of Jordan:

There are only a finite number of conjugate classes of finite subgroups of \(GL(n; \mathbb{Z})\).

This theorem of Jordan follows from the theory of Minkowski-Siegel. Let \(H_n\) be the space of all real symmetric positive definite matrices of degree \(n\). Then \(GL(n; \mathbb{Z})\) acts properly discontinuously on \(H_n\) as follows:

\[ x \to SXS^{-1} \quad \text{for} \quad X \in H_n \] 

Let \(R\) be the subset of \(H_n\) consisting of reduced matrices in the sense of Minkowski. Denote \(SXS^{-1}\) by \(S[X]\). Then

(i) \(U_{S \in GL(n; \mathbb{Z})} S[R] = H_n\);

(ii) The set \(F\) defined by 
\[ F = \{ S \in GL(n; \mathbb{Z}) : S[R] \cap R = \text{non-empty} \} \]
is a finite set.

The first property of \(R\) implies that any finite subgroup \(K\) of \(GL(n; \mathbb{Z})\) is conjugate to a subgroup of \(GL(n; \mathbb{Z})\) contained in \(F\). Indeed, let \(X_0 \in H_n\) be a fixed point of \(K\) (for instance, set \(X_0 = \sum_{a \in K} a AA\)). Let \(S \in GL(n; \mathbb{Z})\) be such that \(S[X_0] \in R\). Then \(S^{-1} K S \subset F\). Since \(F\) is finite, there are only a finite number of conjugate classes of finite subgroups of \(GL(n; \mathbb{Z})\). Q.E.D.

As references we mention Minkowski [1], Bieberbach [2], Bieberbach and Schur [1] and Siegel [1].

Note that (a) implies that two compact flat Riemannian manifolds are equivalent if and only if they are homeomorphic to each other. Although (b) does not hold for non-compact flat Riemannian manifolds, there are only a finite number of homeomorphism classes of complete flat Riemannian manifolds for each dimension (Bieberbach [3]).

For the classification of 3-dimensional complete flat Riemannian manifolds, see Hantzche and Wendt [1] and Nowacki [1].

Most of the results for flat Riemannian manifolds cannot be generalized to flat affine connections, see, for example, Auslander [1].
Note 6. Parallel displacement of curvature

Let \( M \) and \( M' \) be Riemannian manifolds and \( \varphi: M \to M' \) a diffeomorphism which preserves the curvature tensor fields. In general, this does not imply the existence of an isometry of \( M \) onto \( M' \). For instance, let \( M \) be a compact Riemannian manifold obtained by attaching a unit hemisphere to each end of the right circular cylinder \( S^1 \times [0,1] \), where \( S^1 \) is the unit circle, and then smoothing out the corners. Similarly, let \( M' \) be a compact Riemannian manifold obtained by attaching a unit hemisphere to each end of the right circular cylinder \( S^1 \times [0,2] \) and then smoothing out the corners in the same way. Let \( \varphi: M \to M' \) be a diffeomorphism which induces an isometry on the attached hemispheres and their neighborhoods. Since the cylinder parts of \( M \) and \( M' \) are flat, \( \varphi \) preserves the curvature tensor fields. However, \( M \) and \( M' \) cannot be isometric with each other.

Ambrose [1] obtained the following result, which generalizes Theorem 7.4 of Chapter VI in the Riemannian case.

Let \( M \) and \( M' \) be compact, simply connected Riemannian manifolds, \( x \) an arbitrarily fixed point of \( M \) and \( x' \) an arbitrarily fixed point of \( M' \). Let: \( f: T_x(M) \to T_{x'}(M') \) be a fixed orthogonal transformation. Let \( \tau \) be a simply broken geodesic of \( M \) from \( x \) to a point \( y \) and \( \tau' \) the corresponding simply broken geodesic of \( M' \) from \( x' \) to a point \( y' \). The correspondence being given by \( \tau \) through parallel displacement. Let \( p \) (resp. \( p' \)) be a plane in \( T_x(M) \) (resp. \( T_{x'}(M') \)) and \( q \) (resp. \( q' \)) the plane in \( T_{\tau}(M) \) (resp. \( T_{\tau'}(M') \)) obtained from \( p \) (resp. \( p' \)) by parallel displacement along \( \tau \) (resp. \( \tau' \)). Assume that \( p' \) corresponds top \( q \) by \( f \). If the sectional curvature \( K(q) \) is equal to the sectional curvature \( K'(q') \) for all simply broken geodesics \( \tau \) and all planes \( q \) in \( T_{\tau}(M) \), then there exists a unique isometry \( F: M \to M' \) whose differential at \( x \) coincides with \( f \).

Hicks [I] obtained a similar result in the case of affine connection; his result generalizes Theorem 7.4 of Chapter VI.

Note 7. Symmetric spaces

Although the theory of symmetric spaces, in particular, Riemannian symmetric spaces, will be taken up in detail in Volume II, we shall give here its definition and basic properties.

