Nonlinear Analysis and Differential Equations
An Introduction

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Preface

The subject of Differential Equations is a well established part of mathematics and its systematic development goes back to the early days of the development of Calculus. Many recent advances in mathematics, paralleled by a renewed and flourishing interaction between mathematics, the sciences, and engineering, have again shown that many phenomena in the applied sciences, modelled by differential equations will yield some mathematical explanation of these phenomena (at least in some approximate sense).

The intent of this set of notes is to present several of the important existence theorems for solutions of various types of problems associated with differential equations and provide qualitative and quantitative descriptions of solutions. At the same time, we develop methods of analysis which may be applied to carry out the above and which have applications in many other areas of mathematics, as well.

As methods and theories are developed, we shall also pay particular attention to illustrate how these findings may be used and shall throughout consider examples from areas where the theory may be applied.

As differential equations are equations which involve functions and their derivatives as unknowns, we shall adopt throughout the view that differential equations are equations in spaces of functions. We therefore shall, as we progress, develop existence theories for equations defined in various types of function spaces, which usually will be function spaces which are in some sense natural for the given problem.

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Chapter I
Analysis In Banach Spaces

1 Introduction

This chapter is devoted to developing some tools from Banach space valued function theory which will be needed in the following chapters. We first define the concept of a Banach space and introduce a number of examples of such which will be used later. We then discuss the notion of differentiability of Banach–space valued functions and state an infinite dimensional version of Taylor's theorem. As we shall see, a crucial result is the implicit function theorem in Banach spaces, a version of this important result, suitable for our purposes is stated and proved. As a consequence we derive the Inverse Function theorem in Banach spaces and close this chapter with an extension theorem for functions defined on proper subsets of the domain space (the Dugundji extension theorem).

In this chapter we shall mainly be concerned with results for not necessarily linear functions; results about linear operators which are needed in these notes will be quoted as needed.

2 Banach Spaces

Let E be a real (or complex) vector space which is equipped with a norm $\|\cdot\|$, i.e. a function $\|\cdot\| : E \to \mathbb{R}_+$ having the properties:

i) $\|u\| \geq 0$, for every $u \in E$,

ii) $\|u\| = 0$ is equivalent to $u = 0 \in E$,

iii) $\|\lambda u\| = |\lambda|\|u\|$, for every scalar λ and every $u \in E$,

iv) $\|u + v\| \leq \|u\| + \|v\|$, for all $u, v, \in E$ (triangle inequality).

A norm $\|\cdot\|$ defines a metric $d : E \times E \to \mathbb{R}_+$ by $d(u,v) = \|u - v\|$ and $(E, \|\cdot\|)$ or simply E (if it is understood which norm is being used) is called a

Banach space if the metric space (E, d), d defined as above, is complete (i.e. all Cauchy sequences have limits in E).

If E is a real (or complex) vector space which is equipped with an inner product, i.e. a mapping

$$\langle\cdot,\cdot\rangle : E \times E \to \mathbb{R} \text{ (or } \mathbb{C} \quad \text{(the complex numbers))}$$

satisfying

i) $\langle u, v\rangle = \overline{\langle v, u\rangle},\ u, v \in E$

ii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle,\ u, v, w \in E$

iii) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle,\ \lambda \in \mathbb{C},\ u, v, \in E$

iv) $\langle u, u\rangle \ \geq\ 0$, $u \in E$, and $\langle u, u\rangle = 0$ if and only if $u = 0$,

then E is a normed space with the norm defined by

$$\|u\| = \sqrt{\langle u, u\rangle}, \quad u \in E.$$

If E is complete with respect to this norm, then E is called a Hilbert space.

An inner product is a special case of what is known as a conjugate linear form, i.e. a mapping $b : E \times E \to \mathbb{C}$ having the properties (i)–(iv) above (with $\langle\cdot,\cdot\rangle$ replaced by $b(\cdot,\cdot)$); in case E is a real vector space, then b is called a bilinear form.

The following collection of spaces are examples of Banach spaces. They will frequently be employed in the applications presented later. The verification that the spaces defined are Banach spaces may be found in the standard literature on analysis.

2.1 Spaces of continuous functions

Let Ω be an open subset of $\mathbb{R}^n$, define

$$C^0(\Omega, \mathbb{R}^m) = \{f : \Omega \to \mathbb{R}^m \ \text{ such that f is continuous on } \Omega\}.$$

Let

$$\|f\|_0 = \sup_{x\in\Omega} |f(x)|, \tag{1}$$

where $|\cdot|$ is a norm in $\mathbb{R}^m$.

Since the uniform limit of a sequence of continuous functions is again continuous, it follows that the space

$$E = \{f \in C^0(\Omega, \mathbb{R}^m) : \|f\|_0 < \infty\}$$

is a Banach space.

If Ω is as above and Ω' is an open set with $\bar{\Omega} \subset \Omega'$, we let $C^0(\bar{\Omega}, \mathbb{R}^m) = \{$the restriction to $\bar{\Omega}$ of $f \in C^0(\Omega', \mathbb{R}^m)\}$. If Ω is bounded and $f \in C^0(\bar{\Omega}, \mathbb{R}^m)$, then $\|f\|_0 < +\infty$. Hence $C^0(\bar{\Omega}, \mathbb{R}^m)$ is a Banach space.

2.2 Spaces of differentiable functions

Let Ω be an open subset of $\mathbb{R}^n$. Let $\beta = (i_1, \cdots, i_n)$ be a multiindex, i.e. $i_k \in \mathbb{Z}$ (the nonnegative integers), $1 \leq k \leq n$. We let $|\beta| = \sum_{k=1}^n i_k$. Let $f : \Omega \to \mathbb{R}^m$, then the partial derivative of f of order $\beta, D^\beta f(x)$, is given by

$$D^\beta f(x) = \frac{\partial^{|\beta|} f(x)}{\partial^{i_1} x_1 \cdots \partial^{i_n} x_n},$$

where $x = (x_1, \cdots, x_n)$. Define $C^j(\Omega, \mathbb{R}^m) = \{f : \Omega \to \mathbb{R}^m$ such that $D^\beta f$ is continuous for all $\beta,\ |\beta| \leq j\}$.

Let

$$\|f\|_j = \sum_{k=0}^{j} \max_{|\beta| \leq k} \|D^\beta f\|_0. \tag{2}$$

Then, using further convergence results for families of differentiable functions it follows that the space

$$E = \{f \in C^j(\Omega, \mathbb{R}^m) : \|f\|_j < +\infty\}$$

is a Banach space.

The space $C^j(\bar{\Omega}, \mathbb{R}^m)$ is defined in a manner similar to the space $C^0(\bar{\Omega}, \mathbb{R}^m)$ and if Ω is bounded $C^j(\bar{\Omega}, \mathbb{R}^m)$ is a Banach space.

2.3 Hölder spaces

Let Ω be an open set in $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}^m$ is called Hölder continuous with exponent $\alpha,\ 0 < \alpha \leq 1$, at a point $x \in \Omega$, if

$$\sup_{y \neq x} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty,$$

and Hölder continuous with exponent $\alpha, 0 < \alpha \leq 1$, on Ω if it is Hölder continuous with the same exponent Ω at every $x \in \Omega$. For such f we define

$$H_\Omega^\alpha(f) = \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{3}$$

If $f \in C^j(\Omega, \mathbb{R}^m)$ with $D^\beta f,\ |\beta| = j$, is Hölder continuous with exponent α on Ω, we say $f \in C^{j,\alpha}(\Omega, \mathbb{R}^m)$. Let

$$\|f\|_{j,\alpha} = \|f\|_j + \max_{|\beta|=j} H_\Omega^\alpha(D^\beta f),$$

then the space

$$E = \{f \in C^{j,\alpha}(\Omega, \mathbb{R}^m) : \|f\|_{j,\alpha} < \infty\}$$

is a Banach space.

As above, one may define the space $C^{j,\alpha}(\overline{\Omega}, \mathbb{R}^m)$. And again, if Ω is bounded, $C^{j,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ is a Banach space.

We shall also employ the following convention

$$C^{j,0}(\Omega, \mathbb{R}^m) = C^j(\Omega, \mathbb{R}^m)$$

and

$$C^{j,0}(\bar{\Omega}, \mathbb{R}^m) = C^j(\bar{\Omega}, \mathbb{R}^m).$$

2.4 Functions with compact support

Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}^m$ is said to have compact support in Ω if the set

$$\text{supp}\, f = \text{closure}\{x \in \Omega : f(x) = 0\} = \overline{\{x \in \Omega : f(x) = 0\}}$$

is compact.

We let

$$C_0^{j,\alpha}(\Omega, \mathbb{R}^m) = \{f \in C^{j,\alpha}(\Omega, \mathbb{R}^m) : \text{supp}\, f \ \text{ is a compact subset of } \ \Omega\}$$

and define $C_0^{j,\alpha}(\bar{\Omega}, \mathbb{R}^m)$ similarly.

Then, again, if Ω is bounded, the space $C_0^{j,\alpha}(\bar{\Omega}, \mathbb{R}^m)$ is a Banach space and

$$C_0^{j,\alpha}(\bar{\Omega}, \mathbb{R}^m) = \{f \in C^{j,\alpha}(\bar{\Omega}, \mathbb{R}^m) : f(x) = 0, \ x \in \partial\Omega\}.$$

2.5 L^p spaces

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ be a measurable function. Let, for $1 \le p < \infty$,

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p},$$

and for $p = \infty$, let

$$\|f\|_{L^\infty} = \text{essup}_{x \in \Omega} |f(x)|,$$

where essup denotes the essential supremum.

For $1 \le p \le \infty$, let

$$L^p(\Omega, \mathbb{R}^m) = \{f : \|f\|_{L^p} < +\infty\}.$$

Then $L^p(\Omega, \mathbb{R}^m)$ is a Banach space for $1 \le p \le \infty$. The space $L^2(\Omega, \mathbb{R}^m)$ is a Hilbert space with inner product defined by

$$\langle f, g \rangle = \int_\Omega f(x) \cdot g(x) dx,$$

where $f(x) \cdot g(x)$ is the inner product of $f(x)$ and $g(x)$ in the Hilbert space (Euclidean space) $\mathbb{R}^n$.

2.6 Weak derivatives

Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}^m$ is said to belong to class $L^p_{loc}(\Omega, \mathbb{R}^m)$, if for every compact subset $\Omega' \subset \Omega, f \in L^p(\Omega', \mathbb{R}^m)$. Let $\beta = (\beta_1, \dots, \beta_n)$ be a multiindex. Then a locally integrable function v is called the β^{th} weak derivative of f if it satisfies

$$\int_\Omega v\phi dx = (-1)^{|\beta|} \int_\Omega f D^\beta \phi dx, \text{ for all } \phi \in C_0^\infty(\Omega). \tag{4}$$

We write $v = D^\beta f$ and note that, up to a set of measure zero, v is uniquely determined. The concept of weak derivative extends the classical concept of derivative and has many similar properties (see e.g. [11]).

2.7 Sobolev spaces

We say that $f \in W^k(\Omega, \mathbb{R}^m)$, if f has weak derivatives up to order k, and set

$$W^{k,p}(\Omega, \mathbb{R}^m) = \{f \in W^k(\Omega, \mathbb{R}^m) : D^\beta f \in L^p(\Omega, \mathbb{R}^m), |\beta| \leq k\}.$$

Then the vector space $W^{k,p}(\Omega, \mathbb{R}^m)$ equipped with the norm

$$\|f\|_{W^{k,p}} = \left(\int_\Omega \sum_{|\beta| \leq k} |D^\beta f|^p dx \right)^{1/p} \tag{5}$$

is a Banach space. The space $C_0^k(\Omega, \mathbb{R}^m)$ is a subspace of $W^{k,p}(\Omega, \mathbb{R}^m)$, it's closure in $W^{k,p}(\Omega, \mathbb{R}^m)$, denoted by $W_0^{k,p}(\Omega, \mathbb{R}^m)$, is a Banach subspace which, in general, is a proper subspace.

For $p = 2$, the spaces $W^{k,2}(\Omega, \mathbb{R}^m)$ and $W_0^{k,2}(\Omega, \mathbb{R}^m)$ are Hilbert spaces with inner product $\langle f, g \rangle$ given by

$$\langle f, g \rangle = \int_\Omega \sum_{|\alpha| \leq k} D^\alpha f \cdot D^\alpha g dx. \tag{6}$$

These spaces play a special role in the linear theory of partial differential equations, and in case Ω satisfies sufficient regularity conditions (see [1], [26]), they may be identified with the following spaces.

Consider the space $C^k(\bar{\Omega}, \mathbb{R}^m)$ as a normed space using the $\| \cdot \|_{W^{k,p}}$ norm. It's completion is denoted by $H^{k,p}(\Omega, \mathbb{R}^m)$. If $p = 2$ it is a Hilbert space with inner product given by (6). $H_0^{k,p}(\Omega, \mathbb{R}^m)$ is the completion of $C_0^\infty(\Omega, \mathbb{R}^m)$ in $H^{k,p}(\Omega, \mathbb{R}^m)$.

2.8 Spaces of linear operators

Let E and X be normed linear spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_X$, respectively. Let

$$\mathfrak{L}(E; X) = \{f : E \to X \ \text{ such that } \ f \ \text{ is linear and continuous}\}.$$

For $f \in \mathfrak{L}(E;X)$, let

$$\|f\|_{\mathfrak{L}} = \sup_{\|x\|_E \le 1} \|f(x)\|_X. \tag{7}$$

Then $\|\cdot\|_{\mathfrak{L}}$ is a norm for $\mathfrak{L}(E;X)$. This space is a Banach space, whenever X is.

Let $E_1, \dots, E_n$ and X be $n+1$ normed linear spaces, let $\mathfrak{L}(E_1, \dots, E_n; X) = \{f : E_1 \times \cdots \times E_n \to X$ such that f is multilinear (i.e. f is linear in each variable separately) and continuous$\}$. Let

$$\|f\| = \sup\{\|f(x_1, \dots, x_n)\|_X : \|x_1\|_{E_1} \le 1, \dots, \|x_n\|_{E_n} \le 1\}, \tag{8}$$

then $\mathfrak{L}(E_1, \dots, E_n; X)$ with the norm defined by (8) is a normed linear space. It again is a Banach space, whenever X is.

If E and X are normed spaces, one may define the spaces

$$\begin{array}{rcl} \mathfrak{L}_1(E;X) & = & \mathfrak{L}(E;X) \\ \mathfrak{L}_2(E;X) & = & \mathfrak{L}(E;\mathfrak{L}_1(E;X)) \\ \vdots & \vdots & \vdots \\ \mathfrak{L}_n(E;X) & = & \mathfrak{L}(E;\mathfrak{L}_{n-1}(E;X)), \quad n \ge 2. \end{array}$$

We leave it as an exercise to show that the spaces $\mathfrak{L}(E, \dots, E; X)$ (E repeated n times) and $\mathfrak{L}_n(E;X)$ may be identified (i.e. there exists an isomorphism between these spaces which is norm preserving (see e.g. [28]).

3 Differentiability, Taylor's Theorem

3.1 Gâteaux and Fréchet differentiability

Let E and X be Banach spaces and let U be an open subset of E. Let

$$f : U \to X$$

be a function. Let $x_0 \in U$, then f is said to be <u>Gâteaux</u> <u>differentiable</u> (G-differentiable) at x_0 in direction h, if

$$\lim_{t \to 0} \frac{1}{t}\{f(x_0 + th) - f(x_0)\} \tag{9}$$

exists. It said to be <u>Fréchet</u> <u>differentiable</u> (F-differentiable) at x_0, if there exists $T \in \mathfrak{L}(E;X)$ such that

$$f(x_0 + h) - f(x_0) = T(h) + o(\|h\|) \tag{10}$$

for $\|h\|$ small, here $o(\|h\|)$ means that

$$\lim_{\|h\| \to 0} \frac{o(\|h\|)}{\|h\|} = 0.$$

(We shall use the symbol $\|\cdot\|$ to denote both the norm of E and the norm of X, since it will be clear from the context in which space we are working.) We note that Fréchet differentiability is a more restrictive concept.

It follows from this definition that the Fréchet–derivative of f at x_0, if it exists, is unique. We shall use the following symbols interchangeably for the Fréchet–derivative of f at x_0; $Df(x_0)$, $f'(x_0)$, $df(x_0)$, where the latter is usually used in case $X = \mathbb{R}$. We say that f is of class C^1 in a neighborhood of x_0 if f is Fréchet differentiable there and if the mapping

$$Df : x \mapsto Df(x)$$

is a continuous mapping into the Banach space $\mathfrak{L}(E;X)$.

If the mapping

$$Df : U \to \mathfrak{L}(E;X)$$

is Fréchet–differentiable at $x_0 \in U$, we say that f is twice Fréchet–differentiable and we denote the second F–derivative by $D^2f(x_0)$, or $f''(x_0)$, or $d^2f(x_0)$. Thus $D^2f(x_0) \in \mathfrak{L}_2(E;X)$. In an analogous way one defines higher order differentiability.

If $h \in E$, we shall write

$$D^nf(x_0)(h,\cdots,h) \quad \text{as} \quad D^nf(x_0)h^n,$$

(see Subsection 2.8).

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is Fréchet differentiable at x_0, then $Df(x_0)$ is given by the Jacobian matrix

$$Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}|_{x=x_0}\right),$$

and if $f : \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable, then $Df(x_0)$ is represented by the gradient vector $\nabla f(x_0)$, i.e.

$$Df(x_0)(h) = \nabla f(x_0) \cdot h,$$

where "$\cdot$" is the dot product in $\mathbb{R}^n$, and the second derivative $D^2f(x_0)$ is given by the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)|_{x=x_0}$, i.e.

$$D^2f(x_0)h^2 = h^T\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)h,$$

where h^T is the transpose of the vector h.

3.2 Taylor's formula

1 Theorem *Let $f : E \to X$ and all of its Fréchet–derivatives of order less than m, $m > 1$, be of class C^1 on an open set U. Let x and $x + h$ be such that the line segment connecting these points lies in U. Then*

$$f(x+h) - f(x) = \sum_{k=1}^{m-1} \frac{1}{k!}D^kf(x)h^k + \frac{1}{m!}D^mf(z)h^m, \tag{11}$$

where z is a point on the line segment connecting x to $x+h$. The remainder $\frac{1}{m!}D^m f(z)h^m$ is also given by

$$\frac{1}{(m-1)!}\int_0^1 (1-s)^{m-1} D^m f(x_0+sh)h^m ds. \tag{12}$$

We shall not give a proof of this result here, since the proof is similar to the one for functions $f:\mathbb{R}^n \to \mathbb{R}^m$ (see e.g. [8]).

3.3 Euler-Lagrange equations

In this example we shall discuss a fundamental problem of variational calculus to illustrate the concepts of differentiation just introduced; specifically we shall derive the so called Euler-Lagrange differential equations. The equations derived give necessary conditions for the existence of minima (or maxima) of certain funtionals.

Let $g:[a,b]\times\mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Let $E = C_0^2[a,b]$ and let $T:E\to\mathbb{R}$ be given by

$$T(u) = \int_a^b g(t,u(t),u'(t))dt.$$

It then follows from elementary properties of the integral, that T is of class C^1. Let $u_0 \in E$ be such that there exists an open neighborhood U of u_0 such that

$$T(u_0) \le T(u) \tag{13}$$

for all $u \in U$ (u_0 is called an extremal of T). Since T is of class C^1 we obtain that for $u \in U$

$$T(u) = T(u_0) + DT(u_0))(u-u_0) + o(\|u-u_0\|).$$

Hence for fixed $v \in E$ and $\epsilon \in \mathbb{R}$ small,

$$T(u_0+\epsilon v) = T(u_0) + DT(u_0)(\epsilon v) + o(|\epsilon|\|v\|).$$

It follows from (13) that

$$0 \le DT(u_0)(\epsilon v) + o(|\epsilon|\|v\|)$$

and hence, dividing by $\|\epsilon v\|$,

$$0 \le DT(u_0)\left(\frac{v}{\|v\|}\right) + \frac{o(\|\epsilon v\|)}{\|\epsilon v\|},$$

where $\|\cdot\|$ is the norm in E. It therefore follows, letting $\epsilon \to 0$, that for every $v \in E$, $DT(u_0)(v) = 0$. To derive the Euler–Lagrange equation, we must compute $DT(u_0)$. For arbitrary $h \in E$ we have

$$\begin{aligned} T(u_0+h) = &\ \int_a^b g(t,u_0(t)+h(t),u_0'(t)+h'(t))dt \\ = &\ \int_a^b g(t,u_0(t),u_0'(t)dt \\ + &\ \int_a^b \frac{\partial g}{\partial p}(t,u_0(t),u_0'(t))h(t)dt \\ + &\ \int_a^b \frac{\partial g}{\partial q}(t,u_0(t),u_0'(t))h'(t)dt + o(\|h\|), \end{aligned}$$

where p and q denote generic second, respectively, third variables in g. Thus

$$\begin{aligned} DT(u_0)(h) = & \int_a^b \frac{\partial g}{\partial p}(t, u_0(t), u_0'(t))h(t)dt \\ + & \int_a^b \frac{\partial g}{\partial q}(t, u_0(t), u_0'(t))h'(t)dt. \end{aligned}$$

For notation's sake we shall now drop the arguments in g and its partial derivatives. We compute

$$\int_a^b \frac{\partial g}{\partial q} h' dt = \left[\frac{\partial g}{\partial q} h\right]_a^b - \int_a^b h \frac{d}{dt}\left(\frac{\partial g}{\partial q}\right) dt,$$

and since $h \in E$, it follows that

$$DT(u_0)(h) = \int_a^b \left[\frac{\partial g}{\partial p} - \frac{d}{dt}\frac{\partial g}{\partial q}\right] h dt. \tag{14}$$

Since $DT(u_0)(h) = 0$ for all $h \in E$, it follows that

$$\frac{\partial g}{\partial p}(t, u_0(t), u_0'(t)) - \frac{d}{dt}\frac{\partial g}{\partial q}(t, u_0(t), u_0'(t)) = 0, \tag{15}$$

for $a \le t \le b$ (this fact is often referred to as the fundamental lemma of the calculus of variations). Equation (15) is called the <u>Euler-Lagrange</u> equation. If g is twice continuously differentiable (15) becomes

$$\frac{\partial g}{\partial p} - \frac{\partial^2 g}{\partial t \partial q} - \frac{\partial^2 g}{\partial p \partial q} u_0' - \frac{\partial^2 g}{\partial q^2} u_0'' = 0, \tag{16}$$

where it again is understood that all partial derivatives are to be evaluated at $(t, u_0(t), u_0'(t))$. We hence conclude that an extremal $u_0 \in E$ must solve the nonlinear differential equation (16).

4 Some Special Mappings

Throughout our text we shall have occasion to study equations defined by mappings which enjoy special kinds of properties. We shall briefly review some such properties and refer the reader for more detailed discussions to standard texts on analysis and functional analysis (e.g. [8]).

4.1 Completely continuous mappings

Let E and X be Banach spaces and let Ω be an open subset of E, let

$$f : \Omega \to X$$

be a mapping. Then f is called <u>compact</u>, whenever $f(\Omega')$ is precompact in X for every bounded subset Ω' of Ω (i.e. $\overline{f(\Omega')}$ is compact in X). We call f <u>completely continuous</u> whenever f is compact and continuous. We note that if f is linear and compact, then f is completely continuous.

2 Lemma *Let Ω be an open set in E and let $f : \Omega \to X$ be completely continuous, let f be F-differentiable at a point $x_0 \in \Omega$. Then the linear mapping $T = Df(x_0)$ is compact, hence completely continuous.*

PROOF. Since T is linear it suffices to show that $T(\{x : \|x\| \le 1\})$ is precompact in X. (We again shall use the symbol $\|\cdot\|$ to denote both the norm in E and in X.) If this were not the case, there exists $\epsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset E, \|x_n\| \le 1, n = 1, 2, 3, \cdots$ such that

$$\|Tx_n - Tx_m\| \ge \epsilon, n \ne m.$$

Choose $\delta > 0$ such that

$$\|f(x_0 + h) - f(x_0) - Th\| < \frac{\epsilon}{3}\|h\|,$$

for $h \in E, \|h\| \le \delta$. Then for $n \ne m$

$$\begin{aligned} & \|f(x_0 + \delta x_n) - f(x_0 + \delta x_m)\| \ge \delta\|Tx_n - Tx_m\| \\ - \quad & \|f(x_0 + \delta x_n) - f(x_0) - \delta Tx_n\| - \|f(x_0 + \delta x_m) - f(x_0) - \delta Tx_m\| \\ \ge \quad & \delta\epsilon - \tfrac{\delta\epsilon}{3} - \tfrac{\delta\epsilon}{3} = \tfrac{\delta\epsilon}{3}. \end{aligned}$$

Hence the sequence $\{f(x_0 + \delta x_n)\}_{n=1}^{\infty}$ has no convergent subsequence. On the other hand, for $\delta > 0$, small, the set $\{x_0 + \delta x_n\}_{n=1}^{\infty} \subset \Omega$, and is bounded, implying by the complete continuity of f that $\{f(x_0 + \delta x_n)\}_{n=1}^{\infty}$ is precompact. We have hence arrived at a contradiction. $\square$

4.2 Proper mappings

Let $M \subset E$, $Y \subset X$ and let $f : M \to Y$ be continuous, then f is called a <u>proper</u> mapping if for every compact subset K of Y, $f^{-1}(K)$ is compact in M. (Here we consider M and Y as metric spaces with metrics induced by the norms of E and X, respectively.)

3 Lemma *Let $h : E \to X$ be completely continuous and let $g : E \to X$ be proper, then $f = g - h$ is a proper mapping, provided that f is <u>coercive</u>, i.e.*

$$\|f(x)\| \to \infty \quad as \quad \|x\| \to \infty. \tag{17}$$

PROOF. Let K be a compact subset of X and let $N = f^{-1}(K)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in N. Then there exists $\{y_n\}_{n=1}^{\infty} \subset K$ such that

$$y_n = g(x_n) - h(x_n). \tag{18}$$

Since K is compact, the sequence $\{y_n\}_{n=1}^{\infty}$ has a convergent subsequence, and since f is coercive the sequence $\{x_n\}_{n=1}^{\infty}$ must be bounded, further, because h is completely continuous, the sequence $\{h(x_n)\}_{n=1}^{\infty}$ must have a convergent subsequence. It follows that the sequence $\{g(x_n)\}_{n=1}^{\infty}$ has a convergent subsequence. Relabeling, if necessary, we may assume that all three sequences $\{y_n\}_{n=1}^{\infty}$, $\{g(x_n)\}_{n=1}^{\infty}$ and $\{h(x_n)\}_{n=1}^{\infty}$ are convergent. Since

$$g(x_n) = y_n + h(x_n)$$

and g is proper, it follows that $\{x_n\}_{n=1}^{\infty}$ converges also, say $x_n \to x$; hence N is precompact. That N is also closed follows from the fact that g and h are continuous. □

4 **Corollary** *Let $h : E \to E$ be a completely continuous mapping, and let $f =$ $\mathrm{id} - \mathrm{h}$ be coercive, then f is proper (here* id *is the identity mapping).*

PROOF. We note that id : E → E is a proper mapping. □

In finite dimensional spaces the concepts of coercivity and properness are equivalent, i.e. we have:

5 **Lemma** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, then f is proper if and only if f is coercive.*

4.3 Contraction mappings, the Banach fixed point theorem

Let M be a subset of a Banach space E. A function $f : M \to E$ is called a contraction mapping if there exists a constant k, $0 \le k < 1$ such that

$$\|f(x) - f(y)\| \le k\|x - y\|, \quad \text{for all} \quad x, y \in M. \tag{19}$$

6 **Theorem** *Let M be a closed subset of E and $f : M \to M$ be a contraction mapping, then f has a unique fixed point in M; i.e. there exists a unique $x \in M$ such that*

$$f(x) = x \quad . \tag{20}$$

PROOF. If $x, y \in M$ both satisfy (20), then

$$\|x - y\| = \|f(x) - f(y)\| \le k\|x - y\|,$$

hence, since $k < 1$, x must equal y, establishing uniqueness of a fixed point.

To proof existence, we define a sequence $\{x_n\}_{n=0}^{\infty} \subset M$ inductively as follows: Choose $x_0 \in M$ and let

$$x_n = f(x_{n-1}),\ n \ge 1. \tag{21}$$

(21) implies that for any $j \ge 1$

$$\|f(x_j) - f(x_{j-1})\| \le k^j \|x_1 - x_0\|,$$

and hence if $m > n$

$$x_m - x_n = x_m - x_{m-1} + x_{m-1} - \ldots + x_{n+1} - x_n =$$

$$f(x_{m-1}) - f(x_{m-2}) + f(x_{m-2}) - \ldots + f(x_n) - f(x_{n-1})$$

and therefore

$$\|x_m - x_n\| \leq \|x_1 - x_0\|(k^n + \ldots + k^{m-1}) = \frac{k^n - k^m}{1-k}\|x_1 - x_0\|. \quad (22)$$

It follows from (22) that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in E, hence

$$\lim_{n\to\infty} x_n = x$$

exists and since M is closed, $x \in M$. Using (21) we obtain that (20) holds. □

7 Remark We note that the above theorem, Theorem 6, also holds if E is a complete metric space with metric d. This is easily seen by replacing $\|x - y\|$ by $d(x, y)$ in the proof.

In the following example we provide an elementary approach to the existence and uniqueness of a solution of a nonlinear boundary value problem (see [7]). The approach is based on the L^p theory of certain linear differential operators subject to boundary constraints.

Let $T > 0$ be given and let

$$f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

be a mapping satisfying Carathéodory conditions; i.e. $f(t, u, u')$ is continuous in (u, u') for almost all t and measurable in t for fixed (u, u').

We consider the Dirichlet problem, i.e. the problem of finding a function u satisfying the following differential equation subject to boundary conditions

$$\begin{cases} u'' & = f(t, u, u), \ 0 < t < T, \\ u & = 0, \ t \in \{0, T\}. \end{cases} \quad (23)$$

In what is to follow, we shall employ the notation that $|\cdot|$ stands for absolute value in $\mathbb{R}$ and $\|\cdot\|_2$ the norm in $L^2(0, T)$.

We have the following results:

8 Theorem *Let f satisfy*

$$|f(x, u, v) - f(x, \tilde{u}, \tilde{v})| \leq a|u - \tilde{u}| + b|v - \tilde{v}|, \ \forall u, \tilde{u}, v, \tilde{v} \in \mathbb{R}, \ 0 < t < T, \quad (24)$$

where a, b are nonegative constants such that

$$\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1, \quad (25)$$

and λ_1 is the principal eigenvalue of $-u''$ subject to the Dirichlet boundary conditions on $u(0) = 0 = u(T)$. (I.e. the smallest number λ such that the problem

$$\begin{cases} -u'' &= \lambda u,\ 0 < t < T, \\ \quad u &= 0,\ t \in \{0, T\}. \end{cases} \tag{26}$$

has a nontrivial solution.) Then problem (23) has a unique solution $u \in C_0^1([0,T])$, with u' absolutely continuous and the equation (23) being satisfied almost everywhere.

Proof. Results from elementary differential equations tell us that λ_1 is the first positive number λ such that the problem (26) has a nontrivial solution, i.e. $\lambda_1 = \frac{\pi^2}{T^2}$.

To prove the theorem, let us, for $v \in L^1(0,T)$, put

$$Av = f(\cdot, w, w'), \tag{27}$$

where

$$w(t) = -\frac{t}{T}\int_0^T \int_0^\tau v(s)dsd\tau + \int_0^t \int_0^\tau v(s)dsd\tau,$$

which, in turn may be rewritten as

$$w(t) = \int_0^T G(t,s)v(s)ds, \tag{28}$$

where

$$G(t,s) = -\frac{1}{T}\begin{cases} (T-t)s, \text{ if } 0 \le s \le t \\ t(T-s), \text{ if } t \le s \le T. \end{cases} \tag{29}$$

It follows from (24) that the operator A is a mapping of $L^1(0,T)$ to any $L^q(0,T),\ q \ge 1$. On the other hand we have that the imbedding

$$L^q(0,T) \hookrightarrow L^1(0,T),\ q \ge 1,$$

$$u \in L^q(0,T) \mapsto u \in L^1(0,T),$$

is a continuous mapping, since

$$\|u\|_{L^1} \le T^{\frac{q}{q-1}}\|u\|_{L^q}.$$

We hence may consider

$$A : L^q(0,T) \to L^q(0,T),$$

for any $q \ge 1$. In carrying out the computations in the case $q = 2$, the following inequalities will be used; their proofs may be obtained using Fourier series methods, and will be left as an exercise. We have for $w(t) = \int_0^T G(t,s)v(s)ds$ that

$$\|w\|_{L^2} \leq \frac{1}{\lambda_1}\|v\|_{L^2},$$

from which easily follows, via an integration by parts, that

$$\|w'\|_{L^2} \leq \frac{1}{\sqrt{\lambda_1}}\|v\|_{L^2}$$

Using these facts in the computations one obtains the result that A is a contraction mapping.

On the other hand, if $v \in L^2(0,T)$ is a fixed point of A, then

$$u(t) = \int_0^T G(t,s)v(s)ds$$

is in $C_0^1(0,T)$ and $u'' \in L^2(0,T)$ and u solves (23). □

9 Remark It is clear from the proof that in the above the real line $\mathbb{R}$ may be replaced by $\mathbb{R}^m$ thus obtaining a result for systems of boundary value problems.

10 Remark In case $T = \pi$, $\lambda_1 = 1$ and condition (25) becomes

$$a + b < 1,$$

whereas a classical result of Picard requires

$$a\frac{\pi^2}{8} + b\frac{\pi}{2} < 1,$$

(see [14] where also other results are cited).

11 Remark Theorem 8 may be somewhat extended using a result of Opial [21] which says that for $u \in C_0[0,T]$, with u' absolutely continuous, we have that

$$\int_0^T |u(x)||u'(x)|dx \leq \frac{T}{4}\int_0^T |u'(x)|^2 dx. \tag{30}$$

The derivation of such a statement is left as an exercise.

4.4 The implicit function theorem

Let us now assume we have Banach spaces E, X, Λ and let

$$f : U \times V \to X,$$

(where U is open in E, V is open in Λ) be a continuous mapping satisfying the following condition:

- For each $\lambda \in V$ the map $f(\cdot, \lambda) : U \to X$ is Fréchet-differentiable on U with Fréchet derivative

$$D_u f(u, \lambda) \tag{31}$$

and the mapping $(u, \lambda) \mapsto D_u f(u, \lambda)$ is a continuous mapping from $U \times V$ to $\mathfrak{L}(E, X)$.

12 Theorem (Implicit Function Theorem) *Let f satisfy (31) and let there exist $(u_0, \lambda_0) \in U \times V$ such that $D_u f(u_0, \lambda_0)$ is a linear homeomorphism of E onto X (i.e. $D_u f(u_0, \lambda_0) \in \mathfrak{L}(E, X)$ and $[D_u f(u_0, \lambda_0)]^{-1} \in \mathfrak{L}(X, E)$). Then there exist $\delta > 0$ and $r > 0$ and unique mapping $u : B_\delta(\lambda_0) = \{\lambda : \|\lambda - \lambda_0\| \leq \delta\} \to E$ such that*

$$f(u(\lambda), \lambda) = f(u_0, \lambda_0), \tag{32}$$

and $\|u(\lambda) - u_0\| \leq r$, $u(\lambda_0) = u_0$.

Proof. Let us consider the equation

$$f(u, \lambda) = f(u_0, \lambda_0)$$

which is equivalent to

$$[D_u f(u_0, \lambda_0)]^{-1}(f(u, \lambda) - f(u_0, \lambda_0)) = 0, \tag{33}$$

or

$$u = u - [D_u f(u_0, \lambda_0)]^{-1}(f(u, \lambda) - f(u_0, \lambda_0)) \stackrel{\text{def}}{=} G(u, \lambda). \tag{34}$$

The mapping G has the following properties:

i) $G(u_0, \lambda_0) = u_0$,

ii) G and $D_u G$ are continuous in (u, λ),

iii) $D_u G(u_0, \lambda_0) = 0$.

Hence

$$\begin{array}{ll} & \|G(u_1, \lambda) - G(u_2, \lambda)\| \\ \leq & \left(\sup_{0 \leq t \leq 1} \|D_u G(u_1 + t(u_2 - u_1), \lambda)\|\right) \|u_1 - u_2\| \\ \leq & \frac{1}{2}\|u_1 - u_2\|, \end{array} \tag{35}$$

provided $\|u_1 - u_0\| \leq r$, $\|u_2 - u_0\| \leq r$, where r is small enough. Now

$$\|G(u, \lambda) - u_0\| = \|G(u, \lambda) - G(u_0, \lambda_0)\| \leq \|G(u, \lambda) - G(u_0, \lambda)\|$$

$$+\|G(u_0, \lambda) - G(u_0, \lambda_0)\| \leq \frac{1}{2}\|u - u_0\| + \|G(u_0, \lambda) - G(u_0, \lambda_0)\|$$

$$\leq \frac{1}{2}r + \frac{1}{2}r,$$

provided $\|\lambda - \lambda_0\| \leq \delta$ is small enough so that $\|G(u_0, \lambda) - G(u_0, \lambda_0)\| \leq \frac{1}{2}r$.

Let $B_\delta(\lambda_0) = \{\lambda : \|\lambda - \lambda_0\| \leq \delta\}$ and define $M = \{u : B_\delta(\lambda_0) \to E$ such that u is continuous, $u(\lambda_0) = u_0$, $\|u(\lambda_0) - u_0\|_0 \leq r$, and $\|u\|_0 = \sup_{\lambda \in B_\delta(\lambda_0)} \|u(\lambda)\| < +\infty\}$. Then M is a closed subset of a Banach space and (35) defines an equation

$$u(\lambda) = G(u(\lambda), \lambda) \tag{36}$$

in M.

Define g by (here we think of u as an element of M)

$$g(u)(\lambda) = G(u(\lambda), \lambda),$$

then $g : M \to M$ and it follows by (36) that

$$\|g(u) - g(v)\|_0 \leq \frac{1}{2}\|u - v\|_0,$$

hence g has a unique fixed point by the contraction mapping principle (Theorem 6).

□

13 **Remark** If in the implicit function theorem f is k times continuously differentiable, then the mapping $\lambda \mapsto u(\lambda)$ inherits this property.

14 **Example** *As an example let us consider the nonlinear boundary value problem*

$$u'' + \lambda e^u = 0, \ 0 < t < \pi, \quad u(0) = 0 = u(\pi). \tag{37}$$

This is a one space-dimensional mathematical model from the theory of combustion (cf [2]) and u represents a dimensionless temperature. We shall show, by an application of Theorem 12, that for $\lambda \in \mathbb{R}$, in a neighborhood of 0, (37) has a unique solution of small norm in $C^2([0,\pi], \mathbb{R})$.

To this end we define

$$E = C_0^2([0,\pi], \mathbb{R})$$

$$X = C^0[0,\pi]$$

$$\Lambda = \mathbb{R},$$

these spaces being equipped with their usual norms (see earlier examples). Let

$$f : E \times \Lambda \to X$$

be given by

$$f(u, \lambda) = u'' + \lambda e^u.$$

Then f is continuous and $f(0,0)=0$. (When $\lambda=0$ (no heat generation) the unique solution is $u\equiv 0$.) Furthermore, for $u_0\in E$, $D_uf(u_0,\lambda)$ is given by (the reader should carry out the verification)

$$D_uf(u_0,\lambda)v=v''+\lambda e^{u_0(x)}v,$$

and hence the mapping

$$(u,\lambda)\mapsto D_uf(u,\lambda)$$

is continuous. Let us consider the linear mapping

$$T=D_uf(0,0):E\to X.$$

We must show that this mapping is a linear homeomorphism. To see this we note that for every $h\in X$, the unique solution of

$$v''=h(t),\ 0<t<\pi,\quad v(0)=0=v(\pi),$$

is given by (see also (28)

$$v(t)=\int_0^\pi G(t,s)h(s)ds, \tag{38}$$

where

$$G(x,s)=\begin{cases}-\frac{1}{\pi}(\pi-t)s,\ 0\le s\le t\\ -\frac{1}{\pi}t(\pi-s),\ t\le s\le\pi.\end{cases}$$

From the representation (38) we may conclude that there exists a constant c such that

$$\|v\|_2=\|T^{-1}h\|_2\le c\|h\|_0,$$

i.e. T^{-1} is one to one and continuous. Hence all conditions of the implicit function theorem are satisfied and we may conclude that for each λ, λ sufficiently small, (37) has a unique small solution $u\in C^2([0,\pi],\mathbb{R})$, furthermore the map $\lambda\mapsto u(\lambda)$ is continuous from a neighborhood of $0\in\mathbb{R}$ to $C^2([0,\pi],\mathbb{R})$. We later shall show that this 'solution branch' $(\lambda,u(\lambda))$ may be globally continued. To this end we note here that the set $\{\lambda>0:(37)\text{ has a solution }\}$ is bounded above. We observe that if $\lambda>0$ is such that (37) has a solution, then the corresponding solution u must be positive, $u(x)>0$, $0<x<\pi$. Hence

$$0=u''+\lambda e^u>u''+\lambda u. \tag{39}$$

Let $v(t)=\sin t$, then v satisfies

$$v''+v=0,\ 0<t<\pi,\ v(0)=0=v(\pi). \tag{40}$$

From (39) and (40) we obtain

$$0 > \int_0^\pi (u''v - v''u)dt + (\lambda - 1)\int_0^\pi uvdt,$$

and hence, integrating by parts,

$$0 > (\lambda - 1)\int_0^\pi uvdx,$$

implying that $\lambda < 1$.

5 Inverse Function Theorems

We next proceed to the study of the inverse of a given mapping and provide two inverse function theorems. Since the first result is proved in exactly the same way as its finite dimensional analogue (it is an immediate consequence of the implicit function theorem) we shall not prove it here (see again [8]).

15 Theorem *Let E and X be Banach spaces and let U be an open neighborhood of $a \in E$. Let $f : U \to X$ be a C^1 mapping with $Df(a)$ a linear homeomorphism of E onto X. Then there exist open sets U' and V, $a \in U'$, $f(a) \in V$ and a uniquely determined function g such that:*

i) $V = f(U')$,

ii) *f is one to one on U',*

iii) $g : V \to U'$, $g(V) = U'$, $g(f(u)) = u$, *for every* $u \in U'$,

iv) *g is a C^1 function on V and* $Dg(f(a)) = [Df(a)]^{-1}$.

16 Example *Consider the forced nonlinear oscillator (periodic boundary value problem)*

$$u'' + \lambda u + u^2 = g, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \tag{41}$$

where g is a continuous $2\pi -$ periodic function and $\lambda \in \mathbb{R}$, is a parameter. Let $E = C^2([0, 2\pi], \mathbb{R}) \cap \{u : u(0) = u(2\pi), u'(0) = u'(2\pi)\}$, and $X = C^0([0, 2\pi], \mathbb{R})$, where both spaces are equipped with the norms discussed earlier. Then for certain values of λ, (41) has a unique solution for all forcing terms g of small norm.

Let

$$f : E \to X$$

be given by

$$f(u) = u'' + \lambda u + u^2.$$

Then $Df(u)$ is defined by

$$(Df(u))(v) = v'' + \lambda v + 2uv,$$

and hence the mapping

$$u \mapsto Df(u)$$

is a continuous mapping of E to $\mathfrak{L}(E;X)$, i.e. f is a C^1 mapping. It follows from elementary differential equations theory (see eg. [3]) that the problem

$$v'' + \lambda v = h,$$

has a unique 2π–periodic solution for every 2π–periodic h as long as $\lambda \neq n^2$, $n = 1, 2, \ldots$, and that $\|v\|_2 \leq C\|h\|_0$ for some constant C (only depending upon λ). Hence $Df(0)$ is a linear homeomorphism of E onto X. We thus conclude that for given $\lambda \neq n^2$, (32) has a unique solution $u \in E$ of small norm for every $g \in X$ of small norm.

We note that the above example is prototypical for forced nonlinear oscillators. Virtually the same arguments can be applied (the reader might carry out the necessary calculations) to conclude that the forced pendulum equation

$$u'' + \lambda \sin u = g \tag{42}$$

has a 2π- periodic response of small norm for every 2π - periodic forcing term g of small norm, as long as $\lambda \neq n^2$, $n = 1, 2, \ldots$.

In many physical situations (see the example below) it is of interest to know the number of solutions of the equation describing this situation. The following result describes a class of problems where the precise number of solutions (for every given forcing term) may be obtained by simply knowing the number of solutions for some fixed forcing term.

Let M and Y metric spaces (e.g. subsets of Banach spaces with metric induced by the norms).

17 Theorem *Let $f : M \to Y$ be continuous, proper and locally invertible (e.g. Theorem 15 is applicable at each point). For $y \in Y$ let*

$$N(y) = \text{ cardinal number of } \{f^{-1}(y)\} = \#\{f^{-1}(y)\}.$$

Then the mapping

$$y \mapsto N(y)$$

is finite and locally constant.

Proof. We first show that for each $y \in Y$, $N(y)$ is finite. Since $\{y\}$ is compact $\{f^{-1}(y)\}$ is compact also, because f is a proper mapping. Since f is locally invertible, there exists, for each $u \in \{f^{-1}(y)\}$ a neighborhood $\mathcal{O}_u$ such that

$$\mathcal{O}_u \cap (\{f^{-1}(y)\} \setminus \{u\}) = \emptyset,$$

and thus $\{f^{-1}(y)\}$ is a discrete and compact set, hence finite.

We next show that N is a continuous mapping to the nonnegative integers, which will imply that N is constant–valued. Let $y \in Y$ and let $\{f^{-1}(y)\} = \{u_1, \dots, u_n\}$. We choose disjoint open neighborhoods $\mathcal{O}_i$ of $u_i, 1 \leq i \leq n$ and let $I = \bigcap_{i=1}^n f(\mathcal{O}_i)$. Then there exist open sets V_i, $u_i \in V_i$ such that f is a homeomorphism from V_i to I. We next claim that there exists a neighborhood $W \subset I$ of y such that N is constant on W. For if not, there will exist a sequence $\{y_m\}$, with $y_m \to y$, such that as $m \to \infty$, $N(y_m) > N(y)$ (note that for any $v \in I$, v has a preimage in each V_i which implies $N(v) \geq N(y)$, $v \in I$!). Hence there exists a sequence $\{\xi_m\}$, $\xi_m \notin \bigcup_{i_1}^n V_i$, such that $f(\xi_n) = y_n$. Since $f^{-1}(\{y_n\} \cup \{y\})$ is compact, the sequence $\{\xi_n\}$ will have a convergent subsequence, say $\xi_{n_j} \to \xi$. And since f is continuous, $f(\xi) = y$. Hence $\xi = u_i$, for some i, a contradiction to $\xi \notin \bigcup_{i=1}^n V_i$. $\square$

18 Corollary *Assume Y is connected, then $N(Y)$ is constant.*

Examples illustrating this result will be given later in the text.

6 The Dugundji Extension Theorem

In the course of developing the Brouwer and Leray–Schauder degree and in proving some of the classical fixed point theorems we need to extend mappings defined on proper subsets of a Banach space to the whole space in a suitable manner. The result which guarantees the existence of extensions having the desired properties is the Dugundji extension theorem ([9]) which will be established in this section.

In proving the theorem we need a result from general topology which we state here for convenience (see e.g [9]). We first give some terminology.

Let M be a metric space and let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$, where Λ is an index set, be an open cover of M. Then $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is called locally finite if every point $u \in M$ has a neighborhood U such that U intersects at most finitely many elements of $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$.

19 Lemma *Let M be a metric space. Then every open cover of M has a locally finite refinement.*

20 Theorem *Let E and X be Banach spaces and let $f : C \to K$ be a continuous mapping, where C is closed in E and K is convex in X. Then there exists a continuous mapping*

$$\tilde{f} : E \to K$$

such that

$$\tilde{f}(u) = f(u), \ u \in C.$$

PROOF. For each $u \in E\backslash C$ let

$$r_u = \frac{1}{3}\text{dist}(u, C)$$

and

$$B_u = \{v \in E : \|v - u\| < r_u\}.$$

Then

$$\text{diam}B_u \leq \text{dist}(B_u, C).$$

The collection $\{B_u\}_{u \in E\backslash C}$ is an open cover of the metric space $E\backslash C$ and hence has a locally finite refinement $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$, i.e.

i) $\bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda \supset E\backslash C$,

ii) for each $\lambda \in \Lambda$ there exists B_u such that $\mathcal{O}_\lambda \subset B_u$

iii) $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is locally finite.

Define

$$q : E\backslash C \to (0, \infty) \tag{43}$$

by

$$q(u) = \sum_{\lambda\in\Lambda} \text{dist}(u, E\backslash\mathcal{O}_\lambda).$$

The sum in the right hand side of (43) contains only finitely many terms, since $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is locally finite. This also implies that q is a continuous function.

Define

$$\rho_\lambda(u) = \frac{\text{dist}(u, E\backslash\mathcal{O}_\lambda)}{q(u)},\ \lambda \in \Lambda,\ u \in E\backslash C.$$

It follows that

$$0 \leq \rho_\lambda(u) \leq 1,\ \lambda \in \Lambda$$

$$\sum_{\lambda\in\Lambda} \rho_\lambda(u) = 1,\ u \in E\backslash C.$$

For each $\lambda \in \Lambda$ choose $u_\lambda \in C$ such that

$$\text{dist}(u_\lambda, \mathcal{O}_\lambda) \leq 2\text{dist}(C, \mathcal{O}_\lambda)$$

and define

$$\tilde{f}(u) = \begin{cases} f(u),\ u \in C \\ \sum_{\lambda\in\Lambda} \rho_\lambda(u) f(u_\lambda),\ u \notin C. \end{cases}$$

Then $\tilde{f}$ has the following properties:

i) $\tilde{f}$ is defined on E and is an extension of f.

ii) $\tilde{f}$ is continuous on the interior of C.

iii) $\tilde{f}$ is continuous on $E \backslash C$.

These properties follow immediately from the definition of $\tilde{f}$. To show that $\tilde{f}$ is continuous on E it suffices therefore to show that $\tilde{f}$ is continuous on ∂C. Let $u \in \partial C$, then since f is continuous we may, for given $\epsilon > 0$, find $0 < \delta = \delta(u, \epsilon)$ such that

$$\|f(u) - f(v)\| \leq \epsilon, \quad if \quad \|u - v\| \leq \delta, \ v \in C.$$

Now for $v \in E \backslash C$

$$\|\tilde{f}(u) - \tilde{f}(v)\| = \|f(u) - \sum_{\lambda \in \Lambda} \rho_\lambda(v) f(u_\lambda)\| \leq \sum_{\lambda \in \Lambda} \rho_\lambda(v) \|f(u) - f(u_\lambda)\|.$$

If $\rho_\lambda(v) \neq 0$, $\lambda \in \Lambda$, then $\text{dist}(v, E \backslash \mathcal{O}_\lambda) > 0$, i.e. $v \in \mathcal{O}_\lambda$. Hence $\|v - u_\lambda\| \leq \|v - w\| + \|w - u_\lambda\|$ for any $w \in \mathcal{O}_\lambda$. Since $\|v - w\| \leq \text{diam}\mathcal{O}_\lambda$ we may take the infinium for $w \in \mathcal{O}_\lambda$ and obtain

$$\|v - u_\lambda\| \leq \text{diam}\mathcal{O}_\lambda + \text{dist}(u_\lambda, \mathcal{O}_\lambda).$$

Now $\mathcal{O}_\lambda \subset B_{u_1}$ for some $u_1 \in E \backslash C$. Hence, since

$$\text{diam}\mathcal{O}_\lambda \leq \text{diam}B_{u_1} \leq \text{dist}(B_{u_1}, C) \leq \text{dist}(C, \mathcal{O}_\lambda),$$

we get

$$\|v - u_\lambda\| \leq 3\text{dist}(C, \mathcal{O}_\lambda) \leq 3\|v - u\|.$$

Thus for λ such that $\rho_\lambda(v) \neq 0$ we get $\|u - u_\lambda\| \leq \|v - u\| + \|v - u_\lambda\| \leq 4\|u - v\|$. Therefore if $\|u - v\| \leq \delta/4$, then $\|u - u_\lambda\| \leq \delta$, and $\|f(u) - f(u_\lambda)\| \leq \epsilon$, and therefore

$$\|\tilde{f}(u) - \tilde{f}(v)\| \leq \epsilon \sum_{\lambda \in \Lambda} \rho_\lambda(v) = \epsilon.$$

□

21 Corollary *Let E, X be Banach spaces and let $f : C \to X$ be continuous, where C is closed in E. Then f has a continuous extension $\tilde{f}$ to E such that*

$$\tilde{f}(E) \subset \text{co}f(C),$$

where $\text{co}f(C)$ is the convex hull of $f(C)$.

22 Corollary *Let K be a closed convex subset of a Banach space E. Then there exists a continuous mapping $f : E \to K$ such that $f(E) = K$ and $f(u) = u$, $u \in K$, i.e. K is a continuous retract of E.*

Proof. Let $\text{id} : K \to K$ be the identity mapping. This map is continuous. Since K is closed and convex we may apply Corollary 21 to obtain the desired conclusion. □

7 Exercises

1. Supply all the details for the proof of Theorem 8.
2. Compare the requirements discusse in Remark 10.
3. Derive a improved reults as suggested by Remark 11.
4. Establish the assertion of Remark 13.
5. Supply the details od the proof of Example 14.
6. Prove Theorem 15.
7. Carry out the program laid out by Example 16 to discuss the nonlinear oscillator given by (42).

Chapter II
The Method of Lyapunov-Schmidt

1 Introduction

In this chapter we shall develop an approach to bifurcation theory which, is one of the original approaches to the theory. The results obtained, since the implicit function theorem plays an important role, will be of a local nature. We first develop the method of Liapunov–Schmidt and then use it to obtain a local bifurcation result. We then use this result in several examples and also derive a Hopf bifurcation theorem.

2 Splitting Equations

Let X and Y be real Banach spaces and let F be a mapping

$$F : X \times \mathbb{R} \to Y \tag{1}$$

and let F satisfy the following conditions:

$$F(0, \lambda) = 0, \ \forall \lambda \in \mathbb{R}, \tag{2}$$

and

$$F \ \text{ is } C^2 \text{ in a neighborhood of } \ \{0\} \times \mathbb{R}. \tag{3}$$

We shall be interested in obtaining existence of nontrivial solutions (i.e. $u \neq 0$) of the equation

$$F(u, \lambda) = 0 \tag{4}$$

We call λ_0 a <u>bifurcation value</u> or $(0, \lambda_0)$ a <u>bifurcation point</u> for (4) provided every neighborhood of $(0, \lambda_0)$ in $X \times \mathbb{R}$ contains solutions of (4) with $u \neq 0$.

It then follows from the implicit function theorem that the following holds.

1 **Theorem** *If the point $(0, \lambda_0)$ is a bifurcation point for the equation*

$$F(u, \lambda) = 0, \tag{5}$$

then the Fréchet derivative $F_u(0, \lambda_0)$ cannot be a linear homeomorphism of X to Y.

The types of linear operators $F_u(0, \lambda_0)$ we shall consider are so-called Fredholm operators.

2 **Definition** *A linear operator $L : X \to Y$ is called a Fredholm operator provided:*

- *The kernel of L, $\ker L$, is finite dimensional.*
- *The range of L, $\mathrm{im} L$, is closed in Y.*
- *The cokernel of L, $\mathrm{coker} L$, is finite dimensional.*

The following lemma which is a basic result in functional analysis will be important for the development to follow, its proof may be found in any standard text, see e.g. [24].

3 **Lemma** *Let $F_u(0, \lambda_0)$ be a Fredholm operator with kernel V and cokernel Z, then there exists a closed subspace W of X and a closed subspace T of Y such that*

$$\begin{aligned} X &= V \oplus W \\ Y &= Z \oplus T. \end{aligned}$$

The operator $F_u(0, \lambda_0)$ restricted to W, $F_u(0, \lambda_0)|_W : W \to T$, is bijective and since T is closed it has a continuous inverse. Hence $F_u(0, \lambda_0)|_W$ is a linear homeomorphism of W onto T.

We recall that W and Z are not uniquely given.

Using Lemma 3 we may now decompose every $u \in X$ and F uniquely as follows:

$$\begin{aligned} u = u_1 + u_2, &\quad u_1 \in V,\ u_2 \in W, \\ F = F_1 + F_2, &\quad F_1 : X \to Z,\ F_2 : X \to T. \end{aligned} \tag{6}$$

Hence equation (5) is equivalent to the system of equations

$$\begin{aligned} F_1(u_1, u_2, \lambda) &= 0, \\ F_2(u_1, u_2, \lambda) &= 0. \end{aligned} \tag{7}$$

We next let $L = F_u(0, \lambda_0)$ and using a Taylor expansion we may write

$$F(u, \lambda) = F(0, \lambda_0) + F_u(0, \lambda_0)u + N(u, \lambda). \tag{8}$$

or

$$Lu + N(u, \lambda) = 0, \tag{9}$$

where

$$N : X \times \mathbb{R} \to T. \tag{10}$$

Using the decomposition of X we may write equation (9) as

$$Lu_2 + N(u_1 + u_2, \lambda) = 0. \tag{11}$$

Let $Q : Y \to Z$ and $I - Q : Y \to T$ be projections determined by the decomposition, then equation (10) implies that

$$QN(u, \lambda) = 0. \tag{12}$$

Since by Lemma 3, $L|_W : W \to T$ has an inverse $L^{-1} : T \to W$ we obtain from equation (11) the equivalent system

$$u_2 + L^{-1}(I - Q)N(u_1 + u_2, \lambda) = 0. \tag{13}$$

We note, that since Z is finite dimensional, equation (12) is an equation in a finite dimensional space, hence if u_2 can be determined as a function of u_1 and λ, this equation will be a finite set of equations in finitely many variables ($u_1 \in V$, which is also assumed finite dimensional!)

Concerning equation (12) we have the following result.

4 Lemma *Assume that $F_u(0, \lambda_0)$ is a Fredholm operator with W nontrivial. Then there exist $\epsilon > 0$, $\delta > 0$ and a unique solution $u_2(u_1, \lambda)$ of equation (13) defined for $|\lambda - \lambda_0| + \|u_1\| < \epsilon$ with $\|u_2(u_1, \lambda)\| < \delta$. This function solves the equation $F_2(u_1, u_2(u_1, \lambda), \lambda) = 0$.*

PROOF. We employ the implicit function theorem to analyze equation (13). That this may be done follows from the fact that at $u_1 = 0$ and $\lambda = \lambda_0$ equation (13) has the unique solution $u_2 = 0$ and the Fréchet derivative at this point with respect to u_2 is simply the identity mapping on W. □

Hence, using Lemma 4, we will have nontrivial solutions of equation (5) once we can solve

$$F_1(u_1, u_2(u_1, \lambda), \lambda) = QF(u_1, u_2(u_1, \lambda), \lambda) = 0 \tag{14}$$

for u_1, whenever $|\lambda - \lambda_0| + \|u_1\| < \epsilon$. This latter set of equations, usually refered to as the set of bifurcation equations, is, even though a finite set of equations in finitely many unknowns, the more difficult part in the solution of equation (5).

The next sections present situations, where these equations may be solved.

5 Remark We note that in the above considerations at no point was it required that λ be a one dimensional parameter.

3 Bifurcation at a Simple Eigenvalue

In this section we shall consider the analysis of the bifurcation equation (14) in the particular case that the kernel V and the cokernel Z of $F_u(0,\lambda_0)$ both have dimension 1.

We have the following theorem.

6 Theorem *In the notation of the previous section assume that the kernel V and the cokernel Z of $F_u(0,\lambda_0)$ both have dimension 1. Let $V = \text{span}\{\phi\}$ and let Q be a projection of Y onto Z. Furthermore assume that the second Fréchet derivative $F_{u\lambda}$ satisfies*

$$QF_{u\lambda}(0,\lambda_0)(\phi,1) \neq 0. \tag{15}$$

Then $(0,\lambda_0)$ is a bifurcation point and there exists a unique curve

$$u = u(\alpha), \quad \lambda = \lambda(\alpha),$$

defined for $\alpha \in \mathbb{R}$ in a neighborhood of 0 so that

$$u(0) = 0, \ u(\alpha) \neq 0, \ \alpha \neq 0, \ \lambda(0) = \lambda_0$$

and

$$F(u(\alpha), \lambda(\alpha)) = 0.$$

Proof. Since V is one dimensional $u_1 = \alpha\phi$. Hence for $|\alpha|$ small and λ near λ_0 there exists a unique $u_2(\alpha,\lambda)$ such that

$$F_2(\alpha\phi, u_2(\alpha,\lambda),\lambda) = 0.$$

We hence need to solve

$$QF(\alpha\phi, u_2(\alpha,\lambda),\lambda) = 0.$$

We let $\mu = \lambda - \lambda_0$ and define

$$g(\alpha,\mu) = QF(\alpha\phi, u_2(\alpha,\lambda),\lambda).$$

Then g maps a neighborhood of the origin of $\mathbb{R}^2$ into $\mathbb{R}$.

Using Taylor's theorem we may write

$$F(u,\lambda) = F_u u + F_\lambda \mu + \frac{1}{2}\left\{F_{uu}(u,u) + 2F_{u\lambda}(u,\lambda) + F_{\lambda\lambda}(\mu,\mu)\right\} + R, \tag{16}$$

where R contains higher order remainder terms and all Fréchet derivatives above are evaluated at $(0,\lambda_0)$.

Because of (2) we have that F_λ and $F_{\lambda\lambda}$ in the above are the zero operators, hence, by applying Q to (16) we obtain

$$QF(u,\lambda) = \frac{1}{2}\left\{QF_{uu}(u,u) + 2QF_{u\lambda}(u,\mu)\right\} + QR, \tag{17}$$

and for $\alpha \neq 0$

$$\begin{aligned} \frac{g(\alpha,\mu)}{\alpha} &= \frac{1}{2}\Big\{QF_{uu}(\phi + \frac{u_2(\alpha,\lambda)}{\alpha}, \alpha\phi + u_2(\alpha,\lambda)) \\ &+ 2QF_{u\lambda}(\phi + \frac{u_2(\alpha,\lambda)}{\alpha}, \mu)\Big\} + \frac{1}{\alpha}QR. \end{aligned} \tag{18}$$

It follows from Lemma 4 that the term $\frac{u_2(\alpha,\lambda)}{\alpha}$ is bounded for α in a neighborhood of 0. The remainder formula of Taylor's theorem implies a similar statement for the term $\frac{1}{\alpha}QR$. Hence

$$h(\alpha,\mu) = \frac{g(\alpha,\mu}{\alpha} = O(\alpha), \text{ as } \alpha \to 0.$$

We note that in fact $h(0,0) = 0$, and

$$\frac{\partial h(0,0)}{\partial \mu} = QF_{u\lambda}(0,\lambda_0)(\phi,1) \neq 0.$$

We hence conclude by the implicit function theorem that there exists a unique function $\mu = \mu(\alpha)$ defined in a neighborhood of 0 such that

$$h(\alpha,\mu(\alpha)) = 0.$$

We next set

$$u(\alpha) = \alpha\phi + u_2(\alpha,\lambda_0 + \mu(\alpha)), \ \lambda = \lambda_0 + \mu(\alpha).$$

This proves the theorem. □

The following example will serve to illustrate the theorem just established.

7 Example *The point* $(0,0)$ *is a bifurcationn point for the ordinary differential equation*

$$u'' + \lambda(u + u^3) = 0 \tag{19}$$

subject to the periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \tag{20}$$

To see this, we choose

$$X = C^2[0,2\pi] \cap \{u : u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi)\},$$

$$Y = C[0,2\pi] \cap \{u : u(0) = u(2\pi)\},$$

both equipped with the usual norms, and

$$\begin{aligned} &F : X \times \mathbb{R} \to Y \\ &(u,\lambda) \mapsto u'' + \lambda(u + u^3). \end{aligned}$$

Then F belongs to class C^2 with Fréchet derivative

$$F_u(0,\lambda_0)u = u'' + \lambda_0 u. \tag{21}$$

This linear operator has a nontrivial kernel whenever $\lambda_0 = n^2,\ n = 0, 1, \cdots$. The kernel being one dimensional if and only if $\lambda_0 = 0$.

We see that h belongs to the range of $F_u(0,0)$ if and only if $\int_0^{2\pi} h(s)ds = 0$, and hence the cokernel will have dimension 1 also. A projection $Q : Y \to Z$ then is given by $Qh = \frac{1}{2\pi}\int_0^{2\pi} h(s)ds$. Computing further, we find that $F_{u\lambda}(0,0)(u,\lambda) = \lambda u$, and hence, since we may choose $\phi = 1$, $F_{u\lambda}(0,0)(1,1) = 1$. Applying Q we get $Q1 = 1$. We may therefore conclude by Theorem 6 that equation (19) has a solution u satisfying the boundary conditions (20) which is of the form

$$u(\alpha) = \alpha + u_2(\alpha, \lambda(\alpha)).$$

Chapter III
Degree Theory

1 Introduction

In this chapter we shall introduce an important tool for the study of nonlinear equations, the degree of a mapping. We shall mainly follow the analytic development commenced by Heinz in [15] and Nagumo in [20]. For a brief historical account we refer to [27].

2 Definition of the Degree of a Mapping

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $f : \bar{\Omega} \to \mathbb{R}^n$ be a mapping which satisfies

- $$f \in C^1(\Omega, \mathbb{R}^n) \cap C(\bar{\Omega}, \mathbb{R}^n), \tag{1}$$
- $y \in \mathbb{R}^n$ is such that
  $$y \notin f(\partial\Omega), \tag{2}$$
- if $x \in \Omega$ is such that $f(x) = y$ then
  $$f'(x) = Df(x) \tag{3}$$
  is nonsingular.

1 Proposition *If f satisfies (1), (2), (3), then the equation*

$$f(x) = y \tag{4}$$

has at most a finite number of solutions in Ω.

2 Definition *Let f satisfy (1), (2), (3). Define*

$$\mathrm{d}(f, \Omega, y) = \sum_{i=1}^{k} \operatorname{sgn} \det f'(x_i) \tag{5}$$

where $x_1, \cdots, x_k$ are the solutions of (4) in Ω and

$$\operatorname{sgn}\ \det\ f'(x_i) = \begin{cases} +1, & \textit{if } \det\ f'(x_i) > 0 \\ \\ -1, & \textit{if } \det\ f'(x_i) < 0, \quad i = 1, \cdots, k. \end{cases}$$

If (4) has no solutions in Ω we let $\mathrm{d}(f, \Omega, y) = 0$.

The Brouwer degree $\mathrm{d}(f, \Omega, y)$ to be defined for mappings $f \in C(\bar{\Omega}, \mathbb{R}^n)$ which satisfy (2) will coincide with the number just defined in case f satisfies (1), (2), (3). In order to give this definition in the more general case we need a sequence of auxiliary results.

The proof of the first result, which follows readily by making suitable changes of variables, will be left as an exercise.

3 Lemma *Let $\phi : [0, \infty) \to \mathbb{R}$ be continuous and satisfy*

$$\phi(0) = 0, \quad \phi(t) \equiv 0, \quad t \geq r > 0, \quad \int_{\mathbb{R}^n} \phi(|x|)dx = 1. \tag{6}$$

Let f satisfy the conditions (1), (2), (3). Then

$$\mathrm{d}(f, \Omega, y) = \int_\Omega \phi(|f(x) - y|) \det\ f'(x)dx, \tag{7}$$

provided r is sufficiently small.

4 Lemma *Let f satisfy (1) and (2) and let $r > 0$ be such that $|f(x) - y| > r,\ x \in \partial\Omega$. Let $\phi : [0, \infty) \to \mathbb{R}$ be continuous and satisfy:*

$$\phi(s) = 0, \quad s = 0, \quad r \leq s, \quad \text{and} \quad \int_0^\infty s^{n-1}\phi(s)ds = 0. \tag{8}$$

Then

$$\int_\Omega \phi(|f(x) - y|)\ \det\ f'(x)dx = 0. \tag{9}$$

Proof. We note first that it suffices to proof the lemma for functions f which are are of class C^∞ and for functions ϕ that vanish in a neighborhood of 0. We also note that the function $\phi(|f(x) - y|)\ \det\ f'(x)$ vanishes in a neighborhood of $\partial\Omega$, hence we may extend that function to be identically zero outside Ω and

$$\int_\Omega \phi(|f(x) - y|)\ \det\ f'(x)dx = \int_{\Omega'} \phi(|f(x) - y|)\ \det\ f'(x)dx,$$

where Ω' is any domain with smooth boundary containing $\overline{\Omega}$.

We let

$$\psi(s) = \begin{cases} s^{-n}\int_0^s \rho^{n-1}\phi(\rho)d\rho, & 0 < s < \infty \\ 0, & s = 0. \end{cases} \tag{10}$$

Then ψ, so defined is a C^1 function, it vanishes in a neighborhood of 0 and in the interval $[r,\infty)$. Further ψ satifies the differential equation

$$s\psi'(s) + n\psi(s) = \phi(s). \tag{11}$$

It follows that the functions

$$g^j(x) = \psi(|x|)x_j,\ j = 1, \cdots, n$$

belong to class C^1 and

$$g^j(x) = 0,\ |x| \geq r,$$

and furthermore that for $j = 1, \cdots, n$ the functions $g^j(f(x)-y)$ are C^1 functions which vanish in a neighborhood of $\partial\Omega$. If we denote by $a_{ji}(x)$ the cofactor of the element $\frac{\partial f_i}{\partial x_j}$ in the Jacobian matrix $f'(x)$, it follows that

$$\text{div}\,(a_{j1}(x), a_{j2}(x), \cdots, a_{jn}(x)) = 0,\ j = 1, \cdots, n.$$

We next define for $i = 1, \cdots, n$

$$v_i(x) = \sum_{j=1}^{n} a_{ji}(x) g^j(f(x) - y)$$

and show that the function $v = (v_1, v_2, \cdots, v_n)$ has the property that

$$\text{div} v = \phi(|f(x) - y|)\ \det\ f'(x),$$

and hence the result follows from the divergence theorem. □

5 Lemma *Let f satisfy (1) and (2) and let $\phi : [0,\infty) \to \mathbb{R}$ be continuous, $\phi(0) = 0$, $\phi(s) \equiv 0$ for $s \geq r$, where $0 < r \leq \min_{x\in\partial\Omega} |f(x) - y|$, $\int_{\mathbb{R}^n} \phi(|x|)dx = 1$. Then for all such ϕ, the integrals*

$$\int_\Omega \phi(|f(x) - y|)\ \det\ f'(x)dx \tag{12}$$

have a common value.

Proof. Let $\Phi = \{\phi \in C([0,\infty), \mathbb{R}) : \phi(0) = 0,\ \phi(s) \equiv 0, s \geq r\}$. Put

$$L\phi = \int_0^\infty s^{n-1}\phi(s)ds$$

$$M\phi = \int_{\mathbb{R}^n} \phi(|x|)dx$$

$$N\phi = \int_\Omega \phi(|f(x) - y|)\ \det\ f'(x)dx.$$

Then L, M, N are linear functionals. It follows from Lemma 4 and the subsequent remark that $M\phi = 0$ and $N\phi = 0$, whenever $L\phi = 0$. Let $\phi_1,\ \phi_2 \in \Phi$ with $M\phi_1 = M\phi_2 = 1$, then

$$L((L\phi_2)\phi_1 - (L\phi_1)\phi_2) = 0.$$

It follows that

$$(L\phi_2)(M\phi_1) - (L\phi_1)(M\phi_2) = 0,$$

$$L\phi_2 - L\phi_1 = L(\phi_2 - \phi_1) = 0,$$

and

$$N(\phi_2 - \phi_1) = 0,$$

i.e.

$$N\phi_2 = N\phi_1.$$

□

6 Lemma *Let f_1 and f_2 satisfy (1), (2), (3) and let $\epsilon > 0$ be such that*

$$|f_i(x) - y| > 7\epsilon, \quad x \in \partial\Omega, \quad i = 1, 2, \tag{13}$$

$$|f_1(x) - f_2(x)| < \epsilon, \quad x \in \bar{\Omega}, \tag{14}$$

then

$$\mathrm{d}(f_1, \Omega, y) = \mathrm{d}(f_2, \Omega, y).$$

Proof. We may, without loss, assume that $y = 0$, since by Definition 2

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(f - y, \Omega, 0).$$

let $g \in C^1[0, \infty)$ be such that

$$g(s) = 1, \quad 0 \le s \le 2\epsilon$$

$$0 \le g(r) \le 1, \quad 2\epsilon \le r < 3\epsilon$$

$$g(r) = 0, \quad 3\epsilon \le r < \infty. \tag{15}$$

Consider

$$f_3(x) = [1 - g(|f_1(x)|)]f_1(x) + g(|f_1(x)|)f_2(x),$$

then

$$f_3 \in C^1(\Omega, \mathbb{R}^n) \cap C(\bar{\Omega}, \mathbb{R}^n)$$

and

$$|f_i(x) - f_k(x)| < \epsilon, \quad i, k = 1, 2, 3, \ x \in \bar{\Omega}$$

$$|f_i(x)| > 7\epsilon, \quad x \in \partial\Omega, \quad i = 1, 2, 3.$$

Let $\phi_i \in C[0, \infty)$, $i = 1, 2$ be continuous and be such that

$$\phi_1(t) = 0, \quad 0 \leq t \leq 4\epsilon, \quad 5\epsilon \leq t \leq \infty$$

$$\phi_2(t) = 0, \quad \epsilon \leq t < \infty, \quad \phi_2(0) = 0$$

$$\int_{\mathbb{R}^n} \phi_i(|x|)dx = 1, \quad i = 1, 2.$$

We note that

$$f_3 \equiv f_1, \quad \text{if} \quad |f_1| > 3\epsilon$$

$$f_3 \equiv f_2, \quad \text{if} \quad |f_1| < 2\epsilon.$$

Therefore

$$\begin{aligned} &\phi_1(|f_3(x)|)\text{det } f_3'(x) = \phi_1(|f_1(x)|)\text{det } f_1'(x) \\ &\phi_2(|f_3(x)|)det\ f_3'(x) = \phi_2(|f_2(x)|)det\ f_2'(x). \end{aligned} \tag{16}$$

Integrating both sides of (16) over Ω and using Lemmas 4 and 5 we obtain the desired conclusion. □

7 Corollary *Let f satisfy conditions (1), (2), (3), then for $\epsilon > 0$ sufficiently small any function g which also satisfies these conditions and which is such that $|f(x) - g(x)| < \epsilon$, $x \in \bar{\Omega}$, has the property that* $\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, y)$.