Let \( G \) be a connected Lie group with an involutive automorphism \( \sigma (\sigma^2 = 1, \sigma \neq 1) \). Let \( H \) be a closed subgroup which lies between the (closed) subgroup of all fixed points of \( \sigma \) and its identity component. We shall then say that \( G/H \) is a symmetric homogeneous space (defined by \( \sigma \)). Denoting by the same letter \( \sigma \) the involutive automorphism of the Lie algebra \( \mathfrak{g} \) induced by \( \sigma \), we have \( \mathfrak{g} = \mathfrak{m} + \mathfrak{h} \) (direct sum), where \( \mathfrak{h} = \{ X \in \mathfrak{g} : \mathfrak{m} \mathfrak{h} = \mathfrak{h} \} \) coincides with the subalgebra corresponding to \( H \) and \( \mathfrak{m} = \{ X \in \mathfrak{g} : \mathfrak{m} \mathfrak{m} = -\mathfrak{h} \} \). We have obviously \( \mathfrak{h} \subset \mathfrak{m} \) and \( [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \).

The automorphism \( \sigma \) of \( G \) also induces an involutive diffeomorphism \( \sigma_\circ \) of \( G/H \) such that \( \sigma_\circ (\pi x) = \pi (\sigma(x)) \) for every \( x \in G \), where \( \pi \) is the canonical projection of \( G \) onto \( G/H \). The origin \( o = \sigma(1) \) of \( G/H \) is then an isolated fixed point of \( \sigma_\circ \). We call \( o \) the symmetry around \( o \).

By Theorem 1.1 of Chapter II, the bundle \( G(G/H, H) \) admits an invariant connection \( \Gamma \) determined by the subspace \( \mathfrak{m} \). We call this connection the canonical connection in \( G(G/H, H) \).

Theorem 1. For a symmetric space \( G/H \), the canonical connection \( \Gamma \) in \( G(G/H, H) \) has the following properties:

1. \( \Gamma \) is invariant by the automorphism \( \sigma \) of \( G \) (which is a bundle automorphism of \( G(G/H, H) \)).

2. The curvature form \( \Omega \) is given by \( \Omega(X, Y) = \sigma(1/2)[X, Y] \in \mathfrak{h} \), where \( X \) and \( Y \) are arbitrary left invariant vector fields belonging to \( \mathfrak{m} \).

3. For any \( X \in \mathfrak{m} \), let \( \alpha = \exp tX \) and let \( x_t = \sigma(\alpha) = \alpha(\sigma(o)) \). The parallel displacement of the fibre \( H \) along the curve \( x_t \) coincides with the left translation \( 4 = \alpha_h \cdot h \in H \).

Proof. (1) follows easily from \( m^o = m \). (2) is contained in Theorem 11.1 of Chapter II. (3) follows from the fact that \( \alpha_h \cdot h \) for any fixed \( h \in H \) is the horizontal lift through \( h \) of the curve \( x_t \). QED.

The projection \( \pi \) gives a linear isomorphism of the horizontal subspace \( \mathfrak{m} \) at \( o \) in \( \Gamma \) onto the tangent space \( T_o(G/H) \) at the origin \( o \). If \( h \in H \), then \( \alpha \cdot h \) on \( \mathfrak{m} \) corresponds by this isomorphism to the linear isotropy \( h \), i.e., the linear transformation of \( T_o(G/H) \) induced by the transformation \( h \) of \( G/H \) which fixes \( o \).

Now, denoting \( G/H \) by \( M \), we define a mapping \( f \) of \( G \) into the bundle of frames \( L(M) \) over \( M \) as follows. Let \( u_o \in \mathfrak{m} \) be an arbitrarily
fixed frame $X_1, \ldots, X_n$ at $o$, which can be identified with a certain basis of $m$. For any $a \in G$, $f(a)$ is the frame at $a(o)$ consisting of the images of $X_i$ by the differential of $a$. In particular, for $h \in H$, $f(h) = h \cdot u_0 = u_0 \cdot \varphi(h)$, where $\varphi(h) \in GL(n; R)$ is the matrix which represents the linear transformation of $T_o(M)$ induced by $h$ with respect to the basis $u_0$. It is easy to see that if $f$ is a bundle homomorphism of $G$ into $L(M)$ corresponding to the homomorphism $\varphi$ of $H$ to $GL(n; R)$. If $G$ is effective on $G/H$ (or equivalently, if $H$ contains no non-trivial invariant subgroup of $G$), then $f$ and $\varphi$ are isomorphisms.

By Proposition 6.1, of Chapter II, the canonical connection $\Gamma$ in $G(M, H)$ induces a connection in $L(M)$, which we shall call the canonical linear connection on $G/H$ and denote still by $\Gamma$.

**Theorem 2.** The canonical linear connection on a symmetric space $G/H$ has the following properties:

1. $\Gamma$ is invariant by $G$ as well as the symmetry $\sigma_o$ around $o$;
2. The restricted homogeneous holonomy group of $\Gamma$ at $o$ is contained in the linear isotropy group $I_o$;
3. For any $X \in m$, let $a = \exp tX$ and $x_1 = \pi(a_1) = a(o)$. The parallel displacement of vectors along $x_1$ is the same as the transformation by $a$. In particular, $x_1$ is a geodesic;
4. The torsion tensor field is 0;
5. Every $G$-invariant tensor field on $G/H$ is parallel with respect to $\Gamma$.

In particular, the curvature tensor field $R$ is parallel, i.e., $VR = 0$.