Up to now we have shown that if f and g satisfy conditions (1), (2), (3) and if they are sufficiently "close" then they have the same degree. In order to extend this definition to a broader class of functions, namely those which do not satisfy (3) we need a version of Sard's Theorem (Lemma 8) (an important lemma of Differential Topology) whose proof may be found in [25], see also [29].

8 Lemma *If Ω is a bounded open set in $\mathbb{R}^n$, f satisfies (1), (2), and*

$$E = \{x \in \Omega : \text{det } f'(x) = 0\}. \tag{17}$$

Then $f(E)$ does not contain a sphere of the form $\{z : |z - y| < r\}$.

This lemma has as a consequence the obvious corrollary:

9 Corollary *Let*

$$F = \{h \in \mathbb{R}^n : y + h \in f(E)\}, \tag{18}$$

where E is given by (17), then F is dense in a neighborhood of $0 \in \mathbb{R}^n$ and

$$f(x) = y + h, \quad x \in \Omega, \quad h \in F$$

implies that $f'(x)$ is nonsingular.

We thus conclude that for all $\epsilon > 0$, sufficiently small, there exists $h \in F$, $0 < |h| < \epsilon$, such that $\mathrm{d}(f, \Omega, y + h) = \mathrm{d}(f - h, \Omega, y)$ is defined by Definition 2. It also follows from Lemma 6 that for such h, $\mathrm{d}(f, \Omega, y + h)$ is constant. This justifies the following definition.

10 Definition *Let f satisfy (1) and (2). We define*

$$\mathrm{d}(f, \Omega, y) = \lim_{\substack{h \to 0 \\ h \in F}} \mathrm{d}(f - h, \Omega, y). \tag{19}$$

Where F is given by (18) and $\mathrm{d}(f - h, \Omega, y)$ is defined by Definition 2.

We next assume that $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and satisfies (2). Then for $\epsilon > 0$ sufficiently small there exists $g \in C^1(\Omega, \mathbb{R}^n) \bigcap C(\bar{\Omega}, \mathbb{R}^n)$ such that $y \notin g(\partial\Omega)$ and

$$\|f - g\| = \max_{x \in \bar{\Omega}} |f(x) - g(x)| < \epsilon/4$$

and there exists, by Lemma 8, $\tilde{g}$ satisfying (1), (2), (3) such that $\|g - \tilde{g}\| < \epsilon/4$ and if $\tilde{h}$ satisfies (1), (2), (3) and $\|g - \tilde{h}\| < \epsilon/4$, $\|\tilde{g} - \tilde{h}\| < \epsilon/2$, then $\mathrm{d}(\tilde{g}, \Omega, y) = \mathrm{d}(\tilde{h}, \Omega, y)$ provided ϵ is small enough. Thus f may be approximated by functions $\tilde{g}$ satisfying (1), (2), (3) and $\mathrm{d}(\tilde{g}, \Omega, y) = \text{constant}$ provided $\|f - \tilde{g}\|$ is small enough. We therefore may define $\mathrm{d}(f, \Omega, y)$ as follows.

11 Definition *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be such that $y \notin f(\partial\Omega)$. Let*

$$\mathrm{d}(f, \Omega, y) = \lim_{g \to f} \mathrm{d}(g, \Omega, y) \tag{20}$$

where g satisfies (1), (2), (3).

The number defined by (20) is called the Brouwer degree of f at y relative to Ω.

It follows from our considerations above that $\mathrm{d}(f, \Omega, y)$ is also given by formula (7), for any ϕ which satisfies:

$$\phi \in C([0, \infty), \mathbb{R}),\ \phi(0) = 0,\ \phi(s) \equiv 0,\ s \geq r > 0,\ \int_{\mathbb{R}^n} \phi(|x|)dx = 1,$$

where $r < \min_{x \in \partial\Omega} |f(x) - y|$.

3 Properties of the Brouwer Degree

We next proceed to establish some properties of the Brouwer degree of a mapping which will be of use in computing the degree and also in extending the definition to mappings defined in infinite dimensional spaces and in establishing global solution results for parameter dependent equations.

12 **Proposition (Solution property)** *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be such that $y \notin f(\partial\Omega)$ and assume that $\mathrm{d}(f, \Omega, y) \neq 0$. Then the equation*

$$f(x) = y \tag{21}$$

has a solution in Ω.

The proof is a straightforward consequence of Definition 11 and is left as an exercise.

13 **Proposition (Continuity property)** *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ be such that $\mathrm{d}(f, \Omega, y)$ is defined. Then there exists $\epsilon > 0$ such that for all $g \in C(\bar{\Omega}, \mathbb{R}^n)$ and $\hat{y} \in \mathbb{R}$ with $\|f - g\| + |y - \hat{y}| < \epsilon$*

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, \hat{y}).$$

The proof again is left as an exercise.

The proposition has the following important interpretation.

14 **Remark** If we let $C = \{f \in C(\bar{\Omega}, \mathbb{R}^n) : y \notin f(\partial\Omega)\}$ then C is a metric space with metric ρ defined by $\rho(f, g) = \|f - g\|$. If we define the mapping $\mathrm{d} : C \to \mathbb{N}$ (integers) by $\mathrm{d}(f) = \mathrm{d}(f, \Omega, y)$, then the theorem asserts that d is a continuous function from C to $\mathbb{N}$ (equipped with the discrete topology). Thus d will be constant on connected components of C.

Using this remark one may establish the following result.

15 **Proposition (Homotopy invariance property)** *Let $f, g \in C(\bar{\Omega}, \mathbb{R}^n)$ with $f(x)$ and $g(x) \neq y$ for $x \in \partial\Omega$ and let $h : [a, b] \times \bar{\Omega} \to \mathbb{R}^n$ be continuous such that $h(t, x) \neq y$, $(t, x) \in [a, b] \times \partial\Omega$. Further let $h(a, x) = f(x)$, $h(b, x) = g(x)$, $x \in \bar{\Omega}$. Then*

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, y);$$

more generally, $\mathrm{d}(h(t, \cdot), \Omega, y) = \text{constant}$ *for $a \leq t \leq b$.*

The next corollary may be viewed as an extension of Rouché's theorem concerning the equal number of zeros of certain analytic functions. This extension will be the content of one of the exercises at the end of thic chapter.

16 **Corollary** *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be such that $\mathrm{d}(f, \Omega, y)$ is defined. Let $g \in C(\bar{\Omega}, \mathbb{R}^n)$ be such that $|f(x) - g(x)| < |f(x) - y|$, $x \in \partial\Omega$. Then $\mathrm{d}(f, \Omega, y) = \mathrm{d}(g, \Omega, y)$.*

PROOF. For $0 \leq t \leq 1$ and $x \in \partial\Omega$ we have that

$$|y - tg(x) - (1 - t)f(x)| = |(y - f(x)) - t(g(x) - f(x))|$$

$$\geq |y - f(x)| - t|g(x) - f(x)|$$

$$> 0 \quad \text{since} \quad 0 \le t \le 1,$$

hence $h : [0,1] \times \bar{\Omega} \to \mathbb{R}^n$ given by $h(t,x) = tg(x) + (1-t)f(x)$ satisfies the conditions of Proposition 15 and the conclusion follows from that proposition. □ As an immediate corollary we have the following:

17 Corollary *Assume that f and g are mappings such that $f(x) = g(x)$, $x \in \partial\Omega$, then $\mathrm{d}(f,\Omega,y) = \mathrm{d}(g,\Omega,y)$ if the degree is defined, i.e. the degree only depends on the boundary data.*

18 Proposition (Additivity property) *Let Ω be a bounded open set which is the union of m disjoint open sets $\Omega_1, \cdots, \Omega_m$, and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ be such that $y \notin f(\partial\Omega_i)$, $i = 1, \cdots, m$. Then*

$$\mathrm{d}(f,\Omega,y) = \sum_{i=i}^{m} \mathrm{d}(f,\Omega_i,y).$$

19 Proposition (Excision property) *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ and let K be a closed subset of $\bar{\Omega}$ such that $y \notin f(\partial\Omega \cup K)$. Then*

$$\mathrm{d}(f,\Omega,y) = \mathrm{d}(f,\Omega \setminus K, y).$$

20 Proposition (Cartesian product formula) *Assume that $\Omega = \Omega_1 \times \Omega_2$ is a bounded open set in $\mathbb{R}^n$ with Ω_1 open in $\mathbb{R}^p$ and Ω_2 open in $\mathbb{R}^q$, $p + q = n$. For $x \in \mathbb{R}^n$ write $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^q$. Suppose that $f(x) = (f_1(x_1), f_2(x_2))$ where $f_1 : \bar{\Omega}_1, \to \mathbb{R}^p$, $f_2 : \bar{\Omega}_2 \to \mathbb{R}^q$ are continuous. Suppose $y = (y_1, y_2) \in \mathbb{R}^n$ is such that $y_i \notin f_i(\partial\Omega_i), i = 1, 2$. Then*

$$\mathrm{d}(f,\Omega,y) = \mathrm{d}(f_1,\Omega_1,y_1)\mathrm{d}(f_2,\Omega_2,y_2). \tag{22}$$

Proof. Using an approximation argument, we may assume that f, f_1 and f_2 satisfy also (1) and (3) (interpreted appropriately). For such functions we have

$$\mathrm{d}(f,\Omega,y) = \sum_{x \in f^{-1}(y)} \text{sgn det } f'(x)$$

$$= \sum_{x \in f^{-1}(y)} \text{sgn det} \begin{pmatrix} f_1'(x_1) & 0 \\ 0 & f_2'(x_2) \end{pmatrix}$$

$$= \sum_{\substack{x_i \in f^{-1}(y_i) \\ i = 1,2}} \text{sgn det } f_1'(x_1) \text{ sgn det } f_2'(x_2)$$

$$= \prod_{i=1}^{2} \sum_{x_i \in f_i^{-1}(y_i)} \text{sgn det } f_i'(x_i) = \mathrm{d}(f_1,\Omega_1,y_1)\mathrm{d}(f_2,\Omega_2,y_2).$$

□

To give an example to show how the above properties may be used we prove Borsuk's theorem and the Brouwer fixed point theorem.

3.1 The theorems of Borsuk and Brouwer

21 **Theorem (Borsuk)** *Let Ω be a symmetric bounded open neighborhood of $0 \in \mathbb{R}^n$ (i.e. if $x \in \Omega$, then $-x \in \Omega$) and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be an odd mapping (i.e. $f(x) = -f(-x)$). Let $0 \notin f(\partial\Omega)$, then $\mathrm{d}(f, \Omega, 0)$ is an odd integer.*

Proof. Choose $\epsilon > 0$ such that $B_\epsilon(0) = \{x \in \mathbb{R}^n : |x| < \epsilon\} \subset \Omega$. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$\alpha(x) \equiv 1, \quad |x| \leq \epsilon, \quad \alpha(x) = 0, \quad x \in \partial\Omega, \quad 0 \leq \alpha(x) \leq 1, \quad x \in \mathbb{R}^n$$

and put

$$g(x) = \alpha(x)x + (1 - \alpha(x))f(x)$$

$$h(x) = \frac{1}{2}\left[g(x) - g(-x)\right],$$

then $h \in C(\bar{\Omega}, \mathbb{R}^n)$ is odd and $h(x) = f(x)$, $x \in \partial\Omega$, $h(x) = x$, $|x| < \epsilon$. Thus by Corollary 16 and the remark following it

$$\mathrm{d}(f, \Omega, 0) = \mathrm{d}(h, \Omega, 0).$$

On the other hand, the excision and additivity property imply that

$$\mathrm{d}(h, \Omega, 0) = \mathrm{d}(h, B_\epsilon(0), 0) + \mathrm{d}(h, \Omega \setminus \overline{B_\epsilon(0)}, 0),$$

where $\partial B_\epsilon(0)$ has been excised. It follows from Definition 2 that $\mathrm{d}(h, B_\epsilon(0), 0) = 1$, and it therefore suffices to show that (letting $\Theta = \Omega \setminus \overline{B_\epsilon(0)}$) $\mathrm{d}(h, \Theta, 0)$ is an even integer. Since $\bar{\Theta}$ is symmetric, $0 \notin \Theta$, $0 \notin h(\partial\Theta)$, and h is odd one may show (see [25]) that there exists $\tilde{h} \in C(\bar{\Omega}, \mathbb{R}^n)$ which is odd, $\tilde{h}(x) = h(x)$, $x \in \partial\Theta$ and is such that $\tilde{h}(x) \neq 0$, for those $x \in \Theta$ with $x_n = 0$. Hence

$$\mathrm{d}(h, \Theta, 0) = \mathrm{d}(\tilde{h}, \Theta, 0) = \mathrm{d}(\tilde{h}, \Theta \setminus \{x : x_n = 0\}, 0) \tag{23}$$

where we have used the excision property. We let

$$\Theta_1 = \{x \in \Theta : x_n > 0\}, \quad \Theta_2 = \{x \in \Theta : x_n < 0\},$$

then by the additivity property

$$\mathrm{d}(\tilde{h}, \Theta \setminus \{x : x_n = 0\}, 0) = \mathrm{d}(\tilde{h}, \Theta_1, 0) + \mathrm{d}(\tilde{h}, \Theta_2, 0).$$

Since $\Theta_2 = \{x : -x \in \Theta_1\}$ and $\tilde{h}$ is odd one may now employ approximation arguments to conclude that $\mathrm{d}(\tilde{h}, \Theta_1, 0) = \mathrm{d}(\tilde{h}, \Theta_2, 0)$, and hence conclude that the integer given by (23) is even. □

22 **Theorem (Brouwer fixed point theorem)** *Let $f \in C(\bar{\Omega}, \mathbb{R}^n)$, $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, be such that $f : \bar{\Omega} \to \bar{\Omega}$. Then f has a fixed point in Ω, i.e. there exists $x \in \bar{\Omega}$ such that $f(x) = x$.*

Proof. Assume f has no fixed points in $\partial\Omega$. Let $h(t,x) = x - tf(x)$, $0 \le t \le 1$. Then $h(t,x) \ne 0$, $0 \le t \le 1$, $x \in \partial\Omega$ and thus $\mathrm{d}(h(t,0),\Omega,0) = \mathrm{d}(h(0,0),\Omega,0)$ by the homotopy property. Since $\mathrm{d}(\mathrm{id},\Omega,0) = 1$ it follows from the solution property that the equation $x - f(x) = 0$ has a solution in Ω. ∎ Theorem 22 remains valid if the unit ball of $\mathbb{R}^n$ is replaced by any set homeomorphic to the unit ball (replace f by $g^{-1}fg$ where g is the homeomorphism) that the Theorem also remains valid if the unit ball is replaced in arbitrary compact convex set (or a set homeomorphic to it) may be proved using the extension theorem of Dugundji (Theorem I.20).

4 Completely Continuous Perturbations of the Identity in a Banach Space

4.1 Definition of the degree

Let E be a real Banach space with norm $\|\cdot\|$ and let $\Omega \subset E$ be a bounded open set. Let $F : \bar{\Omega} \to E$ be continuous and let $F(\bar{\Omega})$ be contained in a finite dimensional subspace of E. The mapping

$$f(x) = x + F(x) = (\mathrm{id} + F)(x) \tag{24}$$

is called a finite dimensional perturbation of the identity in E.

Let y be a point in E and let $\tilde{E}$ be a finite dimensional subspace of E containing y and $F(\bar{\Omega})$ and assume

$$y \notin f(\partial\Omega). \tag{25}$$

Select a basis $e_1, \cdots, e_n$ of $\tilde{E}$ and define the linear homeomorphism $T : \tilde{E} \to \mathbb{R}^n$ by

$$T\left(\sum_{i=1}^{n} c_i e_i\right) = (c_1, \cdots, c_n) \in \mathbb{R}^n.$$

Consider the mapping

$$TFT^{-1} : T(\bar{\Omega} \cap \tilde{E}) \to \mathbb{R}^n,$$

then, since $y \notin f(\partial\Omega)$, it follows that

$$T(y) \notin TfT^{-1}(T(\partial\Omega \cap \tilde{E})).$$

Let Ω_0 denote the bounded open set $T(\Omega \cap \tilde{E})$ in $\mathbb{R}^n$ and let $\tilde{f} = TfT^{-1}$, $y_0 = T(y)$. Then $\mathrm{d}(\tilde{f}, \Omega_0, y_0)$ is defined.

It is an easy exercise in linear algebra to show that the following lemma holds.

23 Lemma *The integer $\mathrm{d}(\tilde{f}, \Omega_0, y_0)$ calculated above is independent of the choice the finite dimensional space $\tilde{E}$ containing y and $F(\bar{\Omega})$ and the choice of basis of $\tilde{E}$.*

We hence may define

$$\mathrm{d}(f,\Omega,y)=\mathrm{d}(\tilde{f},\Omega_0,y_0),$$

where $\tilde{f},\Omega_0,y_0$ *are as above.*

Recall that a mapping $F:\bar{\Omega}\to E$ is called completely continuous if F is continuous and $F(\bar{\Omega})$ is precompact in E (i.e. $\overline{F(\Omega)}$ is compact). More generally if D is any subset of E and $F\in C(D,E)$, then F is called completely continuous if $F(V)$ is precompact for any bounded subset V of D.

We shall now demonstrate that if $f=\mathrm{id}+F$, with F completely continuous (f is called a completely continuous perturbation of the identity) and $y\notin f(\partial\Omega)$, then an integer valued function $\mathrm{d}(f,\Omega,y)$ (the Leray Schauder degree of f at y relative to Ω) may be defined having much the same properties as the Brouwer degree. In order to accomplish this we need the following lemma.

24 Lemma *Let* $f:\bar{\Omega}\to E$ *be a completely continuous perturbation of the identity and let* $y\notin f(\partial\Omega)$*. Then there exists an integer* d *with the following property: If* $h:\bar{\Omega}\to E$ *is a finite dimensional continuous perturbation of the identity such that*

$$\sup_{x\in\Omega}\|f(x)-h(x)\|<\inf_{x\in\partial\Omega}\|f(x)-y\|, \tag{26}$$

then $y\notin h(\partial\Omega)$ *and* $\mathrm{d}(h,\Omega,y)=d$*.*

Proof. That $y\notin h(\partial\Omega)$ follows from (26). Let h_1 and h_2 be any two such mappings. Let $k(t,x)=th_1(x)+(1-t)h_2(x)$, $0\le t\le 1$, $x\in\bar{\Omega}$, then if $t\in(0,1)$ and $x\in\Omega$ is such that $k(t,x)=y$ it follows that

$$\begin{aligned}
&\|f(x)-y\|=\|f(x)-th_1(x)-(1-t)h_2(x)\|\\
=\;&\|t(f-(x)-h_1(x))+(1-t)(f(x)-h_2(x))\|\\
\le\;&t\|f(x)-h_1(x)\|+(1-t)\|f(x)-h_2(x)\|\\
<\;&\inf\nolimits_{x\in\partial\Omega}\|f(x)-y\|\quad(\text{see }(26)),
\end{aligned}$$

hence $x\notin\partial\Omega$. Let $\tilde{E}$ be a finite dimensional subspace containing y and $(h_i-\mathrm{id})(\bar{\Omega})$, then $\mathrm{d}(h_i,\Omega,y)=\mathrm{d}(Th_iT^{-1},T(\Omega\cap\tilde{E}),T(y))$, where T is given as above. Then $(k(t,\cdot)-\mathrm{id})(\bar{\Omega})$ is contained in $\tilde{E}$ and $Tk(t,\cdot)T^{-1}(x)\ne T(y)$, $x\in T(\partial\Omega\cap\tilde{E})$. Hence we may use the homotopy invariance property of Brouwer degree to conclude that

$$\mathrm{d}(Tk(t,\cdot)T^{-1},T(\partial\Omega\cap\tilde{E}),T(y))=\text{ constant},$$

i.e.,

$$\mathrm{d}(h_1,\Omega,y)=\mathrm{d}(h_2,\Omega,y).$$

□

It follows from Lemma 24 that if a finite dimensional perturbation of the identity h exists which satisfies (26), then we may define $\mathrm{d}(f,\Omega,y)=d$, where d

is the integer whose existence follows from this lemma. In order to accomplish this we need an approximation result.

Let M be a compact subset of E. Then for every $\epsilon > 0$ there exists a finite covering of M by spheres of radius ϵ with centers at $y_1, \cdots, y_n \in M$. Define $\mu_i : M \to [0, \infty)$ by

$$\begin{aligned} \mu_i(y) &= \epsilon - \|y - y_i\|, \text{ if } \|y - y_i\| \leq \epsilon \\ &= 0, \quad \text{otherwise} \end{aligned}$$

and let

$$\lambda_i(y) = \frac{\mu_i(y)}{\sum_{j=1}^n \mu_j(y)}, \quad 1 \leq i \leq n.$$

Since not all μ_i vanish simultaneously, $\lambda_i(y)$ is non–negative and continuous on M and further $\sum_{i=1}^n \lambda_i(y) = 1$. The operator P_ϵ defined by

$$P_\epsilon(y) = \sum_{i=1}^n \lambda_i(y) y_i \tag{27}$$

is called a <u>Schauder projection operator</u> on M determined by ϵ, and $y_1, \cdots, y_n$. Such an operator has the following properties:

25 Lemma •
$$P_\epsilon : M \to \mathrm{co}\{y_1, \cdots, y_n\}$$
(the convex hull of $y_1, \cdots, y_n$) is continuous.

- *$P_\epsilon(M)$ is contained in a finite dimensional subspace of E.*
- $$\|P_\epsilon y - y\| \leq \epsilon, \quad y \in M.$$

26 Lemma *Let $f : \bar{\Omega} \to E$ be a completely continuous perturbation of the identity. Let $y \notin f(\partial\Omega)$. Let $\epsilon > 0$ be such that $\epsilon < \inf_{x \in \partial\Omega} \|f(x) - y\|$. Let P_ϵ be a Schauder projection operator determined by ϵ and points $\{y_1, \cdots, y_n\} \subset \overline{(f - \mathrm{id})(\bar{\Omega})}$. Then $\mathrm{d}(\mathrm{id} + P_\epsilon F, \Omega, y) = d$, where d is the integer whose existence is established by Lemma 24.*

Proof. The properties of P_ϵ (cf Lemma 25) imply that for each $x \in \bar{\Omega}$

$$\|P_\epsilon F(x) - F(x)\| \leq \epsilon \inf_{x \in \partial\Omega} \|f(x) - y\|,$$

and the mapping $\mathrm{id} + P_\epsilon F$ is a continuous finite dimensional perturbation of the identity. $\square$

27 Definition *The integer d whose existence has been esatblished by Lemma 26 is called the <u>Leray-Schauder degree</u> of f relative to Ω and the point y and is denoted by*

$$\mathrm{d}(f, \Omega, y).$$

4.2 Properties of the degree

28 **Proposition** *The Leray–Schauder degree has the solution, continuity, homotopy invariance, additivity, and excision properties similar to the Brouwer degree; the Cartesian product formula also holds.*

PROOF. (solution property). We let Ω be a bounded open set in E and $f : \bar{\Omega} \to E$ be a completely continuous perturbation of the identity, y a point in E with $y \notin f(\partial\Omega)$ and $\mathrm{d}(f, \Omega, y) \neq 0$. We claim that the equation $f(x) = y$ has a solution in E. To see this let $\{\epsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} \epsilon_n = 0$ and $\epsilon_1 < \inf_{x\in\partial\Omega} \|f(x)-y\|$. Let P_{ϵ_i} be associated Schauder projection operators. Then

$$\mathrm{d}(f, \Omega, y) = \mathrm{d}(\mathrm{id} + P_{\epsilon_n} F, \Omega, y), \quad n = 1, 2, \cdots,$$

where

$$\mathrm{d}(\mathrm{id} + P_{\epsilon_n} F, \Omega, y) = \mathrm{d}(\mathrm{id} + T_n P_{\epsilon_n} F T_n^{-1}, T_n(\Omega \cap \tilde{E}_n), T_n(y)),$$

and the spaces $\tilde{E}_n$ are finite dimensional. Hence the solution property of Brouwer degree implies the existence of a solution $z_n \in T_n(\Omega \cap \tilde{E}_n)$ of the equation

$$z + T_n P_{\epsilon_n} F T_n^{-1}(z) = T(y),$$

or equivalently a solution $x_n \in \Omega \cap \tilde{E}_n$ of

$$x + P_{\epsilon_n} F(x) = y.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence ($\{x_n\}_{n=1}^{\infty} \subset \Omega$), thus, since F is completely continuous there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ such that $F(x_{n_i}) \to u \in F(\bar{\Omega})$. We relabel the subsequence and call it again $\{x_n\}_{n=1}^{\infty}$. Then

$$\begin{array}{rc} & \|x_n - x_m\| = \|P_{\epsilon_n} F(x_n) - P_{\epsilon_m} F(x_m)\| \\ \leq & \|P_{\epsilon_n} F(x_n) - F(x_n)\| + \|P_{\epsilon_m} F(x_m) - F(x_m)\| \\ + & \|F(x_n) - F(x_m)\| < \epsilon_n + \epsilon_m + \|F(x_n) - F(x_m)\|. \end{array}$$

We let $\epsilon > 0$ be given and choose N such that $n, m \geq N$ imply $\epsilon_n, \epsilon_m < \epsilon/3$ and $\|F(x_n) - F(x_m)\| < \epsilon/3$. Thus $\|x_n - x_m\| < \epsilon$. The sequence $\{x_n\}_{n=1}^{\infty}$ therefore is a Cauchy sequence, hence has a limit, say x. It now follows that $x \in \bar{\Omega}$ and solves the equation $f(x) = y$, hence, since $y \notin f(\partial\Omega)$ we have that $x \in \Omega$. □

4.3 Borsuk's theorem and fixed point theorems

29 **Theorem (Borsuk's theorem)** *Let Ω be a bounded symmetric open neighborhood of $0 \in E$ and let $f : \bar{\Omega} \to E$ be a completely continuous odd perturbation of the identity with $0 \notin f(\partial\Omega)$. Then $\mathrm{d}(f, \Omega, 0)$ is an odd integer.*

PROOF. Let $\epsilon > 0$ be such that $\epsilon < \inf_{x\in\partial\Omega} \|f(x)\|$ and let P_ϵ be an associated Schauder projection operator. Let $f_\epsilon = \mathrm{id} + P_\epsilon f$ and put $h(x) = 1/2[f_\epsilon(x) -$

$f_\epsilon(-x)]$. Then h is a finite dimensional perturbation of the identity which is odd and

$$\|h(x) - f(x)\| \leq \epsilon.$$

Thus $\mathrm{d}(f, \Omega, 0) = \mathrm{d}(h, \Omega \cap \tilde{E}, 0)$, but

$$\mathrm{d}(h, \Omega, 0) = \mathrm{d}(ThT^{-1}, T(\Omega \cap \tilde{E}), 0).$$

On the other hand $T(\Omega \cap \tilde{E})$ is a symmetric bounded open neighborhood of $0 \in T(\Omega \cap \tilde{E})$ and ThT^{-1} is an odd mapping, hence the result follows from Theorem 21. □

We next establish extensions of the Brouwer fixed point theorem to Banach spaces.

30 Theorem (Schauder fixed point theorem) *Let K be a compact convex subset of E and let $F : K \to K$ be continuous. Then F has a fixed point in K.*

Proof. Since K is a compact there exists $r > 0$ such that $K \subset B_r(0) = \{x \in E : \|x\| < r\}$. Using the Extension Theorem (Theorem I.20) we may continuously extend F to $\overline{B_r(0)}$. Call the extension $\tilde{F}$. Then $\tilde{F}(\overline{B_r(0)}) \subset \mathrm{co}F(K) \subset K$, where $\mathrm{co}F(K)$ is the convex hull of $F(K)$, i.e. the smallest convex set containing $F(K)$. Hence $\tilde{F}$ is completely continuous. Consider the homotopy

$$k(t, x) = x - t\tilde{F}(x),\ 0 \leq t \leq 1.$$

Since $t\tilde{F}(K) \subset tK \subset B_r(0)$, $0 \leq t \leq 1$, $x \in \overline{B_r(0)}$, it follows that $k(t, x) \neq 0$, $0 \leq t \leq 1$, $x \in \partial B_r(0)$. Hence by the homotopy invariance property of the Leray–Schauder degree

$$\text{constant} \ = \mathrm{d}(k(t, \cdot), B_r(0), 0) = \mathrm{d}(id, B_r(0), 0) = 1.$$

The solution property of degree therefore implies that the equation

$$x - \tilde{F}(x) = 0$$

has a solution in $x \in B_r(0)$, and hence in K, i.e.

$$x - F(x) = 0.$$

□

In many applications the mapping F is known to be completely continuous but it is difficult to find a compact convex set K such that $F : K \to K$, whereas closed convex sets K having this property are more easily found. In such a case the following result may be applied.

31 Theorem (Schauder) *Let K be a closed, bounded, convex subset of E and let F be a completely continuous mapping such that $F : K \to K$. Then F has a fixed point in K.*

Proof. Let $\tilde{K} = \overline{\mathrm{co}\overline{F(K)}}$. Then since $\overline{F(K)}$ is compact it follows from a theorem of Mazur (see eg. [10], [24], and [28]) that $\tilde{K}$ is compact and $\tilde{K} \subset K$. Thus $F : \tilde{K} \to \tilde{K}$ and F has fixed point in $\tilde{K}$ by Theorem 30. Therefore F has a fixed point in K. □

5 Exercises

1. Let $[a,b]$ be a compact interval in $\mathbb{R}$ and let $f:[a,b]\to\mathbb{R}$ be a continuous function such that $f(a)f(b)\neq 0$. Verify the following.

   (i) If $f(b)>0>f(a)$, then $d(f,(a,b),0)=1$

   (ii) If $f(b)<0<f(a)$, then $d(f,(a,b),0)=-1$

   (iii) If $f(a)f(b)>0$, then $d(f,(a,b),0)=0$.

2. Identify $\mathbb{R}^2$ with the complex plane. Let Ω be a bounded open subset of $\mathbb{R}^2$ and let f and g be functions which are analytic in Ω and continuous on $\bar{\Omega}$. Let $f(z)\neq 0$, $z\in\partial\Omega$ and assume that $|f(z)-g(z)|<|f(z)|$, $z\in\partial\Omega$. Show that f and g have precisely the same number of zeros, counting multiplicities, in Ω. This result is called Rouch'e's theorem.

3. Let Ω be a bounded open subset of $\mathbb{R}^n$ and let

$$f,g\in C(\bar{\Omega},\mathbb{R}^n) \text{ with } f(x)\neq 0\neq g(x),\ x\in\partial\Omega.$$

   Further assume that

$$\frac{f(x)}{|f(x)|}\neq\frac{-g(x)}{|g(x)|},\ x\in\partial\Omega.$$

   Show that $d(f,\Omega,0)=d(g,\Omega,0)$.

4. Let $\Omega\subset\mathbb{R}^n$ be a bounded open neighborhood of $0\in\mathbb{R}^n$, let $f\in C(\Omega,\mathbb{R}^n)$ be such that $0\notin f(\partial\Omega)$ and either

$$f(x)\neq\frac{x}{|x|}|f(x)|,\quad x\in\partial\Omega$$

   or

$$f(x)\neq-\frac{x}{|x|}|f(x)|,\quad x\in\partial\Omega.$$

   Show that the equation $f(x)=0$ has a solution in Ω.

5. Let Ω be as in Exercise 4 and let $f\in C(\bar{\Omega},\mathbb{R}^n)$ be such that $0\notin f(\partial\Omega)$. Let n be odd. Show there exists λ $(\lambda\neq 0)\in\mathbb{R}$ and $x\in\partial\Omega$ such that $f(x)=\lambda x$. (This is commonly called the hedgehog theorem.)

6. Let $B^n=\{x\in\mathbb{R}^n:|x|<1\}$, $S^{n-1}=\partial B^n$. Let $f,g\in C(\overline{B^n},\mathbb{R}^n)$ be such that $f(S^{n-1})$, $g(S^{n-1})\subset S^{n-1}$ and $|f(x)-g(x)|<2$, $x\in S^{n-1}$. Show that $d(f,B^n,0)=d(g,B^n,0)$.

7. Let f be as in Exercise 6 and assume that $f(S^{n-1})$ does not equal S^{n-1}. Show that $d(f,B^n,0)=0$.

8. Let A be an $n \times n$ real matrix for which 1 is not an eigenvalue. Let Ω be a bounded open neighborhood of $0 \in \mathbb{R}^n$. Show, using linear algebra methods, that

$$d\,(\mathrm{id} - A, \Omega, 0) = (-1)^{\beta},$$

where β equals the sum of the algebraic multiplicities of all real eigenvalues μ of A with $\mu > 1$.

9. Let $\Omega \subset \mathbb{R}^n$ be a symmetric bounded open neighborhood of $0 \in \mathbb{R}^n$ and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be such that $0 \notin f(\partial\Omega)$. Also assume that

$$\frac{f(x)}{|f(x)|} \neq \frac{f(-x)}{|f(-x)|}, \quad x \in \partial\Omega.$$

Show that $d(f, \Omega, 0)$ is an odd integer.

10. Let Ω be as in Exercise 9 and let $f \in C(\bar{\Omega}, \mathbb{R}^n)$ be an odd function such that $f(\partial\Omega) \subset \mathbb{R}^m$, where $m < n$. Show there exists $x \in \partial\Omega$ such that $f(x) = 0$.

11. Let f and Ω be as in Exercise 10 except that f is not necessarily odd. Show there exists $x \in \partial\Omega$ such that $f(x) = f(-x)$.

12. Let K be a bounded, open, convex subset of E. Let $F : \bar{K} \to E$ be completely continuous and be such that $F(\partial K) \subset K$. Then F has a fixed point in K.

13. Let Ω be a bounded open set in E with $0 \in \Omega$. Let $F : \bar{\Omega} \to E$ be completely continuous and satisfy

$$\|x - F(x)\|^2 \geq \|F(x)\|^2, \quad x \in \partial\Omega.$$

then F has a fixed point in $\bar{\Omega}$.

14. Provide detailed proofs of the results of Section 3.

15. Provide detailed proofs of the results of Section 4.

Chapter IV
Global Solution Theorems

1 Introduction

In this chapter we shall consider a globalization of the *implicit function theorem* (see Chapter I) and provide some global bifurcation results. Our main tools in establishing such global results will be the properties of the *Leray Schauder degree* and a topological lemma concerning continua in compact metric spaces.

2 The Continuation Principle of Leray-Schauder

In this section we shall extend the homotopy property of Leray-Schauder degree (Proposition III.28) to homotopy cylinders having variable cross sections and from it deduce the Leray-Schauder continuation principle. As will be seen in later sections, this result also allows us to derive a globalization of the implicit function theorem and results about global bifurcation in nonlinear equations.

Let O be a bounded open (in the relative topology) subset of $E \times [a,b]$, where E is a real Banach space, and let

$$F : \bar{O} \to E$$

be a completely continuous mapping. Let

$$f(u,\lambda) = u - F(u,\lambda) \tag{1}$$

and assume that

$$f(u,\lambda) \neq 0, \quad (u,\lambda) \in \partial O \tag{2}$$

(here ∂O is the boundary of O in $E \times [a,b]$).

1 **Theorem (The generalized homotopy principle)** *Let f be given by (1) and satisfy (2). Then for $a \leq \lambda \leq b$,*

$$\mathrm{d}(f(\cdot,\lambda), O_\lambda, 0) = \text{constant},$$

(here $O_\lambda = \{u \in E : (u,\lambda) \in O\}$).

Proof. We may assume that $O \neq \emptyset$ and that

$$a = \inf\{\lambda : O_\lambda \neq \emptyset\}, \quad b = \sup\{\lambda : O_\lambda \neq \emptyset\}.$$

We let

$$\hat{O} = O \cup O_a \times (a-\epsilon, a] \cup O_b \times [b, b+\epsilon),$$

where $\epsilon > 0$ is fixed. Then $\hat{O}$ is a bounded open subset of $E \times \mathbb{R}$. Let $\tilde{F}$ be the extension of F to $E \times \mathbb{R}$ whose existence is guaranteed by the Dugundji extension theorem (Theorem I.20). Let

$$\tilde{f}(u, \lambda) = (u - \tilde{F}(u, \lambda), \lambda - \lambda^*),$$

where $a \leq \lambda^* \leq b$ is fixed. Then $\tilde{f}$ is a completely continuous perturbation of the identity in $E \times \mathbb{R}$.