Proof. (1), (2) and (3) follow from the corresponding properties in Theorem 1. (4) follows from (1); since the torsion tensor field $T$ is invariant by $\sigma_o$, we have $T(X, Y) = (T(X, Y))^o = -T(X, Y)$ and hence $T(X, Y) = 0$ for any $X$ and $Y$ in $T_o(M)$. Thus $T = 0$ at $o$ and hence everywhere. (5) follows from (3). In fact, if $K$ is a $G$-invariant tensor field, then $\nabla_{x_1} K = 0$ for any $X \in T_o(M)$, since there exists $X \in m$ such that $x_1$ in (3) has the initial tangent vector $X_o$. QED.

Remark. $\Gamma$ is the unique linear connection on $G/H$ which has property (1). This justifies the name of canonical linear connection.

Let $G/H$ be a symmetric space with compact $H$. There exists a $G$-invariant Riemannian metric on $G/H$. For any such metric $g$, the Riemannian connection coincides with $\Gamma$. In fact, the metric tensor field $g$ is parallel with respect to $\Gamma$ by (5). Since $\Gamma$ has zero torsion, it is the Riemannian connection by the uniqueness (Theorem 2.2, Chapter II).

**Example.** In $G = SO(n + 1)$, let $\sigma$ be the involutive automorphism $A \in SO(n + 1) \to SAS^{-1} \in SO(n + 1)$ where $S$ is the matrix of the form $\begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$ with identity matrix $I_n$ of degree $n$.

The identity component $H^0$ of the subgroup $H$ of fixed points of $\sigma$ consists of all matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, where $B \cdot SO(n)$. We shall write $SO(n)$ for $H^0$ with this understanding. The symmetric homogeneous space $SO(n + 1)/SO(n)$ is naturally diffeomorphic with the unit sphere $S^n$ in $R^{n+1}$. In fact, let $e_0, e_1, \ldots, e_n$ be the standard orthonormal basis in $R^{n+1}$. The mapping $A \in SO(n + 1) \to Ae_0 \in S^n$ induces a diffeomorphism of $SO(n + 1)/SO(n)$ onto $S^n$. The set of vectors $Ae_0, \ldots, Ae_n$ can be considered as an orthonormal frame at the point $Ae_0$ of $S^n$. This gives an isomorphism of the bundle $SO(n + 1)$ over $SO(n + 1)/SO(n)$ onto the bundle of orthonormal frames over $S^n$. The canonical linear connection on $SO(n + 1)/SO(n)$ coincides with the Riemannian connection of $S^n$ with respect to the Riemannian metric of $S^n$ as imbedded submanifold of $R^{n+1}$.

A linear connection $\Gamma$ on a differentiable manifold $M$ is said to be locally symmetric at $x \in M$, if there exists an involutive affine transformation of an open neighborhood $U$ of $x$ which has $x$ as an isolated fixed point. This local symmetry at $x$, if it exists, must be of the form $(x') \to (-x')$ with respect to any normal coordinate system with origin $x$, since it induces the linear transformation $X \to -X$ in $T_x(M)$. We say that $\Gamma$ is locally symmetric, if it is locally symmetric at every point $x$ of $M$.

**Theorem 3.** A linear connection $\Gamma$ on $M$ is locally symmetric if and only if $T = 0$ and $VR = 0$.

Proof. If $\Gamma$ is locally symmetric, then any tensor field of type $(r, s)$ with odd $r + s$ which is invariant by the local symmetry at $x$ is 0 at $x$. Hence $T = 0$ and $VR = 0$ on $M$. The converse follows from Theorem 7.4 of Chapter VI. QED.
Theorem 4. Let $\Gamma$ be a \textit{locally} symmetric linear connection on $M$. If $M$ is connected, simply connected and complete, then the group $\Pi(M)$ of all affine connections is transitive on $M$. Let $G = V(M)$. Then $M = G/H$ is a symmetric space for which $\Gamma$ is the canonical linear connection.

Proof. The first assertion follows from Corollary 7.9 of Chapter VI. Let $\sigma_0$ be the local symmetry at a point $o$ of $M$. By Corollary 6.2, $\sigma_0$ can be extended to an affine transformation of $M$ onto itself which is involutive. Define an involutive automorphism of $G$ by $a'' = \sigma_0 \circ a \circ \sigma_0$. Then $H$ lies between the subgroup of all fixed elements of $\sigma$ and its identity component. QED.

The Riemannian versions of Theorems 3 and 4 are obvious.

The Riemannian symmetric spaces were introduced and studied extensively by Cartan [7]. For the canonical linear connection on symmetric $G/H$, see Nomizu [2] and Kobayashi [3]. Nomizu [4, 6] proved the converse of (2) of Theorem 2 that if the restricted linear holonomy group of a complete Riemannian manifold $M$ is contained in the linear isotropy group at every point, then $M$ is locally symmetric. Simons [1] has a similar theorem.

Nomizu and Ozeki [3] proved that, for any complete Riemannian manifold $M$, the condition $\nabla^m R = 0$ for some $m > 1$ implies $\nabla^m R = 0$. (This was known by Lichnerowicz [3, p. 4] when $M$ is compact.) They remarked later that the assumption of completeness is not necessary.

Note 8. Linear connections with recurrent curvature

Let $M$ be an n-dimensional manifold with a linear connection $\Gamma$. A non-zero tensor field $K$ of type $(r, s)$ on $M$ is said to be \textit{recurrent} if there exists a $1$-form $\alpha$ such that $VK = K \otimes \alpha$. The following result is due to Wong [1].

Theorem 1. In the notation of §5 of Chapter III, let $f: L(M) \rightarrow T^*_1(R^n)$ be the mapping which corresponds to a given tensor field $K$ of type $(r, s)$. Then $K$ is recurrent if and only if, for the holonomy bundle $P(u_0)$ through any $u_0 \in L(M)$, there exists a differentiable function $\varphi(u)$ with no zero on $P(u_0)$ such that

$$f(u) = \varphi(u) \cdot f(u_0) \quad \text{for} \quad u \in P(u_0).$$

As a special case, $K$ is parallel if and only if $f(u)$ is constant on $P(u_0)$.

Using this result and the holonomy theorem (Theorem 8.1 of Chapter II), Wong obtained

Theorem 2. Let $\Gamma$ be a linear 'connection on $M$ with recurrent curvature tensor $R$. Then the Lie algebra of its linear holonomy group $\psi(u_0)$ is spanned by all \textit{elements} of the form $\Omega_{u_0}(X, Y)$, where $\Omega$ is the curvature form and $X$ and $Y$ are horizontal vectors at $u_0$. In particular, we have

$$\dim \psi(u_0) \leq \frac{1}{2}n(n - 1).$$

As an application of Theorem 1, we shall sketch the proof of the following

\textit{Theorem 3.} For a Riemannian manifold $M$ with recurrent curvature tensor whose restricted linear holonomy group is irreducible, the curvature tensor is necessarily parallel provided that $\dim M \geq 3$.

Proof. Let $R_{ijkl}$ be the components of the $T^*_1(R^n)$-valued function on $O(M)$ which corresponds to the curvature tensor field $R$. We apply Theorem 1 to $R$. Since $\Sigma_{i,j,k,l}(R_{ijkl})^2$ is constant on each fibre of $O(M)$, $\Omega_{u_0}^2$ is constant each fibre of $P(u_0)$. Since $\varphi$ never vanishes on $P(u_0)$, it is either always positive or always negative. Hence $\varphi$ itself is constant on each fibre of $P(u_0)$. Let $\lambda$ be the function on $M$ defined' by $\lambda(x) = 1/\varphi(u)$, where $x = \pi(u) \in M$. Then $\lambda R$ is a parallel tensor field. If we denote by $S$ the Ricci tensor field, then $\lambda S$ is also parallel. The irreducibility of $M$ implies that $\lambda S = c \cdot g$, where $c$ is a constant and $g$ is the metric tensor (cf. Theorem 1 of Appendix 5). If $\dim M \geq 3$ and if the Ricci tensor $S$ is non-trivial, then $\lambda$ is a constant function by Theorem 1 of Note 3. Since $\lambda R$ is parallel and since $\lambda$ is a constant, $R$ is parallel.

Next we shall consider the case where the Ricci tensor $S$ vanishes identically. Let $\nabla R = R \otimes a'$ and let $R_{ijkl}$ and $a_*$ be the components of $R$ and $a$ with respect to a local coordinate system $x^1, \ldots, x^n$. By Bianchi's second identity (Theorem 5.3 of Chapter III; see also Note 3), we have

$$R_{ijkl}a_m + R_{jim}a_k + R_{jmk}a_i = 0.$$

Multiply by $g^{jm}$ and sum with respect to $j$ and $m$. Since the Ricci tensor vanishes identically, we have $\Sigma_{j,m}g^{jm}R_{jim} = \Sigma_{j,m}g^{jm}R_{jmk} = 0$. Hence,

$$\Sigma_j R_{ijkl}a_j = 0, \quad \text{where} \quad \alpha_j = \Sigma_m g^{jm}a_m.$$
This equation has the following geometric implication. Let \( x \) be an arbitrarily fixed point of \( M \) and let \( X \) and \( Y \) be any vectors at \( x \). If we denote by \( V \) the vector at \( x \) with components \( \varepsilon_i(x) \), then the linear transformation \( R(X, Y) : T_x(M) \to T_x(M) \) maps \( V \) into the zero vector. On the other hand, by the Holonomy Theorem (Theorem 8.1 of Chapter II) and Theorem 1 of this Note (see also Wong [1]), the Lie algebra of the linear holonomy group \( \Psi(x) \) is spanned by the set of all endomorphisms of \( T_x(M) \) given by \( R(X, Y) \) with \( X, Y \in T_x(M) \). It follows that \( V \) is invariant by \( \Psi(x) \) and hence is zero by the irreducibility of \( Y(x) \). Consequently, \( \nabla R \) vanishes at \( x \). Since \( x \) is an arbitrary point of \( M \), \( R \) is parallel. QED.

On the other hand, every non-flat 2-dimensional Riemannian manifold is of recurrent curvature if the sectional curvature does not vanish anywhere.

**COROLLARY.** If \( M \) is a complete Riemannian manifold with recurrent curvature tensor, then the universal covering manifold \( \tilde{M} \) of \( M \) is either a symmetric space or a direct product of the Euclidean space \( \mathbb{R}^n \) and a 2-dimensional Riemannian manifold.