Furthermore for any such λ^*

$$\tilde{f}(u, \lambda) \neq 0, \quad (u, \lambda) \in \partial\hat{O},$$

and hence $\mathrm{d}(\tilde{f}, \hat{O}, 0)$ is defined and constant (for such λ^*). Let $0 \leq t \leq 1$, and consider the vector field

$$\tilde{f}_t(u, \lambda) = (u - t\tilde{F}(u, \lambda) - (1-t)\tilde{F}(u, \lambda^*), \lambda - \lambda^*),$$

then $\tilde{f}_t(u, \lambda) = 0$ if and only if $\lambda = \lambda^*$ and $u = \tilde{F}(u, \lambda^*)$. Thus, our hypotheses imply that $\tilde{f}_t(u, \lambda) \neq 0$ for $(u, \lambda) \in \partial\hat{O}$ and $t \in [0, 1]$. By the homotopy invariance principle (Proposition III.28) we therefore conclude that

$$\mathrm{d}(\tilde{f}_1, \hat{O}, 0) = \mathrm{d}(\tilde{f}, \hat{O}, 0) = \mathrm{d}(\tilde{f}_0, \hat{O}, 0).$$

On the other hand,

$$\mathrm{d}(\tilde{f}_0, \hat{O}, 0) = \mathrm{d}(\tilde{f}_0, O_{\lambda^*} \times (a-\epsilon, b+\epsilon), 0),$$

by the excision property of degree (Proposition III.28). Using the Cartesian product formula (Proposition III.28), we obtain

$$\mathrm{d}(\tilde{f}_0, O_{\lambda^*} \times (a-\epsilon, b+\epsilon), 0) = \mathrm{d}(f(\cdot, \lambda^*), O_{\lambda^*}, 0).$$

This completes the proof. □

As an immediate consequence we obtain the continuation principle of Leray-Schauder.

2 Theorem (Leray–Schauder Continuation Theorem) *Let O be a bounded open subset of $E \times [a, b]$ and let $f : \bar{O} \to E$ be given by (1) and satisfy (2). Furthermore assume that*

$$\mathrm{d}(f(\cdot, a), O_a, 0) \neq 0.$$

Let

$$S = \{(u, \lambda) \in \bar{O} : f(u, \lambda) = 0\}.$$

Then there exists a closed connected set C in S such that

$$C_a \cap O_a \neq \emptyset \neq C_b \cap O_b.$$

PROOF. It follows from Theorem 1 that

$$\mathrm{d}(f(\cdot,a),O_a,0)=\mathrm{d}(f(\cdot,b),O_b,0).$$

Hence

$$S_a\times\{a\}=A\neq\emptyset\neq B=S_b\times\{b\}.$$

Using the complete continuity of F we may conclude that S is a compact metric subspace of $E\times[a,b]$. We now apply Whyburn's lemma (see [29]) with $X=S$. If there is no such continuum (as asserted above) there will exist compact sets X_A, X_B in X such that

$$A\subset X_A,\quad B\subset X_B,\quad X_A\cap X_B=\emptyset,\quad X_A\cup X_B=X.$$

We hence may find an open set $U\subset E\times[a,b]$ such that $A\subset U\cap O=V$ and $S\cap\partial V=\emptyset=V_b$. Therefore

$$\mathrm{d}(f(\cdot,\lambda),V_\lambda,0)=\text{constant},\quad \lambda\geq a.$$

On the other hand, the excision principle implies that

$$\mathrm{d}(f(\cdot,a),V_a,0)=\mathrm{d}(f(\cdot,a),O_a,0).$$

Since $V_b=\emptyset$, these equalities yield a contradiction, and there exists a continuum as asserted. □

In the following examples we shall develop, as an application of the above results, some basic existence results for the existence of solutions of nonlinear boundary value problems.

3 Example *Let $I=[0,1]$ and let $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous. Consider the nonlinear Dirichlet problem*

$$\begin{cases} u''+g(x,u) &= 0, \quad \text{in } I\\ u &= 0, \quad \text{on } \partial I.\end{cases}\tag{3}$$

Let there exist constants $a<0<b$ such that

$$g(x,a)>0>g(x,b),\quad x\in\Omega.$$

Then (3) has a solution $u\in C^2([0,1],\mathbb{R})$ such that

$$a<u(x)<b,\quad x\in I.$$

PROOF. To see this, we consider the one parameter family of problems

$$\begin{cases} u''+\lambda g(x,u) &= 0, \quad \text{in } I\\ u &= 0, \quad \text{on } \partial I.\end{cases}\tag{4}$$

Let G be defined by

$$G(u)(x)=g(x,u(x)),$$

then (4) is equivalent to the operator equation

$$u = \lambda LG(u), \quad u \in C([0,1],\mathbb{R}) = E, \tag{5}$$

where for each $v \in E$, $w = LG(v)$ is the unique solution of

$$\begin{aligned} w'' + g(x,v) &= 0, \quad \textit{in } I \\ w &= 0, \quad \textit{on } \partial I. \end{aligned}$$

It follows that for each $v \in E$, $LG(v) \in C^2(I)$ and since $C^2(I)$ is compactly embedded in E that

$$LG(\cdot) : E \to E$$

is a completely continuous operator. Let $O = \{(u,\lambda) : u \in E,\ a < u(x) < b,\ x \in I,\ 0 \le \lambda \le 1\}$. Then O is an open and bounded set in $E \times [0,1]$. If $(u,\lambda) \in \partial O$ is a solution of (4), then there will either exist $x \in I$ such that $u(x) = b$ or there exists $x \in I$ such that $u(x) = a$ and $\lambda > 0$. In either case, (3) yields, via elementary calculus, a contradiction. Hence (4) has no solutions in ∂O. Therefore

$$\mathrm{d}(\mathrm{id} - \lambda LG, O_\lambda, 0) = \mathrm{d}(\mathrm{id}, O_0, 0) = 1,$$

and Theorem 2 implies the existence of a continuum C of solutions of (5), hence of (4), such that $C \cap E \times \{0\} = \{0\}$ and $C \cap E \times \{1\} \neq \emptyset$. □

3 A Globalization of the Implicit Function Theorem

Assume that

$$F : E \times \mathbb{R} \to E$$

is a completely continuous mapping and consider the equation

$$f(u,\lambda) = u - F(u,\lambda) = 0. \tag{6}$$

Let (u_0,λ_0) be a solution of (6) such that the condition of the *implicit function theorem* (Theorem I.12) hold at (u_0,λ_0). Then there is a solution curve $\{(u(\lambda),\lambda)\}$ of (6) defined in a neighborhood of λ_0, passing through (u_0,λ_0). Furthermore the conditions of Theorem I.12 imply that the solution u_0 is an isolated solution of (6) at $\lambda = \lambda_0$, and if O is an isolating neighborhood, we have that

$$\mathrm{d}(f(\cdot,\lambda_0), O, 0) \neq 0. \tag{7}$$

We shall now show that condition (7) alone suffices to guarantee that equation (6) has a global solution branch in the half spaces $E \times [\lambda_0,\infty)$ and $E \times (-\infty,\lambda_0]$.

4 **Theorem** *Let O be a bounded open subset of E and assume that for $\lambda = \lambda_0$ equation (6) has a unique solution in O and let (7) hold. Let*

$$\mathfrak{S}^+ = \{(u,\lambda) \in E \times [\lambda_0, \infty) : (u,\lambda) \ \ \textit{solves} \ \ (6)\}$$

and

$$\mathfrak{S}^- = \{(u,\lambda) \in E \times (-\infty, \lambda_0] : (u,\lambda) \ \ \textit{solves} \ \ (6)\}.$$

Then there exists a continuum $C^+ \subset \mathfrak{S}^+$ ($C^- \subset \mathfrak{S}^-$) such that:

1. *$C^+_{\lambda_0} \cap O = \{u_0\}$ $(C^-_{\lambda_0} \cap O = \{u_0\})$,*
2. *C^+ is either unbounded in $E \times [\lambda_0, \infty)$ (C^- is unbounded in $E \times (-\infty, \lambda_0]$) or $C^+_{\lambda_0} \cap (E \backslash \overline{O}) \neq \emptyset$ ($C^-_{\lambda_0} \cap (E\backslash\overline{O}) \neq \emptyset$).*

PROOF. Let C^+ be the maximal connected subset of $\mathfrak{S}^+$ such that 1. above holds. Assume that $C^+ \cap (E\backslash\overline{O}) = \emptyset$ and that C^+ is bounded in $E \times [\lambda_0, \infty)$. Then there exists a constant $R > 0$ such that for each $(u,\lambda) \in C^+$ we have that $||u|| + |\lambda| < R$. Let

$$\mathfrak{S}^+_{2R} = \{(u,\lambda) \in \mathfrak{S}^+ : ||u|| + |\lambda| \leq 2R\},$$

then $\mathfrak{S}^+_{2R}$ is a compact subset of $E \times [\lambda_0, \infty)$, and hence is a compact metric space. There are two possibilities: Either $\mathfrak{S}^+_{2R} = C^+$ or else there exists $(u,\lambda) \in \mathfrak{S}^+_{2R}$ such that $(u,\lambda) \notin C^+$. In either case, we may find a bounded open set $U \subset E \times [\lambda_0, \infty)$ such that $U_{\lambda_0} = O$, $\mathfrak{S}^+_{2R} \cap \partial U = \emptyset$, $C^+ \subset U$. It therefore follows from Theorem 1 that

$$\mathrm{d}(f(\cdot,\lambda_0), U_{\lambda_0}, 0) = \text{constant}, \quad \lambda \geq \lambda_0,$$

where this constant is given by

$$\mathrm{d}(f(\cdot,\lambda_0), U_{\lambda_0}, 0) = \mathrm{d}(f(\cdot,\lambda_0), O, 0)$$

which is nonzero, because of (7). On the other hand, there exists $\lambda^* > \lambda_0$ such that U_{λ^*} contains no solutions of (6) and hence $\mathrm{d}(f(\cdot,\lambda^*), U_{\lambda^*}, 0) = 0$, contradicting (7). (To obtain the existence of an open set U with properties given above, we employ again Whyburn's lemma ([29]).)

The existence of C^- with the above listed properties is demonstrated in a similar manner. □

5 **Remark** The assumption of Theorem 4 that u_0 is the unique solution of (6) inside the set O, was made for convenience of proof. If one only assumes (7), one may obtain the conclusion that the set of all such continua is either bounded in the right (left) half space, or else there exists one such continuum which meets the $\lambda = \lambda_0$ hyperplane outside the set $\overline{O}$.

6 **Remark** If the component C^+ of Theorem 4 is bounded and $\tilde{O}$ is an isolating neighborhood of $C^+ \cap (E\backslash O) \times \{\lambda_0\}$, then it follows from the excision property, Whyburn's lemma, and the generalized homotopy principle that

$$\mathrm{d}(f(\cdot,\lambda_0), O, 0) = -\mathrm{d}(f(\cdot,\lambda_0), \tilde{O}, 0).$$

This observation has the following important consequence. If equation (6) has, for $\lambda = \lambda_0$ only isolated solutions and if the integer given by (7) has the same sign with respect to isolating neighborhoods O for all such solutions where (7) holds, then all continua C^+ must be unbounded.

7 Example *Let $p(z)$, $z \in \mathbb{C}$, be a polynomial of degree n whose leading coefficient is (without loss in generality) assumed to be 1 and let $q(z) = \prod_{i=1}^{n}(z-a_i)$, where $a_1, \ldots, a_n$ are distinct complex numbers. Let*

$$f(z, \lambda) = \lambda p(z) + (1-\lambda)q(z).$$

Then f may be considered as a continuous mapping

$$f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2.$$

Furthermore for $\lambda \in [0, r]$, $r > 0$, there exists a constant R such that any solution of

$$f(z, \lambda) = 0, \tag{8}$$

satisfies $|z| < R$. For all $\lambda \geq 0$, (8) has only isolated solutions and for $\lambda = 0$ each such solution has the property that

$$\mathrm{d}(f(\cdot, 0), O_i, 0) = 1,$$

where O_i is an isolating neighborhood of a_i. Hence, for each i, there exists a continuum C_i^+ of solutions of (8) which is unbounded with respect to λ, and must therefore reach every $\lambda-$level, in particular, the level $\lambda = 1$. We conclude that each zero of $p(z)$ must be connected to some a_i (apply the above argument backwards from the $\lambda = 1-$level, if need be).

4 The Theorem of Krein-Rutman

In this section we shall employ Theorem 4 to prove an extension of the *Perron-Frobenius* theorem about eigenvalues for positive matrices. The *Krein-Rutman* theorem [18] is a generalization of this classical result to positive compact operators on a not necessarily finite dimensional Banach space.

Let E be a real Banach space and let K be a cone in E, i.e., a closed convex subset of E with the properties:

- For all $u \in K$, $t \geq 0$, $tu \in K$.
- $K \cap \{-K\} = \{0\}$.

It is an elementary exercise to show that a cone K induces a partial order $\leq$ on E by the convention $u \leq v$ if and only if $v - u \in K$. A linear operator $L : E \to E$ is called *positive* whenever K is an invariant set for L, i.e. $L : K \to K$. If K is a cone whose interior $\mathrm{int}K$ is nonempty, we call L a *strongly positive* operator, whenever $L : K \backslash \{0\} \to \mathrm{int}K$.

8 Theorem *Let E be a real Banach space with a cone K and let $L : E \to E$ be a positive compact linear operator. Assume there exists $w \in K$, $w \neq 0$ and a constant $m > 0$ such that*

$$w \leq mLw, \tag{9}$$

where $\leq$ is the partial order induced by K. Then there exists $\lambda_0 > 0$ and $u \in K$, $||u|| = 1$, such that

$$u = \lambda_0 Lu. \tag{10}$$

PROOF. Restrict the operator L to the cone K and denote by $\tilde{L}$ the Dugundji extension of this operator to E. Since L is a compact linear operator, the operator $\tilde{L}$ is a completely continuous mapping with $\tilde{L}(E) \subset K$. Choose $\epsilon > 0$ and consider the equation

$$u - \lambda\tilde{L}(u + \epsilon w) = 0. \tag{11}$$

For $\lambda = 0$, equation (11) has the unique solution $u = 0$ and we may apply Theorem 4 to obtain an unbounded continuum $C_\epsilon^+ \subset E \times [0, \infty)$ of solutions of (11). Since $\tilde{L}(E) \subset K$, we have that $u \in K$, whenever $(u, \lambda) \in C_\epsilon^+$, and therefore $u = \lambda L(u + \epsilon w)$. Thus

$$\lambda Lu \leq u, \quad \frac{\lambda\epsilon}{m} \leq \lambda\epsilon Lw \leq u.$$

Applying L to this last inequality repeatedly, we obtain by induction that

$$\left(\frac{\lambda}{m}\right)^n \epsilon w \leq u. \tag{12}$$

Since $w \neq 0$, by assumption, it follows from (12) that $\lambda \leq m$. Thus, if $(u, \lambda) \in C_\epsilon^+$, it must be case that $\lambda \leq m$, and hence that $C_\epsilon^+ \subset K \times [0, m]$. Since C_ϵ^+ is unbounded, we conclude that for each $\epsilon > 0$, there exists $\lambda_\epsilon > 0$, $u_\epsilon \in K$, $||u_\epsilon|| = 1$, such that

$$u_\epsilon = \lambda_\epsilon L(u_\epsilon + \epsilon w).$$

Since L is compact, the set $\{(u_\epsilon, \lambda_\epsilon)\}$ will contain a convergent subsequence (letting $\epsilon \to 0$), converging to, say, (u, λ_0). Since clearly $||u|| = 1$, it follows that $\lambda_0 > 0$. □

If it is the case that L is a strongly positive compact linear operator, much more can be asserted; this will be done in the theorem of Krein-Rutman which we shall establish as a corollary of Theorem 8.

9 Theorem *Let E have a cone K, whose interior, $\text{int}K \neq \emptyset$. Let L be a strongly positive compact linear operator. Then there exists a unique $\lambda_0 > 0$ with the following properties:*

1. *There exists $u \in \text{int}K$, with $u = \lambda_0 Lu$.*

2. *If $\lambda(\in \mathbb{R}) \neq \lambda_0$ is such that there exists $v \in E$, $v \neq 0$, with $v = \lambda L v$, then $v \notin K \cup \{-K\}$ and $\lambda_0 < |\lambda|$.*

PROOF. Choose $w \in K \backslash \{0\}$, then, since $Lw \in \text{int}K$, there exists $\delta > 0$, small such that $Lw - \delta w \in \text{int}K$, i.e., in terms of the partial order $\delta w \leq Lw$. We therefore may apply Theorem 8 to obtain $\lambda_0 > 0$ and $u \in K$ such that $u = \lambda_0 L u$. Since L is strongly positive, we must have that $u \in \text{int}K$. If $(v, \lambda) \in (K \backslash \{0\}) \times (0, \infty)$ is such that $v = \lambda L v$, then $v \in intK$. Hence, for all $\delta > 0$, sufficiently small, we have that $u - \delta v \in intK$. Consequently, there exists a maximal $\delta^* > 0$, such that $u - \delta^* v \in K$, i.e. $u - rv \notin K$, $r > \delta^*$. Now

$$L(u - \delta^* v) = \frac{1}{\lambda_0}(u - \frac{\lambda_0}{\lambda}\delta^* v),$$

which implies that $u - \frac{\lambda_0}{\lambda}\delta^* v \in \text{int}K$, unless $u - \delta^* v = 0$. If the latter holds, then $\lambda_0 = \lambda$, if not, then $\lambda_0 < \lambda$, because δ^* is maximal. If $\lambda_0 < \lambda$, we may reverse the role of u and v and also obtain $\lambda < \lambda_0$, a contradiction. Hence it must be the case that $\lambda = \lambda_0$. We have therefore proved that λ_0 is the only characteristic value of L having an *eigendirection* in the cone K and further that any other eigenvector corresponding to λ_0 must be a constant multiple of u, i.e. λ_0 is a characteristic value of L of geometric multiplicity one, i.e the dimension of the kernel of $\text{id} - \lambda_0 L$ equals one.

Next let $\lambda \neq \lambda_0$ be another characteristic value of L and let $v \neq 0$ be such that $v \notin K \cup \{-K\}$. Again, for $|\delta|$ small, $u - \delta v \in \text{int}K$ and there exists $\delta^* > 0$, maximal, such that $u - \delta^* v \in K$, and there exists $\delta_* < 0$, minimal, such that $u - \delta_* v \in K$. Now

$$L(u - \delta^* v) = \frac{1}{\lambda_0}(u - \frac{\lambda_0}{\lambda}\delta^* v) \in K,$$

and

$$L(u - \delta_* v) = \frac{1}{\lambda_0}(u - \frac{\lambda_0}{\lambda}\delta_* v) \in K.$$

Thus, if $\lambda > 0$, we conclude that $\lambda_0 < \lambda$, whereas, if $\lambda < 0$, we get that $\lambda_0 \delta^* < \lambda \delta_*$, and $\lambda_0 \delta_* > \lambda \delta^*$, i.e $\lambda_0^2 < \lambda^2$. □

As observed above we have that λ_0 is a characteristic value of geometric multiplicity one. Before giving an application of the above result, we shall establish that λ_0, in fact also has algebraic multiplicity one. Recall from the *Riesz* theory of compact linear operators (viz. [19], [28]) that the operator $id - \lambda_0 L$ has the following property:

> There exists a minimal integer n such that
>
> $$\ker(\text{id} - \lambda_0 L)^n = \ker(\text{id} - \lambda_0 L)^{n+1} = \ker(\text{id} - \lambda_0 L)^{n+2} = \ldots,$$
>
> and the dimension of the generalized eigenspace $\ker(\text{id} - \lambda_0 L)^n$ is called the *algebraic multiplicity* of λ_0.

With this terminology, we have the following addition to Theorem 9.

10 Theorem *Assume the conditions of Theorem 9 and let λ_0 be the characteristic value of L, whose existence is established there. Then λ_0 is a characteristic value of L of algebraic multiplicity one.*

PROOF. We assume the contrary. Then, since $\ker(\mathrm{id} - \lambda_0 L)$ has dimension one (Theorem 9), it follows that there exists a smallest integer $n > 1$ such that the generalized eigenspace is given by $\ker(\mathrm{id} - \lambda_0 L)^n$. Hence, there exists a nonzero $v \in E$ such that $(\mathrm{id} - \lambda_0 L)^n v = 0$ and $(id - \lambda_0 L)^{n-1} v = w \neq 0$. It follows from Theorem 9 and its proof that $w = ku$, where u is given by Theorem 9 and k may assumed to be positive. Let $z = (\mathrm{id} - \lambda_0 L)^{n-2} v$, then $z - \lambda_0 Lz = ku$, and hence, by induction, we get that $\lambda_0^m L^m z = z - mku$, for any positive integer m. It follows therefore that $z \notin K$, for otherwise $\frac{1}{m} z - ku \in K$, for any integer m, implying that $-ku \in K$, a contradiction. Since $u \in \mathrm{int} K$, there exist $\alpha > 0$ and $y \in K$ such that $z = \alpha u - y$. Then $\lambda_0^m L^m z = \alpha u - \lambda_0^m L^m y$, or $\lambda_0^m L^m y = y + mku$. Choose $\beta > 0$, such that $y \leq \beta u$, then $\lambda_0^m L^m y \leq \beta u$, and by the above we see that $y + mku \leq \beta u$. Dividing this inequality my m and letting $m \to \infty$, we obtain that $ku \in -K$, a contradiction. □

11 Remark It may be the case that, aside from real characteristic values, L also has complex ones. If μ is such a characteristic value, then it may be shown that $|\mu| > \lambda_0$, where λ_0 is as in Theorem 10. We refer the interested reader to Krasnosel'skii [17] for a verification.

5 Global Bifurcation

As before, let E be a real Banach space and let $f : E \times \mathbb{R} \to E$ have the form

$$f(u, \lambda) = u - F(u, \lambda), \tag{13}$$

where $F : E \times \mathbb{R} \to E$ is completely continuous. We shall now assume that

$$F(0, \lambda) \equiv 0, \quad \lambda \in \mathbb{R}, \tag{14}$$

and hence that the equation

$$f(u, \lambda) = 0, \tag{15}$$

has the trivial solution for all values of λ. We shall now consider the question of bifurcation from this trivial branch of solutions and demonstrate the existence of global *branches* of nontrivial solutions bifurcating from the trivial branch. Our main tools will again be the properties of the Leray-Schauder degree and Whyburn's lemma.

We shall see that this result is an extension of the local bifurcation theorem, Theorem II.6.

12 Theorem *Let there exist $a, b \in \mathbb{R}$ with $a < b$, such that $u = 0$ is an isolated solution of (15) for $\lambda = a$ and $\lambda = b$, where a, b are not bifurcation points, furthermore assume that*

$$\mathrm{d}(f(\cdot, a), B_r(0), 0) \neq \mathrm{d}(f(\cdot, b), B_r(0), 0), \tag{16}$$

where $B_r(0) = \{u \in E : ||u|| < r\}$ is an isolating neighborhood of the trivial solution. Let

$$\mathfrak{S} = \overline{\{(u, \lambda) : (u, \lambda) \ \ solves \ \ (15) \ \ with \ \ u \neq 0\}} \cup \{0\} \times [a, b]$$

and let $\mathfrak{C} \subset \mathfrak{S}$ be the maximal connected subset of $\mathfrak{S}$ which contains $\{0\} \times [a, b]$. Then either

(i) *$\mathfrak{C}$ is unbounded in $E \times \mathbb{R}$,*

or else

(ii) *$\mathfrak{C} \cap \{0\} \times (\mathbb{R} \backslash [a, b]) \neq \emptyset$.*

Proof. Define a class $\mathfrak{U}$ of subsets of $E \times \mathbb{R}$ as follows

$$\mathfrak{U} = \{\Omega \subset E \times \mathbb{R} : \Omega = \Omega_0 \cup \Omega_\infty\},$$

where $\Omega_0 = B_r(0) \times [a, b]$, and Ω_∞ is a bounded open subset of $(E \backslash \{0\}) \times \mathbb{R}$. We shall first show that (15) has a nontrivial solution $(u, \lambda) \in \partial\Omega$ for any such $\Omega \in \mathfrak{U}$. To accomplish this, let us consider the following sets:

$$\begin{cases} K &= f^{-1}(0) \cap \overline{\Omega}, \\ A &= \{0\} \times [a, b], \\ B &= f^{-1}(0) \cap (\partial\Omega \backslash (B_r(0) \times \{a\} \cup B_r(0) \times \{b\})). \end{cases} \tag{17}$$

We observe that K may be regarded as a compact metric space and A and B are compact subsets of K. We hence may apply Whyburn's lemma to deduce that either there exists a continuum in K connecting A to B or else, there is a separation $K_A, \ \ K_B$ of K, with $A \subset K_A, \ \ B \subset K_B$. If the latter holds, we may find open sets $U, \ \ V$ in $E \times \mathbb{R}$ such that $K_A \subset U, \ \ K_B \subset V$, with $U \cap V = \emptyset$. We let $\Omega^* = \Omega \cap (U \cup V)$ and observe that $\Omega^* \in \mathfrak{U}$. It follows, by construction, that there are no nontrivial solutions of (15) which belong to $\partial\Omega^*$; this, however, is impossible, since, it would imply, by the generalized homotopy and the excision principle of Leray-Schauder degree, that $\mathrm{d}(f(\cdot, a), B_r(0), 0) = \mathrm{d}(f(\cdot, b), B_r(0), 0)$, contradicting (16). We hence have that for each $\Omega \in \mathfrak{U}$ there is a continuum C of solutions of (15) which intersects $\partial\Omega$ in a nontrivial solution.

We assume now that neither of the alternatives of the theorem hold, i.e we assume that $\mathfrak{C}$ is bounded and $\mathfrak{C} \cap \{0\} \times (\mathbb{R} \backslash [a, b]) = \emptyset$. In this case, we may, using the boundednes of $\mathfrak{C}$, construct a set $\Omega \in \mathfrak{U}$, containing no nontrivial solutions in its boundary, thus arriving once more at a contradiction. $\square$

We shall, throughout this text, apply the above theorem to several problems for nonlinear differential equations. Here we shall, for the sake of illustration provide two simple one dimensional examples.

13 **Example** *Let* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, be given by*

$$f(u, \lambda) = u(u^2 + \lambda^2 - 1).$$

It is easy to see that $\mathfrak{S}$ *is given by*

$$\mathfrak{S} = \{(u, \lambda) : u^2 + \lambda^2 = 1\},$$

and hence that $(0, -1)$ *and* $(0, 1)$ *are the only bifurcation points from the trivial solution. Furthermore, the bifurcating continuum is bounded. Also one may quickly check that (16) holds with* a, b *chosen in a neighborhood of* $\lambda = -1$ *and also in a neighborhood of* $\lambda = 1$.

14 **Example** *Let* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be given by*

$$f(u, \lambda) = (1 - \lambda)u + u \sin \frac{1}{u}.$$

In this case $\mathfrak{S}$ *is given by*

$$\mathfrak{S} = \{(u, \lambda) : \lambda - 1 = \sin \frac{1}{u}\} \cup \{0\} \times [0, 2],$$

which is an unbounded set, and we may check that (16) holds, by chosing $a < 0$ *and* $b > 2$.

In many interesting cases the nonlinear mapping F is of the special form

$$F(u, \lambda) = \lambda B u + o(||u||), \quad as\ ||u|| \to 0, \tag{18}$$

where B is a compact linear operator. In this case bifurcation points from the trivial solution are isolated, in fact one has the following necessary conditions for bifurcation.

15 **Proposition** *Assume that* F *has the form (18), where* B *is the Fréchet derivative of* F*. If* $(0, \lambda_0)$ *is a bifurcation point from the trivial solution for equation (15), then* λ_0 *is a characteristic value of* B.

Using this result, Theorem 12, and the Leray-Schauder formula for computing the degree of a compact linear perturbation of the identity (an extension to infinite dimensions of Exercise 8 of Chapter III, we obtain the following result.

16 **Theorem** *Assume that* F *has the form (18) and let* λ_0 *be a characteristic value of* B *which is of odd algebraic multiplicity. Then there exists a continuum* $\mathfrak{C}$ *of nontrivial solutions of (15) which bifurcates from the set of trivial solutions at* $(0, \lambda_0)$ *and* $\mathfrak{C}$ *is either unbounded in* $E \times \mathbb{R}$ *or else* $\mathfrak{C}$ *also bifurcates from the trivial solution set at* $(0, \lambda_1)$*, where* λ_1 *is another characteristic value of* B.

PROOF. Since λ_0 is isolated as a characteristic value, we may find $a < \lambda_0 < b$ such that the interval $[a, b]$ contains, besides λ_0, no other characteristic values. It follows that the trivial solution is an isolated solution (in E) of (15) for $\lambda = a$ and $\lambda = b$. Hence, $\mathrm{d}(f(\cdot, a), B_r(0), 0)$ and $\mathrm{d}(f(\cdot, b), B_r(0), 0)$ are defined for r, sufficiently small and are, respectively, given by $\mathrm{d}(id - aB, B_r(0), 0)$, and $\mathrm{d}(id - bB, B_r(0), 0)$. On the other hand,

$$\mathrm{d}(\mathrm{id} - aB, B_r(0), 0) = (-1)^{\beta}\mathrm{d}(\mathrm{id} - bB, B_r(0), 0),$$

where β equals the algebraic multiplicity of λ_0 as a characteristic value of B. Since β is odd, by assumption, the result follows from Theorem 12 and Proposition 15. $\square$

The following example serves to demonstrate that, in general, not every characteristic value will yield a bifurcation point.

17 Example *The system of scalar equations*

$$\begin{cases} x &= \lambda x + y^3 \\ y &= \lambda y - x^3 \end{cases} \tag{19}$$

has only the trivial solution $x = 0 = y$ for all values of λ. We note, that $\lambda_0 = 1$ is a characteristic value of the Fréchet derivative of multiplicity two.

As a further example let us consider a boundary value problem for a second order ordinary differential equation, the pendulum equation.

18 Example *Consider the boundary value problem*

$$\begin{cases} u'' + \lambda \sin u = 0, \quad x \in [0, \pi] \\ u(0) = 0, \ u(\pi) = 0. \end{cases} \tag{20}$$

As already observed this problem is equivalent to an operator equation

$$u = \lambda F(u),$$

where

$$F : C[0, \pi] \to C[0, \pi]$$

is a completely continuous operator which is continuously Fréchet differentiable with Fréchet derivative $F'(0)$. Thus to find the bifurcation points for (20) we must compute the eigenvalues of $F'(0)$. On the other hand, to find these eigenvalues is equivalent to finding the values of λ for which

$$\begin{cases} u'' + \lambda u = 0, \quad x \in [0, \pi] \\ u(0) = 0, \ u(\pi) = 0 \end{cases} \tag{21}$$

has nontrivial solutions. These values are given by

$$\lambda = 1, 4, \cdots k^2, \cdots, \ k \in \mathbb{N}.$$

Furthermore we know from elementary differential equations that each such eigenvalue has a one-dimensonal eigenspace and one may convince oneself that the above theorem may be applied at each such eigenvalue and conclude that each value

$$(0, k^2), \ k \in \mathbb{N}$$

is a bifurcation point for (20).

6 Exercises

1. Prove Proposition 15.
2. Supply the details for the proof of Theorem 12.
3. Perform the calculations indicated in Example 13.
4. Perform the calculations indicated in Example **??**.
5. Prove Proposition 15.
6. Supply the details for the proof of Theorem 16.
7. Prove the assertion of Example 17.
8. Provide the detials for Example 18.
9. In Example 18 show that the second alternative of Theorem 16 cannot hold.

Part II

Ordinary Differential Equations

Chapter V
Existence and Uniqueness Theorems

1 Introduction

In this chapter, we shall present the basic existence and uniqueness theorems for solutions of initial value problems for systems of ordinary differential equations. To this end let D be an open connected subset of $\mathbb{R} \times \mathbb{R}^N,\ N \geq 1$, and let

$$f : D \to \mathbb{R}^N$$

be a continuous mapping.

We consider the differential equation

$$u' = f(t, u),\ ' = \frac{d}{dt}. \tag{1}$$

and seek sufficient conditions for the existence of solutions of (1), where $u \in C^1(I, \mathbb{R}^N)$, with I an interval, $I \subset \mathbb{R}$, is called a solution, if $(t, u(t)) \in D,\ t \in I$ and

$$u'(t) = f(t, u(t)),\ t \in I.$$

Simple examples tell us that a given differential equation may have a multitude of solutions, in general, whereas some constraints on the solutions sought might provide existence and uniqueness of the solution. The most basic such constraints are given by fixing an initial value of a solution. By an *initial value problem* we mean the following:

- Given a point $(t_0, u_0) \in D$ we seek a solution u of (1) such that

$$u(t_0) = u_0. \tag{2}$$

We have the following proposition whose proof is straightforward:

1 **Proposition** *A function $u \in C^1(I, \mathbb{R}^N)$, with $I \subset \mathbb{R}$, and I an interval containing t_0 is a solution of the initial value problem (1), satisfying the initial condition (2) if and only if $(t, u(t)) \in D,\ t \in I$ and*

$$u(t) = u(t_0) + \int_{t_0}^{t} f(s, u(s))ds. \tag{3}$$

We shall now, using Proposition 1, establish some of the classical and basic existence and existence/uniqueness theorems.

2 The Picard-Lindelöf Theorem

We say that f satisfies a *local Lipschitz* condition on the domain D, provided for every compact set $K \subset D$, there exists a constant $L = L(K)$, such that for all $(t, u_1),\ (t, u_2) \in K$

$$|f(t, u_1) - f(t, u_2)| \leq L|u_1 - u_2|.$$

For such functions, one has the following existence and uniqueness theorem. This result is usually called the Picard-Lindelöf theorem

2 **Theorem** *Assume that $f : D \to \mathbb{R}^N$ satisfies a local Lipschitz condition on the domain D, then for every $(t_0, u_0) \in D$ equation (1) has a unique solution satisfying the initial condition (2) on some interval I.*

We remark that the theorem as stated is a *local* existence and uniqueness theorem, in the sense that the interval I, where the solution exists will depend upon the initial condition. Global results will follow from this result, by extending solutions to maximal intervals of existence, as will be seen in a subsequent section.
Proof. Let $(t_0, u_0) \in D$, then, since D is open, there exist positive constants a and b such that

$$Q = \{(t, u) : |t - t_0| \leq a,\ |u - u_0| \leq b\} \subset D.$$

Let L be the Lipschitz constant for f associated with the set Q. Further let

$$\begin{array}{rcl} m & = & \max_{(t,u)\in Q} |f(t, u)|, \\ \alpha & = & \min\{a, \frac{b}{m}\}. \end{array}$$

Let $\tilde{L}$ be any constant, $\tilde{L} > L$, and define

$$M = \{u \in C\left([t_0 - \alpha, t_0 + \alpha], \mathbb{R}^N\right) : |u(t) - u_0| \leq b,\ |t - t_0| \leq \alpha\}.$$

In $C\left([t_0 - \alpha, t_0 + \alpha], \mathbb{R}^N\right)$ we define a new norm as follows:

$$\|u\| = \max_{|t-t_0|\leq\alpha} e^{-\tilde{L}|t-t_0|}|u(t)|.$$

And we let $\rho(u,v) = \|u - v\|$, then (M, ρ) is a complete metric space. Next define the operator T on M by:

$$(Tu)(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \ |t - t_0| \le \alpha. \tag{4}$$

Then

$$|(Tu)(t) - u_0| \le |\int_{t_0}^{t} |f(s, u(s))|ds|,$$

and, since $u \in M$,

$$|(Tu)(t) - u_0| \le \alpha m \le b.$$

Hence

$$T : M \to M.$$

Computing further, we obtain, for $u, v \in M$ that

$$\begin{array}{rcl} |(Tu)(t) - (Tv)(t)| & \le & |\int_{t_0}^{t} |f(s, u(s)) - f(s, v(s))|ds| \\ & \le & L|\int_{t_0}^{t} |u(s) - v(s)|ds|, \end{array}$$

and hence

$$\begin{array}{rcl} e^{-\tilde{L}|t-t_0|}|(Tu)(t) - (Tv)(t)| & \le & e^{-\tilde{L}|t-t_0|}L|\int_{t_0}^{t} |u(s) - v(s)|ds| \\ & \le & \frac{L}{\tilde{L}}\|u - v\|, \end{array}$$

and hence

$$\rho(Tu, Tv) \le \frac{L}{\tilde{L}}\rho(u, v),$$

proving that T is a contraction mapping. The result therefore follows from the contraction mapping principle, Theorem I.6. □

We remark that, since T is a contraction mapping, the contraction mapping theorem gives a constructive means for the solution of the initial value problem in Theorem 2 and the solution may in fact be obtained via an iteration procedure. This procedure is known as *Picard iteration.*

In the next section, we shall show, that without the assumption of a local Lipschitz condition, we still get the existence of solutions.