Proof. Use the decomposition theorem of de Rham (Theorem 6.2 of Chapter IV) and Theorem 3 above together with the following fact which can be verified easily. Let \( M \) and \( M' \) be manifolds with linear connections and let \( R \) and \( R' \) be their curvature tensors, respectively. If the curvature tensor of \( M \times M' \) is recurrent, then there are only three possibilities: (1) \( \nabla R = 0 \) and \( \nabla R' = 0 \); (2) \( R = 0 \) and \( \nabla R' \neq 0 \); (3) \( \nabla R \neq 0 \) and \( \nabla R' = 0 \).

(See also Walker [2].)

**Note 9. The automorphism group of a geometric structure**

Given a differentiable manifold \( M \), the group of all differentiable transformations of \( M \) is a very large group. However, the group of differentiable transformations of \( M \) leaving a certain geometric structure is often a Lie group. The first result of this nature was given by H. Cartan [1] who proved that the group of all complex analytic transformations of a bounded domain in \( \mathbb{C}^n \) is a Lie group. Myers and Steenrod [1] proved that the group of all isometries of a Riemannian manifold is a Lie group. Bochner and Montgomery [1, 2] proved that the group of all complex analytic transformations of a compact complex manifold is a complex Lie group; they made use of a general theorem concerning locally compact group of differentiable transformations which is now known to be valid, & the form of Theorem 4.6, Chapter I. The theorem that the group of all affine transformations of an affinely connected manifold is a Lie group was first proved by Nomizu [1] under the assumption of completeness; this assumption was later removed by L. Hano and Motimoto [1]. Kobayashi (1, 8, 6) proved that the group of all automorphisms of an absolute parallelism is a Lie group by imbedding it into the manifold. This method can be applied to the absolute parallelism of the bundle of frames \( \pi_x(M) \) of an affinely connected manifold \( M \) (cf. Proposition 2.6 of Chapter III and Theorem 1.5 of Chapter VI).

Automorphisms of a complex structure and a Kählerian structure will be discussed in Volume II.

A global theory of Lie transformation groups was studied in Palais [1]. We shall here state one theorem which has a direct bearing on us. Let \( G \) be a certain group of differentiable transformations acting on a differentiable manifold \( M \). Let \( \mathfrak{g} \) be the set of all vector fields \( X \) on \( M \) which generate a global 1-parameter group of transformations which belong to the given group \( G \). Let \( \mathfrak{g} \) be the Lie subalgebra of the Lie algebra \( \mathfrak{g}(M) \) generated by \( \mathfrak{g} \).

**Theorem.** If \( \mathfrak{g} \) is finite-dimensional, then \( G \) admits a Lie group structure (such that the mapping \( G \times M \to M \) is differentiable) and \( \mathfrak{g} = \mathfrak{g}' \). The Lie algebra of \( G \) is naturally isomorphic with \( \mathfrak{g} \).

We have the following applications of this result. If \( G \) is the group of all affine transformations (resp. isometries) of an affinely connected (resp. Riemannian) manifold \( M \), then \( \mathfrak{g} \) is the set of all infinitesimal affine transformations (resp. infinitesimal isometries) which are globally integrable (note that if \( M \) is complete, these infinitesimal transformations are always globally integrable by Theorem 2.4 of Chapter VI). By virtue of Theorem 2.3 (resp. Theorem 3.3) of Chapter VI, it follows that \( \mathfrak{g} \) is finite-dimensional.

By the theorem above, \( G \) is a Lie group.

The Lie algebra \( \mathfrak{g}(M) \) of all infinitesimal isometries of a Riemannian manifold \( M \) was studied in detail by Nomizu [8, 9]. At each
point $x$ of $M$, a certain Lie algebra $i(x)$ is constructed by using the curvature tensor field and its covariant differentials. If $M$ is simply connected and analytic together with the metric, then $i(M)$ is naturally isomorphic with $i(x)$, where $x$ is an arbitrary point.

**Note 10. Groups of isometries and affine transformations with maximum dimensions**

In Theorem 3.3 of Chapter VI, we proved that the group $\mathfrak{S}(M)$ of isometries of a connected, $n$-dimensional Riemannian manifold $M$ is of dimension at most $\frac{1}{2}n(n+1)$ and that if $\dim \mathfrak{S}(M) = \frac{1}{2}n(n+1)$, then $M$ is a space of constant curvature. We shall outline the proof of the following theorem.

**Theorem 1.** Let $M$ be a connected, $n$-dimensional Riemannian manifold. If $\dim \mathfrak{S}(M) = \frac{1}{2}n(n+1)$, then $M$ is isometric to one of the following spaces of constant curvature:

(a) An $n$-dimensional Euclidean space $\mathbb{R}^n$;
(b) An $n$-dimensional sphere $S^n$;
(c) An $n$-dimensional real projective space $\mathbb{P}^n$;
(d) An $n$-dimensional, simply connected hyperbolic space.