3 The Cauchy-Peano Theorem

The following result, called the Cauchy-Peano theorem provides the local solvabilty of initial value problems.

3 Theorem *Assume that $f : D \to \mathbb{R}^N$ is continuous. Then for every $(t_0, u_0) \in D$ the initial value problem (1), (2) has a solution on some interval I, $t_0 \in I$.*

PROOF. Let $(t_0, u_0) \in D$, and let Q, α, m be as in the proof of Theorem 2. Consider the space $E = C([t_0 - \alpha, t_0 + \alpha], \mathbb{R}^N)$ with norm $\|u\| = \max_{|t-t_0| \le \alpha} |u(t)|$. Then E is a Banach space. We let M be as defined in the proof of Theorem 2 and note that M is a closed, bounded convex subset of E and further that $T : M \to M$. We hence may apply the Schauder fixed point theorem (Theorem III.30) once we verify that T is completely continuous on M. To see this we note, that, since f is continuous, it follows that T is continuous. On the other hand, if $\{u_n\} \subset M$, then

$$\begin{array}{rcl} |(Tu_n)(t) - (Tu_n)(\bar{t})| & \le & |\int_{\bar{t}}^{t} |f(s, u_n(s))| ds| \\ & \le & m|t - \bar{t}|. \end{array}$$

Hence $\{Tu_n\} \subset M$, is a uniformly bounded and equicontinuous family in E. It therefore has a uniformly convergent subsequence (as follows from the theorem of Ascoli-Arzèla [22]), showing that $\{Tu_n\}$ is precompact and hence T is completely continuous. This completes the proof. □

We note from the above proofs (of Theorems 2 and 3) that for a solution u thus obtained, both $(t_0 \pm \alpha, u(t_0 \pm \alpha)) \in D$. We hence may reapply these theorems with initial conditions given at $t_0 \pm \alpha$ and conditions $u(t_0 \pm \alpha)$ and thus continue solutions to larger intervals (in the case of Theorem 2 uniquely and in the case of Theorem 3 not necessarily so.) We shall prove below that this continuation process leads to maximal intervals of existence and also describes the behavior of solutions as one approaches the endpoints of such maximal intervals.

3.1 Carathéodory equations

In many situations the nonlinear term f is not continuous as assumed above but satisfies the so-called Carathéodory conditions on any parallelepiped $Q \subset D$, where Q is as given in the proof of Theorem 2, i.e.,

- f is measurable in t for each fixed u and continuous in u for almost all t,

- for each Q there exists a function $m \in L^1(t_0 - a, t_0 + a)$ such that

$$|f(t, u)| \le m(t), \ (t, u) \in Q.$$

Under such assumptions we have the following extension of the Cauchy-Peano theorem, Theorem 3:

4 **Theorem** *Let f satisfy the Carathéodory conditions on D. Then for every $(t_0, u_0) \in D$ the initial value problem (1), (2) has a solution on some interval I, $t_0 \in I$, in the sense that there exists an absolutely continuous function $u : I \to \mathbb{R}^N$ which satisfies the initial condition (2) and the differential equation (1) a.e. in I.*

PROOF. Let $(t_0, u_0) \in D$, and choose

$$Q = \{(t,u) : |t - t_0| \le a,\ |u - u_0| \le b\} \subset D.$$

Let

$$M = \{u \in C\left([t_0 - \alpha, t_0 + \alpha], \mathbb{R}^N\right) : |u(t) - u_0| \le b,\ |t - t_0| \le \alpha\},$$

where $\alpha \le a$ is to be determined.

Next define the operator T on M by:

$$(Tu)(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds,\ |t - t_0| \le \alpha.$$

Then, because of the Carathéodory conditions, Tu is a a continuous function and

$$|(Tu)(t) - u_0| \le |\int_{t_0}^{t} |f(s, u(s))|ds|.$$

Further, since $u \in M$,

$$|(Tu)(t) - u_0| \le \int_{t_0-\alpha}^{t_0+\alpha} m(s)ds \le \|m\|_{L^1[t_0-\alpha, t_0+\alpha]} \le b,$$

for α small enough. Hence

$$T : M \to M.$$

One next shows (see the Exercise 14 below) that T is a completely continuous mapping, hence will have a fixed point in M by the Schauder Fixed Point Theorem. That fixed points of T are solutions of the initial value problem (1), (2), in the sense given in the theorem, is immediate. □

4 Extension Theorems

In this section we establish a basic result about maximal intervals of existence of solutions of initial value problems. We first prove the following lemma.

5 Lemma *Assume that $f : D \to \mathbb{R}^N$ is continuous and let $\tilde{D}$ be a subdomain of D, with f bounded on $\tilde{D}$. Further let u be a solution of (1) defined on a bounded interval (a, b) with $(t, u(t)) \in \tilde{D},\ t \in (a, b)$. Then the limits*

$$\lim_{t \to a+} u(t),\ \lim_{t \to b-} u(t)$$

exist.

PROOF. Let $t_0 \in (a,b)$, then $u(t) = u(t_0) + \int_{t_0}^t f(s,u(s))ds$. Hence for $t_1, t_2 \in (a,b)$, we obtain

$$|u(t_1) - u(t_2)| \le m|t_1 - t_2|,$$

where m is a bound on f on $\tilde{D}$. Hence the above limits exist. □

We may therefore, as indicated above, continue the solution beyond the interval (a,b), to the left of a and the right of b.

We say that a solution u of (1) has maximal interval of existence (ω_-, ω_+), provided u cannot be continued as a solution of (1) to the right of ω_+ nor to the left of ω_-.

The following theorem holds.

6 Theorem *Assume that $f : D \to \mathbb{R}^N$ is continuous and let u be a solution of (1) defined on some interval I. Then u may be extended as a solution of (1) to a maximal interval of existence (ω_-, ω_+) and $(t, u(t)) \to \partial D$ as $t \to \omega_\pm$.*

PROOF. We establish the existence of a right maximal interval of existence; a similar argument will yield the existence of a left maximal one and together the two will imply the existence of a maximal interval of existence.

Let u be a solution of (1) with $u(t_0) = u_0$ defined on an interval $I = [t_0, a_u)$. We say that two solutions v, w of (1), (2) satisfy

$$v \preceq w, \tag{5}$$

if and only if:

- $u \equiv v \equiv w$ on $[t_0, a_u)$,
- $$\begin{array}{ll} v & \text{is defined on } I_v = [t_0, a_v),\ a_v \ge a_u, \\ w & \text{is defined on } I_w = [t_0, a_w),\ a_w \ge a_u, \end{array}$$
- $a_w \ge a_v$,
- $v \equiv w$ on I_v.

We see that $\preceq$ is a partial order on the set of all solutions S of (1), (2) which agree with u on I. One next verifies that the conditions of the Hausdorff maximum principle (see [23]) hold and hence that S contains a maximal element, $\tilde{u}$. This maximal element $\tilde{u}$ cannot be further extended to the right.

Next let u be a solution of (1), (2) with right maximal interval of existence $[t_0, \omega_+)$. We must show that $(t, u(t)) \to \partial D$ as $t \to \omega_+$, i.e., given any compact set $K \subset D$, there exists t_K, such that $(t, u(t)) \notin K$, for $t > t_K$. If $\omega_+ = \infty$, the conclusion clearly holds. On the other hand, if $\omega_+ < \infty$, we proceed indirectly. In which case there exists a compact set $K \subset D$, such that for every $n = 1, 2, \cdots$ there exists t_n, $0 < \omega_+ - t_n < \frac{1}{n}$, and $(t_n, u(t_n)) \in K$. Since K is compact, there will be a subsequence, call it again $\{(t_n, u(t_n))\}$ such that $\{(t_n, u(t_n))\}$ converges to, say, $(\omega_+, u^*) \in K$. Since $(\omega_+, u^*) \in K$, it is an interior point of D. We may therefore choose a constant $a > 0$, such that $Q = \{(t,u) : |\omega_+ - t| \le a,\ |u - u^*| \le$

$a\} \subset D$, and thus for n large $(t_n, u(t_n)) \in Q$. Let $m = \max_{(t,u)\in Q} |f(t,u)|$, and let n be so large that

$$0 < \omega_+ - t_n \leq \frac{a}{2m},\ |u(t_n) - u^*| \leq \frac{a}{2}.$$

Then

$$|u(t_n) - u(t)| < m(\omega_+ - t_n) \leq \frac{a}{2},\ \text{for } t_{\leq} t < \omega_+,$$

as an easy indirect argument shows.

It therefore follows that

$$\lim_{t\to\omega_+} u(t) = u^*,$$

and we may extend u to the right of ω_+ contradicting the maximality of u. □

7 Corollary *Assume that $f : [t_0, t_0 + a] \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and let u be a solution of (1) defined on some right maximal interval of existence $I \subset [t_0, t_0+a]$. Then, either $I = [t_0, t_0 + a]$, or else $I = [t_0, \omega_+)$, $\omega_+ < t_0 + a$, and*

$$\lim_{t\to\omega_+} |u(t)| = \infty.$$

We also consider the following corollary, which is of importance for differential equations whose right hand side have at most linear growth. I.e., we assume that the following growth condition holds:

$$|f(t,u)| \leq \alpha(t)|u| + \beta(t). \tag{6}$$

8 Corollary *Assume that $f : (a, b)\times\mathbb{R}^N \to \mathbb{R}^N$ is continuous and let f satisfy (6), where $\alpha, \beta \in L^1(a, b)$ are nonnegative continuous functions. Then the maximal interval of existence (ω_-, ω_+) is (a, b) for any solution of (1).*

Proof. If u is a solution of (1), then u satisfies the integral equation (3) and hence, because of (6), we obtain

$$|u(t)| \leq |u(t_0)| + |\int_{t_0}^{t} |[\alpha(s)|u(s)| + \beta(s)]ds. \tag{7}$$

Considering the case $t \geq t_0$, the other case being similar, we let

$$v(t) = \int_{t_0}^{t} [|\alpha(s)||u(s)| + |\beta(s)|]ds,$$

and $c = |u(t_0)|$. Then an easy calculation yields

$$v' - \alpha(t)v \leq \alpha(t)c + \beta(t),$$

and hence

$$v(t) \leq e^{\int_{t_0}^{t} \alpha(\tau)d\tau} \int_{t_0}^{t} e^{\int_{t_0}^{s} \alpha(\tau)d\tau}[\alpha(s)c + \beta(s)]ds,$$

from which, using Corollary 7, follows that $\omega_+ = b$. □

5 Dependence upon Initial Conditions

Let again D be an open connected subset of $\mathbb{R} \times \mathbb{R}^N$, $N \geq 1$, and let

$$f : D \to \mathbb{R}^N$$

be a continuous mapping.

We consider the initial value problem

$$u' = f(t,u),\ u(t_0) = u_0, \tag{8}$$

and assume we have conditions which guarantee that (8) has a unique solution

$$u(t) = u(t, t_0, u_0), \tag{9}$$

for every $(t_0, u_0) \in D$. We shall now present conditions which guarantee that $u(t, t_0, u_0)$ depends either continuously or smoothly on the initial condition (t_0, u_0).

A somewhat more general situation occurs frequently, where the function f also depends upon parameters, $\lambda \in \mathbb{R}^m$, i.e. (8) is replaced by the parameter dependent problem

$$u' = f(t,u,\lambda),\ u(t_0) = u_0, \tag{10}$$

and solutions u then are functions of the type

$$u(t) = u(t, t_0, u_0, \lambda), \tag{11}$$

provided (10) is uniquely solvable. This situation is a special case of the above, as we may augment the original system (10) as

$$\begin{aligned} u' &= f(t,u,\lambda),\ u(t_0) = u_0, \\ \lambda' &= 0,\ \lambda(t_0) = \lambda, \end{aligned} \tag{12}$$

and obtain an initial value problem for a system of equations of higher dimension which does not depend upon parameters.

5.1 Continuous dependence

We first prove the following proposition.

9 **Theorem** *Assume that $f : D \to \mathbb{R}^N$ is a continuous mapping and that (8) has a unique solution $u(t) = u(t, t_0, u_0)$, for every $(t_0, u_0) \in D$. Then the solution depends continuously on (t_0, u_0), in the following sense: If $\{(t_n, u_n)\} \subset D$ converges to $(t_0, u_0) \in D$, then given $\epsilon > 0$, there exists n_ϵ and an interval I_ϵ such that for all $n \geq n_\epsilon$, the solution $u_n(t) = u(t, t_n, u_n)$, exists on I_ϵ and*

$$\max_{t \in I_\epsilon} |u(t) - u_n(t)| \leq \epsilon.$$

PROOF. We rely on the proof of Theorem 3 and find that for given $\epsilon > 0$, there exists n_ϵ such that $\{(t_n, u_n)\} \subset \tilde{Q}$, where

$$\tilde{Q} = \{(t,u) : |t - t_0| \le \frac{a}{2},\ |u - u_0| \le \frac{b}{2}\} \subset Q,$$

where Q is the set given in the proof of Theorems 2 and 3. Using the proof of Theorem 3 we obtain a common compact interval I_ϵ of existence of the sequence $\{u_n\}$ and $\{(t, u_n(t))\} \subset Q$, for $t \in I_\epsilon$. The sequence $\{u_n\}$ hence will be uniformly bounded and equicontinuous on I_ϵ and will therefore have a subsequence converging uniformly on I_ϵ. Employing the integral equation (3) we see that the limit must be a solution of (8) and hence, by the uniqueness assumption must equal u. Since this is true for every subsequence, the whole sequence must converge to u, completing the proof. □

We have the following corollary, which asserts continuity of solutions with respect to the differential equation. The proof is similar to the above and will hence be omitted.

10 Corollary *Assume that $f_n : D \to \mathbb{R}^N$, $n = 1, 2, \cdots$, are continuous mappings and that (8) (with $f = f_n$) has a unique solution $u_n(t) = u(t, t_n, u_n)$, for every $(t_n, u_n) \in D$. Then the solution depends continuously on (t_0, u_0), in the following sense: If $\{(t_n, u_n)\} \subset D$ converges to $(t_0, u_0) \in D$, and f_n converges to f, uniformly on compact subsets of D, then given $\epsilon > 0$, there exists n_ϵ and an interval I_ϵ such that for all $n \ge n_\epsilon$, the solution $u_n(t) = u(t, t_n, u_n)$, exists on I_ϵ and*

$$\max_{t \in I_\epsilon} |u(t) - u_n(t)| \le \epsilon.$$

5.2 Differentiability with respect to initial conditions

In the following we shall employ the convention $u = (u^1, u^2, \cdots, u^N)$. We have the following theorem.

11 Theorem *Assume that $f : D \to \mathbb{R}^N$ is a continuous mapping and that the partial derivatives $\frac{\partial f}{\partial u^i}$, $1 \le i \le N$ are continuous also. Then the solution $u(t) = u(t, t_0, u_0)$, of (8) is of class C^1 in the variable u_0. Further, if $J(t)$ is the Jacobian matrix*

$$J(t) = J(t, t_0, u_0) = \left(\frac{\partial f}{\partial u}\right)_{u = u(t, t_0, u_0)},$$

then

$$y(t) = \frac{\partial u}{\partial u^i}(t, t_0, u_0)$$

is the solution of the initial value problem

$$y' = J(t)y,\ y(t_0) = e_i,\ 1 \le i \le N,$$

where $e_i \in \mathbb{R}^n$ is given by $e_i^k = \delta_{ik}$, with δ_{ik} the Kronecker delta.

PROOF. Let e_i be given as above and let $u(t) = u(t, t_0, u_0)$, $u_h(t) = u(t, t_0, u_0 + he_i)$, where $|h|$ is sufficiently small so that u_h exists. We note that u and u_h will have a common interval of existence, whenever $|h|$ is sufficiently small. We compute

$$(u_h(t) - u(t))' = f(t, u_h(t)) - f(t, u(t)).$$

Letting

$$y_h(t) = \frac{u_h(t) - u(t)}{h},$$

we get $y_h(t_0) = e_i$. If we let

$$G(t, y_1, y_2) = \int_0^1 \frac{\partial f}{\partial u}(t, sy_1 + (1-s)y_2)ds,$$

we obtain that y_h is the unique solution of

$$y' = G(t, u(t), u_h(t))y,\ y(t_0) = e_i.$$

Since $G(t, u(t), u_h(t)) \to J(t)$ as $h \to 0$, we may apply Corollary 10 to conclude that $y_h \to y$ uniformly on a neighborhood of t_0. □

6 Differential Inequalities

We consider in $\mathbb{R}^N$ the following partial orders:

$$\begin{array}{lcl} x \le y & \Leftrightarrow & x^i \le y^i,\ 1 \le i \le N, \\ x < y & \Leftrightarrow & x^i < y^i,\ 1 \le i \le N. \end{array}$$

For a function $u : I \to \mathbb{R}^N$, where I is an open interval, we consider the Dini derivatives

$$\begin{array}{lcl} D^+u(t) & = & \limsup_{h\to 0+} \frac{u(t+h)-u(t)}{h}, \\ D_+u(t) & = & \liminf_{h\to 0+} \frac{u(t+h)-u(t)}{h}, \\ D^-u(t) & = & \limsup_{h\to 0-} \frac{u(t+h)-u(t)}{h}, \\ D_-u(t) & = & \liminf_{h\to 0-} \frac{u(t+h)-u(t)}{h}, \end{array}$$

where lim sup and lim inf are taken componentwise.

12 **Definition** *A function $f : \mathbb{R}^N \to \mathbb{R}^N$ is said to be of type K (after Kamke [16]) on a set $S \subset \mathbb{R}^N$, whenever*

$$f^i(x) \le f^i(y),\ \forall x, y \in S,\ x \le y,\ x^i = y^i.$$

The following theorem on differential inequalities is of use in obtaining estimates on solutions.

13 **Theorem** *Assume that* $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ *is a continuous mapping which is of type* K *for each fixed* t. *Let* $u : [a,b] \to \mathbb{R}^N$ *be a solution of (1) and let* $v : [a,b] \to \mathbb{R}^N$ *be continuous and satisfy*

$$\begin{array}{rcl} D^{-}v(t) & > & f(t,v(t)),\ a < t \leq b, \\ v(a) & > & u(a), \end{array} \tag{13}$$

then $v(t) > u(t),\ a \leq t \leq b$.

If $z : [a,b] \to \mathbb{R}^N$ *is continuous and satisfies*

$$\begin{array}{rcl} D_{-}z(t) & < & f(t,z(t)),\ a \leq t < b, \\ z(a) & < & u(a), \end{array} \tag{14}$$

then $z(t) < u(t),\ a \leq t \leq b$.

Proof. We prove the first part of the theorem. The second part follows along the same line of reasoning. By continuity of u and v, there exists $\delta > 0$, such that

$$v(t) > u(t),\ a \leq t \leq a + \delta.$$

If the inequality does not hold throughout $[a,b]$, there will exist a first point c and an index i such that

$$v(t) > u(t),\ a \leq t < c,\ v(c) \geq u(c),\ v^i(c) = u^i(c).$$

Hence (since f is of type K)

$$D^{-}v^i(c) > f^i(c,v(c)) \geq f^i(c,u(c)) = u^{i\prime}(c).$$

On the other hand,

$$\begin{array}{rcl} D^{-}v^i(c) & = & \limsup_{h\to 0-} \frac{v^i(c+h)-v^i(c)}{h} \\ & \leq & \limsup_{h\to 0-} \frac{u^i(c+h)-u^i(c)}{h} = u^{i\prime}(c), \end{array}$$

a contradiction. □

14 **Definition** *A solution* u^* *of (1) is called a right maximal solution on an interval* I, *if for every* $t_0 \in I$ *and any solution* u *of (1) such that*

$$u(t_0) \leq u^*(t_0),$$

it follows that

$$u(t) \leq u^*(t),\ t_0 \leq t \in I.$$

Right minimal solutions are defined similarly.

15 **Theorem** *Assume that* $f : D \to \mathbb{R}^N$ *is a continuous mapping which is of type* K *for each* t. *Then the initial value problem(8) has a unique right maximal (minimal) solution for each* $(t_0, u_0) \in D$.

PROOF. That maximal and minimal solutions are unique follows from the definition. Choose $0 < \epsilon \in \mathbb{R}^N$ and let v_n be any solution of

$$u' = f(t,u,\lambda) + \frac{\epsilon}{n},\ u(t_0) = u_0 + \frac{\epsilon}{n}. \tag{15}$$

Then, given a neighborhood U of $(t_0, u_0) \in D$, there exists an interval $[t_0, t_1]$ of positive length such that all v_n are defined on this interval with $\{(t, v_n(t))\} \subset U,\ t_0 \le t \le t_1$, for all n sufficiently large. On the other hand it follows from Theorem 13 that

$$v_n(t) < v_m(t),\ t_0 \le t \le t_1,\ n < m.$$

The sequence $\{v_n\}$ is therefore unformly bounded and equicontinuous on $[t_0, t_1]$, hence will have a subsequence which converges uniformly to a solution u^* of (8). Since the sequence is monotone, the whole sequence will, in fact, converge to u^*. Applying Theorem 13 once more, we obtain that u^* is right maximal on $[t_0, t_1]$, and extending this solution to a right maximal interval of existence as a right maximal solution, completes the proof. $\square$

We next prove an existence theorem for initial value problems which allows for estimates of the solution in terms of given solutions of related differential inequalities.

16 Theorem *Assume that $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping which is of type K for each fixed t. Let $v, z : [a,b] \to \mathbb{R}^N$ be continuous and satisfy*

$$\begin{aligned} D^+v(t) &\ge f(t, v(t)),\ a \le t < b, \\ D_+z(t) &\le f(t, z(t)),\ a \le t < b, \\ z(t) &\le v(t),\ a \le t \le b. \end{aligned} \tag{16}$$

Then for every $u_0,\ z(a) \le u_0 \le v(a)$, there exists a solution u of (8) (with $t_0 = a$) such that

$$z(t) \le u(t) \le v(t),\ a \le t \le b.$$

The functions z and v are called, respectively, sub- and supersolutions of (8).
PROOF. Define $\bar f(t,x) = f(t, \bar x)$, where for $1 \le i \le N$,

$$\bar x^i = \begin{cases} v^i(t) & \text{, if} \quad x^i > v^i(t), \\ x^i & \text{, if} \quad z^i(t) \le x^i \le v^i(t), \\ z^i(t) & \text{, if} \quad x^i < z^i(t). \end{cases}$$

Then $\bar f$ is bounded and continuous, hence the initial value problem (8), with f replaced by $\bar f$ has a solution u that extends to $[a,b]$. We show that $z(t) \le u(t) \le v(t),\ a \le t \le b$, and hence may conclude that u solves the original initial value problem (8). To see this, we argue indirectly and suppose there exists c and i such that

$$u^i(c) = v^i(c),\ u^i(t) > v^i(t),\ c < t \le t_1 \le b.$$

Since $\bar{u}^i(t) = v^i(t),\ c < t \le t_1$ and $\bar{u}^k(t) \le v^k(t),\ c < t \le t_1,\ k \ne i$, we get that

$$\bar{f}^i(t, u(t)) \le \bar{f}^i(t, v(t)),\ c < t \le t_1,$$

and hence

$$\begin{aligned} u^i(t_1) - v^i(c) &= \int_c^{t_1} \bar{f}^i(t, u(t))dt \le \int_c^{t_1} \bar{f}^i(t, v(t))dt \\ &= \int_c^{t_1} f^i(t, v(t))dt \le v^i(t_1) - v^i(c), \end{aligned}$$

a contradiction. The other case is argued similarly. $\square$

17 Corollary *Assume the hypotheses of Theorem 16 and that f satisfies a local Lipschitz condition. Assume furthermore that*

$$z(a) \le z(b),\ v(a) \ge v(b).$$

Then the problem

$$u' = f(t, u),\ u(a) = u(b) \tag{17}$$

has a solution u with

$$z(t) \le u(t) \le v(t),\ a \le t \le b.$$

Proof. Since f satisfies a local Lipschitz condition, initial value problems are uniquely solvable. Hence for every $u_0,\ z(a) \le u_0 \le v(a)$, there exists a unique solution $u(t, u_0)$ of (8) (with $t_0 = a$) such that

$$z(t) \le u(t) \le v(t),\ a \le t \le b,$$

as follows from Theorem 16. Define the mapping

$$T : \{x : z(a) \le x \le v(a)\} \to \{x : z(b) \le x \le v(b)\}$$

by

$$Tx = u(b, x),$$

then, since by hypothesis $\{x : z(a) \le x \le v(a)\} \supset \{x : z(b) \le x \le v(b)\}$ and since $\{x : z(a) \le x \le v(a)\}$ is convex, it follows by Brouwer's fixed point theorem (Theorem III.22) and the fact that T is continuous (Theorem 9) that T has a fixed point, completing the proof. $\square$

An important consequence of this corollary is that if in addition f is a function which is periodic in t with period $b - a$, then Corollary 17 asserts the existence of a periodic solution (of period $b - a$).

The following results use comparison and differential inequality arguments to provide a priori bounds and extendability results.

18 Theorem *Assume that $F : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous mapping and that $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous also and*

$$|f(t,x)| \le F(t,|x|),\ a \le t \le b,\ x \in \mathbb{R}^N.$$

Let $u : [a,b] \to \mathbb{R}^N$ be a solution of (1) and let $v : [a,b] \to \mathbb{R}_+$ be the continuous and right maximal solution of

$$\begin{aligned} v'(t) &= F(t,v(t)),\ a \le t \le b, \\ v(a) &\ge |u(a)|, \end{aligned} \tag{18}$$

then $v(t) \ge |u(t)|,\ a \le t \le b$.

Proof. Let $z(t) = |u(t)|$, then z is continuous and $D_-z(t) = D^-z(t)$. Further

$$\begin{aligned} D_-z(t) &= \liminf_{h\to 0+} \tfrac{|u(t)|-|u(t-h)|}{h} \\ &\le \lim_{h\to 0+} \tfrac{|u(t-h)-u(t)|}{h} \\ &= |f(t,u(t))| \le F(t,z(t)). \end{aligned}$$

Hence, by Theorem 15 (actually its corollary (Exercise 6)), we conclude that $v(t) \ge |u(t)|,\ a \le t \le b$. □

7 Uniqueness Theorems

In this section we provide supplementary conditions which guarantee the uniqueness of solutions of ivp's.

19 Theorem *Assume that $F : (a,b) \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous mapping and that $f : (a,b) \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous also and*

$$|f(t,x) - f(t,y)| \le F(t,|x-y|),\ a \le t \le b,\ x,y \in \mathbb{R}^N.$$

Let $F(t,0) \equiv 0$ and let, for any $c \in (a,b)$, $w \equiv 0$ be the only solution of $w' = F(t,w)$ on (a,c) such that $w(t) = 0(\mu(t)),\ t \to a$ where μ is a given positive and continuous function. Then (1) cannot have distinct solutions such that $|u(t) - v(t)| = 0(\mu(t)),\ t \to a$.

Proof. Let u, v be distinct solutions of (1) such that $|u(t)-v(t)| = 0(\mu(t)),\ t \to a$. Let $z(t) = |u(t) - v(t)|$. Then z is continuous and

$$D^+z(t) \le |f(t,u(t)) - f(t,v(t)| \le F(t,z(t)).$$

The proof is completed by employing arguments like those used in the proof of Theorem 18. □

20 Remark Theorem 19 does not require that f be defined for $t = a$. The advantage of this may be that $a = -\infty$ or that f may be singular there. A similar result, of course, holds for $t \to b-$.

8 Exercises

1. Prove Proposition 1.
2. Prove Corollary 10.
3. Verify that the space (M, ρ) in the proof of Theorem 2 is a complete metric space.
4. Complete the details in the proof of Theorem 11.
5. State and prove a theorem similar to Theorem 11, providing a differential equation for $\frac{\partial u}{\partial \lambda}$ whenever $f = f(t, u, \lambda)$ also depends upon a parameter λ.
6. Prove the following result: Assume that $f : [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping which is of type K for each fixed t. Let $u : [a, b] \to \mathbb{R}^N$ be a right maximal solution of (1) and let $z : [a, b] \to \mathbb{R}^N$ be continuous and satisfy

$$\begin{aligned} D_- z(t) &\leq f(t, z(t)),\ a < t \leq b, \\ z(a) &\leq u(a), \end{aligned} \tag{19}$$

then $z(t) \leq u(t),\ a \leq t \leq b$.

7. Show that a real valued continuous function $z(t)$ is nonincreasing on an interval $[a, b]$ if and only if $D_- z \leq 0$ on $(a, b]$.
8. Assume that $f : [a, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping such that

$$|f(t, x)| \leq M(t) L(|x|),\ a \leq t < \infty,\ x \in \mathbb{R}^N,$$

where M and L are continuous functions on their respective domains and

$$\int^\infty \frac{ds}{L(s)} = \infty.$$

Prove that $\omega_+ = \infty$ for all solutions of (1).

9. Give the details of the proof of Theorem 18.
10. Assume that $f : [a, b) \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping and assume the conditions of Theorem 19 with $\mu \equiv 1$. The the initial value problem

$$u' = f(t, u),\ u(a) = u_0 \tag{20}$$

has at most one solution.

11. Assume that $f : [a, b) \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping and that

$$|f(t, x) - f(t, y)| \leq c \frac{|x - y|}{t - a},\ t > a,\ x, y \in \mathbb{R}^N,\ 0 < c < 1 \tag{21}$$

Then the initial value problem (20) has at most one solution.

12. The previous exercise remains valid if (21) is replaced by

$$(f(t,x) - f(t,y)) \cdot (x - y) \le c\frac{|x-y|^2}{t-a},\ t > a,\ x, y \in \mathbb{R}^N.$$

13. Assume that $f : [a, b) \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping and that

$$(f(t,x) - f(t,y)) \cdot (x - y) \le 0,\ t \ge a,\ x, y \in \mathbb{R}^N.$$

    Then every initial value problem is uniquely solvable to the right. This exercise, of course follows from the previous one. Give a more elementary and direct proof. Note that unique solvability to the left of an initial point is not guaranteed. How must the above condition be modified to guarantee uniqueness to the left of an initial point?

14. Provide the details in the proof of Theorem 4.

15. Establish a result similar to Theorem 6 assuming that f satisfies Carthéodory conditions.

Chapter VI
Linear Ordinary Differential Equations

1 Introduction

In this chapter we shall employ what has been developed to give a brief overview of the theory of linear ordinary differential equations. The results obtained will be useful in the study of stability of solutions of nonlinear differential equations as well as bifurcation theory for periodic orbits and many other facets where linearization techniques are of importance. The results are also of interest in their own right.

2 Preliminaries

Let $I \subset \mathbb{R}$ be a real interval and let

$$A : I \to \mathfrak{L}(\mathbb{R}^N, \mathbb{R}^N)$$

$$f : I \to \mathbb{R}^N$$

be continuous functions. We consider here the system of ordinary differential equations

$$u' = A(t)u + f(t),\ t \in I, \tag{1}$$

and

$$u' = A(t)u,\ t \in I. \tag{2}$$

Using earlier results we may establish the following basic proposition (see Exercise 1).

1 Proposition *For any given f, initial value problems for (1) are uniquely solvable and solutions are defined on all of I.*

2 Remark More generally we may assume that A and f are measurable on I and locally integrable there, in which case the conclusion of Proposition 1 still holds. We shall not go into details for this more general situation, but leave it to the reader to present a parallel development.

3 Proposition *The set of solutions of (2) is a vector space of dimension N.*

PROOF. That the solution set forms a vector space is left as an exercise (Exercise 2, below). To show that the dimension of this space is N, we employ the uniqueness principle above. Thus let $t_0 \in I$, and let $u_k(t),\ k = 1, \cdots, N$ be the solution of (2) such that

$$u_k(t_0) = e_k,\ e_k^i = \delta_{ki} \text{ (Kronecker delta)}. \tag{3}$$

It follows that for any set of constants $a_1, \cdots, a_N$,

$$u(t) = \sum_1^N a_i u_i(t) \tag{4}$$

is a solution of (2). Further, for given $\xi \in \mathbb{R}^N$, the solution u of (2) such that $u(t_0) = \xi$ is given by (4) with $a_i = \xi^i,\ i = 1, \cdots, N$.

Let the $N \times N$ matrix function Φ be defined by

$$\Phi(t) = (u_j^i(t)),\ 1 \le i, j \le N, \tag{5}$$

i.e., the columns of Φ are solutions of (2). Then (4) takes the form

$$u(t) = \Phi(t)a,\ a = (a^1, \cdots, a^N)^T. \tag{6}$$

Hence for given $\xi \in \mathbb{R}^N$, the solution u of (2) such that $u(t_1) = \xi,\ t_1 \in I$, is given by (6) provided that $\Phi(t_1)$ is a nonsingular matrix, in which case a may be uniquely determined. That this matrix is never singular, provided it is nonsingular at some point, is known as the Abel-Liouville lemma, whose proof is left as an exercise below. $\square$

4 Lemma *If $g(t) = \det\Phi(t)$, then g satisfies*

$$g(t) = g(t_0)e^{\int_{t_0}^t \text{trace} A(s)} ds. \tag{7}$$

Hence, if Φ is defined by (5), where $u_1, \cdots, u_N$ are solutions of (2), then $\Phi(t)$ is nonsingular for all $t \in I$ if and only if $\Phi(t_0)$ is nonsingular for some $t_0 \in I$.

2.1 Fundamental solutions

A nonsingular $N \times N$ matrix function Ψ whose columns are solutions of (2) is called a fundamental matrix solution or a fundamental system of (2). Such a matrix is a nonsingular solution of the matrix differential equation

$$\Psi' = A(t)\Psi. \tag{8}$$

The following proposition characterizes the set of fundamental solutions; its proof is again left as an exercise.

5 **Proposition** *Let Φ be a given fundamental matrix solution of (2). Then every other fundamental matrix solution has the form $\Psi = \Phi C$, where C is a constant nonsingular $N \times N$ matrix. Furthermore the set of all solutions of (2) is given by*

$$\{\Phi c : c \in \mathbb{R}^N\},$$

where Φ is a fundamental system.

2.2 Variation of constants

It follows from Propositions 3 and 5 that all solutions of (1) are given by

$$\{\Phi(t)c + u_p(t) : c \in \mathbb{R}^N\},$$

where Φ is a fundamental system of (2) and u_p is some particular solution of (1). Hence the problem of finding all solutions of (1) is solved once a fundamental system of (2) is known and some particular solution of (1) has been found. The following formula, known as the variation of constants formula, shows that a particular solution of (1) may be obtained from a fundamental system.

6 **Proposition** *Let Φ be a fundamental matrix solution of (2) and let $t_0 \in I$. Then*

$$u_p(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(s) ds \tag{9}$$

is a solution of (1). Hence the set of all solutions of (1) is given by

$$\{\Phi(t) \left(c + \int_{t_0}^{t} \Phi^{-1}(s) f(s) ds \right) : c \in \mathbb{R}^N\},$$

where Φ is a fundamental system of (2).

3 Constant Coefficient Systems

In this section we shall assume that the matrix A is a constant matrix and thus have that solutions of (2) are defined for all $t \in \mathbb{R}$. In this case a fundamental matrix solution Φ is given by

$$\Phi(t) = e^{tA} C, \tag{10}$$

where C is a nonsingular constant $N \times N$ matrix and

$$e^{tA} = \sum_{0}^{\infty} \frac{t^n A^n}{n!}.$$

Thus the solution u of (2) with $u(t_0) = \xi$ is given by

$$u(t) = e^{(t-t_0)A} \xi.$$

To compute e^{tA} we use the (complex) Jordan canonical form J of A. Since A and J are similar, there exists a nonsingular matrix P such that $A = PJP^{-1}$ and hence $e^{tA} = Pe^{tJ}P^{-1}$. We therefore compute e^{tJ}. On the other hand J has the form

$$J = \begin{pmatrix} J_0 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix},$$

where

$$J_0 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix},$$

is a $q \times q$ diagonal matrix whose entries are the simple (algebraically) and semisimple eigenvalues of A, repeated according to their multiplicities, and for $1 \leq i \leq s$,

$$J_i = \begin{pmatrix} \lambda_{q+i} & 1 & & & \\ & \lambda_{q+i} & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & \lambda_{q+i} \end{pmatrix}$$

is a $q_i \times q_i$ matrix, with

$$q + \sum_1^s q_i = N.$$

By the laws of matrix multiplication it follows that

$$e^{tJ} = \begin{pmatrix} e^{tJ_0} & & & \\ & e^{tJ_1} & & \\ & & \ddots & \\ & & & e^{tJ_s} \end{pmatrix},$$

and

$$e^{tJ_0} = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_q t} \end{pmatrix}.$$

Further, since $J_i = \lambda_{q+i} I_{r_i} + Z_i$, where I_{r_i} is the $r_i \times r_i$ identity matrix and Z_i is given by

$$Z_i = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix},$$

we obtain that

$$e^{tJ_i} = e^{t\lambda_{q+i} I_{r_i}} e^{tZ_i} = e^{t\lambda_{q+i}} e^{tZ_i}.$$

An easy computation now shows that

$$e^{tZ_i} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{r_i-1}}{r_{i-1}!} \\ 0 & 1 & t & \cdots & \frac{t^{r_i-2}}{r_{i-2}!} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Since P is a nonsingular matrix $e^{tA}P = Pe^{tJ}$ is a fundamental matrix solution as well. Also, since J and P may be complex we obtain the set of all real solutions as

$$\{\mathrm{Re} Pe^{tJ}c,\ \mathrm{Im} Pe^{tJ}c : c \in \mathbb{C}^N\}.$$

The above considerations have the following proposition as a consequence.

7 **Proposition** *Let A be an $N \times N$ constant matrix and consider the differential equation*

$$u' = Au. \tag{11}$$

Then:

1. *All solutions u of (11) satisfy $u(t) \to 0$, as $t \to \infty$, if and only if $\mathrm{Re}\lambda < 0$, for all eigenvalues λ of A.*

2. *All solutions u of (11) are bounded on $[0, \infty)$, if and only if $\mathrm{Re}\lambda \le 0$, for all eigenvalues λ of A and those with zero real part are semisimple.*

4 Floquet Theory

Let $A(t),\ t \in \mathbb{R}$ be an $N \times N$ continuous matrix which is periodic with respect to t of period T, i.e., $A(t+T) = A(t),\ -\infty < t < \infty$, and consider the differential equation

$$u' = A(t)u. \tag{12}$$

We shall associate to (12) a constant coefficient system which determines the asymptotic behavior of solutions of (12). To this end we first establish some facts about fundamental solutions of (12).

8 **Proposition** *Let $\Phi(t)$ be a fundamental matrix solution of (12), then so is $\Psi(t) = \Phi(t+T)$.*

Proof. Since Φ is a fundamental matrix it is nonsingular for all t, hence Ψ is nonsingular. Further

$$\begin{aligned} \Psi'(t) = \Phi'(t+T) \quad &= A(t+T)\Phi(t+T) \\ &= A(t)\Phi(t+T) \\ &= A(t)\Psi(t). \end{aligned}$$

□

It follows by our earlier considerations that there exists a nonsingular constant matrix Q such that

$$\Phi(t+T) = \Phi(t)Q.$$

Since Q is nonsingular, there exists a matrix R such that

$$Q = e^{TR}.$$

Letting $C(t) = \Phi(t)e^{-tR}$ we compute

$$\begin{aligned} C(t+T) \quad &= \Phi(t+T)e^{-(t+T)R} \\ &= \Phi(t)Qe^{-TR}e^{-tR} \\ &= \Phi(t)e^{-tR} = C(t). \end{aligned}$$

We have proved the following proposition.

9 **Proposition** *Let $\Phi(t)$ be a fundamental matrix solution of (12), then there exists a nonsingular periodic (of period T) matrix C and a constant matrix R such that*

$$\Phi(t) = C(t)e^{tR}. \tag{13}$$

From this representation we may immediately deduce conditions which guarantee the existence of nontrivial $T-$ periodic and $mT-$ periodic (subharmonics) solutions of (12).

10 **Corollary** *For any positive integer m (12) has a nontrivial $mT-$periodic solution if and only if $\Phi^{-1}(0)\Phi(T)$ has an $m-$th root of unity as an eigenvalue, where Φ is a fundamental matrix solution of (12).*

Proof. The properties of fundamental matrix solutions guarantee that the matrix $\Phi^{-1}(0)\Phi(T)$ is uniquely determined by the equation and Proposition 9 implies that

$$\Phi^{-1}(0)\Phi(T) = e^{TR}.$$

On the other hand a solution u of (12) is given by

$$u(t) = C(t)e^{tR}d,$$

where $u(0) = C(0)d$. Hence u is periodic of period mT if and only if

$$u(mT) = C(mT)e^{mTR}d = C(0)e^{mTR}d = C(0)d.$$

Which is the case if and only if $e^{mTR} = \left(e^{TR}\right)^m$ has 1 as an eigenvalue. □

Let us apply these results to the second order scalar equation

$$y'' + p(t)y = 0, \tag{14}$$

(Hill's equation) where $p : \mathbb{R} \to \mathbb{R}$ is a T-periodic function. Equation (14) may be rewritten as the system

$$u' = \begin{pmatrix} 0 & 1 \\ -p(t) & 0 \end{pmatrix} u. \tag{15}$$

Let y_1 be the solution of (14) such that

$$y_1(0) = 1,\ y_1'(0) = 0,$$

and y_2 the solution of (14) such that

$$y_2(0) = 0,\ y_2'(0) = 1.$$

Then

$$\Phi(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

will be a fundamental solution of (15)and

$$\det\Phi(t) = e^{\int_0^t \text{trace}A(s)ds} = 1,$$

by the Abel-Liouville formula (7). Hence (14), or equivalently (15), will have a $mT-$periodic solution if and only if $\Phi(T)$ has an eigenvalue λ which is an $m-$th root of unity. The eigenvalues of $\Phi(T)$ are solutions of the equation

$$\det \begin{pmatrix} y_1(T) - \lambda & y_2(T) \\ y_1'(T) & y_2'(T) - \lambda \end{pmatrix} = 0,$$

or

$$\lambda^2 - a\lambda + 1 = 0,$$

where

$$a = y_1(T) + y_2'(T).$$

Therefore

$$\lambda = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

5 Exercises

1. Prove Proposition 1.

2. Prove that the solution set of (2) forms a vector space over either the real field or the field of complex numbers.

3. Verify the Abel-Liouville formula (7).

4. Prove Proposition 5. Also give an example to show that for a nonsingular $N \times N$ constant matrix C, and a fundamental solution Φ, $\Psi = C\Phi$ need not be a fundamental solution.

5. Let $[0, \infty) \subset I$ and assume that all solutions of (2) are bounded on $[0, \infty)$. Let Φ be a fundamental matrix solution of (2). Show that $\Phi^{-1}(t)$ is bounded on $[0, \infty)$ if and only if $\int_0^t A(s)ds$ is bounded from below. If this is the case, prove that no solution u of (2) may satisfy $u(t) \to 0$ as $t \to \infty$ unless $u \equiv 0$.

6. Let $[0, \infty) \subset I$ and assume that all solutions of (2) are bounded on $[0, \infty)$. Further assume that the matrix $\Phi^{-1}(t)$ is bounded on $[0, \infty)$. Let $B : [0, \infty) \to \mathbb{R}^{N \times N}$ be continuous and such that

$$\int_0^\infty |A(s) - B(s)| ds < \infty.$$

   Then all solutions of

$$u' = B(t)u \tag{16}$$

   are bounded on $[0, \infty)$.

7. Assume the conditions of the previous exercise. Show that corresponding to every solution u of (2) there exists a unique solution v of (16) such that

$$\lim_{t \to \infty} |u(t) - v(t)| = 0.$$

8. Assume that

$$\int_0^\infty |B(s)| ds < \infty.$$

   Show that any solution, not the trivial solution, of (16) tends to a nonzero limit as $t \to \infty$ and for any $c \in \mathbb{R}^N$, there exists a solution v of (16) such that

$$\lim_{t \to \infty} v(t) = c.$$

9. Prove Proposition 7.

10. Give necessary and sufficient conditions in order that all solutions u of (12) satisfy

$$\lim_{t\to\infty} u(t) = 0.$$

11. Show that there exists a nonsingular C^1 matrix $L(t)$ such that the substitution $u = L(t)v$ reduces (12) to a constant coefficient system $v' = Bv$.

12. Provide conditions on $a = y_1(T) + y_2'(T)$ in order that (14) have a

$$T,\ 2T,\ \cdots,\ mT - \text{periodic}$$

solution, where the period should be the minimal period.

13. Consider equation (1), where both A and f are $T-$periodic. Use the variation of constants formula to discuss the existence of $T-$periodic solutions of (1).

Chapter VII
Periodic Solutions

1 Introduction

In this chapter we shall develop, using the linear theory developed in the previous chapter together with the implicit function theorem and degree theory, some of the basic existence results about periodic solutions of periodic nonlinear systems of ordinary differential equations. In particular, we shall mainly be concerned with systems of the form

$$u' = A(t)u + f(t,u), \tag{1}$$

where

$$\begin{array}{l} A : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N \\ f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \end{array}$$

are continuous and $T-$periodic with respect to t. We call the equation *nonresonant* provided the linear system

$$u' = A(t)u \tag{2}$$

has as its only $T-$ periodic solution the trivial one; we call it *resonant,* otherwise.

2 Preliminaries

We recall from Chapter **??** that the set of all solutions of the equation

$$u' = A(t)u + g(t), \tag{3}$$

is given by

$$u(t) = \Phi(t)c + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds,\ c \in \mathbb{R}^N, \tag{4}$$

where Φ is a fundamental matrix solution of the linear system (2). On the other hand it follows from Floquet theory (Section VI.4) that Φ has the form

$\Phi(t) = C(t)e^{tR}$, where C is a continuous nonsingular periodic matrix of period T and R is a constant matrix (viz. Proposition VI.9). As we may choose Φ such that $\Phi(0) = I$ (the $N \times N$ identity matrix), it follows that (with this choice)

$$C(0) = C(T) = I$$

and

$$u(T) = e^{TR}\left(c + \int_0^T \Phi^{-1}(s)g(s)ds,\ c \in \mathbb{R}^N\right),$$

hence $u(0) = u(T)$ if and only if

$$c = e^{TR}\left(c + \int_0^T \Phi^{-1}(s)g(s)ds,\ c \in \mathbb{R}^N\right). \tag{5}$$

We note that equation (5) is uniquely solvable for every g, if and only if

$$I - e^{TR}$$

is a nonsingular matrix. I.e. we have the following result:

1 Proposition *Equation (3) has a $T-$periodic solution for every $T-$periodic forcing term g if and only if $e^{TR} - I$ is a nonsingular matrix. If this is the case, the periodic solution u is given by the following formula:*

$$u(t) = \Phi(t)\left((I - e^{TR})^{-1}\int_0^T \Phi^{-1}(s)g(s)ds + \int_0^t \Phi^{-1}(s)g(s)ds\right). \tag{6}$$

This proposition allows us to formulate a fixed point equation whose solution will determine $T-$ periodic solutions of equation (1). The following section is devoted to results of this type.

3 Perturbations of Nonresonant Equations

In the following let

$$E = \{u \in C([0,T], \mathbb{R}^N) : u(0) = u(T)\}$$

with $\|u\| = \max_{t\in[0,T]} |u(t)|$ and let $S : E \to E$ be given by

$$\begin{aligned}(Su)(t) \quad &= \Phi(t)\Big((I - e^{TR})^{-1}\textstyle\int_0^T \Phi^{-1}(s)f(s,u(s))ds \\ &\quad + \textstyle\int_0^t \Phi^{-1}(s)f(s,u(s))ds\Big).\end{aligned} \tag{7}$$

2 Proposition *Assume that $I - e^{TR}$ is nonsingular, then (1) has a $T-$ periodic solution u, whenever the operator S given by equation (7) has a fixed point in the space E.*

For f as given above let us define

$$P(r) = \max\{|f(t,u)| : 0 \le t \le T,\ |u| \le r\}. \tag{8}$$

We have the following theorem.

3 **Theorem** *Assume that A and f are as above and that $I - e^{TR}$ is nonsingular, then (1) has a $T-$ periodic solution u, whenever*

$$\liminf_{r\to\infty} \frac{P(r)}{r} = 0, \tag{9}$$

where P is defined by (8).

Proof. Let us define

$$B(r) = \{u \in E : \|u\| \le r\},$$

then for $u \in B(r)$ we obtain

$$\|Su\| \le KP(r),$$

where S is the operator defined by equation (7) and K is a constant that depends only on the matrix A. Hence

$$S : B(r) \to B(r),$$

provided that

$$KP(r) \le r,$$

which holds, by condition (9), for some r sufficiently large. Since S is completely continuous the result follows from the Schauder fixed point theorem (Theorem III.30). □

As a corollary we immediately obtain:

4 **Corollary** *Assume that A and f are as above and that $I - e^{TR}$ is nonsingular, then*

$$u' = A(t)u + \epsilon f(t,u) \tag{10}$$

has a $T-$ periodic solution u, provided that ϵ is sufficiently small.

Proof. Using the operator S associated with equation (10) we obtain for $u \in B(r)$

$$\|Su\| \le |\epsilon| KP(r),$$

thus for given $r > 0$, there exists $\epsilon \ne 0$ such that

$$|\epsilon| KP(r) \le r,$$

and S will have a fixed point in $B(r)$. □

The above corollary may be considerably extended using the global continuation theorem Theorem IV.4. Namely we have the following result.

5 Theorem *Assume that A and f are as above and that $I - e^{TR}$ is nonsingular. Let*

$$\mathfrak{S}^{+} = \{(u,\epsilon) \in E \times [0,\infty) : (u,\epsilon) \ \ \textit{solves} \ \ (10)\}$$

and

$$\mathfrak{S}^{-} = \{(u,\epsilon) \in E \times (-\infty,0] : (u,\epsilon) \ \ \textit{solves} \ \ (10)\}.$$

Then there exists a continuum $C^{+} \subset \mathfrak{S}^{+}$ ($C^{-} \subset \mathfrak{S}^{-}$) such that:

1. *$C_0^{+} \cap E = \{0\}$ $\ (C_0^{-} \cap E = \{0\})$,*
2. *C^{+} is unbounded in $E \times [0,\infty)$ (C^{-} is unbounded in $E \times (-\infty,0]$).*

Proof. The proof follows immediately from Theorem IV.4 by noting that the existence of $T-$periodic solutions of equation (10) is equivalent to the existence of solutions of the operator equation

$$u - S(\epsilon,u) = 0,$$

where

$$\begin{aligned} S(\epsilon,u)(t) \quad &= \Phi(t)\left((I - e^{TR})^{-1} \int_0^T \Phi^{-1}(s)\epsilon f(s,u(s))ds \right. \\ &\left. + \int_0^t \Phi^{-1}(s)\epsilon f(s,u(s))ds\right). \end{aligned}$$

□

4 Resonant Equations

4.1 Preliminaries

We shall now consider the equation subject to constraint

$$\begin{aligned} u' &= f(t,u), \\ u(0) &= u(T), \end{aligned} \tag{11}$$

where

$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is continuous. Should f be $T-$periodic with respect to t, then a $T-$periodic extension of a solution of (11) will be a $T-$ periodic solution of the equation. We view (11) as a perturbation of the equation $u' = 0$, i.e. we are in the case of resonance.

We now let

$$E = \{u \in C([0,T],\mathbb{R}^N)\}$$

with $\|u\| = \max_{t\in[0,T]} |u(t)|$ and let $S : E \to E$ be given by

$$(Su)(t) = u(T) + \int_0^t f(s,u(s))ds, \tag{12}$$

then clearly $S : E \to E$ is a completely continuous mapping because of the the continuity assumption on f.

The following lemma holds.

6 Lemma *An element $u \in E$ is a solution of (11) if and only if it is a fixed point of the operator S given by (12).*

This lemma, whose proof is immediate, gives us an operator equation in the space E whose solutions are solutions of the problem (11).

4.2 Homotopy methods

We shall next impose conditions on the finite dimensional vector field

$$\begin{array}{rcl} x \in \mathbb{R}^N & \mapsto & g(x) \\ g(x) & = & -\int_0^T f(s,x)ds, \end{array} \tag{13}$$

which will guarantee the existence of solutions of an associated problem

$$\begin{array}{l} u' = \epsilon f(t,u), \\ u(0) = u(T), \end{array} \tag{14}$$

where ϵ is a small parameter. We have the following theorem.

7 Theorem *Assume that f is continuous and there exists a bounded open set $\Omega \subset \mathbb{R}^N$ such that the mapping g defined by (13) does not vanish on $\partial\Omega$. Further assume that*

$$\mathrm{d}(g, \Omega, 0) \neq 0, \tag{15}$$

where $\mathrm{d}(g,\Omega,0)$ is the Brouwer degree. Then problem (14) has a solution for all sufficiently small ϵ.

Proof. We define the bounded open set $G \subset E$ by

$$G = \{u \in E : u : [0,T] \to \Omega\}. \tag{16}$$

For $u \in \bar{G}$ define

$$u(t,\lambda) = \lambda u(t) + (1-\lambda)u(T),\ 0 \leq \lambda \leq 1, \tag{17}$$

and let

$$a(t,\lambda) = \lambda t + (1-\lambda)T,\ 0 \leq \lambda \leq 1. \tag{18}$$

For $0 \leq \lambda \leq 1,\ 0 \leq \epsilon \leq 1$, define $S : E \times [0,1] \times [0,1] \to E$ by

$$S(u,\lambda,\epsilon)(t) = u(T) + \epsilon \int_0^{a(t,\lambda)} f(s, u(s,\lambda))ds. \tag{19}$$

Then S is a completely continuous mapping and the theorem will be proved once we show that

$$\mathrm{d}(\mathrm{id} - S(\cdot, 1, \epsilon), G, 0) \neq 0, \tag{20}$$

for ϵ sufficiently small, for if this is the case, $S(\cdot, 1, \epsilon)$ has a fixed point in G which is equivalent to the assertion of the theorem.

To show that (20) holds we first show that $S(\cdot, \lambda, \epsilon)$ has no zeros on ∂G for all $\lambda \in [0, 1]$ and ϵ sufficiently small. This we argue indirectly and hence obtain sequences $\{u_n\} \subset \partial G$, $\{\lambda_n\} \subset [0, 1]$, and $\{\epsilon_n\}$, $\epsilon_n \to 0$, such that

$$u_n(T) + \epsilon_n \int_0^{a(t,\lambda_n)} f(s, u_n(s, \lambda_n))ds \equiv u_n(t), \ 0 \leq t \leq T,$$

and hence

$$\int_0^T f(s, u_n(s, \lambda_n))ds = 0, \ n = 1, 2, \cdots.$$

Without loss in generality, we may assume that the sequences mentioned converge to, say, u and λ_0 and the following must hold:

$$u(t) \equiv u(T) = a \in \partial\Omega.$$

Hence also

$$u_n(t, \lambda_n) \to u(t, \lambda_0) \equiv u(T),$$

which further implies that

$$\int_0^T f(s, a)ds = 0,$$

where $a \in \partial\Omega$, in contradiction to the assumptions of the theorem. Thus

$$\mathrm{d}(\mathrm{id} - S(\cdot, 0, \epsilon), G, 0) = \mathrm{d}(\mathrm{id} - S(\cdot, 1, \epsilon), G, 0)$$

by the homotopy invariance property of Leray-Schauder degree, for all ϵ sufficiently small. On the other hand

$$\begin{aligned} \mathrm{d}(\mathrm{id} - S(\cdot, 0, \epsilon), G, 0) &= \mathrm{d}(\mathrm{id} - S(\cdot, 0, \epsilon), G \cap \mathbb{R}^N, 0) \\ &= \mathrm{d}(\mathrm{id} - S(\cdot, 0, \epsilon), \Omega, 0) \\ &= \mathrm{d}(g, \Omega, 0) \neq 0, \end{aligned}$$

if $\epsilon > 0$ and $(-1)^N \mathrm{d}(g, \Omega, 0)$, if $\epsilon < 0$, completing the proof. □

8 Corollary *Assume the hypotheses of Theorem 7 and assume that all possible solutions u, for $0 < \epsilon \leq 1$, of equation (14) are such that $u \notin \partial G$, where G is given by (16). Then (11) has a $T-$ periodic solution.*

4.3 A Liénard type equation

In this section we apply Corollary 8 to prove the existence of periodic solutions of Liénard type oscillators of the form

$$x'' + h(x)x' + x = e(t), \tag{21}$$

where

$$e : \mathbb{R} \to \mathbb{R}$$

is a continuous $T-$ periodic forcing term and

$$h : \mathbb{R} \to \mathbb{R}$$

is a continuous mapping. We shall prove the following result.

9 Theorem *Assume that $T < 2\pi$. Then for every continuous $T-$ periodic forcing term e, equation (21) has a $T-$ periodic response x.*

We note that, since aside from the continuity assumption, nothing else is assumed about h, we may, without loss in generality, assume that $\int_0^T e(s)ds = 0$, as follows from the substitution

$$y = x - \int_0^T e(s)ds.$$

We hence shall make that assumption. In order to apply our earlier results, we convert (21) into a system as follows:

$$\begin{aligned} x' &= y \\ y' &= -h(x)y - x - e(t), \end{aligned} \tag{22}$$

and put

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(t,u) = \begin{pmatrix} y \\ -h(x)y - x - e(t) \end{pmatrix}. \tag{23}$$

We next shall show that the hypotheses of Theorem 7 and Corollary 8 may be satisfied by choosing

$$\Omega = \left\{ u = \begin{pmatrix} x \\ y \end{pmatrix} : |x| < R,\ |y| < R \right\}, \tag{24}$$

where R is a sufficiently large constant. We note that (15) holds for such choices of Ω for any $R > 0$. Hence, if we are able to provide a priori bounds for solutions of equation (14) for $0 < \epsilon \le 1$ for f given as above, the result will follow. Now

$$u = \begin{pmatrix} x \\ y \end{pmatrix},$$

is a solution of (14) whenever x satisfies

$$x'' + \epsilon h(x)x' + \epsilon^2 x = \epsilon^2 e(t). \tag{25}$$

Integrating (25) from 0 to T, we find that

$$\int_0^T x(s)ds = 0.$$

Multiplying (25) by x and integrating we obtain

$$-\|x'\|_{L^2}^2 + \epsilon^2\|x\|_{L^2}^2 = \epsilon^2\langle x, e\rangle_{L^2}, \tag{26}$$

where $\langle x, e\rangle_{L^2} = \int_0^T x(s)e(s)ds$. Now, since

$$\|x\|_{L^2}^2 \le \frac{T^2}{4\pi^2}\|x'\|_{L^2}^2, \tag{27}$$

we obtain from (26)

$$\left(1 - \frac{T^2}{4\pi^2}\right)\|x'\|_{L^2}^2 \le -\epsilon^2\langle x, e\rangle_{L^2}, \tag{28}$$

from which follows that

$$\|x'\|_{L^2} \le \left(\frac{2\pi T}{4\pi^2 - T^2}\right)\|e\|_{L^2}, \tag{29}$$

from which, in turn, we obtain

$$\|x\|_\infty \le \sqrt{\frac{T}{12}}\left(\frac{2\pi T}{4\pi^2 - T^2}\right)\|e\|_{L^2}, \tag{30}$$

providing an a priori bound on $\|x\|_\infty$. We let

$$\left(\frac{2\pi T}{4\pi^2 - T^2}\right)\|e\|_{L^2} = M,$$

$$q = \max_{|x| \le M}|h(x)|, \quad p = T\|e\|_\infty.$$

Then

$$\|x''\|_\infty \le \epsilon q\|x'\|_\infty + \epsilon^2(M + p).$$

Hence, by Landau' s inequality (Exercise 6, below), we obtain

$$\|x'\|_\infty^2 \le 4M(\epsilon q\|x'\|_\infty + \epsilon^2(M + p)),$$

from which follows a bound on $\|x'\|_\infty$ which is independent of ϵ, for $0 \le \epsilon \le 1$. These considerations complete the proof Theorem 9.

4.4 Partial resonance

This section is a continuation of of what has been discussed in Subsection 4.2. We shall impose conditions on the finite dimensional vector field

$$\begin{array}{rcl} x \in \mathbb{R}^p & \mapsto & g(x) \\ g(x) & = & -\int_0^T f(s,x,0)ds, \end{array} \tag{31}$$

which will guarantee the existence of solutions of an associated problem

$$\begin{array}{l} u' = \epsilon f(t,u,v), \\ v' = By + \epsilon h(t,u,v) \\ u(0) = u(T),\ v(0) = v(T), \end{array} \tag{32}$$

where ϵ is a small parameter and

$$\begin{array}{l} f : \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \\ h : \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q \end{array}$$

are continuous and $T-$periodic with respect to t, $p+q=N$. Further B is a $q \times q$ constant matrix with the property that the system $v' = Bv$ is nonresonant, i.e. only has the trivial solution as a $T-$periodic solution. We have the following theorem.

10 Theorem *Assume the above and there exists a bounded open set $\Omega \subset \mathbb{R}^p$ such that the mapping g defined by (31) does not vanish on $\partial\Omega$. Further assume that*

$$\mathrm{d}(g,\Omega,0) \neq 0, \tag{33}$$

where $\mathrm{d}(g,\Omega,0)$ is the Brouwer degree. Then problem (32) has a solution for all sufficiently small ϵ.

Proof. To prove the existence of a $T-$ periodic solution (u,v) of equation (32) is equivalent to establishing the existence of a solution of

$$\begin{array}{l} u(t) = u(T) + \epsilon \int_0^t f(s,u(s),v(s))ds \\ v(t) = e^{Bt}v(T) + \epsilon e^{Bt} \int_0^t e^{-Bs}h(s,u(s),v(s))ds. \end{array} \tag{34}$$

We consider equation (34) as an equation in the Banach space $E = C([0,T], \mathbb{R}^p \times \mathbb{R}^q)$. Let Λ be a bounded neighborhood of $0 \in \mathbb{R}^q$. We define the bounded open set $G \subset E$ by

$$G = \{(u,v) \in E : u : [0,T] \to \Omega,\ v : [0,T] \to \Lambda\}. \tag{35}$$

For $(u,v) \in \bar{G}$, define as in Subsection 4.2,

$$u(t,\lambda) = \lambda u(t) + (1-\lambda)u(T),\ 0 \leq \lambda \leq 1, \tag{36}$$

and let

$$a(t,\lambda) = \lambda t + (1-\lambda)T,\ 0 \leq \lambda \leq 1. \tag{37}$$

For $0 \le \lambda \le 1,\ 0 \le \epsilon \le 1$ define $S = (S_1, S_2) : E \times [0,1] \times [0,1] \to E$ by

$$\begin{array}{l} S_1(u,v,\lambda,\epsilon)(t) = u(T) + \epsilon \int_0^{a(t,\lambda)} f(s,u(s,\lambda),\lambda v(s))ds \\ S_2(u,v,\lambda,\epsilon)(t) = e^{Bt}v(T) + \lambda\epsilon e^{Bt}\int_0^t e^{-Bs}h(s,u(s),v(s))ds \end{array} . \tag{38}$$

Then S is a completely continuous mapping and the theorem will be proved once we show that

$$\mathrm{d}(\mathrm{id} - S(\cdot,\cdot,1,\epsilon), G, 0) \neq 0, \tag{39}$$

for ϵ sufficiently small, for if this is the case, $S(\cdot,1,\epsilon)$ has a fixed point in G which is equivalent to the assertion of the theorem.

To show that (39) holds we first show that $S(\cdot,\lambda,\epsilon)$ has no zeros on ∂G for all $\lambda \in [0,1]$ and ϵ sufficiently small. This we argue in a manner similar to the proof of Theorem 7. Hence

$$\mathrm{d}(\mathrm{id} - S(\cdot,\cdot,0,\epsilon), G, 0) = \mathrm{d}(\mathrm{id} - S(\cdot,\cdot,1,\epsilon), G, 0)$$

by the homotopy invariance property of Leray-Schauder degree, for all ϵ sufficiently small. On the other hand

$$\begin{array}{rl} & \mathrm{d}(\mathrm{id} - S(\cdot,\cdot,0,\epsilon), G, 0) \\ = & \mathrm{d}((-\epsilon\int_0^T f(s,\cdot,0)ds, \mathrm{id} - S_2(\cdot,\cdot 0,\epsilon), G, (0,0)) \\ = & \mathrm{d}(-\epsilon\int_0^T f(s,\cdot,0)ds, \Omega, 0)\mathrm{d}(\mathrm{id} - S(\cdot,\cdot 0,\epsilon), \tilde{G}, 0) \\ = & \mathrm{sgn}\ \det(I - e^{BT})\mathrm{d}(-\epsilon\int_0^T f(s,\cdot,0)ds, \Omega, 0) \neq 0, \end{array}$$

where $\tilde{G} = C([0,T],\Lambda)$, completing the proof. □

As before, we obtain the following corollary.

11 Corollary *Assume the hypotheses of Theorem 10 and assume that all possible solutions u, v, for $0 < \epsilon \le 1$, of equation (32) are such that $u, v \notin \partial G$, where G is given by (36). Then (36) has a $T-$ periodic solution.*

5 Exercises

1. Consider equation (1) with A a constant matrix. Give conditions that $I - e^{TR}$ be nonsingular, where $I - e^{TR}$ is given as in Theorem 3. Use Theorem 5 to show that equation (1) has a $T-$periodic solution provided the set of $T-$periodic solutions of (10) is a priori bounded for $0 \le \epsilon < 1$.

2. Prove Corollary 8.

3. Let f satisfy for some $R > 0$

$$f(t,x) \cdot x \neq 0,\ |x| = R,\ 0 \le t \le T.$$

Prove that (11) has a $T-$periodic solution u with $|u(t)| < R,\ 0 \le t \le T$.

4. Let $\Omega \subset \mathbb{R}^N$ be an open convex set with $0 \in \Omega$ and let f satisfy

$$f(t,x) \cdot n(x) \neq 0, \ x \in \partial\Omega, \ 0 \leq t \leq T,$$

where for each $x \in \partial\Omega$, $n(x)$ is an outer normal vector to Ω at x. Prove that (11) has a $T-$periodic solution $u : [0,T] \to \Omega$.

5. Verify inequality (27).

6. Let $x \in C^2[0,\infty)$. Use Taylor expansions to prove Landau's inequality

$$\|x'\|_\infty^2 \leq 4\|x\|_\infty \|x''\|_\infty.$$

7. Complete the details in the proof of Theorem 10.

8. Assume that the unforced Liénard equation (i.e. equation (21) with $e \equiv 0$) has a nontrivial $T-$periodic solution x. Show that $T \geq 2\pi$.

Chapter VIII
Stability Theory

1 Introduction

In Chapter VI we studied in detail linear and perturbed linear systems of differential equations. In the case of constant or periodic coefficients we found criteria which describe the asymptotic behavior of solutions (viz. Proposition 7 and Exercise 10 of Chapter **??**.) In this chapter we shall consider similar problems for general systems of the from

$$u' = f(t, u), \tag{1}$$

where

$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a continuous function.

If $u : [t_0, \infty) \to \mathbb{R}^N$ is a given solution of (1), then discussing the behavior of another solution v of this equation relative to the solution u, i.e. discussing the behavior of the difference $v - u$ is equivalent to studying the behavior of the solution $z = v - u$ of the equation

$$z' = f(t, z + u(t)) - f(t, u(t)), \tag{2}$$

relative to the trivial solution $z \equiv 0$. Thus we may, without loss in generality, assume that (1) has the trivial solution as a reference solution, i.e.

$$f(t, 0) \equiv 0,$$

an assumption we shall henceforth make.

2 Stability Concepts

There are various stability concepts which are important in the asymptotic behavior of systems of differential equations. We shall discuss here some of them and their interrelationships.

1 Definition *We say that the trivial solution of (1) is:*

(i) stable (s) on $[t_0, \infty)$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that any solution v of (1) with $|v(t_0)| < \delta$ exists on $[t_0, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_0 \leq t < \infty$;

(ii) asymptotically stable (a.s) on $[t_0, \infty)$, if it is stable and $\lim_{t\to\infty} v(t) = 0$, where v is as in (i);

(iii) unstable (us), if it is not stable;

(iv) uniformly stable (u.s) on $[t_0, \infty)$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that any solution v of (1) with $|v(t_1)| < \delta,\ t_1 \geq t_0$ exists on $[t_1, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_1 \leq t < \infty$;

(v) uniformly asymptotically stable (u.a.s), if it is uniformly stable and there exists $\delta > 0$ such that for all $\epsilon > 0$ there exists $T = T(\epsilon)$ such that any solution v of (1) with $|v(t_1)| < \delta,\ t_1 \geq t_0$ exists on $[t_1, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_1 + T \leq t < \infty$;

(v) strongly stable (s.s) on $[t_0, \infty)$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that any solution v of (1) with $|v(t_1)| < \delta$ exists on $[t_0, \infty)$ and satisfies $|v(t)| < \epsilon,\ t_0 \leq t < \infty$.

2 Proposition *The following implications are valid:*

$$\begin{array}{ccccc} & & \text{u.a.s} & \Rightarrow & \text{a.s} \\ & & \Downarrow & & \Downarrow \\ \text{s.s} & \Rightarrow & \text{u.s} & \Rightarrow & \text{s.} \end{array}$$

If the equation (1) is autonomous, i.e. f is independent of t, then the above implications take the from

$$\begin{array}{ccc} \text{u.a.s} & \Leftrightarrow & \text{a.s} \\ \Downarrow & & \Downarrow \\ \text{u.s} & \Leftrightarrow & \text{s.} \end{array}$$

The following examples of scalar differential equations will serve to illustrate the various concepts.

3 Example

1. *The zero solution of $u' = 0$ is stable but not asymptotically stable.*

2. *The zero solution of $u' = u^2$ is unstable.*

3. *The zero solution of $u' = -u$ is uniformly asymptotically stable.*

4. *The zero solution of $u' = a(t)u$ is asymptotically stable if and only if $\lim_{t\to\infty} \int_{t_0}^{t} a(s)ds = -\infty$. It is uniformly stable if and only if $\int_{t_1}^{t} a(s)ds$ is bounded above for $t \geq t_1 \geq t_0$. Letting $a(t) = \sin\log t + \cos\log t - \alpha$ one sees that asymptotic stability holds but uniform stability does not.*

3 Stability of Linear Equations

In the case of a linear system ($A \in C(\mathbb{R} \to \mathfrak{L}(\mathbb{R}^N, \mathbb{R}^N))$)

$$u' = A(t)u, \tag{3}$$

a particular stability property of any solution is equivalent to that stability property of the trivial solution. Thus one may ascribe that property to the equation and talk about the equation (3) being stable, uniformly stable, etc. The stability concepts may be expressed in terms of conditions imposed on a fundamental matrix Φ.

4 **Theorem** *Let Φ be a fundamental matrix solution of (3). Then equation (3) is :*

(i) stable if and only if there exists $K > 0$ such that

$$|\Phi(t)| \le K, \; t_0 \le t < \infty; \tag{4}$$

(ii) uniformly stable if and only if there exists $K > 0$ such that

$$|\Phi(t)\Phi^{-1}(s)| \le K, \; t_0 \le s \le t < \infty; \tag{5}$$

(iii) strongly stable if and only if there exists $K > 0$ such that

$$|\Phi(t)| \le K, \; |\Phi^{-1}(t)| \le K, \; t_0 \le t < \infty; \tag{6}$$

(iv) asymptotically stable if and only if

$$\lim_{t\to\infty} |\Phi(t)| = 0; \tag{7}$$

(v) uniformly asymptotically stable if and only if there exist $K > 0, \; \alpha > 0$ such that

$$|\Phi(t)\Phi^{-1}(s)| \le Ke^{-\alpha(t-s)}, \; t_0 \le s \le t < \infty. \tag{8}$$

PROOF. We shall demonstrate the last part of the theorem and leave the demonstration of the remaining part as an exercise. We may assume without loss in generality that $\Phi(t_0) = I$, the $N \times N$ identity matrix, since conditions (4) - (8) will hold for any fundamental matrix if and only if they hold for a particular one. Thus, let us assume that (8) holds. Then the second part of the theorem guarantees that the equation is uniformly stable. Moreover, for each ϵ, $0 < \epsilon < K$, if we put $T = -\frac{1}{\alpha}\log\frac{\epsilon}{K}$, then for $\xi \in \mathbb{R}^N, \; |\xi| \le 1$ we have $|\Phi(t)\Phi^{-1}(t_1)\xi| \le Ke^{-\alpha(t-t_1)}$, if $t_1 + T < t$. Thus we have uniform asymptotic stability.