Proof. From the proof of Theorem 3.3 of Chapter VI, we see that $M$ is homogeneous and hence is complete. The universal covering space $\tilde{M}$ of $M$ is isometric to one of (a), (b) and (d) above (cf. Theorem 7.10 of Chapter VI). Every infinitesimal isometry $\tilde{X}$ of $\tilde{M}$ induces an infinitesimal isometry $X$ of $M$. Hence, $\frac{1}{2}n(n+1) = \dim \mathfrak{S}(M) \leq \dim \mathfrak{S}(\tilde{M}) \leq \frac{1}{2}n(n+1)$, which implies that every infinitesimal isometry $\tilde{X}$ of $\tilde{M}$ is induced by an infinitesimal isometry $X$ of $M$. If $\tilde{M}$ is isometric to (a) or (d), then there exists an infinitesimal isometry $\tilde{X}$ of $\tilde{M}$ which vanishes only at a single point of $\tilde{M}$. Hence $\tilde{M}$ is simply connected in case the curvature is nonpositive. If $\tilde{M}$ is isometric to a sphere $S^n$ for any antipodal points $x$ and $x'$, there exists an infinitesimal isometry $\tilde{X}$ of $\tilde{M}$ which vanishes only at $x$ and $x'$. This implies that $M = S^n$ or $M = S^n/(\pm 1)$. We see easily that if $M$ is isomorphic to the projective space $\mathbb{P}^n/(\pm 1)$, then $3(M)$ is isomorphic to $O(n+1)$ modulo its center and hence of dimension $\frac{1}{2}n(n+1)$. QED.

In Theorem 2.3 of Chapter VI, we proved that the group $\mathfrak{A}(M)$ of affine transformations of a connected, $n$-dimensional manifold $M$ with an affine connection is of dimension at most $n^2 + n$ and that if $\dim \mathfrak{A}(M) = n^2 + n$, then the connection is flat. We prove

**Theorem 2.** If $\dim X(M) = n^2 + n$, then $M$ is an ordinary affine space with the natural flat affine connection.

Proof. Every element of $\mathfrak{t}(M)$ induces a transformation of $L(M)$ leaving the canonical form and the connection form invariant (cf. §6 of Chapter VI). From the fact that $\mathfrak{A}(M)$ acts freely on $L(M)$ and from the assumption that $\dim \mathfrak{A}(M) = n^2 + n \equiv \dim L(M)$, it follows that $\mathfrak{A}(M)$ is transitive on each connected component of $L(M)$. This implies that every standard horizontal vector field on $L(M)$ is complete, the proof is similar to that of Theorem 2.4 of Chapter VI. In other words, the connection is complete. By Theorem 4.2 of Chapter V or by Theorem 7.8 of Chapter VI, the universal covering space $\tilde{M}$ of $M$ is an ordinary affine space. Finally, the fact that $\tilde{M} = M$ can be proved in the same way as Theorem 1 above. QED.

Theorems 2.3 and 3.3 are classical (see, for instance Eisenhart [1,1]).

Riemannian manifolds and affine connections admitting very large groups of automorphisms have been studied by Egorov Wang, Yano and others. The reader will find references on this subject in the book of Yano [2].

**Note 11. Conformal transformations of a Riemannian manifold**

Let $M$ be a Riemannian manifold with metric tensor $g$. A transformation $\varphi$ of $M$ is said to be conformal if $g = \rho g$, where $\rho$ is a positive function on $M$. If $\rho$ is a constant function, $\varphi$ is a homothetic transformation. If $\rho$ is identically equal to 1, $g$ is nothing but an isometry. An infinitesimal transformation $X$ of $M$ is said to be conformal if $\mathcal{L}_X g = \sigma g$, where $\sigma$ is a function on $M$. It is homothetic if $\sigma$ is a constant function, and it is isometric if $\sigma \equiv 0$. The local $1$-Parameter group of local transformations generated by an infinitesimal transformation $X$ is conformal if and only if $X$ is conformal.
1. The group of conformal transformations of a connected, n-dimensional Riemannian manifold $M$ is a Lie transformation group of dimension at most $\frac{1}{2}(n+1)(n+2)$, provided $n \geq 3$.

This can be proved along the following line. The integrability conditions of $L_X g = \sigma g$ imply that the Lie algebra of infinitesimal conformal transformation $X$ is of dimension at most $\frac{1}{2}(n+1)(n+2)$ (cf. for instance, Eisenhart [1, p. 285]). By the theorem of Palais cited in Note 9, the group of conformal transformations is a Lie transformation group.

In §3 of Chapter VI we showed that, for almost all Riemannian manifolds $M$, the largest connected group $\mathfrak{X}(M)$ of affine transformations of $M$ coincides with the largest connected group $\mathfrak{O}(M)$ of isometries of $M$. For the largest connected group $\mathfrak{C}^0(M)$ of conformal transformations of $M$, we have the following several results in the same direction.

**Theorem 2.** Let $A_4$ be a connected $n$-dimensional Riemannian manifold for which $\mathfrak{C}^0(M) \neq \mathfrak{Y}(M)$. Then,

1. If $M$ is compact, there is no harmonic $p$-form of constant length for $1 \leq p < n$ (Goldberg and Kobayashi [1]);
2. If $M$ is compact and homogeneous, then $M$ is isometric to a sphere provided $n > 3$ (Goldberg and Kobayashi [2]);
3. If $M$ is a complete Riemannian manifold of dimension $n \geq 3$ with parallel Ricci tensor, then $M$ is isometric to a sphere (Nagano [1]);
4. $M$ cannot be a compact Riemannian manifold with constant nonpositive scalar curvature (Yano [2; p. 279] and Lichnerowicz [3; p. 134]).