Conversely, if the equation is uniformly asymptotically stable, then there exists $\delta > 0$ and for all $\epsilon, \; 0 < \epsilon < \delta$, there exists $T = T(\epsilon) > 0$ such that if $|\xi| < \delta$, then

$$|\Phi(t)\Phi^{-1}(t_1)\xi| < \epsilon, \; t \ge t_1 + T, \; t_1 \ge t_0.$$

In particular

$$|\Phi(t+T)\Phi^{-1}(t)\xi| < \epsilon,\ |\xi| < \delta,$$

or

$$|\Phi(t+T)\Phi^{-1}(t)\frac{\xi}{\delta}| < \frac{\epsilon}{\delta}$$

and thus

$$|\Phi(t+T)\Phi^{-1}(t)| \le \frac{\epsilon}{\delta} < 1,\ t \ge t_0.$$

Furthermore, since we have uniform stability,

$$|\Phi(t+h)\Phi^{-1}(t)| \le K,\ t_0 \le t,\ 0 \le h \le T.$$

If $t \ge t_1$, we obtain for some integer n, that $t_1 + nT \le t < t_1 + (n+1)T$, and

$$\begin{aligned} |\Phi(t)\Phi^{-1}(t_1)| \quad & \le |\Phi(t)\Phi^{-1}(t_1+nT)||\Phi(t_1+nT)\Phi^{-1}(t_1)| \\ & \le K|\Phi(t_1+nT)\Phi^{-1}(t_1+(n-1)T)| \\ & \cdots |\Phi(t_1+T)\Phi^{-1}(t_1)| \\ & \le K\left(\tfrac{\epsilon}{\delta}\right)^n. \end{aligned}$$

Letting $\alpha = -\frac{1}{T}\log\frac{\epsilon}{\delta}$, we get

$$\begin{aligned} |\Phi(t)\Phi^{-1}(t_1)| \le Ke^{-n\alpha T} \quad & = Ke^{-n\alpha T}e^{-\alpha T}\tfrac{\delta}{\epsilon} \\ & < K\tfrac{\delta}{\epsilon}e^{-\alpha(t-t_1)},\ t \ge t_1 \ge t_0. \end{aligned}$$

$\square$

In the case that the matrix A is independent of t one obtains the following corollary.

5 Corollary *Equation (3) is stable if and only if every eigenvalue of A has non-positive real part and those with zero real part are semisimple. It is strongly stable if and only if all eigenvalues of A have zero real part and are semisimple. It is asymptotically stable if and only if all eigenvalues have negative real part.*

Using the Abel-Liouville formula, we obtain the following result.

6 Theorem *Equation (3) is unstable whenever*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\mathrm{trace}A(s)ds = \infty. \tag{9}$$

If (3) is stable, then it is strongly stable if and only if

$$\liminf_{t\to\infty}\int_{t_0}^{t}\mathrm{trace}A(s)ds > -\infty. \tag{10}$$

Additional stability criteria for linear systems abound. The following concept of the *measure* of a matrix due to Lozinskii and Dahlquist (see [6]) is particularly useful in numerical computations. We provide a brief discussion.

7 Definition *For an $N \times N$ matrix A we define*

$$\mu(A) = \lim_{h \to 0+} \frac{|I + hA| - |I|}{h}, \tag{11}$$

where $|\cdot|$ is a matrix norm induced by a norm $|\cdot|$ in $\mathbb{R}^N$ and I is the $N \times N$ identity matrix.

We have the following proposition:

8 Proposition *For any $N \times N$ matrix A, $\mu(A)$ exists and satisfies:*

1. $\mu(\alpha A) = \alpha \mu(A), \ \alpha \geq 0;$
2. $|\mu(A)| \leq |A|;$
3. $\mu(A + B) \leq \mu(A) + \mu(B);$
4. $|\mu(A) - \mu(B)| \leq |A - B|.$

If u is a solution of equation (3), then the function $r(t) = |u(t)|$ has a right derivative $r'_+(t)$ at every point t for every norm $|\cdot|$ in $\mathbb{R}^N$ and $r'_+(t)$ satisfies

$$r'_+(t) - \mu(A(t))r(t) \leq 0. \tag{12}$$

Using this inequality we obtain the following proposition.

9 Proposition *Let $A : [t_0, \infty) \to \mathbb{R}^{N \times N}$ be a continuous matrix and let u be a solution of (3) on $[t_0, \infty)$. Then*

$$|u(t)| e^{-\int_{t_0}^t \mu(A(s))ds}, \ t_0 \leq t < \infty \tag{13}$$

is a nonincreasing function of t and

$$|u(t)| e^{\int_{t_0}^t \mu(-A(s))ds}, \ t_0 \leq t < \infty \tag{14}$$

is a nondecreasing function of t. Furthermore

$$|u(t_0)| e^{-\int_{t_0}^t \mu(-A(s))ds} \leq |u(t)| \leq |u(t_0)| e^{\int_{t_0}^t \mu(A(s))ds}. \tag{15}$$

This proposition has, for constant matrices A, the immediate corollary.

10 Corollary *For any $N \times N$ constant matrix A the following inequalities hold.*

$$e^{-t\mu(-A)} \leq \left| e^{tA} \right| \leq e^{t\mu(A)}.$$

The following theorem provides stability criteria for the system (3) in terms of conditions on the measure of the coefficient matrix. Its proof is again left as an exercise.

11 Theorem *The system (3) is:*

1. *unstable, if*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \mu(-A(s))ds = -\infty;$$

2. *stable, if*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \mu(A(s))ds < \infty;$$

3. *asymptotically stable, if*

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu(-A(s))ds = -\infty;$$

4. *uniformly stable, if*

$$\mu(A(t)) \leq 0, \ t \geq t_0;$$

5. *uniformly asymptotically stable, if*

$$\mu(A(t)) \leq -\alpha < 0, \ t \geq t_0.$$

4 Stability of Nonlinear Equations

In this section we shall consider stability properties of nonlinear equations of the form

$$u' = A(t)u + f(t,u), \tag{16}$$

where

$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a continuous function with

$$|f(t,x)| \leq \gamma(t)|x|, \ x \in \mathbb{R}^N, \tag{17}$$

where γ is some positive continuous function and A is a continuous $N \times N$ matrix defined on $\mathbb{R}$. The results to be discussed are consequences of the variation of constants formula (Proposition VI.6) and the stability theorem Theorem 4. We shall only present a sample of results. We refer to [6], [4], [13], and [14], where further results are given. See also the exercises below.

Throughout this section $\Phi(t)$ will denote a fundamental matrix solution of the homogeneous (unperturbed) linear problem (3). We hence know that if u is a solution of equation (16), then u satisfies the integral equation

$$u(t) = \Phi(t)\left(\Phi^{-1}(t_0)u(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)f(s,u(s))ds\right). \tag{18}$$

The following theorem will have uniform and strong stability as a consequence.

12 Theorem *Let f satisfy (17) with $\int^\infty \gamma(s)ds < \infty$. Further assume that*

$$|\Phi(t)\Phi^{-1}(s)| \le K,\ t_0 \le s \le t < \infty.$$

Then there exists a positive constant $L = L(t_0)$ such that any solution u of (16) is defined for $t \ge t_0$ and satisfies

$$|u(t)| \le L|u(t_1)|,\ t \ge t_1 \ge t_0.$$

If in addition $\Phi(t) \to 0$ as $t \to \infty$, then $u(t) \to 0$ as $t \to \infty$.

Proof. If u is a solution of of (16) it also satisfies (18). Hence

$$|u(t)| \le K|u(t_1)| + K\int_{t_1}^t \gamma(s)|u(s)|ds,\ t \ge t_1,$$

and

$$|u(t)| \le K|u(t_1)|e^{K\int_{t_1}^t \gamma(s)ds} \le L|u(t_1)|,\ t \ge t_1,$$

where

$$L = Ke^{K\int_{t_0}^\infty \gamma(s)ds}.$$

To prove the remaining part of the theorem, we note from (1) that

$$\begin{aligned} |u(t)| &\le |\Phi(t)| \left(|\Phi^{-1}(t_0)u(t_0)| + |\textstyle\int_{t_0}^{t_1} \Phi^{-1}(s)f(s,u(s))ds|\right) \\ &\quad + |\textstyle\int_{t_1}^t \Phi(t)\Phi^{-1}(s)f(s,u(s))ds| \end{aligned}$$

Now

$$\left|\int_{t_1}^t \Phi(t)\Phi^{-1}(s)f(s,u(s))ds\right| \le KL|u(t_0)|\int_{t_1}^\infty \gamma(s)ds.$$

This together with the fact that $\Phi(t) \to 0$ as $t \to \infty$ completes the proof. □

13 Remark It follows from Theorem 4 that the conditions of the above theorem imply that the perturbed system (16) is uniformly (asymptotically) stable whenever the unperturbed system is uniformly (asymptotically) stable. The conditions of Theorem 12 also imply that the zero solution of the perturbed system is strongly stable, whenever the unperturbed system is strongly stable.

A further stability result is the following. Its proof is delegated to the exercises.

14 Theorem *Assume that there exist $K > 0,\ \alpha > 0$ such that*

$$|\Phi(t)\Phi^{-1}(s)| \le Ke^{-\alpha(t-s)},\ t_0 \le s \le t < \infty. \tag{19}$$

and f satisfies (17) with $\gamma < \frac{\alpha}{K}$ a constant. Then any solution u of (16) is defined for $t \ge t_0$ and satisfies

$$|u(t)| \le Ke^{-\beta(t-t_1)}|u(t_1)|,\ t \ge t_1 \ge t_0,$$

where $\beta = \alpha - \gamma K > 0$.

15 **Remark** It again follows from Theorem 4 that the conditions of the above theorem imply that the perturbed system (16) is uniformly asymptotically stable whenever the unperturbed system is uniformly asymptotically stable.

5 Lyapunov Stability

5.1 Introdcution

In this section we shall introduce some geometric ideas, which were first formulated by Poincaré and Lyapunov, about the stability of constant solutions of systems of the form

$$u' = f(t, u), \tag{20}$$

where

$$f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a continuous function. As observed earlier, we may assume that $f(t, 0) = 0$ and we discuss the stability properties of the trivial solution $u \equiv 0$. The geometric ideas amount to constructing level surfaces in $\mathbb{R}^N$ which shrink to 0 and which have the property that orbits associated with (20) cross these level surfaces transversally toward the origin, thus being guided to the origin. To illustrate, let us consider the following example.

$$\begin{aligned} x' &= ax - y + kx\left(x^2 + y^2\right) \\ y' &= x - ay + ky\left(x^2 + y^2\right), \end{aligned} \tag{21}$$

where a is a constant $|a| < 1$. Obviously $x = 0 = y$ is a stationary solution of (21). Now consider the family of curves in $\mathbb{R}^2$ given by

$$v(x, y) = x^2 - 2axy + y^2 = \text{constant}, \tag{22}$$

a family of ellipses which share the origin as a common center. If we consider an orbit $\{(x(t), y(t)) : t \geq t_0\}$ associated with (21), then as t varies, the orbit will cross members of the above family of ellipses. Computing the scalar product of the tangent vector of an orbit with the gradient to an ellipse we find:

$$\nabla v \cdot (x', y') = 2k\left(x^2 + y^2\right)\left(x^2 + y^2 - 2axy\right),$$

which is clearly negative, whenever k is (and positive if k is). Of course, we may view $v(x(t), y(t))$ as a norm of the point $(x(t), y(t))$ and $\nabla v \cdot (x', y') = \frac{d}{dt} v(x(t), y(t))$. Thus, if $k < 0$, $v(x(t), y(t))$ will be a strictly decreasing function and hence should approach a limit as $t \to \infty$. That this limit must be zero follows by an indirect argument. I.e. the orbit tends to the origin and the zero solution attracts all orbits, i.e. it appears asymptotically stable.

5.2 Lyapunov functions

Let

$$v : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R},\ v(t,0) = 0,\ t \in \mathbb{R}$$

be a continuous functional. We shall introduce the following terminology.

16 Definition *The functional v is called:*

1. *positive definite, if there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$, with $\phi(0) = 0,\ \phi(r) \neq 0,\ r \neq 0$ and*

$$\phi(|x|) \leq v(t,x),\ x \in \mathbb{R}^N,\ t \geq t_0;$$

2. *radially unbounded, if it is positive definite and*

$$\lim_{r\to\infty} \phi(r) = \infty;$$

3. *decrescent, if it is positive definite and there exists a continuous increasing function $\psi : [0,\infty) \to [0,\infty)$, with $\psi(0) = 0$, and*

$$\psi(|x|) \geq v(t,x),\ x \in \mathbb{R}^N,\ t \geq t_0.$$

We have the following stability criteria.

17 Theorem *Let there exist a positive definite functional v and $\delta_0 > 0$ such that for every solution u of (20) with $|u(t_0)| \leq \delta_0$, the function $v^*(t) = v(t,u(t))$ is nonincreasing with respect to t, then the trivial solution of (20) is stable.*

Proof. Let $0 < \epsilon \leq \delta_0$ be given and choose $\delta = \delta(\epsilon)$ such that $v(t_0,u_0) < \phi(\epsilon)$, for $|u_0| < \delta$. Let u be a solution of (20) with $u(t_0) = u_0$. Then $v^*(t) = v(t,u(t))$ is nonincreasing with respect to t, and hence

$$v^*(t) \leq v^*(t_0) = v(t_0,u_0).$$

Therefore

$$\phi(|u(t)|) \leq v(t,u(t)) = v^*(t) \leq v^*(t_0) = v(t_0,u_0) < \phi(\epsilon).$$

Since ϕ is nondecreasing, the result follows. □

If v is also decrescent we obtain the stronger result.

18 Theorem *Let there exist a positive definite functional v which is decrescent and $\delta_0 > 0$ such that for every solution u of (20) with $|u(t_1)| \leq \delta_0,\ t_1 \geq t_0$ the function $v^*(t) = v(t,u(t))$ is nonincreasing with respect to t, then the trivial solution of (20) is uniformly stable.*

PROOF. We have already shown that the trivial solution is stable. Let $0 < \epsilon \le \delta_0$ be given and let $\delta = \psi^{-1}(\phi(\epsilon))$. Let u be a solution of (20) with $u(t_1) = u_0$ with $|u_0| < \delta$. Then

$$\begin{aligned} \phi(|u(t)|) \le v(t, u(t)) \quad &= v^*(t) \le v^*(t_1) \\ &= v(t_1, u_0) \le \psi(|u_0|) \\ &< \psi(\delta) = \phi(\epsilon). \end{aligned}$$

The result now follows from the monotonicity assumption on ϕ. $\square$

As an example we consider the two dimensional system

$$\begin{aligned} x' &= a(t)y + b(t)x\left(x^2 + y^2\right) \\ y' &= -a(t)x + b(t)y\left(x^2 + y^2\right), \end{aligned} \tag{23}$$

where the coefficient functions a and b are continuous with $b \le 0$. We choose $v(x, y) = x^2 + y^2$, then, since

$$\frac{dv^*}{dt} = \frac{\partial v}{\partial t} + \nabla v \cdot f(t, u).$$

we obtain

$$\frac{dv^*}{dt} = 2b(t)\left(x^2 + y^2\right)^2 \le 0.$$

Since v is positive definite and decrescent, it follows from Theorem 18 that the trivial solution of (23) is uniformly stable.

We next prove an instability theorem.

19 Theorem *Assume there exists a continuous functional*

$$v : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

with the properties:

1. *there exists a continuous increasing function* $\psi : [0, \infty) \to [0, \infty)$, *such that* $\psi(0) = 0$ *and*

$$|v(t, x)| \le \psi(|x|);$$

2. *for all* $\delta > 0$, *and* $t_1 \ge t_0$, *there exists* x_0, $|x_0| < \delta$ *such that* $v(t_1, x_0) < 0$;

3. *if* u *is any solution of (23) with* $u(t) = x$, *then*

$$\lim_{h \to 0+} \frac{v(t + h, u(t + h)) - v(t, x)}{h} \le -c(|x|),$$

where c *is a continuous increasing function with* $c(0) = 0$.

Then the trivial solution of (23) is unstable.

PROOF. Assume the trivial solution is stable. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that any solution u of (23) with $|u(t_0)| < \delta$ exists on $[t_0, \infty)$ and satisfies $|u(t)| < \epsilon,\ t_0 \le t < \infty$. Choose $x_0,\ |x_0| < \delta$ such that $v(t_0, x_0) < 0$ and let u be a solution with $u(t_0) = x_0$. Then

$$|v(t, u(t))| \le \psi(|u(t)|) \le \psi(\epsilon). \tag{24}$$

The third property above implies that $v(t, u(t))$ is nonincreasing and hence for $t \ge t_0$,

$$v(t, u(t)) \le v(t_0, x_0) < 0.$$

Thus

$$|v(t_0, x_0)| \le \psi(|u(t)|)$$

and

$$\psi^{-1}(|v(t_0, x_0)|) \le |u(t)|.$$

We therefore have

$$v(t, u(t)) \le v(t_0, x_0) - \int_{t_0}^{t} c(|u(s)|) ds,$$

which by (24) implies that

$$v(t, u(t)) \le v(t_0, x_0) - (t - t_0) c(\psi^{-1}(|v(t_0, x_0)|),$$

and thus

$$\lim_{t\to\infty} v(t, u(t)) = -\infty,$$

contradicting (24). □

We finally develop some asymptotic stability criteria.

20 Theorem *Let there exist a positive definite functional $v(t, x)$ such that*

$$\frac{dv^*}{dt} = \frac{dv(t, u(t))}{dt} \le -c(v(t, u(t)),$$

for every solution u of (20) with $|u(t_0)| \le \delta_0$, with c a continuous increasing function and $c(0) = 0$. Then the trivial solution is asymptotically stable. If v is also decrescent. Then the asymptotic stability is uniform.

PROOF. The hypotheses imply that the trivial solution is stable, as was proved earlier. Hence, if u is a solution of (20) with $|u(t)| \le \delta_0$, then

$$v_0 = \lim_{t\to\infty} v(t, u(t))$$

exists. An easy indirect argument shows that $v_0 = 0$. Hence, since v is positive definite,

$$\lim_{t\to\infty} \phi(|u(t)|) = 0,$$

implying that

$$\lim_{t\to\infty} |u(t)| = 0,$$

since ϕ is increasing. The proof that the trivial solution is uniformly asymptotically stable, whenever v is also decrescent, is left as an exercise. $\square$

In the next section we shall employ the stability criteria just derived, to, once more study perturbed linear systems, where the associated linear system is a constant coefficient system. This we do by showing how appropriate Lyapunov functionals may be constructed. The construction is an exercise in linear algebra.

5.3 Perturbed linear systems

Let A be a constant $N \times N$ matrix and let $g : [t_0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function such that

$$g(t, x) = o(|x|),$$

uniformly with respect to $t \in [t_0, \infty)$. We shall consider the equation

$$u' = Au + g(t, u) \tag{25}$$

and show how to construct Lyapunov functionals to test the stability of the trivial solution of this system.

The type of Lyapunov functional we shall be looking for are of the form

$$v(x) = x^T Bx,$$

where B is a constant $N \times N$ matrix, i.e. we are looking for v as a quadratic form. If u is a solution of (25) then

$$\begin{aligned} \frac{dv^*}{dt} &= \frac{dv(t, u(t))}{dt} \\ &= u^T \left(A^T B + BA\right) u + g^T(t, u)Bu + u^T Bg(t, u). \end{aligned} \tag{26}$$

Hence, given A, if B can be found so that $C = A^T B + BA$ has certain definiteness properties, then the results of the previous section may be applied to determine stability or instability of the trivial solution. To proceed along these lines we need some linear algebra results.

21 Proposition *Let A be a constant $N \times N$ matrix having the property that for any eigenvalue λ of A, $-\lambda$ is not an eigenvalue. Then for any $N \times N$ matrix C, there exists a unique $N \times N$ matrix B such that $C = A^T B + BA$.*

Proof. On the space of $N \times N$ matrices define the bounded linear operator L by

$$L(B) = A^T B + BA.$$

Then L may be viewed as a bounded linear operator of $\mathbb{R}^{N\times N}$ to itself, hence it will be a bijection provided it does not have 0 as an eigenvalue. Once we show the latter, the result follows. Thus let μ be an eigenvalue of L, i.e., there exists a nonzero matrix B such that

$$L(B) = A^T B + BA = \mu B.$$

Hence

$$A^T B + B(A - \mu I) = 0.$$

From this follows

$$B(A - \mu I)^n = \left(-A^T\right)^n B,$$

for any integer $n \geq 1$, hence for any polynomial p

$$Bp(A - \mu I) = p(-A^T)B. \tag{27}$$

Since, on the other hand, if F and G are two matrices with no common eigenvalues, there exists a polynomial p such that $p(F) = I,\ p(G) = 0$, (27) implies that $A - \mu I$ and $-A^T$ must have a common eigenvalue. From which follows that μ is the sum of two eigenvalues of A, which, by hypothesis cannot equal 0. □

This proposition has the following corollary.

22 Corollary *Let A be a constant $N \times N$ matrix. Then for any $N \times N$ matrix C, there exists $\mu > 0$ and a unique $N \times N$ matrix B such that $2\mu B + C = A^T B + BA$.*

Proof. Let

$$S = \{\lambda \in \mathbb{C} : \lambda = \lambda_1 + \lambda_2\},$$

where λ_1 and λ_2 are eigenvalues of A. Since S is a finite set, there exists $r_0 > 0$, such that $\lambda(\neq 0) \in S$ implies that $|\lambda| > r_0$. Choose $0 < \mu \leq r_0$ and consider the matrix $A_1 = A - \mu I$. We may now apply Proposition 21 to the matrix A_1 and find for a given matrix C, a unique matrix B such that $C = A_1^T B + BA_1$, i.e. $2\mu BC = A^T B + BA$. □

23 Corollary *Let A be a constant $N \times N$ matrix having the property that all eigenvalues λ of A have negative real parts. Then for any negative definite $N \times N$ matrix C, there exists a unique positive definite $N \times N$ matrix B such that $C = A^T B + BA$.*

Proof. Let C be a negative definite matrix and let B be given by Proposition 21, which may be applied since all eigenvalues of A have negative real part. Let $v(x) = x^T Bx$, and let u be a solution of $u' = Au,\ u(0) = x_0 \neq 0$. Then

$$\frac{dv^*}{dt} = \frac{dv(u(t))}{dt} = u^T\left(A^T B + BA\right)u = u^T Cu \leq -\mu|u|^2,$$

since C is negative definite. Since $\lim_{t\to\infty} u(t) = 0$ (all eigenvalues of A have negative real part!), it follows that $\lim_{t\to\infty} v(u(t)) = 0$. We also have

$$v(u(t)) \leq v(x_0) - \int_0^t \mu |u(s)|^2 ds,$$

from which follows that $v(x_0) > 0$. Hence B is positive definite. $\square$

The next corollary follows from stability theory for linear equations and what has just been discussed.

24 Corollary *A necessary and sufficient condition that an $N \times N$ matrix A have all of its eigenvalues with negative real part is that there exists a unique positive definite matrix B such that*

$$A^T B + BA = -I.$$

We next consider the nonlinear problem (25) with

$$g(t,x) = o(|x|), \tag{28}$$

uniformly with respect to $t \in [t_0, \infty)$, and show that for certain types of matrices A the trivial solution of the perturbed system has the same stability property as that of the unperturbed problem. The class of matrices we shall consider is the following.

25 Definition *We call an $N \times N$ matrix A <u>critical</u> if all its eigenvalues have non-positive real part and there exits at least one eigenvalue with zero real part. We call it <u>noncritical</u> otherwise.*

26 Theorem *Assume A is a noncritical $N \times N$ matrix and let g satisfy (28). Then the stability behavior of the trivial solution of (25) is the same as that of the trivial solution of $u' = Au$, i.e the trivial solution of (25) is uniformly asymptotically stable if all eigenvalues of A have negative real part and it is unstable if A has an eigenvalue with positive real part.*

Proof. (i) Assume all eigenvalues of A have negative real part. By the above exists a unique positive definite matrix B such that

$$A^T B + BA = -I.$$

Let $v(x) = x^T Bx$. Then v is positive definite and if u is a solution of (25) it satisfies

$$\begin{aligned} \frac{dv^*}{dt} &= \frac{dv(u(t))}{dt} \\ &= -|u(t)|^2 + g^T(t,u)Bu + u^T Bg(t,u). \end{aligned} \tag{29}$$

Now

$$|g^T(t,u)Bu + u^T Bg(t,u)| \leq 2|g(t,u)||B||u|.$$

Choose $r > 0$ such that $|x| \leq r$ implies

$$|g(t,u)| \leq \frac{1}{4}|B|^{-1}|u|,$$

then

$$\frac{dv^*}{dt} \leq -\frac{1}{2}|u(t)|^2,$$

as long as $|u(t)| \leq r$. The result now follows from Theorem 20.

(ii) Let A have an eigenvalue with positive real part. Then there exists $\mu > 0$ such that $2\mu < |\lambda_i + \lambda_j|$, for all eigenvalues $\lambda_i,\ \lambda_j$ of A, and a unique matrix B such that

$$A^T B + BA = 2\mu B - I,$$

as follows from Corollary 22. We note that B cannot be positive definite nor positive semidefinite for otherwise we must have, letting $v(x) = x^T Bx$,

$$\frac{dv^*}{dt} = 2\mu v^*(t) - |u(t)|^2,$$

or

$$e^{-2\mu t}v^*(t) - v^*(0) = -\int_0^t e^{-2\mu s}|u(s)|^2 ds$$

i.e

$$0 \leq v^*(0) - \int_0^t e^{-2\mu s}|u(s)|^2 ds$$

for any solution u, contradicting the fact that solutions u exist for which

$$\int_0^t e^{-2\mu s}|u(s)|^2 ds$$

becomes unbounded as $t \to \infty$ (see Chapter **??**). Hence there exists $x_0 \neq 0$, of arbitrarily small norm, so that $v(x_0) < 0$. Let u be a solution of (25). If the trivial solution were stable, then $|u(t)| \leq r$ for some $r > 0$. Again letting $v(x) = x^T Bx$ we obtain

$$\frac{dv^*}{dt} = 2\mu v^*(t) - |u(t)|^2 + g^T(t,u)Bu + u^T Bg(t,u).$$

We can choose r so small that

$$2|g(t,u)||B||u| \leq \frac{1}{2}|u|^2,\ |u| \leq r,$$

hence

$$e^{-2\mu t}v^*(t) - v^*(0) \leq -\frac{1}{2}\int_0^t e^{-2\mu s}|u(s)|^2 ds,$$

i.e.

$$v^*(t) \le e^{2\mu t} v^*(0) \to -\infty,$$

contradicting that $v^*(t)$ is bounded for bounded u. Hence u cannot stay bounded, and we have instability. $\square$

If it is the case that the matrix A is a critical matrix, the trivial solution of the linear system may still be stable or it may be unstable. In either case, one may construct examples, where the trivial solution of the perturbed problem has either the same or opposite stability behavior as the unperturbed system.

6 Exercises

1. Prove Proposition 2.
2. Verify the last part of Example 3.
3. Prove Theorem 4.
4. Establish a result similar to Corollary 5 for linear periodic systems.
5. Prove Theorem 6.
6. Show that the zero solution of the scalar equation

   $$x'' + a(t)x = 0$$

   is strongly stable, whenever it is stable.
7. Prove Proposition 8.
8. Establish inequality (12).
9. Prove Proposition 9.
10. Prove Corollary 22.
11. Prove Theorem 11.
12. Show that if $\int^\infty \mu(A(s))ds$ exists, then any nonzero solution u of (3) satisfies

    $$\limsup_{t\to\infty} |u(s)| < \infty.$$

    On the other hand, if $\int^\infty \mu(-A(s))ds$ exists, then

    $$0 \neq \liminf_{t\to\infty} |u(s)| \le \infty.$$

13. If A is a constant $N \times N$ matrix show that $\mu(A)$ is an upper bound for the real parts of the eigenvalues of A.

14. Verify the following table:

$\lvert x\rvert$	$\lvert A\rvert$	$\mu(A)$
$\max_i \lvert x^i\rvert$	$\max_i \sum_k \lvert a_{ik}\rvert$	$\max_i \left(a_{ii} + \sum_{k\neq i} \lvert a_{ik}\rvert\right)$
$\sum_i \lvert x^i\rvert$	$\max_k \sum_i \lvert a_{ik}\rvert$	$\max_k \left(a_{kk} + \sum_{i\neq k} \lvert a_{ik}\rvert\right)$
$\sqrt{\sum_i \lvert x^i\rvert^2}$	λ^*	λ_*

where $\lambda^*,\ \lambda_*$ are, respectively, the square root of the largest eigenvalue of A^TA and the largest eigenvalue of $\frac{1}{2}\left(A^T + A\right)$.

15. If the linear system (3) is uniformly (asymptotically) stable and

$$B : [t_0, \infty) \to \mathbb{R}^{N\times N}$$

is continuous and satisfies

$$\int^{\infty} |B(s)|ds < \infty,$$

then the system

$$u' = (A(t) + B(t))u$$

is uniformly (asymptotically) stable.

16. Prove Theorem 14.

17. If the linear system (3) is uniformly asymptotically stable and $B : [t_0, \infty) \to \mathbb{R}^{N\times N}$ is continuous and satisfies

$$\lim_{t\to\infty} |B(t)| = 0,$$

then the system

$$u' = (A(t) + B(t))u$$

is also uniformly asymptotically stable.

18. Complete the proof of Theorem 20.

19. Consider the Liénard oscillator

$$x'' + f(x)x' + g(x) = 0,$$

where f and g are continuous functions with $g(0) = 0$. Assume that there exist $\alpha > 0,\ \beta > 0$ such that

$$\int_0^x g(s)ds < \beta \ \Rightarrow\ |x| < \alpha,$$

and

$$0 < |x| < \alpha \;\Rightarrow\; g(x)F(x) > 0,$$

where $F(x) = \int_0^x f(s)ds$. The equation is equivalent to the system

$$x' = y - F(x),\; y' = -g(x).$$

Using the functional $v(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)ds$ prove that the trivial solution is asymptotically stable.

20. Let a be a positive constant. Use the functional $v(x,y) = \frac{1}{2}\left(y^2 + x^2\right)$ to show that the trivial solution of

$$x'' + a\left(1 - x^2\right)x' + x = 0,$$

is asymptotically stable. What can one say about the zero solution of

$$x'' + a\left(x^2 - 1\right)x' + x = 0,$$

where again a is a positive constant?

21. Prove Corollary 23.

22. Consider the situation of Proposition 21 and assume all eigenvalues of A have negative real part. Show that for given C the matrix B of the Proposition is given by

$$B = \int_0^\infty e^{A^T t} C e^{At} dt.$$

Hint: Find a differential equation that is satisfied by the matrix

$$P(t) = \int_t^\infty e^{A^T(\tau - t)} C e^{A(\tau - t)} d\tau$$

and show that it is a constant matrix.

23. Consider the system

$$\begin{aligned} x' &= y + ax\left(x^2 + y^2\right) \\ y' &= -x + ay\left(x^2 + y^2\right). \end{aligned}$$

Show that the trivial solution is stable if $a > 0$ and unstable if $a < 0$. Contrast this with Theorem 26.

24. Show that the trivial solution of

$$\begin{aligned} x' &= -2y^3 \\ y' &= x \end{aligned}$$

is stable. Contrast this with Theorem 26.

Chapter IX
Invariant Sets

1 Introduction

In this chapter, we shall present some of the basic results about invariant sets for solutions of initial value problems for systems of autonomous (i.e. time independent) ordinary differential equations. We let D be an open connected subset of $\mathbb{R}^N$, $N \geq 1$, and let

$$f : D \to \mathbb{R}^N$$

be a locally Lipschitz continuous mapping.

We consider the initial value problem

$$\begin{aligned} u' &= f(u) \\ u(0) &= u_0 \in D \end{aligned} \tag{1}$$

and seek sufficient conditions on subsets $M \subset D$ for solutions of (1) to have the property that $\{u(t)\} \subset M,\ t \in I$, whenever $u_0 \in M$, where I is the maximal interval of existence of the solution u. We note that the initial value problem (1) is uniquely solvable since f satisfies a local Lipschitz condition (viz. Chapter **??**). Under these assumptions, we have for each $u_0 \in D$ a maximal interval of existence I_{u_0} and a function

$$\begin{aligned} u : I_{u_0} &\to D \\ t &\mapsto u(t, u_0), \end{aligned} \tag{2}$$

i.e., we obtain a mapping

$$\begin{aligned} &u : I_{u_0} \times D \to D \\ &(t, u_0) \mapsto u(t, u_0). \end{aligned} \tag{3}$$

Thus if we let

$$U = \cup_{v \in D} I_v \times \{v\} \subset \mathbb{R} \times \mathbb{R}^N,$$

we obtain a mapping

$$u : U \to D,$$

which has the following properties:

1 Lemma *1. u is continuous,*

2. $$u(0, u_0) = u_0,\ \forall u_0 \in D$$

3. for each $u_0 \in D$, $s \in I_{u_0}$ and $t \in I_{u(s,u_0)}$ we have $s + t \in I_{u_0}$ and $u(s + t, u_0) = u(t, u(s, u_0))$.

A mapping having the above properties is called a *flow* on D and we shall henceforth, in this chapter, use this term freely and call u the *flow determined by f.*

2 Orbits and Flows

Let u be the flow determined by f (with regard to the initial value problem (1)). We shall use the following (standard) convention. If $S \subset I_{u_0} = (t^-_{u_0}, t^+_{u_0})$,

$$u(S, u_0) = \{v : v = u(t, u_0),\ t \in I_{u_0}\}.$$

We shall call

$$\gamma(v) = u(I_v, v)$$

the orbit of v,

$$\gamma^+(v) = u([0, t^+_v), v)$$

the positive semiorbit of v and

$$\gamma^-(v) = u((t^-_v, 0]), v)$$

the negative semiorbit of v.

Furthermore, we call $v \in D$ a stationary or critical point of the flow, whenever $f(v) = 0$. It is, of course immediate, that if $v \in D$ is a stationary point, then $I_v = \mathbb{R}$ and

$$\gamma(v) = \gamma^+(v) = \gamma^-(v) = \{v\}.$$

We call $v \in D$ a periodic point of period T of the flow, whenever there exists $T > 0$, such that $u(0, v) = u(T, v)$. If v is a periodic point which is not a critical point, one calls $T > 0$ its *minimal* period, provided $u(0, v) \neq u(t, v),\ 0 < t < T$.

We have the following proposition.

2 Proposition *Let u be the flow determined by f and let $v \in D$. Then either:*

1. *v is a stationary point;*
2. *v is a periodic point having a minimal positive period;*
3. *the flow $u(\cdot, v)$ is injective;*
4. *if $\gamma^+(v)$ is relatively compact, then $t^+(v) = +\infty$, if $\gamma^-(v)$ is relatively compact, then $t^-(v) = -\infty$, whereas if $\gamma(v)$ is relatively compact, then $I_v = \mathbb{R}$.*

3 Invariant Sets

A subset $M \subset D$ is called positively invariant with respect to the the flow u determined by f, whenever

$$\gamma^+(v) \subset M, \ \forall v \in M,$$

i.e.,

$$\gamma^+(M) \subset M.$$

We similarly call $M \subset D$ negatively invariant provided

$$\gamma^-(M) \subset M,$$

and invariant provided

$$\gamma(M) \subset M.$$

We note that a set M is invariant if and only if it is both positively and negatively invariant.

We have the following proposition:

3 **Proposition** *Let u be the flow determined by f and let $V \subset D$. Then there exists a smallest positively invariant subset M, $V \subset M \subset D$ and there exists a largest invariant set $\tilde{M}$, $\tilde{M} \subset V$. Also there exists a largest negatively invariant subset M, $V \supset M$ and there exists a smallest invariant set $\tilde{M}$, $\tilde{M} \subset V$. As a consequence V contains a largest invariant subset and is contained in a smallest invariant set.*

As a corollary we obtain:

4 **Corollary** *(i) If a set M is positively invariant with respect to the flow u, then so are $\overline{M}$ and $\mathrm{int}M$.*
(ii) A closed set M is positively invariant with respect to the flow u if and only if for every $v \in \partial M$ there exists $\epsilon > 0$ such that $u([0,\epsilon), v) \subset M$.
(iii) A set M is positively invariant if and only if $\mathrm{comp}M$ (the complement of M) is negatively invariant.
(iv) If a set M is invariant, then so is ∂M. If ∂M is invariant, then so are $\overline{M}$ and $\mathrm{int}M$.

We now provide a geometric condition on ∂M which will guarantee the invariance of a set M and, in particular will aid us in finding invariant sets.

We have the following theorem, which provides a relationship between the vector field f and the set M (usually called the subtangent condition) in order that invariance holds.

5 **Theorem** *Let $M \subset D$ be a closed set. Then M is positively invariant with respect to the flow u determined by f if and only if for every $v \in M$*

$$\liminf_{t \to 0+} \frac{\mathrm{dist}(v + tf(v), M)}{t} = 0. \tag{4}$$

PROOF. Let $v \in M$, then

$$u(t,v) = v + tf(v) + o(t),\ t > 0.$$

Hence, if M is positively invariant

$$\mathrm{dist}(v + tf(v), M) \le |u(t,v) - v + tf(v)| = o(t),$$

proving the necessity of (4).