(3) is an improvement of the result of Nagano and Yano [1] to the effect that if $M$ is a complete Einstein space of dimension $\geq 3$ for which $\mathfrak{C}^0(M) \neq \mathfrak{Y}(M)$, then $M$ is isometric to a sphere. Nagano [1] made use of a result of Tanaka [1].

On the other hand, it is easy to construct Riemannian manifolds (other than spheres) for which $\mathfrak{C}^0(M) \neq \mathfrak{Y}(M)$. Indeed, let $M$ be a Riemannian manifold with metric tensor $g$ which admits a 1-parameter group of isometries. Let $\rho$ be a positive function on $M$ which is not invariant by this 1-parameter group of isometries. Then, with respect to the new metric $\rho g$, this group is a 1-parameter group of non-isometric, conformal transformations.

To show that $\dim \mathfrak{C}^0(M) = \frac{1}{2}(n+1)(n+2)$ for a sphere $M$ of dimension $n$, we imbed $M$ into the real projective space of dimension $n+1$. Let $x^0, x^1, \ldots, x^{n+1}$ be an homogeneous coordinate system of the real projective space $P_{n+1}$ of dimension $n+1$. Let $M$ be the $n$-dimensional sphere in $\mathbb{R}^{n+1}$ defined by $(y^1)^2 + \ldots + (y^{n+1})^2 = 1$. We imbed $M$ into $P_{n+1}$ by means of the mapping defined by

$$x_0 = \frac{1}{\sqrt{2}} (1 + y^{n+1}), \quad x^1 = y^1, \ldots, x^n = y^n, \quad x^{n+1} = \frac{1}{\sqrt{2}} (1 - y^{n+1}).$$

The image of $M$ in $P_{n+1}$ is given by

$$(x^1)^2 + \ldots + (x^n)^2 - 2x^0x^{n+1} = 0.$$ 

Let $h$ be the Riemannian metric on $P_{n+1}$ given by

$$b^* h = 2 \frac{(\Sigma_{i=0}^{n+1} (x^i)^2)(\Sigma_{i=0}^{n+1} (dx^i)^2)}{(\Sigma_{i=0}^{n+1} (x^{n+1}))^2},$$

where $p$ is the natural projection from $\mathbb{R}^{n+2} - \{0\}$ onto $P_{n+1}$. Then the imbedding $M \rightarrow P_{n+1}$ is isometric. Let $G$ be the group of linear transformations of $\mathbb{R}^{n+2}$ leaving the quadratic form $(x^1)^2 + \ldots + (x^n)^2 - 2x^0x^{n+1}$ invariant. Then $G$ maps the image of $M$ in $P_{n+1}$ onto itself. It is easy to verify that, considered as a transformation group acting on $P_{n+1}$, $G$ is a group of conformal transformations of dimension $\frac{1}{2}(n+1)(n+2)$.

The case $n = 2$ is exceptional in most of the problems concerning conformal transformations for the following reason. Let $M$ be a complex manifold of complex dimension 1 with a local coordinate system $z = x + iy$. Let $g$ be a Riemannian metric on $M$ which is of the form

$$f(dx^2 + dy^2) = dz d\bar{z},$$

where $f$ is a positive function on $M$. Then every complex analytic transformation of $M$ is conformal.
SUMMARY OF BASIC NOTATIONS

We summarize only those basic notations which are used most frequently throughout the book.

1. $\Sigma$, $\Sigma_{i,j}$, etc., stand for the summation taken over $i$ or $i,j,\ldots$, where the range of indices is generally clear from the context.

2. $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex number fields, respectively.
   
   $\mathbb{R}^n$: vector space of $n$-tuples of real numbers $(x^1, \ldots, x^n)$
   
   $\mathbb{C}^n$: vector space of $n$-tuples of complex numbers $(z^1, \ldots, z^n)$
   
   $(x,y)$ : standard inner product $\Sigma_x x^i y^i$ in $\mathbb{R}^n$ ($\Sigma_x x^i y^i$ in $\mathbb{C}^n$)
   
   $GL(n; \mathbb{R})$: general linear group acting on $\mathbb{R}^n$
   
   $gl(n; \mathbb{R})$: Lie algebra of $GL(n; \mathbb{R})$
   
   $GL(n; \mathbb{C})$: general linear group acting on $\mathbb{C}^n$
   
   $gl(n; \mathbb{C})$: Lie algebra of $GL(n; \mathbb{C})$
   
   $O(n)$: orthogonal group
   
   $o(n)$: Lie algebra of $O(n)$
   
   $U(n)$: unitary group
   
   $u(n)$: Lie algebra of $U(n)$
   
   $T^n(V)$: tensor space of type $(r,s)$ over a vector space $V$
   
   $T(V)$: tensor algebra over $V$
   
   $A^n$: space $\mathbb{R}^n$ regarded as an affine space
   
   $\mathcal{A}(n; \mathbb{R})$: group of affine transformations of $A^n$
   
   $\mathfrak{a}(n; \mathbb{R})$: Lie algebra of $\mathcal{A}(n; \mathbb{R})$

3. $M$ denotes an $n$-dimensional differentiable manifold.
   
   $T_x(M)$: tangent space of $M$ at $x$
   
   $\mathcal{F}(M)$: algebra of differentiable functions on $M$
   
   $\mathfrak{f}(M)$: Lie algebra of vector fields on $M$
   
   $\mathfrak{g}(M)$: algebra of tensor fields on $M$
   
   $\mathfrak{D}(M)$: algebra of differential forms on $M$
   
   $T(M)$: tangent bundle of $M$
   
   $\mathcal{L}(M)$: bundle of linear frames of $M$
O(M) : bundle of orthonormal frames of M (with respect to a given Riemannian metric)
θ = (θ^i) : canonical 1-form on L(M) or O(M)
A(M) : bundle of affine frames of M
T^r_s (M) : tensor bundle of type (r, s) of M
J^f : differential of a differentiable mapping
f_* ω : the transform of a differential form ω by f
x_t : tangent vector of a curve x(t), 0 ≤ t ≤ 1, at the point x_t
L_X : Lie differentiation with respect to a vector field X

4. For a Lie group G, G^0 denotes the identity component and \mathfrak{g}
   the Lie algebra of G.
   L_e : left translation by e \in G
   R_e : right translation by e \in G
   ad a : inner automorphism by ε \in G; also adjoint representation
   P(M, G) : principal fibre bundle over M with structure group G
   A^* : fundamental vector field corresponding to A \in \mathfrak{g}
   ω = (ω^i) : connection form
   Ω = (Ω^i) : curvature form
   E (M, F, G, P) : bundle associated to P(M, G) with fibre F

5. For an affine (linear) connection Γ on M,
   Θ = (Θ^i) : torsion form
   Γ^i_jk : Christoffel's symbols
   Ψ(x) : linear holonomy group at x \in M
   Φ(x) : affine holonomy group at x \in M
   \nabla_X : covariant differentiation with respect to a vector (field) X
   R : curvature tensor field (with components R^i_{jk})
   T : torsion tensor field (with components T^i_{jk})
   S : Ricci tensor field (with components R^i_{ii})
   A(M) : group of all affine transformations
   a(\mathfrak{g}) : Lie algebra of all infinitesimal affine transformations
   Λ(M) : group of all isometries
   i(\mathfrak{g}) : Lie algebra of all infinitesimal isometries

BIBLIOGRAPHY

Amrose, W.

Amrose, W. and Singer, I.-M.

Arkin, E.

Arens, R.

Auslander, L.

Auslander, L. and Kuranishi, M.

Auslander, L. and Markus, L.

Berger, M.

Bernard, D.

Bieberrbach, L.


Bieberrbach, L. and Schur, I.

Bochner, S.
Bochner, S. and Montgomery, D.


Borel, A. and Lichnerowicz, A.


Calabi, E. and Markus, L.

1. Relativistic space forms, Ann. of Math. 75 (1962), 63-76.

Cartan, E.

5. La Geometrie des Espaces de Riemann, Memoire des Sciences Math., Vol. 9, 1925.

Cartan, H.

2. Notion d'algèbre differentielle; applications aux groupes de Lie et aux varietes oh opère un groupe de Lie, Colloque de Topologie, Bruxelles (1950), 15-27.
3. La transgression dans un groupe de Lie et dans un espace firé principal, Colloque de Topologie, Bruxelles (1950), 57-71.

Chern, S. S.

[1] HANO, J. and MOMORIMOTO, A. 

[1] HANO, J. and OZEKI, H. 

[1] HANTZSCHE, W. and WENDT, W. 

[1] HELGASON, S. 
Some remarks on the exponential mapping for an affine connection, Math. Scand. 9 (1961), 129-146.

[1] HICKS, N. 

[1] HILBERT, D. 


[1] IWASAWA, K. 
On some types of topological groups, Ann. of Math. 50 (1949), 507-558.

[1] KELLEY, J. L. 

[1] KILLING, W. 
Die nicht-euklidischen Raumformen in analytische Behandlung, Teubner, Leipzig, 1885.

[1] KLEIN, F. 

[1] BIBLIOGRAPHY

319

[1] Klingenberg, W. 

[1] KOBAYASHI, S. 
Le groupe de transformations qui laissent invariant un parallélisme, Colloque de Topologie, Strasbourg (1954).

[1] LEVI-CIVITA, T. 
Nozione di parallelismo in una varietà qualunque c conseguente specificazione geometrica delta curvature Riemanniana. Rend. Palermo 42 (1917), 73-205.

[1] LICHNEROWICZ, A. 
Espaces homogènes kählériens, Colloque de Géométrie Différentielle, Strasbourg (1955), 171-201.

[1] MINKOWSKI, H. 
Diskontinuitätsbereich für arithmetische Äquivalenz, J. Reine Angew. Math. 129 (1905), 220-274.
Montgomery, D. and Zippin, L.


Myers, S. B. and Steenrod, N.


Nagano, T.


Nagano, T. and Yano, K.


Nomizu, K.


BIBLIOGRAPHY

Nomizu, K. and Ozeki, H.


Nowacki, W.


Ozeki, H.


Palais, R. S.


Pontrjagin, L.


Reinhart, B.


De Rham, G.


Riemann, B.


Saito, M.


Schouten, J. A. and Struik, D. J.


Schur, F.


Seifert, H. and Threlfall, W.

Sibuya, Y.  

Siegel, C. L.  

Sierpinski, W.  

Simons, J.  

Smith, J. W.  

Steenrod, N.  

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