Next, let $v \in M$, and assume (4) holds. Choose a compact neighborhood B of v such that $B \subset D$ and choose $\epsilon > 0$ so that $u([0,\epsilon],v) \subset B$. Let $w(t) = \mathrm{dist}(u(t,v), M)$, then for each $t \in [0,\epsilon]$ there exists $v_t \in M$ such that $w(t) = |u(t,v) - v_t)|$ and $\lim_{t\to 0+} v_t = v$. It follows that

$$\begin{aligned} w(t+s) \quad &= |u(t+s,v) - v_{t+s}| \\ &\le w(t) + |u(t+s,v) - u(t,v) - sf(u(t,v))| \\ &+ s|f(u(t,v)) - f(v_t)| + |v_t + sf(v_t) - v_{t+s}| \\ &\le w(t) + sLw(t) + \mathrm{dist}(v_t + tf(v_t), M) \end{aligned}$$

since f satisfies a local Lipschitz condition. Hence

$$D_+ w(t) \le Lw(t),\ 0 \le t < \epsilon,\ w(0) = 0,$$

or

$$D_+ \left(e^{-Lt} w(t)\right) \le 0,\ 0 \le t < \epsilon,\ w(0) = 0,$$

which implies $w(t) \le 0,\ \ 0 \le t < \epsilon$, i.e. $w(t) \equiv 0$. Completing the proof. □

We remark here that condition (4) only needs to be checked for points $v \in \partial M$ since it obviously holds for interior points.

We next consider some special cases where the set M is given as a smooth manifold. We consider the case where we have a function $\phi \in C^1(D,\mathbb{R})$ which is such that every value $v \in \phi^{-1}(0)$ is a regular value, i.e. $\nabla\phi(v) \ne 0$. Let $M = \phi^{-1}(-\infty, 0]$, then $\partial M = \phi^{-1}(0)$. We have the following theorem.

6 Theorem *Let M be given as above, then M is positively invariant with respect to the flow determined by f if and only if*

$$\nabla\phi(v) \cdot f(v) \le 0,\ \forall v \in \partial M = \phi^{-1}(0). \tag{5}$$

We leave the proof as an exercise. We remark that the set M given in the previous theorem will be negatively invariant provided the reverse inequality holds and hence invariant if and only if

$$\nabla\phi(v) \cdot f(v) = 0,\ \forall v \in \partial M = \phi^{-1}(0),$$

in which case $\phi(u(t,v)) \equiv 0,\ \forall v \in \partial M$, i.e. ϕ is a first integral for the differential equation.

4 Limit Sets

In this section we shall consider semiorbits and study their limit sets.

Let $\gamma^+(v),\ v \in D$ be the positive semiorbit associated with $v \in D$. We define the set $?^+(v)$ as follows:

$$?^+(v) = \{w : \exists\{t_n\},\ t_n \to t_v^+,\ u(t_n, v) \to w\} \tag{6}$$

(i.e. $?^+(v)$ is the set of limit points of $\gamma^+(v)$) and we call it the positive limit set of v. We have the following proposition:

7 Proposition *(i)* $\overline{\gamma^+(v)} = \gamma^+(v) \cup ?^+(v)$.

(ii) $?^+(v) = \cap_{w \in \gamma^+(v)} \overline{\gamma^+(w)}.$

(iii) If $\gamma^+(v)$ is bounded, then $?^+(v) \neq \emptyset$ and compact.

(iv) If $?^+(v) \neq \emptyset$ and bounded, then

$$\lim_{t \to t_v^+} \operatorname{dist}\big(u(t,v), ?^+(v)\big) = 0.$$

(v) $?^+(v) \cap D$ is an invariant set.

8 Theorem *If $\gamma^+(v)$ is contained in a compact subset $K \subset D$, then $?^+(v) \neq \emptyset$ is a compact connected set, i.e a continuum.*

Proof. We already know (cf. Proposition 7) that $?^+(v)$ is a compact set. Thus we need to show it is connected. Suppose it is not. Then there exist nonempty disjoint compact sets M and N such that

$$?^+(v) = M \cup N.$$

Let $\delta = \inf\{|v - w| : v \in M,\ w \in N\} > 0$. Since $M \subset ?^+(v)$ and $N \subset ?^+(v)$, there exist values of t arbitrarily large (note $t_v^+ = \infty$ in this case) such that $\operatorname{dist}(u(t,v), M) < \frac{\delta}{2}$ and values of t arbitrarily large such that $\operatorname{dist}(u(t,v), N) < \frac{\delta}{2}$ and hence there exists a sequence $\{t_n \to \infty\}$ such that $\operatorname{dist}(u(t_n,v), M) = \frac{\delta}{2}$. The sequence $\{u(t_n,v)\}$ must have a convergent subsequence and hence has a limit point which is in neither M nor N, a contradiction. □

4.1 LaSalle's theorem

In this section we shall return again to invariant sets and consider Lyapunov type functions and their use in determining invariant sets.

Thus let $\phi : D \to \mathbb{R}$ be a C^1 function. We shall denote by $\phi'(v) = \nabla\phi(v) \cdot f(v)$.

9 Lemma *Assume that $\phi'(v) \leq 0,\ \forall v \in D$. Then for all $v \in D$ ϕ is constant on the set $?^+(v) \cap D$.*

10 Theorem *Let there exist a compact set $K \subset D$ such that $\phi'(v) \le 0,\ \forall v \in K$. Let*

$$\tilde{K} = \{v \in K : \phi'(v) = 0\}$$

and let M be the largest invariant set contained in $\tilde{K}$. Then for all $v \in D$ such that $\gamma^+(v) \subset K$

$$\lim_{t\to\infty} \operatorname{dist}(u(t,v), M) = 0.$$

PROOF. Let $v \in D$ such that $\gamma^+(v) \subset K$, then, using the previous lemma, we have that ϕ is constant on ?$^+(v)$, which is an invariant set and hence contained in M. $\square$

11 Theorem (Lasalle's Theorem) *Assume that $D = \mathbb{R}^N$ and let $\phi'(x) \le 0,\ \forall x \in \mathbb{R}^N$. Furthermore suppose that ϕ is bounded below and that $\phi(x) \to \infty$ as $|x| \to \infty$. Let $E = \{v : \phi'(v) = 0\}$, then*

$$\lim_{t\to\infty} \operatorname{dist}(u(t,v), M) = 0,$$

for all $v \in \mathbb{R}^N$, where M is the largest invariant set contained in E.

As an example to illustrate the last result consider the oscillator

$$mx'' + hx' + kx = 0,$$

where m, h, k are positive constants. This equation may be written as

$$\begin{aligned} x' &= y \\ y' &= -\tfrac{k}{m}x - \tfrac{h}{m}y. \end{aligned}$$

We choose

$$\phi(x,y) = \frac{1}{2}\left(my^2 + kx^2\right)$$

and obtain that

$$\phi'(x,y) = -hy^2.$$

Hence $E = \{(x,y) : y = 0\}$. The largest invariant set contained in E is the origin, hence all solution orbits tend to the origin.

5 Two Dimensional Systems

In this section we analyze limit sets for two dimensional systems in somewhat more detail and prove a classical theorem (the Theorem of Poincaré-Bendixson) about periodic orbits of such systems. Thus we shall assume throughout this section that $N = 2$.

Let $v \in D$ be a regular (i.e. not critical) point of f. We call a compact straight line segment $l \subset D$ through v a transversal through v, provided it contains only regular points.

We shall need the following observation.

12 Lemma *Let $v \in D$ be a regular point of f. Then there exists a transversal l containing v in its relative interior. Furthermore, the direction of l may be chosen any direction not parallel to $f(v)$. Also every orbit associated with f crosses l in the same direction.*

PROOF. Let v be a regular point of f. Choose a neighborhood V of v consisting of regualr points only. Let $\eta \in \mathbb{R}^2$ be any direction not parallel to $f(v)$, i.e. $\eta \times f(v) \neq 0$, (here $\times$ is the cross product in $\mathbb{R}^3$). We may restrict V further such that $\eta \times f(w) \neq 0 \ \forall w \in V$. We then may take l to be the intersection of the straight line through v with direction η and V. The remaining part of the proof is left as an exercise. □

13 Lemma *Let v be an interior point of some transversal l. Then for every $\epsilon > 0$ there exists a circular disc D_ϵ with center at v such that for every $v_1 \in D_\epsilon$, $u(t, v_1)$ will cross l in time t, $|t| < \epsilon$.*

PROOF. Let $v \in \text{int}\, l$ and let

$$l = \{w : w = v + s\eta, \ s_0 \leq s \leq s_1\}.$$

Let B be a disc centered at v containing only regular points of f. Let $L(t,v) = au^1(t,v) + bu^2(t,v) + c$, where $u(t,v)$ is the solution with initial condition v and $au^1 + u^2 + c = 0$ is the equation of l. Then $L(0,v) = 0$, and $\frac{\partial L}{\partial t}(0,v) = (a,b) \cdot f(v) \neq 0$. We hence may apply the implicit function theorem to complete the proof. □

14 Lemma *Let l be a transversal and let $? = \{w = u(t,v) : a \leq t \leq b\}$ be a closed arc of an orbit u associated with f which has the property that $u(t_1, v) \neq u(t_2, v)$, $a \leq t_1 < t_2 \leq b$. Then if ? intersects l it does so at a finite number of points whose order on ? is the same as the order on l. If the orbit is periodic it intersects l at most once.*

The proof relies on the Jordan curve theorem and is left as an exercise.

15 Lemma *Let $\gamma^+(v)$ be a semiorbit which does not intersect itself and let $w \in ?^+(v)$ be a regular point of f. Then any transversal containing w in its interior contains no other points of $?^+(v)$ in its interior.*

16 Lemma *Let $\gamma^+(v)$ be a semiorbit which does not intersect itself which is contained in a compact set $K \subset D$ and let all points in $?^+(v)$ be regular points of f. Then $?^+(v)$ contains a periodic orbit.*

17 Theorem (Poincaré-Bendixson) *Assume the hypotheses of Lemma 16. Then $?^+(v)$ is the orbit of a periodic solution u_T with smallest positive period T.*

18 Theorem *Let ? be a closed orbit of (1) which together with its interior is contained in a compact set $K \subset D$. Then there exists at least one singular point of f in the interior of D.*

6 Exercises

1. Prove Lemma 1. Also provide conditions in order that the mapping u be smooth, say of class C^1.

2. Let u be the flow determined by f (see Lemma 1). Show that if $I_{u_0} = (t^-_{u_0}, t^+_{u_0})$, then $-t^-_{u_0} : D \to [0, \infty]$ and $t^+_{u_0} : D \to [0, \infty]$ are lower semicontinuous functions of u_0.

3. Prove Proposition 2.

4. Prove Proposition 3 and Corollary 4.

5. Prove Theorem 6.

6. Consider the three dimensional system of equations (the Lorenz system)

$$\begin{aligned} x' &= -\sigma x + \sigma y \\ y' &= rx - y - xz \\ z' &= -bz + xy, \end{aligned}$$

   where σ, r, b are positive parameters.

   Find a family of ellipsoids which are positively invariant sets for the flow determined by the system.

7. Prove Theorem 6.

8. Extend Theorem 6 to the case where

$$M = \cap_{i=1}^m v_i^{-1}(-\infty, 0].$$

   Consider the special case where each ϕ_i is affine linear, i.e M is a parallelepiped.

9. Prove Proposition 7.

10. Prove Lemma 9.

11. Prove Theorem 17.

12. Prove Theorem 18.

13. Complete the proof of Lemma 12.

14. Prove Lemma 14.

15. Let u be a solution whose interval of existence is $\mathbb{R}$ which is not periodic and let $\gamma^+(u(0))$ and $\gamma^-(u(0))$ be contained in a compact set $K \subset D$. Prove that $?^+(u(0))$ and $?^-(u(0))$ are distinct periodic orbits provided they contain regular points only.

Chapter X
Hopf Bifurcation

1 Introduction

This chapter is devoted to a version of the classical Hopf bifurcation theorem which establishes the existence of nontrivial periodic orbits of autonomous systems of differential equations which depend upon a parameter and for which the stability properties of the trivial solution changes as the parameter is varied. The proof we give is base on the method of Lyapunov-Schmidt presented in Chapter II.

2 A Hopf Bifurcation Theorem

Let

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,$$

be a C^2 mapping which is such that

$$f(0, \alpha) = 0, \text{ all } \alpha \in \mathbb{R}.$$

We consider the system of ordinary differential equations depending on a parameter α

$$\frac{du}{dt} + f(u, \alpha) = 0, \tag{1}$$

and prove a theorem about the existence of nontrivial periodic solutions of this system. Results of the type proved here are referred to as *Hopf bifurcation* theorems.

We establish the following theorem.

1 Theorem *Assume that f satisfies the following conditions:*

1. *For some given value of $\alpha = \alpha_0$, $i = \sqrt{-1}$ and $-i$ are eigenvalues of $f_u(0, \alpha_0)$ and $\pm ni$, $n = 0, 2, 3, \cdots$, are not eigenvalues of $f_u(0, \alpha_0)$;*

2. in a neighborhood of α_0 there is a curve of eigenvalues and eigenvectors

$$\begin{array}{l} f_u(0,\alpha)a(\alpha) = \beta(\alpha)a(\alpha) \\ a(\alpha_0) \neq 0,\ \beta(\alpha_0) = i,\ \mathrm{Re}\frac{d\beta}{d\alpha}|_{\alpha_0} \neq 0. \end{array} \tag{2}$$

Then there exist postive numbers ϵ and η and a C^1 function

$$(u, \rho, \alpha) : (-\eta, \eta) \to C^1_{2\pi} \times \mathbb{R} \times \mathbb{R},$$

where $C^1_{2\pi}$ is the space of 2π periodic C^1 $\mathbb{R}^n-$ valued functions, such that $(u(s), \rho(s), \alpha(s))$ solves the equation

$$\frac{du}{d\tau} + \rho f(u, \alpha) = 0, \tag{3}$$

nontrivially, i.e. $u(s) \neq 0,\ s \neq 0$ and

$$\rho(0) = 1,\ \alpha(0) = \alpha_0,\ u(0) = 0. \tag{4}$$

Furthermore, if (u_1, α_1) is a nontrivial solution of (1) of period $2\pi\rho_1$, with $|\rho_1 - 1| < \epsilon$, $|\alpha_1 - \alpha_0| < \epsilon$, $|u_1(t)| < \epsilon$, $t \in [0, 2\pi\rho_1]$, then there exists $s \in (-\eta, \eta)$ such that $\rho_1 = \rho(s)$, $\alpha_1 = \alpha(s)$ and $u_1(\rho_1 t) = u(s)(\tau + \theta)$, $\tau = \rho_1 t \in [0, 2\pi]$, $\theta \in [0, 2\pi)$.

Proof. We note that $u(t)$ will be a solution of (1) of period $2\pi\rho$, whenever $u(\tau)$, $\tau = \rho t$ is a solution of period 2π of (3). We thus let $X = C^1_{2\pi}$ and $Y = C_{2\pi}$ be Banach spaces of C^1, respectively, continuous $2\pi-$ periodic functions endowed with the usual norms and define an operator

$$\begin{array}{rl} F : X \times \mathbb{R} \times \mathbb{R} & \to Y \\ (u, \rho, \alpha) & \mapsto u' + \rho f(u, \alpha),\ \ ' = \frac{d}{d\tau}. \end{array} \tag{5}$$

Then F belongs to class C^2 and we seek nontrivial solutions of the equation

$$F(u, \rho, \alpha) = 0, \tag{6}$$

with values of ρ close to 1, α close to α_0, and $u \neq 0$.

We note that

$$F(0, \rho, \alpha) = 0, \quad \text{for all } \rho \in \mathbb{R},\ \alpha \in \mathbb{R}.$$

Thus the claim is that the value $(1, \alpha_0)$ of the two dimensional parameter (ρ, α) is a bifurcation value. Theorem II.1 tells us that

$$u' + f_u(0, \alpha_0)u$$

cannot be a linear homeomorphism of X onto Y. This is guaranteed by the assumptions, since the functions

$$\phi_0 = \mathrm{Re}(e^{i\tau}a(\alpha_0)), \quad \phi_1 = \mathrm{Im}(e^{i\tau}a(\alpha_0))$$

are $2\pi-$ periodic solutions of

$$u' + f_u(0, \alpha_0)u = 0, \tag{7}$$

and they span the the kernel of $F_u(0,1,\alpha_0)$,

$$\ker F_u(0,1,\alpha_0) = \{\phi_0, \phi_1 = -\phi_0'\}.$$

It follows from the theory of linear differential equations that the image, $\mathrm{im} F_u(0,1,\alpha_0)$, is closed in Y and that

$$\mathrm{im} F_u(0,1,\alpha_0) = \{f \in Y : \langle f, \psi_i \rangle = 0, i = 0,1\},$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product and $\{\psi_0, \psi_1\}$ forms a basis for $\ker\{-u' + f_u^T(0,\alpha_0)u\}$, the differential equation adjoint operator of $u' + f_u(0,\alpha_0)u$. In fact $\psi_1 = \psi_0'$ and $\langle \phi_i, \psi_j \rangle = \delta_{ij}$, the Kronecker delta. Thus $F_u(0,1,\alpha_0)$ is a linear Fredholm mapping from X to Y having a two dimensional kernel as well as a two dimensional cokernel. We now write, as in the beginning of Chapter II,

$$\begin{aligned} X &= V \oplus W \\ Y &= Z \oplus T. \end{aligned}$$

We let U be a neighborhood of $(0,1,\alpha_0,0)$ in $V \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and define G on U as follows

$$G(v,\rho,\alpha,s) = \begin{cases} \frac{1}{s} F(s(\phi_0 + v), \rho, \alpha), & s \neq 0 \\ F_u(0,\rho,\alpha)(\phi_0 + v), & s = 0. \end{cases}$$

We now want to solve the equation

$$G(v,\rho,\alpha,s) = 0,$$

for v, ρ, α in terms of s in a neighborhood of $0 \in \mathbb{R}$. We note that G is C^1 and

$$G(v,\rho,\alpha,0) = (\phi_0 + v)' + \rho f_u(0,\alpha)(\phi_0 + v).$$

Hence

$$G(0,\rho,\alpha,0) = \phi_0' + \rho f_u(0,\alpha)\phi_0.$$

Thus, in order to be able to apply the implicit function theorem, we need to differentiate the map

$$(v,\rho,\alpha) \mapsto G(v,\rho,\alpha,s)$$

evaluate the result at $(0,1,\alpha_0,0)$ and show that this derivative is a linear homeomorphism of $V \times \mathbb{R} \times \mathbb{R}$ onto Y.

Computing the Taylor expansion, we obtain

$$\begin{aligned} G(v,\rho,\alpha,s) &= G(0,1,\alpha_0,s) + G_\rho(0,1,\alpha_0,s)(\rho - 1) \\ &\quad G_\alpha(0,1,\alpha_0,s)(\alpha - \alpha_0) + G_v(0,1,\alpha_0,s)v + \cdots, \end{aligned} \tag{8}$$

and evaluating at $s = 0$ we get

$$\begin{aligned} G_{v,\rho,\alpha}(0,1,\alpha_0,0)(v,\rho - 1,\alpha - \alpha_0) &= f_v(0,\alpha_0)\phi_0(\rho - 1) \\ &\quad + f_{v\alpha}(0,\alpha_0)\phi_0(\alpha - \alpha_0) \\ &\quad + (v' + f_v(0,\alpha_0)v). \end{aligned} \tag{9}$$

Note that the mapping

$$v \mapsto v' + f_v(0, \alpha_0)v$$

is a linear homeomorphism of V onto T. Thus we need to show that the map

$$(\rho, \alpha) \mapsto f_v(0, \alpha_0)\phi_0(\rho - 1) + f_{v\alpha}(0, \alpha_0)\phi_0(\alpha - \alpha_0)$$

only belongs to T if $\rho = 1$ and $\alpha = \alpha_0$ and for all $\psi \in Z$ there exists a unique (ρ, α) such that

$$f_v(0, \alpha_0)\phi_0(\rho - 1) + f_{v\alpha}(0, \alpha_0)\phi_0(\alpha - \alpha_0) = \psi.$$

By the characterization of T, we have that

$$f_v(0, \alpha_0)\phi_0(\rho - 1) + f_{v\alpha}(0, \alpha_0)\phi_0(\alpha - \alpha_0) \in T$$

if and only if

$$\begin{array}{l} < f_v(0, \alpha_0)\phi_0, \psi_i > (\rho - 1) + < f_{v\alpha}(0, \alpha_0)\phi_0, \psi_i > (\alpha - \alpha_0) = 0, \\ i = 1, 2. \end{array} \tag{10}$$

Since

$$f_{v\alpha}(0, \alpha_0)\phi_0 = -\phi_0' = \phi_1,$$

we may write equation (10) as two equations in the two unknowns $\rho - 1$ and $\alpha - \alpha_0$,

$$\begin{array}{rcl} < f_{v\alpha}(0, \alpha_0)\phi_0, \psi_0 > (\alpha - \alpha_0) & = & 0 \\ (\rho - 1) + < f_{v\alpha}(0, \alpha_0)\phi_0, \psi_1 > (\alpha - \alpha_0) & = & 0, \end{array} \tag{11}$$

which has only the trivial solution if and only if

$$< f_{v\alpha}(0, \alpha_0)\phi_0, \psi_0 > \neq 0.$$

Computing this latter expression, one obtains

$$< f_{v\alpha}(0, \alpha_0)\phi_0, \psi_0 > = \mathrm{Re}\beta'(0),$$

which by hypothesis is not zero. The uniqueness assertion we leave as an exercise. □

For much further discussion of Hopf bifurcation we refer to [12].

The following example of the classical *Van der Pol* oscillator from nonlinear electrical circuit theory (see [12]) will serve to illustrate the applicability of the theorem.

2 Example *Consider the nonlinear oscillator*

$$x'' + x - \alpha(1 - x^2)x' = 0. \tag{12}$$

This equation has for for certain small values of α nontrivial periodic solutions with periods close to 2π.

We first transform (12) into a system by setting

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

and obtain

$$u' + \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ u_1^2 u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

We hence obtain that

$$f(u,\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ u_1^2 u_2 \end{pmatrix}$$

and

$$f_u(0,\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix},$$

whose eigenvalues satisfy

$$\beta(\alpha+\beta)+1=0.$$

Letting $\alpha_0 = 0$, we get $\beta(0) = \pm i$, and computing $\frac{d\beta}{d\alpha} = \beta'$ we obtain $2\beta\beta' + \beta'\alpha + \beta = 0$, or $\beta' = \frac{-\beta}{\alpha + 2\beta} = -\frac{1}{2}$, for $\alpha = 0$. Thus by Theorem II.1 there exists $\eta > 0$ and continuous functions $\alpha(s),\ \rho(s),\ s \in (-\eta,\eta)$ such that $\alpha(0)=0,\ \rho(0)=1$ and (12) has for $s \neq 0$ a nontrivial solution $x(s)$ with period $2\pi\rho(s)$. This solution is unique up to phase shift.

Chapter XI
Sturm-Liouville Boundary Value Problems

1 Introduction

In this chapter we shall study a very classical problem in the theory of ordinary differential equations, namely linear second order differential equations which are parameter dependent and are subject to boundary conditions. While the existence of *eigenvalues* (parameter values for which nontrivial solutions exist) and *eigenfunctions* (corresponding nontrivial solutions) follows easily from the abstract Riesz spectral theory for compact linear operators, it is instructive to deduce the same conclusions using some of the results we have developed up to now for ordinary differential equations. While the theory presented below is for some rather specific cases, much more general problems and various other cases may be considered and similar theorems may be established. We refer to the books [4], [5] and [14] where the subject is studied in some more detail.

2 Linear Boundary Value Problems

Let $I = [a, b]$ be a compact interval and let $p, q, r \in C(I, \mathbb{R})$, with p, r positive on I. Consider the linear differential equation

$$(p(t)x')' + (\lambda r(t) + q(t))x = 0, \ t \in I, \tag{1}$$

where λ is a complex parameter.

We seek parameter values (*eigenvalues*) for which (1) has nontrivial solutions (*eigensolutions or eigenfunctions*) when it is subject to the set of boundary conditions

$$\begin{aligned} x(a)\cos\alpha - p(a)x'(a)\sin\alpha = 0 \\ x(b)\cos\beta - p(b)x'(b)\sin\beta = 0, \end{aligned} \tag{2}$$

where α and β are given constants and (without loss in generality, $0 \le \alpha < \pi$, $0 < \beta \le \pi$). Such a boundary value problem is called a *Sturm-Liouville* boundary value problem.

We note that (2) is equivalent to the requirement

$$\begin{aligned} c_1x(a) + c_2x'(a) = 0,\ |c_1| + |c_2| \neq 0 \\ c_3x(b) + c_4x'(b) = 0,\ |c_3| + |c_4| \neq 0. \end{aligned} \tag{3}$$

We have the following lemma.

1 Lemma *Every eigenvalue of equation (1) subject to the boundary conditions (2) is real.*

Proof. Let the differential operator L be defined by

$$L(x) = (px')' + qx.$$

Then, if λ is an eigenvalue

$$L(x) = -\lambda rx,$$

for some nontrivial x which satisfies the boundary conditions. Hence also

$$L(\bar{x}) = -\overline{\lambda rx} = -\bar{\lambda}r\bar{x}.$$

Therefore

$$\bar{x}L(x) - xL(\bar{x}) = -(\lambda - \bar{\lambda})rx\bar{x}.$$

Hence

$$\int_a^b (\bar{x}L(x) - xL(\bar{x}))\,dt = -(\lambda - \bar{\lambda}) \int_a^b rx\bar{x}dt.$$

Integrating the latter expression and using the fact that both x and $\bar{x}$ satisfy the boundary conditions we obtain the value 0 for this expression and hence $\lambda = \bar{\lambda}$. □

We next let $u(t, \lambda) = u(t)$ be the solution of (1) which satisfies the (initial) conditions

$$u(a) = \sin\alpha,\ p(a)u'(a) = \cos\alpha,$$

then $u \neq 0$ and satisfies the first set of boundary conditions. We introduce the following transformation (*Prüfer transformation*)

$$\rho = \sqrt{u^2 + p^2(u')^2},\ \phi = \arctan\frac{u}{pu'}.$$

Then ρ and ϕ are solutions of the differential equations

$$\rho' = -\left[(\lambda r + q) - \frac{1}{p}\right]\rho\sin\phi\cos\phi \tag{4}$$

and

$$\phi' = \frac{1}{p}\cos^2\phi + (\lambda r + q)\sin^2\phi. \tag{5}$$

Further $\phi(a) = \alpha$. (Note that the second equation depends upon ϕ only, hence, once ϕ is known, ρ may be determined by integrating a linear equation and hence u is determined.

We have the following lemma describing the dependence of ϕ upon λ.

2 Lemma *Let ϕ be the solution of (5) such that $\phi(a) = \alpha$. Then ϕ satisfies the following conditions:*

1. *$\phi(b, \lambda)$ is a continuous strictly increasing function of λ;*
2. $\lim_{\lambda\to\infty} \phi(b, \lambda) = \infty$;
3. $\lim_{\lambda\to-\infty} \phi(b, \lambda) = 0$.

Proof. The first part follows immediately from the discussion in Sections V.5 and V.6. To prove the other parts of the lemma, we find it convenient to make the change of independent variable

$$s = \int_a^t \frac{d\tau}{p(\tau)},$$

which transforms equation (1) to

$$x'' + p(\lambda r + q)x = 0, \ ' = \frac{d}{ds}. \tag{6}$$

We now apply the Prüfer transformation to (6) and use the comparison theorems in Section V.6 to deduce the remaining parts of the lemma. □

Using the above lemma we obtain the existence of eigenvalues, namely we have the following theorem.

3 Theorem *The boundary value problem (1)-(2) has an unbounded infinite sequence of eigenvalues*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

with

$$\lim_{n\to\infty} \lambda_n = \infty.$$

The eigenspace associated with each eigenvalue is one dimensional and the eigenfunctions associated with λ_k have precisely k simple zeros in (a, b).

Proof. The equation

$$\beta + k\pi = \phi(b, \lambda)$$

has a unique solution λ_k, for $k = 0, 1, \cdots$. This set $\{\lambda_k\}_{k=0}^{\infty}$ has the desired properties. □

We also have the following lemma.

4 Lemma *Let $u_i,\ i=j,k,\ j\neq k$ be eigenfunctions of the boundary value problem (1)-(2) corresponding to the eigenvalues λ_j and λ_k. Then u_j and u_k are orthogonal with respect to the weight function r, i.e.*

$$\langle u_j,u_k\rangle=\int_a^b ru_ju_k=0. \tag{7}$$

In what is to follow we denote by $\{u_i\}_{i=0}^{\infty}$ the set of eigenfunctions whose existence is guaranteed by Theorem 3 with u_i an eigenfunction corresponding to $\lambda_i,\ i=0,1,\cdots$ which has been normalized so that

$$\int_a^b ru_i^2=1. \tag{8}$$

We also consider the nonhomogeneous boundary value problem

$$(p(t)x')'+(\lambda r(t)+q(t))x=rh,\ t\in I, \tag{9}$$

where $h\in L^2(a,b)$ is a given function, the equation being subject to the boundary conditions (2) and solutions being interpreted in the Carathéodory sense.

We have the following result.

5 Lemma *For $\lambda=\lambda_k$ equation (9) has a solution subject to the boundary conditions (2) if and only if*

$$\int_a^b ru_kh=0.$$

If this is the case, and w is a particular solution of (9)-(2), then any other solution has the form $w+cu_k$, where c is an arbitrary constant.

Proof. Let v be a solution of $(p(t)x')'+(\lambda_kr(t)+q(t))x=0$, which is linearly independent of u_k, then

$$(u_kv'-u_k'v)=\frac{c}{p},$$

where c is a nonzero constant. One verifies that

$$w(t)=\frac{1}{c}\left(v(t)\int_a^t ru_kh+u_k\int_t^b rvh\right)$$

is a solution of (9)-(2) (for $\lambda=\lambda_k$), whenever $\int_a^b ru_kh=0$ holds. $\square$

3 Completeness of Eigenfunctions

We note that it follows from Lemma 5 that (9)-(2) has a solution for every $\lambda_k,\ k=0,1,2,\cdots$ if and only if $\int_a^b ru_kh=0$, for $k=0,1,2,\cdots$. Hence, since $\{u_i\}_{i=0}^{\infty}$ forms an orthonormal system for the Hilbert space $L_r^2(a,b)$ (i.e. $L^2(a,b)$ with weight function r defining the inner product), $\{u_i\}_{i=0}^{\infty}$ will be a complete orthonormal system, once we can show that $\int_a^b u_k h = 0$, for $k = 0, 1, 2, \cdots$, implies that $h = 0$ (see [23]). The aim of this section is to prove completeness.

The following lemma will be needed in this discussion.

6 Lemma *If $\lambda \neq \lambda_k,\ k = 0, 1, \cdots$ (9) has a solution subject to the boundary conditions (2) for every $h \in L^2(a, b)$.*

PROOF. For $\lambda \neq \lambda_k,\ k = 0, 1, \cdots$ we let u be a nontrivial solution of (1) which satisfies the first boundary condition of (2) and let v be a nontrivial solution of (1) which satisfies the second boundary condition of (2). Then

$$uv' - u'v = \frac{c}{p}$$

with c a nonzero constant. Define the *Green's* function

$$G(t, s) = \frac{1}{c}\begin{cases} v(t)u(s), & a \leq s \leq t \\ v(s)u(t), & t \leq s \leq b. \end{cases} \tag{10}$$

Then

$$w(t) = \int_a^b G(t, s)r(s)h(s)ds$$

is the unique solution of (9) - (2). □

We have the following corollary.

7 Corollary *Lemma 6 defines a continuous mapping*

$$G : L^2(a, b) \to C^1[a, b], \tag{11}$$

by

$$h \mapsto G(h) = w.$$

Further

$$\langle Gh, w\rangle = \langle h, Gw\rangle.$$

PROOF. We merely need to examine the definition of $G(t, s)$ as given by equation (10). □

Let us now let

$$S = \{w \in L^2(a, b) : \langle u_i, h\rangle = 0,\ i = 0, 1, 2, \cdots\}. \tag{12}$$

Using the definition of G we obtain the lemma.

8 Lemma $\quad G : S \to S.$

We note that S is a linear manifold in $L^2(a, b)$ which is weakly closed, i.e. if $\{x_n\} \subset S$ is a sequence sucht that

$$\langle x_n, h\rangle \to \langle x, h\rangle,\ \forall h \in L^2(a, b),$$

then $x \in S$.

9 Lemma *If $S \neq \{0\}$, then there exists $x \in S$ such that*

$$\langle G(x), x\rangle \neq 0.$$

PROOF. If $\langle G(x), x\rangle = 0$ for all $x \in S$, then, since S is a linear manifold, we have for all $x, y \in S$ and $\alpha \in \mathbb{R}$

$$\begin{aligned} 0 &= \langle G(x+\alpha y), x+\alpha y\rangle \\ &= 2\alpha\langle G(y), x\rangle, \end{aligned}$$

in particular, choosing $x = G(y)$ we obtain a contradiction, since for $y \neq 0$ $G(y) \neq 0$. □

10 Lemma *If $S \neq \{0\}$, then there exists $x \in S\backslash\{0\}$ and $\mu \neq 0$ such that*

$$G(x) = \mu x.$$

PROOF. Since there exists $x \in S$ such that $\langle G(x), x\rangle \neq 0$ we set

$$\mu = \begin{cases} \inf\{\langle G(x), x\rangle : x \in S,\ \|x\| = 1,\ \text{if } \langle G(x), x\rangle \leq 0,\ \forall x \in S\} \\ \sup\{\langle G(x), x\rangle : x \in S,\ \|x\| = 1,\ \text{if } \langle G(x), x\rangle \leq 0, \text{for some } x \in S\}. \end{cases}$$

We easily see that there exists $x_0 \in S$, $\|x_0\| = 1$ such that $\langle G(x_0), x_0\rangle = \mu \neq 0$. If S is one dimensional, then $G(x_0) = \mu x_0$. If S has dimension greater than 1, then there exists $0 \neq y \in S$ such that $\langle y, x_0\rangle = 0$. Letting $z = \frac{x_0+\epsilon y}{\sqrt{1+\epsilon^2}}$ we find that $\langle G(z), z\rangle$ has an extremum at $\epsilon = 0$ and thus obtain that $\langle G(x_0), y\rangle = 0$, for any $y \in S$ with $\langle y, x_0\rangle = 0$. Hence since $\langle G(x_0) - \mu x_0, x_0\rangle = 0$ it follows that $\langle G(x_0), G(x_0) - \mu x_0\rangle = 0$ and thus $\langle G(x_0) - \mu x_0, G(x_0) - \mu x_0\rangle = 0$, proving that μ is an eigenvalue. □

Combining the above results we obtain the following completeness theorem.

11 Theorem *The set of eigenfunctions $\{u_i\}_{i=0}^{\infty}$ forms acomplete orthonormal system for the Hilbert space $L_r^2(a, b)$.*

PROOF. Following the above reasoning, we merely need to show that $S = \{0\}$. If this is not the case, we obtain, by Lemma 10, a nonzero element $h \in S$ and a nonzero number μ such that $G(h) = \mu h$. On the other hand $w = G(h)$ satisfies the boundary conditions (2) and solves (9); hence h satisfies the boundary conditions and solves the equation

$$(p(t)h')' + (\lambda r(t) + q(t))h = \frac{r}{\mu}h,\ t \in I, \tag{13}$$

i.e. $\lambda - \frac{1}{\mu} = \lambda_j$ for some j. Hence $h = cu_j$ for some nonzero constant c, contradicting that $h \in S$. □

4 Exercises

1. Find the set of eigenvalues and eigenfunctions for the boundary value problem

$$x'' + \lambda x = 0$$
$$x(0) = 0 = x'(0).$$

2. Supply the details for the proof of Lemma 2.

3. Prove Lemma 4.

4. Prove that the Green's function given by (10) is continuous on the square $[a,b]^2$ and that $\frac{\partial G(t,s)}{\partial t}$ is continuous for $t \neq s$. Discuss the behavior of this derivative as $t \to s$.

5. Provide the details of the proof of Corollary 7. Also prove that $G : L^2(a,b) \to L^2(a,b)$ is a compact mapping.

6. Let $G(t,s)$ be defined by equation (10). Show that

$$G(t,s) = \sum_{i=0}^{\infty} \frac{u_i(t)u_i(s)}{\lambda - \lambda_i},$$

where the convergence is in the L^2 norm.

7. Replace the boundary conditions (2) by the periodic boundary conditions

$$x(a) = x(b),\ x'(a) = x'(b).$$

Prove that the existence and completeness part of the above theory may be established provided the functions satisfy $p(a) = p(b),\ q(a) = q(b),\ r(a) = r(b)$.

8. Apply the previous exercise to the problem

$$x'' + \lambda x = 0,$$
$$x(0) = x(2\pi)$$
$$x'(0) = x'(2\pi).$$

9. Let the differential operator L be given by

$$L(x) = (tx')' + \frac{m^2}{t}x,\ 0 < t < 1.$$

and consider the eigenvalue problem

$$L(x) = -\lambda t x.$$

In this case the hypotheses imposed earlier are not applicable and other types of boundary conditions than those given by (3) must be sought in order that a development parrallel to that given in Section 2 may be made. Establish such a theory for this *singular* problem. Extend this to more general singular problems.

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